

Automatic Sequences and Zip-Specifications

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Abstract

We consider infinite sequences of symbols, also known as streams, and the decidability question for equality of streams defined in a restricted format. This restricted format consists of prefixing a symbol at the head of a stream, of the stream function ‘zip’, and recursion variables. Here ‘zip’ interleaves the elements of two streams in alternating order, starting with the first stream; e.g., for the streams defined by $\{\text{zeros} = 0 : \text{zeros}, \text{ones} = 1 : \text{ones}, \text{alt} = 0 : 1 : \text{alt}\}$ we have $\text{zip}(\text{zeros}, \text{ones}) = \text{alt}$. The celebrated Thue–Morse sequence is obtained by the succinct ‘zip-specification’

$$M = 0 : X \quad X = 1 : \text{zip}(X, Y) \quad Y = 0 : \text{zip}(Y, X)$$

Our analysis of such systems employs both term rewriting and coalgebraic techniques. We establish decidability for these zip-specifications, employing bisimilarity of observation graphs based on a suitably chosen cobasis. The importance of zip-specifications resides in their intimate connection with automatic sequences. The analysis leading to the decidability proof of the ‘infinite word problem’ for zip-specifications, yields a new and simple characterization of automatic sequences. Thus we obtain for the binary **zip** that a stream is 2-automatic iff its observation graph using the cobasis $\langle \text{hd}, \text{even}, \text{odd} \rangle$ is finite. Here **odd** and **even** have the usual recursive definition: $\text{even}(a : s) = a : \text{odd}(s)$, and $\text{odd}(a : s) = \text{even}(s)$. The generalization to zip- k specifications and their relation to k -automaticity is straightforward. In fact, zip-specifications can be perceived as a term rewriting syntax for automatic sequences. Our study of zip-specifications is placed in an even wider perspective by employing the observation graphs in a dynamic logic setting, leading to an alternative characterization of automatic sequences.

We further obtain a natural extension of the class of automatic sequences, obtained by ‘zip-mix’ specifications that use zips of different arities in one specification (recursion system). The corresponding notion of automaton employs a state-dependent input-alphabet, with a number representation $(n)_A = d_m \dots d_0$ where the base of digit d_i is determined by the automaton A on input $d_{i-1} \dots d_0$.

We also show that equivalence is undecidable for a simple extension of the zip-mix format with projections like **even** and **odd**. However, it remains open whether zip-mix specifications have a decidable equivalence problem.

1 Introduction

Infinite sequences of symbols, also called ‘streams’, are a playground of common interest for logic, computer science (functional programming, formal languages, combinatorics on infinite words), mathematics (numerations and number theory, fractals) and physics (signal processing). For logic and theoretical computer science this interest focuses in particular on unique solvability of systems of recursion equations defining streams, on expressivity (what scope does a definition or specification format have), and productivity (does a stream specification indeed unfold to its intended infinite result without stagnation). In addition, there is the ‘infinitary word problem’: when do two stream specifications over a first-order signature define the same stream? And, is that question decidable? If not, what is the logical complexity?

Against this background, we can now situate our present paper. In the landscape of streams there are some well-known families, with automatic sequences [2] as a prominent family, including members such as the Thue–Morse sequence [1]. Such sequences are defined in first-order signature that includes some basic stream functions such as `hd` (head), `tl` (tail), `·` (prefixing a symbol to an infinite stream), `even`, `odd`; all these are familiar from any functional programming language.

One stream function in particular is frequently used in stream specifications. This is the `zip` function, that ‘zips’ the elements of two streams in alternating order, starting with the first stream. Now there is an elegant definition of the Thue–Morse sequence `M` using only this function `zip`, next to prefixing an element, and of course recursion variables:

$$M = 0 : X \qquad X = 1 : \text{zip}(X, Y) \qquad Y = 0 : \text{zip}(Y, X) \qquad (1)$$

For general term rewrite systems, stream equality is easily seen to be undecidable [17], just as most interesting properties of streams. But by adopting some restrictions in the definitional format, decidability may hold.

Thus we consider the problem whether definitions like the one of `M`, using only `zip` next to prefixing and recursion, are still within the realm of decidability. Answering this question positively turned out to be rewarding. In addition to solving the technical problem, the analysis leading to the solution had a useful surprise in petto: it entailed a new and simple characterization of the important notion of k -automaticity of streams. (The same ‘aha-insight’ was independently obtained by Kupke and Rutten, preliminary reported in [14].)

The remainder of the paper is devoted to an elaboration of several aspects concerning `zip`-specifications and automaticity. First, we treat a representation of automatic sequences in a framework of propositional dynamic logic, employing cobases and the ensuing observation graphs (used before for the decidability of equivalence) as the underlying semantics for a dynamic logic formula characterizing the automaticity of a stream. Second, we are led to a natural generalization of automatic sequences, corresponding to mixed `zip`-specifications that contain `zip` operators of different arities. The corresponding type of automaton

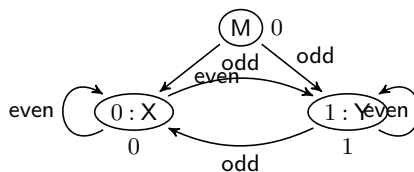


Figure 1: Observation graph for the specification (1) of the Thue-Morse sequence M .

employs a state-dependent alphabet. Third, we show that stream equality for a slight extension of zip-specifications is Π_1^0 ; the latter via a reduction from the halting problem of Fractran programs [7].

Let us now describe somewhat informally the key method that we employ to solve the equivalence problem for zip-specifications. To that end, consider the specification (1) above with root variable M . This specification is productive [19, 8, 10] and defines the Thue-Morse sequence:

$$M \rightarrow^\omega 0 : 1 : 1 : 0 : 1 : 0 : 0 : 1 : 1 : 0 : 0 : 1 : 0 : 1 : 1 : 0 : \dots,$$

that is, by repeatedly applying rewrite rules that arise by orienting the equations for M , X and Y from left to right, M rewrites in the limit to the Thue-Morse sequence [1].

We will construct so-called ‘observation graphs’ based on the stream cobasis $\langle \text{hd}, \text{even}, \text{odd} \rangle$ where all nodes have a double label: inside, a term corresponding to a stream (such as M and $0 : X$ in Figure 1) and outside, the head of that stream. The nodes have outgoing edges to their *even*- and *odd*-derivatives. An example is shown in Figure 1.

So, the problem of equivalence of zip-specifications reduces to the problem of bisimilarity of their observation graphs, which we prove to be finite. This does not hold for observation graphs of zip-specifications with respect to the cobasis $\langle \text{hd}, \text{tl} \rangle$: for this cobasis, the above specification would yield an infinite observation graph. (The same would hold for any stream which is not eventually periodic.)

The observation graph in Figure 1 evokes the ‘aha-insight’ mentioned above: it can be recognized as a DFAO¹ (deterministic finite automaton with output) that witnesses the fact that M is a 2-automatic sequence [2].

We will exhibit the close connection between zip-specifications and automatic sequences, residing in the coincidence of DFAOs and observation graphs.

2 Zip-Specifications

For term rewriting notions see further [21]. For $k \in \mathbb{N}$ we define $\mathbb{N}_{<k} = \{0, 1, \dots, k - 1\}$. Let Δ be a finite alphabet of at least two symbols, and \mathcal{X} a finite set of recursion variables.

¹The bisimulation collapse of the graph in Fig. 1 identifies the states labeled M and $0 : X$, giving rise to the familiar (minimal) DFAO for M .

Definition 1. The set Δ^ω of *streams over* Δ is defined by $\Delta^\omega = \{\sigma \mid \sigma : \mathbb{N} \rightarrow \Delta\}$.

We write $a : \sigma$ for the stream τ defined by $\tau(0) = a$ and $\tau(n+1) = \sigma(n)$ for all $n \in \mathbb{N}$. We define $\text{hd} : \Delta^\omega \rightarrow \Delta$ and $\text{tl} : \Delta^\omega \rightarrow \Delta^\omega$ by $\text{hd}(x : \sigma) = x$ and $\text{tl}(x : \sigma) = \sigma$.

Convention 2. We usually mix notations for syntax (term rewriting) and semantics ('real' functions). Whenever confusion is possible, we use fonts *fun*, and *fun* to distinguish between functions, and term rewrite symbols, respectively.

Definition 3. For $k \in \mathbb{N}_{>0}$, the function $\text{zip}_k : (\Delta^\omega)^k \rightarrow \Delta^\omega$ is defined by the following rewrite rule:

$$\text{zip}_k(x : \sigma_0, \sigma_1, \dots, \sigma_{k-1}) \rightarrow x : \text{zip}_k(\sigma_1, \dots, \sigma_{k-1}, \sigma_0)$$

Thus zip_k interleaves its argument streams:

$$\text{zip}_k(\sigma_0, \dots, \sigma_{k-1})(kn + i) = \sigma_i(n) \quad (0 \leq i < k)$$

Definition 4. The set $\mathcal{Z}(\Delta, \mathcal{X})$ of *zip-terms* over $\langle \Delta, \mathcal{X} \rangle$ is defined by the grammar:

$$Z ::= X \mid a : Z \mid \underbrace{\text{zip}_k(Z, \dots, Z)}_{k \text{ times}} \quad (X \in \mathcal{X}, a \in \Delta, k \in \mathbb{N})$$

A *zip-specification* \mathcal{S} over $\langle \Delta, \mathcal{X} \rangle$ consists of a distinguished variable $X_0 \in \mathcal{X}$ called the *root of* \mathcal{S} , and for every $X \in \mathcal{X}$ a pair $\langle X, t \rangle$ with $t \in \mathcal{Z}(\Delta, \mathcal{X})$ a zip-term. We treat these pairs as term rewrite rules, and write them as equations $X = t$.

Definition 5. For $k \in \mathbb{N}$, the set $\mathcal{Z}_k(\Delta, \mathcal{X})$ of *zip-k terms* is the restriction of $\mathcal{Z}(\Delta, \mathcal{X})$ to terms where for every occurrence of a symbol zip_ℓ ($\ell \in \mathbb{N}$) it holds that $\ell = k$.

A *zip-k specification* is a zip-specification such that for all equations $X = t$ it holds that $t \in \mathcal{Z}_k(\Delta, \mathcal{X})$.

We always assume for zip-specifications \mathcal{S} that every recursion variable is *reachable from the root* X_0 .

2.1 Unique Solvability, Productivity and Leftmost Cycles

Definition 6. A *valuation* is a mapping $\alpha : \mathcal{X} \rightarrow \Delta^\omega$. Such a valuation α extends to $\llbracket \cdot \rrbracket_\alpha : \mathcal{Z}(\Delta, \mathcal{X}) \rightarrow \Delta^\omega$ as follows:

$$\begin{aligned} \llbracket X \rrbracket_\alpha &= \alpha(X) \\ \llbracket a : t \rrbracket_\alpha &= a : \llbracket t \rrbracket_\alpha \\ \llbracket \text{zip}_k(t_1, \dots, t_k) \rrbracket_\alpha &= \text{zip}_k(\llbracket t_1 \rrbracket_\alpha, \dots, \llbracket t_k \rrbracket_\alpha) \end{aligned}$$

A *solution* for a zip-specification \mathcal{S} is a valuation $\alpha : \mathcal{X} \rightarrow \Delta^\omega$, denoted $\alpha \models \mathcal{S}$, such that $\llbracket X \rrbracket_\alpha = \llbracket t \rrbracket_\alpha$ for all $X = t \in \mathcal{S}$.

A zip-specification \mathcal{S} is *uniquely solvable* if there is a unique solution α for \mathcal{S} ; then we let $\llbracket \cdot \rrbracket^{\mathcal{S}} = \alpha$ denote this solution.

Definition 7. Let \mathcal{S} and \mathcal{S}' be zip-specifications with roots X_0 and X'_0 , respectively. Then \mathcal{S} is called *equivalent* to \mathcal{S}' if they have the same set of solutions for their roots:

$$\{ \llbracket X_0 \rrbracket_\alpha \mid \alpha \models \mathcal{S} \} = \{ \llbracket X'_0 \rrbracket_{\alpha'} \mid \alpha' \models \mathcal{S}' \}$$

Definition 8. A zip-specification \mathcal{S} with root X_0 is *productive* if there exists a reduction of the form $X_0 \rightarrow^* a_1 : \dots : a_n : t$ for all $n \in \mathbb{N}$. If a zip-specification \mathcal{S} is productive, then \mathcal{S} is said to *define the stream* $\llbracket X_0 \rrbracket^{\mathcal{S}}$ where X_0 is the root of \mathcal{S} .

Note that if a specification is productive, then by confluence of orthogonal term rewrite systems [21], there exists a rewrite sequence of length ω that converges towards an infinite stream term $a_1 : a_2 : a_3 : \dots$ in the limit.

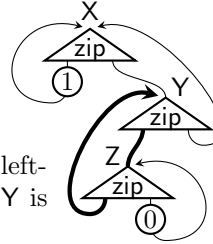
While productivity is undecidable [9, 20] for term rewrite systems in general, zip-specifications fall into the class of ‘pure stream specifications’ [8, 10] for which (automated) decision procedures exist. However, the latter would be taking a sledgehammer to crack a nut. For zip-specifications, productivity boils down to a simple syntactic criterion.

Definition 9. Let \mathcal{S} be a zip-specification. A *step in \mathcal{S}* is pair of terms $\langle s, t \rangle$, denoted by $s \rightsquigarrow t$, such that (a) $s \rightarrow t \in \mathcal{S}$, (b) $s = a : t$, or (c) $s = \text{zip}_k(\dots, t, \dots)$. A *guard* is a step of form (b). A *left-step $s \rightsquigarrow_\ell t$ in \mathcal{S}* is a step $s \rightsquigarrow t$ of the form (a), (b) or (c') $s = \text{zip}_k(t, \dots)$.

A *cycle in \mathcal{S}* is a sequence t_1, t_2, \dots, t_n such that $t_1 = t_n \in \mathcal{X}$ and $t_i \rightsquigarrow t_{i+1}$ for $1 \leq i < n$. A *leftmost cycle in \mathcal{S}* is a cycle t_1, t_2, \dots, t_n such that $t_i \rightsquigarrow_\ell t_{i+1}$ for $1 \leq i < n$.

Example 10. Consider the following specification

$$\begin{aligned} X &= \text{zip}(1 : X, Y) \\ Y &= \text{zip}(Z, X) \\ Z &= \text{zip}(Y, 0 : Z) \end{aligned}$$



visualized as the cyclic term graph on the right. The leftmost cycle $Y \rightsquigarrow_\ell \text{zip}(Z, X) \rightsquigarrow_\ell Z \rightsquigarrow_\ell \text{zip}(Y, 0 : Z) \rightsquigarrow_\ell Y$ is not guarded.

For term rewriting systems in general, productivity implies the uniqueness of solutions, but unique solvability is not sufficient for productivity. For zip-specifications it turns out that both concepts coincide. Here we need that Δ is not a singleton — otherwise every specification has a unique solution.

Theorem 11. *For zip-specifications \mathcal{S} these are equivalent:*

- (i) \mathcal{S} is uniquely solvable.
- (ii) \mathcal{S} is productive.
- (iii) \mathcal{S} has a guard on every leftmost cycle.

2.2 Evolving and Solving Zip-Specifications

The key to the proof of Theorem 11 consists of a transformation of zip-specifications by (i) simple equational logic steps, and (ii) internal rewrite steps.

Definition 12. For zip-specifications $\mathcal{S}, \mathcal{S}'$ we say \mathcal{S} evolves to \mathcal{S}' , denoted by $\mathcal{S} \circ \mathcal{S}'$, if one of the conditions holds:

- (i) \mathcal{S} contains an equation $X = a : t$ with $X \neq X_0$ and \mathcal{S}' is obtained from \mathcal{S} by: let X' be fresh and
 - (a) exchange the equation $X = a : t$ for $X' = t$, then
 - (b) replace all X in all right-hand sides by $a : X'$, and
 - (c) finally rename X' to X (X is no longer used).
- (ii) \mathcal{S} contains an equation $X = t$ such that t rewrites to t' via a zip-rule (Definition 3), and \mathcal{S}' is obtained from \mathcal{S} by replacing the equation $X = t$ with $X = t'$.

The condition $X \neq X_0$ in clause (i) guarantees that the meaning (its solution) is preserved under evolving. It prevents transforming a specification like $X_0 = 0 : 1 : X_0$ into $X_0 = 1 : 0 : X_0$ which clearly has a different solution.

Lemma 13. *Let $\mathcal{S} \circ \mathcal{S}'$. Then for every $\alpha : \mathcal{X} \rightarrow \Delta^\omega$ it holds that α is a solution of \mathcal{S} if and only if α is a solution of \mathcal{S}' . Moreover, if \mathcal{S} is productive then so is \mathcal{S}' .*

Definition 14. A zip-specification \mathcal{S} is said to have a *free root* if the root X_0 of \mathcal{S} does not occur in any right-hand side of \mathcal{S} .

Lemma 15. *Every zip-specification can be transformed into an equivalent one with free root.*

The following lemma relates rewriting to evolving:

Lemma 16. *Let \mathcal{S} be a zip-specification with free root X_0 . There exists a reduction $X_0 \rightarrow^* a_1 : \dots : a_n : t$ in \mathcal{S} if and only if there exists a zip-specification \mathcal{S}' such that $\mathcal{S} \circ^* \mathcal{S}'$ and \mathcal{S}' contains an equation of the form $X_0 = a_1 : \dots : a_n : t'$.*

Example 17. We evolve the following specification:

$$\begin{aligned}
X &= \text{zip}(1 : X, Y) \quad Y = \underline{0} : \text{tl}(\text{zip}(Z, X)) \quad Z = \text{zip}(Y, 0 : Z) \\
X &= \text{zip}(1 : X, \overline{0} : Y) \quad Y = \text{tl}(\text{zip}(Z, X)) \quad Z = \text{zip}(\overline{0} : Y, 0 : Z) \\
&\dots \quad Y = \text{tl}(\text{zip}(Z, X)) \quad Z = \overline{0} : \text{zip}(0 : Z, Y) \\
&\dots \quad Y = \text{tl}(\text{zip}(\overline{0} : Z, X)) \quad Z = \text{zip}(0 : \overline{0} : Z, Y) \\
&\dots \quad Y = \text{tl}(\overline{0} : \text{zip}(X, Z)) \quad Z = \text{zip}(0 : 0 : Z, Y) \\
X &= \text{zip}(1 : X, 0 : Y) \quad Y = \text{zip}(X, Z) \quad Z = \text{zip}(0 : 0 : Z, Y)
\end{aligned}$$

Note that the contracted redexes are underlined and the created symbols are overlined. Also note that invoking a free root is not needed for the evolution above.

Strictly speaking, the last step in the above example is not covered by Definition 12 since the rule for ‘tl’ is not included. We have chosen this example to demonstrate another principle. The specification we started from is obtained from Example 10 by inserting $0 : \text{tl}(\dots)$ on an unguarded leftmost cycle. Evolving has resulted in a productive zip-specification (now every leftmost cycle is guarded) that represents a solution of the original specification. Similarly, by inserting $1 : \text{tl}(\dots)$, we obtain the solution:

$$X = \text{zip}(1 : X, 1 : Y) \quad Y = \text{zip}(X, Z) \quad Z = \text{zip}(0 : 1 : Z, Y)$$

The insertion of $0 : \text{tl}(\dots)$ and $1 : \text{tl}(\dots)$ corresponds to choosing whether we are interested in a solution for Y starting with head 0 or 1. To see that the result of the insertions are valid solutions it is crucial to observe that the symbol ‘tl’ in the inserted $a : \text{tl}(\dots)$ disappears by consuming a ‘descendant’ of the element $a \in \Delta$. In general we have:

Lemma 18. *Let \mathcal{S} be a zip-specification. Define the set $\{Y_1, \dots, Y_m\}$ to contain precisely one recursion variable from every unguarded leftmost cycle from \mathcal{S} .*

Let $\vec{a} = \langle a_1, \dots, a_m \rangle \in \Delta^m$ and define $\mathcal{S}_{\vec{a}}$ to be obtained from \mathcal{S} by replacing each equation $Y_i = t_i$ by $Y_i = a_i : \text{tl}(t_i)$. Subsequently, we can by the evolving procedure eliminate the occurrences of the symbol tl as in Example 17. Then $\mathcal{S}_{\vec{a}}$ is productive, and the unique solution $\llbracket \cdot \rrbracket^{\mathcal{S}_{\vec{a}}} : \mathcal{X} \rightarrow \Delta^\omega$ is a solution of \mathcal{S} . Hence, $\{\mathcal{S}_{\vec{a}} \mid \vec{a} \in \Delta^m\}$ is the set of all solutions of \mathcal{S} , in particular, \mathcal{S} has $|\Delta|^m$ different solutions.

2.3 Formats of Zip-Specifications

Definition 19. A zip-specification \mathcal{S} is called *flat* if each of its equations is of the form:

$$X_i = c_{i,1} : \dots : c_{i,m_i} : \text{zip}_{k_i}(X_{i,1}, \dots, X_{i,k_i}) \quad (0 \leq i < n)$$

for $m_i, k_i \in \mathbb{N}$, $k_i \geq 2$, recursion variables $X_i, X_{i,1}, \dots, X_{i,k_i}$ and data constants $c_{i,1}, \dots, c_{i,m_i}$.

Zip-free cycles correspond to periodic sequences, and these can be specified by flat zip- k specifications. Together with unfolding and introduction of fresh variables we then obtain:

Lemma 20. *Every productive zip- k specification can be transformed into an equivalent productive, flat zip- k specification.*

3 Zip-Specifications and Observation Graphs

For the decidability result and the connection with automaticity we need to observe streams and compare them. This is done with observations in terms of a cobasis and bisimulations to compare the resulting graphs.

3.1 Cobases, Observation Graphs, and Bisimulation

For general introductions to coalgebra we refer to [4, 18]. We first introduce the notion of ‘cobasis’ [16, 13]. For the sake of simplicity, we restrict to the single observation hd .

Definition 21. A *stream cobasis* $\mathcal{B} = \langle \text{hd}, \langle \gamma_1, \dots, \gamma_k \rangle \rangle$ is a tuple consisting of operations $\gamma_i : \Delta^\omega \rightarrow \Delta^\omega$ ($1 \leq i \leq k$) such that for all $\sigma, \tau \in \Delta^\omega$ it holds that $\sigma = \tau$ whenever

$$\text{hd}(\gamma_{i_1}(\dots(\gamma_{i_n}(\sigma))\dots)) = \text{hd}(\gamma_{i_1}(\dots(\gamma_{i_n}(\tau))\dots))$$

for all $n \in \mathbb{N}$ and $1 \leq i_1, \dots, i_n \leq k$.

As hd is integral part of every stream cobasis, we suppress hd and write $\langle \gamma_1, \dots, \gamma_k \rangle$ as shorthand for $\langle \text{hd}, \langle \gamma_1, \dots, \gamma_k \rangle \rangle$.

Definition 22. For $i \in \mathbb{N}$, $k \in \mathbb{N}_{>0}$ define $\pi_{i,k} : \Delta^\omega \rightarrow \Delta^\omega$:

$$\begin{aligned} \pi_{0,k}(x : \sigma) &\rightarrow x : \pi_{k-1,k}(\sigma) \\ \pi_{i+1,k}(x : \sigma) &\rightarrow \pi_{i,k}(\sigma) \end{aligned}$$

For every $k \geq 2$ we define two stream cobases:

$$\mathcal{N}_k = \langle \pi_{0,k}, \dots, \pi_{k-1,k} \rangle \quad \mathcal{O}_k = \langle \pi_{1,k}, \dots, \pi_{k,k} \rangle$$

Note that $\pi_{i,k}(\sigma)$ selects an arithmetic subsequence of σ ; it picks every k -th element beginning from index i : $\pi_{i,k}(\sigma)(n) = \sigma(kn+i)$. The $\pi_{i,k}$ are generalized even and odd functions, in particular we have: $\text{tl} = \pi_{1,1}$, $\text{even} = \pi_{0,2}$ and $\text{odd} = \pi_{1,2}$.

Observe that \mathcal{N}_k and \mathcal{O}_k are cobases, that is, every element of a stream can be observed. The main difference between \mathcal{N}_k and \mathcal{O}_k is that \mathcal{N}_k has an ambiguity in naming stream entries: $\text{hd}(\sigma) = \text{hd}(\text{even}(\sigma))$. On the other hand, \mathcal{O}_k is an orthogonal basis, names of stream entries are unambiguous.

We employ the following simple coinduction principle:

Definition 23. Let $\mathcal{B} = \langle \gamma_1, \dots, \gamma_k \rangle$ be a cobasis. A \mathcal{B} -bisimulation is a relation $R \subseteq \Delta^\omega \times \Delta^\omega$ s.t. $\langle \sigma, \tau \rangle \in R$ implies $\text{hd}(\sigma) = \text{hd}(\tau)$ and $\langle \gamma_i(\sigma), \gamma_i(\tau) \rangle \in R$ for $1 \leq i \leq k$.

Lemma 24. For all $\sigma, \tau \in \Delta^\omega$ it holds that $\sigma = \tau$ if and only if there exists a \mathcal{B} -bisimulation R such that $\langle \sigma, \tau \rangle \in R$. \square

We now further elaborate the coalgebraic perspective. The following definition formalizes ‘ \mathcal{B} -observation graphs’ where $\mathcal{B} = \langle \gamma_1, \dots, \gamma_k \rangle$ is a cobasis. Every node n will represent the stream $\llbracket n \rrbracket \in \Delta^\omega$, and if the i -th outgoing edge of n points to node m then $\gamma_i(\llbracket n \rrbracket) = \llbracket m \rrbracket$.

Definition 25. Let $\mathcal{B} = \langle \text{hd}, \langle \gamma_1, \dots, \gamma_k \rangle \rangle$ be a stream cobasis, and let F be the functor $F(X) = \Delta \times X^k$.

A \mathcal{B} -observation graph is an F -coalgebra $\mathcal{G} = \langle S, \langle o, n \rangle \rangle$ with a distinguished root element $r \in S$, such that there exists an F -homomorphism $\llbracket \cdot \rrbracket : S \rightarrow \Delta^\omega$ from \mathcal{G} to the F -coalgebra $\langle \Delta^\omega, \mathcal{B} \rangle$ of all streams with respect to \mathcal{B} :

$$\begin{array}{ccc} S & \xrightarrow{\llbracket \cdot \rrbracket} & \Delta^\omega \\ \langle o, n \rangle \downarrow & \text{id} \times \llbracket \cdot \rrbracket^k & \downarrow \mathcal{B} \\ \Delta \times S^k & \xrightarrow{\quad} & \Delta \times (\Delta^\omega)^k \end{array}$$

The observation graph \mathcal{G} is said to *define* the stream $\llbracket r \rrbracket \in \Delta^\omega$. (We note that $\llbracket \cdot \rrbracket$ is unique by Lemma 26, below.)

Let $\sigma \in \Delta^\omega$. The *canonical \mathcal{B} -observation graph of σ* is defined as the subcoalgebra of the F -coalgebra $\langle \Delta^\omega, \mathcal{B} \rangle$ generated by σ , that is, the observation graph $\langle T, \mathcal{B} \rangle$ with root σ where $T \subseteq \Delta^\omega$ is the least set containing σ that is closed under $\gamma_1, \dots, \gamma_k$. The set $\partial_{\mathcal{B}}(\sigma)$ of \mathcal{B} -derivatives of σ is the set of elements of the canonical observation graph of σ .

Lemma 26. For every \mathcal{B} -observation graph the mapping $\llbracket \cdot \rrbracket$ is unique whenever it exists.

For the cobasis \mathcal{O}_k , the existence of $\llbracket \cdot \rrbracket$ is guaranteed:

Proposition 27. The stream coalgebra $\langle \Delta^\omega, \mathcal{O}_k \rangle$ is final for the functor $F(X) = \Delta \times X^k$. As a consequence, we have that every F -coalgebra is an \mathcal{O}_k -observation graph.

In contrast, the existence of $\llbracket \cdot \rrbracket$ is *not* guaranteed for \mathcal{N}_k . The coalgebra $\langle \Delta^\omega, \mathcal{N}_k \rangle$ is final for a subset of F -coalgebras, called *zero-consistent*, see further [14].

Definition 28. Let $\mathcal{B} = \langle \text{hd}, \langle \gamma_1, \dots, \gamma_k \rangle \rangle$ be a stream cobasis. A *bisimulation* between \mathcal{B} -observation graphs $\mathcal{G} = \langle S, \langle o, n \rangle \rangle$ and $\mathcal{G}' = \langle S', \langle o', n' \rangle \rangle$ is a relation $R \subseteq S \times S'$ such that for all $\langle s, s' \rangle \in R$ we have that $o(s) = o'(s')$ and $\langle n_i(s), n'_i(s') \rangle \in R$ for all $1 \leq i \leq k$, where n_i denotes the i -th projection on n . Two observation graphs are *bisimilar* if there is a bisimulation relating their roots.

For deterministic transition systems, such as observation graphs, bisimilarity coincides with trace equivalence. As a consequence, the algorithm of Hopcroft–Karp [11] is applicable.

Proposition 29. *Bisimilarity of finite \mathcal{B} -observation graphs is decidable (in linear time with respect to the sum of the number of vertices).*

Proposition 30. *Let \mathcal{B} be a stream cobasis. Two \mathcal{B} -observation graphs define the same stream if and only if they are bisimilar.*

3.2 From Zip- k Specifications To Observation Graphs

We construct observation graphs for zip- k specifications.

Definition 31. Let $\mathcal{X} = \{X_0, \dots, X_{n-1}\}$ be a set of recursion variables and Δ a finite set of data-constants. Let $k \in \mathbb{N}$, and \mathcal{S} be a zip- k specification over $\langle \Delta, \mathcal{X} \rangle$. We define the orthogonal term rewrite system $\mathcal{R}_k(\mathcal{S})$ to consist of the following rules:

$$\begin{aligned} \text{hd}(a : \sigma) &\rightarrow a \\ \pi_{0,k}(a : \sigma) &\rightarrow a : \pi_{k-1,k}(\sigma) \\ \pi_{i+1,k}(a : \sigma) &\rightarrow \pi_{i,k}(\sigma) && (0 \leq i < k + 1) \\ \text{hd}(\text{zip}_k(\sigma_0, \dots, \sigma_{k-1})) &\rightarrow \text{hd}(\sigma_0) \\ \pi_{i,k}(\text{zip}_k(\sigma_0, \dots, \sigma_{k-1})) &\rightarrow \sigma_i && (0 \leq i < k) \end{aligned}$$

and additionally for every equation $X_j = t$ of \mathcal{S} the rules

$$\begin{aligned} \text{hd}(X_j) &\rightarrow \text{hd}(t) \\ \pi_{i,k}(X_j) &\rightarrow \pi_{i,k}(t) && (0 \leq i \leq k + 1) \end{aligned}$$

where the X_j are treated as constant symbols.

Whenever \mathcal{S} is clear from the context, then by $t \downarrow$ we denote the unique normal form of term t with respect to $\mathcal{R}_k(\mathcal{S})$.

Definition 32. Let \mathcal{S} be a productive, flat zip- k specification with root X_0 . The set $\delta_k(\mathcal{S})$ is the least set containing X_0 that is closed under $\lambda t. (\pi_{i,k}(t) \downarrow)$ for every $0 \leq i < k$.

Definition 33. Let \mathcal{S} be a productive, flat zip- k specification with root X_0 . The \mathcal{N}_k -observation graph $\mathcal{G}(\mathcal{S})$ is defined as:

$$\begin{aligned} \mathcal{G}(\mathcal{S}) &= \langle \delta_k(\mathcal{S}), \langle o, n \rangle \rangle && o(t) = \text{hd}(t) \downarrow \\ &&& n(t) = \langle \pi_{0,k}(t) \downarrow, \dots, \pi_{k-1,k}(t) \downarrow \rangle \end{aligned}$$

with root X_0 . In words: every node t has

- (i) the observation $\text{hd}(t) \downarrow$ (the label), and

(ii) outgoing edges to $\pi_{0,k}(t)\downarrow, \dots, \pi_{k-1,k}(t)\downarrow$ (in this order).

Lemma 34. *Let \mathcal{S} be a productive, flat zip- k specification with root X_0 . There exists $m \in \mathbb{N}$ such that every term in $\delta_k(\mathcal{S})$ is of the form $d_0 : \dots : d_{\ell-1} : X_j$ with $\ell \leq m$, $d_0, \dots, d_{\ell-1} \in \Delta$ and $X_j \in \mathcal{X}$. As a consequence $\delta_k(\mathcal{S})$ and $\mathcal{G}(\mathcal{S})$ are finite.*

Proof Sketch. The equations of \mathcal{S} are of the form:

$$X_j = c_{j,0} : \dots : c_{j,m_j-1} : \text{zip}_k(X_{j,0}, \dots, X_{j,k-1}) \quad (0 \leq j < n)$$

Let $m := \max\{m_i \mid 0 \leq i < n\}$. It suffices that the claimed shape is closed under $\lambda s. \pi_{i,k}(s)\downarrow$ for $0 \leq i < k$. This follows by a straightforward application of Definition 31 together with a precise counting of the ‘produced’ elements. \square

We need to ensure that the rewrite system from Definition 31 implements (is sound for) the intended semantics; recall that \mathcal{S} has a unique solution $[[\cdot]]^{\mathcal{S}} : \mathcal{X} \rightarrow \Delta^\omega$ due to productivity:

Lemma 35. *Let \mathcal{S} be a productive, flat zip- k specification with root X_0 . For every $t \in \delta_k(\mathcal{S})$ and $0 \leq i < k$ we have that $\text{hd}(t) \rightarrow^* \text{hd}([[t]])$ and $[[\pi_{i,k}(t)\downarrow]] = \pi_{i,k}([[t]])$. Hence, the graph $\mathcal{G}(\mathcal{S})$ is an \mathcal{N}_k -observation graph defining $[[X_0]]^{\mathcal{S}}$.*

Proof. The extension of $[[\cdot]]_\alpha$ from Definition 6, interpreting the symbols $\pi_{i,k}$ by the stream function $\pi_{i,k} : \Delta^\omega \rightarrow \Delta^\omega$ for every $0 \leq i < k$, is a model of $\mathcal{R}_k(\mathcal{S})$. \square

As an application of Lemmas 20 and 35 we get

Lemma 36. *For every productive zip- k specification with root X_0 we can construct an \mathcal{N}_k -observation graph defining the stream $[[X_0]]^{\mathcal{S}}$.*

We arrive at our first main result:

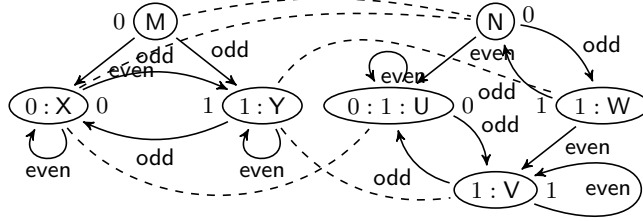
Theorem 37. *Equivalence of zip- k specifications is decidable.*

Proof. Lemma 18 allows to reduce the equivalence problem for unproductive zip- k specifications to a finite number of equivalence problems for productive zip- k specifications. Propositions 30, 29 and Lemma 36 imply decidability of equivalence for productive zip- k specifications. \square

Proposition 38. *Equivalence of productive, flat zip-specifications is decidable in quadratic time.*

Example 39. Consider the zip-2 specification with root N :

$$\begin{aligned} N &= 0 : \text{zip}(1 : W, 1 : U) & U &= 1 : \text{zip}(V, U) \\ V &= 0 : \text{zip}(V, 1 : U) & W &= \text{zip}(N, V) \end{aligned}$$



Its \mathcal{N}_2 -observation graph is depicted on the right above. The dashed lines indicate a bisimulation with the observation graph from Fig. 1 here depicted on the left.

3.3 From Observation Graphs To Zip- k Specifications

Lemma 40. *The canonical \mathcal{O}_k -observation graph of a stream $\sigma \in \Delta^\omega$ is finite if and only if σ can be defined by a zip- k specification consisting of equations of the form:*

$$X_i = a_i : \text{zip}_k(X_{i,1}, X_{i,2}, \dots, X_{i,k})$$

Proof. For the translation forth and back, it suffices to observe the correspondence between an equation $Y = a : \text{zip}_k(Y_1, \dots, Y_k)$ and its semantics $\text{hd}(\llbracket Y \rrbracket) = a$, $\pi_{1,k}(\llbracket Y \rrbracket) = \llbracket Y_1 \rrbracket$, \dots , $\pi_{k,k}(\llbracket Y \rrbracket) = \llbracket Y_k \rrbracket$. \square

Lemma 41. *The canonical \mathcal{N}_k -observation graph of a stream $\sigma \in \Delta^\omega$ is finite if and only if σ can be defined by a zip- k specification consisting of pairs of equations of the form:*

$$\begin{aligned} X_i &= a_i : X'_i \\ X'_i &= \text{zip}_k(X_{f(i,1)}, X_{f(i,2)}, \dots, X_{f(i,k-1)}, X'_{f(i,0)}) \end{aligned}$$

over recursion variables $\mathcal{X} \cup \mathcal{X}'$ where $\mathcal{X} = \{X_0, \dots, X_{n-1}\}$ and $\mathcal{X}' = \{X'_i \mid X_i \in \mathcal{X}\}$, and $f : \mathbb{N}_{<n} \times \mathbb{N}_{<k} \rightarrow \mathbb{N}_{<n}$ such that $a_{f(i,0)} = a_i$ for all $i \in \mathbb{N}_{<n}$.

Proof. If $Y = a : Y'$ and $Y' = \text{zip}_k(Y_1, \dots, Y_{k-1}, Y'_0)$ then $\text{hd}(\llbracket Y \rrbracket) = a$, $\pi_{0,k}(\llbracket Y \rrbracket) = a : \llbracket Y'_0 \rrbracket$, and $\pi_{i,k}(\llbracket Y \rrbracket) = \llbracket Y_i \rrbracket$ ($1 \leq i < k$). Since there also is an equation $Y_0 = a : Y'_0$, it holds that $\llbracket Y'_0 \rrbracket = \text{tl}(\llbracket Y_0 \rrbracket)$ and hence $\pi_{0,k}(\llbracket Y \rrbracket) = \llbracket Y_0 \rrbracket$. \square

4 Automaticity and Observation Graphs

After our first main result (Theorem 37) we proceed with connecting zip- k specifications to k -automatic sequences.

4.1 Automatic Sequences

Definition 42 ([2]). A *deterministic finite automaton with output (DFAO)* is a tuple $\langle Q, \Sigma, \delta, q_0, \Delta, \lambda \rangle$ where

- Q is a finite set of states,
- Σ a finite input alphabet,
- $\delta : Q \times \Sigma \rightarrow Q$ a transition function,
- $q_0 \in Q$ the initial state,
- Δ an output alphabet, and
- $\lambda : Q \rightarrow \Delta$ an output function.

We extend δ to words over Σ as follows:

$$\begin{aligned} \delta(q, \varepsilon) &= q && \text{for } q \in Q \\ \delta(q, wa) &= \delta(\delta(q, a), w) && \text{for } q \in Q, a \in \Sigma, w \in \Sigma^* \end{aligned}$$

and we write $\delta(w)$ as shorthand for $\delta(q_0, w)$.

For $n, k \in \mathbb{N}$, $k \geq 2$, we use $(n)_k$ to denote the representation of n with respect to the base k (without leading zeros). More precisely, for $n > 0$ we have $(n)_k = n_m n_{m-1} \dots n_0$ where $0 \leq n_m, \dots, n_0 < k$, $n_m > 0$ and $n = \sum_{i=0}^m n_i k^i$; for $n = 0$ we fix $(n)_k = \varepsilon$.

Definition 43. A k -DFAO A is a DFAO $\langle Q, \Sigma, \delta, q_0, \Delta, \lambda \rangle$ with input alphabet $\Sigma = \mathbb{N}_{<k}$. For $q \in Q$, we define a stream $\zeta(A, q)$ by: $\zeta(A, q)(n) = \lambda(\delta(q, (n)_k))$ for every $n \in \mathbb{N}$.

We write $\zeta(A)$ as shorthand for $\zeta(A, q_0)$. Moreover, we say that the automaton A *generates* the stream $\zeta(A)$.

Definition 44. A stream $\sigma : \Delta^\omega$ is called *k -automatic* if there exists a k -DFAO that generates σ . A stream is called *automatic* if it is k -automatic for some $k \geq 2$.

The exclusion of leading zeros in the number representation $(n)_k$ is not crucial for the definition of automatic sequences. Every k -DFAO can be transformed into an equivalent k -DFAO that ignores leading zeros:

Definition 45. A k -DFAO $\langle Q, \Sigma, \delta, q_0, \Delta, \lambda \rangle$ is called *invariant under leading zeros* if for all $q \in Q$: $\lambda(q) = \lambda(\delta(q, 0))$.

Lemma 46 ([2, Theorem 5.2.1 with Corollary 4.3.4]). *For every k -DFAO A there is a k -DFAO A' that is invariant under leading zeros and generates the same stream ($\zeta(A) = \zeta(A')$).*

Automatic sequences can be characterized in terms of their ‘kernels’ being finite. Kernels of a stream σ are sets of arithmetic subsequences of σ , defined as follows.

Definition 47. The *k -kernel* of a stream $\sigma \in \Delta^\omega$ is the set of subsequences $\{\pi_{i, k^p}(\sigma) \mid p \in \mathbb{N}, i < k^p\}$.

Lemma 48 ([2, Theorem 6.6.2]). *A stream σ is k -automatic if and only if the k -kernel of σ is finite.*

4.2 Observation Graphs and Automatic Sequences

There is a close correspondence between observation graphs with respect to the cobasis \mathcal{N}_k and k -DFAOs. For k -DFAOs A that are invariant under leading zeros an edge $q \rightarrow p$ labeled i implies that the stream generated by p is the $\pi_{i,k}$ -projection of the stream generated by q , that is, $\zeta(A, p) = \pi_{i,k}(\zeta(A, q))$. The following lemma treats the case of general k -DFAOs.

Lemma 49. *Let $A = \langle Q, \Sigma, \delta, q_0, \Delta, \lambda \rangle$ be a k -DFAO. Then for every $q \in Q$ we have: $\text{tl}(\zeta(A, \delta(q, 0))) = \text{tl}(\pi_{0,k}(\zeta(A, q)))$ and for all $1 \leq i < k$:*

$$\zeta(A, \delta(q, i)) = \pi_{i,k}(\zeta(A, q)) \quad (2)$$

Hence, if A is invariant under leading zeros, then property (2) holds for all $0 \leq i < k$.

Proof. Follows immediately from $(kn + i)_k = (n)_k i$ for all $n \in \mathbb{N}$ and $0 \leq i < k$ such that $n \neq 0$ or $i \neq 0$. \square

As a consequence of Lemma 49 we have that k -DFAOs, that are invariant under leading zeros, are \mathcal{N}_k -observation graphs for the streams they define, and vice versa. Formally, this is just a simple change of notation²:

Definition 50. Let $A = \langle Q, \Sigma, \delta, q_0, \Delta, \lambda \rangle$ be a k -DFAO that is invariant under leading zeros. We define the \mathcal{N}_k -observation graph $\mathcal{G}(A) = \langle Q, \langle o, n \rangle \rangle$ with root q_0 where for every $q \in Q$: $o(q) = \lambda(q)$, $n_i(q) = \delta(q, i)$ for $i < k$, and $\llbracket q \rrbracket = \zeta(A, q)$.

Let $\mathcal{G} = \langle S, \langle o, n \rangle \rangle$ be an \mathcal{N}_k -observation graph over Δ with root $r \in S$. Then we define a k -DFAO $A(\mathcal{G})$ as follows: $A(\mathcal{G}) = \langle Q, \mathbb{N}_{<k}, \delta, q_0, \Delta, \lambda \rangle$ where $Q = S$, $q_0 = r$, and for every $s \in S$: $\lambda(s) = o(s)$, and $\delta(s, i) = n_i(s)$ for $i < k$.

Proposition 51. *For every k -DFAO A that is invariant under leading zeros, the \mathcal{N}_k -observation graph $\mathcal{G}(A)$ defines the stream that is generated by A .*

Conversely, we have for every \mathcal{N}_k -observation graph \mathcal{G} , that the k -DFAO $A(\mathcal{G})$ is invariant under zeros and generates the stream defined by \mathcal{G} .

Another way to see the correspondence between automatic sequences and their finite, canonical \mathcal{N}_k -observation graphs is as follows. The elements of the canonical observation graph of a stream σ , that is, the set of $\{\pi_{0,k}, \dots, \pi_{k-1,k}\}$ -derivatives of σ , coincide with the elements of the k -kernel of σ . This is used in the proof of the following theorem.

Proposition 52. *For streams $\sigma \in \Delta^\omega$ the following properties are equivalent:*

- (i) *The stream σ is k -automatic.*
- (ii) *The canonical \mathcal{N}_k -observation graph of σ is finite.*

²Note that even this small change of notation can be avoided by introducing k -DFAOs as coalgebras over the functor $F(X) = \Delta \times X^k$ as well.

Proof. The equivalence of (i) and (ii) is a consequence of Lemma 48 in combination with the observation that the set of functions $\{\pi_{i,k^p} \mid p \in \mathbb{N}, i < k^p\}$ coincides with the set of functions obtained from arbitrary iterations of functions $\pi_{0,k}, \dots, \pi_{k-1,k}$ (that is, function compositions $\gamma_1 \dots \gamma_n$ with $n \in \mathbb{N}$ and $\gamma_i \in \{\pi_{0,k}, \dots, \pi_{k-1,k}\}$). \square

Proposition 52 gives a coalgebraic perspective on automatic sequences. Moreover, it frequently allows for simpler proofs or disproofs of automaticity than existing characterizations. For example, in the following sections we will derive observation graphs for streams that are specified by zip-specifications. Then it is easier to stepwise iterate the finite set of functions $\{\pi_{0,k}, \dots, \pi_{k-1,k}\}$ than to reason about infinitely many subsequences in the kernel $\{\pi_{i,k^p}(\sigma) \mid p \in \mathbb{N}, i < k^p\}$.

Proposition 52 was independently found by Kupke and Rutten, see Theorem 8 in their recent report [14].

We arrive at our second main result:

Theorem 53. *For streams $\sigma \in \Delta^\omega$ the following properties are equivalent:*

- (i) *The stream σ is k -automatic.*
- (ii) *The stream σ can be defined by a zip- k specification.*
- (iii) *The canonical \mathcal{N}_k -observation graph of σ is finite.*
- (iv) *The canonical \mathcal{O}_k -observation graph of σ is finite.*

Proof. We have that (i) \Leftrightarrow (iii) by Theorem 52, (iii) \Rightarrow (ii) by Lemma 41, and (ii) \Rightarrow (iii) by Lemma 36. Moreover, it holds that (iv) \Rightarrow (ii) by Lemma 40.

Finally, we show (iii) \Rightarrow (iv). Assume $\mathcal{G} = \langle S, \langle o, n \rangle \rangle$ is a finite \mathcal{N}_k -observation graph with root r defining σ and let $\llbracket \cdot \rrbracket_{\mathcal{G}} : S \rightarrow \Delta^\omega$ be the unique F -homomorphism into $\langle \Delta^\omega, \mathcal{N}_k \rangle$. Let $n = \langle n_1, \dots, n_k \rangle$. Then $o(s) = \text{hd}(\llbracket s \rrbracket_{\mathcal{G}})$ and $\llbracket n_i(s) \rrbracket_{\mathcal{G}} = \pi_{i-1,k}(\llbracket s \rrbracket_{\mathcal{G}})$ for all $1 \leq i \leq k$ and $s \in S$. We define $\mathcal{G}' = \langle S', \langle o', n' \rangle \rangle$ where $S' = S \cup \{\underline{\text{tl}}(s) \mid s \in S\}$, $o'(s) = o(s)$, $o'(\underline{\text{tl}}(s)) = o(n_2(s))$, $n'_i(s) = n_{i+1}(s)$ for $1 \leq i < k$, $n'_k(s) = \underline{\text{tl}}(n_1(s))$, $n'_i(\underline{\text{tl}}(s)) = n_{i+2}(s)$ for $1 \leq i \leq k-2$, $n'_{k-1}(\underline{\text{tl}}(s)) = \underline{\text{tl}}(n_1(s))$ and $n'_k(\underline{\text{tl}}(s)) = \underline{\text{tl}}(n_2(s))$ with root $r \in S'$. Let $\llbracket \cdot \rrbracket_{\mathcal{G}'} : S' \rightarrow \Delta^\omega$ be defined by $\llbracket s \rrbracket_{\mathcal{G}'} = \llbracket s \rrbracket_{\mathcal{G}}$ and $\llbracket \underline{\text{tl}}(s) \rrbracket_{\mathcal{G}'} = \underline{\text{tl}}(\llbracket s \rrbracket_{\mathcal{G}})$. It can be checked that $\llbracket \cdot \rrbracket_{\mathcal{G}'}$ is an F -homomorphism into $\langle \Delta^\omega, \mathcal{O}_k \rangle$ with $\sigma = \llbracket r \rrbracket_{\mathcal{G}'}$. Hence \mathcal{G}' is an \mathcal{O}_k -observation graph defining σ . \square

5 A Dynamic Logic Representation of Automatic Sequences

This section connects automatic sequences with expressivity in a *propositional dynamic logic* (PDL) derived from the cobases \mathcal{N}_k and \mathcal{O}_k . For simplicity, we shall restrict attention to the case of $\Delta = \{0, 1\}$ and $\mathcal{N}_2 = \langle \text{hd}, \text{even}, \text{odd} \rangle$.

The set of *sentences* φ and *programs* π of our version of PDL is given by the following BNF grammar:

$$\begin{aligned}\varphi &::= 0 \mid 1 \mid \neg\varphi \mid \varphi \wedge \varphi \mid [\pi]\varphi \\ \pi &::= \text{even} \mid \text{odd} \mid \pi; \pi \mid \pi \sqcup \pi \mid \pi^*\end{aligned}$$

We interpret PDL in an arbitrary F -coalgebra $\mathcal{G} = \langle S, \langle o, n \rangle \rangle$. Actually, we can be more liberal and interpret PDL in *models* of the form

$$\mathcal{G} = \langle S, 0, 1, \text{even}, \text{odd} \rangle$$

where $0 \subseteq S$, $1 \subseteq S$, $\text{even} \subseteq S^2$, and $\text{odd} \subseteq S^2$. These are more general than F -coalgebras because we do not insist that $0 \cap 1 = \emptyset$, or that even and odd be interpreted as functions. Nevertheless, these extra properties do hold in the intended model

$$\langle \Delta^\omega, 0, 1, \text{even}, \text{odd} \rangle$$

where 0 is the set of streams whose head is the number 0 , and similarly for 1 ; $(\sigma, \tau) \in \text{even}$ iff $\tau = (\sigma_0, \sigma_2, \sigma_4, \dots)$, and similarly for odd .

The interpretation of each sentence φ is a subset of S ; the interpretation of each program π is a relation on S , that is, a subset of $S \times S$. The definition is as usual for PDL:

$$\begin{aligned}[[0]] &= \{x \in S : x \in 0\} & [[\text{even}]] &= \text{even} \\ [[1]] &= \{x \in S : x \in 1\} & [[\text{odd}]] &= \text{odd} \\ [[\varphi \wedge \psi]] &= [[\varphi]] \cap [[\psi]] & [[\pi_1; \pi_2]] &= [[\pi_1]]; [[\pi_2]] \\ [[\neg\varphi]] &= S \setminus [[\varphi]] & [[\pi_1 \sqcup \pi_2]] &= [[\pi_1]] \cup [[\pi_2]] \\ & & [[\pi^*]] &= [[\pi]]^* \\ [[[\pi]\varphi]] &= \{x : (\forall y)(\langle x, y \rangle \in [[\pi]] \rightarrow y \in [[\varphi]])\}\end{aligned}$$

In words, we interpret even and odd by themselves that correspond in the given model. We interpret $;$ by relational composition, \sqcup by union of relations, $*$ by Kleene star (= reflexive-transitive closure) of relations, and we use the usual boolean operations and dynamic modality $[\pi]\varphi$.

We use the standard boolean abbreviations for $\varphi \rightarrow \psi$ and $\varphi \leftrightarrow \psi$, and of course we use the standard semantics. We also write $\langle \pi \rangle \varphi$ for $\neg[\pi]\neg\varphi$; again this is standard.

For example, let χ be the sentence $[(\text{even} \sqcup \text{odd})^*](0 \leftrightarrow \neg 1)$. Then in any model \mathcal{G} , a point x has $x \models \chi$ iff for all points y reachable from x in zero or more steps in the relation $\text{even} \cup \text{odd}$, y satisfies exactly one of 0 or 1 .

Proposition 54. *If $f : M \rightarrow N$ is a morphism of models and $x \models \varphi$ in M , then $f(x) \models \varphi$ in N .*

Proposition 55. *For every finite pointed model $\langle \mathcal{G}, x \rangle$ there is a sentence φ_x of PDL so that for all (finite or infinite) F -coalgebras $\langle \mathcal{H}, y \rangle$, the following are equivalent:*

(i) $y \models \varphi_x$ in \mathcal{H} .

(ii) There is a bisimulation between \mathcal{G} and \mathcal{H} which relates x to y .

We call φ_x the characterizing sentence of x .

For infinitary modal logic, this result was shown in [4], and the result here for PDL is a refinement of it.

For example, we construct a characterizing sentence for the Thue–Morse sequence \mathbf{M} , see Fig. 1. Let φ and ψ be given by

$$\begin{aligned}\varphi &= 0 \wedge \neg 1 \wedge \langle \text{even} \rangle 0 \wedge [\text{even}] 0 \wedge \langle \text{odd} \rangle 1 \wedge [\text{odd}] 1 \\ \psi &= \neg 0 \wedge 1 \wedge \langle \text{even} \rangle 1 \wedge [\text{even}] 1 \wedge \langle \text{odd} \rangle 0 \wedge [\text{odd}] 0\end{aligned}$$

Then $\varphi_{\mathbf{M}} = \varphi \wedge [(\text{even} \sqcup \text{odd})^*](\varphi \vee \psi)$ is a characteristic sentence of the top node in Fig. 1; $\varphi_{\mathbf{M}}$ also characterizes \mathbf{M} in the following sense: the only stream σ such that $\sigma \models \varphi_{\mathbf{M}}$ is \mathbf{M} .

Proposition 56. *The following finite model properties hold:*

(i) *If a sentence φ has a model, it has a finite model [12].*

(ii) *If φ has a model in which even and odd are total functions, then it has a finite model with these properties [5].*

Remark 57. Our statement of the second result is a slight variation of what appears in [5].

We arrive at our third main result:

Theorem 58. *The following are equivalent for $\sigma \in \Delta^\omega$:*

(i) *σ is 2-automatic.*

(ii) *There is a sentence φ such that for all $\tau \in \Delta^\omega$, $\tau \models \varphi$ in $\langle \Delta^\omega, \langle \text{hd}, \text{even}, \text{odd} \rangle \rangle$ iff $\tau = \sigma$.*

Proof. (i) \Rightarrow (ii): Let σ be automatic, and let M be a finite F -coalgebra and $x \in M$ be such that the unique coalgebra morphism $f : M \rightarrow \Delta^\omega$ has $f(x) = \sigma$. Let φ_x be the characterizing sentence of x in M , using Proposition 55. By Proposition 54, $\sigma \models \varphi_x$ in Δ^ω . Now suppose that $\tau \models \varphi_x$ in Δ^ω . Since φ_x is a characterizing sentence, there is a bisimulation on Δ^ω relating σ to τ . By Lemma 24, $\sigma = \tau$.

(ii) \Rightarrow (i): Let φ be a sentence with the property that σ is the only stream which satisfies φ . Since σ has a model, it has a finite model, by Proposition 56. Moreover, this model M may be taken to be a finite F -coalgebra with a distinguished point x . By [14, Theorem 5] let $\varphi : M \rightarrow \Delta^\omega$ be the unique coalgebra morphism. Let $\tau = \varphi(x)$. Since M is finite, τ is automatic. By Proposition 55, $\tau \models \varphi$ in Δ^ω . But by the uniqueness assertion in part (2) of our theorem, we must have $\tau = \sigma$. Therefore σ is automatic. \square

6 Mix-Automaticity

The zip-specifications considered so far were uniform, all zip-operations in a zip- k specification have the same arity k . Now we admit different arities of zip in one zip-specification (Definition 4). To emphasize the difference with zip- k specifications we will here speak of *zip-mix* specifications. This extension leads to a proper extension of automatic sequences and some delicate decidability problems.

Definition 59. A *state-dependent-alphabet DFAO* is a tuple $\langle Q, \Sigma, \delta, q_0, \Delta, \lambda \rangle$, where

- Q is a finite set of states,
- $\Sigma = \{\Sigma_q\}_{q \in Q}$ a family of input alphabets,
- $\delta = \{\delta_q : \Sigma_q \rightarrow Q\}_{q \in Q}$ a family of transition functions,
- $q_0 \in Q$ the initial state,
- Δ an output alphabet, and
- $\lambda : Q \rightarrow \Delta$ an output function.

We write $\delta(q, i)$ for $\delta_q(i)$ iff $i \in \Sigma_q$, and extend δ to words as follows: Let $q \in Q$ and $w = a_{n-1} \dots a_0$ where $a_i \in \Sigma_{r_i}$ ($0 \leq i < n$) with $r_i \in Q$ defined by: $r_0 = q$ and $r_{i+1} = \delta(r_i, a_i)$. Then we let $\delta(q, w) = r_n$.

A state-dependent-alphabet DFAO can be seen as a DFAO whose transition function is a *partial* map $\delta : Q \times \bigcup \Sigma \rightarrow Q$ such that $\delta(q, a)$ is defined iff $a \in \Sigma_q$.

We use this concept to generalize k -DFAOs where the input format are numbers in base k by the following two-tiered construction. We define P -DFAOs where P is a DFAO determining the base of each digit depending on the digits read before. Thus P can be seen as fixing a variadic numeration system. For example, for ordinary base k numbers, we define P to consist of a single state q with output k and edges $0, \dots, k-1$ looping to itself.

Definition 60. A *base determiner* P is a state-dependent-alphabet DFAO of the form $P = \langle Q, \{\mathbb{N}_{<\beta(q)}\}_q, \delta, q_0, \mathbb{N}, \beta \rangle$. The *base- P representation* of $n \in \mathbb{N}$ is defined by

$$(n)_P = (n)_{P, q_0} \quad \text{where} \quad (n)_{P, q} = (n')_{P, \delta(q, d)} \cdot d$$

with $n' = \lfloor \frac{n}{\beta(q)} \rfloor$ and $d = [n]_{\beta(q)}$, the quotient and the remainder of division of n by $\beta(q)$, respectively.

A P -DFAO A is a state-dependent-alphabet DFAO

$$A = \langle Q', \{\mathbb{N}_{<\beta'(q')}\}_{q' \in Q'}, \delta', q'_0, \Delta, \lambda \rangle$$

compatible with P , i.e. $\langle Q', \{\mathbb{N}_{<\beta'(q')}\}_{q' \in Q'}, \delta', q'_0, \mathbb{N}, \beta' \rangle$ and P are bisimilar.

A *mix-DFAO* is a P -DFAO for some base determiner P .

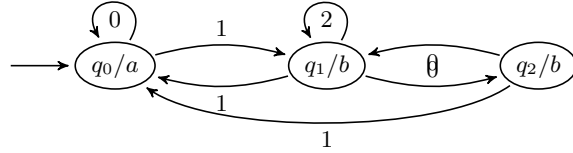
Note that the output alphabet of a base determiner can be taken to be finite as the range of β . The compatibility of A with P entails that A reads the number format defined by P . Moreover, every mix-DFAO $A = \langle Q, \{\mathbb{N}_{<\beta(q)}\}_{q \in Q}, \delta, q_0, \Delta, \lambda \rangle$ is a P_A -DFAO where $P_A = \langle Q, \{\mathbb{N}_{<\beta(q)}\}_{q \in Q}, \delta, q_0, \mathbb{N}, \beta \rangle$.

These DFAOs introduce a new class of sequences, which we call ‘mix-automatic’ in order to emphasize the connection with zip-mix specifications.

Definition 61. Let P be a base determiner, and $A = \langle Q, \{\mathbb{N}_{<\beta(q)}\}_{q \in Q}, \delta, q_0, \Delta, \lambda \rangle$ a P -DFAO. For states $q \in Q$, we define $\zeta(A, q) \in \Delta^\omega$ by: $\zeta(A, q)(n) = \lambda(\delta(q, (n)_P))$ for all $n \in \mathbb{N}$. We define $\zeta(A) = \zeta(A, q_0)$, and say A *generates* the stream $\zeta(A)$.

A sequence $\sigma \in \Delta^\omega$ is *P-automatic* if there is a P -DFAO A such that $\sigma = \zeta(A)$. A stream is called *mix-automatic* if it is P -automatic for some base determiner P .

Example 62. Consider the following mix-DFAO A :



We note that A is a P -DFAO where P is the base determiner obtained from A by redefining the output for q_0 , q_1 and q_2 as the number of their outgoing edges 2, 3 and 2, respectively.

As an example, we compute $(5)_A$, and $(23)_A$ as follows:

$$(5)_{q_0} = (2)_{q_1} 1 = (0)_{q_2} 21 = 21$$

$$(23)_{q_0} = (11)_{q_1} 1 = (3)_{q_1} 21 = (1)_{q_2} 021 = (0)_{q_0} 1021 = 1021$$

where $(n)_q$ denotes $(n)_{A,q}$. The sequence $\zeta(A)$ begins with

$a : b : b : a : b : b : a : a : b : b : b : a : a : a : \underline{a} : b : b : b : b : b : b : a : a : \underline{a} : a : b : a : b : \dots$

with entries 5 and 23 underlined. E.g. $\lambda(\delta(q_0, 1021)) = a$ since starting from q_0 and reading 1021 from right to left brings you back at state q_0 with output a .

We briefly indicate how to see that mix-automaticity properly extends automaticity. Let σ and τ be k - and ℓ -automatic sequences. Then the stream $\text{zip}(\sigma, \tau)$ is mix-automatic, but not necessarily automatic, by Cobham’s Theorem [6].

Proposition 63. *The class of mix-automatic sequences extends that of automatic sequences.*

Definition 64. Let $\kappa : \Delta^\omega \rightarrow \mathbb{N}_{>1}$, and let G be the functor $G(X) = \sum_{k=2}^\infty \Delta \times X^k$. We define the cobasis

$$\mathcal{N}_\kappa = \langle \text{hd}, \lambda\sigma. \langle \pi_{0, \kappa(\sigma)}(\sigma), \dots, \pi_{\kappa(\sigma)-1, \kappa(\sigma)}(\sigma) \rangle \rangle$$

An \mathcal{N}_κ -observation graph is a G -coalgebra $\mathcal{G} = \langle S, \langle o, n \rangle \rangle$ with a distinguished root element $r \in S$, such that there exists a G -homomorphism $\llbracket \cdot \rrbracket : S \rightarrow \Delta^\omega$ from \mathcal{G} to the G -coalgebra $\langle \Delta^\omega, \mathcal{N}_\kappa \rangle$ of all streams with respect to \mathcal{N}_κ :

$$\begin{array}{ccc}
S & \xrightarrow{\llbracket \cdot \rrbracket} & \Delta^\omega \\
\langle o, n \rangle \downarrow & & \downarrow \mathcal{N}_\kappa \\
\sum_{k=2}^\infty \Delta \times S^k & \xrightarrow{\sum_{k=2}^\infty \text{id} \times \llbracket \cdot \rrbracket^k} & \sum_{k=2}^\infty \Delta \times (\Delta^\omega)^k
\end{array}$$

The observation graph \mathcal{G} defines the stream $\llbracket r \rrbracket \in \Delta^\omega$. A *mix-observation graph* is an \mathcal{N}_κ -observation graph for some κ .

The following result is a generalization of Theorem 53. The key idea is to adapt Definition 33 by computing the derivatives $\pi_{0,k}(t)\downarrow, \dots, \pi_{k-1,k}(t)\downarrow$ of a zip-term t where now k is the arity of the first zip-symbol in the tree unfolding of t . Moreover, we note that mix-DFAOs yield mix-observation graphs by collapsing states that generate the same stream (for each of the equivalence classes one representative and its outgoing edges is chosen). This collapse caters for mix-DFAOs which employ different bases for states that generate the same stream.

Theorem 65. *For streams $\sigma \in \Delta^\omega$ the following properties are equivalent:*

- (i) *The stream σ is mix-automatic.*
- (ii) *The stream σ can be defined by a zip-mix specification.*
- (iii) *There exists a finite mix-observation graph defining σ .*

Example 66. The zip-mix specification corresponding to the mix-automaton from Example 62 is:

$$\begin{array}{ll}
X_0 = a : X'_0 & X'_0 = \text{zip}_2(X_1, X'_0) \\
X_1 = b : X'_1 & X'_1 = \text{zip}_3(X_0, X_1, X'_2) \\
X_2 = b : X'_2 & X'_2 = \text{zip}_2(X_0, X'_1)
\end{array}$$

We have seen that equivalence for zip- k specifications is decidable (Theorem 37), and it can be shown that comparing zip- k with zip-mix is decidable as well. In the next section we show that equivalence becomes undecidable when zip-mix specifications are extended with projections $\pi_{i,k}$. But what about zip-mix specifications?

Question 67. Is equivalence decidable for zip-mix specifications?

7 Stream Equality is Π_1^0 -complete

In this section, we show that the decidability results for the equality of zip- k specifications are on the verge of undecidability. To this end we consider an extension of the format of zip-specifications with the projections $\pi_{i,k}$.

Definition 68. The set $\mathcal{Z}^\pi(\Delta, \mathcal{X})$ of *zip $^\pi$ -terms* over $\langle \Delta, \mathcal{X} \rangle$ is defined by the grammar:

$$Z ::= X \mid a : Z \mid \text{zip}_k \overbrace{(Z, \dots, Z)}^{k \text{ times}} \mid \pi_{i,k}(Z)$$

where $X \in \mathcal{X}$, $a \in \Delta$, $i, k \in \mathbb{N}$. A *zip $^\pi$ -specification* consists for every $X \in \mathcal{X}$ of an equation $X = t$ where $t \in \mathcal{Z}^\pi(\Delta, \mathcal{X})$.

The class of zip $^\pi$ -specifications forms a subclass of pure specifications [10], and hence their productivity is decidable. In contrast, the equivalence of zip $^\pi$ -specifications turns out to be undecidable (even for productive specifications).

Theorem 69. *The problem of deciding the equality of streams defined by productive zip $^\pi$ -specifications is Π_1^0 -complete.*

For the proof of the theorem, we devise a reduction from the halting problem of Fractran programs (on the input 2) to an equivalence problem of zip $^\pi$ -specifications. Fractran [7] is a Turing-complete programming language. As intermediate step of the reduction we employ an extension of Fractran programs with output (and immediate termination):

Definition 70. An *Fractran program with output* consists of:

- a list of fractions $\frac{p_1}{q_1}, \dots, \frac{p_k}{q_k}$ ($k, p_1, q_1, \dots, p_k, q_k \in \mathbb{N}_{>0}$),
- a partial *step output* function $\lambda : \{1, \dots, k\} \rightarrow \Gamma$

where Γ is a finite output alphabet. A *Fractran program* is a Fractran program with output for which $\lambda(1)\uparrow, \dots, \lambda(k)\uparrow$.

Let F be a Fractran program with output as above. Then we define the partial function $\langle \cdot \rangle : \mathbb{N} \rightarrow \{1, \dots, k\}$ that for every $n \in \mathbb{N}$ selects the index $\langle n \rangle$ of the first applicable fraction by:

$$\langle n \rangle = \min \{ i \mid 1 \leq i \leq k, n \cdot \frac{p_i}{q_i} \in \mathbb{N} \}$$

where we fix $(\min \emptyset)\uparrow$. We define $f_F : \mathbb{N} \rightarrow \mathbb{N} \cup \Gamma \cup \{\perp\}$ by:

$$f_F(n) = \begin{cases} n \cdot \frac{p_{\langle n \rangle}}{q_{\langle n \rangle}} & \text{if } \langle n \rangle \downarrow \text{ and } \lambda(\langle n \rangle) \uparrow \\ \lambda(\langle n \rangle) & \text{if } \langle n \rangle \downarrow \text{ and } \lambda(\langle n \rangle) \downarrow \\ \perp & \text{if } \langle n \rangle \uparrow \end{cases}$$

for all $n \in \mathbb{N}$. The first case is a *computation step*, the latter two are *termination with* and *without output*, respectively.

We define the *output function* $\lambda_F^* : \mathbb{N} \rightarrow \Gamma \cup \{\perp\}$ of F by

$$\lambda_F^*(n) = \begin{cases} \gamma & \text{if } \gamma = f_F^i(n) \in \Gamma \cup \{\perp\} \text{ for some } i \in \mathbb{N} \\ \uparrow & \text{if no such } i \text{ exists} \end{cases}$$

If $\lambda_F^*(n)\downarrow$ then F is said to *halt on n with output $\lambda_F^*(n)$* . Then F is called *universally halting* if F halts on every $n \in \mathbb{N}_{>0}$, and *decreasing* if $p_i < q_i$ for every $1 \leq i \leq k$ with $\lambda(i)\uparrow$.

For convenience, we denote Fractran programs with output by lists of annotated fractions where $\lambda(i)\uparrow$ is represented by the empty word (no annotation):

$$\frac{p_1}{q_1}\lambda(1), \dots, \frac{p_k}{q_k}\lambda(k)$$

Lemma 71 ([7]). *The problem of deciding on the input of a Fractran program whether it halts on 2 is Σ_1^0 -complete.*

We transform Fractran programs F into two decreasing (and therefore universally halting) Fractran programs F_0 and F_1 with output such that F halts on input 2 if and only if there exists $n \in \mathbb{N}$ such that the outputs of F_0 and F_1 differ on n .

Definition 72. Let $F = \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k}$ be a Fractran program. Let $a_1 < \dots < a_m$ be the primes occurring in the factorizations of $p_1, \dots, p_k, q_1, \dots, q_k$. Let z_1, z_2, c be primes such that $z_1, z_2, c > \prod_{0 \leq i \leq k} p_i \cdot q_i$, and $z_1 > z_2$ and $z_1 > 2 \cdot c$.

We define the Fractran program F^0 with output as:

$$\begin{array}{c} \begin{array}{ccc} \text{simulate } F & & \text{cleanup} \\ \frac{p_1}{q_1 \cdot z_2}, \dots, \frac{p_k}{q_k \cdot z_2}, & \frac{1}{a_1}, \dots, \frac{1}{a_m}, & \\ \frac{1}{c \cdot z_2} \chi_a, \frac{1}{c}, \frac{z_2}{z_1 \cdot z_1}, \frac{2 \cdot c}{z_1}, & \frac{1}{1} \chi_b & \\ \underbrace{\hspace{1.5cm}}_{F \text{ halted}} & \underbrace{\hspace{1.5cm}}_{\text{initialization}} & \underbrace{\hspace{1.5cm}}_{F \text{ did not halt}} \end{array} \end{array}$$

Let F^1 be obtained from F^0 by dropping $\frac{z_2}{z_1 \cdot z_1}$ and $\frac{2 \cdot c}{z_1}$.

Lemma 73. *The programs F^0, F^1 are decreasing and universally halting, and $\lambda_{F^i}^*(n) \in \{\chi_a, \chi_b\}$ for all $n \in \mathbb{N}$, $i \in \{0, 1\}$.*

Lemma 74. *The following statements are equivalent:*

- (i) $\lambda_{F^0}^*(n) = \lambda_{F^1}^*(n)$ for all $n \in \mathbb{N}_{>0}$.
- (ii) $\lambda_{F^0}^*(z_1^{e_1} \cdot z_2^{e_2}) = \lambda_{F^1}^*(z_1^{e_1} \cdot z_2^{e_2})$ for all $e_1, e_2 \in \mathbb{N}$.
- (iii) *The Fractran program F does not halt on 2.*

Next, we translate Fractran programs to zip $^\pi$ -specifications.

Definition 75. Let $F = \frac{p_1}{q_1}\lambda(1), \dots, \frac{p_k}{q_k}\lambda(k)$ be a decreasing Fractran program with output.

Let $d := \text{lcm}(q_1, \dots, q_k)$, and define $p'_n = d \cdot p_{\langle n \rangle} / q_{\langle n \rangle}$ and $b_n = n \cdot p_{\langle n \rangle} / q_{\langle n \rangle}$ for $1 \leq n \leq d$; if $\langle n \rangle \uparrow$, let $p'_n \uparrow$ and $b_n \uparrow$. We define the zip $^\pi$ -specification $\mathcal{S}(F)$ for $1 \leq n \leq d$ by:

$$\begin{array}{ll} \mathbf{X}_0 = \text{zip}_d(\mathbf{X}_1, \dots, \mathbf{X}_d) & \\ \mathbf{X}_n = \pi_{b_n-1, p'_n}(\mathbf{X}_0) & \text{if } \langle n \rangle \downarrow \text{ and } \lambda(\langle n \rangle) \uparrow \\ \mathbf{X}_n = \lambda(\langle n \rangle) : \mathbf{X}_n & \text{if } \langle n \rangle \downarrow \text{ and } \lambda(\langle n \rangle) \downarrow \\ \mathbf{X}_n = \perp : \mathbf{X}_n & \text{if } \langle n \rangle \uparrow \end{array}$$

Lemma 76. *Let F be a decreasing Fractran program with output. The zip^π -specification $\mathcal{S}(F)$ is productive and it holds that $\llbracket X_0 \rrbracket^{\mathcal{S}(F)}(n) = \lambda_{F^*}^*(n+1)$ for every $n \in \mathbb{N}$.*

Proof of Theorem 69. We reduce the *complement* of the halting problem of Fractran programs on input 2 (which is Π_1^0 -complete by Lemma 71) to equivalence of zip^π -specifications.

Let F be a Fractran program. Define F^0, F^1 as in Definition 72. By Lemma 73 both are decreasing. By Lemma 76 $\mathcal{S}(F^i)$ is productive, and we have $\llbracket X_0 \rrbracket^{\mathcal{S}(F^i)}(n) = \lambda_{F^i}^*(n+1)$ for every $n \in \mathbb{N}$ and $i \in \{0, 1\}$. Finally, by Lemma 74 it follows that $\mathcal{S}(F^0)$ and $\mathcal{S}(F^1)$ are equivalent if and only if F halts on 2.

The equivalence problem of productive specifications is obviously in Π_1^0 since every element can be evaluated. \square

Related work The complexity of deciding the equality of streams defined by systems of equations has been considered in [17] and [3]. In [17], Roşu shows Π_2^0 -completeness of the problem for (unrestricted) stream equations. In [3], Balestrieri strengthens the result to polymorphic stream equations. However, both results depend on the use of ill-defined (non-productive) specifications that do not uniquely define a stream. The Π_2^0 -hardness proofs employ stream specifications for which productivity coincides with unique solvability. As a consequence, both results depend crucially on the notion of equivalence for specifications without unique solutions.

In contrast to [17] and [3], we are concerned with productive specifications, that is, every element of which can be evaluated constructively. Then equality is obviously in Π_1^0 . We show that equality is Π_1^0 -hard even for a restricted class of polymorphic, productive stream specifications.

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A Appendix

Proof of Lemma 15. Introduce a fresh root X'_0 and add the equation $X'_0 = X_0$. \square

Definition 77. A zip-specification \mathcal{S} is called *zip-guarded* if every cycle in \mathcal{S} contains an occurrence of `zip`.

Example 78. Specification (1) of the Thue–Morse stream is zip-guarded. In contrast, the zip-specification:

$$\text{alt} = \text{zip}(\text{zeros}, \text{ones}) \quad \text{zeros} = 0 : \text{zeros} \quad \text{ones} = 1 : \text{ones},$$

specifying the stream $0 : 1 : 0 : 1 : 0 \dots$ of alternating zeros and ones, is not zip-guarded.

It is an easy exercise to show that zip-free cycles correspond to periodic sequences, and periodic sequences can be specified by zip-guarded zip- k specification (for arbitrary k). Hence:

Lemma 79. *Every zip- k specification can be transformed into an equivalent, zip-guarded zip- k specification.*

Proof. Every zip-free cycle $M = \dots = c_1 : \dots : c_n : M$ characterizes a periodic sequence $\sigma = uuu \dots$ with $u \in \Delta^*$. Note that for every $i, k \in \mathbb{N}$, $\pi_{i,k}(\sigma)$ is again periodic with a period length $\leq |u|$. Thus the \mathcal{N}_k -observation graph is finite (for every k) and hence by Lemma 41 we have a zip-guarded specification for σ . \square

Proof of Lemma 20. Let \mathcal{S} be a zip-specification. Using Lemma 79 let \mathcal{S} be zip-guarded. If \mathcal{S} contains an equation of the form:

$$X = c_1 : \dots : c_m : \text{zip}_k(s_1, \dots, s_i, \dots, s_k) \quad (*)$$

such that $s_i \notin \mathcal{X}$ for some $1 \leq i \leq k$, then we pick a fresh X' and replace the equation by:

$$\begin{aligned} X &= c_1 : \dots : c_m : \text{zip}_k(s_1, \dots, X', \dots, s_k) \\ X' &= s_i \end{aligned}$$

Clearly, the resulting specification is equivalent to the original, and still zip-guarded. We repeat this transformation step until there are no equations of form $(*)$ left, that is, the arguments of every occurrence of `zipk`-symbols are only recursion variables.

Next, we replace equations of the form:

$$X = c_1 : \dots : c_m : Y \quad (\dagger)$$

with Y a recursion variable, by (unfolding Y):

$$X = c_1 : \dots : c_m : r$$

where the defining equation for Y is $Y = r$. The obtained specification is equivalent and remains zip-guarded. Again, we repeat this step until there no longer are equations of form (\dagger) . This process is guaranteed to terminate since the specification is zip-guarded.

In the final specification, every right-hand side of an equation contains a zip_k (for some $k \geq 2$), and this zip_k is applied to recursion variables only. Hence the final specification is flat.

Furthermore, note that the resulting specification contains only zip_k -symbols for $k \geq 2$ for which a zip_k also occurs in the original specification. As a consequence, the transformation preserves zip- k specifications. \square

Proof of Lemma 26. Let $\mathcal{G} = \langle S, \langle o, n \rangle \rangle$ be a \mathcal{B} -observation graph, and let $[\cdot]_1, [\cdot]_2 : S \rightarrow \Delta^\omega$ be two F -homomorphisms from \mathcal{G} to $\mathcal{S}_{\mathcal{B}} = \langle \Delta^\omega, \mathcal{B} \rangle$. Define the relation $R \subseteq \Delta^\omega \times \Delta^\omega$ by $[[t]]_1 R [[t]]_2$ for all $t \in S$. It is easy to check that R is a \mathcal{B} -bisimulation, and hence $[[s]]_1 = [[s]]_2$ for all $s \in S$ by Lemma 24. \square

Proof of Proposition 27. The final coalgebra for F is the k -automaton of the Δ -weighted languages (or of the k -ary trees with labels in Δ) $\langle \Delta^{\mathbf{k}^*}, \langle o, n \rangle \rangle$ where $\mathbf{k} := \{0, 1, \dots, k\}$, $o : \Delta^{\mathbf{k}^*} \rightarrow \Delta$, $L \mapsto L(\varepsilon)$, and $n : \Delta^{\mathbf{k}^*} \rightarrow (\Delta^{\mathbf{k}^*})^k$, $n(L) \mapsto \langle L_1, \dots, L_k \rangle$ with L_i defined by $L_i(w) = L(i \cdot w)$ for all $w \in \mathbf{k}^*$ (for the case $k = 2$ see [14, Thm. 3]). The function $f : \mathbf{k}^* \rightarrow \omega$ defined by $\varepsilon \mapsto 0$, and $a_0 a_1 \dots a_n \mapsto \sum_{j=0}^n (1 + a_j) k^j$ is bijective and induces an isomorphism from $\langle \Delta^\omega, \mathcal{O}_k \rangle$ to $\langle \Delta^{\mathbf{k}^*}, \langle o, n \rangle \rangle$, because it has the property that $f(i \cdot w) = kf(w) + i$ holds for all $i \in \mathbf{k}$ and $w \in \mathbf{k}^*$. \square

Proof of Lemma 34. The equations of \mathcal{S} are of the form:

$$X_j = c_{j,0} : \dots : c_{j,m_j-1} : \text{zip}_k(X_{j,0}, \dots, X_{j,k-1}) \quad (0 \leq j < n)$$

We define $m := \max \{ m_i \mid 0 \leq i < n \}$.

Note that the root X_0 is of the claimed form. Thus it suffices that $\pi_{i,k}(s) \downarrow$ is the shape whenever $s \in S$ is and $0 \leq i < k$. Let $s = d_0 : \dots : d_{\ell-1} : X_j \in S$ with $\ell \leq m$, and let $0 \leq i < k$. Then it holds that:

$$\begin{aligned} \pi_{i,k}(s) &\rightarrow^* \underbrace{d_i : d_{i+k} : \dots : d_{i+ak}}_{\text{abbreviate as } D} : \pi_{i',k}(X_j) \\ &\rightarrow^* D \underbrace{[c_{j,i'} : c_{j,i'+k} : \dots : c_{j,i'+bk} : X_{j,i''}]}_{\text{abbreviate as } C} \end{aligned}$$

where a , b , i' and i'' are defined by:

$$\begin{aligned} a &= \lfloor (\ell - 1 - i)/k \rfloor & i' &= [\ell - 1 - i]_k \\ b &= \lfloor (m_j - 1 - i')/k \rfloor & i'' &= [m_j - 1 - i']_k \end{aligned}$$

The number of elements in $D[C[\square]]$ is at most $\lfloor (\ell + m_j + 1)/k \rfloor$ (since $\pi_{i,k}$ ‘walks’ over $\ell + m_j$ elements). Hence $D[C[X_{j,i''}]]$ is a normal form of the claimed form with a stream prefix of length $\leq \lfloor (\ell + m_j + 1)/k \rfloor \leq \lfloor (2m + 1)/k \rfloor \leq m$. \square

Lemma 80. *For every productive, flat zip-specification \mathcal{S} with root X_0 it holds that:*

$$|\delta_{\mathcal{S}}(X_0)| \leq 2 \cdot (|\Sigma| + 1) \cdot m \cdot n + 4 \cdot m,$$

where n is the number of recursion variables and m the longest prefix (as defined in the proof of Lemma 34).

Proof. We use the notation from the proof of Lemma 34. Here we consider only case $\mathcal{N}_2 = \langle \text{even}, \text{odd} \rangle$. The proof of the general case works analogous. We define

$$T = \{c_1 : \dots : c_k : X_i \mid k \leq m, c_i \text{ data elements}, i \leq n\}$$

We have $\delta_{\mathcal{S}}(X_0) \subseteq T$ by the proof of Lemma 34.

For a set S of terms we define the one step derivatives $\delta(S)$ as follows:

$$\delta(S) = \{\text{even}(s)\downarrow, \text{odd}(s)\downarrow \mid s \in S\}$$

We consider a term $t \in T$:

$$t = c_1 : \dots : c_k : X_i$$

with $k \leq m$. Then $\text{even}(t)\downarrow$ and $\text{odd}(t)\downarrow$ are of the form:

$$c_{i_1} : \dots : c_{i_{k'}} : t' \quad \text{with} \quad t' \in \delta(X_i) \tag{3}$$

where $k' \leq \lceil k/2 \rceil$ and $c_{i_1}, \dots, c_{i_{k'}}$ are residuals of c_1, \dots, c_k .

Then by induction it follows that every term $s \in \delta^{\lceil \log_2 k \rceil}(t)$ has the form:

$$s = t' \quad \text{or} \quad s = c_j : t' \quad \text{with} \quad t' \in \delta^{\lceil \log_2 k \rceil}(X_i) \tag{4}$$

with $1 \leq j \leq k$. Furthermore, for $l \geq k$, every $s \in \delta^{\lceil \log_2 l \rceil}(t)$ has the form (4) with $j \in \{1, \dots, k\}$. Hence it follows:

$$\delta^{\lceil \log_2 m \rceil}(T) \subseteq \bigcup_{i=0}^{n-1} \left(\delta^{\lceil \log_2 m \rceil}(X_i) \cup \bigcup_{c \in \Sigma} c : \delta^{\lceil \log_2 m \rceil}(X_i) \right)$$

and therefore:

$$|\delta^{\lceil \log_2 m \rceil}(T)| \leq (|\Sigma| + 1) \cdot \left| \bigcup_{i=0}^{n-1} \delta^{\lceil \log_2 m \rceil}(X_i) \right|$$

Since $|\delta^{\lceil \log_2 m \rceil}(\mathbf{X}_i)| = 2^{\lceil \log_2 m \rceil}$ we get:

$$\begin{aligned} |\delta^{\lceil \log_2 m \rceil}(T)| &\leq (|\Sigma| + 1) \cdot 2^{\lceil \log_2 m \rceil} \cdot n \\ &\leq (|\Sigma| + 1) \cdot 2 \cdot m \cdot n \end{aligned}$$

Let $\delta^{\leq i}(t) = \bigcup_{j \leq i} \delta^j(t)$. Then $\delta_S(\mathbf{X}_0) \subseteq \delta^{\leq \lceil \log_2 m \rceil}(\mathbf{X}_0) \cup \delta^{\lceil \log_2 k \rceil}(T)$ and hence:

$$\begin{aligned} |\partial_S(\mathbf{X}_0)| &\leq |\delta^{\leq \lceil \log_2 m \rceil}(\mathbf{X}_0)| + |\delta^{\lceil \log_2 k \rceil}(T)| \\ &\leq (1 + 2 + 4 + \dots + 2^{\lceil \log_2 m \rceil}) + (|\Sigma| + 1) \cdot 2mn \\ &\leq 4 \cdot m + (|\Sigma| + 1) \cdot 2 \cdot m \cdot n \end{aligned} \quad \square$$

Proof of Proposition 38. By Lemmas 34 and 80 the observation graphs of productive and flat zip-specifications are finite and have polynomial size. By Proposition 29 we can decide bisimilarity of observation graphs in linear time. Consequently equality of zip-specifications is decidable in polynomial time. \square

Proof of Proposition 55. First, we have a construction that works for any model M . For every point $a \in M$ and every h , we define the formula $\varphi_{M,a}^h$. The definition is by recursion on h (simultaneously for all $x \in M$) as follows: $\varphi_{M,a}^0$ is the conjunction of all atomic propositions (0 or 1) satisfied by a and all negations of atomic propositions not satisfied by a . Given $\varphi_{M,b}^h$ for all $b \in M$, we define

$$\begin{aligned} \varphi_{M,a}^{h+1} &= \bigwedge_{(a,b) \in \text{even}} \langle \text{even} \rangle \varphi_{M,b}^h \wedge [\text{even}] \bigvee_{(a,b) \in \text{even}} \varphi_{M,b}^h \\ &\quad \wedge \bigwedge_{(a,b) \in \text{odd}} \langle \text{odd} \rangle \varphi_{M,b}^h \wedge [\text{odd}] \bigvee_{(a,b) \in \text{odd}} \varphi_{M,b}^h \\ &\quad \wedge \varphi_{M,a}^0 \end{aligned}$$

We always identify sentences up to logical equivalence.

As M ranges over all models and a over the points of M , we call the sentences $\varphi_{M,a}^h$ the *canonical sentences of height h* .

Lemma 81. *The following hold:*

- (i) *For all h , there are only finitely many sentences $\varphi_{M,a}^h$.*
- (ii) *For every h , every world of every model satisfies a unique canonical sentence of height h .*
- (iii) *If R is a bisimulation relation between models M and N , and if $a R b$, then $\varphi_{M,a}^h = \varphi_{N,b}^h$.*

(iv) If M and N are finitely branching models, then the converse of part (iii) holds: the largest bisimulation between M and N is the relation R defined by

$$a R b \text{ iff for all } h, \varphi_{M,a}^h = \varphi_{N,b}^h \quad (5)$$

Proof. For parts (i) and (ii), see Proposition 2.5 and Lemma 2.6 of [15]. (The arguments there are for one modality, but the results extend in an obvious way to our setting.) Part (iii) is a standard fact, as is the “Hennessy–Milner” result in part (iv). \square

Now we return to Proposition 55. We fix a finite model M and some point x in it. Let R be the largest bisimulation on M . It is a general fact that R is an equivalence relation, and indeed this also follows from the characterization in (5). For all a, b in M , if $\neg(a R b)$, then there is some natural number h so that $\varphi_{M,a}^h \neq \varphi_{M,b}^h$. Since $M \times M$ is a finite set, there is some fixed h^* so that for all $a, b \in M$, if $\neg(a R b)$, then $\varphi_{M,a}^{h^*} \neq \varphi_{M,b}^{h^*}$. The key consequence of this is that for all $a, b \in M$,

$$\text{if } \varphi_{M,a}^{h^*} = \varphi_{M,b}^{h^*}, \text{ then also } \varphi_{M,a}^{h^*+1} = \varphi_{M,b}^{h^*+1} \quad (6)$$

From this and Lemma 81, part (ii), it follows that for all $a, b \in M$,

$$\text{if } a \models \varphi_{M,b}^{h^*}, \text{ then also } a \models \varphi_{M,b}^{h^*+1} \quad (7)$$

For $a \in M$, let ψ_a denote the formula

$$\begin{aligned} \varphi_{M,a}^{h^*} \rightarrow & \left(\bigwedge_{(a,b) \in \text{even}} \langle \text{even} \rangle \varphi_{M,b}^{h^*} \wedge [\text{even}] \bigvee_{(a,b) \in \text{even}} \varphi_{M,b}^{h^*} \right. \\ & \left. \wedge \bigwedge_{(a,b) \in \text{odd}} \langle \text{odd} \rangle \varphi_{M,b}^{h^*} \wedge [\text{odd}] \bigvee_{(a,b) \in \text{odd}} \varphi_{M,b}^{h^*} \right) \end{aligned}$$

Using (7), we see that for all $a, b \in M$, $a \models \psi_b$.

We now finish the proof of Proposition 55. We have our model M and a point $x \in M$. We take the characterizing sentence of x in M to be

$$\varphi_x = \varphi_{M,x}^{h^*} \wedge [(\text{even} \sqcup \text{odd})^*] \bigwedge_{a \in M} \psi_a$$

It is easy to see that $x \models \varphi_x$. To end our proof, suppose that N is any model, and $y \in N$ satisfies φ_x . We define a bisimulation R between M and N which relates x to y :

$$\begin{aligned} a R b \text{ iff } & a \text{ is reachable in } M \text{ from } x \text{ using } (\text{even} \sqcup \text{odd})^*, \\ & b \text{ is reachable in } N \text{ from } y \text{ using } (\text{even} \sqcup \text{odd})^*, \\ & \text{and } \varphi_{M,a}^{h^*} = \varphi_{N,b}^{h^*} \end{aligned}$$

The definition of ψ_a ensures that R is a bisimulation.

This completes the proof. \square

As an example of the construction in the above proof we obtain the following sentence φ_M characterizing the Thue–Morse sequence M : Let $\{\pi\}\varphi$ abbreviate $\langle \pi \rangle \varphi \wedge [\pi] \varphi$, and let

$$\begin{aligned}\varphi &= 0 \wedge \neg 1 \wedge \{\text{even}\}0 \wedge \{\text{odd}\}1 \\ \psi &= \neg 0 \wedge 1 \wedge \{\text{even}\}1 \wedge \{\text{odd}\}0\end{aligned}$$

Then

$$\begin{aligned}\varphi_M &= \varphi \wedge [(\text{even} \sqcup \text{odd})^*] \left((\varphi \rightarrow \{\text{even}\}\varphi \wedge \{\text{odd}\}\psi) \right. \\ &\quad \wedge (\psi \rightarrow \{\text{even}\}\psi \wedge \{\text{odd}\}\varphi) \\ &\quad \wedge (0 \rightarrow \{\text{even}\}0) \\ &\quad \left. \wedge (1 \rightarrow \{\text{even}\}1) \right)\end{aligned}$$

Theorem 82 (Cobham [6]). *Let $k, \ell \geq 2$ be multiplicatively independent (i.e., $k^a \neq \ell^b$, for all $a, b > 0$), and let $\sigma \in \Delta^\omega$ be both k - and ℓ -automatic. Then σ is eventually periodic.*

Proof of Proposition 63. Let k, ℓ be multiplicatively independent integers, and let $\sigma \in \Delta_A^\omega$ and $\tau \in \Delta_B^\omega$ be k - and ℓ -automatic sequences, generated by DFAOs $A = \langle Q_A, \mathbb{N}_{<k}, \delta_A, q_{A,0}, \Delta_A, \lambda_A \rangle$ and $B = \langle Q_B, \mathbb{N}_{<\ell}, \delta_B, q_{B,0}, \Delta_B, \lambda_B \rangle$, respectively. Assume that both σ and τ are not eventually periodic.

We show that the sequence $v = \text{zip}(\sigma, \tau) \in (\Delta_A \cup \Delta_B)^\omega$ is mix-automatic, but that there is no integer $m \geq 2$ such that v is m -automatic. For the first, note that v is generated by the mix-DFAO $C = \langle S, \{\mathbb{N}_{<\beta(s)}\}_{s \in S}, \delta, s_0, \Delta_A \cup \Delta_B, \lambda \rangle$, where $S = \{s_0\} \cup Q_A \cup Q_B$, and with β, δ and λ defined by

$$\begin{array}{lll}\beta(s_0) = 2 & \delta(s_0, 0) = q_0 & \lambda(s_0) = \lambda_A(q_0) \\ & \delta(s_0, 1) = r_0 & \\ \beta(q) = k & \delta(q, a) = \delta_A(q, a) & \lambda(q) = \lambda_A(q) \\ \beta(r) = \ell & \delta(r, b) = \delta_B(r, b) & \lambda(r) = \lambda_B(r)\end{array}$$

for all $q \in Q_A, r \in Q_B, a \in \mathbb{N}_{<k}$ and $b \in \mathbb{N}_{<\ell}$. Now suppose v is m -automatic, for some $m \geq 2$. Then, as m -automaticity is closed under arithmetic subsequences, also σ and τ are m -automatic. By Theorem 82 and the assumption that σ, τ are not eventually periodic, it follows that k and m are not multiplicatively independent, and likewise for ℓ and m . But then k and ℓ are not multiplicatively independent, contradicting our assumption. \square

Lemma 83. *Every decreasing Fractran program with output is universally halting.*

Proof of Lemma 83. Then $f_F(n) < n$ or $f_F(n) \in \Gamma \cup \{\perp\}$ for every $n \in \mathbb{N}$. Hence F is universally halting. \square

Proof of Lemma 73. By the choice of z_1, z_2 and c we have $q > p$ for all annotated-free fractions $\frac{p}{q}$ from F^0 and F^0 . Hence F^0 and F^0 are decreasing and by Lemma 83 universally halting.

Since F^0 and F^1 contain $\frac{1}{1}\chi_b$, it follows that the programs always terminate with output, that is, either χ_a or χ_b . \square

The following lemmas use the notation from Definition 72:

Lemma 84. *For every $n \in \mathbb{N}$ we have $\lambda_{F^0}^*(2 \cdot c \cdot z_2^n) = \chi_a$ if and only if the Fractran program F halts on input 2 within n steps (that is, $\exists n' \leq n. f_F^{n'}(2) = \perp$).*

Proof of Lemma 84. On inputs of the form $2 \cdot c \cdot z_2^n$ the program F^0 behaves as F except for removing a prime z_2 in each step. By definition of F^0 and induction we obtain

$$f_{F^0}^i(2 \cdot c \cdot z_2^n) = f_F^i(2) \cdot c \cdot z_2^{n-i}$$

for every $i \leq n$ such that $\forall j \leq i. f_F^j(2) \neq \perp$.

Assume that there is $0 < n' \leq n$ such that $f_F^{n'-1}(2) \in \mathbb{N}$ and $f_F^{n'}(2) = \perp$, that is, F halts in precisely n' steps. Then in F^0 to the value $f_{F^0}^{n'-1}(2 \cdot c \cdot z_2^n) = f_F^{n'-1}(2) \cdot c \cdot z_2^{n-(n'-1)}$ none of the ‘simulate F ’-fractions is applicable, and after the ‘cleanup’-fractions have removed all primes occurring in F , the fraction $\frac{1}{c \cdot z_2} \chi_a$ will result in termination with output χ_a (the fraction is applicable since $n - (n' - 1) > 0$).

On the other hand assume $f_F^{n'}(2) \in \mathbb{N}$ for all $n' \leq n$, that is, F does not terminate within n steps. Then $f_{F^0}^n(2 \cdot c \cdot z_2^n) = f_F^n(2) \cdot c$. As a consequence, in F^0 the ‘simulate F ’-fractions and likewise $\frac{1}{c \cdot z_2} \chi_a$ are not applicable (lacking z_2), and thus the ‘cleanup’-fractions will remove all primes occurring in F and then $\frac{1}{c}$ removes the c . Finally, only $\frac{1}{1} \chi_b$ is applicable, resulting in termination with output χ_b . \square

Lemma 85. *It holds that $f_{F^0}(n) = f_{F^1}(n)$ for all $n \in \mathbb{N}$ such that any of $a_1 \dots, a_m$ or c divides n .*

Proof of Lemma 85. For every $p \in \{a_1 \dots, a_m, c\}$ there is a fraction $\frac{1}{p}$ in the common prefix of F^0 and F^1 . \square

Proof of Lemma 74. The equivalence of (i) and (ii) is a consequence of Lemma 85 since z_1 and z_2 are the only remaining primes from F^0 and F^1 not covered by this lemma.

Consider (ii) \Rightarrow (iii). Let $n \in \mathbb{N}$, we show that F does not halt on 2 within n steps. We have $f_{F^0}(z_1 \cdot z_2^n) = 2 \cdot c \cdot z_2^n$ and $f_{F^1}(z_1 \cdot z_2^n) = \chi_b$, and thus $\lambda_{F^0}^*(2 \cdot c \cdot z_2^n) = \chi_b$ by (ii). We conclude this case with an appeal to Lemma 84.

For (iii) \Rightarrow (ii), let $e_1, e_2 \in \mathbb{N}$. We have $\lambda_{F^1}^*(z_1^{e_1} \cdot z_2^{e_2}) = \chi_b$. Assume $e_1 = 2 \cdot n$, $n \in \mathbb{N}$ then $f_{F^0}^n(z_1^{e_1} \cdot z_2^{e_2}) = z_2^{n+e_2}$ and hence $\lambda_{F^0}^*(z_1^{e_1} \cdot z_2^{e_2}) = \chi_b$ (by definition of F^0). Otherwise $e_1 = 2 \cdot n + 1$ and we obtain $f_{F^0}^n(z_1^{e_1} \cdot z_2^{e_2}) = 2 \cdot c \cdot z_2^{n+e_2}$. Then it follows $\lambda_{F^0}^*(z_1^{e_1} \cdot z_2^{e_2}) = \chi_b$ by Lemma 84. \square

Proof of Lemma 76. Let $\llbracket \cdot \rrbracket$ be a solution for $\mathcal{S}(F)$. We use the notation from Definition 75. We prove

$$\llbracket X_0 \rrbracket^{\mathcal{S}(F)}(n) = \lambda_F^*(n+1)$$

by induction on $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and $m \in N_{<d}$, $o \in \mathbb{N}$ such that $n = m + od$. For every $1 \leq i \leq k$: $(n+1) \cdot \frac{p_i}{q_i} = (m+1) \cdot \frac{p_i}{q_i} + od \cdot \frac{p_i}{q_i}$. By choice of d : $d \cdot \frac{p_i}{q_i} \in \mathbb{N}$, hence $(n+1) \cdot \frac{p_i}{q_i} \in \mathbb{N} \Leftrightarrow (m+1) \cdot \frac{p_i}{q_i} \in \mathbb{N}$ and thus $\langle n+1 \rangle = \langle m+1 \rangle$ and $\lambda(\langle n+1 \rangle) = \lambda(\langle m+1 \rangle)$.

We have $\llbracket X_0 \rrbracket(n) = \llbracket X_{m+1} \rrbracket(o)$. If $\langle n+1 \rangle \uparrow$ then $\langle m+1 \rangle \uparrow$ and hence $\llbracket X_0 \rrbracket(n) = \llbracket X_{m+1} \rrbracket(o) = \perp$. Likewise for $\langle n+1 \rangle \downarrow$ and $\lambda(\langle n+1 \rangle) \downarrow$ we obtain $\llbracket X_0 \rrbracket(n) = \lambda(\langle n+1 \rangle)$.

The remaining case is $\langle n+1 \rangle \downarrow$ and $\lambda(\langle n+1 \rangle) \uparrow$. Then

$$\begin{aligned} \llbracket X_0 \rrbracket(n) &= (\pi_{b_{m+1}-1, p'_{m+1}}(\llbracket X_0 \rrbracket))(o) \\ &= \llbracket X_0 \rrbracket(b_{m+1} - 1 + p'_{m+1} \cdot o) \\ &= \llbracket X_0 \rrbracket((m+1) \cdot \frac{p_{\langle n+1 \rangle}}{q_{\langle n+1 \rangle}} - 1 + d \cdot \frac{p_{\langle n+1 \rangle}}{q_{\langle n+1 \rangle}} \cdot o) \\ &= \llbracket X_0 \rrbracket((m+1 + do) \cdot \frac{p_{\langle n+1 \rangle}}{q_{\langle n+1 \rangle}} - 1) \\ &= \llbracket X_0 \rrbracket(f_F(n+1) - 1) \\ &= \lambda_F^*(f_F(n+1)) \quad \text{by induction hypothesis} \\ &= \lambda_F^*(n+1) \end{aligned}$$

Finally, we note that since the derivations from $\llbracket X_0 \rrbracket(n)$ to $\llbracket X_0 \rrbracket(f_F(n+1) - 1)$ also exist on the level of rewrite sequences in the zip-specification, and by decreasingness it holds $f_F(n+1) - 1 < n$, it follows that the specification is productive. \square