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$M/G/\infty$ TANDEM QUEUES

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We consider a series of queues with Poisson input. Each queueing system contains an infinite number of service channels. The service times in each channel have a general distribution.

For this $M/G/\infty$ tandem model we obtain the joint time-dependent distribution of queue length and residual service times at each queue. This leads to an expression for the joint stationary distribution of the number of customers in various queues at the arrival epochs of a tagged customer at those queues.

$M/G/\infty$ queue * queue length * tandem queue * correlation coefficient * time-dependent analysis

1. Introduction

One of the main interests in present-day queueing literature concerns the analysis of queueing networks for which the stationary distribution of the number of customers at each queue is of a special type: the so-called product form (see e.g. Baskett e.a. [1], Cohen [2]). One of the models belonging to this class is the model of a network of $M/G/\infty$ queues: queues with independent Poisson arrival processes, an infinite number of servers at each node and general service time distributions.

Although analysis of the stationary queue length distribution in such product form networks is simple, a complete analysis of the relevant queueing characteristics is in general not easy. In particular, as is observed by Kreinin and Vainshtein [3], not much is known about the distribution of, e.g., the process $(\boldsymbol{X}_1(k), \boldsymbol{X}_2(k), \ldots, \boldsymbol{X}_n(k))$, where $\boldsymbol{X}_j(k)$ is some characteristic of the jth queue *at the arrival epoch of the k-th customer at that queue*, like waiting time, response time or queue length.

In $M/G/\infty$ networks waiting time and response time distributions are clearly trivial; but the determination of the joint distribution of the queue lengths at the various nodes at the arrival epochs of a tagged customer in those nodes presents an interesting problem. The present paper is devoted to this problem.

In order not to obscure the basic ideas we restrict ourself to the case of a tandem connection of n $M/G/\infty$ nodes with one Poisson input stream, at the first queue (note that it is justified to speak of $M/G/\infty$ nodes since the departure process of each queue again is a Poisson process). A remark at the end of the paper explains how our results can be extended to more general models. Section 2 is devoted to the case $n = 2$, and Section 3 to the case of arbitrary n. In both cases we determine the joint stationary distribution of the queue lengths at each node and the residual service times at each server. This leads to our main result: *the joint stationary distribution of the successive queue lengths a particular customer finds at his arrival epochs at two queues of the tandem connection.*

We thus extend results in [6] ($n = 2$) and [3] (n arbitrary), obtained for the special case of a tandem connection of $M/M/\infty$ queues. In [3] much use has been made of combinatorial arguments. We follow a different approach, which is more direct and which clearly expresses the meaning of the intermediate and resulting formulas.

Finally some notation: $Q_1, \ldots, Q_n$ are n queues in series with an infinite number of servers at each queue. Customers arrive at Q_1 according to a Poisson process with intensity λ. Service times at $Q_1, Q_2, \ldots, Q_n$ are independent, identically distributed stochastic variables $\boldsymbol{\tau}_i^{(1)}, \boldsymbol{\tau}_i^{(2)}, \ldots, \boldsymbol{\tau}_i^{(n)}$ with distribution $B_1(\cdot)$, $B_2(\cdot), \ldots, B_n(\cdot)$ with first moment $\beta_1, \beta_2, \ldots, \beta_n$. In the following it will be assumed that $B_i(0+) = 0$ and $\beta_i < \infty$, $i = 1, \ldots, n$.

2. Two $M/G/\infty$ queues in series

Let $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t)$ denote the queue lengths at Q_1 and Q_2 at time t and under the condition that $\boldsymbol{x}_1(t) = l_1, \boldsymbol{x}_2(t) = l_2$, let $\boldsymbol{\sigma}_1^{(1)}, \ldots, \boldsymbol{\sigma}_{l_1}^{(1)}, \boldsymbol{\sigma}_1^{(2)}, \ldots, \boldsymbol{\sigma}_{l_2}^{(2)}$ denote the residual service times of the customers in service, i.e. the still required service times. Clearly

$$(\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \boldsymbol{\sigma}_1^{(1)}, \ldots, \boldsymbol{\sigma}_{\boldsymbol{x}_1(t)}^{(1)}, \boldsymbol{\sigma}_1^{(2)}, \ldots, \boldsymbol{\sigma}_{\boldsymbol{x}_2(t)}^{(2)})$$

constitutes a Markov process. We shall analyse this process, following the elegant approach of Takács [4, 5] for the $M/G/\infty$ queue. The results in Theorem 2.1 below are of independent interest, but they also provide the basis for our further investigations.

Theorem 2.1

$$\begin{aligned}
&\Pr\{\boldsymbol{x}_1(t) = l_1, \boldsymbol{x}_2(t) = l_2, \boldsymbol{\sigma}_1^{(1)} \le x_1, \ldots, \boldsymbol{\sigma}_{l_1}^{(1)} \le x_{l_1}, \\
&\quad \boldsymbol{\sigma}_1^{(2)} \le y_1, \ldots, \boldsymbol{\sigma}_{l_2}^{(2)} \le y_{l_2} \mid \boldsymbol{x}_1(0) = 0, \boldsymbol{x}_2(0) = 0\} \\
&= \exp\left(-\lambda \int_0^t (1 - B_1 * B_2(x))\, \mathrm{d}x\right) \frac{\lambda^{l_1}}{l_1!} \prod_{i=1}^{l_1} \left\{\int_0^t (B_1(x + x_i) - B_1(x))\, \mathrm{d}x\right\} \\
&\quad \times \frac{\lambda^{l_2}}{l_2!} \prod_{j=1}^{l_2} \left\{\int_0^t \int_0^x \{B_2(x - u + y_j) - B_2(x - u)\}\, \mathrm{d}B_1(u)\, \mathrm{d}x\right\}, \\
&\quad t \ge 0;\ l_1, l_2 = 0, 1, \ldots;\quad x_1, \ldots, x_{l_1}, y_1, \ldots, y_{l_2} \ge 0. \qquad (2.1)
\end{aligned}$$

If $\beta_1 < \infty, \beta_2 < \infty$ then

$$\lim_{t\to\infty} \Pr\{x_1(t) = l_1, x_2(t) = l_2, \sigma_1^{(1)} \le x_1, \dots, \sigma_{l_1}^{(1)} \le x_{l_1}, \sigma_1^{(2)} \le y_1, \dots, \sigma_{l_2}^{(2)} \le y_{l_2}\}$$
$$= e^{-\lambda\beta_1} \frac{(\lambda\beta_1)^{l_1}}{l_1!} e^{-\lambda\beta_2} \frac{(\lambda\beta_2)^{l_2}}{l_2!} \prod_{i=1}^{l_1} \left\{ \int_0^{x_i} \frac{1-B_1(x)}{\beta_1} dx \right\} \prod_{j=1}^{l_2} \left\{ \int_0^{y_j} \frac{1-B_2(x)}{\beta_2} dx \right\}, \tag{2.2}$$

and the limiting distribution is independent of the initial state.

Proof. Assume that in the interval $(0, t)$ n customers arrive ($n \ge l_1 + l_2$). It is well known that under the condition that in the Poisson process n customers arrive in $(0, t)$, the joint distribution of the epochs of these arrivals agrees with the joint distribution of n independent random points distributed uniformly in $(0, t)$ (see e.g. [4, 5]). If a particular customer arrived at epoch $t - x$, then

$B_1(x + x_i) - B_1(x)$ denotes the probability he is still in Q_1 at t, with residual service time at most x_i;

$\int_0^x \{B_2(x-u+y_j) - B_2(x-u)\}\, dB_1(u)$ denotes the probability he has left Q_1, but he is still in Q_2 at t, with residual service time at most y_j;

$B_1 * B_2(x)$ denotes the probability he has left Q_2 before t.

Hence the lefthand side of (2.1) can be written as

$$\sum_{n=l_1+l_2}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \frac{n!}{l_1!\, l_2!\, (n-l_1-l_2)!} \prod_{i=1}^{l_1} \left\{ \frac{1}{t} \int_0^t (B_1(x+x_i) - B_1(x))\, dx \right\}$$
$$\times \prod_{j=1}^{l_2} \left\{ \frac{1}{t} \int_0^t \int_0^x (B_2(x-u+y_j) - B_2(x-u))\, dB_1(u)\, dx \right\}$$
$$\times \prod_{k=1}^{n-l_1-l_2} \left\{ \frac{1}{t} \int_0^t B_1 * B_2(x)\, dx \right\}, \tag{2.3}$$

which easily leads to (2.1). If we let $t \to \infty$ then we obtain (2.2) after some rearrangements of integrations. The argument can be adapted for general $x_1(0)$, $x_2(0)$ to show that the limiting distribution is independent of the initial distribution. □

Remark 2.1. It follows from (2.1) that at any time t, $x_1(t)$ and $x_2(t)$ are independent Poisson distributed random variables. This is a result which has been known in far greater generality for some time, cf. Kingman [9, Section 4] and Harrison and Lemoine [8]. Observe that (2.2) yields the well-known product form for the joint stationary queue length distribution. Cohen [2] gives a detailed analysis of the joint distribution of queue lengths and attained (or residual) service times in networks of queues with the so-called generalized processor sharing discipline; (2.2) can be viewed as a limiting case of one of his models (cf. [2, Section 7]).

We are now ready to study the joint stationary distribution of the queue lengths L_1, L_2 which a particular customer sees in Q_1 and in Q_2 at his successive arrival epochs at these queues. For simplicity of notation assume that the tagged customer arrives at Q_1 at time 0, finding the system in equilibrium. Applying Wolff's [7]

PASTA property, which states that Poisson arrivals see time averages, it follows that the stationary joint distribution of queue lengths and residual service times just before the arrival epoch of the tagged customer at Q_1 is given by (2.2).

Theorem 2.2. *The generating function of the joint stationary distribution of the queue lengths $\boldsymbol{L}_1$, $\boldsymbol{L}_2$ is given by*

$$E[z_1^{\boldsymbol{L}_1} z_2^{\boldsymbol{L}_2}] = e^{\lambda\beta_1(z_1-1)+\lambda\beta_2(z_2-1)} \int_0^\infty e^{\lambda(1-z_1)(1-z_2)Q(t)}\, dB_1(t), \quad |z_1| \leq 1,\ |z_2| \leq 1, \tag{2.4}$$

with

$$Q(t) := \int_0^t (1-B_1(\tau))(1-B_2(t-\tau))\, d\tau, \quad t \geq 0; \tag{2.5}$$

$$\operatorname{cov}(\boldsymbol{L}_1, \boldsymbol{L}_2) = \lambda \int_0^\infty Q(t)\, dB_1(t), \qquad \operatorname{corr}(\boldsymbol{L}_1, \boldsymbol{L}_2) = \frac{1}{\sqrt{\beta_1\beta_2}} \int_0^\infty Q(t)\, dB_1(t). \tag{2.6}$$

Proof.

$$E[z_1^{\boldsymbol{L}_1} z_2^{\boldsymbol{L}_2}] = \int_{t=0}^\infty dB_1(t) \sum_{l_1=0}^\infty z_1^{l_1} \sum_{l_2=0}^\infty z_2^{l_2} \sum_{n_2=0}^\infty \int_{x_1=0}^\infty \cdots$$
$$\times \int_{y_{n_2}=0}^\infty \Pr\{\boldsymbol{x}_2(t) = l_2 \mid \boldsymbol{x}_1(0) = l_1, \boldsymbol{x}_2(0) = n_2, \boldsymbol{\tau} = t, \boldsymbol{\sigma}_1^{(1)} = x_1, \ldots, \boldsymbol{\sigma}_{l_1}^{(1)} = x_{l_1},$$
$$\boldsymbol{\sigma}_1^{(2)} = y_1, \ldots, \boldsymbol{\sigma}_{n_2}^{(2)} = y_{n_2}\}.$$
$$\times e^{-\lambda\beta_1} \frac{(\lambda\beta_1)^{l_1}}{l_1!} e^{-\lambda\beta_2} \frac{(\lambda\beta_2)^{n_2}}{n_2!}$$
$$\times \prod_{i=1}^{l_1} \left\{\frac{1-B_1(x_i)}{\beta_1}\right\} \prod_{j=1}^{n_2} \left\{\frac{1-B_2(y_j)}{\beta_2}\right\} dx_1 \cdots dy_{n_2},$$
$$|z_1| \leq 1,\ |z_2| \leq 1. \tag{2.7}$$

The essential observation now is that $\boldsymbol{x}_2(t)$ is composed of three independent terms:

$$\boldsymbol{x}_2(t) = \boldsymbol{u}_0(t) + \boldsymbol{u}_1(t) + \boldsymbol{u}_2(t),$$

$\boldsymbol{u}_0(t)$:= number of customers in Q_2 at t who were not yet present in the system at 0;
$\boldsymbol{u}_1(t)$:= number of customers in Q_2 at t who were at Q_1 at 0; (2.8)
$\boldsymbol{u}_2(t)$:= number of customers in Q_2 at t who were at Q_2 at 0.

Note that $\boldsymbol{u}_1(t)$ depends on l_1 and $\boldsymbol{\sigma}_1^{(1)}, \ldots, \boldsymbol{\sigma}_{l_1}^{(1)}$, while $\boldsymbol{u}_2(t)$ depends on n_2 and $\boldsymbol{\sigma}_1^{(2)}, \ldots, \boldsymbol{\sigma}_{n_2}^{(2)}$.

Once more applying the argument concerning the uniform distribution of Poisson arrivals in $(0, t)$ we can write:

$$E[z_2^{\boldsymbol{u}_0(t)}] = \sum_{j=0}^\infty z_2^j \sum_{n=j}^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} \binom{n}{j} \left(\frac{p_0}{t}\right)^j \left(\frac{t-p_0}{t}\right)^{n-j} = e^{-\lambda p_0(1-z_2)}, \quad |z_2| \leq 1, \tag{2.9}$$

with

$$p_0 := \int_0^t (B_1(x) - B_1 * B_2(x))\,\mathrm{d}x; \tag{2.10}$$

the interpretation is that a customer arrived at Q_1 at $(t-x)$, got completely served in Q_1 in $(t-x, t)$, but not yet in Q_2.

Next consider $\boldsymbol{u}_1(t)$:

$$\int_{x_1=0}^{\infty} \cdots \int_{x_{l_1}=0}^{\infty} E[z_2^{\boldsymbol{u}_1(t)} | \boldsymbol{x}_1(0) = l_1, \boldsymbol{\sigma}_1^{(1)} = x_1, \ldots, \boldsymbol{\sigma}_{l_1}^{(1)} = x_{l_1}]$$
$$\times \prod_{i=1}^{l_1} \left(\frac{1 - B_1(x_i)}{\beta_1}\right) \mathrm{d}x_1 \cdots \mathrm{d}x_{l_1}$$
$$= \sum_{j=0}^{\infty} z_2^j \binom{l_1}{j} p_1^j (1-p_1)^{l_1 - j} = (1 - p_1 + p_1 z_2)^{l_1}, \quad |z_2| \leq 1, \tag{2.11}$$

with

$$p_1 := \int_0^t \frac{1 - B_1(x)}{\beta_1} \{1 - B_2(t-x)\}\,\mathrm{d}x; \tag{2.12}$$

the interpretation is that a customer has residual service time $x < t$ in Q_1, while his service time in Q_2 exceeds $t - x$.

Finally consider $\boldsymbol{u}_2(t)$:

$$\int_{y_1=0}^{\infty} \cdots \int_{y_{n_2}=0}^{\infty} E[z_2^{\boldsymbol{u}_2(t)} | \boldsymbol{x}_2(0) = n_2, \boldsymbol{\sigma}_1^{(2)} = y_1, \ldots, \boldsymbol{\sigma}_{n_2}^{(2)} = y_{n_2}]$$
$$\times \prod_{j=1}^{n_2} \left(\frac{1 - B_2(y_j)}{\beta_2}\right) \mathrm{d}y_1 \cdots \mathrm{d}y_{n_2}$$
$$= \sum_{k=0}^{\infty} z_2^k \binom{n_2}{k} p_2^k (1-p_2)^{n_2 - k} = (1 - p_2 + p_2 z_2)^{n_2}, \quad |z_2| \leq 1, \tag{2.13}$$

with

$$p_2 := \int_t^{\infty} \frac{1 - B_2(y)}{\beta_2}\,\mathrm{d}y; \tag{2.14}$$

the interpretation is that a customer has residual service time in Q_2 exceeding t.

Combining (2.7), (2.9), (2.11) and (2.13) yields:

$$E[z_1^{\boldsymbol{L}_1} z_2^{\boldsymbol{L}_2}] = \int_{t=0}^{\infty} \mathrm{d}B_1(t) \sum_{l_1=0}^{\infty} z_1^{l_1} \sum_{n_2=0}^{\infty} \mathrm{e}^{-\lambda\beta_1} \frac{(\lambda\beta_1)^{l_1}}{l_1!} \mathrm{e}^{-\lambda\beta_2} \frac{(\lambda\beta_2)^{n_2}}{n_2!} \mathrm{e}^{-\lambda p_0(1-z_2)}$$
$$\times (1 - p_1 + p_1 z_2)^{l_1} (1 - p_2 + p_2 z_2)^{n_2}$$
$$= \mathrm{e}^{\lambda\beta_1(z_1 - 1) + \lambda\beta_2(z_2 - 1)} \int_0^{\infty} \mathrm{e}^{-\lambda\beta_1 z_1 p_1 (1 - z_2)} \mathrm{e}^{\lambda\beta_2(1-p_2)(1-z_2) - \lambda p_0(1-z_2)}\,\mathrm{d}B_1(t),$$
$$|z_1| \leq 1, \quad |z_2| \leq 1. \tag{2.15}$$

Finally (2.4) follows from the observation that

$$\beta_1 p_1 = \int_0^t (1 - B_1(x))(1 - B_2(t-x))\,\mathrm{d}x = Q(t), \tag{2.16}$$

and

$$-p_0 + \beta_2(1 - p_2) = -\int_0^t (B_1(x) - B_1 * B_2(x))\,\mathrm{d}x + \int_0^t (1 - B_2(x))\,\mathrm{d}x = Q(t). \tag{2.17}$$

Expressions (2.6) for the covariance and correlation coefficient of $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are an immediate consequence of (2.4). □

Remark 2.2. According to (2.16) $Q(t)$ has the following interpretation:

$$Q(t) = \beta_1 \Pr\{\boldsymbol{\sigma}^{(1)} < t < \boldsymbol{\sigma}^{(1)} + \boldsymbol{\tau}^{(2)}\}, \tag{2.18}$$

with $\boldsymbol{\sigma}^{(1)}$ a residual service time in Q_1 and $\boldsymbol{\tau}^{(2)}$ the successive service time in Q_2. Equality of expressions such as $\beta_1 p_1$ and $-p_0 + \beta_2(1 - p_2)$ plays a central role in the analysis of the case of an *arbitrary* number of queues in series. To prove such equalities note that, e.g.

$$\int_{x=0}^t (B_1 * B_2(x))\,\mathrm{d}x = \int_{x=0}^t \int_{u=0}^x \mathrm{d}B_2(u) B_1(x-u)\,\mathrm{d}x = \int_{y=0}^t B_1(y) B_2(t-y)\,\mathrm{d}y, \tag{2.19}$$

after some rearrangement of integrations; now compare (2.16) and (2.17).

Remark 2.3. In the special case that $B_1(\cdot)$ and $B_2(\cdot)$ are negative exponential distributions, $E[z_1^{\boldsymbol{L}_1} z_2^{\boldsymbol{L}_2}]$ has been obtained by Vainshtein e.a. [6]. In this case

$$Q(t) = (\mathrm{e}^{-t/\beta_1} - \mathrm{e}^{-t/\beta_2}) \Big/ \left(\frac{1}{\beta_2} - \frac{1}{\beta_1}\right),$$

and

$$\operatorname{cov}(\boldsymbol{L}_1, \boldsymbol{L}_2) = \frac{\lambda}{2} \frac{\beta_1 \beta_2}{\beta_1 + \beta_2}, \qquad \operatorname{corr}(\boldsymbol{L}_1, \boldsymbol{L}_2) = \frac{\sqrt{\beta_1 \beta_2}}{2(\beta_1 + \beta_2)}.$$

Remark 2.4. Although the present model is one which allows overtaking of customers at each node (a fact which usually strongly complicates the analysis of characteristic queueing quantities at succesive nodes), the fact that no customer ever has to wait renders a complete analysis possible.

Expression (2.4) exhibits a nice symmetry in Q_1 and Q_2 (apart from the integral w.r.t. $B_1(\cdot)$); this should not be surprising in view of the reversibility property of this model.

A trivial extension of (2.7) and (2.15) yields, with $\boldsymbol{n}_2$ the number of customers in Q_2 at the arrival epoch of the tagged customer in Q_1:

$$E[z_1^{\boldsymbol{L}_1} z_2^{\boldsymbol{L}_2} z^{\boldsymbol{n}_2}] = \mathrm{e}^{\lambda\beta_1(z_1-1)+\lambda\beta_2(z_2-1)+\lambda\beta_2(z-1)} \times \int_0^\infty \mathrm{e}^{\lambda(1-z_1)(1-z_2)\beta_1 p_1 + \lambda(1-z_2)(1-z)\beta_2 p_2}\, \mathrm{d}B_1(t),$$

$$|z_1| \le 1,\ |z_2| \le 1,\ |z| \le 1. \tag{2.20}$$

$z=1$ yields (2.4); $z_2 = 1$ yields the well-known generating function of the joint stationary distribution of the queue lengths at Q_1 and Q_2 at an arbitrary (or arrival) epoch. $z_1 = 1$ yields the generating function of the joint stationary distribution of the queue lengths at Q_2 at the arrival epochs of the tagged customer in Q_1 and in Q_2, respectively (an expression which is completely symmetric in z_2 and z); their correlation coefficient is given by

$$\mathrm{corr}(\boldsymbol{L}_2, \boldsymbol{n}_2) = \frac{1}{\beta_2}\int_0^\infty \beta_2 p_2\, \mathrm{d}B_1(t) = \int_0^\infty B_1(y)\left(\frac{1-B_2(y)}{\beta_2}\right) \mathrm{d}y = \Pr\{\boldsymbol{\tau}^{(1)} < \boldsymbol{\sigma}^{(2)}\}. \tag{2.21}$$

Further note that

$$E[z_2^{\boldsymbol{L}_2} | \boldsymbol{L}_1 = k] = \mathrm{e}^{\lambda\beta_2(z_2-1)} \int_0^\infty \mathrm{e}^{\lambda(1-z_2)Q(t)}\left(1-(1-z_2)\frac{Q(t)}{\beta_1}\right)^k \mathrm{d}B_1(t),$$

$$|z_2| \le 1,\ k = 0,1,\ldots, \tag{2.22}$$

hence

$$E[\boldsymbol{L}_2 | \boldsymbol{L}_1 = k] = \lambda\beta_2 + (k - \lambda\beta_1)\int_0^\infty \frac{Q(t)}{\beta_1}\, \mathrm{d}B_1(t)$$

$$= \lambda\beta_2 + (k-\lambda\beta_1)\Pr\{\boldsymbol{\sigma}^{(1)} < \tilde{\boldsymbol{\tau}}^{(1)} < \boldsymbol{\sigma}^{(1)} + \boldsymbol{\tau}^{(2)}\}$$

$$= E[\boldsymbol{L}_2] + (k - E[\boldsymbol{L}_1])\Pr\{\boldsymbol{\sigma}^{(1)} < \tilde{\boldsymbol{\tau}}^{(1)} < \boldsymbol{\sigma}^{(1)} + \boldsymbol{\tau}^{(2)}\}, \quad k = 0,1,\ldots, \tag{2.23}$$

$\tilde{\boldsymbol{\tau}}^{(1)}$ denoting the service time of the tagged customer in Q_1; similarly

$$E[z_2^{\boldsymbol{L}_2} | \boldsymbol{n}_2 = k] = \mathrm{e}^{\lambda\beta_2(z_2-1)} \int_0^\infty \mathrm{e}^{\lambda(1-z_2)\beta_2 p_2}(1-(1-z_2)p_2)^k\, \mathrm{d}B_1(t),$$

$$|z_2| \le 1,\ k = 0, 1, \ldots, \tag{2.24}$$

hence

$$E[\boldsymbol{L}_2 | \boldsymbol{n}_2 = k] = \lambda\beta_2 + (k-\lambda\beta_2)\int_0^\infty B_1(y)\frac{1-B_2(y)}{\beta_2}\, \mathrm{d}y$$

$$= E[\boldsymbol{L}_2] + (k - E[\boldsymbol{n}_2])\Pr\{\tilde{\boldsymbol{\tau}}^{(1)} < \boldsymbol{\sigma}^{(2)}\}, \quad k = 0,1,\ldots. \tag{2.25}$$

The interpretation of (2.23) and (2.25) is obvious.

The second terms in the r.h.s. represent the influence of $\boldsymbol{L}_1$ and $\boldsymbol{n}_2$, resp. E.g., in (2.25) $\Pr\{\tilde{\boldsymbol{\tau}}^{(1)} < \boldsymbol{\sigma}^{(2)}\}$ is the probability that a customer, who is present in Q_2 at 0, is still in Q_2 at the end of the service of the tagged customer in Q_1.

3. The general tandem case

Let $\boldsymbol{x}_1(t), \ldots, \boldsymbol{x}_n(t)$ denote the queue lengths at $Q_1, \ldots, Q_n$ at time t, and, under the condition that $\boldsymbol{x}_1(t) = l_1, \ldots, \boldsymbol{x}_n(t) = l_n$, let $\boldsymbol{\sigma}_1^{(1)}, \ldots, \boldsymbol{\sigma}_{l_1}^{(1)}, \ldots, \boldsymbol{\sigma}_1^{(n)}, \ldots, \boldsymbol{\sigma}_{l_n}^{(n)}$ denote the residual service times of the customers in service. Analogously to Theorem 2.1 one can derive the joint time-dependent distribution of $\boldsymbol{x}_1(t), \ldots, \boldsymbol{x}_n(t)$, $\boldsymbol{\sigma}_1^{(1)}, \ldots, \boldsymbol{\sigma}_{\boldsymbol{x}_n(t)}^{(n)}$;the stationary distribution is, analogously to (2.2), given by:

If $\beta_1 < \infty, \ldots, \beta_n < \infty$, then

$$\lim_{t \to \infty} \Pr\{\boldsymbol{x}_1(t) = l_1, \ldots, \boldsymbol{x}_n(t) = l_n, \boldsymbol{\sigma}_1^{(1)} \leq x_1^{(1)}, \ldots, \boldsymbol{\sigma}_{l_n}^{(n)} \leq x_{l_n}^{(n)}\}$$

$$= \prod_{j=1}^{n} \left[e^{-\lambda\beta_j} \frac{(\lambda\beta_j)^{l_j}}{l_j!} \prod_{i=1}^{l_j} \left\{ \int_0^{x_i^{(j)}} \frac{1 - B_j(x)}{\beta_j} \, dx \right\} \right]. \tag{3.1}$$

Starting from this expression one might investigate the joint distribution of $\boldsymbol{L}_1, \ldots, \boldsymbol{L}_m$, the queue lengths found in $Q_1, \ldots, Q_n$ by a tagged customer upon his course through those queues. Following Kreinin and Vainshtein [3] we shall restrict ourself to the simpler matter of the determination of the joint distribution of $\boldsymbol{L}_1$ and $\boldsymbol{L}_m$. Analysis of the general case proceeds in a very similar manner, but the calculations become rather lengthy.

Theorem 3.1

$$E[z_1^{\boldsymbol{L}_1} z_m^{\boldsymbol{L}_m}] = e^{\lambda\beta_1(z_1 - 1) + \lambda\beta_m(z_m - 1)} \int_0^{\infty} e^{\lambda(1 - z_1)(1 - z_m)Q_m(t)} \, d(B_1 * B_2 * \cdots * B_{m-1})(t),$$

$$|z_1| \leq 1, \ |z_m| \leq 1, \ m = 2, 3, \ldots, \tag{3.2}$$

with

$$Q_m(t) := \int_0^t (1 - B_1(\tau))[(B_2 * \cdots * B_{m-1})(t - \tau) - (B_2 * \cdots * B_m)(t - \tau)] \, d\tau$$

$$= \beta_1 \Pr\{\boldsymbol{\sigma}^{(1)} + \boldsymbol{\tau}^{(2)} + \cdots + \boldsymbol{\tau}^{(m-1)} < t < \boldsymbol{\sigma}^{(1)} + \boldsymbol{\tau}^{(2)} + \cdots + \boldsymbol{\tau}^{(m)}\}, \quad t \geq 0; \tag{3.3}$$

$$\operatorname{cov}(\boldsymbol{L}_1, \boldsymbol{L}_m) = \lambda \int_0^{\infty} Q_m(t) d(B_1 * \cdots * B_{m-1})(t),$$

$$\operatorname{corr}(\boldsymbol{L}_1, \boldsymbol{L}_m) = \frac{1}{\sqrt{\beta_1 \beta_m}} \int_0^{\infty} Q_m(t) \, d(B_1 * \cdots * B_{m-1})(t). \tag{3.4}$$

Proof. The proof will not be given in full detail, since it proceeds in the same way as the proof of Theorem 2.2. Assume that the tagged customer arrives at Q_1 at time 0, finding the system in equilibrium. Condition on the event that the sum of his sojourn times (= service times) in $Q_1, \ldots, Q_{m-1}$ equals t. The essential observation is that $\boldsymbol{x}_m(t)$ is composed of $m+1$ independent terms:

$$\boldsymbol{x}_m(t) = \boldsymbol{u}_0(t) + \boldsymbol{u}_1(t) + \cdots + \boldsymbol{u}_m(t),$$

with

$$\begin{aligned} \boldsymbol{u}_0(t) &:= \text{number of customers in } Q_m \text{ at } t \text{ who were not yet} \\ &\qquad \text{present in the system at } 0; \\ \boldsymbol{u}_j(t) &:= \text{number of customers in } Q_m \text{ at } t \text{ who were at } Q_j \text{ at} \\ &\qquad 0,\ j=1,\ldots,m. \end{aligned} \tag{3.5}$$

Note that $\boldsymbol{u}_j(t)$ depends on $\boldsymbol{x}_j(0)$ and the residual service times at Q_j at time 0. Defining

$$\begin{aligned} p_0 &:= \int_0^t \Pr\{\boldsymbol{\tau}^{(1)} + \cdots + \boldsymbol{\tau}^{(m-1)} < x < \boldsymbol{\tau}^{(1)} + \cdots + \boldsymbol{\tau}^{(m)}\}\,\mathrm{d}x \\ &= \int_0^t \{(B_1 * \cdots * B_{m-1})(x) - (B_1 * \cdots * B_m)(x)\}\,\mathrm{d}x, \\ p_j &:= \Pr\{\boldsymbol{\sigma}^{(j)} + \boldsymbol{\tau}^{(j+1)} + \cdots + \boldsymbol{\tau}^{(m-1)} < t < \boldsymbol{\sigma}^{(j)} + \boldsymbol{\tau}^{(j+1)} + \cdots + \boldsymbol{\tau}^{(m)}\} \\ &= \int_0^t \frac{1 - B_j(x)}{\beta_j} \{(B_{j+1} * \cdots * B_{m-1})(t-x) - (B_{j+1} * \cdots * B_m)(t-x)\}\,\mathrm{d}x, \\ &\qquad\qquad j = 1, \ldots, m-2, \\ p_{m-1} &:= \Pr\{\boldsymbol{\sigma}^{(m-1)} < t < \boldsymbol{\sigma}^{(m-1)} + \boldsymbol{\tau}^{(m)}\} = \int_0^t \frac{1 - B_{m-1}(x)}{\beta_{m-1}} (1 - B_m(t-x))\,\mathrm{d}x, \\ p_m &:= \Pr\{\boldsymbol{\sigma}^{(m)} > t\} = \int_t^\infty \frac{1 - B_m(x)}{\beta_m}\,\mathrm{d}x, \end{aligned} \tag{3.6}$$

one can now prove, analogously to (2.15), that

$$\begin{aligned} E[z_1^{\boldsymbol{L}_1} z_m^{\boldsymbol{L}_m}] &= \int_0^\infty \mathrm{d}(B_1 * \cdots * B_{m-1})(t)\, \mathrm{e}^{-\lambda p_0(1-z_m)}\, \mathrm{e}^{-\lambda\beta_1 + \lambda\beta_1 z_1(1-p_1+p_1 z_m)} \\ &\qquad \times \mathrm{e}^{-\lambda\beta_2 + \lambda\beta_2(1-p_2+p_2 z_m)} \cdot \cdots \cdot \mathrm{e}^{-\lambda\beta_m + \lambda\beta_m(1-p_m+p_m z_m)}, \\ & |z_1| \leqslant 1,\ |z_m| \leqslant 1. \end{aligned} \tag{3.7}$$

Note that

$$\beta_1 p_1 = Q_m(t). \tag{3.8}$$

The theorem is proved if we show that

$$-p_0-\beta_2 p_2-\cdots-\beta_{m-1}p_{m-1}+\beta_m(1-p_m)=Q_m(t). \tag{3.9}$$

This is easily established, proceeding as in (2.19) and noting that the terms in the left-hand side of (3.9) cancel out almost completely:

$$-\int_0^t ((B_1*\cdots*B_{m-1})(x)-(B_1*\cdots*B_m)(x))\,\mathrm{d}x$$

$$-\int_0^t (1-B_2(x))((B_3*\cdots*B_{m-1})(t-x)-(B_3*\cdots*B_m)(t-x))\,\mathrm{d}x$$

$$-\cdots-\int_0^t (1-B_{m-1}(x))(1-B_m(t-x))\,\mathrm{d}x+\int_0^t (1-B_m(x))\,\mathrm{d}x$$

$$=-\int_0^t (B_1*\cdots*B_{m-1})(x)\,\mathrm{d}x+\int_0^t (B_1*\cdots*B_m)(x)\,\mathrm{d}x$$

$$+\int_0^t (B_2*\cdots*B_{m-1})(x)\,\mathrm{d}x-\int_0^t (B_2*\cdots*B_m)(x)\,\mathrm{d}x$$

$$=Q_m(t).$$

Expressions (3.4) for the covariance and correlation coefficient of $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are an immediate consequence of (3.2). □

Remark 3.1. In the special case that $B_i(\cdot)$ is a negative exponential distribution for all i, $E[z_1^{\boldsymbol{L}_1}z_m^{\boldsymbol{L}_m}]$ has been obtained by Kreinin and Vainshtein [3], using a different method. If, moreover, all m service time distributions are identical, they show that the correlation coefficient of $\boldsymbol{L}_1$ and $\boldsymbol{L}_m$ tends to zero as $1/\sqrt{m}$ for $m\to\infty$. If all service times are identical constants β, then it is immediately clear from (3.4) that $\mathrm{corr}(\boldsymbol{L}_1,\boldsymbol{L}_m)=1$ (as it should be).

Remark 3.2. Since the arrival process at each queue is Poisson with intensity λ, it can be proved that a similar expression as (3.2) holds for $E[z_1^{\boldsymbol{L}_1}z_m^{\boldsymbol{L}_{m-1}}]$; cf. [3].

Remark 3.3. It follows from (3.4) that if $\beta_1=\beta_m$ then (see also (3.3)),

$$\mathrm{corr}(\boldsymbol{L}_1,\boldsymbol{L}_m)=\int_0^\infty \frac{Q_m(t)}{\beta_1}\,\mathrm{d}(B_1*\cdots*B_{m-1})(t)$$

$$=Pr\{\boldsymbol{\sigma}^{(1)}+\boldsymbol{\tau}^{(2)}+\cdots+\boldsymbol{\tau}^{(m-1)}<\tilde{\boldsymbol{\tau}}^{(1)}+\cdots+\tilde{\boldsymbol{\tau}}^{(m-1)}$$

$$<\boldsymbol{\sigma}^{(1)}+\boldsymbol{\tau}^{(2)}+\cdots+\boldsymbol{\tau}^{(m)}\}, \tag{3.10}$$

$\tilde{\boldsymbol{\tau}}^{(j)}$ denoting the service time of the tagged customer in Q_j, $j=1,\ldots,m-1$.

It follows from (3.2) that (cf. (2.22) and (2.23)).

$$E[z_m^{L_m} \mid L_1 = k] = e^{\lambda\beta_m(z_m-1)} \int_0^\infty e^{\lambda(1-z_m)Q_m(t)} \times\left(1-(1-z_m)\frac{Q_m(t)}{\beta_1}\right)^k d(B_1*\cdots*B_{m-1})(t),$$

$$|z_m| \le 1,\ k=0,1,\ldots, \tag{3.11}$$

hence

$$E[L_m \mid L_1 = k] = \lambda\beta_m + (k-\lambda\beta_1)\int_0^\infty \frac{Q_m(t)}{\beta_1} d(B_1*\cdots*B_{m-1})(t)$$
$$= E[L_m] + (k - E[L_1]) \Pr\{\sigma^{(1)} + \tau^{(2)} + \cdots + \tau^{(m-1)} < \tilde{\tau}^{(1)} + \cdots + \tilde{\tau}^{(m-1)} < \sigma^{(1)} + \tau^{(2)} + \cdots + \tau^{(m)}\}, \tag{3.12}$$

with an obvious interpretation.

Example. Consider the case that the service times in all queues are Erlang-k distributed with mean β. A straightforward calculation yields

$$\operatorname{corr}(L_1, L_m) = \frac{1}{k}\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}(\tfrac{1}{2})^{(2m-3)k+i+j+1}\binom{(2m-3)k+i+j}{(m-1)k-1}.$$

Some numerical results are presented in Table 1. Note that for $k\to\infty$ the Erlang-k distribution approaches the deterministic distribution, and indeed $\operatorname{corr}(L_1, L_m)$ approaches the value 1. For fixed k, and $m\to\infty$, it easily follows using Stirling's formula, that $\operatorname{corr}(L_1, L_m)\to 0$ as $m^{-1/2}$.

Table 1 $\operatorname{corr}(L_1, L_m)$ for the case that all service times are E_k distributed with mean β

m \ k	1	2	3	4	5	25	125
2	0.250	0.375	0.449	0.501	0.540	0.776	0.899
5	0.137	0.194	0.236	0.271	0.300	0.567	0.798
10	0.093	0.131	0.160	0.184	0.205	0.423	0.698

Remark 3.4. The derivation of Theorem 3.1 shows that it is possible to obtain the joint stationary queue length distribution of a tagged customer at his arrival epochs at two queues of a general *network* of $M/G/\infty$ queues; one may even allow independent Poisson arrival processes at each queue. The basic steps are the same as before:

(i) determine a product-form expression of the type (3.1);

(ii) use the PASTA property;

(iii) decompose the queue length $x_m(t)$ into independent terms corresponding to the position of a customer at time 0.

In fact one may even allow different classes of customers.

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