STATE VECTOR SPLITTING FOR THE EULER EQUATIONS OF GASDYNAMICS*

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Abstract

A new upwind scheme is introduced for the Euler equations of gasdynamics in multi-dimensions. Its relation to Steger-Warming Flux Vector Splitting is discussed. Implementation of the conservative boundary conditions on solid walls is also given. The method is intuitive, easy to implement and does not require the generation of elaborate grids for complicated geometries. Several numerical examples demonstrating this advantage are presented.

1 Introduction

The efficient computation of compressible fluid flow with shock waves has been a significant challenge for decades. In recent years, important progress has been made in dealing with discontinuities in a numerical solution. A general strategy for a solution has been to consider a simple model problem in one space dimension, and to develop new algorithms or to improve the existing schemes by using physical insight and/or mathematical results given for a restricted class of equations (Riemann Solvers, TVD schemes etc.). In generalizing to multi-dimensions, it is assumed that the problem can be treated as a combination of one dimensional problems in each coordinate direction . A crucial assumption is made without justification: waves propagate along coordinate lines (Finite Difference) or normal to cell interfaces (Finite Volume). This assumption is not critical,however, if the region of rapid variation, such as shock waves, is aligned with one of the coordinate lines. This straightforward generalization can lead to either more restrictive stability bounds in multi-dimensions or degraded accuracy. Currently, there is a considerable interest in developing genuinely multidimensional upwind schemes. For a critical discussion of these issues, see the review paper by Roe [8] and a recent monograph by LeVeque [7].

In his exploratory paper [9], Roe suggested looking at the local gradients to choose the upwind direction. The essential feature of his model is the grid independent character of the directions in which information can propagate. Similar approaches based on looking at the local gradients for multidimensional upwinding have been employed by many authors [1] [2] [3].

In this paper, a new multidimensional up-

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wind scheme which uses only local data rather than its gradients for choosing the upwind direction is presented.

2 State Vector Splitting

The Euler equations can be written in the following conservative form

$$\mathbf{U}_t + \mathbf{F}_x + \mathbf{G}_y + \mathbf{H}_z = 0,$$

where $\mathbf{U}$ is the state vector and $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$ are the flux vectors in the corresponding coordinate directions. The central idea of the state vector splitting is to find a wave decomposition of the state vector in state space such that following consistency conditions are satisfied

$$\begin{aligned} \mathbf{U} &= \sum_k \mathbf{R}_k \\ [\mathbf{F}, \mathbf{G}, \mathbf{H}] &= \sum_k [\alpha_k, \beta_k, \gamma_k] \mathbf{R}_k \end{aligned}$$

where $\mathbf{R}_k$ is the wave component in state space, propagating with the velocity $[\alpha_k, \beta_k, \gamma_k]$ in physical space. In general, this decomposition is not unique. In one dimension, a decomposition with three components is known. For the 2D model to be given, there are five components. However, the author believes that there are decompositions with smaller number of wave components. Finding the minimum number of waves for the given number of space dimensions that would result in a consistent difference equation is presently under investigation.

Another interpretation for this decomposition comes from the microscopic point of view. In this description, each wave component represents a group of particles having the same momentum and energy. Therefore, the consistency conditions become moment equations of the Boltzmann equation for a discrete velocity distribution. Harten *et al.* defined a general class of difference equations based on a discrete velocity model, and called them the Boltzmann-type schemes [5]. State vector splitting can be considered as a multidimensional Boltzmann-type scheme.

3 State Vector Splitting in One Space Dimension

In one space dimension, the eigenvalues and the eigenvectors of the flux jacobian can be used to find a consistent decomposition, although other decompositions are possible. In 1D, there are three characteristic speeds,

$$\begin{aligned} \alpha_1 &= u - c \\ \alpha_2 &= u \\ \alpha_3 &= u + c \end{aligned}$$

and the state vector can be decomposed into eigenvectors of the flux jacobian.

$$\begin{aligned} \mathbf{R}_1 &= \frac{\rho}{2\gamma} \begin{bmatrix} 1 \\ u - c \\ h - uc \end{bmatrix} \\ \mathbf{R}_2 &= \frac{\rho(\gamma - 1)}{\gamma} \begin{bmatrix} 1 \\ u \\ u^2/2 \end{bmatrix} \\ \mathbf{R}_3 &= \frac{\rho}{2\gamma} \begin{bmatrix} 1 \\ u + c \\ h + uc \end{bmatrix} \end{aligned}$$

where

$$h = (e + p)/\rho$$

with ρ density, u velocity, c the speed of sound, h specific enthalpy, e total energy and p pressure.

This decomposition satisfies the consistency conditions

$$\begin{aligned} \mathbf{R}_1 + \mathbf{R}_2 + \mathbf{R}_3 &= \mathbf{U} \\ \alpha_1 \mathbf{R}_1 + \alpha_2 \mathbf{R}_2 + \alpha_3 \mathbf{R}_3 &= \mathbf{F} \end{aligned}$$

where

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (e+p)u \end{bmatrix}$$

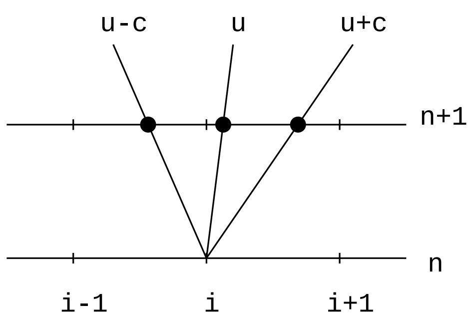

Figure 1: Wave Decomposition in One Dimension

The first stage of the numerical implementation is to approximate the initial data by a piecewise constant vector valued grid function. Then, this approximation is decomposed into three wave components and shifted according to corresponding wave speeds before combining them at the next time level (Figure 1). Of course, in order to continue this process, we have to average this approximate solution to obtain piecewise constant data on the same grid. If we use a simple averaging (Figure 2), the update formula can be expressed as

$$\mathbf{U}_i^{n+1} = (\sum_{+}(1-c_k)\mathbf{R}_k)_i + (\sum_{+}(c_k)\mathbf{R}_k)_{i-1} + (\sum_{-}(1+c_k)\mathbf{R}_k)_i + (\sum_{-}(-c_k)\mathbf{R}_k)_{i+1}$$

where $c_k = \alpha_k \Delta t/\Delta x$ and α_k is the kth eigenvalue of the flux jacobian. The symbols $\sum_+$, $\sum_-$ indicate summation over positive and negative eigenvalues. This type of averaging guarantees that the scheme is conservative, and positivity preserving, since we are not creating or destroying any one of the conserved variables; we are splitting them in one time level, moving them according to corresponding wave speeds, and recombining them at the next time level. This particular averaging for the given decomposition[1] in 1D leads to a scheme which is equivalent to the original form of the Steger-Warming Flux Vector splitting [10]. This can be easily proven with spectral decomposition of the flux jacobian.

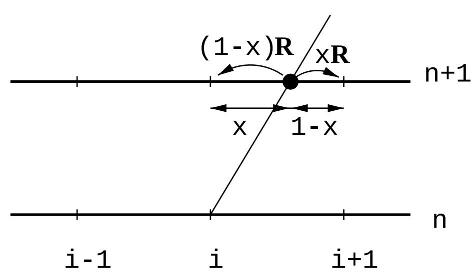

Figure 2: Averaging Stage of the algorithm

[1] This particular decomposition can also be interpreted as a numerical implementation of an analytical solution technique–the method of decomposition into plane waves.

4 State Vector Splitting in Two Dimensions

The straightforward generalization of Steger-Warming method to multi-dimensions leads to a scheme in which there are three waves in each coordinate direction (Figure 3a). Information can propagate only in those directions between time steps, and this brings a more restrictive stability bound with the increasing number of dimensions.

In two dimensional case, it is possible to obtain a consistent difference equation by using only five vectors[2]. The new scheme decomposes the initial data into five waves in 2D, satisfying the consistency conditions, such that information can propagate in the directions dictated by the local data, as opposed to arbitrarily chosen coordinate directions (Figure 3b). Velocities of these waves, and the fraction of mass, momentum, and energy they carry are given as follows.

k	$[\alpha_k, \beta_k]$
1	$[u+c, v+c]$
2	$[u-c, v+c]$
3	$[u+c, v-c]$
4	$[u-c, v-c]$
5	$[u, v]$

$$\mathbf{R}_1 = \frac{\rho}{4\gamma}\begin{bmatrix} 1 \\ u+c \\ v+c \\ h+(u+v)c \end{bmatrix}$$

$$\mathbf{R}_2 = \frac{\rho}{4\gamma}\begin{bmatrix} 1 \\ u-c \\ v+c \\ h-(u-v)c \end{bmatrix}$$

[2]not necessarily the eigenvectors of the flux jacobians

$$\mathbf{R}_3 = \frac{\rho}{4\gamma}\begin{bmatrix} 1 \\ u+c \\ v-c \\ h+(u-v)c \end{bmatrix}$$

$$\mathbf{R}_4 = \frac{\rho}{4\gamma}\begin{bmatrix} 1 \\ u-c \\ v-c \\ h-(u+v)c \end{bmatrix}$$

$$\mathbf{R}_5 = \frac{\rho(\gamma-1)}{\gamma}\begin{bmatrix} 1 \\ u \\ v \\ (u^2+v^2)/2 \end{bmatrix}$$

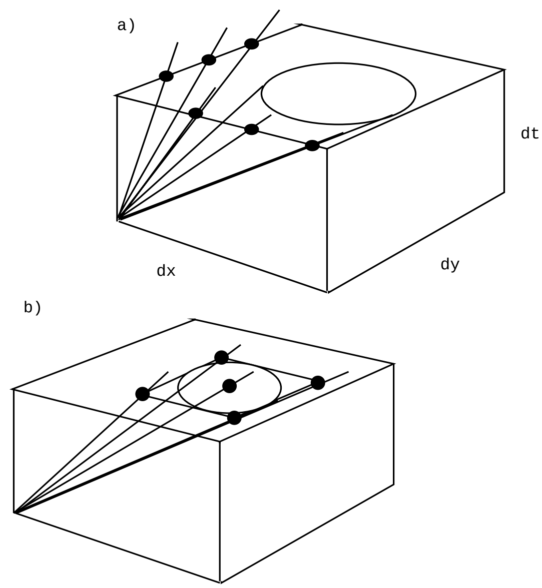

Figure 3: a) Waves in Two Dimensional Steger-Warming Flux Vector Splitting; b) State Vector Splitting in 2D

This new model takes into account the multidimensional nature of the domain of influence. The waves/particles now follow the physical paths rather than an arbitrary coordinate system. If we let the waves propagate to the nearest neighbors, we obtain the following CFL restriction on cartesian grids

$$\Delta t < min(\frac{\Delta x}{|u|+c}, \frac{\Delta y}{|v|+c})$$

5 Boundary Conditions on Solid Walls

In order to develop appropriate boundary conditions on solid walls, it is more convenient to consider the waves as groups of particles. In this description, the idea of specular reflection can be used to conserve the tangential momentum and to reverse the normal momentum (Figure 4).

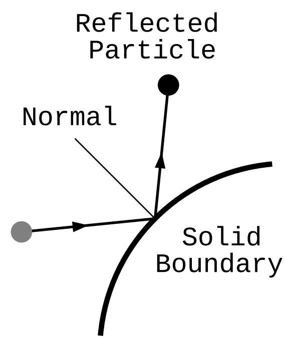

Figure 4: Specular Reflection from a Solid Boundary

In this way, we can implement *the conservative boundary conditions* consistent with the internal scheme without using non-conserved variables like pressure. This implementation is also grid independent so that solid boundaries do not need to coincide with the coordinate lines. That means,*we can solve complicated geometry problems (including moving boundaries) on simple regular grids.*

6 Numerical Examples

To demonstrate the advantages of the scheme, several numerical examples will be given. All of the computations were done on cartesian grids. The grid independent nature of the scheme allows us to compute the flow around an arbitrary geometry without a need for body-fitted or shock-aligned grids. Of course, exploiting specific information about the problem at hand and adjusting the grid accordingly would increase the efficiency of a computer code for that specific problem; but , it is not a crucial ingredient for obtaining a good numerical solution with this scheme. By getting rid of the need for human intervention for choosing a "good" grid, one can get one step closer to a robust scheme.

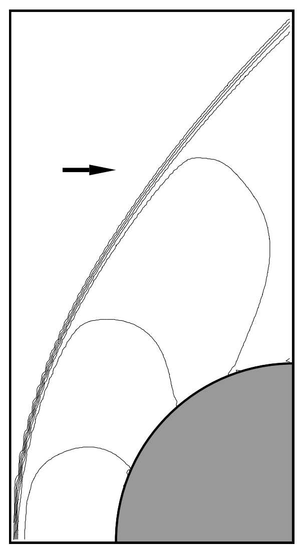

Figure 5: Mach Contours for Supersonic Flow around a Cylinder (64 × 120 cartesian grid, $M_\infty = 4.$).

The first example is the supersonic flow around a cylinder, the classical blunt body problem. Mach number contours are given for free-stream Mach number 4. (Figure 5). The solution is in good agreement with the experimental data and the previous numerical results which use problem specific grid topology [6](Page 126).

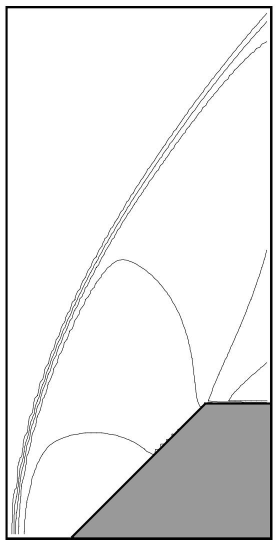

Figure 6: Mach Contours for the Wedge of 45^o half angle (50×100 cartesian grid, $M_\infty = 2.5$).

The second problem is the flow around a wedge with a detached shock wave. For this case, $M_\infty = 2.5$ and the wedge has 45^o half angle (Figure 6). Numerical results are also in good agreement with the experimental data [4](Page 141).

Two other examples are given to show the generality of the current implementation as far as geometry is concerned (Figure 7 8). With conventional schemes, these problems require the generation of sophisticated grids for obtaining a reasonable solution. The present scheme overcomes this difficulty.

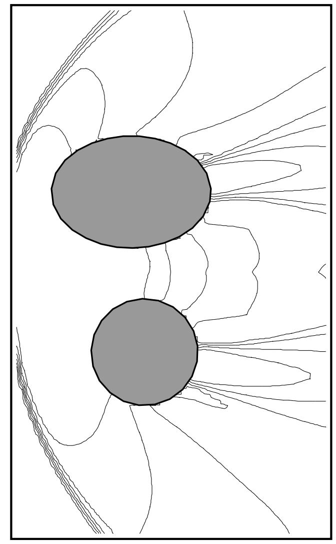

Figure 7: Mach Contours for a multi-body configuration (60×100 cartesian grid, $M_\infty =$ 4.).

7 Conclusion

A new multidimensional upwind scheme for the equations of unsteady gasdynamics have been proposed. This new scheme makes possible to write a code for handling arbitrary geometries on cartesian grids. It is robust in the sense that numerical solution is not critically depend upon whether the grid is aligned with discontinuities or boundaries. Its feasibility has been demonstrated in 2D; generalization to 3D will be straightforward. Grid refinement strategies (cartesian) and generalization to moving boundary problems are going to be the next stages in development.

Supersonic inflow and outflow boundary conditions are trivial and no special treatment is necessary; the internal scheme satisfies the physical boundary conditions without any modification. Subsonic boundary conditions have not been considered yet.

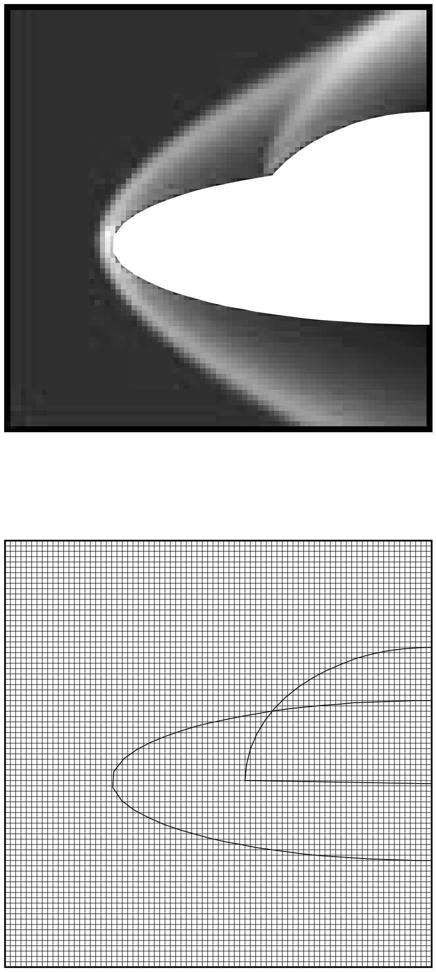

Figure 8: Density Plot for a Double Ellipse Configuration and 80×80 cartesian grid used in computation ($M_\infty = 7.$).

Of course, the ultimate purpose is to solve the Navier-Stokes equations. For this generalization, the wave description is not so meaningful. However, there is a possibility of using the particle description for the inclusion of viscous terms.

Final remarks:

- For the given type of averaging, the method is first order accurate and diffusive. It also preserves the positivity of density and energy.

- The possibility of increasing the order of accuracy by changing the averaging needs to be explored.

- The convection stage of the algorithm is independent of the coordinate system being used. Only at the averaging stage, it uses grid points. In other words, it can be implemented on arbitrary meshes by using an appropriate averaging.

- The method uses the information about the local state of the flow rather than its gradients to choose the upwind direction. This is because the waves represent fluxes instead of flux differences.

- This explicit method is suitable for parallel computation because of the uniform treatment of internal and boundary cells. It also requires a relatively small amount of communication between grid points.

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