UNIFICATION IN TRANSITIVE REFLEXIVE MODAL LOGICS

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Abstract. This paper contains a proof-theoretic account of unification in transitive reflexive modal logic, which means that the reasoning is syntactic and uses as little semantics as possible. New proofs of theorems on unification types are given and these results are extended to negationless fragments. In particular, a syntactic proof of Ghilardi's result that $\mathsf{S4}$ has finitary unification is provided. In this approach the relation between classical valuations, projective unifiers and admissible rules is clarified.

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1. Introduction

When restricted to propositional logic, unification theory is concerned with the problem whether a given formula becomes derivable under a substitution. In general, a unification problem asks for the unifier for a pair of terms, or collection of pairs of terms, which in the context of a logic is a substitution under which two formulas become equivalent in the logic. This, however, can be reformulated as the problem of finding a substitution under which a formula becomes derivable. Such substitutions are called the *unifiers* of a formula.

In classical propositional logic every consistent formula has a unifier, because every satisfying valuation corresponds to a *ground unifier* that replaces the atoms in the formula by $\top$ or $\bot$. A substitution is a *maximal unifier (mu)* of a formula if among the unifiers of the formula it is maximal in the following ordering:

$$\tau \leqslant \sigma \equiv_{def} \exists \pi(\tau =_{\mathsf{L}} \pi\sigma),$$

and it is a *most general unifier (mgu)* if it is also unique modulo $=$, which is the intersection of $\leqslant$ and $\geqslant$. Here $=_{\mathsf{L}}$ is the equivalence relation on substitutions associated with the logic: $\sigma =_{\mathsf{L}} \tau$ if and only if $\sigma(p) \leftrightarrow \tau(p)$ is derivable for all atoms p. If $\tau \leqslant \sigma$ we say that τ is *less general* than σ.

Mgus generate all unifiers of a formula, which is the reason that they play an important role in unification theory. In classical propositional logic every unifiable formula has a mgu, but this no longer holds for intermediate and modal logics, as was first observed by Ghilardi [8, 9]. For modal logics, which will be the logics this paper is concerned with, the formula $\Box p \vee \Box\neg p$ is an example of a formula that has two unifiers such that neither one is less general than the other, namely $\sigma_0(p) = \top$ and $\sigma_1(p) = \bot$. Thus this formula has no mgu. But, as Ghilardi showed in [9], for many transitive modal logics, something almost as good holds: instead of unitary unification these logics have finitary unification, which is defined as follows.

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A *complete* set of unifiers for a formula is a set of unifiers such that every unifier of the formula is less general than a unifier in the set. It is *minimal* if no two unifiers in the set are comparable with respect to $\leqslant$. A logic has *unification type*

- *unitary* if every unifiable formula has a mgu,
- *finitary* if every unifiable formula has a finite complete set of mus,
- *infinitary* if every unifiable formula has a (in)finite complete set of mus,
- *nullary* if none of the above.

The classes are meant to be disjunct. For example, in a logic of unification type infinitary there exists at least one formula that has no finite complete set of mus. As was mentioned above, classical logic has unitary unification type, and several transitive modal logics, including the well-known logics $\mathsf{K4}$, $\mathsf{S4}$, and GL, have finitary unification. For example, in the example above $\{\sigma_0, \sigma_1\}$ is a finite complete set of mus for $\Box p \vee \Box\neg p$ in $\mathsf{K4}$, $\mathsf{S4}$, as well as GL.

In this paper we extend these results to the negationless fragment of $\mathsf{S4}$. However, our aim is not so much to extend Ghilardi's results to this fragment, an extension that is not a terribly interesting one and might have been obtained from existing work on $\mathsf{S4}$ anyway, but rather to give a proof-theoretic analysis of unification in transitive modal logic.

Let us first see how Ghilardi proves the finitary unification of $\mathsf{S4}$ and several other modal logics. In [9] Ghilardi first shows that if A satisfies a certain semantical property (the extension property), it has a mgu σ_A. Then he shows that for every formula A there exists a finite set of formulas with the extension property, forming the *projective approximation* Π_A of A, such that every unifier of A is less general than one of the mgus of the formulas in Π_A. These two theorems then establish the finitary unification of $\mathsf{S4}$.

Ghilardi uses semantics in the form of Kripke models to prove these theorems (in fact, his results stem from a categorical approach to unification in logic). Our Theorems 1 and 3 and Lemma 7 can be viewed as proof-theoretic equivalents of these theorems. They provide a syntactic closure condition on formulas which is sufficient for having a mgu. And they show that in $\mathsf{S4}$ and its negationless fragment, there is for every formula A a finite set of formulas that satisfy the closure condition and such that every unifier of A is less general than one of the mgus of the formulas in that set. Observe that this proves that these logics have finitary unification (Corollary 2), by proof-theoretic means.

Besides providing a proof-theoretic treatment of unification, another aim is to clarify the relation between unifiers and valuations. The mgus that play an important role in unification in modal logic often are *projective*, where a unifier σ of a formula A is called projective if $A \vdash \sigma(p) \leftrightarrow p$ for all atoms p, that is, if A implies that the substitution is the identity. The projective unifiers that Ghilardi introduced in [9] are compositions of substitutions of the form

$$\sigma_I(p) \equiv_{def} \begin{cases} A \wedge p & \text{if } p \notin I \\ A \to p & \text{if } p \in I, \end{cases}$$

where I is a set of atoms. It is not difficult to see that σ_I is a projective unifier of A in classical propositional logic whenever A is valid under the valuation

$$v_I(p) \equiv_{def} \begin{cases} 0 & \text{if } p \notin I \\ 1 & \text{if } p \in I. \end{cases}$$

One could view Theorem 1 below as an analogue of this fact for modal logic.

At the end of the paper we apply these results to admissible rules, which are the rules under which a logic is closed. Jeřábek proved in [19] that the modal Visser rules V° (definition in Section 6) form a basis for the admissible rules of any extension of S4 in which they are admissible. In Theorem 4 we show that this result can be obtained via syntactic methods as well and extend it to the negationless fragment of S4.

The restriction in this paper to reflexive logics is, we think, not essential for a proof-theoretic treatment of unification, but it seems to simplify the reasoning at some points, and we therefore leave the general case (fragments of K4) for future work.

Finally, let us briefly discuss other work on unification and admissible rules in modal logic. We restrict ourselves to those results that are directly related to the content of this paper, and will therefore not discuss intermediate logics or multi-modal logics. Rybakov was the first to prove the decidability of admissibility for various modal logics, including S4. Chagrov constructed a decidable modal logic which admissibility problem is undecidable [2], and Wolter and Zakharyaschev did the same for the unification problem [28]. As mentioned above, Ghilardi introduced the notion of projectivity for formulas and unifiers and proved that various modal and intermediate logics have finitary unification [9]. He showed that projective approximations can be found effectively, and a similar thing holds for the irreducible projective approximations from [14] that we use in this paper. Ghilardi also provided an elegant algorithm for deciding admissibility of several modal logics. Jeřábek in [19] gave a basis for the admissible rules of various modal logics, including S4. In [20] he showed that the admissibility problem of S4 and various other logics is *coNEXP*-complete. Iemhoff and Metcalfe in [14, 15] developed proof systems for admissibility for K4, S4, and GL.

Dzik in several papers studied the lattice of transitive reflexive modal logics. In [6] he showed that one can split the lattice in two parts in such a way that one part, those logics that contain S4.2, contains all extensions of S4 that have unitary unification, and that the other part contains all extensions of S4 that have finitary unification. Dzik and Wojtylak showed in [7] that every logic containing S4 has projective unification if and only if it contains S4.3, where a logic has *projective unification* if every unifiable formula has a projective unifier. In the same paper they also show that among the extensions of S4.3 those that are extensions of S4.1 are exactly those that are structurally complete. This is a brief summary of part of the literature on unification in modal transitive reflexive logics. For further references, see [1].

The inspiration for this paper is the proof-theoretic approach to unification in intuitionistic logic as presented by Rozière in [26]. In [16] we have extended these results to intermediate logics. I would like to thank Emil Jeřábek, George Metcalfe, and Paul Rozière for helpful remarks along the way, and an anonymous referee for comments that helped improve the paper.

2. The logics

The logics we consider are normal transitive modal logics that contain S4 and the negationless fragments of such logics, which means without $\bot$ and $\neg$. The results in this paper are proved for the full logics, but the extension to the negationless

fragments is straightforward: inspection of the proofs shows that only implication and conjunction are explicitly used.
$\mathcal{P} = \{p_1, p_2, \dots\}$ is the set of atoms, and p, q, r, s denote arbitrary elements of $\mathcal{P}$. In the case that $\bot$ is part of the language, p, q, r, s range over $\mathcal{P} \cup \{\bot\}$. A, B, C denote formulas.
We use Γ, Δ without further comment to denote finite sets of formulas. *Sequents* are expressions $\Gamma \Rightarrow \Delta$, thus pairs of finite sets of formulas. In the case that $\bot$ and negation do not belong to the language, we require that Δ is not empty. S ranges over sequents. A sequent is *irreducible* if it only contains atoms, boxed atoms ($\Box p$ for an atom p), and $\bot$. $\mathcal{S}, \mathcal{G}, \mathcal{H}$ range over finite sets of sequents.
$\mathcal{L}_{\mathcal{G}}$ denotes the set of atoms that occur in $\mathcal{G}$, and if $\bot$ it is present and occurs in $\mathcal{G}$, $\mathcal{L}_{\mathcal{G}}$ also contains $\bot$. $n_{\mathcal{G}}$ is the minimal n for which all atoms in $\mathcal{G}$ are among $p_1, \dots, p_n$, and For a sequent S, $b(S)$ denotes the sum of the number of atoms in S plus the number of occurrences of boxes in S, and similarly for formulas.
We need the following notation, where v stands for variable, b for box, i for interior, a for assumption, and c for conclusion:

$$\Gamma^v \equiv_{def} \{p \mid p \in \Gamma\} \quad \Gamma^b \equiv_{def} \{\Box p \mid \Box p \in \Gamma\} \quad \Gamma^i \equiv_{def} \{p \mid \Box p \in \Gamma)\}$$

$$(\Gamma \Rightarrow \Delta)^a \equiv_{def} \Gamma \quad (\Gamma \Rightarrow \Delta)^c \equiv_{def} \Delta$$

$$S^{kl} \equiv_{def} (S^k)^l \qquad k \in \{a, c\} \ \ l \in \{a, c, v, b, i\}.$$

For example, S^{ab} is the set of boxed atoms in the antecedent of S. Sequents are interpreted as formulas in the usual way: $I(S) = (\bigwedge S^a \to \bigvee S^c)$. For notational convenience we sometimes write S for $I(S)$, for example in $\vdash S$, which thus should be read as $\vdash \bigwedge S^a \to \bigvee S^c$. The following sets play an important role in the theorems to come.

$$\mathcal{B}_{\mathcal{G}} \equiv_{def} \bigcup \{S^{ab} \mid S \in \mathcal{G}\} \qquad \Sigma^{\mathcal{G}}_{\mathcal{S}} \equiv_{def} \{p \mid \mathcal{G} \vdash I(\mathcal{B}_{\mathcal{S}} \Rightarrow p)\}.$$

Sets of sequents are interpreted as conjunctions and we sometimes use the noncalligraphic version of a letter to denote the corresponding formula:

$$I(\mathcal{G}) \equiv_{def} \bigwedge_{S \in \mathcal{G}} I(S) \qquad G \equiv_{def} \Box I(\mathcal{G}).$$

When we speak of the unifiability of $\mathcal{G}$, we mean the unifiability of G. Note that reflexivity implies that $\vdash G \to I(\mathcal{G})$.
We assume that the logics are given by finitary structural consequence relations. In the setting of rules it is convenient to consider *multi-conclusion finitary structural consequence relations*, which are relations $\vdash$ between finite sets of formulas satisfying

reflexivity	$A \vdash A$,
weakening	if $\Gamma \vdash \Delta$, then $\Gamma', \Gamma \vdash \Delta, \Delta'$,
transitivity	if $\Gamma \vdash \Delta, A$ and $\Gamma', A \vdash \Delta'$, then $\Gamma', \Gamma \vdash \Delta, \Delta'$,
structurality	if $\Gamma \vdash \Delta$, then $\sigma\Gamma \vdash \sigma\Delta$ for all substitutions σ.

A *finitary single-conclusion consequence relation* is a relation between finite sets of formulas and formulas satisfying the single-conclusion variants of the four properties above. Thus for single-conclusion consequence relations the conclusion of a rule cannot be empty.
The *theorems* of (the logic given by) a consequence relation $\vdash$ are those A for which $\emptyset \vdash A$, which we denote by $\vdash A$, holds. There are many consequence relations that correspond to a single set of theorems. Here we do not require much of the

consequence relation, except that if $\bigwedge\Gamma \to A$ holds in the logic, then $\Gamma \vdash_{\mathsf{L}} A$ holds for the consequence relation.

A *(multi-conclusion) rule* is an expression of the form Γ/Δ. It is *derivable* in a logic given by consequence relation $\vdash_{\mathsf{L}}$ if $\Gamma \vdash_{\mathsf{L}} \Delta$, and admissible, written $\Gamma \mathrel{|\!\!\sim}_{\mathsf{L}} \Delta$, if for all substitutions σ, if $\sigma\Gamma$ consists of theorems of L, then $\sigma\Delta$ contains a theorem of L. Note that a logic has the modal disjunction property ($\vdash_{\mathsf{L}} \Box A \vee \Box B$ implies $\vdash_{\mathsf{L}} A$ or $\vdash_{\mathsf{L}} B$) if and only if $\{\Box p \vee \Box q\}/\{p,q\}$ is admissible. Given a rule R, $\vdash_{\mathsf{L}}^{R}$ is the extension of the logic by the rule R. For details see [18].

3. Proof sketch

Given a formula A and a subset I of the atoms in A, consider the valuation v_I and substitution σ_I^A given in the introduction:

$$v_I(p) \equiv_{def} \begin{cases} 1 & \text{if } p \in I \\ 0 & \text{if } p \notin I \end{cases} \qquad \sigma_I^A(p) \equiv_{def} \begin{cases} A \to p & \text{if } p \in I \\ A \wedge p & \text{if } p \notin I. \end{cases}$$

It is not difficult to see that if S consists of atoms, then for $A = I(S)$, if $v_I(A) = 1$, then $\vdash_{\mathsf{L}} \sigma_I^A(A)$. Also, $A \vdash_{\mathsf{L}} \sigma_I^A(B) \leftrightarrow B$ for all B. Therefore, in case v_I satisfies A, σ_I^A is a most general unifier of A in L. For if $\vdash_{\mathsf{L}} \tau A$, then as $\tau A \vdash_{\mathsf{L}} \tau\sigma_I^A(B) \leftrightarrow \tau B$, also $\vdash_{\mathsf{L}} \tau\sigma_I^A(B) \leftrightarrow \tau B$. That is, $\tau \leqslant \sigma_I^A$.

Because the logics contain (the negationless fragment of) $\mathsf{S4}$, the above arguments extends in the following way to irreducible sequents S: if $v_I(S^{av} \cup S^{ai} \Rightarrow S^{cv}) = 1$, then $\sigma_I^{I(S)}$ is a most general unifier of $I(S)$.

One of the key observations in the results below, Corollary 1, states that a set of irreducible sequents $\mathcal{G}$ closed under the rules V° is projective. The projective unifier of the formula G, where $G = I(\mathcal{G})$, is a composition of substitutions of the form σ_I^G, for some I. The main part of the proof of Corollary 1 is to show that such a composition is a unifier for the formula, as the argument above implies that if so, it is a most general one.

The proof that a certain composition $\sigma = \sigma_n \dots \sigma_1$ of substitutions of the form σ_I^G is a unifier for G is based on the following simple observation. Writing $\overline{\sigma}_i$ for $\sigma_n \dots \sigma_i$, to prove $\vdash_{\mathsf{L}} \sigma G$, one has to show that $\vdash_{\mathsf{L}} I(\overline{\sigma}_1 S_1^a \Rightarrow \overline{\sigma}_1 S_1^v)$ for all $S_1 \in \mathcal{G}$. For this it suffices to show that $\vdash_{\mathsf{L}} I(\overline{\sigma}_1 S_1^a \Rightarrow I(\overline{\sigma}_2 S_2))$ for all $S_2 \in \mathcal{G}$, as this would imply that $\vdash_{\mathsf{L}} \overline{\sigma}_1 S_1^a \Rightarrow \overline{\sigma}_2 G$. And as $\vdash_{\mathsf{L}} G \to \sigma_j G$ for all j, and therefore $\vdash_{\mathsf{L}} I(\overline{\sigma}_{j-1} G \Rightarrow \overline{\sigma}_j G)$, this gives $\vdash_{\mathsf{L}} I(\overline{\sigma}_1 S_1^a \Rightarrow \overline{\sigma}_1 G)$, which implies $\vdash_{\mathsf{L}} I(\overline{\sigma}_1 S_1^a \Rightarrow \overline{\sigma}_1 S_1)$, as $S_1 \in \mathcal{G}$. And thus $\vdash_{\mathsf{L}} I(\overline{\sigma}_1 S_1^a \Rightarrow \overline{\sigma}_1 S_1^v)$, as $A \to (A \to B)$ implies $A \to B$.

Continuing this argument one sees that it suffices to show that for all possible sequences of sequents from $\mathcal{G}$ there is an i such that

$$\forall S \in \mathcal{G} : \vdash_{\mathsf{L}} I(\overline{\sigma}_1 S_1^a, \dots, \overline{\sigma}_i S_i^a \Rightarrow I(\overline{\sigma}_{i+1} S)). \tag{1}$$

Reasoning as above in the simpler case, one sees that if for $\mathcal{S} = \{S_1, \dots, S_i\}$, I would be such that $\sigma_{i+1} = \sigma_I^G$ and v_I satisfies $I(S^{av} \cup S^{ai} \Rightarrow S^{cv} \cup (S^{ci} \cap \Sigma_{\mathcal{S}}^{\mathcal{G}}))$, then (1) holds. This explains the notion of strong satisfiability introduced below, which requires that v_I satifsies $I(S^{av} \cup S^{ai} \Rightarrow S^{cv} \cup (S^{ci} \cap \Sigma_{\mathcal{S}}^{\mathcal{G}}))$ for all $\mathcal{S} \subseteq \mathcal{G}$.

The proof of Corollary 1 therefore consists of two parts: Lemma 7 stating that closure under the rules V° implies strong satisfiability and Theorem 1 stating that strong satisfiability implies projectivity. The rest of the paper shows how to apply Corollary 1 to prove that certain (fragments of) logics have finitary unification type and V° as a basis for admissibility.

4. Substitutions and valuations

The discussion above serves as a background for the definitions given below. In this and the next section we consider an arbitrary finite set $\mathcal{G}$ of *irreducible* sequents, and corresponding formula $G = I(\mathcal{G})$, and assume the atoms that occur in $\mathcal{G}$ to be $\{p_1, \ldots, p_{n_\mathcal{G}}\}$. Most definitions are relative to $\mathcal{G}$ but for simplicity we do not always indicate this in our notation.

We fix an arbitrary enumeration $J_1, \ldots, J_{2^{n_\mathcal{G}}}$ of all subsets of $\mathcal{P}$. In what follows I ranges over subsets of $\{p_1, \ldots, p_{n_\mathcal{G}}\}$. Given a set I the valuation v_I has been defined above, and σ_I denotes σ_I^G, also defined at the beginning of Section 3. We extend it to a valuation for sequents S relative to a set of sequents $\mathcal{S} \subseteq \mathcal{G}$: S is *strongly satisfiable* with respect to $\mathcal{S}$ if

$$\overline{v}_I(S \mid \mathcal{S}) \equiv_{def} v_I\big(S^{av} \cup S^{ai} \Rightarrow S^{cv} \cup (S^{ci} \cap \Sigma_\mathcal{S}^\mathcal{G})\big) = 1.$$

The empty sequent is interpreted as $\bot$ and thus has no satisfying valuation. The valuations are extended to sets of sequents in the usual way: $\overline{v}_I(\mathcal{S}' \mid \mathcal{S}) = 1$ if and only if $\overline{v}_I(S \mid \mathcal{S}) = 1$ for all $S \in \mathcal{S}'$. We write $\overline{v}_I(\mathcal{S})$ for $\overline{v}_I(\mathcal{S} \mid \mathcal{S})$. $\mathcal{G}$ is *strongly satisfiable* if for all $\mathcal{S} \subseteq \mathcal{G}$ there is an I such that $\overline{v}_I(\mathcal{S}) = 1$.

σ and τ range over substitutions that assign propositional formulas in the language of L to atoms, and ι is the identity substitution. As usual, $\tau\Gamma = \{\tau A \mid A \in \Gamma\}$ and $\tau S = (\tau S^a \Rightarrow \tau S^c)$. The substitutions that we consider have finite domains, where $dom(\sigma)$ denotes the domain of σ. We use the following notation:

$$\sigma \leftrightarrow \tau \equiv_{def} \bigwedge_{p \in dom(\sigma) \cup dom(\tau)} \big(\sigma(p) \leftrightarrow \tau(p)\big).$$

Observe that $\sigma \leftrightarrow \tau$ is a propositional formula, and that

$$\vdash \sigma \leftrightarrow \tau \text{ implies } \vdash \sigma A \leftrightarrow \tau A.$$

I ranges over subsets of $\mathcal{L}_\mathcal{G}$. Given a set I, the substitution σ_I is defined as follows:

$$\sigma_I(p) \equiv_{def} \begin{cases} G \to p & \text{if } p \in I \\ G \wedge p & \text{if } p \notin I \end{cases}$$

$$\overline{\sigma} \equiv_{def} \sigma_{J_{2^{n_\mathcal{G}}}} \ldots \sigma_{J_1} \qquad \sigma_G \equiv_{def} \overline{\sigma}^{|\mathcal{G}|+1}.$$

Write g for $2^{n_\mathcal{G}}$. Thus σ_G is the composition of $g(|\mathcal{G}|+1)$ substitutions. The i-th substitution in σ_G (reading from right to left) is denoted by σ_i and for $i < j$, $\sigma_j \ldots \sigma_i$ is denoted by $\sigma_{j,i}$. We denote $\sigma_{2^{n_\mathcal{G}}(|\mathcal{G}|+1),i} = \sigma_{2^{n_\mathcal{G}}(|\mathcal{G}|+1)} \ldots \sigma_i$ by $\overline{\sigma}_i$. For example, $\sigma_2 = \sigma_{2^{n_\mathcal{G}}+2} = \cdots = \sigma_{2^{n_\mathcal{G}}|\mathcal{G}|+2} = \sigma_{J_2}$, $\overline{\sigma}_1 = \sigma_G$, and $\overline{\sigma}_{2^{n_\mathcal{G}}+1} = \overline{\sigma}^{|\mathcal{G}|}$. We denote by I_i the set J_j such that $\sigma_i = \sigma_{J_j}$. Thus $i = k2^{n_\mathcal{G}} + j$ for some k. For valuations we define:

$$v_i \equiv_{def} v_{I_i}.$$

The rest of this section contains technical lemmas that we will need later on.

Lemma 1. For all m and $i < j$: $\vdash G \to \Box\big(\iota \leftrightarrow \sigma_i \leftrightarrow \sigma_{j,i}\big)$ and $\vdash \overline{\sigma}_j G \to \overline{\sigma}_i G$.

Proof. Observe that $\vdash G \to \Box G$ holds because the logic is transitive. The first equivalence in the first statement immediately follows from this. The second equivalence follows from this and the fact that $\vdash \Box(B \leftrightarrow C) \to (A[B/p] \leftrightarrow A[C/p])$ for any atom p.

The first statement implies that $\vdash G \to \sigma_{j-1,i}G$, which implies $\vdash \overline{\sigma}_j G \to \overline{\sigma}_i G$. $\Box$

This lemma immediately implies the following lemma.

Lemma 2. For all $S \in \mathcal{G}$ for which $(S^{cv} \cap I)$ or $(S^{av} \cup S^{ai}) \backslash I$ is not empty, $\vdash \sigma_I S$.

Proof. We treat the case that $S^{ai} \backslash I$ is not empty, say it contains the atom p. Thus $\Box p$ belongs to S^a and since the logic is reflexive, S^a implies p. p is under σ_I replaced by $G \wedge p$. Thus $\sigma_I S^a$ implies G, and Lemma 1 and the fact that $S \in \mathcal{G}$ prove that it implies S^c, and $\sigma_I S^c$ as well, which gives the result. □

Define

$$F(i_1, \ldots, i_j, S_1, \ldots, S_j, A) \equiv_{def} I(\overline{\sigma}_{i_1} S_1^a, \overline{\sigma}_{i_2} S_2^a, \ldots \overline{\sigma}_{i_j} S_j^a \Rightarrow A) \tag{2}$$

and recall that g is short for $2^{n_{\mathcal{G}}}$.

Proposition 1. For all $\mathcal{S} = \{S_1, \ldots, S_j\} \subseteq \mathcal{G}$ and all $1 \leq i_1, \ldots, i_j \leq g(|\mathcal{G}| + 1)$, if $\overline{v}_I(\mathcal{S}) = 1$, then for all $S \in \mathcal{S}$: $\vdash F(i_1, \ldots, i_j, S_1, \ldots, S_j, \sigma_I S)$.

Proof. Consider $S \in \mathcal{S}$. If $(S^{cv} \cap I)$ or $(S^{av} \cup S^{ai}) \backslash I$ is not empty, the previous lemma applies. Otherwise for some $p \in \Sigma_{\mathcal{S}}^{\mathcal{G}} \cap I$, $\Box p \in S^c$. Since $\mathcal{G} \vdash I(B_{\mathcal{S}} \Rightarrow p)$ and p is under σ_I replaced by $G \rightarrow p$, it follows that $\vdash I(\overline{\sigma}_{i_1} S_1^{ab}, \ldots \overline{\sigma}_{i_j} S_j^{ab} \Rightarrow \sigma_I p)$. Therefore $\vdash F(i_1, \ldots, i_j, S_1, \ldots, S_j, \sigma_I \Box p)$ and thus $\vdash F(i_1, \ldots, i_j, S_1, \ldots, S_j, \sigma_I S)$. □

5. Unifiers

In this section we show that strong satisfiability implies projectivity. The proof of this fact is syntactic and does not use models. The definitions below are relative to $\mathcal{G}$, but we do not indicate this in our notation. Substitutions σ and σ_i have been defined in the previous section. We need some terminology to be able to prove this theorem by backwards induction. A sequence of m numbers followed by m sequents $i_1, \ldots, i_m, S_1, \ldots, S_m$ is *appropriate* if $m \leq |\mathcal{G}|$,

$$1 = i_1 \leq g < i_2 \leq 2g \leq \cdots < i_m \leq mg,$$

and the sequents are distinct and belong to $\mathcal{G}$. It is *G-sufficient* if for all numbers j such that $mg < j \leq (m+1)g$ and $\overline{v}_j(\{S_1, \ldots, S_m\}) = 1$, the formula $F(i_1, \ldots, i_m, S_1, \ldots, S_m, \overline{\sigma}_j G)$ is derivable, where F is defined in (2).

Lemma 3. If $\mathcal{G}$ is strongly satisfiable, then for any number $k > 0$ and every appropriate sequence $i_1, \ldots, i_m, S_1, \ldots, S_m$ there exists a natural number h such that $kg < h \leq (k+1)g$ and $\overline{v}_h(\{S_1, \ldots, S_m\}) = 1$.

Proof. As $\mathcal{G}$ is strongly satisfiable, there is a $j \leq g$ such that $\overline{v}_j(\{S_1, \ldots, S_m\})$ equals 1. Since $v_j = v_{kg+j}$, the lemma follows. □

Lemma 4. If $\mathcal{G}$ is strongly satisfiable then for all $m \leq |\mathcal{G}|$: if all appropriate sequences of length $2m$ are G-sufficient, then so are all appropriate sequences of length $2m - 2$.

Proof. Consider an appropriate $i_1, \ldots, i_{m-1}, S_1, \ldots, S_{m-1}$ and let j be such that $(m-1)g < j \leq mg$ and $\overline{v}_j(\{S_1, \ldots, S_{m-1}\}) = 1$. We have to show that for all $S \in \mathcal{G}$:

$$\vdash F(i_1, \ldots, i_{m-1}, S_1, \ldots, S_{m-1}, \overline{\sigma}_j S) \tag{3}$$

If $S \in \{S_1, \ldots, S_{m-1}\}$, then (3) follows from Proposition 1. If, on the other hand, $S \notin \{S_1, \ldots, S_{m-1}\}$, then $i_1, \ldots, i_{m-1}, j, S_1, \ldots, S_{m-1}, S$ is an appropriate sequence of length $2m$. By Lemma 3 there exists a number h such that $mg < h \leq (m+1)g$ and $\overline{v}_h(\{S_1, \ldots, S_{m-1}, S\}) = 1$. Therefore by G-sufficiency

$$\vdash F(i_1, \ldots, i_{m-1}, j, S_1, \ldots, S_{m-1}, S, \overline{\sigma}_h G).$$

Since $\vdash \overline{\sigma}_h G \rightarrow \overline{\sigma}_j G$ and $S \in \mathcal{G}$, this implies that

$$\vdash F(i_1, \ldots, i_{m-1}, j, S_1, \ldots, S_{m-1}, S, \overline{\sigma}_j S).$$

Hence $\vdash F(i_1, \ldots, i_{m-1}, S_1, \ldots, S_{m-1}, \overline{\sigma}_j S)$, which is what we had to show. □

Lemma 5. If $S \in \mathcal{G}$ and $1, S$ is G-sufficient, then $\vdash \overline{\sigma} S$.

Proof. By Lemma 3 there exists an $i \leq 2g$ such that $\overline{v}_i(\{S\}) = 1$. Therefore $\vdash \bigwedge \overline{\sigma}_1 S^a \rightarrow \overline{\sigma}_i G$. Since $\vdash \overline{\sigma}_i G \rightarrow \overline{\sigma}_1 G$ by Lemma 1, this gives $\vdash \bigwedge \overline{\sigma}_1 S^a \rightarrow \overline{\sigma}_1 G$. As $S \in \mathcal{G}$, $\vdash \overline{\sigma}_1 S$ follows, that is, $\vdash \sigma_G S$. □

Lemma 6. Every appropriate sequence of length $2|\mathcal{G}|$ is G-sufficient.

Proof. Let $|\mathcal{G}| = m$ and consider an appropriate sequence $i_1, \ldots, i_m, S_1, \ldots, S_m$ and let j be such that $mg < j \leq (m+1)g$ and $\overline{v}_j(\{S_1, \ldots, S_m\}) = 1$. Because $m = |\mathcal{G}|$ and the S_i are distinct, $\{S_1, \ldots, S_m\} = \mathcal{G}$. Therefore by Proposition 1, $\vdash F(i_1, \ldots, i_m, S_1, \ldots, S_m, \overline{\sigma}_j S)$ for all $S \in \mathcal{G}$. This implies that the sequence is G-sufficient. □

Theorem 1. If $\mathcal{G}$ is strongly satisfiable, then $\vdash \sigma_G G$.

Proof. By Lemma 6 every appropriate sequence of length $2|\mathcal{G}|$ is G-sufficient. By repeated application of Lemma 4 it follows that $1, S$ is G-sufficient for every $S \in \mathcal{G}$. This implies $\vdash \sigma_G S$ by Lemma 5. Hence $\vdash \sigma_G G$. □

6. Rules and satisfiability

In the following we use $\Gamma, \Box A \equiv A \Rightarrow \Delta$ as an abbreviation for the sequence of two sequents $(\Gamma, \Box A, A \Rightarrow \Delta), (\Gamma \Rightarrow A, \Box A, \Delta)$, and $\Box\{A_1, \ldots, A_n\} \equiv \{A_1, \ldots, A_n\}$ for $\Box A_1 \equiv A_1, \ldots, \Box A_n \equiv A_n$. Furthermore, *resolution refutations* are sequent derivations in which every sequent contains only atoms, and every inference is a cut.

Jeřábek in [19] showed that the following rule is a basis for the admissible rules of $\mathsf{S4}$, and obtained similar results for other modal logics.

$$\frac{\{\Box\Gamma \equiv \Gamma \Rightarrow \Box\Delta\}}{\{\Box\Gamma \Rightarrow p \mid p \in \Delta\}} \; \mathsf{V}^\circ$$

We provide another proof of this fact and extend it to the negationless fragment of $\mathsf{S4}$. We prove it by showing that closure under V° is a sufficient condition for strong satisfiability.

Lemma 7. If $\mathcal{G}$ is a consistent set of irreducible sequents closed under V°, then $\mathcal{G}$ is strongly satisfiable.

Proof. Arguing by contraposition, suppose that for some $\mathcal{S} \subseteq \mathcal{G}$, $\overline{v}_I(\mathcal{S}) = 0$ for all I. Thus there exists a resolution proof from the set of sequents

$$\{S^{av} \cup S^{ai} \Rightarrow S^{cv} \cup (S^{ci} \cap \Sigma^{\mathcal{G}}_{\mathcal{S}}) \mid S \in \mathcal{S}\}$$

that ends in the empty sequent. For a clause C, C^+ denotes the set of atoms in C and C^- the set of atoms that occur negated in C. We can assume that no clause contains an atom and its negation. We are going to associate with every clause C in the refutation a sequent S_C derivable from $\mathcal{G}$ such that

$$S_C^{av} \cup S_C^{ai} \subseteq C^- \qquad S_C^{cv} \cup (S_C^{ci} \cap \Sigma_{\mathcal{S}}^{\mathcal{G}}) \subseteq C^+.$$

The antecedent of such a sequent can contain atoms, boxed atoms, and formulas of the form $p \equiv \Box p$, and the succedent consists of atoms and boxed atoms only. For the initial clauses C, S_C is the sequent to which C corresponds. For a cut on clauses C_1 and C_2 with corresponding sequents S_1 and S_2 there are the following possibilities. Let C be the clause resulting from the cut. If $p \in S_1^c$ and $p \in S_2^a$, then S_C is the result of applying a cut to the sequents S_1 and S_2 on p. Similarly for $\Box p$. If $\Box p \in S_1^c$ and $p \in S_2^a$, then because of reflexivity, $\mathcal{G}$ derives $S_1' = (S_1^a \Rightarrow p, S_1^c\backslash\{\Box p\})$, and S_C is the result of a cut on S_1' and S_2 with cutformula p. In the remaining case, $p \in S_1^c$ and $\Box p \in S_2^a$, we put

$$S_C = S_2^a\backslash\{\Box p\} \cup S_1^a \cup \{\Box p \equiv p\} \Rightarrow S_1^c\backslash\{p\} \cup S_2^c.$$

Note that S_C is derivable from $\mathcal{G}$ if S_1 and S_2 are. Also note that for all $\Box p \equiv p$ that occur in S_C, $\Box p \in \mathcal{B}_{\mathcal{G}}$.
Now $S_\emptyset$ is of the form $\Box\Gamma \equiv \Gamma \Rightarrow \Box\Delta$, for which $\Delta \cap \Sigma_{\mathcal{S}}^{\mathcal{G}}$ is empty. If Δ is empty, then $\mathcal{G}$ derives $\Box\Gamma \equiv \Gamma \Rightarrow$, which would make $\mathcal{G}$ inconsistent. Therefore Δ is not empty. As $\mathcal{G}$ is closed under V° there exists a $p \in \Delta$ such that $\mathcal{G}$ derives $(\Box\Gamma \Rightarrow p)$. Hence $p \in \Delta \cap \Sigma_{\mathcal{S}}^{\mathcal{G}}$, contradicting $\Delta \cap \Sigma_{\mathcal{S}}^{\mathcal{G}} = \emptyset$. □

Combining the previous lemma with Theorem 1 gives a necessary condition for projectivity.

Corollary 1. If $\mathcal{G}$ is a consistent set of irreducible sequents closed under V°, then G is projective.

Theorem 2. If V° is admissible in L and $\mathcal{G}$ is a consistent set of irreducible sequents, then $\mathcal{G}$ is closed under V° if and only if G is projective if and only if σ_G is a unifier of G if and only if $\mathcal{G}$ is strongly satisfiable.

Proof. We prove the first equivalence. The direction from left to right is Corollary 1. For the other direction, let σ be a projective unifier of G and suppose that $\mathcal{G}$ derives $(\Box\Gamma \equiv \Gamma \Rightarrow \Box\Delta)$, meaning the conjunction of all the sequents of the sequence $(\Box\Gamma \equiv \Gamma \Rightarrow \Box\Delta)$. Thus $\sigma(\Box\Gamma \equiv \Gamma \Rightarrow \Box\Delta)$ is derivable in L. Hence so is $\sigma(\Box\Gamma \Rightarrow p)$ for some p in Δ. Therefore $\mathcal{G}$ derives $(\Box\Gamma \Rightarrow p)$. □

7. Unification types

In this section we use the previous results to show that in $\mathsf{S4}$ and in all extensions thereof, admissibility of V° implies finitary unification. To show this we need the following generalizations of the notions of derivability and admissibility. Formula A *broadly derives* formula B, $A \Vdash B$, if and only if there is a substitution σ that is the identity on $\mathcal{L}_A$ such that $A \vdash \sigma B$. We need this notion because in the construction of the approximation fresh atoms may be introduced. The corresponding notion for admissibility is *broad admissibility* defined as: $\Gamma \Vdash\!\sim \Delta$ if and only if every unifier of all formulas in Γ can be extended to a unifier of at least one formula in Δ.
Recall from the preliminaries that if $\vdash$ stand for derivability in logic L, then $\vdash^R$ stands for derivability in L extended by the rule R. $\Vdash^R$ stands for broad derivability in L extended by the rule R.

Lemma 8. For every set of sequents $\mathcal{S}$ there exists a finite set of irreducible sequents $\mathcal{G}$ such that $\Box I(\mathcal{G}) \vdash I(\mathcal{S}) \Vdash \Box I(\mathcal{G})$.

Proof. We follow the method of proof of a similar lemma in [4]. The *length* of a formula is the number of symbols occurring in it. Let $ml(\mathcal{S})$ be the multiset of the lengths of the formulas in the sequents in $\mathcal{S}$. We prove the lemma by induction on $ml(\mathcal{S})$, using the multiset ordering. If $ml(\mathcal{S}) = 0$, all sequents in $\mathcal{S}$ are irreducible. Suppose there is a sequent $S \in \mathcal{S}$ that is not irreducible, and consider A in S that is not an atom or a boxed atom. If $A = (B \wedge C)$ and $A \in S^a$, we replace S by $(S^a \backslash \{A\}, B, C \Rightarrow S^c)$, and if $A \in S^c$ we replace S by $(S^a \Rightarrow S^c \backslash \{A\}, B)$ and $(S^a \Rightarrow S^c \backslash \{A\}, C)$. Similarly if A is a disjunction or an implication.
Suppose $A = \Box B$. If $A \in S^c$ we choose a fresh atom p and replace S by $S_1 = (S^a \Rightarrow S^c \backslash \{A\}, \Box p)$ and $S_2 = (p \Rightarrow B)$. If $A \in S^a$, S is replaced by $S_1 = (S^a \backslash \{A\}, \Box p \Rightarrow S^c)$ and $S_2 = (B \Rightarrow p)$. Note that in both cases we have $\Box I(S_1) \wedge \Box I(S_2) \mathrel{|\!\!\sim} I(S) \Vdash \Box I(S_1) \wedge \Box I(S_2)$. $I(S) \Vdash \Box I(S_1) \wedge \Box I(S_2)$ holds because every unifier τ of S can be extended to a unifier of S_1 and S_2 by putting $\tau(p) = B$. And $\Box I(S_1) \wedge \Box I(S_2) \vdash I(S)$ holds because our logics are reflexive, so $\Box I(S_i) \vdash I(S_i)$. $\Box$

The lemma above implies that for every formula A we can construct a finite set of irreducible sequents that is admissibly equivalent to it. Dzik and Wojtylak use a similar method in [7]. A set of finite sets of irreducible sequents $\{\mathcal{G}_1, \dots, \mathcal{G}_m\}$ such that the G_i are projective and $\bigvee G_i \vdash A \mathrel{|\!\!\!\sim} \{G_1, \dots, G_m\}$ is an *irreducible projective approximation* of A.
To obtain approximations we need the following relation $\overset{n}{\rightsquigarrow}$ on sets of irreducible sequents. For every set of irreducible sequents $\mathcal{G}$ and any irreducible sequent $\Box\Gamma \equiv \Gamma \Rightarrow \Box\Delta$ in $\mathcal{L}_\mathcal{G}$ with Γ of size at most n such that $\mathcal{G} \vdash (\Box\Gamma \equiv \Gamma \Rightarrow \Box\Delta)$ and for every $p \in \Delta$ we set

$$\mathcal{G} \overset{n}{\rightsquigarrow} \big(\mathcal{G} \cup \{(\Box\Gamma \equiv \Gamma \Rightarrow \Box\Delta), (\boxdot\Gamma \Rightarrow p)\} \big).$$

Slightly ambiguous, we also use $\overset{n}{\rightsquigarrow}$ for the transitive closure of this relation. A set of sequents $\mathcal{G}$ is in $\overset{n}{\rightsquigarrow}$*-normal form* if there is no $\mathcal{H} \supset \mathcal{G}$ such that $\mathcal{G} \overset{n}{\rightsquigarrow} \mathcal{H}$. Note that $\overset{n}{\rightsquigarrow}$ is neither deterministic nor confluent, but it terminates in the following sense. For full $\mathsf{S4}$, this theorem has been proved in [14], and here it is extended to the negationless fragment of $\mathsf{S4}$.

Theorem 3. If $\mathsf{V}^\circ_{b(A)}$ is admissible and A is consistent in L, then A has an irreducible projective approximation.

Proof. Let $\mathcal{G}$ be the set of irreducible sequents such that $\Box I(\mathcal{G}) \vdash A \Vdash \Box I(\mathcal{G})$ that exists by Lemma 8. Let $n = b(A)$ and note that because of the construction we can assume that the number of different atoms in $\mathcal{G}$ is n. Therefore all irreducible sequents in $\mathcal{L}_\mathcal{G}$ have at most n boxed atoms at the right.
Let $\mathcal{G}_1, \dots, \mathcal{G}_m$ be all sets in normal form such that $\mathcal{G} \overset{n}{\rightsquigarrow} \mathcal{G}_i$. Note that the $\mathcal{G}_i$ are consistent because A is. As all $\mathcal{G}_i$ are closed under V°, the corresponding formulas G_i are projective by Corollary 1. Because $\mathsf{V}^\circ_{b(A)}$ is admissible, the construction of the G_i implies that $G \mathrel{|\!\!\sim} \{G_1, \dots, G_m\}$. Hence $A \mathrel{|\!\!\!\sim} \{G_1, \dots, G_m\}$. By definition of $\overset{n}{\rightsquigarrow}$, $\bigvee G_i \vdash G$. Hence $\bigvee G_i \vdash A$.
This proves that $\{G_1, \dots, G_m\}$ is an irreducible projective approximation of A. $\Box$

Corollary 2. If V°_n is admissible in L, then the n-unification type of L is finitary.

Proof. This follows from Theorem 3 and the observation that if the number of symbols in A is n, then $b(A) \leq n$. □

The previous corollary implies the following corollary, which for full $\mathsf{S4}$ has been proved by Ghilardi in [9].

Corollary 3. $\mathsf{S4}$ and its negationless fragment have finitary unification.

8. Admissible rules

This last section of the paper contains some applications of the previous results to admissible rules. A set of rules $\mathcal{R}$ is a *basis* for the admissible rules of a logic L if

$$\Gamma \mathrel{|\!\sim}_{\mathsf{L}} \Delta \;\Leftrightarrow\; \Gamma \vdash_{\mathsf{L}}^{\mathcal{R}} \Delta,$$

where $\vdash_{\mathsf{L}}^{\mathcal{R}}$ is the smallest finitary structural multi-conclusion consequence relation that extends $\vdash_{\mathsf{L}}$ in which all rules in $\mathcal{R}$ are derivable. Thus intuitively, $\mathcal{R}$ is a basis if all admissible rules can derived from those in $\mathcal{R}$.

In intermediate logics all consistent formulas are unifiable, but this is no longer the case in modal logic. This leads to the notion of *passive* admissible rules, which are admissible rules for which the hypothesis $(\bigwedge \Gamma)$ has no unifier. $\bot/A$ is a typical example of such a rule, and $(\Gamma \equiv \Box\Gamma \Rightarrow)/A$ is so in reflexive logics.

A logic is *structurally complete* if all single-conclusion admissible rules are derivable, and *almost structurally complete* if all nonpassive single-conclusion admissible rules are derivable [7]. A logic is *hereditarily (almost) structurally complete* if all its extensions, including the logic itself, are (almost) structurally complete.

Jeřábek has proved the following theorem for full $\mathsf{S4}$ [19]. Using the techniques in this paper it can also be proved in the following way, also for the negationless fragment.

Theorem 4. In any extension of $\mathsf{S4}$ or its negationless fragment, the rules V° form a basis for the admissible rules once they are admissible.

Proof. Suppose that V° is admissible and that $\Gamma \mathrel{|\!\sim}_{\mathsf{L}} \Delta$. Let $A = \bigwedge \Gamma$. If A is inconsistent, $\Gamma \vdash_{\mathsf{L}} \Delta$ clearly holds. Otherwise, let $\{G_1, \ldots, G_n\}$ be the projective approximation of A which exists by Theorem 3. As $G_i \vdash A$ for all i, also $G_i \mathrel{|\!\sim} \Delta$. Because G_i is projective, this implies $G_i \vdash \Delta$. The construction in the proof of Theorem 3 shows that $A \Vdash^{\mathsf{V}^\circ} \{G_1, \ldots, G_n\}$. Hence $\Gamma \vdash^{\mathsf{V}^\circ} \Delta$. □

Corollary 4. V° is a basis for the admissible rules of $\mathsf{S4}$ as well as for its negationless fragment.

Dzik and Wojtylak prove in [7] that any extension of $\mathsf{S4}$ has projective unification if and only if it contains $\mathsf{S4.3}$, where $\mathsf{S4.3}$ is the logic $\mathsf{S4}$ extended by the principle $\Box(\Box A \to \Box B) \vee \Box(\Box B \to \Box A)$. This implies that $\mathsf{S4.3}$ is hereditarily almost structurally complete. Here we provide another proof of the last result and extend it to fragments.

Theorem 5. $\mathsf{S4.3}$ is hereditarily almost structurally complete. Its negationless fragment is hereditarily structurally complete.

Proof. Let L be an extension of a fragment of $\mathsf{S4.3}$. The fact that $\mathsf{S4.3}$ is complete with respect to transitive reflexive Kripke frames in which every two nodes are compatible (xRy or yRx holds) is easily seen to imply that all non passive instances

of V° are derivable in L. Theorem 4 now shows that all non passive admissible rules are derivable.
If L does not contain $\bot$, then there are no passive admissible rules. Hence the hereditarily structural completeness. □

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