THE FOKKER-PLANCK EQUATION FOR RAY DISPERSION IN GYROTROPIC STRATIFIED MEDIA

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The Hamilton equations of geometrical optics determine the rays of the relevant wave field in the short wavelength. We give a systematic derivation of the Fokker-Planck equation for the joint probability density of the position and unit direction vector of rays propagating in a gyrotropic stratified medium with random inhomogeneities. The results are a generalisation of previous work for the case of an isotropic medium.

1. Introduction

In our previous publication on the same subject [1] it has been shown that the process of ray scattering in random media can be regarded as approximately markovian, in which the role of time is played by the path length traversed by the ray. We gave a systematic derivation of the Fokker-Planck equation (FPE) for the joint probability density of the position and the direction vector of the rays. We were concerned with the propagation in isotropic media like a troposphere or an isotropic ionosphere.

The earth's ionosphere, however, is a gyrotropic medium. In order to solve the problem in this case it is necessary to take into account the influence of the magnetic field $\boldsymbol{H}^0$ of the earth. When $\boldsymbol{H}^0 \neq 0$ the picture of ray propagation becomes much more complicated. We know only of one attempt [2] to describe the problem in a Markov approximation. But the FPE for the unit direction vector obtained in [2] was restricted to the particular case of the small angle scattering in a random gyrotropic medium without refraction.

In this paper we shall extend our preceding results to a general case of ray propagation in a gyrotropic stratified medium with random inhomogeneities.

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2. Ray equation in a gyrotropic medium

We begin our derivation with the Hamilton equations of geometrical optics [3]

$$\mathrm{d}\boldsymbol{r}(\sigma)/\mathrm{d}\sigma = \boldsymbol{S}, \qquad \mathrm{d}[n\boldsymbol{K}(\sigma)]/\mathrm{d}\sigma = \nabla_r \mu. \tag{2.1}$$

Here the independent variable σ is the path length of the ray; $\boldsymbol{r}$ defines the position of the ray; $\boldsymbol{K}$ ($|\boldsymbol{K}|^2 = 1$) is the unit direction vector perpendicular to the wave front; $\boldsymbol{S}$ ($|\boldsymbol{S}|^2 = 1$) is the unit ray vector, which coincides with the direction of the time-average Poynting vector or the group velocity vector. In the general case of wave propagation in a gyrotropic medium the directions of $\boldsymbol{K}$ and $\boldsymbol{S}$ are different, i.e. $(\boldsymbol{K}\cdot\boldsymbol{S}) = \cos\psi$. The functions $n = n(\boldsymbol{r}, \boldsymbol{S})$ and $\mu = \mu(\boldsymbol{r}, \boldsymbol{S})$ in (2.1) are the refractive indexes along the directions of $\boldsymbol{K}$ and $\boldsymbol{S}$ respectively. If $\boldsymbol{r}$ is fixed, their values are linked by the equality [3]

$$\mu(\boldsymbol{r}, \boldsymbol{S}) = n(\boldsymbol{r}, \boldsymbol{S}) \cos\psi. \tag{2.2}$$

In what follows we consider the ray propagation in a magnetoactive medium such as the ionosphere with a constant magnetic field and without absorption. Neglecting the influence of ions, we make use of the formulas [4–6]

$$n^2_{1,2} = 1 - 2v(1-v)\{2(1-v) - u(1-q^2) \pm [u^2(1-q^2) + 4u(1-v)^2q^2]^{1/2}\}^{-1}, \quad (2.3)$$

$$\cos\psi = [1 + \gamma^2(1-q^2)]^{-1/2}, \quad (2.4)$$

$$\boldsymbol{S} = [\boldsymbol{K} + \gamma \boldsymbol{l}]\cos\psi. \quad (2.5)$$

In (2.3–2.5) the following notations are introduced

$$u^{1/2} = \omega_H/\omega, \quad \omega_H = |e|H^0/mc,$$

$$v = \omega_0^2/\omega^2, \quad \omega_0^2 = 4\pi e^2 \mathcal{N}/m,$$

$$q = (\boldsymbol{K}\cdot\boldsymbol{h}) = \cos(\boldsymbol{K},\boldsymbol{h}), \quad \boldsymbol{h} = \boldsymbol{H}^0/H^0,$$

$$\boldsymbol{l} = q\boldsymbol{K} - \boldsymbol{h}, \quad \gamma = n^{-1}\partial n/\partial q,$$

where ω is the frequency of the wave field in question; ω_H is the cyclotron frequency for the electrons; ω_0 is the plasma frequency; e and m are the charge and the mass of the electrons, c is the velocity of light in vacuum; $\mathcal{N} = \mathcal{N}(\boldsymbol{r})$ is the electron concentration, whose fluctuations are responsible for the appearance of random inhomogeneities in the medium. The two indices and two signs for the square root in (2.3) correspond to the "ordinary" and "extraordinary" waves. We shall drop these indices later on and shall obtain the general result, which will be valid for both waves.

Substituting (2.5) into the first equation (2.1) and transforming the second one with the use of the identity $\boldsymbol{S}\nabla_r n = \boldsymbol{K}\nabla_r \mu$, we rewrite our initial equations in the more convenient form

$$\mathrm{d}\boldsymbol{r}(\sigma)/\mathrm{d}\sigma = [\boldsymbol{K} + \gamma\boldsymbol{l}]\cos\psi \equiv \boldsymbol{f}[v(\boldsymbol{r}),\boldsymbol{K}],$$

$$\mathrm{d}\boldsymbol{K}(\sigma)/\mathrm{d}\sigma = n^{-1}[\nabla_r\mu - \boldsymbol{K}(\boldsymbol{K}\nabla_r\mu)] \equiv \boldsymbol{g}[v(\boldsymbol{r}),\boldsymbol{K}]. \quad (2.6)$$

3. Derivation of the FPE

We restrict ourselves to the case of ray propagation in a horizontally stratified medium

$$\mathcal{N}(\boldsymbol{r}) = \mathcal{N}_0(z) + \alpha\mathcal{N}_1(\boldsymbol{r}), \quad \langle\mathcal{N}_1(\boldsymbol{r})\rangle = 0 \quad (3.1)$$

with the sure part of the electron concentration $\mathcal{N}_0$, which depends only on the height z, and zero average of random inhomogeneities $\mathcal{N}_1$. Here α is the small dimensionless parameter determining the size of the fluctuations of the process under consideration. According to (3.1) and (2.2–2.4) we obtain

$$v(\boldsymbol{r}) = v_0(z) + \alpha v_1(\boldsymbol{r}),$$

$$v_1(\boldsymbol{r}) = 4\pi e^2\mathcal{N}_1(\boldsymbol{r})/m\omega^2, \quad \langle v_1(\boldsymbol{r})\rangle = 0,$$

$$n(\boldsymbol{r},q) = n_0(z,q) + \alpha n_1(\boldsymbol{r},q),$$

$$n_1 = \partial n/\partial v|_{v_0} v_1 = (\partial n_0/\partial v_0)v_1, \quad \langle n_1(\boldsymbol{r},q)\rangle = 0,$$

$$u(\boldsymbol{r},q) = \mu_0(z,q) + \alpha\mu_1(\boldsymbol{r},q),$$

$$\mu_1 = \partial\mu/\partial v|_{v_0} v_1 = (\partial\mu_0/\partial v_0)v_1, \quad \langle\mu_1(\boldsymbol{r},q)\rangle = 0.$$

Here and later on subscript "0" means the sure part of the function in question.

For brevity we introduce the six-vector $u = \{\boldsymbol{r},\boldsymbol{K}\}$ and divide the rhs of (2.6) into the sure part

$$F_0 = \{f_0, g_0\} = \{[\boldsymbol{K} + \gamma_0\boldsymbol{l}]\cos\psi_0,\ n_0^{-1}[\nabla_r\mu_0 - \boldsymbol{K}(\boldsymbol{K}\nabla_r\mu_0)]\}$$

and a fluctuating part

$$F_1 = \{f_1, g_1\} = \{(\partial f_0/\partial v_0)v_1, (\partial g_0/\partial v_0)v_1\}.$$

Then the system (2.6) can be rewritten in the universal form of stochastic nonlinear differential equations

$$\mathrm{d}u/\mathrm{d}\sigma = F_0(u) + \alpha F_1(u), \quad \langle F_1(u)\rangle = 0.$$

In order to derive the corresponding FPE we use the method developed in [7]. Applying a simple device for reducing the nonlinear problem to the linear case, one can obtain the stochastic Liouville equation for the probability density $P(u,\sigma) = P(\boldsymbol{r},\boldsymbol{K},\sigma)$ that, after having propagated over a distance σ, the ray has arrived in a point $\boldsymbol{r}$ with a normal vector $\boldsymbol{K}$

$$\partial P(u,\sigma)/\partial\sigma = -\nabla\{F_0(u) + \alpha F_1(u)\}P(u,\sigma). \quad (3.2)$$

Here ∇ is used for the operator that differentiates everything that comes after it with respect to u.

Following [7] we expand the rhs of (3.2) in successive powers of a new dimensionless parameter $\epsilon = \alpha r_c\nabla\mu_1/n_0 \ll 1$ (and as a consequence $\alpha\mu_1/n_0 \sim \epsilon$, $\alpha n_1/n_0 \sim \epsilon$, $\alpha n_0^{-1}\partial n_1/\partial q \sim \epsilon$), where r_c is the spatial correlation radius of the inhomogeneities. Disregarding in the expansion terms of order α^3 we obtain

$$\frac{\partial P(u,\sigma)}{\partial\sigma} = \nabla\left\{-F_0(u) + \alpha^2 \int_0^\infty \frac{d(u^{-\sigma})}{d(u)} \times \langle F_1(u)\nabla_{-\sigma}F_1(u^{-\sigma},\sigma)\rangle \frac{d(u)}{d(u^{-\sigma})}\, d\sigma\right\} P(u,\sigma). \tag{3.3}$$

The six-vector $u^{-\sigma}$ is defined for fixed σ by means of a mapping from the initial $u(\sigma = 0)$ into $u(\sigma)$, i.e. $u \to u^\sigma$ with inverse $(u^\sigma)^{-\sigma} = u$. The operator $\nabla_{-\sigma}$ denotes differentiation with respect to $u^{-\sigma}$. The first and the last terms in the integrand of (3.3) are the jacobian determinants of the mapping. To determine this mapping one should solve the unperturbed equations (2.6) for the medium without inhomogeneities. But in the presence of refraction their exact solution is possible only for some special functions $n_0(z)$ and moreover only for some particular cases of ray propagation, e.g. for the longitudinal or the transverse propagation.

To proceed further we introduce the new small parameter $n_0^{-1}(\partial\mu_0/\partial z)r_c \ll 1$, which is equivalent to two new conditions imposed on functions $n_0(z)$ according to (2.2–2.4)

$$n_0^{-1}(\partial n_0/\partial z)r_c \ll 1,$$

$$n_0^{-1}(\partial^2 n_0/\partial q\partial z)r_c \ll 1. \tag{3.4}$$

Physically this means that the change of $n_0^{-1}\partial\mu_0/\partial z$ over the correlation distance can be neglected, which is true in many applications. To put it differently, we consider the unperturbed trajectories inside the inhomogeneities as straight lines. Under the conditions (3.4) one can obtain for $\sigma \lesssim r_c$

$$u^{-\sigma} = \{r - \sigma f_0(z, K), K\},$$

$$d(u^{-\sigma})/d(u) = d(u)/d(u^{-\sigma}) = 1,$$

$$\partial/\partial r_i^{-\sigma} = \partial/\partial r_i, \quad \partial/\partial K_i^{-\sigma} = \partial/\partial K_i + \sigma(\partial f_0/\partial K_i)\partial/\partial r, \quad (i = 1, 2, 3) \tag{3.5}$$

It should be stressed that the different directions of K inside the inhomogeneities define the different unperturbed trajectories. So we have no restrictions with respect to the angle of scattering.

Bearing in mind (3.5) it is a rather straightforward matter to transform (3.3) into the usual form of the FPE (summation over repeated indices is always applied)

$$\partial P(u,\sigma)/\partial\sigma = -(\partial/\partial u_\nu)\{[F_{0\nu}(u) + C_\nu(u)]P(u,\sigma)\} + \tfrac{1}{2}(\partial^2/\partial u_\nu\partial u_\mu)\{C_{\nu\mu}(u)P(u,\sigma)\}$$

$$(\nu, \mu = 1, 2, ..., 6). \tag{3.6}$$

The coefficients in the diffusion and convection terms are

$$C_{\nu\mu}(u) = 2\int_0^\infty \langle F_{1\nu}(u)\tilde{F}_{1\mu}(u^{-\sigma},\sigma)\rangle d\sigma,$$

$$C_\nu(u) = \int_0^\infty \langle(\partial F_{1\nu}(u)/\partial u_\mu)\tilde{F}_{1\mu}(u^{-\sigma},\sigma)\rangle d\sigma, \tag{3.7}$$

where the components of the six-vectors $F_1 = \{f_1, g_1\}$ and $\tilde{F}_1 = \{\boldsymbol{\beta}_1, g_1\}$ can be defined with the use of (2.6), (2.2), (3.4) and (3.5) as

$$f_{1i} = n_0^{-1}[l_i\cos\psi_0(\partial n_1/\partial q - \gamma_0 n_1) + (K_i + \gamma_0 l_i)(\mu_1 - n_1\cos\psi_0)],$$

$$g_1 = n_0^{-1}[\nabla_r\mu_1 - K(K\nabla_r\mu_1)],$$

$$\beta_{1i} = f_{1i} + \sigma(\partial f_{0i}/\partial K)g_1, \quad (i = 1, 2, 3) \tag{3.8}$$

Here we made use of the relation $n_0(\partial\cos\psi_0/\partial v_0)v_1 = \mu_1 - n_1\cos\psi_0$ and obtained the shortest form of these functions. But according to (2.2) one can get another expression more convenient for our purpose. The explicit form of the coefficients (3.7) depends on the statistical properties of the inhomogeneities in a medium.

The basic assumption of our derivation of the FPE (3.6) is

$$\langle F_{1\nu}(u)\tilde{F}_{1\mu}(u^{-\sigma})\rangle \approx 0 \quad \text{for } \sigma > r_c$$

and similarly for higher cumulants.

4. Equation for the angular distribution function

For solving some of the practical problems it is sufficient to know the angular distribution function $\mathcal{W}(K,\sigma)$ [8] only, which can be defined by

$$\mathcal{W}(K,\sigma) = \int P(r, K, \sigma)d^3r. \tag{4.1}$$

We shall derive the equation for $\mathcal{W}(K, \sigma)$, using our main result (3.6). But it turns out that the limits of validity for this equation are different for a medium without refraction and a stratified medium.

Lets us at first concentrate on a gyrotropic medium with $\mathcal{N}_0$ = const., i.e. the average electron concentration does not depend on position and $g_0 \equiv 0$. If $\mathcal{N}_1 = \mathcal{N}_1(r)$ is a homogeneous random field, than the coefficients of the FPE (3.6) are independent of r. Integrating (3.6) over r one can reduce this equation to the equation for the angular distribution function alone, without any additional restrictions.

On the other hand in the presence of refractions we encounter the difficulty that the coefficients (3.7) have a spatial dependence even in the case of homogeneous field $\mathcal{N}_1(r)$. To proceed further we have to impose new limitations.

It is necessary to notice that so far we have no restrictions on the angle of scattering. Now we make use of the small-angle approximation, which was suggested in [9] for the vertical propagations in an isotropic medium. We assume the angle of ray scattering to be small, i.e. the small fluctuations of K inside the inhomogeneities do not change the position of the ray. This positions is defined by the unique unperturbed trajectory with given initial conditions. This permits us to consider the sure parts of the refractive indexes n_0 and μ_0 as functions of σ corresponding to the unique unperturbed trajectory and to integrate the FPE (3.6) over r. Such an approach is a generalization of the small-angle approximation [9] for the case of a stratified medium.

As consequence of this procedure we obtain the FPE for the angular distribution function (4.1)

$$\partial \mathcal{W}(K,\sigma)/\partial\sigma = -(\partial/\partial K_i) \times \{[g_{0i}(K,z) + C_{i+3}(K,z)\mathcal{W}(K,\sigma)]\} + \tfrac{1}{2}(\partial^2/\partial K_i \partial K_j)\{C_{i+3,j+3}(K,z)\mathcal{W}(K,\sigma)\},$$

$$(i,j = 1,2,3), \qquad (4.2)$$

where z is a function of σ thanks to the solution on unperturbed equations with given initial conditions. The drift and diffusion coefficients are determined by (3.7).

As an example of calculating the explicit form of the coefficients (3.7) for the FPE (4.2) we consider the simple case that $\mathcal{N}_1(r)$ is a gaussian homogeneous isomeric random field, viz.

$$\langle \mathcal{N}_1(r)\, \mathcal{N}_1(r')\rangle = \langle \mathcal{N}_1^2\rangle \exp(-|r-r'|^2/a^2), \qquad (4.3)$$

where a is the characteristic scale of the inhomogeneities, which is the same order as their spatial correlation radius r_c. Taking into account (3.7), (3.8) and (4.3), we obtain for the coefficients in the diffusion terms

$$C_{i+3,j+3}(K,z) = (2D_1/n_0^2)[(\delta_{ij} - K_iK_j) - (\gamma_0^2/p_0^2)X_iX_j\cos^2\psi_0], \qquad (i,j = 1,2,3).$$

Here we used the following notations

$$X_i = -(\delta_{ij} - K_iK_j)h_j, \qquad p_0^2 = [1 + \gamma_0^2(1-q^2)]\cos^2\psi_0,$$

$$D_1 = D_0(\partial\mu_0/\partial v_0)^2$$

and $D_0 = \sqrt{\pi}\langle v_1^2\rangle\,(ap_0)^{-1}$ plays the role of a ray diffusion coefficient [8].

The expressions for the drift coefficients (3.7) are rather unwieldy, and we do not write them down for the sake of space.

5. Conclusion

First of all we discuss the conditions for applicability of the FPE (3.6). We expanded the rhs of (3.2) in successive powers of $\epsilon = \alpha(\nabla_r\mu_1/n_0)r_c$. Here $(\alpha\nabla_r\mu_1/n_0)^{-1} = \Delta L$ is the scale on which u varies and r_c is the scale on which the random nature of the function $\mathcal{N}_1(r)$ becomes appreciable. If ϵ is small ($\epsilon \ll 1$) it is possible to subdivide the path length σ in intervals $\Delta\sigma$, such that $\Delta\sigma \gg r_c$, and yet $\alpha(\nabla_r\mu_1/n_0)\Delta\sigma \ll 1$. That is, u does not vary much during an interval $\Delta\sigma$ in which $\mathcal{N}_1(r)$ has forgotten its past. Combining these inequalities

$$\Delta L \gg \Delta\sigma \gg r_c, \qquad (5.1)$$

we come to the conclusion that on the course-grained level determined by $\Delta\sigma$ the process under consideration is (approximately) markovian. Hence the application of the FPE in a medium without refraction is justified under the condition (5.1) alone. In the presence of refraction one has to bear in mind the additional restrictions (3.4).

The condition (5.1) and the first inequality in (3.4) coincide with the analogous conditions in an isotropic medium. The second condition in (3.4) is new. It reflects that in the presence of a magnetic field the refracture index n_0 depends on the direction of the ray and it imposes the limitation on this dependence.

In the derivation of the FPE (3.6) we took into account that different values of the direction vector K corresponded to different unperturbed trajectories of rays inside the inhomogeneity. To put it otherwise, we did not ignore the deviation of the actual ray from the unperturbed one. In consequence of this we do not need the additional restriction with respect to the angle of scattering. This refutes the conclusion in [9,10] that the FPE for ray diffusion can be justified only in the small-angle approximation and has no preference over other perturbation methods. It follows from the present analysis that such statement is true only for the FPE defining the angular distribution function (4.1) in the case of ray propagation in the presence of refraction.

The FPE (3.6) describing the process of ray scattering in a gyrotropic stratified medium includes all equations considered earlier, both for a gyrotropic medium [2] and for an isotropic medium [1,8–13]. An application of the present approach to the particular cases of ray dispersion in a gyrotropic medium, such as the quasi-longitudinal or quasi-transverse propagation, will be given elsewhere.

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References

[1] S.M. Golynski and N.G. van Kampen, Phys. Lett. A (to be published).
[2] V.D. Gusev and O.K. Vlasova, Geomagnetizm i Aeronomiya 9 (1969) 828 (in russian).
[3] P. Frank and R. von Mizes, Die Differential- und Integralgleichungen, V. 2 (Druck und Verlag von Friedr. Vieweg & John Akt.-Ges., Braunschweig, 1935) (in German).
[4] K.G. Budden, Radio waves in the ionosphere (University Press, Cambridge, 1961).
[5] V.L. Ginzburg, Propagation of electromagnetic waves in plasma (Gordon and Breach, Science Publishers, Inc., New York, 1961).
[6] Ya.L. Alpert, Radio wave propagation and the conosphere (Gordon and Breach, Science Publishers, Inc., New York, 1963).
[7] N.G. van Kampen, Phys. Rep. 24C (1976) 171.
[8] L.A. Chernov, Wave propagation in a random media (McGraw-Hill Book Company, New York-Toronto-London, 1961).
[9] V.I. Klyatskin and V.I. Iatarski, Sov. Phys.-Usp. 16 (1974) 494.
[10] S.M. Rytov, Introduction to a statistical radiophysics, part I (Nauka, Moskow, 1976) (in russian).
[11] V.M. Komissarov, Izv. Vuzov (Radiofizika) 9 (1966) 292 (in russian).
[12] C.H. Liu and R.C. Yeh, IEEE Trans. Ant. and Propag. AP-16 (1968) 678.
[13] U. Frisch, in: Probabilistic methods in applied mathematics, V.1, ed. A.T. Bharucha-Reid (Academic Press, New York and London, 1968) p. 75.