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ON THE THEORY OF BETA-RADIOACTIVITY II

A THEORETICAL DISCUSSION OF THE POLARIZATION OF ELECTRON BEAMS AND ITS OBSERVATION *)

by H. A. TOLHOEK and S. R. DE GROOT

Instituut voor theoretische natuurkunde, Universiteit, Utrecht, Nederland

Synopsis

A number of properties of polarized electron beams are investigated in view of an application to polarized beta-rays emitted from nuclei with aligned spins. The state of polarization of electron beams, polarized or unpolarized, can be characterized by a density matrix ϱ with two rows and colums

$$\varrho = \begin{Vmatrix} \varrho_{11} & \varrho_{12} \\ \varrho_{21} & \varrho_{22} \end{Vmatrix},$$

for a certain pair of fundamental (orthogonal) states (ψ_1, ψ_2), which serve to characterize the spin orientation. As *orientation coefficient* with respect to (ψ_1, ψ_2) we define $P(\psi_1, \psi_2) = \varrho_{11} - \varrho_{22}$; the *degree of polarization* is defined as $P = |P(\psi_1, \psi_2)|$ if ϱ is in diagonal form for the basis (ψ_1, ψ_2).

It is proved that scattering experiments can give an observation of $P(\psi_1, \psi_2)$ for certain pairs of fundamental states (ψ_1, ψ_2).

In a single-scattering experiment of an entirely polarized beam we give the intensity ratio of two beams in opposite directions, obtained after scattering over a right angle by $(1 + a)/(1 - a)$. The intensity ratio of the final beams (in opposite directions) in a double-scattering experiment of an unpolarized beam is written as $(1 + \delta)/(1 - \delta)$. It is shown that we have the relation $a^2 = \delta$.

Further it is found that in order to determine completely the polarization of a beam the determination of three independent orientation coefficients is necessary. The polarizations of light and of electron beams have been compared.

§ 1. *Introduction.* In this paper we give a discussion of some phenomena concerning the polarization of electron beams. In the following paper [2]) we shall consider the behaviour of polarized electron beams

*) Formulae from the first paper of this series [1]) will be quoted as, say, I (33).

in electric and magnetic fields. These two papers serve as a basis for the treatment of the phenomenon of polarization of β-radiation emitted by nuclei with aligned spins which will be discussed in the fourth paper of this series [3]), cf. also [4]). These polarization phenomena of β-rays may become important since the experimental methods for alignment of nuclear spins are now being developed [5]).

With the term *polarization* of β-rays we point to the fact that the spin orientation of an electron gives properties to electron waves, which are to a certain extent similar to the polarization of light. That polarization of electron beams could exist was clear since the discovery of the electron spin. Theoretical considerations on possibilities for direct experimental detection of polarization phenomena have been given since 1928 [6]), [7]), [8]), [9]), [10]) (reviews in [11]), [12]), and [13])). It was, however, not before 1942 that satisfactory experimental results gave a confirmation of the theoretical considerations [14]), [15]), [16]). At present the production and detection of polarized electron beams by scattering by thin foils is well established. It is, however, the only method that has as yet been experimentally successful for this purpose.

The alignment of β-radioactive nuclei provides in principle a new means of obtaining polarized electron beams [3]), [4]). The direction of the spin of the emitted electrons may have any orientation with respect to the momentum of the electron, depending on the direction of emission. The spin direction may e.g. be the same as the direction of the electron momentum or may be perpendicular to it, these cases corresponding to different types of polarization.

In this paper and the following, we try to find whether and how the types of polarization mentioned in the foregoing paragraph can be studied experimentally with the aid of scattering experiments.

§ 2. *The fundamental notions concerning the polarization of light and of electron beams.* In this section a number of fundamental notions on polarization will be developed to have a basis for further considerations. We shall be concerned only with the polarization aspect of the waves, so that we have to compare only plane waves with the same momentum $\mathbf{p}$ (and hence the same energy E). In this case of a definite momentum the polarization degree of freedom is still arbitrary and a wave function for light or electrons can be written as

$$\psi = c_1\psi_1 + c_2\psi_2, \tag{1}$$

in which ψ_1 and ψ_2 are two orthogonal wave functions for the case of of *light*, e.g., *a*) two plane polarized waves with perpendicular polarization planes or, *b*) two right and left circularly polarized waves; for the case of *electrons*, e.g., *a*) two opposite spin orientations perpendicular to the momentum **p** or, *b*) two opposite spin directions parallel and antiparallel to **p**.

A wave function with arbitrary polarization can be written down in the form (1). If we have, however, an unpolarized beam we cannot represent this beam by (1): such a beam has to be considered as an "ensemble" of wave functions (1) not as a single wave function. In quantummechanics the appropriate means to describe such an "ensemble" is a *density matrix* or *statistical operator* ϱ (cf., e.g., [17]), [18]) or [19])). In our case of two fundamental states ϱ has the shape

$$\| \varrho_{rs} \| = \begin{Vmatrix} \varrho_{11} & \varrho_{12} \\ \varrho_{21} & \varrho_{22} \end{Vmatrix}. \tag{2}$$

ϱ is hermitian, semi-definite and normalized in such a way that

$$\varrho_{11} + \varrho_{22} = 1. \tag{3}$$

Depending on the values of the matrix elements, the ensemble described by ϱ may represent a totally polarized or a partially polarized or an unpolarized beam.

The *special case of a totally polarized beam* (1) could be described as well by a single wave function; ϱ becomes for this special case (pure state)

$$\varrho = \begin{Vmatrix} |c_1|^2 & c_1c_2^* \\ c_1^*c_2 & |c_2|^2 \end{Vmatrix}. \tag{4}$$

By a unitary transformation this ϱ can be brought into the simple shape

$$\varrho = \begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix}. \tag{5}$$

In the *general case of an arbitrary beam* it is not possible to attain this simple shape; we have a quantummechanical "mixture". ϱ can always be brought into diagonal form (with e.g. $\varrho' \geqslant \varrho''$)

$$\varrho = \begin{Vmatrix} \varrho' & 0 \\ 0 & \varrho'' \end{Vmatrix}. \tag{6}$$

If we pass from one representation with ψ_1 and ψ_2 to another with ψ_1' and ψ_2' as fundamental states, it follows immediately from the properties of unitary transformations that if ϱ has diagonal form with $\varrho' = \varrho''$ for two fundamental states ψ_1 and ψ_2, the same is true for any other pair of fundamental states ψ_1' and ψ_2'. In this case we speak by definition of an *unpolarized beam.*

In the general case ($\varrho' \geqslant \varrho''$) we define the *degree of polarization* as

$$P = \varrho' - \varrho''. \tag{7}$$

Hence P is real and $0 \leqslant P \leqslant 1$. If $0 < P < 1$, we speak of a partially polarized beam, if $P = 0$ of an unpolarized beam, if $P = 1$ of a totally polarized beam.

Characterization of a beam. For an arbitrary beam we can write ϱ in the shape

$$\varrho = \begin{Vmatrix} \varrho' & 0 \\ 0 & \varrho'' \end{Vmatrix} = \varrho'' \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} + (\varrho' - \varrho'') \begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix}, \tag{8}$$

which means that the beam can be regarded as a superposition of an unpolarized beam and a totally polarized beam. P determines the degree in which unpolarized and totally polarized light are mixed. Two parameters describe the polarization of the totally polarized component, as this component, like a pure wave function (1), can be described by a complex ratio c_2/c_1 (cf. (4)). Hence 3 *parameters* are needed to describe completely the state of polarization of an arbitrary beam.

The definition of P is independent of the choice of the fundamental states. We also introduce a quantity which does depend on this choice: we define as the *orientation coefficient* $P(\psi_1, \psi_2)$ with respect to two fundamental states ψ_1 and ψ_2, the real number

$$P(\psi_1, \psi_2) = \varrho_{11} - \varrho_{22}, \tag{9}$$

if ϱ is given by (2).

Remarks: It is easy to prove that we have a totally polarized beam ($P = 1$) if $P(\psi_1, \psi_2) = 1$ or -1; if $P(\psi_1, \psi_2) = 0$ we have an unpolarized beam ($P = 0$) only, if ϱ were in the diagonal form. If $-1 < P(\psi_1, \psi_2) < 1$ it is still quite possible that we have a totally polarized beam. If ϱ is in diagonal form for the fundamental states ψ_1 and ψ_2, we have

$$P = | P(\psi_1, \psi_2) |. \tag{10}$$

Characterization of totally polarized beams. These can be described by a single wave function according to (1). The polarization of light can be plane, circular or elliptic, as is well-known. The states of polarization of an electron wave can be specified by the direction of the electron spin. We must, however, define more precisely what we mean by the "spin direction". By this term we shall understand the direction of the spin angular momentum of the electron in the coordinate system in which the electron is at rest. For the solution I (12) of the Dirac equation I (8) this direction is given by [6], [7])

$$B/A = \{\mathrm{tg}\,(\chi/2)\} \exp{(i\omega)}. \tag{11}$$

The angles (χ, ω) in this formula are defined as follows: if $(\zeta_x, \zeta_y, \zeta_z)$ are the components of the unit vector $\boldsymbol{\zeta}$ in the direction of the spin angular momentum we have

$$\zeta_x = \sin\chi \cos\omega, \;\; \zeta_y = \sin\chi \sin\omega, \;\; \zeta_z = \cos\chi. \tag{12}$$

We give the direction of the spin angular momentum, not of the magnetic moment as in [7]). These directions are opposite for negatons and are the same for positons. The notation (11) will be the most adequate, in our case, where we have to deal both with negatons and positons.

The directions of the spin angular momentum are given in the coordinate system in which the electron is at rest, not in the laboratory system; this has the following advantages:

a) For the calculation of (χ, ω) according to (11) only the ψ_3 and ψ_4 components of a Dirac wave function are needed.

b) It can be proved (cf. the following paper) that the direction (11) remains unchanged if an electron is accelerated by electric fields; this would not be the case for two other possibilities of defining the "spin direction", namely as the direction of the spin angular momentum or the magnetic moment of the electron in the laboratory system.

c) For an unpolarized beam the directions according to (11) point isotropically to all directions, which is not true for the two other possibilities mentioned under *b*) which could have been taken for the spin direction.

The formula (11) means that we take the direction of the vector $\psi^*[\frac{1}{2}(1 - \beta)\boldsymbol{\sigma}]\,\psi$, instead of $\psi^*\boldsymbol{\sigma}\psi$ for the spin angular momentum or $\psi^*(-\beta\boldsymbol{\sigma})\,\psi$ for the magnetic moment in the laboratory system (omitting some constant factors); the latter two vectors are parts of 4-dimensional relativistic covariants: an axial vector and a tensor. $\psi^*[\frac{1}{2}(1 - \beta)\,\boldsymbol{\sigma}]\,\psi$ behaves as a vector for rotations but is not a part of a relativistic covariant. Hence the operators for infinitesimal rotations operating on $\begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix}$ are given as in other cases by

$$-(i/2)\sigma_x,\ -(i/2)\sigma_y,\ -(i/2)\sigma_z. \tag{13}$$

In the subsequent sections we shall also use the notion of direction of the spin in a certain point, even if we have wave functions, that are not entirely plane waves, but if the wave function can be approximated locally by such plane waves. We shall speak of *transverse polarization* of an electron beam if the direction of the spin is perpendicular to the momentum; of *longitudinal polarization* if the direction of the spin is parallel or antiparallel to the momentum.

We may remark here that the characterization of the polarization of an electron beam is a problem equivalent to the description of the spin orientations of an ensemble of particles of spin $\frac{1}{2}$, for we can consider the beam in a coordinate system in which the particles are at rest (we supposed a uniform momentum). In this coordinate system we have for an unpolarized beam an ensemble that is invariant for rotations of space [20]).

It is characteristic for particles with spin $\frac{1}{2}$ that a partially polarized beam can be characterized by the degree of polarization and the description of the totally polarized wave. For particles of higher spin, the density matrix can still be brought into the diagonal form, but it can no longer be regarded in every case as the superposition of an unpolarized and a completely polarized beam.

We will now compare the observation of the polarization of light and of electrons. The familiar method to study the *polarization of light* is with a Nicol prism, in which the beam of light is separated into two parts that are linearly polarized in two perpendicular planes, of which one is absorbed and the other observed. We shall consider the case that both components are observed, which is simpler from a theoretical point of view: if ψ_1 and ψ_2 are the wave functions of the two linearly polarized waves, which are not divided into two parts by the double-refracting crystal but which are transmitted as a whole, the relative intensities I_1 and I_2 which are measured, are related with the matrix elements of (2) by

$$I_1/I_2 = \varrho_{11}/\varrho_{22}. \tag{14}$$

Hence:
$$P(\psi_1, \psi_2) = (I_1 - I_2)/(I_1 + I_2), \tag{15}$$

and we can say that a measurement of I_1/I_2 gives a measurement of an orientation coefficient of the beam.

We can consider an operator Ω with ψ_1 and ψ_2 as eigenfunctions for the eigenvalues 1 and —1. $P(\psi_1, \psi_2)$ is then the measured mean value of Ω for the beam described by ϱ. Of course a measurement of $P(\psi_1, \psi_2)$ is not yet sufficient to describe the state of polarization of a beam, even if we determine $P(\psi_1, \psi_2)$ for any two perpendicular states of linear polarization. For if we have, e.g., circularly polarized light, $P(\psi_1, \psi_2) = 0$ for any two states of linear polarization. The experimental means to determine the state of polarization completely, consists in inserting plates of double-refracting crystals before the analyzer, which are in such a position that they transform states of elliptic or circular polarizarion into states of linear polarization. Is is easily checked that the density matrix (2) can be determined in this way by three determinations of $P(\psi_1, \psi_2)$. For (ψ_1, ψ_2) we can take for example: *a*) two states of linear polarization, with perpendicular planes of polarization, *b*) two other states of linear polarization making angles of $\pi/4$ with the planes of polarization in *a*), *c*) the two states of left and right circularly polarized light.

Another possible effect of crystal plates consists in rotating the polarization plane through a certain angle for any linearly polarized beam (e.g., quartz plates, cut perpendicular to the optical axis). With such a plate the analyzer could remain in the same position for *a*) and *b*), but this is of course, experimentally, no simplification.

We shall show that the problem of determination of the *polarization of electron beams* is to a high extent analogous to the determination of the polarization of a light beam from a theoretical point of view.

In the case of electrons we can determine the density matrix (2) by three determinations of $P(\psi_1, \psi_2)$ with for ψ_1 and ψ_2, e.g., *a*) two states of transverse polarization with opposite spin direction, *b*) two other states of transverse polarization turned over an angle of $\pi/2$ relative to *a*), *c*) the two opposite states of longitudinal polarization (parallel and antiparallel).

If the results of the measurements of the orientation coefficients are respectively t, t' and l and if the pairs of fundamental states in the three cases are called (ψ_1, ψ_2), (ψ_1', ψ_2') and (ψ_1'', ψ_2''), we can choose the ψ's in such a way that

$$\left.\begin{aligned} \psi_1' &= \tfrac{1}{2}\sqrt{2}\,(\psi_1 + \psi_2), \\ \psi_2' &= \tfrac{1}{2}\sqrt{2}\,(\psi_1 - \psi_2), \end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned} \psi_1'' &= \tfrac{1}{2}\sqrt{2}\,(\psi_1 + i\psi_2), \\ \psi_2'' &= \tfrac{1}{2}\sqrt{2}\,(\psi_1 - i\psi_2). \end{aligned}\right\} \tag{17}$$

By a straightforward calculation we obtain the following formula for the orientation coefficient $P(\tilde{\psi}_1, \tilde{\psi}_2)$, where $\tilde{\psi}_1$ is the state for which the electron spin has the direction $\boldsymbol{\zeta}$, according to (12) and $\tilde{\psi}_2$ the state with the opposite spin direction

$$P(\tilde{\psi}_1, \tilde{\psi}_2) = \varrho_{11}\zeta_z + \varrho_{12}(\zeta_x - i\zeta_y) + \varrho_{21}(\zeta_x + i\zeta_y) - \varrho_{22}\zeta_z. \quad (18)$$

The general shape (2) has been assumed for ϱ. With the aid of this formula the elements of ϱ can easily be expressed in terms of t, t' and l; we obtain for ϱ in the (ψ_1, ψ_2) representation

$$\|\varrho_{rs}\| = \tfrac{1}{2} \left\| \begin{matrix} 1 + t & t' + il \\ t' - il & 1 - t \end{matrix} \right\|. \quad (19)$$

If we consider the cases *a*), *b*) and *c*) for light in the order in which they were mentioned above the states can again be given by (ψ_1, ψ_2) (ψ_1', ψ_2') and (ψ_1'', ψ_2'') according to (16) and (17) and ϱ is expressed by the same formula (19) in terms of the orientation coefficients t, t' and l.

Of course the experimental technique for determination of $P(\psi_1, \psi_2)$ is entirely different for light and electrons. In the next section it will be shown that $P(\psi_1, \psi_2)$ can be measured for transverse states of polarization with the aid of a scattering experiment. The observation of $P(\psi_1, \psi_2)$ for states of longitudinal polarization will be discussed in the following paper [2]).

§ 3. *Scattering experiments as observation of the orientation coefficient.* We give in this section a discussion of some aspects of single electron scattering starting from the considerations of M o t t [7]), who calculated scattering of electrons in a Coulomb field. We call the intensities of the incident and scattered wave I and S respectively. The complete wave function is given by (ϑ and φ are the polar coordinates of the direction of scattering; cf. fig. 1)

$$\psi_\lambda = a_\lambda I + S u_\lambda(\vartheta, \varphi). \quad (20)$$

We put $a_3 = A$, $a_4 = B$ (in accordance with I (12)) and give the wave functions for the two orthogonal cases [7])

$$a)\ A = 1, B = 0: \left.\begin{aligned} \psi_3 &\sim I + S f(\vartheta) , \\ \psi_4 &\sim \qquad S\, g(\vartheta) \exp(i\varphi), \end{aligned}\right\} \quad (21)$$

$$b)\ A = 0, B = 1: \left.\begin{aligned} \psi_3 &\sim \quad - S\, g(\vartheta) \exp(-i\varphi), \\ \psi_4 &\sim I + S\, f(\vartheta). \end{aligned}\right\} \quad (22)$$

These expressions are valid for scattering, according to the Dirac equation, at a spherically symmetrical potential, not necessarily a Coulomb potential.

From this it follows that for the general case (arbitrary A and B)

$$\left.\begin{aligned} u_3(\vartheta, \varphi) &= Af - Bg \exp(-i\varphi), \\ u_4(\vartheta, \varphi) &= Bf + Ag \exp(i\varphi). \end{aligned}\right\} \quad (23)$$

If we put the scattered intensity in the solid angle $\sin\vartheta \, d\vartheta \, d\varphi$ proportional to

$$P(\vartheta, \varphi) \sin\vartheta \, d\vartheta \, d\varphi, \quad (24)$$

it is found that

$$P(\vartheta, \varphi) = \bar{P}(\vartheta) - D(\vartheta) \sin\chi \sin(\omega - \varphi), \quad (25)$$

with

$$\bar{P}(\vartheta) = |f|^2 + |g|^2, \quad (26)$$

$$D(\vartheta) = i(fg^* - f^*g), \quad (27)$$

and χ and ω according to (11).

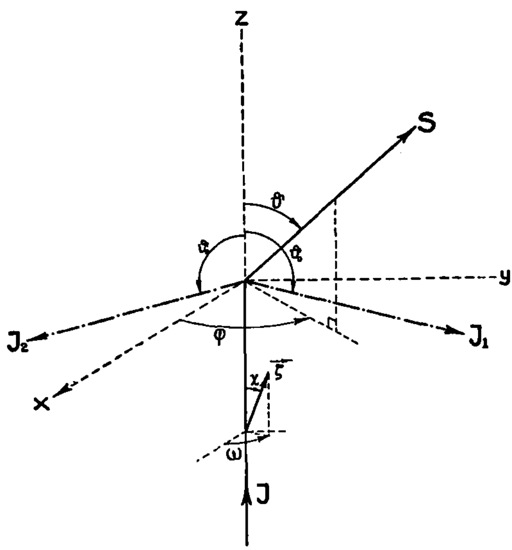

Fig. 1. Scattering of an electron beam. I direction of incident wave. (ϑ, φ) determine the direction of scattering S. (χ, ω) give the spin direction, according to (11).

Starting with these formule, we shall prove:

I. Scattering experiments can be used as observations of the orientation coefficient with respect to certain fundamental states.

II. By scattering a transverse state of polarization is not altered, if the spin orientation is perpendicular to the plane through the directions of incidence and scattering.

We now give the proofs of I and II.

I. It must be investigated to which extent a measurement of the intensity ratio I_1/I_2 can serve as an observation of an orientation coefficient. Take the situation of fig. 1; I_1 and I_2 are the intensities scattered in the directions ($\vartheta = \vartheta_0$, $\varphi = \pi/2$) and ($\vartheta = \vartheta_0$, $\varphi = -\pi/2$) respectively. As fundamental states ψ_+ and ψ_- will be used: waves with momentum along the z-axis and spin along the positive and negative x-axis respectively; we normalize according to $|A|^2 + |B|^2 = 1$ (cf. I (12)), so that according to (11)

$$\left.\begin{aligned} \psi_+ &: \ \chi = \pi/2, \quad \omega = 0; \quad A = B = \tfrac{1}{2}\sqrt{2}, \\ \psi_- &: \ \chi = \pi/2, \quad \omega = \pi; \quad A = -B = \tfrac{1}{2}\sqrt{2}. \end{aligned}\right\} \tag{28}$$

According to (25), we have

$$\left.\begin{aligned} I_1/I_2 &= (1 + a)/(1 - a) \quad \text{for } \psi_+, \\ I_1/I_2 &= (1 - a)/(1 + a) \quad \text{for } \psi_-, \end{aligned}\right\} \tag{29}$$

with $a = D(\vartheta_0)/\bar{P}(\vartheta_0)$. (30)

If we have an incident beam given by a wave function ψ, and if it is observed that for this beam

$$I_1/I_2 = (1 + a\varepsilon)/(1 - a\varepsilon) \quad (|\varepsilon| \leq 1), \tag{31}$$

we shall prove that we can write

$$\psi = \alpha\psi_+ + \beta\psi_- \tag{32}$$

with

$$(|\alpha|^2 - |\beta|^2)/(|\alpha|^2 + |\beta|^2) = \varepsilon. \tag{33}$$

(If ψ_+, ψ_- and ψ are normalized, we have $|\alpha|^2 + |\beta|^2 = 1$).

To give the proof, we conclude first from (25) and the definition of I_1 and I_2 that for ψ

$$I_1/I_2 = (1 + a \sin\chi \cos\omega)/(1 - a \sin\chi \cos\omega). \tag{34}$$

For the wave function ψ according to (32) the constants A and B have the values (cf. (28))

$$\left.\begin{aligned} A &= \tfrac{1}{2}\sqrt{2}\,(\alpha + \beta), \\ B &= \tfrac{1}{2}\sqrt{2}\,(\alpha - \beta). \end{aligned}\right\} \tag{35}$$

According to (11) and (35), we have for ψ

$$(\alpha - \beta)/(\alpha + \beta) = \{\mathrm{tg}\,(\chi/2)\} \exp(i\omega). \tag{36}$$

From (36) we can obtain $(|\alpha|^2 - |\beta|^2)/(|\alpha|^2 + |\beta|^2)$ according to an elementary calculation, which gives

$$(|\alpha|^2 - |\beta|^2)/(|\alpha|^2 + |\beta|^2) = \sin\chi\cos\omega. \tag{37}$$

Hence we get, comparing (31), (34) and (37), the formula (33) as a result

$$\varepsilon = \sin\chi\cos\omega = (|\alpha|^2 - |\beta|^2)/(|\alpha|^2 + |\beta|^2). \tag{38}$$

In this way I is proved for the case that the beam can be described as a pure state. It is easy to generalize the result for the case that we have a mixture. Suppose the mixture is described by (2) if we have the fundamental states ψ_+ and ψ_-. ϱ can always be brought into the form (6). Say the fundamental states are ψ' and ψ'' for this case with

$$\left.\begin{aligned} \psi' &= \alpha'\psi_+ + \beta'\psi_-, \\ \psi'' &= \alpha''\psi_+ + \beta''\psi_-. \end{aligned}\right\} \tag{39}$$

If ϱ^{I} and ϱ^{II} are the density matrices describing ψ' and ψ'', we have in the representation with ψ' and ψ'' as fundamental states

$$\varrho = \begin{Vmatrix} \varrho' & 0 \\ 0 & \varrho'' \end{Vmatrix} = \varrho' \begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix} + \varrho'' \begin{Vmatrix} 0 & 0 \\ 0 & 1 \end{Vmatrix} = \varrho'\varrho^{\mathrm{I}} + \varrho''\varrho^{\mathrm{II}}; \tag{40}$$

using the (ψ_+, ψ_-) representation we have

$$\varrho = \begin{Vmatrix} \varrho_{11} & \varrho_{12} \\ \varrho_{21} & \varrho_{22} \end{Vmatrix} = \varrho' \begin{Vmatrix} |\alpha'|^2 & \alpha'\beta'^* \\ \alpha'^*\beta' & |\beta'|^2 \end{Vmatrix} + \varrho'' \begin{Vmatrix} |\alpha''|^2 & \alpha''\beta''^* \\ \alpha''^*\beta'' & |\beta''|^2 \end{Vmatrix} \tag{41}$$

hence

$$\left.\begin{aligned} \varrho_{11} &= \varrho'|\alpha'|^2 + \varrho''|\alpha''|^2, \\ \varrho_{22} &= \varrho'|\beta'|^2 + \varrho''|\beta''|^2. \end{aligned}\right\} \tag{42}$$

If the wave functions are normalized, the result ε of a measurement of I_1/I_2 gives for the beams described by ϱ^{I} and ϱ^{II} according to (38) respectively

$$\left.\begin{aligned} \varepsilon' &= |\alpha'|^2 - |\beta'|^2, \\ \varepsilon'' &= |\alpha''|^2 - |\beta''|^2. \end{aligned}\right\} \tag{43}$$

The result ε for the beam described by ϱ is, according to the general theory of the density matrix

$$\begin{aligned} \varepsilon = \varrho'\varepsilon' + \varrho''\varepsilon'' &= [\varrho'|\alpha'|^2 + \varrho''|\alpha''|^2] - [\varrho'|\beta'|^2 + \varrho''|\beta''|^2] = \\ &= \varrho_{11} - \varrho_{22} = P(\psi_+, \psi_-). \end{aligned} \tag{44}$$

Hence I is proved for the general case.

II. If we consider transverse states of polarization, we have $\chi = \pi/2$, hence, according to (11)

$$|A/B| = 1, \; B = A \exp(i\omega). \tag{45}$$

According to (23) u_3 and u_4 then have the expressions

$$\left.\begin{aligned} u_3(\vartheta, \varphi) &= Af - Ag \exp[i(\omega - \varphi)], \\ u_4(\vartheta, \varphi) &= Af \exp(i\omega) + Ag \exp(i\varphi) = \\ &= \{Af + Ag \exp[i(\varphi - \omega)]\} \exp(i\omega). \end{aligned}\right\} \tag{46}$$

If we take $\varphi - \omega = \pm \pi/2$, (46) is reduced to

$$\left.\begin{aligned} u_3(\vartheta, \varphi) &= A(f \pm ig), \\ u_4(\vartheta, \varphi) &= A(f \pm ig) \exp(i\omega). \end{aligned}\right\} \tag{47}$$

Hence if $(\bar{\chi}, \bar{\omega})$ determines the polarization of the electrons scattered in the direction (ϑ, φ) we have

$$\exp(i\bar{\omega}) \operatorname{tg}(\bar{\chi}/2) = \exp(i\omega), \tag{48}$$

hence

$$\bar{\chi} = \pi/2 = \chi; \; \bar{\omega} = \omega, \tag{49}$$

which is the mathematical expression of II.

Roughly speaking it can be said that I and II express that the parts ψ_+ and ψ_- of $\psi = \alpha\psi_+ + \beta\psi_-$ *are scattered independently* if we consider scattering directions in the yz-plane, a property which for an arbitrary pair of fundamental states is entirely false.

§ 4. *The relation between single and double-scattering experiments.* We shall consider in this section the relation between:

a) the asymmetry produced by a single-scattering of a transversely polarized electron beam,

b) the asymmetry in a double-scattering experiment of an unpolarized beam.

For simplification we use everywhere as scattering angle $\pi/2$. We use the properties of § 3 and consider double-scattering for transversely polarized beams in the situations of fig. 2*a*) and *b*). We take the points Q, R, S, T, U and V in the same plane. The beams are scattered at R and T. The intensities are measured at U and V; the intensities of the different beams are given in the figures. I, I' and I'' are constants. a is the constant from (30) for $\vartheta_0 = \pi/2$.

The relative intensities in these situations are an immediate consequence of (29) and the property II from § 3. If we start with an unpolarized beam in Q (fig. 2) it follows from property I § 3 that the

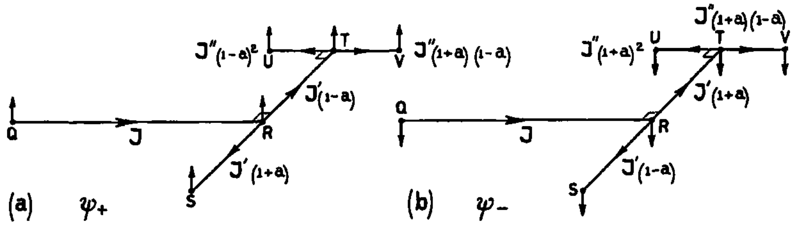

Fig. 2. Intensities in a double scattering experiment. The beams are scattered at R and T. ↑ gives the orientations of the spin for the two cases ψ_+ and ψ_-.

intensities in U and V are obtained by averaging the intensities for ψ_+ and ψ_-; hence we get for these intensities

$$\left.\begin{aligned} I_U &= \tfrac{1}{2}I''\,[(1+a)^2 + (1-a)^2] = I''\,(1+a)^2, \\ I_V &= \tfrac{1}{2}I''\,[(1+a)\,(1-a) + (1+a)\,(1-a)] = I''(1-a^2). \end{aligned}\right\} \quad (50)$$

So the intensity ratio becomes

$$I_U/I_V = (1+a^2)/(1-a^2). \quad (51)$$

M o t t [7] [8]) has introduced as a measure for the asymmetry in double-scattering experiments a quantity δ, defined by

$$I_U/I_V = (1+\delta)/(1-\delta). \quad (52)$$

Hence

$$\delta = a^2 \quad (53)$$

gives the relation between the asymmetry in the mentioned single and double-scattering experiments. While a can be calculated from (30), we have from (30) and (53)

$$\delta = [D(\vartheta_0)/\bar{P}(\vartheta_0)]^2. \quad (54)$$

The absolute value of a can be drawn from the value of δ, not its sign. An alternative deduction of (53) could have been given, without use of the property II of § 3, if we had deduced (54) directly from (21) and (22) cf. [7]), [8])).

The importance of the relation (53) consists in the fact that it could be used to determine the constant a, which is necessary if we have to measure polarization by single-scattering, from δ. The quantity δ can be determined by direct experiments. In this way a is found from experiments. Such results are probably more reliable than the theoretical values, as these may need some corrections, because the scattering field is not exactly a Coulomb field so that approxim-

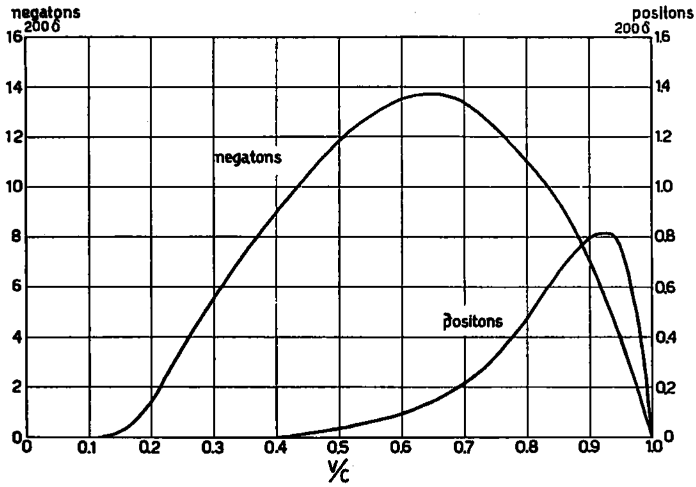

Fig. 3. The asymmetry percentage 200 δ in a double-scattering experiment as a function of v/c for negatons and positons ($Z = 80$).

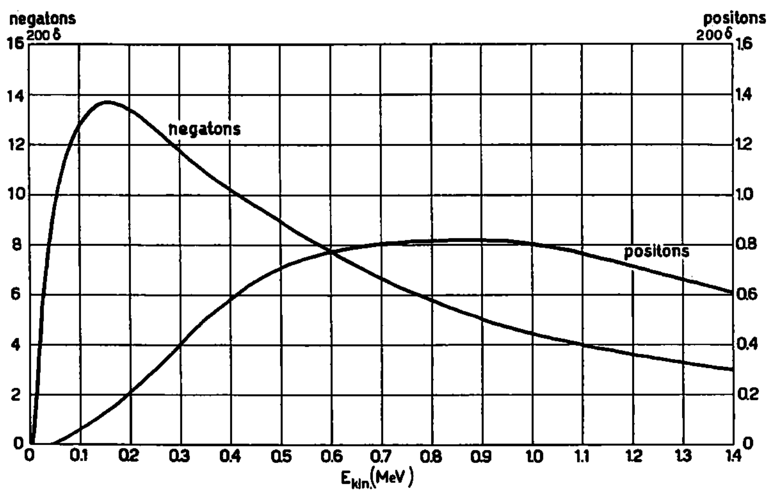

Fig. 4. The asymmetry percentage 200δ in a double-scattering experiment as a function of E_{kin} for negatons and positons ($Z = 80$).

ate methods must be used to take the influence of the atomic electrons into account. The starting point (21), (22) for our deductions, which leads to (53) is, however, valid for scattering at an arbitrary spherically symmetrical potential.

The value of δ for negaton scattering at Au ($Z = 79$) has been calculated by Mott [7]) and [8]). Later corrections have been given for the screening of the nucleus [9]). These corrections appear to be rather small. For positons δ has been calculated by Massey [10]). The calculations have been made either for Au ($Z = 79$) either for Hg ($Z = 80$). In experiments Au can be used in the shape of thin gold foils; Hg in the shape of mercury vapour. In fig. 3 we give δ as a function of v/c; in fig. 4 as a function of the kinetic energy E_{kin}. In fig. 5 we give the asymmetry a in a single-scattering experiment for a totally polarized beam as a function of E_{kin}. All curves have been drawn for $Z = 80$, using the results of Massey [10]), Bartlett and Watson [9]). The asymmetry has been expressed as the ratio of the difference of the beams that are compared and the mean value of both beams. We have, e.g.,

$$(I_U - I_V)/[\tfrac{1}{2}(I_U + I_V)] = 2\delta. \tag{55}$$

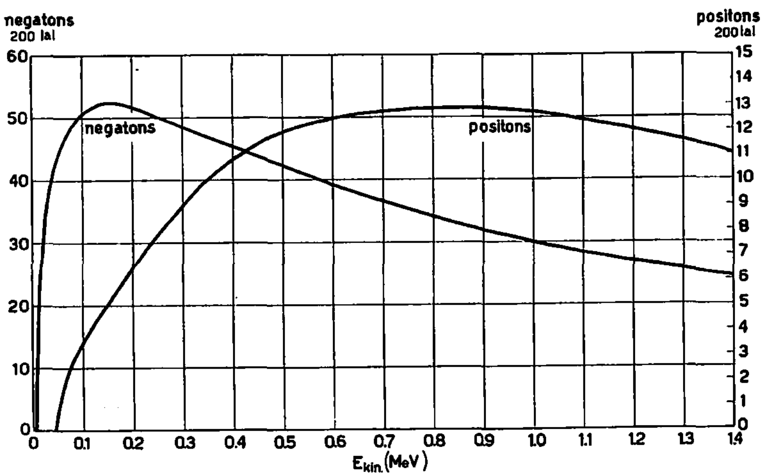

Fig. 5. The asymmetry percentage 200 $|a|$ in a single-scattering experiment of a totally polarized beam as a function of E_{kin} for negatons and positons ($Z = 80$).

It is seen that the detection of the asymmetry in a double-scattering experiment for positons must experimentally be nearly impossible; even the detection of the polarization of a positon beam by a single-scattering experiment will be difficult.

The asymmetry in a double-scattering experiment with gold foils for negatons has been confirmed experimentally [14], [15]) for 400 keV electrons. A value of 12 ± 2 was measured for 200 δ in agreement with the calculated value. For a long time experiments had been unsuccessful to detect the asymmetry especially as a result of multiple scattering (as discussed in [16]).

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