Extremal points of infinite clusters in stationary percolation

Ronald Meester

Abstract

It is well known that in stationary percolation, an infinite component cannot have a finite number of extremal points in a certain direction. In this note, we investigate whether or not an infinite cluster can have infinitely many extremal points in a certain direction. To make this question at all interesting, it is necessary (and natural) to simultaneously ask for an infinite path in the opposite direction. It turns out that the answer depend on the dimension of the model, and on the question whether or not the model has so-called finite range.

1 Properties of infinite clusters

In this note, we shall look at certain characteristics of infinite clusters in stationary d-dimensional percolation. We will restrict ourselves to the d-dimensional integer lattice, but this is mostly for convenience. The set up is the following. Denote by $\mathbf{E}^d$ the set of undirected edges $\{\{z_i, z_j\} : z_i, z_j \in \mathbf{Z}^d\}$. That is, $\mathbf{E}^d$ consists of *all* edges, not only nearest neighbour edges. We equip $\Omega := \{0,1\}^{\mathbf{E}^d}$ with the usual sigma field, and μ denotes a stationary measure on this space, i.e. μ is invariant under translations. Two points z

and z' are connected in $\omega \in \Omega$ if there is a sequence $(z_1 = z, z_2, \ldots, z_k = z')$ of vertices such that $\omega(\{z_n, z_{n+1}\}) = 1$ for $n = 1, \ldots, k-1$. An edge with label 1 will be referred to as being *open*, other edges are called *closed*. A cluster of a realisation ω is a maximal set of connected vertices. We say that μ *percolates* if μ assigns positive probability to the event that the origin is contained in an infinite cluster. We are mostly interested in measures μ that percolate.

A *selection rule* is a measurable function $s : \Omega \to \{0,1\}^{\mathbf{Z}^d}$ with the property that for each cluster C of ω, there is exactly one vertex $z \in C$ with $s(\omega)(z) = 1$. We say that a selection rule is stationary if the induced measure $\mu \circ s^{-1}$ on $\{0,1\}^{\mathbf{Z}^d}$ is stationary. We shall abuse notation: if C is a cluster of ω, we write $s(C)$ for the unique vertex z of C for which $s(\omega)(z) = 1$.

For instance, the vertex of C closest to the origin (with a certain predetermined decision rule in case of ties) is a selection rule that is not stationary. If all clusters are finite a.s., then putting $s(C)$ equal to the left-lowest vertex of C yields a stationary selection rule.

One of the more useful facts about infinite clusters is the following. Although typically not stated in this form, versions of this result are well known. We shall sketch a modern proof, using the idea of 'mass transport' which was introduced in the percolation literature in Häggström (1997).

Lemma 1.1 *If μ percolates, then there are no stationary selection rules.*

Proof Suppose μ percolates and suppose that s is a stationary selection rule. Denote the cluster that contains z by $C(z)$. Denoting cardinality by #, the process $(1_{s(\omega)(z)=1}, \#(C(z)))$ is jointly stationary, and therefore also the process $1_{\{s(\omega)(z)=1, \#(C(z))=\infty\}}$ is stationary. Imagine that each vertex has 'mass' 1. We now redistribute all these masses in a stationary way as follows: each vertex z in an infinite cluster sends its mass to $s(C)$. Other than that, nothing changes. Denote, for each vertex z, the mass sent away by $M_{\text{out}}(z)$ and the mass received by $M_{\text{in}}(z)$. No mass gets lost, and therefore it follows by stationarity and the ergodic theorem that $\mathbf{E}M_{\text{out}}(z) = \mathbf{E}M_{\text{in}}(z)$. But clearly, $\mathbf{E}M_{\text{out}}(z) \leq 1$, and at the same time, the probability to receive an infinite amount of mass is positive, hence $\mathbf{E}M_{\text{in}}(z) = \infty$, a contradiction. $\square$.

Lemma 1.1 might look a bit abstract, but it really tells a lot about the geometry of infinite clusters. It is one of the most important steps in the modern proof that in independent percolation there can be at most one infinite cluster (see Burton and Keane 1989, 1991). The question addressed in this paper is not interesting in independent percolation. Here are some other consequences of Lemma 1.1:

(1) Infinite clusters either have no lowest point, or infinitely many lowest points. To see this, suppose that an infinite cluster has, say, three lowest points with positive probability. We can change the configuration in a stationary way by removing all infinite clusters which do not have three lowest points. In the resulting configuration, we then put $s(C)$ equal to the left-lowest point of C, and this would be a stationary selection rule, contradicting Lemma 1.1.

(2) Infinite clusters cannot be rooted binary trees, since we could put $s(C)$ equal to this (unique) root.

We see that the general principle of non-existence of stationary selection rules exludes certain topological and geometrical possibilities for infinite clusters. But questions remain. For instance, is it possible for an infinite cluster to have infinitely many lowest vertices? (Here and in what follows, the use of the word 'lowest' refers to an extreme point in any particular direction.) With a little thought it is easy to see that this is the wrong question: the measure μ (in two dimensions) that makes all horizontal edges open and all remaining edges closed has of course infinitely many infinite clusters with infinitely many lowest points. So we have to ask another question: is it possible to have an infinite cluster with infinitely many lowest points which is unbounded in the opposite direction? We shall see below that this question is not so interesting either. The most interesting question in this context is the following. Is it possible for an infinite cluster to have infinitely many lowest vertices, and at the same time to have an infinite path which goes to infinity in the opposite direction? (It might take a little thought to understand that this is really a different question.) We shall see that the answer depends on the dimension and the so called *range* of μ. The range of μ is defined as $\sup\{|z_i - z_j| : \mu(\{z_i, z_j\} = 1) > 0\}$, i.e. the range is the length of the longest possible open edge. Choosing one specific direction for definiteness, we call a cluster C *special* if

- The set $\{z = (z(1), \ldots, z(d)) : z(1) = \min\{z'(1) : z' \in C\}$ contains infinitely many elements;
- The set $\{z(1) : z \in C\}$ is unbounded above.

We call a cluster C *very special* if

- The set $\{z = (z(1), \ldots, z(d)) : z(1) = \min\{z'(1) : z' \in C\}$ contains infinitely many elements;
- There is an infinite path $(z_1, z_2, \ldots)$ in C such that $\lim_{n \to \infty} z_n(1) = \infty$.

Theorem 1.2 *Let μ be a (stationary) measure in two dimensions with bounded range. Then no very special clusters exist μ - a.s.*

We shall see that both conditions (dimension and range) are needed for the conclusion of the theorem. The theorem is also no longer true if we replace 'very special' by 'special'. We give examples in the next section. Before we proceed with the proof of the theorem, we make a few more definitions. For the rest of this section, we are in two dimensions.

The *density* $\rho_n(A)$ of a subset $A \subseteq \{z \in \mathbf{Z}^2 : z(1) = n\}$ is the limit

$$\lim_{k\to\infty} \frac{\#(A \cap \{z : -k \leq z(2) \leq k\})}{2k+1},$$

if this limit exists. Here $\#(\cdot)$ denotes cardinality.

Denote by $\mathcal{C}(n)$ the union of all infinite clusters C for which $\min\{z(1) : z \in C\} = n$. Furthermore, for all $k \geq 0$, we denote the set $\mathcal{C}(n) \cap \{z : z(1) = n + k\}$ by $\mathcal{C}_k(n)$.

It is clear from the stationarity of μ that $\mathcal{C}_k(n)$ forms a stationary process with respect to all vertical translations, i.e., the process $(W_\ell)_{\ell\in\mathbf{Z}}$ defined by $W_\ell = 1$ if $(n+k, \ell) \in \mathcal{C}_k(n)$ and $W_\ell = 0$ otherwise, is stationary. It then follows from the ergodic theorem that $\mathcal{C}_k(n)$ has a (random) density $\rho_{n+k}(\mathcal{C}_k(n))$ which we denote by $D_k(n)$. The sequence $(D_k(n))_n$ for fixed k is stationary, and therefore the expectation of $D_k(n)$ with respect to μ is independent of n and denoted by e_k.

Lemma 1.3 *It is the case that*

$$\sum_{n=0}^{\infty} e_n \leq 1. \tag{1}$$

Furthermore, for all m and n we have,

$$\mu\left((n+k, m) \in \mathcal{C}_k(n) \textit{ for infinitely many } k\right) = 0. \tag{2}$$

Proof Since the sets $\mathcal{C}_n(-n)$ are all subsets of the y-axis, and are mutually disjoint by construction, it follows that

$$\sum_{n=0}^{\infty} D_n(-n) \leq 1,$$

surely. Hence, by taking expectations, we find (1). It is a simple consequence of the ergodic theorem that for all m, the vertex $(n+k, m)$ is contained in $\mathcal{C}_k(n)$ with probability e_k. From (1) and the Borel-Cantelli lemma, (2) now follows. $\square$

Proof of Theorem 1.2 Suppose that very special clusters exist with positive probability. Then $\mathcal{C}(0)$ contains a very special cluster with positive probability. If this is the case, this implies that for any $k \geq 0$, the line $\{z : z(1) = n+k\}$ contains vertices which are contained in an infinite cluster of the halfspace $\{z : z(1) \geq n+k\}$. (Note that this would not necessarily be true for special clusters instead of very special clusters.) Now let $R < \infty$ be the range of μ, and consider the set $S = \{z : z(1) \geq 0, 0 \leq z(2) \leq R\}$. According to (1), only finitely many vertices in S belong to $\mathcal{C}(0)$ a.s. This implies that there is a finite (random) number M so that $S \cap \{z : z(1) \geq M\} \cap \mathcal{C}(0) = \emptyset$. Let m_0 be such that $\mu(M = m_0) > 0$. (Note that m_0 is not random.) Next we consider the following map g from $\Omega \to \Omega$: $g(\omega)(e) = 0$ if each of the following is true:

- $\omega(e) = 1$,
- both endpoints of e are in a very special cluster C with $\inf\{z(1) : z \in C\} = n_e$ for some n_e, and
- both endpoints of e are contained in $\{z : z(1) \leq n_e + m_0\}$;

in all other cases, $g(\omega)(e) = \omega(e)$. In words, g eliminates all open edges of very special clusters between their left boundary $\{z : z(1) = n\}$ and the line $\{z : z(1) = n + m_0\}$.

Define $\mu' = \mu \circ g^{-1}$. It is clear from the construction that μ' is stationary. It is also clear that μ' assigns positive probability to infinite clusters. But μ' has the additional property that with positive probability, a realisation chosen according to μ' contains a very special cluster with empty intersection with the strip S. Let C be such a very special cluster, with $\inf\{z(1) : z \in C\} = 0$, say. Since $C \cap S = \emptyset$, and the range of μ' is at most R (the range of μ), this implies that C is either completely above S or completely below S. In the former case, C has a left lower vertex, in the latter case, it has a left upper vertex. Both conclusions contradict Example (1) following Lemma 1.1. $\square$

2 Counterexamples

Next we show that both conditions are needed for the theorem to be true. We first construct an example of a two-dimensional measure μ with infinite range with very special clusters.

Example 1 We label each vertex z of $\mathbf{Z}^2$ with a label $c(e)$ from the set $\{1, 2, \ldots\}$ in such a way that all labels are independent and indentically distributed and such that the probability of label m equals 2^{-m}. For any vertex z we find the nearest vertex z' with the following properties: (i) $z'(1) = z(1) + 1$, (ii) $c(z') = c(z) + 1$. If there is more than one nearest z' with these properties we choose one randomly. Next we declare the edge between z and z' open. We repeat this procedure for every vertex z. Edges that are not declared open are declared closed. It is clear that this yields a stationary probability measure on Ω. Furthermore, all clusters are very special clusters 'starting' at vertices with label 1. Finally, the constructed measure obviously has infinite range.

Our next example shows that special clusters with bounded range can exist in two dimensions.

Example 2 Consider a discrete time (indexed by $\mathbf{Z}$), regenerative, stationary stochastic process taking values in $\{0, 1, 2, \ldots\}$ and making steps of size 1 only and which is a.s. unbounded above. For instance, we could take a one-sided simple random walk on the positive line with negative drift. Draw the path of this process in a space-time diagram, connecting consecutive points by edges of length $\sqrt{2}$. Suppose that time is depicted vertically and space horizontally. For any $n \in \mathbf{Z}$, the row $(-\infty, \infty) \times [n, n+1]$ contains exactly one edge of the path of our process, and we declare this edge open, together with all its horizontal translates. All other edges are closed. The measure μ corresponding to this construction is stationary and has infinitely many special clusters a.s.

Our final example shows that bounded range in dimension three is not enough to rule out very special clusters. The construction can be seen as a three-dimensional version of Example 1.

Example 3 We start our description with the square lattice $\mathbf{Z}^2$. We first give a geometrical description; probability comes in later. Tile the plane with adjacent 2×2 squares $S(i, j) = [0, 2] \times [0, 2] + (2i, 2j)$. Now for every second square (in both directions, starting at an arbitrary one), we label the centre of the square 0. The vertices labelled 0 are the corners of another tiling of the plane with squares of size 4×4. We again consider every second square in both directions among these and label the centres of these squares with a 1. The vertices labelled 1 again define a tiling of the plane. The centres of every second square in this tiling is labelled 2, and so on.

We can make this labelling stationary (in the two-dimensional sense) as follows: when we tile the plane initially with 2 by 2 squares, we have four possibilities of doing that, and we choose one of them with uniform probabilities. Then, at each stage, we have two possibilities: either all centres in the 'even' squares, or all centres in the 'odd' squares are labelled. Each time we choose one of these possibilities with equal probability. The result of this is a stationary labelling of $\mathbf{Z}^2$. We make this into a labelling of $\mathbf{Z}^3$ by copying this labelling in all layers $\{z : z(1) = U + 10k\}$, $k \in \mathbf{Z}$, where U is an independent uniform random variable on $\{0, 1, \ldots, 9\}$.

Finally, we connect a vertex z in the layer $\{z : z(1) = U + 10k_0\}$ with label m with a vertex with the nearest vertex with label $m+1$ in the layer $\{z : z(1) = U + 10(k_0 + 1)\}$ with a nearest neighbour path which is completely between these layers, in such a way that any path from a label m to a label $m+1$ is disjoint from any path from label ℓ to label $\ell + 1$, when $m \neq \ell$. It is easy to see that in three-dimensional space there is enough room to do this. This yields a configuration with only very special clusters.

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University of Utrecht, Mathematical Institute, P.O. Box 80.010, 3508 TA Utrecht, The Netherlands, `meester@math.uu.nl`