Symmetry and integrability in Hamiltonian normal forms

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1 Introduction

In studies in the natural sciences assumptions of symmetries abound: spherical symmetry, cylindrical symmetry, reflection symmetry etc. This is partly because such an assumption simplifies the mathematical analysis but it is also based on the observation that equilibrium states as the outcome of a transient, evolutionary process, usually display symmetries.

In the sequel we will study the consequences of symmetry assumptions for finite-dimensional Hamiltonian systems by normal form methods. It should be noted that such techniques are approximation techniques; they are related to, in fact can be viewed as equivalent to, averaging and multiple time-scale methods. Now and then we shall refer to the relation with averaging results. We shall not refer to multiple time-scale methods; these are, generally speaking, useful techniques but they are not very efficient in the case of Hamiltonian systems.
Symmetry assumptions are causing degenerations which necessitate the calculation of higher order normal forms. In mathematics, the analysis is usually

focused on generic cases but in fields of application like mechanical engineering or nonlinear wave mechanics, symmetric cases arise which are not generic at all in the mathematical sense.
In the last section we shall return to the problem of *evolution towards symmetry*. Apart from the additional complications of time-dependence we have that analysis by normal forms is not sufficient: normalisation involves localisation in some sense while time-evolution is interesting when including global dynamics. We shall review a number of case studies with rather intriguing results.

1.1 Formulation

Time-independent Hamiltonian systems are obtained as a result of formulations of conservative dynamics which is dynamics where friction and in general energy losses play no part. Although in reality mechanical systems always will have some form of energy dissipation, a study of more realistic systems usually starts with trying to understand the basic, underlying Hamiltonian system. Also, in a number of applications (in particular celestial mechanics but also in engineering mechanics) friction-less dynamics may serve as a good approximation for a long time.
With the exception of the last section we shall consider time-independent Hamiltonian systems near equilibrium, both with and without symmetries.
We start with introducing a smooth function of $2n$ variables: $H(q,p)$ in which $q = (q_1, q_2, \ldots, q_n)$ and $p = (p_1, p_2, \ldots, p_n)$ are local coordinates. The function H is called *Hamilton function* or *Hamiltonian*, the number n is called the number of degrees of freedom of the system. From H we derive the $2n$ equations of motion (in short-hand notation)

$$\dot{q} = \partial H/\partial p, \quad \dot{p} = -\partial H/\partial q. \tag{1}$$

The phase-flow in $2n$-space induced by the equations of motion is *volume-* or *measure-preserving* with as a consequence the absence of attracting equilibria or attracting periodic solutions, unless there are additional constraints.
There exists a wealth of literature on Hamiltonian systems; for introductions we refer the reader to Goldstein (1980), Arnold (1978), Abraham and Marsden (1978), Broer (1991), Lazutkin (1993); a useful reprint collection has been edited by MacKay and Meiss (1987).

1.2 The ingredients of phase-space

What we would like, for any given Hamiltonian system, is a description of phase space and phase orbits in terms of equilibrium points, periodic and quasi-periodic solutions, invariant manifolds with corresponding details like periods and behaviour with time, boundedness of orbits, stability. Together, these ingredients are called the *regular component*. Such a description is available for instance for the harmonic oscillator with an arbitrary number of degrees of freedom and for the Newtonian two-body problem. In the last case this is highly nontrivial as the equations are nonlinear and the number of degrees of freedom is 6 so the dimension of phase-space is 12.

There is however in general an *irregular component* which consists of orbits in phase-space indicated by *irregular, chaotic, stochastic, wild*, terms displaying the difficulties to characterize the nature of these orbits. By "in general" we mean that by writing down an arbitrary Hamiltonian function with at least two degrees of freedom, we will more often than not come across the presence of such an irregular component. Indeed, such irregularities will generally dominate the flow.

In studying the regular component, the presence of (first) integrals is very important. An integral $F(q,p)$ of the equations of motion (1) is a function of $2n$ variables such that the orbital derivative $L_t F(q,p) = \partial F/\partial q\, \dot{q} + \partial F/\partial p\, \dot{p} = 0$; see for such definitions also Verhulst (1996). It is easy to see that the Hamiltonian function $H(q,p)$ is an integral corresponding with a one-parameter set of manifolds in phase-space. If we would find more integrals, independent of H, we would have more invariant manifolds consisting of sets of orbits, thus restricting the space for irregular motion. To study integrals the Poisson bracket is useful.

Consider two smooth functions F and G of $2n$ variables and introduce the bracket $\{\ ,\ \}$ by

$$\{F,G\} = \sum_{i=1}^{n}\left(\frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i} - \frac{\partial G}{\partial q_i}\frac{\partial F}{\partial p_i}\right)$$

There are a number of important properties of the Poisson bracket of which we need a few here. Consider a function $F(q,p)$ which is an integral of the equations of motion (1). Using these equations we easily find

$$\{F,H\} = -\{H,F\} = 0$$

We say that F and H are *in involution.*

1.3 Integrable systems

An n degrees of freedom Hamiltonian system is called (Liouville or completely) integrable if n independent integrals in involution exist. In this case one can give a complete description of the topology of phase-space, there is no irregular component.
The Hamiltonian H is always an integral when considering time-independent Hamiltonians, so one degree of freedom systems are integrable. The analysis in this case, reduces to a study of planar flow with centre and saddle points, homoclinic and heteroclinic connections, continuous families of periodic solutions.
For two degrees of freedom systems we need already an additional integral and there are a number of examples from classical mechanics where such a second integral is present; for instance the spherical pendulum where the second integral is the angular momentum with respect to the vertical axis.
Until the end of the 19th century most scientists thought that the equations of classical mechanics were integrable and that finding enough integrals was a matter of ingenuity only. However this is not correct. An example is the simple looking Hénon-Heiles family of two degrees of freedom systems of which only a few are integrable (see Bountis, Segur and Vivaldi, 1982). In fact the study of the Newtonian two-body problem and a few other integrable cases in mechanics has been both instructive and misleading. Instructive as it led us to the many intricate formalisms of Hamiltonian mechanics, misleading because in general Hamiltonian systems are non-integrable.

In this context it should be noted that symmetry assumptions, like translational invariance, time reversal symmetry, reflection symmetry and rotational symmetry, may add to the number of integrals. In general one has that each one-parameter group of symmetries of a Hamiltonian system corresponds with a conserved quantity (Noether's theorem). The usual formulation for this is to derive the Jacobi identity from the Poisson bracket together with some other simple properties and to associate the system with a Lie algebra; see the introductory literature, in particular Abraham and Marsden (1978), also Cushman and Bates (1997).

Examples

- In the case of the Newtonian two-body problem one has translational

symmetry which reduces the number of degrees of freedom from 6 to 3, then rotational symmetry adds three scalar angular momentum integrals (one vector). Together with the Hamiltonian, the energy, we have even more integrals than we need for integrability: the Newtonian two-body problem is degenerate.
As discovered by Henri Poincaré, adding a third body makes the system non-integrable.

- We mentioned already the spherical pendulum where the symmetry with respect to the vertical axis triggers off the existence of the angular momentum integral.

- Another example is the Lagrange top which is a solid body without external forces with one axis of symmetry; again we have energy and angular momentum conservation.

1.4 Non-integrability

To determine whether a Hamiltonian system is integrable is not easy. The earliest proofs show that integrals of a certain kind are not present and this is still a useful line to take, for instance by showing that algebraic integrals to a certain degree are absent.
Also, if a system is non-integrable, one does not know what to expect. There will be an irregular component, probably chaos, but we have no classification of irregularity, even in the case of two degrees of freedom. One powerful and general criterion for chaos is to show that a horseshoe map is imbedded in the phase flow. The presence of a horseshoe involves locally an infinite number of unstable periodic solutions and sensitive dependence on initial values, i.e. chaotic motion. This is exploited by Devaney (1976) who uses Šilnikov-bifurcation in the Hamiltonian context. One needs for this a complex unstable (four conjugate complex eigenvalues) equilibrium point or periodic solution and an isolated homoclinic orbit.
Other approaches are by Ziglin (1982, '83) who establishes a relation with, what he calls, the order of integrability of the monodromy group and the number of independent integrals, and Duistermaat (1984) who proves non-integrability by complex continuation and showing infinite branching.
From the 19th century dates the technique of using the Painlevé conjecture

which asserts that if in the Laurent expansions of the solutions, the singularities are no worse than poles, the system is integrable. The paper by Bountis, Segur and Vivaldi (1982) is based on this approach; for introductions see Ercolani and Siggia (1991), Flaschka, Newell and Tabor (1991).
All these methods have a somewhat local character; a global character has the method of Fomenko (1987) who gives a topological classification of integrable Hamiltonian systems. The level sets of such Hamiltonians can be built from a few basic components and when the Hamiltonian changes only a few "transitions" are allowed to preserve integrability.
We shall return to some of these methods later on.

In the sequel we shall focus on Hamiltonian systems close to integrable ones, also called near-integrable systems. To characterize such systems it is useful to use *canonical variables*, in particular action-angle coordinates. These variables, which are related to polar variables, have the property that when introducing them by a transformation, one conserves the Hamiltonian character of the system, i.e. the transformed equations of motion are tied in again with the transformed Hamiltonian as in (1). One calls such transformations *canonical*.
The idea is, when transforming $p, q \rightarrow I, \varphi$, to use a generating function $S(I, q)$ with the property

$$p = \frac{\partial S}{\partial q}, \ \varphi = \frac{\partial S}{\partial I}.$$

If the system is integrable with integrals $(I_1, I_2, \cdots, I_n)$ we find

$$H(q,p) = H(q, \frac{\partial S}{\partial q}) \ \rightarrow \ H_0(I).$$

The corresponding equations of motion are

$$\dot{I} = 0, \quad \dot{\varphi} = \frac{\partial H_0}{\partial I} = \omega(I).$$

Suppose now that we perturb the Hamiltonian by considering

$$H = H_0 + \varepsilon H_1 \tag{2}$$

with ε a small parameter. Introduction of action-angle coordinates then leads to the system

$$\dot{I} = \varepsilon f(I, \varphi), \quad \dot{\varphi} = \omega(I) + \varepsilon g(I, \varphi).$$

If $\varepsilon = 0$ the system is integrable, if $0 < \varepsilon \ll 1$ the system is called near-integrable. As we shall show in the next sections, near-integrable systems can be studied by normalisation techniques as averaging and Birkhoff-Gustavson normalisation.

2 Normal forms

We shall consider Hamiltonians with equilibrium $(q, p) = (0, 0)$, a non-degenerate critical point (no eigenvalues zero) of the equations of motion, which have a series expansion, at least to a certain degree, of the form

$$H(q, p) = H_2 + H_3 + H_4 + \cdots \tag{3}$$

where

$$H_2 = \frac{1}{2} \sum_{i=1}^{n} \omega_i (p_i^2 + q_i^2), \tag{4}$$

H_3 represents the cubic terms, H_4 the quartic terms etc. In the sequel we shall restrict ourselves to the case that H_2 is a positive definite quadratic form, the equilibrium $(0, 0)$ is stable.

The idea is to introduce a near-identity, polynomial transformation which simplifies the Hamiltonian while preserving its canonical character. This is called *normalisation*. In the 19th century the intention was to prove in this way integrability and construct the integrals. Clear descriptions of this technique can be found in Born (1927) and Birkhoff(1927). In the sixties this approach got a new and more useful twist to obtain qualitative and quantitative information on non-integrable systems.

More information is given below, detailed accounts can be found in Arnold (1978, 1983) and Verhulst (1996). A survey of results and a discussion of various normalisation methods can be found in Verhulst (1983).

2.1 Resonance

In the actual constructions of the near-identity transformations a crucial part is played by resonance. The frequencies $\omega_1, \ldots, \omega_n$ are satisfying a resonance relation of order $k \in \mathcal{N}$ if there exist numbers $k_1, \ldots, k_n \in \mathcal{Z}$ such that

$$k_1\omega_1 + k_2\omega_2 + \cdots + k_n\omega_n = 0 \tag{5}$$

with $k = |k_1| + |k_2| + \cdots + |k_n|$.
We shall take $I_i = \frac{1}{2}(p_i^2 + q_i^2), i = 1, \ldots, n$; we say that the Hamiltonian H is in *Birkhoff normal form to degree* d if H can be written as $H = H_d(I) + R$ with H_d of degree d and R representing the higher order terms. A Hamiltonian H_d in Birkhoff normal form is integrable.
For the construction by normalisation we have the following important theorem:

Theorem
Consider the Hamiltonian $H(q,p) = H_2 + H_3 + H_4 + \cdots$ and suppose the frequencies $\omega_1, \ldots, \omega_n$ do not satisfy a resonance relation of order $\leq k$, then there is exists a canonical transformation by polynomials such that the new Hamiltonian is in Birkhoff normal form to degree k.

Note that the canonical transformation is not unique. After carrying out this "Birkhoff normalisation" additional transformations may be helpful.
Suppose a Hamiltonian is in Birkhoff normal form to degree k, but the frequencies are satisfying a resonance relation of order $k+1$. This means that H_{k+1}, H_{k+2} etc. may contain resonant terms which cannot be transformed away. Part of H_{k+1}, H_{k+2} is non-resonant, for instance the terms which are already in Birkhoff normal form. The procedure is now to split H_{k+1} etc. in resonant terms and terms to which the Birkhoff normalisation process can be applied. The resulting normal form will also contain resonant terms and is called a *Birkhoff-Gustavson normal form*. It will contain terms dependent on the actions I and on combination angles of the form $\chi = k_1\varphi_1 + \cdots + k_n\varphi_n$.

The near-identity transformation leaves H_2 invariant. More explicitly, if $\{\ ,\ \}$ indicates again the Poisson bracket, the normalisation procedure implies that a term f in H_k is resonant if

$$\{H_2, f\} = 0.$$

It follows from the actual construction in the normalisation process that the resonant terms cannot be removed by transformation. An important

consequence is that H_2 is in involution with the Hamiltonian normal form, i.e. H_2 is an integral of the induced normalized equations of motion.
This normal form is usually truncated at a certain level, say $m \geq k+1$ and then analysed to study periodic solutions, invariant manifolds and other ingredients. A number of qualitative and quantitative results can be obtained from the truncated Birkhoff-Gustavson normal form which are valid for the original Hamiltonian system, for references see Verhulst (1983). Of course there are many open questions in this field.

2.2 Integrals of the normal form

To obtain estimates of the approximations by normalisation, it is useful to make explicit that we are considering a neighbourhood of an equilibrium point. We introduce a small, positive parameter ε and we scale

$$q = \varepsilon\,\tilde{q}, p = \varepsilon\,\tilde{p}.$$

So ε^2 is a measure for the energy relative to the equilibrium energy. Introducing this scaling, dividing by ε^2 and dropping the tildes, produces

$$H(q,p) = H_2 + \varepsilon H_3 + \varepsilon^2 H_4 + \cdots + \varepsilon^{m-2} H_m + \cdots \tag{6}$$

with H_2 as in (4). Suppose we have normalised to degree $m \geq 3$, the Hamiltonian in normal form is

$$\bar{H}(q,p) = H_2 + \varepsilon \bar{H}_3 + \varepsilon^2 \bar{H}_4 + \cdots + \varepsilon^{m-2} \bar{H}_m + \cdots$$

Until now the description is exact: solving the equations of motion corresponding with this normalised Hamiltonian and inverting the transformations, produces the solutions corresponding with the original Hamiltonian.
The next step introduces an approximation, we truncate $\bar{H}$ at the level of terms of degree m

$$\hat{H}(q,p) = H_2 + \varepsilon \bar{H}_3 + \varepsilon^2 \bar{H}_4 + \cdots + \varepsilon^{m-2} \bar{H}_m \tag{7}$$

Again, by construction, $\hat{H}$ is an integral of the phase-flow induced by the truncated Hamiltonian and so is H_2. *If at least one of the normal form expressions $\bar{H}_k, k = 3, \ldots$ is independent of H_2, we have two independent integrals of the truncated Hamiltonian in normal form.* Using the stability of the equilibrium point, it is easy to prove the following lemma:

Lemma 1 The integral $\hat{H}$ is conserved for the original Hamiltonian system 6 with error $O(\varepsilon^{m-1})$ *for all time*; the integral H_2 is conserved for the original Hamiltonian system with error $O(\varepsilon)$ *for all time*.

If the equations of motion, induced by the truncated Hamiltonian $\hat{H}$ have more than two independent integrals, we have for such additional integrals slightly weaker estimates:

Lemma 2
Suppose $F(q,p)$ is an integral of motion of the truncated Hamiltonian system (7), F_0 its value for given initial conditions, then we have for F as an approximation of the original Hamiltonian system

$$F(q,p) - F_0 = O(\varepsilon^{m-1}t)$$

Proof
We calculate the orbital derivative of F by splitting the Poisson bracket using $\bar{H} = \hat{H} + \varepsilon^{m-1}\bar{H}_{m+1} + \cdots$ and the fact that the orbits are bounded. We find

$$\frac{dF}{dt} = \{F, \bar{H}\} = \{F, \hat{H}\} + O(\varepsilon^{m-1}) = O(\varepsilon^{m-1})$$

Integration produces the estimate.

Note that in the worst case $m = 3$, the expression $F(q,p)$ is conserved for the flow induced by the original Hamiltonian (6) with error $O(\varepsilon)$ on the time-scale $1/\varepsilon$. In this sense the (exact) integral of the truncated, normalised Hamiltonian $\hat{H}$ (7), is an approximate integral of the original Hamiltonian system (6).
An important consequence is the following statement. If the phase-flow induced by the truncated Hamiltonian $\hat{H}$ (7) is (completely) integrable, the flow of the original Hamiltonian (6) is approximately integrable in the sense described above. This means that the irregular component in such a system is limited by the given error estimates and must be a small-scale phenomenon on a long time-scale.

3 Two degrees of freedom

The truncated normal form of a Hamiltonian system near stable equilibrium has two independent integrals which makes the system integrable in the case of two degrees of freedom. The integrals are $\hat{H}$ and H_2 or suitable combinations of these, see section 2.2. The energy manifold is diffeomorphic to the sphere S^3 in 4-dimensional phase-space and the first integral can be considered to describe the energy manifold, the second one corresponds with the KAM-foliation of the energy manifold. As a consequence, reducing the system to $H_2 = constant$ and considering a Poincaré-map, one finds critical points and curves in a 2-dimensional map. The critical points, corresponding with periodic solutions, will in general be saddles or centres; if the critical points are not Morse (see for instance Verhulst, 1996), this may be caused by a too heavy truncation of the normal form or it may be a genuine degeneracy. As we shall see, symmetries trigger off degeneracies.
We shall review the possible resonances after which we discuss as an example the elastic pendulum.

3.1 Resonance

It follows from (5), section 2.1 that the appropriate resonance relation is

$$k_1\omega_1 + k_2\omega_2 = 0,\ k_1, k_2 \in \mathcal{Z}.$$

(ω_1, ω_2) is called the frequency-vector, (k_1, k_2) the annihilation-vector. We consider rational frequency ratio's and include the irrational ratio's by perturbation theory. An irrational ratio r can always be approximated arbitrarily close by a rational number m/n with $m, n \in \mathcal{Z}$ and relative prime. While doing this, we introduce the "detuning" $r - m/n$ as a perturbation factor. The actual choice of m and n is determined by r and the energy level we are considering. For smaller values of the energy we have to take a more accurate approximation of r. Put in a different way: increasing the energy around equilibrium, more important resonances may be encountered as $m + n$ can be smaller. An extensive discussion can be found in Sanders and Verhulst (1983).
The presence of resonances is indicated by the orthogonality of frequency and annihilation vectors. As the quadratic part of the Hamiltonian H_2 is already

in normal form, the first resonance is found for $k = 3$:

$$(\omega_1, \omega_2) = (1, 2) \ or \ (2, 1); (k_1, k_2) = (2, -1) \ resp. \ (1, -2)$$

This resonance is sometimes called the first order resonance and it is active at H_3-level if the Hamiltonian is generic; as we shall see symmetry may destroy this.
At H_4-level we have $k = 4$ and the possibilities

$$(\omega_1, \omega_2) = (1, 3) \ or \ (3, 1) \ or \ (1, 1).$$

This is sometimes called a second order resonance. For the $(1, 1)$ resonance we have apart from the annihilation-vector $(2, -2)$ also $(1, -1)$, but as H_2 is already in normal form, the last one plays no part. The implication is that for second order resonances we have

$$\bar{H}_3 = 0.$$

Higher order resonances are defined as

$$(\omega_1, \omega_2) = (m, n) \ with \ m + n \geq 5$$

with m, n relative prime.
The main structure of phase-space for first and second order resonances at one side and higher order resonances at the other side are rather different. We shall return to this difference as the introduction of symmetries tends to shift a system to higher order resonance.

3.2 The elastic pendulum

Consider a helical spring fixed at one side and with mass m at the other side. The spring may swing in a plane like a pendulum and oscillate at the same time, see figure (1). We shall neglect friction and we consider a linear spring. The Hamiltonian system describing this system will be discrete symmetric with respect to the vertical axis. If φ is the angle of the spring with the vertical, the Hamiltonian depends (as far as φ is concerned) on $\dot{\varphi}^2$ (or the corresponding momentum) and $\cos\varphi$. We shall explore the consequences of this "mirror symmetry".
If g is the gravitational constant, l the length of the spring under load in

Figure 1: The elastic pendulum.

static position, l_0 the length of the spring without load, r the dynamically changing length of the spring, s the spring constant, we have with momenta

$$p_r = \dot{r},\ p_\varphi = r^2\dot{\varphi}$$

for the Hamiltonian

$$H = \frac{1}{2}(p_r^2 + \frac{p_\varphi^2}{r^2}) + \frac{s}{2}(r - l_0)^2 - mgr\cos\varphi.$$

Introducing the elongation of the spring

$$z = \frac{r - l}{l}$$

the point $(z, \dot{z}, \varphi, \dot{\varphi}) = (0, 0, 0, 0)$ corresponds with the equilibrium position of the spring hanging vertically at rest. We indicate the frequencies near equilibrium (obtained by linearisation) by ω_z, ω_φ. Assuming $3l = 4l_0$ and expanding the Hamiltonian with respect to z and φ we obtain, after some rescaling, (3) with

$$\begin{aligned} H_2 &= \frac{1}{2}\omega_z^2(z^2 + p_z^2) + \frac{1}{2}\omega_\varphi^2(\varphi^2 + p_\varphi^2) \\ H_3 &= a_1 z p_\varphi^2 + a_2 z \varphi^2 \\ H_4 &= b_1 z^2 p_\varphi^2 + b_2 \varphi^4 \end{aligned}$$

where the constants are not independent, they are determined by the physical parameters.
An interesting phenomenon called autoparametric excitation, arises as follows. Choose the physical parameters such that

$$\omega_z = 2\omega_\varphi.$$

Thus, in the linearised system, one oscillation in the φ-direction corresponds with two oscillations in the z-direction. Oscillation of the spring in the vertical position is a normal mode solution of the nonlinear equations of motion. This solution however, is unstable. Starting very near to the vertical, the mass transfers gradually more energy into the φ-direction until the vertical motion is nearly absent; then energy is transferred again to the z-direction until, after some time, we are again near to vertical motion. Then the process repeats itself more or less.
The phenomenon of autoparametric excitation which is destabilisation of a normal mode by coupling to another mode, is well understood, see van der Burgh (1968) for a detailed description. For our purpose we summarize this as follows.
For each value of the (small) energy we have three periodic solutions, so actually when using the energy as a parameter, we have three families of periodic solutions. The first one is the normal mode in the vertical direction which is unstable. Note that the other normal mode (in the φ-direction) does not exist. The other two periodic solutions are stable, in both of them one oscillation in the φ-direction corresponds *exactly* with two oscillations in the z-direction; in one of them the oscillations are in-phase, in the other one out-phase. This is well described by the Birkhoff-Gustavson normal form $\hat{H} = H_2 + \bar{H}_3$. Also we find from the normal form, the quasi-periodic motion on the tori around the stable periodic solutions. As has been explained before, the normal form is integrable.
The instability of the normal mode triggers off the autoparametric resonance. Starting near the normal mode we move away on a "large" torus around a stable periodic solution, but because of the recurrence in a conservative system, we return again after some time to a neighbourhood of the starting point.

The discrete symmetry does not seem to have any effect on the normal form of the Hamiltonian. However we shall show that we owe this to the special resonance relation; we shall explore this now.
Consider the other first resonance case

$$2\omega_z = \omega_\varphi \ or \ \omega_z : \omega_\varphi = 1 : 2.$$

Inspection of H_3 reveals that there are no resonant terms, so

$$\bar{H}_3 = 0.$$

To find non-trivial results we have to calculate $\bar{H}_4$. This has been carried out by van der Burgh (1975) using averaging as a normalisation technique. As we have no resonance at $k = 4$, $\bar{H}_4$ will be in Birkhoff normal form. The result of the calculation is very surprising:

$$\bar{H}_4 = 0$$

This comes as a surprise as H_4 contains the resonant term starting with b_2. It turns out however that the effect of the transformed H_3 terms annihilates this resonance. We conclude that, in the case of the 1 : 2-resonance, the discrete symmetry causes a very drastic degeneration of the normal form and so of the dynamics of the phase flow.
We note that a complete analysis of the 1 : 2-resonance for the elastic pendulum which involves terms until at least H_6, has not been carried out yet.

We consider now the second order resonances of the elastic pendulum. In these cases

$$\bar{H}_3 = 0$$

and we consider H_4. First we have

$$\omega_z : \omega_\varphi = 1 : 3 \textit{ or } 3 : 1$$

Inspection of H_4 reveals that the only possible resonant term is the one with coefficient b_2. There will be no interaction between the two modes in the dynamics generated by the normal form $\hat{H} = H_2 + \bar{H}_4$; the normal mode in the vertical position is stable in this approximation. The implication is that we have to treat the 1 : 3- and 3 : 1-resonances of the elastic pendulum as higher order resonances. We suspect that we have the same degeneration as in the case of the 1 : 2-resonance, but a complete analysis is lacking.

Consider now the case

$$\omega_z : \omega_\varphi = 1 : 1.$$

For this resonance we expect full interaction of the two modes as the term with coefficient b_1 is clearly resonant. The calculation was carried out by van der Burgh (1975) and he finds a partial degeneration. $\bar{H}_4 \neq 0$ and it is in Birkhoff normal form, so there is no energy exchange between the modes, there are some phase-interactions only. This is again an example of a degeneration of a first or second order resonance to a higher order one and we return to such a situation at the end of this section.

The analysis of the elastic pendulum can be accompanied by experiments which show that indeed the $2:1$-resonance is the only case with large-scale exchange of energy between the two modes and autoparametric instability of the vertical position.

These results for the elastic pendulum are typical for two degrees of freedom systems with a discrete symmetric Hamiltonian.

4 Three degrees of freedom

Increasing the number of degrees of freedom is very important for applications in for instance mechanical engineering and wave mechanics; it produces an immediate increase of qualitative and quantitative problems.
The phase-flow in three degrees of freedom systems near stable equilibrium takes place on the hypersphere S^5. Even in integrable cases to obtain a picture of the phase-flow is a non-trivial affair as this requires a study of all possible foliations of S^5 in a symplectic context.
In this section we shall discuss mainly first order resonances and some aspects of higher order resonance which will concern us in applications to wave equations. We focus on the question of the integrability of normal forms and the location of periodic solutions in the presence of discrete symmetries. The normal forms will be given until level k, the index of H_k, where at least two combination angles are present. Below this level we have integrability of the normal form.
One reason to take a closer look at the role of symmetries is, that genericity conditions are rarely satisfied in applications.

4.1 Resonance

In a three degrees of freedom system with

$$H_2 = \frac{1}{2}\omega_1(q_1^2 + p_1^2) + \frac{1}{2}\omega_2(q_2^2 + p_2^2) + \frac{1}{2}\omega_3(q_3^2 + p_3^2)$$

we have as basic resonance relation from (5)

$$k_1\omega_1 + k_2\omega_2 + k_3\omega_3 = 0.$$

We shall indicate $\omega_1, \omega_2, \omega_3$ as a first order resonance if at H_3-level ($k = 3$) we have at least two different resonance relations.
If there is no resonance at $k = 3$, $\bar{H}_3$ is in Birkhoff normal form, see section 2.1. If there is one resonance relation, $\hat{H} = H_2 + \bar{H}_3$ depends on the three actions and one combination angle only. In these two cases the normal form to this order is integrable.
We find the following first order resonances:

- $1 : 2 : 1$
- $1 : 2 : 2$
- $1 : 2 : 3$
- $1 : 2 : 4$

As we have noted before, $\hat{H}$ and H_2 are integrals of the equations of motion. So if we can find a third integral, the equations of motion generated by the normal form are integrable and in that case the irregular component of the phase-flow near stable equilibrium is small in size (as before, scaling with respect to the small parameter ε is the best way to make this statement more precise).
The integrability of the normal form in three degrees of freedom systems turns out to be a difficult question. Also, in the case of non-integrability we still have little insight in the consequences of this for the phase-flow. Although all the expressions are much more complicated than in the two degrees of freedom case, the analysis of periodic solutions is still possible.
In the sequel we discuss questions of integrability and periodic solutions of normal forms, both in the general case and in cases with discrete symmetry.

Figure 2: The action simplex for the $1:2:1$-resonance

4.2 The $1:2:1$-resonance: periodic solutions and integrability

The annihilation vectors at H_3-level are $(2,-1,0)$, $(0,-1,2)$ and $(1,-1,1)$. This means that $\hat{H}_3$ is in Birkhoff-Gustavson normal form with three combination angles; in action-angle coordinates it becomes with the scaling as in (7)

$$\begin{aligned}\hat{H} \;&=\; I_1 + 2I_2 + I_3 + \varepsilon 2\sqrt{2I_2}[a_1 I_1 \cos(2\phi_1 - \phi_2 - a_2) \\ &\quad + a_3\sqrt{I_1 I_3}\cos(\phi_1 - \phi_2 + \phi_3 - a_4) + a_5 I_3 \cos(2\phi_3 - \phi_2 - a_6)] \qquad (8)\end{aligned}$$

The normal form (8) and the corresponding equations of motion have been analysed by van der Aa (1983). She finds periodic solutions by looking for critical points of the equations of motion. Generically there are 7 families of (short-)periodic solutions which have been depicted in the action simplex in figure (2). In such a simplex we present, out of the 6 variables, the 3 actions only; as H_2 is an integral of the normal form, for a fixed value of the energy the periodic solutions are located in the front plane

$$I_1 + 2I_2 + I_3 = constant$$

In figure (2) the stability, in terms of the eigenvalues of the system linearised around the periodic solutions, is indicated by letters E (elliptic), H (hyperbolic) and C (complex). Note that there are 6 eigenvalues, conjugate in pairs of 2, of which at least 2 are zero. E indicates a pair of purely imaginary eigenvalues, H indicates a pair of real eigenvalues (one positive, one negative) and C a pair of complex eigenvalues (non-vanishing real and imaginary part). It is clear from figure (2) that only one normal mode survives which is unstable.

The integrability of the normal form (8) has been settled by Duistermaat (1984). First it is noted that a number of isolated cases, special choices of the coefficients in the normal form, yield integrable cases. Also that the hypersurface $\bar{H}_3 = 0$ constitutes an invariant manifold on which all the solutions are periodic. One considers then the period function of these periodic solutions by complex continuation which turns out to show infinite branching at certain points. This excludes the existence of a third analytic integral of the normalized Hamiltonian $\hat{H} = H_2 + \varepsilon \bar{H}_3$ (8). It is still an open question what this means for the dynamics of the system.
Apart from this proof it is possible to give a dynamic characterisation of the phase-flow in a rather symmetric case. Restriction to manifolds $\bar{H}_3 =$ *constant* yields a reduction to a two degrees of freedom Hamiltonian system. With certain assumptions on the coefficients in (8) and on adding $\bar{H}_4$ tot the normal form, chosen from an open set of normalized quartic polynomials, it is proved that a homoclinic spiral exists. This enables us to apply Devaney's (1976) theorem which implies the presence of a horseshoe map in the flow, an infinite number of unstable periodic orbits and chaotic behaviour in the normal form $\hat{H} = H_2 + \varepsilon \bar{H}_3 + \varepsilon^2 \bar{H}_4$.

4.3 The $1:2:1$-resonance: discrete symmetry

The symmetries which arise in applications may destroy the results discussed above. Assume first that we have discrete symmetry in the first or third degree of freedom (or both).
In the normal form (8) this implies $a_3 = 0$ and the analysis was given by Verhulst (1984). The periodic solutions, taking into account various coordinate transformtions if $I_1 I_2 I_3 = 0$, all move to the coordinate planes , in particular the periodic orbits in general position move to the hyperplanes $I_1 = 0$ respectively $I_3 = 0$. At $a_1 = a_5$ there is an exchange of stability between solutions in the $I_1 = 0$ and $I_3 = 0$ hyperplanes. There is still only one normal mode, see figure (3).

Figure 3: The action simplex for the $1:2:1$-resonance with discrete symmetry in the first and,or the third degree of freedom.

It is interesting to repeat Duistermaat's (1984) analysis in the case of this symmetry. To study the flow at $\bar{H}_3 = 0$, a linear symplectic transformation is used which leaves H_2 invariant. This reduces the normal form (8) exactly to the discrete symmetric Hamiltonian which we are considering. We conclude that even on assuming discrete symmetry in the first or third degree of freedom, in general no analytic third integral exists (apart from the cases of special coefficients).
An example of such a symmetric system is the elastic spherical pendulum, an extension of the pendulum in section 3.2, where the frequency 2 characterizes the vertical motion, the frequency 1 applies to motion in the two other directions. However, this sytem has an extra symmetry which evokes the existence of the angular momentum integral with respect to the vertical axis. So the normal form is integrable. One may conjecture that if one applies an small perturbation by a gravitational or electromagnetic force in one horizontal direction, this will cause non-integrability.

The assumption of discrete symmetry in the second degree of freedom changes the analysis dramatically. We find

$$\bar{H}_3 = 0$$

and we have to normalize to H_4. The normal form is

$$\hat{H} = I_1 + 2I_2 + I_3 + \varepsilon^2 \bar{H}_4(I_1, I_2, I_3, \chi)$$

Figure 4: The action simplex for the $1:2:2$-resonance

with $\chi = 2(\phi_1 - \phi_3)$. It is clear that $I_2 = constant$ is a third integral of motion. To this order of approximation, i.e. to $O(\varepsilon^2)$ on the time-scale $1/\varepsilon^2$, the system acts as a two degrees of freedom system in $1:1$-resonance and an uncoupled anharmonic one degree of freedom oscillator. Non-trivial behaviour on a longer time-scale can be studied by calculating higher order normal forms.

4.4 The $1:2:2$-resonance: periodic solutions and integrability

This resonance behaves rather differently from the preceding one. The annihilation vectors at H_3-level are $(2,-1,0)$ and $(2,0,-1)$. This means that $\hat{H} = H_2 + \bar{H}_3$ is in Birkhoff-Gustavson normal form with two combination angles; in action-angle coordinates (**??**) it becomes

$$\begin{aligned}\hat{H} &= I_1 + 2I_2 + 2I_3 + \\ &\quad +2\varepsilon I_1[a_1\sqrt{2I_2}\cos(2\phi_1 - \phi_2 - a_2) + a_3\sqrt{2I_3}\cos(2\phi_1 - \phi_3 - a_4)] \quad (9)\end{aligned}$$

The normal form (9) and the corresponding equations of motion have been analysed by van der Aa (1983) and more extensively by van der Aa and Verhulst (1984). One finds again periodic solutions by looking for critical points of the equations of motion. Generically there are 2 general position families of (short-)periodic solutions which have been depicted in the action simplex, see figure (4).

However, the hyperplane $I_1 = 0$ contains a continuous family of periodic solutions which is non-generic. It turns out that in normalising to H_4, this manifold of periodic solutions breaks up into 6 periodic solutions among which two unstable normal modes (in the second and third degree of freedom); see figure (4).

The integrability has been studied by van der Aa and Verhulst (1984), inspired by an application in astrophysics (Martinet, Magnenat and Verhulst, 1981). They find that the normal form $\hat{H} = H_2 + \varepsilon \bar{H}_3$ contains a certain symmetry which enables reduction to a two degrees of freedom system. In fact it is this symmetry which gives rise to the presence of a manifold of periodic solutions. By the reduction a third integral of motion is found.
Note that, referring to the discussion about the approximation character of normal forms in section 2.2, this implies that the chaotic motion which is present in the original Hamiltonian, must be a small-scale affair according to our estimates, i.e. at most $O(\varepsilon)$ on the time-scale $1/\varepsilon$.
The integrability of the normal form to $\bar{H}_3$ gives rise to foliations of S^5 which have not been studied up till now. Another open question concerns what will happen to integrability on adding the $\bar{H}_4$ normal form terms. It turns out that the reduction applied by van der Aa and Verhulst (1984) does not carry over.

4.5 The $1:2:2$-resonance: discrete symmetry

First we note that the discrete symmetric terms in the first degree of freedom in H_3 are resonant, the asymmetric terms are not, so the assumption of discrete symmetry in the first direction does not change or simplify the picture sketched above.
The assumption of discrete symmetry in the second or third degree of freedom however, yields a drastic reduction as in this case the Birkhoff-Gustavson normal form to degree 6 becomes

$$\hat{H} = H_2 + \varepsilon^2 \bar{H}_4(I_1, I_2, I_3, \chi_1) + \varepsilon^4 \bar{H}_6(I_1, I_2, I_3, \chi_2, \chi_3)$$

with $\chi_1 = 2(\phi_2 - \phi_3), \chi_2 = 4\phi_1 - 2\phi_2, \chi_3 = 4\phi_1 - 2\phi_3$. So this normal form is still integrable when truncated at $\bar{H}_5$ and chaos in the original Hamiltonian has to be a small-scale affair, size $O(\varepsilon^4 t)$. The system behaves to a good approximation as a $1:1$-resonant system in the second and third degree of

Figure 5: The action simplex for the $1:2:3$-resonance

freedom and a decoupled first degree of freedom. Interactions between the three degrees of freedom arise only through H_6.

4.6 The $1:2:3$-resonance: periodic solutions and integrability

The Hamiltonian with this resonance behaves differently again. The annihilation vectors at H_3-level are $(2,-1,0)$ and $(1,1,-1)$. This means that $\hat{H} = H_2 + \varepsilon \bar{H}_3$ is in Birkhoff-Gustavson normal form with two combination angles; in action-angle coordinates (**??**) it becomes

$$\begin{aligned}\hat{H} &= I_1 + 2I_2 + 3I_3 + 2\varepsilon\sqrt{2I_1I_2}[a_1\sqrt{I_1}\cos(2\phi_1 - \phi_2 - a_2) + \\ &\quad a_3\sqrt{I_3}\cos(\phi_1 + \phi_2 - \phi_3 - a_4)] \qquad (10)\end{aligned}$$

The normal form (10) and the corresponding equations of motion have been analysed by van der Aa (1983) who finds find 7 families of short-periodic solutions. One finds two unstable normal modes (in the second and third direction), four general position periodic orbits of which two are stable, two unstable. There is one stable periodic solution in the $I_2 = 0$ hyperplane; see figure (5).

The integrability of the normal form (10) has been studied by Hoveijn and Verhulst (1990) where geometric characterization and nonlinear analysis both play a part. They focus on the case that the normal mode π_2 in the second

Figure 6: Projection of the level set N containing periodic, homoclinic and heteroclinic solutions.

degree of freedom is complex unstable; this case arises for an open set of coefficients in H_3. One can study the phase flow on the invariant level set N defined by $H_2 = constant, \bar{H}_3 = 0$. The level set N contains three periodic solutions: π_2, π_3, π_4, see figure (6) and one can find heteroclinic orbits between π_2 and π_3; also one can identify submanifolds which behave like separatrix manifolds and which contain a one-parameter family of orbits homoclinic to the complex unstable solution π_2. This is an obstruction to applying Devaney's (1976) result to prove non-integrability as for this one needs a transverse homoclinic to a complex unstable periodic solution.

The next step is to look at the consequences of adding the $\bar{H}_4$-terms. One expects a heteroclinic orbit and a one-parameter family of homoclinic orbits to break up under small perturbation. Indeed, the heteroclinic connection between π_2 and π_3 does not longer exist on adding $\bar{H}_4$. It turns out that

the one-parameter family of homoclinic solutions breaks up to produce a tranverse homoclinic solution. The break-up is demonstrated numerically by Hoveijn and Verhulst (1990), a proof using the Melnikov function has been given by Hoveijn (1992). This implies that Devaney's (1976) results applies, i.e. a horseshoe map is imbedded in the flow near the homoclinic orbit of the normal form to degree 4.
An open question is still the integrability of $\hat{H} = H_2 + \bar{H}_3$ (10). Numerical analysis suggests non-integrability but a proof is lacking. Also, Hoveijn and Verhulst (1990) assumed that the normal mode π_2 is complex unstable (C); it is an open question what happens if we have the other instability case (HH).

4.7 The $1:2:3$-resonance: discrete symmetry

The assumption of discrete symmetry in the first direction produces the Birkhoff-Gustavson normal form

$$\hat{H} = H_2 + \varepsilon \bar{H}_3(I_1, I_2, I_3, \chi_1) + \varepsilon^2 \bar{H}_4(I_1, I_2, I_3) + \varepsilon^3 \bar{H}_5(I_1, I_2, I_3, \chi_2)$$

with $\chi_1 = 2\phi_1 - \phi_2, \chi_2 = 3\phi_2 - 2\phi_3$. Until $k = 4$, the system behaves like a two degrees of freedom system (first and second direction) and an uncoupled one degree of freedom system.
The assumption of discrete symmetry in the second direction produces the normal form

$$\hat{H} = H_2 + \varepsilon^2 \bar{H}_4(I_1, I_2, I_3, \chi_1, \chi_2)$$

with $\chi_1 = 3\phi_1 - \phi_3, \chi_2 = \phi_1 - 2\phi_2 + \phi_3$. Interaction between the three degrees of freedom takes place at a more prominent level than in the preceding (and next) case.
Assumption of discrete symmetry in the third direction yields

$$\hat{H} = H_2 + \varepsilon \bar{H}_3(I_1, I_2, I_3, \chi_1) + \varepsilon^2 \bar{H}_4(I_1, I_2, I_3) + \varepsilon^3 \bar{H}_5(I_1, I_2, I_3, \chi_2)$$

with $\chi_1 = 2\phi_1 - \phi_2, \chi_2 = 3\phi_2 - 2\phi_3$. As in the first case, until $k = 4$, the system behaves like a two degrees of freedom system (first and second direction) and an uncoupled one degree of freedom system.
The integrability questions of the Birkhoff-Gustavson normal forms in these symmetric cases have not been resolved.

Figure 7: The action simplex for the 1 : 2 : 4-resonance

4.8 The 1 : 2 : 4-resonance: periodic solutions and integrability

The Hamiltonian with this resonance has not been studied extensively. The annihilation vectors at H_3-level are $(2, -1, 0)$ and $(0, 2, -1)$. This means that $\hat{H} = H_2 + \bar{H}_3$ is in Birkhoff-Gustavson normal form with two combination angles; in action-angle coordinates (**??**) it becomes

$$\begin{aligned} \hat{H} &= I_1 + 2I_2 + 4I_3 + \\ &\quad 2\varepsilon[a_1 I_1 \sqrt{2I_2} \cos(2\phi_1 - \phi_2 - a_2) + a_3 I_2 \sqrt{2I_3} \cos(2\phi_2 - \phi_3 - a_4)] \end{aligned} \tag{11}$$

The normal form (11) and the corresponding equations of motion have been analysed by van der Aa (1983) who finds only three families of short-periodic solutions. There is one unstable normal mode in the second direction with two real eigenvalues and four eigenvalues zero, so one should study the effect of including $\bar{H}_4$. There are two general position periodic orbits which are stable (EE) or unstable (EH), depending on the parameters; see figure (7).

The question of integrability of the normal form (11) has not been settled. Using the Poisson bracket (section 2.1), van der Aa (1983) has shown that no independent quadratic or cubic third integral exists.

4.9 The 1 : 2 : 4-resonance: discrete symmetry

The assumptions of discrete symmetry produce a variety of results. With discrete symmetry in the first direction, the Birkhoff-Gustavson normal form

(11) does not change and we have the analysis as before.
With discrete symmetry in the second direction we have

$$\hat{H} = H_2 + \varepsilon \bar{H}_3(I_1, I_2, I_3, \chi_1) + \varepsilon^2 \bar{H}_4(I_1, I_2, I_3) + \varepsilon^3 \bar{H}_5(I_1, I_2, I_3, \chi_2)$$

with $\chi_1 = 2\phi_2 - \phi_3, \chi_2 = 4\phi_1 - \phi_3$. So the normal form is integrable to $\bar{H}_4$-level, the analysis to $\bar{H}_5$-level is still open.
A different case again is produced by discrete symmetry in the third direction. We find

$$\hat{H} = H_2 + \varepsilon \bar{H}_3(I_1, I_2, I_3, \chi_1) + \varepsilon^2 \bar{H}_4(I_1, I_2, I_3) + \varepsilon^4 \bar{H}_6(I_1, I_2, I_3, \chi_1, \chi_2)$$

with $\chi_1 = 2\phi_1 - \phi_2, \chi_2 = 4\phi_2 - 2\phi_3$. So the normal form is integrable to $\bar{H}_5$-level.

4.10 Second and higher order resonance

We have a second order resonance if $H_2 + \varepsilon \bar{H}_3$ has less than two combination angles and $H_2 + \varepsilon \bar{H}_3 + \varepsilon^2 \bar{H}_4$ has at least two combination angles. If we need normal forms of degree higher than four to obtain two resonant combination angles, we will call this higher order resonance. It does not make sense to try to have a systematic study of all the possible cases, there are just too many. Also, in applications we always have degeneracies and higher order resonance is the rule more than the exception.
We shall illustrate degeneracy for the 1 : 3 : 7-resonance and we discuss briefly two types of resonances arising frequently in applications.

The 1 : 3 : 7-resonance is a second order one as we have

$$\hat{H} = H_2 + \varepsilon^2 \bar{H}_4(I_1, I_2, I_3, \chi_1, \chi_2)$$

with $\chi_1 = 3\phi_1 - \phi_2, \chi_2 = \phi_1 + 2\phi_2 - \phi_3$. This is however the generic case; if we have for instance discrete symmetry in the first degree of freedom, we have to normalize to H_6. Moreover this symmetric normal form contains a four dimensional invariant subset, the analysis of which requires normalization to H_{10}. Verhulst and Hoveijn (1992) show for this resonance, that certain natural symmetry assumptions require normalization to H_{20}. This demonstrates again that a study of such normal forms makes sense while keeping an eye on possible applications.

Among second order resonances, the $1:1:1$-resonance is the most complicated one, at H_4-level there are six combination angles. In applications it arises in chains of identical oscillators and in astrophysics. The last application involves the motion of stars in elliptical galaxies which have planes with reflection symmetry. The results are a classification of periodic orbits and integrability of the normal form. The integrability is of importance with regards to the statistical mechanics of such stellar systems. See de Zeeuw (1982, 1984) for details.

An interesting frequency resonance is $1:2:m$ with for the natural number $m, m \geq 5$. If $m = 5, 6$, we have a second order resonance. The resonance arises in nonlinear chains and in wave equations in the following way. Suppose we have a prominent, first order resonance like $1:2$; in wave theory this would produce a two waves or two modes interaction. Is it possible to influence, even to destabilize this two waves system by a resonant higher order mode?
This type of resonance has been systematically analysed by van der Aa and de Winkel (1988, 1994), we summarize their results. Some aspects of these resonances have also been studied by Wang, Bosley and Kevorkian (1995).
It turns out, the analysis is different in the four cases $m = 5, 6$ and if $m \geq 7$, whether m is even or odd. Intuitively one would expect that if we would put no energy in the third degree of freedom, the system behaves like the $1:2$-resonance case treated in section 3; this is nearly correct: if $I_3 = 0$ we have an unstable normal mode in the second direction except if $m = 6$. In all the cases we find two stable out of phase periodic solutions, see the simplex in figure (8).
Also intuitively one would expect that putting no energy in the first degree of freedom, the system would behave like a higher order resonance as discussed in section 3.6. This picture looks roughly like it but there are distinct changes.

Apart from the case $m = 6$ where no normal mode in the second direction exists, we have a normal mode in the second direction which is unstable. Also the resonance manifold which, if it exists, contains two periodic solutions in $2:m$-resonance, is of a different topological structure as both periodic so-

Figure 8: Energy simplex for the $1 : 2 : m$-resonance, $m \geq 5, m \neq 6$. There are four to six periodic solutions, depending on the parameters, and two normal modes of which the one in the third direction is stable.

lutions are unstable (with corresponding stable and unstable 2-dimensional invariant manifolds). However, there exists a stable normal mode in the third direction. Depending on the parameters of the Hamiltonian, the resonance manifold moves between the two normal modes. There are no general position periodic orbits.
We return now to the question whether it is possible to destabilize a system in $1 : 2$-resonance by adding a higher order mode, even if it has small energy. The energy simplex suggests that the answer is yes if we do not choose the initial conditions near one of the stable periodic solutions of the $1 : 2$-resonance.

Van der Aa and de Winkel (1988, 1994) also discuss the cases of discrete symmetry which lead to strong degenerations.

5 A case with n degrees of freedom

Most of the attention to nonlinear chains or lattices in the literature, is by investigations using numerical techniques. The difficulty of analyzing normal forms for nonlinear chains or lattices arises from the presence of resonances or near-resonances. In the case of resonance, an increase of the number of degrees of freedom makes the mathematical analysis of generic normal forms (in resonance) very complicated. An exception is the $1 : 2 : 2 : \cdots : 2$-resonance which has been studied by van der Aa and Verhulst (1984).

Consider the Hamiltonian with n degrees of freedom (6) where we have

$$H_2 = \sum_{j=1}^{n} \frac{1}{2}(\omega_j^2 q_j^2 + p_j^2)$$

and $\omega_1 = 1, \omega_2 = \cdots = \omega_n = 2$. On calculating the normal form $\hat{H} = H_2 + \varepsilon \bar{H}_3$, we find the same symmetry as for the $1:2:2$-resonance. In this normal form we have in action variables

$$H_2 = I_1 + 2\sum_{j=2}^{n} I_j \tag{12}$$

$$\bar{H}_3 = 4I_1 \sum_{j=2}^{n} a_{2j-3}\sqrt{I_j}\cos\chi_j \tag{13}$$

with for the combination angles $\chi_j = 2\phi_1 - \phi_j - a_{2j-2}, j = 2, \cdots, n$.
The normal form is completely integrable with $n-1$ quadratic first integrals and one cubic integral, which are found in the same way as for the $1:2:2$-resonance. Also, as in section 4.4, we find two stable general position periodic orbits (found as relative equilibria of $\bar{H}_3$ restricted to H_2).
It is easily seen that putting $I_1 = 0$ produces an invariant subset in which the dynamics is governed by a $(n-1)$ degrees of freedom Hamiltonian system in $1:1:\cdots:1$-resonance. We expect that, upon adding the $\bar{H}_4$ terms, this set will break up to produce a.o. a number of periodic solutions and unstable normal modes.It is instructive to consider the recurrence of the actions when starting near an unstable normal mode; van der Aa and Verhulst (1984) produce such a picture, see figure (9) for a five degrees of freedom system in $1:2:2:2:2$-resonance.

Both the existence of the integrals and the invariant subset are triggered off by the symmetry of the normal form of degree 3. Note that, in contrast, the $2:1:1:\cdots:1$-resonance behaves differently. The normal form in that case contains more combination angles, lacks a special symmetry and is generally non-integrable.

6 Evolution towards symmetry

In evolution problems we have explicit time-dependence without the usual assumption of periodicity in time; this poses hard problems. Still there are

Figure 9: The actions $I_1, \cdots, I_5$ as functions of time for a system with five degrees of freedom in $1:2:2:2:2$-resonance. The solutions were obtained by numerical integration with $\varepsilon = .1$; the initial conditions are $q_2(0) = 1, q_1(0) = q_3(0) = q_4(0) = q_5(0) = .1, p_1(0) = \cdots = p_5(0) = 0$. Note the recurrence on the time-scale $1/\varepsilon$.

mathematical techniques to handle this and we shall discuss studies of evolution towards symmetry in one and two degrees of freedom problems. We start however with an evolution problem in a different formulation i.e. tidal evolution. The surprise in this case is that the outcome of evolution is rather symmetric.

6.1 Tidal evolution in the two-body problem

Consider the Newtonian two-body problem where the bodies are stars, star and planet or planet and satellite. We assume that the masses are outside the Roche limit which determines the break-up of a body by tidal forces. The bodies contain fluid (water, gases etc.) which is affected by gravitation, resulting in tidal bulges. The basic mass distribution of the bodies is spherical apart from the fluid bulges; of course the bulges in their turn influence the motion but this is a higher order effect.
In such a mechanical system we have energy dissipation of a form which depends strongly on the actual physical conditions: the location of continents, the nature of the fluid etc. Apart from energy dissipation the tidal bulges

produce a torque which results in an exchange of orbital and spin angular momenta.
Because of tidal evolution the bodies will slowly spiral in and remarkably enough, we can predict the possible outcome of evolution in such a system *without* a more detailed specification of the physical mechanism. This is important as the long time behaviour of for instance the Solar System, cannot be understood without taking into account tidal evolution.
The analysis which we present is based on Counselman (1973) where the two-dimensional case is treated, and Hut (1980) which deals with the full three-dimensional case. For the slowly changing energy we have in relative coordinates

$$E = -G\frac{Mm}{2a} + \frac{1}{2}I_1|\mathbf{\Omega}|^2 + \frac{1}{2}I_2|\omega|^2$$

where G is the gravitational constant, M and m the masses of the bodies, a the length of the (slowly changing) semi-major axis, I_1 and I_2 the moments of inertia, $\mathbf{\Omega}$ and ω the respective angular velocities of rotation. The total angular momentum $\mathbf{L}$ will be conserved

$$\mathbf{L} = \mathbf{h} + I_1\mathbf{\Omega} + I_2\omega$$

with $\mathbf{h}$ the orbital angular momentum.
To find the possible equilibrium states we determine the critical points of E under the constraint that $\mathbf{L}$ is constant in a six-dimensional subspace (obtained after some reductions) with nine parameters. The result is that we have

- no equilibrium state if $L < L_{crit}$ with $L = |\mathbf{L}|$ and L_{crit} a positive number depending on the masses and the moments of inertia.
- two equilibrium states if $L > L_{crit}$ which correspond with orbits which are coplanar, circular and corotating. A very symmetric end stage.

To determine the stability of the equilibrium states one considers the second order variation of the energy E under the constraint $\mathbf{L}$ is constant. One finds that for stability the orbital angular momentum has to exceed a critical value; this means that the equilibrium state corresponding with the widest orbit is stable, the other one is unstable.
We note that it is quite remarkable that without specification of the mechanism of tidal dissipation we can determine the outcome of evolution. Also, that the equilibrium states clearly display a certain symmetry.

6.2 Evolution towards symmetry in one degree of freedom

Studies of the evolution of actual physical systems are difficult and so relatively rare. We propose therefore to ignore, at least for the time being, the actual physical mechanisms and to consider systems described by a simple looking Hamiltonian of the form

$$H(p, q, \epsilon t) = \frac{1}{2}(p^2 + q^2) - \frac{1}{3}a(\varepsilon t)q^3 \tag{14}$$

where the asymmetric part is slowly vanishing as we put for $a(\varepsilon t)$ a smooth, monotonically decreasing function for which

$$a(0) = 1, \lim_{t\to\infty} a(\varepsilon t) = 0, 0 < \varepsilon \ll 1 \tag{15}$$

The problem was analysed by Huveneers and Verhulst (1997). Putting $p = \dot{x}$, $q = x$ we have the equation of motion

$$\ddot{x} + x = a(\varepsilon t)x^2 \tag{16}$$

We note that in the autonomous system (16), $\varepsilon = 0$, there are basically two regions: within the homoclinic solution the orbits are bounded, outside the homoclinic solution the orbits diverge to infinity (with the exception of the stable manifold and the saddle point itself). In system (16) for $\varepsilon > 0$ we have no fixed saddle point, still it turns out that we have two separate regions of initial values in which the orbits are bounded or diverge to infinity. It is instructive (though slightly wrong) to view system (16) as having a saddle point moving slowly towards infinity and having a slowly expanding homoclinic orbit. In this picture, an orbit can remain bounded in two ways, either by starting inside the homoclinic orbit, or by getting "captured" by the slowly expanding homoclinic orbit, which can only happen if the orbit starts sufficiently close to the stable manifold of the saddle point. To make these statements mathematically correct, one should use the concept of *normally hyperbolic motion*. One of the aims of this study is then to obtain adiabatic invariants characterizing the dynamics of the problem.
Two key ideas play a part in the paper by Huveneers and Verhulst (1997). First, by using a simple transformation a direct relation to dissipative mechanics is established. Secondly the subsequent analysis in this paper is

based on averaging methods using elliptic and hypergeometric functions but, because of its relation to dissipative mechanics and "crossing of separatrix" aspects it clearly profits from the results by Haberman (1983), Robinson (1983), Haberman and Ho (1994,1995) and Bourland and Haberman (1990, 1991). Rand (1990) used a different approach (Jacobian elliptic functions) to study a similar class of dynamical systems. We also mention Neishtadt (1987) and a nice survey of the theory of adiabatic invariants by Henrard (1993).
The key step in analyzing system (16) is performing the transformation

$$y = a(\varepsilon t)x \tag{17}$$

The idea behind this transformation is to fix the normally hyperbolic motion of system (16). Since we want to study system (16) *for all time*, this time-dependent rescaling of the coordinates enables us to study a *bounded* domain, which simplifies the calculations considerably. This transformation has also disadvantages, in particular the loss of the Hamiltonian structure. The special choice $a(\varepsilon t) = e^{-\varepsilon t}$ will be used to show some of the more general results; for the general treatment see Huveneers and Verhulst (1997). With this choice for $a(\epsilon t)$, system (16) becomes

$$\ddot{y} + y - y^2 = -2\varepsilon\dot{y} - \varepsilon^2 y \tag{18}$$

From the original equation describing evolution towards symmetry, we have now obtained a dissipative system. The region of attraction of system (18) is bounded by the stable manifold of the saddle point.
It is instructive to combine the picture with the homoclinic orbit of the unperturbed system (18), $\varepsilon = 0$, with the perturbed system, $\varepsilon = 0.1$, see figure (10).
By averaging one determines an adiabatic invariant inside the homoclinic solution which takes a different form in different domains of the phase-plane. The location of the stable manifold of the saddle point of system (18) is calculated by considering the variation of the energy along the stable manifold. Since this variation is an $O(\varepsilon)$ effect, we may use the unperturbed stable manifold in this calculation, which involves elliptic functions. One of the conclusions is:
There exists a global adiabatic invariant inside the homoclinic orbit of the unperturbed system with the exclusion of an exponentially thin boundary layer,

Figure 10: Left the dynamics of the unperturbed system, right the stable and unstable manifold of system (18) with $\varepsilon = 0.1$

valid for all time.

An explicit calculation of the adiabatic invariant and transforming back to the original $x, \dot{x}$ variables produces an interesting result. The level curves of the adiabatic invariant *for a fixed time* "resemble" ellipses, of which the long axis and the short axis differ by an $O(\varepsilon)$ amount. and which are rotated around the origin, causing asymmetry.

This behaviour persists when t tends to infinity. Put in other words, when t goes to infinity, our dynamical system (16) becomes symmetric (with respect to x and $\dot{x}$), but the level curves of the adiabatic invariant remain asymmetric. So we have reached the following conclusion:

The evolution of an ensemble of phase points towards a symmetric potential will show significant (i.e. $O(\varepsilon)$) traces of its asymmetrical past, for all time.

So there is a sort of hysteresis effect present: although the system becomes symmetric, it still "knows" that it was asymmetric in the past. To demonstrate this phenomenon visually, we have to take ε not too small, so taking $\varepsilon = \frac{1}{4}$ figure (11) shows a few level curves of the adiabatic invariant for $a(\epsilon t) = e^{-\epsilon t}$ and t fixed at infinity. The asymmetric effect is clearly present. We expected to see the influence of the slowly decaying asymmetry in the neighbourhood of the boundary layer separating the stable and unstable region, but it turns out that there are also $(O(\varepsilon))$ effects close to the origin for all time.

Figure 11: A few level curves of the adiabatic invariant for t fixed at infinity

6.3 Evolution towards symmetry in two degrees of freedom

A natural question is how to extend the preceding results to Hamiltonian systems with two degrees of freedom. It turns out that this is quite difficult; however, with some restrictive assumptions, Huveneers (1997) obtained results which we shall review. The difficulties can be summarised as follows:

- There is very little experience with a global (i.e. global in the energy) analysis of Hamiltonian systems, time-dependent or not.
- For two and more degrees of freedom there are resonance manifolds in phase space which pose special obstructions.
- The Hamiltonian may be integrable, near-integrable or non-integrable which requires different treatments.
- In the integrable case we should employ action-angle variables but these are very difficult to construct globally.

We start again with

$$H = H_{sym} + a(\varepsilon t)H_{asym}, \tag{19}$$

where $H_{sym}$is the symmetric part of the Hamiltonian, H_{asym} the asymmetric part and a decreases again monotonically and smoothly from 1 to 0. The condition on ε ensures adiabatic evolution to symmetry.

A second degree of freedom introduces an important complication: *resonance*. It is clear from normal form theory that in order to study a non-trivial system, we must choose the linear part to be in low order resonance, like 1:2, 1:1 or 1:3. Here the 1:2 resonance has been chosen, since the first resonant nonlinear terms appear already at third order in this case. Actually, the author notes that the calculations for the 1:1 resonance are easier (for various mathematical reasons), although one has to take fourth order terms into account. The calculations for the 1:3 resonance are more elaborate, on the other hand. We will concentrate on the case in which the unperturbed system, $\varepsilon = 0$,is integrable:

$$H = \frac{1}{2}p_1^2 + 2x_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}x_2^2 + a(\varepsilon t)\left\{2x_1^3 + x_1x_2^2\right\} \tag{20}$$

for which the unperturbed system ($\varepsilon = 0$)

$$H_0 = \frac{1}{2}p_1^2 + 2x_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}x_2^2 + 2x_1^3 + x_1x_2^2 \tag{21}$$

possesses the two first integrals

$$E_1 = \frac{1}{2}p_1^2 + 2x_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}x_2^2 + \left(2x_1^3 + x_1x_2^2\right) \tag{22}$$

$$E_2 = x_2p_1p_2 + x_1x_2^2 - x_1p_2^2 + \left(\frac{1}{4}x_2^4 + x_1^2x_2^2\right) \tag{23}$$

Note that the unperturbed potential is discrete symmetric with respect to x_2, but asymmetric with respect to x_1. The corresponding equations of motion are

$$\left\{\begin{array}{rcl} \dot{x}_1 &=& p_1 \\ \dot{p}_1 &=& -4x_1 - x_2^2 - 6x_1^2 \\ \dot{x}_2 &=& p_2 \\ \dot{p}_2 &=& -x_2 - 2x_1x_2 \end{array}\right.$$

Figure 12: The equipotential curves of the unperturbed system.

Identifying $2x_1^2 + \frac{1}{2}x_2^2 + 2x_1^3 + x_1 x_2^2$ as the potential energy of the unperturbed system, we can draw the equipotential curves which give a crude qualitative description of the dynamics, figure (12).

From figure (12) it is clear that the neighbourhood of the origin contains bounded solutions which live on compact tori due to the Liouville-Arnold theorem. Near the two saddle (hyperbolic) points the energy manifold loses compactness, the tori break open, resulting in unbounded and less interesting dynamics. With this picture in mind, we can give a crude qualitative description of the perturbed system, for which the coefficient a decreases slowly to zero. Indeed, on an $O(1)$ timescale the system will behave like depicted in figure (12), but due to the slow decrease in a, the hyperbolic points are moving slowly towards infinity, resulting in capturing of nearby orbits; see for comparison similar behaviour in the one-degree-of-freedom problem discussed above. The unperturbed system possesses a heteroclinic orbit connecting the two hyperbolic points. The heteroclinic orbit is important for our analysis, since it may be regarded as a periodic orbit with an infinitely large period, so averaging is due to fail near this heteroclinic orbit.
To do any computations, one must know the range of values which (E_1, E_2)

Figure 13: The possible values of E_1 and E_2

can take (in the unperturbed problem). The range of E_1 is easily determined from the compactness requirement to be $\left[0, \frac{1}{4}\right]$. Determining the range for E_2 is more subtle since it depends on E_1. In figure (13) the range of (E_1, E_2) values is drawn. It is worthwile to note that at $E_{2,min}$ and $E_{2,max}$ there is only one value of x_1 at which the orbit is allowed to intersect the hyperplane $x_2 = 0$. This implies that these orbits are *periodic*. The one-parameter (E_1) family of these orbits constitutes a *manifold of relative equilibria* , see also Derks and Valkering (1992), Derks (1992), Zeegers (1993), Mitropolsky (1963).

There are two more sources of periodic orbits. The first one are the resonant tori and the second one is the normal mode $x_2 = p_2 = 0$, i.e. $E_2 = 0$ which is unstable (hyperbolic) for any $E_1 \leq \frac{1}{4}$. The main difficulty in analysing the perturbed system (20) is that the interesting dynamics takes place on a slowly expanding subset of phase space. Analogous to the 1-degree of freedom case (Huveneers and Verhulst, 1997) this difficulty is removed by rescaling the space variables adiabatically:

$$\begin{cases} x_1 &= \frac{1}{a(\epsilon t)} X_1 \\ x_2 &= \frac{1}{a(\epsilon t)} X_2 \end{cases} \tag{24}$$

The conjugated momentum variables (p_1, p_2) are mixed with (x_1, x_2) and transformed into (P_1, P_2) defined by $P_i = \dot{X}_i$. This explicit equivalence of adiabatic Hamiltonian perturbation and small dissipation is very helpful both in understanding the dynamics and in doing calculations, since we can use ideas and techniques from both fields.
We shall summarise the analysis. The first goal is to show that the perturbed system possesses two *adiabatic invariants*. To do this, one uses a theorem due to Neihstadt (1987) which requires the unperturbed system to be in action-angle form. This is also necessary for locating the resonances as the main problem in averaging systems with more than one degree of freedom is the presence of resonant tori. Near these tori a linear combination of the two angular coordinates becomes slowly dependent on time; this normally prohibits averaging over all angular coordinates near these tori. However, for the case of two degrees of freedom a more accurate result is known. Due to the perturbation, the coordinates (I_1, I_2) (which determine the tori) will slowly change in time. If this flow crosses the main resonant tori transversally everywhere, one is still allowed to apply averaging. The idea behind this statement is that although orbits cross resonant tori occasionally, they are only near these tori during a "short" (i.e. $O(\frac{1}{\sqrt{\varepsilon}})$) time-interval, which gives an $O(\sqrt{\varepsilon})$ contribution to the total error. This is the effect of the resonant tori: instead of an $O(\epsilon)$ approximation on a $\frac{1}{\epsilon}$ timescale, we now get an $O(\sqrt{\varepsilon})$ approximation on a $\frac{1}{\varepsilon}$ timescale. This is sufficient to give a constructive proof of the existence of two adiabatic invariants.
For a description of the phenomenon of *capturing* the reader is referred to Huveneers' (1997) paper.

7 References

1. R.H. Abraham and J.E. Marsden, *Foundations of Mechanics* 2d ed., Benjamin, New York (1978).

2. V.I. Arnold, *Proof of a theorem by A.N. Kolmogorov on the invariance of quasiperiodic motion under small perturbations of the Hamiltonian*, Uspehi. Math. Nauk 18, pp. 13-40 (1963); transl. Russ. Math. Surveys 18, pp. 9-36 (1963).

3. V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York etc. (1978) (Russian edition, Moscow, 1974).

4. V.I. Arnold, *Geometrical methods in the theory of ordinary differential equations*, Springer-Verlag, New York etc. (1983).

5. G.D. Birkhoff, *Dynamical Systems*, AMS, Providence, RI (1927), (reprinted by AMS with addendum and notes, 1979).

6. M. Born, *The Mechanics of the Atom*, G. Bell and Sons, Ltd, London (1927).

7. F.J. Bourland and R. Haberman, *Separatrix crossing: Time-invariant potentials with dissipation*, SIAM J.Appl.Math. 50, pp. 1716-1744 (1990).

8. F.J. Bourland and R. Haberman, *Connection across a separatrix with dissipation*, Studies Appl.Math. 91, pp. 95-124 (1994).

9. H.W. Broer, *Conservative dynamical systems*, in *Structures in Dynamics* pp. 267-302, (H.W. Broer, F. Dumortier, S.J. van Strien, F Takens ,eds.), North-Holland, Amsterdam (1991).

10. T. Bountis, H. Segur and F. Vivaldi, *Integrable Hamiltonian Systems and the Painlevé property*, Phys. Review A, 25(3), pp. 1257-1264 (1982).

11. R.C. Churchill, M. Kummer and D.L. Rod, *On Averaging, Reduction and Symmetry in Hamiltonian Systems*, J. Differential Equations 49, pp. 359-414 (1983).

12. C.C. Counselman, *Outcomes of tidal evolution*, Astrophys. J. 180, pp. 307-314 (1973).

13. R.H. Cushman and L.M. Bates, *Global Aspects of Classical Integrable Systems*, Birkhäuser Verlag, Basel etc. (1997).

14. G. Derks and T.P. Valkering, *Approximation in a damped hamiltonian system by successive relative equilibria*, Japan J.Ind.Appl.Math. 9, pp. 141-161 (1992).

15. G. Derks, *Coherent structures in the dynamics of perturbed hamiltonian systems*, PhD thesis, University of Twente (1992).

16. T. de Zeeuw, *Periodic orbits in triaxial galaxies*, in Proc. CECAM Workshop on *Structure, Formation and Evolution of Galaxies*, (J. Audouze and C. Norman, eds.), Paris (1982).

17. T. de Zeeuw, *Dynamics of Triaxial Stellar Systems*, PhD thesis, University of Leiden (1984).

18. R.L. Devaney, *Homoclinic orbits in Hamiltonian Systems*, J.Diff.Eqs. 21, pp. 431-438 (1976).

19. J.J. Duistermaat, *Non-integrability of the 1:1:2-resonance*, Ergod. Th. & Dynam. Sys. 4, pp. 553-568 (1984).

20. N. Ercolani and E.D. Siggia, *Painlevé Property and Integrability*, in *What is Integrability*, (Zakharov, ed.) pp. 63-72, Springer Series in Nonlinear Dynamics, Springer-Verlag, Berlin etc. (1991).

21. H. Flaschka, A.C. Newell and M. Tabor, *Integrability*, in *What is Integrability*, (Zakharov, ed.) pp. 73-114, Springer Series in Nonlinear Dynamics, Springer-Verlag, Berlin etc. (1991).

22. E. Fermi, J. Pasta and S. Ulam, *Studies of nonlinear problems*, Los Alamos Report LA-1940 (1955); reprinted in *Nonlinear wave motion*, A.C. Newell, ed., Lectures Appl. Math. 15, pp. 143-156, AMS (1974).

23. A.T. Fomenko, *The topology of surfaces of constant energy in integrable Hamiltonian systems, and obstructions to integrability*, Math. USSR Izvestiya 29, pp. 629-658 (1987).

24. F.R. Gantmacher and M.G. Krein, *Oszillationsmatrizen, Oszillationskerne und kleine Schwingungen mechanischer Systeme*, Akademie-Verlag, Berlin (1960) (original Russian ed. Moskau-Leningrad, 1950).

25. H. Goldstein, *Classical Mechanics*, Addison-Wesley, Reading (1980).

26. F.G. Gustavson, *On constructing formal integrals for an autonomous Hamiltonian system near an equilibrium point*, Astron. J. 71, pp. 670-686 (1966).

27. R. Haberman, *Energy bounds for the slow capture by a center in sustained resonance*, SIAM J. Appl.Math. 43, pp. 244-256 (1983).

28. R. Haberman and E.K. Ho, *Higher-order change of the hamiltonian (energy) for nearly homoclinic orbits*, Methods and Applications of Analysis 1, pp. 446-464 (1994).

29. R. Haberman and E.K. Ho, *Logarithmic correction to the probability of capture for dissipatively perturbed hamiltonian systems*, Chaos 5, pp. 374-384 (1995).

30. M. Hénon and C. Heiles, *The applicability of the third integral of motion; some numerical experiments*, Astron. J. 69, pp. 73-79 (1964).

31. J. Henrard, *The adiabatic invariant in classical mechanics*, Dynamics Reported 2 (New Series), pp. 117-235, Springer-Verlag (1993).

32. I. Hoveijn and F. Verhulst, *Chaos in the 1:2:3 Hamiltonian normal form*, Physica D 44, pp. 397-406 (1990).

33. I. Hoveijn, *Aspects of resonance in dynamical systems*, PhD thesis, University of Utrecht (1992).

34. P. Hut, *Stability of Tidal Equilibrium*, Astron. Astrophys. 92, pp. 167-170 (1980).

35. R.J.A.G. Huveneers and F. Verhulst, *A Metaphor for Adiabatic Evolution to Symmetry*, SIAM J.Appl.Math. 57, pp. 1421-1442 (1997).

36. R.J.A.G. Huveneers, *Slow evolution towards symmetry*, PhD thesis University of Utrecht (1997).

37. E. A. Jackson, *Nonlinearity and irreversibility in lattice dynamics*, The Rocky Mountain J. of Mathematics 8, pp. 127-196 (1978).

38. P.S. Landa, *Nonlinear Oscillations and Waves in Dynamical Systems*, Kluwer Academic Publishers, Dordrecht etc. (1996).

39. V.F. Lazutkin, *KAM Theory and Semiclassical Approximations to Eigenfunctions*, Springer-Verlag, Berlin etc. (1993).

40. G.H. Lunsford and Ford, *On the stability of periodic orbits for nonlinear oscillator systems in regions exhibiting stochastic behavior*, J. Math. Phys. 13, pp. 700-705 (1972).

41. R.S. MacKay and J.D. Meiss, *Hamiltonian Dynamical Systems, a reprint section*, Adam Hilger, Bristol and Philadelphia (1987).

42. Y.A. Mitropolsky, *The method of integral manifolds in the theory of nonlinear oscillators*, Int. Symp.Nonlinear Diff.Eqs. (J.P.Lasalle and S. Lefschetz, eds.), Academic Press (1963).

43. J. Moser, *On invariant curves of area preserving mappings of an annulus*, Nachr. Akad. Wiss. Göttingen, Math. Phys. Kl., pp. 1-20 (1962).

44. J. Moser, *Lectures on Hamiltonian systems*, Mem. AMS 81, pp. 1-57 (1968).

45. A.I. Neishtadt, *On the change in the adiabatic invariant on crossing a separatrix in systems with two degrees of freedom*, J.Appl.Math.Mech. 51, pp. 586-592 (1987).

46. R.H. Rand, *Symbolic computations and their impact on dynamics*, Am.Soc.Mech.Eng. 205 (1990).

47. C. Robinson, *Sustained resonance for nonlinear system with slowly varying coefficients*, SIAM J.Math.Anal. 14, pp. 847-860 (1983).

48. J.A. Sanders, *On the theory of nonlinear resonance*, PhD thesis, University of Utrecht (1978).

49. J.A. Sanders and F. Verhulst, *Approximations of higher order resonances with an application to Contopoulos' model problem*, in *Asymptotic Analysis, from theory to application*, (F. Verhulst, ed.),

Lecture Notes Math. 711, pp. 209-228, Springer-Verlag, Heidelberg etc. (1979).

50. J.A. Sanders and F. Verhulst, *Averaging methods in nonlinear dynamical systems*, Applied Math. Sciences 59, Springer-Verlag, New York etc. (1985).

51. D.S. Schmidt, *Periodic solutions near a resonant equilibrium of a Hamiltonian system*, Celestial Mech. 9, pp. 81-103 (1974).

52. E. van der Aa, *First order resonances in three-degrees-of-freedom systems*, Celest. Mech. 31, pp. 163-191 (1983).

53. E. van der Aa and F. Verhulst, *Asymptotic integrability and periodic solutions of a Hamiltonian system in 1:2:2-resonance*, SIAM J. Math. Anal. 15, pp. 890-911 (1984).

54. E. van der Aa and M. de Winkel, *Hamiltonian Systems in* $1:2:\omega-$*resonance* ($\omega \geq 7$), Preprint 514, Dept. Mathematics, University of Utrecht (1988).

55. E. van der Aa and M. de Winkel, *Hamiltonian Systems in* $1:2:\omega-$*resonance* ($\omega = 5 or 6$), International J. Nonlinear Mechanics, 29, pp. 261-270 (1994).

56. A.H.P. van der Burgh, *On the asymptotic solutions of the differential equations of the elastic pendulum*, J. de Mécanique 7, pp. 507-520 (1968).

57. A.H.P. van der Burgh, *On the higher order asymptotic approximations for the solutions of the equations of motion of an elastic pendulum*, Journal of Sound and Vibration 42, pp. 463-475 (1975).

58. F. Verhulst, *Discrete-symmetric dynamical systems at the main resonances with applications to axi-symmetric galaxies*, Phil. Trans. Roy. Soc. London A, 290 pp. 435-465 (1979).

59. F. Verhulst, *Asymptotic Analysis of Hamiltonian Systems*, in *Asymptotic Analysis II* (F. Verhulst, ed.), Lecture Notes in Math. 985, pp. 137-183, Springer-Verlag, Heidelberg etc. (1983).

60. F. Verhulst, *Invariant manifolds and periodic solutions of three degrees of freedom Hamiltonian systems*, Proc. 5th Symposium *Trends and Aplications of Pure Mathematics to Mechanics*, Palaiseau 1983, (P.G.Ciarlet and M. Roseau, eds.), Lecture Notes Physics, 195, pp. 413-422, Springer-Verlag, Heidelberg etc. (1984).

61. F. Verhulst and I. Hoveijn, *Integrability and chaos in Hamiltonian normal forms*, in *Geometry and Analysis in Nonlinear Dynamics*, sc H.W. Broer and F. Takens, eds., Pitman Research Notes Mathematics Series 222, pp. 114-134, Longman (1992)

62. F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems*, 2d ed., Springer-Verlag, Heidelberg etc. (1996).

63. L. Wang, D.L. Bosley and J. Kevorkian, *Asymptotic analysis of a class of three-degree-of-freedom Hamiltonian systems near stable equilibria*, Physica D 88, pp. 87-115 (1995).

64. T. Zeegers, *Low-dimensional phenomena in dissipative hamiltonian systems*, PhD thesis University of Twente (1993).

65. S.L. Ziglin, *Branching of Solutions and the Nonexistence of First Integrals in Hamiltonian Mechanics. I*, Funct. Anal. Applic. 16, pp. 181-189 (1982).

66. S.L. Ziglin, *Branching of Solutions and the Nonexistence of First Integrals in Hamiltonian Mechanics. II*, Funct. Anal. Applic. 17, pp. 6-17 (1983).