

Isomorphic Actions of Group Extensions on a Measure Space

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ABSTRACT

Let $E = E(G, A)$ be a group extension of an abelian l.c.s.c. group A by an amenable l.c.s.c. group G . An ergodic action V of A is said to be extendible to an action W of E if $V(A)$ is isomorphic to the restriction of W onto the subgroup $A \subset E$. The extension property is described and studied in terms of cocycles over a skew product with values in A . Several examples of \mathbb{R} -actions are considered. We answer the question of when two isomorphic actions of A can be extended to isomorphic actions of $E(G, A)$.

INTRODUCTION

Let A be an abelian locally compact second countable (l.c.s.c.) group and let G be an amenable l.c.s.c. group acting on A by group automorphisms. Denote by E the group extension of A by G . Then A can be identified with a normal subgroup of E . The group extension concept becomes more transparent in case of topologically trivial group extensions $E_f(G, A)$ constructed by a 2-cocycle $f : G \times G \rightarrow A$. An action V of A on a measure space is called extendible to an action W of E if $V(A)$ is isomorphic to $W(A)$. In [B], the question of when an action V of A can be extended to an action W of E_f was answered. It turns out that the extendibility property can be reformulated in terms of properties of cocycles with values in A . In the present paper, we study a circle of problems that is concentrated around actions of group extensions. It is worth noting that we are mainly interested in topologically trivial group extensions E_f because in

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this case one can prove deeper results. On the other hand, we believe that the theorems may be generalized to arbitrary group extensions as was done in [Dan] where some of the results of [B] were extended. In the present paper we will first find the criterion of extendibility of an action $V(A)$ to an action $W(E)$, when $E = E_f(G, A)$ (Theorem 2.3 and Proposition 2.6). One of the consequences is that if $V(A)$ is extendible to $W(E)$, then the actions $V(A)$ and $V(g \cdot A)$ are isomorphic for all $g \in G$. This observation makes it possible for us to show the existence of non-extendible actions given at the end of Section 2 in the case $A = \mathbb{R}$ and $G = \mathbb{R}_+^*$. We remark that these examples can have either zero or positive entropy. Secondly, we will show that extension property is not generic in some sense. Precisely, we consider a subset $I(c)$ in the set $Z^1(X \times \Gamma, A)$ of all cocycles over an approximately finite group Γ of measure preserving automorphisms with values in A such that the Mackey action of A generated by a cocycle from $I(c)$ is explicitly extendible to an action of $E_f(G, A)$. Assuming that G is countable, we prove in Theorem 2.10 that the set $I(c)$ is nowhere dense when $Z^1(X \times \Gamma, A)$ is equipped with the topology of convergence in measure. Finally, for a countable group G , we will answer the question of when two isomorphic actions of A are extendible to the isomorphic actions of E_f (Theorems 3.2 and 3.4).

Our study is based on two key results about actions of amenable groups proved in [AHS, BG1, GS1, GS2]. The first result says that any ergodic non-singular action of an amenable l.c.s.c. group is isomorphic to the Mackey action of this group defined by an ergodic countable approximately finite (a.f.) group Γ of measure preserving automorphisms and a recurrent cocycle over Γ . Moreover since all such automorphism groups are orbit equivalent, we can fix the group Γ , and the variety of Mackey actions is determined, up to isomorphism, by classes of weakly equivalent cocycles. The other result states that, roughly speaking, two Mackey actions are isomorphic if and only if the corresponding cocycles are weakly equivalent (see Section 1 for exact definitions and references).

The outline of the paper is as follows. To make the article self-contained, we collect in Section 1 all necessary definitions and facts that are applied later on in the paper. Section 2 contains the basic results about extendible and non-extendible actions. We give also several examples of such actions of $ax + b$ -group. The last section is devoted to the solution of the following problem: find necessary and sufficient conditions under which two isomorphic (extendible) actions of A can be extended to isomorphic actions of E_f .

We will use freely the notions of the full group and its normalizer, approximative finiteness, cocycles, Mackey actions. The necessary definitions can be found, for example, in [HO, Sch]. All equalities below hold a.e. on the appropriate measure space.

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1. PRELIMINARIES

We establish the following **notation** which are used throughout the paper.

- A is an abelian l.c.s.c. group that will be written additively;
- G is an amenable l.c.s.c. group with the identity e ;
- $(g, a) \xrightarrow{R} g \cdot a : G \times A \xrightarrow{R} A$ denotes a Borel action of G on A by group automorphisms. R is jointly continuous by a theorem from [M];
- Γ is a countable ergodic group of automorphisms of a measure space (X, \mathcal{B}, μ) (as a rule, Γ is measure preserving and approximately finite);
- $Z^1(X \times \Gamma, G)$ stands for the set of G -valued cocycles over Γ and $B^1(X \times \Gamma, G) \subset Z^1(X \times \Gamma, G)$ stands for the set of all Γ -coboundaries (recall $c \in Z^1(X \times \Gamma, G)$ if $c(x, \gamma_2 \gamma_1) = c(\gamma_1 x, \gamma_2) c(x, \gamma_1)$ for any $\gamma_1, \gamma_2 \in \Gamma$ and a.e. $x \in X$).

Let

$$(1.1) \quad 1 \longrightarrow B \xrightarrow{i} E \xrightarrow{j} G \longrightarrow 1$$

be a topological group extension of a l.c.s.c. group B by G . This means that (i) (1.1) is a short exact sequence where i is a homeomorphism from B onto a normal closed subgroup $i(B) \subset E$, (ii) j is a homomorphism of E onto G which induces a homeomorphism of $E/i(B)$ and G such that a natural action of G by conjugation on $i(B) \simeq B$ coincides with the given action of G on B . Throughout the paper we will identify B and $i(B)$ and refer to E as a *group extension*. Let q be a normalized Borel section from G into E , that is $j \circ q = \text{id}$ and $q(e) = e$. Then every $k \in E$ can be uniquely represented as $k = q(g)b$ where $b \in B$. If q can be chosen as a group homomorphism from G into E , then we say that E *splits*. Given (1.1) and a Borel section q , we can define a map $f : G \times G \rightarrow B$, called a 2-cocycle, by:

$$f(g, h) = q(gh)^{-1} q(g) q(h).$$

The above definitions become simpler in the case of *topologically trivial group extensions* of an abelian group A by G . In this settings, we introduce the set $Z^2(G, A)$ of continuous 2-cocycles: $f \in Z^2(G, A)$ if $f(g, e) = f(e, g) = 0$ and

$$(1.2) \quad g_3^{-1} \cdot f(g_1, g_2) + f(g_1 g_2, g_3) = f(g_2, g_3) + f(g_1, g_2 g_3),$$

where $g, g_1, g_2, g_3 \in G$. We may equip $E = G \times A$ with the product topology and for each $f \in Z^2(G, A)$ we define a group structure on E as follows:

$$(1.3) \quad (g_1, a_1)(g_2, a_2) = (g_1 g_2, f(g_1, g_2) + g_2^{-1} \cdot a_1 + a_2),$$

$$(1.4) \quad (g, a)^{-1} = (g^{-1}, -f(g, g^{-1}) - g \cdot a).$$

The set E , equipped with the group structure (1.3) and (1.4), is called a *topologically trivial group extension* of A by means of G and denoted by $E_f(G, A)$ (or simply E_f). For a continuous map $p : G \rightarrow A$, $p(e) = 0$, define the 2-cocycle $f_p \in Z^2(G, A)$ by

$$(1.5) \quad f_p(g_1, g_2) = -g_2^{-1} \cdot p(g_1) + p(g_1 g_2) - p(g_2).$$

Then f_p is called a 2-coboundary. The set of all 2-coboundaries is denoted by $B^2(G, A)$. The quotient $H_c^2(G, A) = Z^2(G, A)/B^2(G, A)$ is the group of continuous 2-cohomologies. It is well known that $H_c^2(G, A)$ is isomorphic to $Ext_t(G, A)$, the group of equivalence classes of topologically trivial group extensions. Note that $E_f(G, A)$ is isomorphic to $E_{f'}(G, A)$ if and only if $f - f'$ is a 2-coboundary. Note that the map implementing such an isomorphism acts trivially on A . The case when $f = 0$ (or f is a 2-coboundary) is of a crucial importance. The group extension $E_0(G, A)$ is called a *semi-direct product* of G and A . The notation $G \rtimes A$ is also used for $E_0(G, A)$.

Later we will use the following statement. Its proof is a slight modification of an argument given by Banach [Ba] and is left to the reader.

Lemma 1.1 *Let p be a normalized Borel map from G into A and let f_p be defined by (1.5). If $f_p : G \times G \rightarrow A$ is separately continuous, then p is also continuous.*

Let $0 \rightarrow A \xrightarrow{i} E \xrightarrow{j} G \rightarrow 1$ be a group extension of an abelian group A by G . If $q : G \rightarrow E$ is a normalized section of E , $j \circ q = \text{id}$, $q(e) = e$, then we set $g \cdot a = q(g)aq(g)^{-1}$. Describe the group $\text{Aut}(E; A)$ of all Borel (continuous) group automorphisms of E that leave A fixed. Let $Z^1(G, A)$ denote the group of algebraic 1-cocycles, i.e. $p \in Z^1(G, A)$ if and only if $p(gh) = h^{-1} \cdot p(g)p(h)$ and $p(e) = e$. The proof of the next proposition is straightforward.

Proposition 1.2 *Assume that G acts freely on A , i.e. if $g \cdot a = a$ for some $a \neq 0$, then $g = e$. Then $\theta \in \text{Aut}(E; A)$ if and only if there exists $p \in Z^1(G, A)$ such that*

$$\theta(g, a) = (g, a + p(g)).$$

We will use the notions of cocycles and H -cocycles over an automorphism group Γ of (X, \mathcal{B}, μ) . The notion of H -cocycles appeared first in [U] and then was studied in [B, DaD, Da1, Da2]).

Definition 1.3 *Let $f \in Z^2(G, A)$ and $c \in Z^1(X \times \Gamma, G)$. A measurable map $\alpha : X \times \Gamma \rightarrow A$ is called an H -cocycles if it satisfies the following conditions for $\gamma_1, \gamma_2 \in \Gamma$ and a.e. $x \in X$:*

$$(1.6) \quad \begin{aligned} \alpha(x, \mathbb{I}) &= 0, \\ \alpha(x, \gamma_2 \gamma_1) &= f(c(\gamma_1 x, \gamma_2), c(x, \gamma_1)) + c(x, \gamma_1)^{-1} \cdot \alpha(\gamma_1 x, \gamma_2) + \alpha(x, \gamma_1) \end{aligned}$$

where \mathbb{I} is the identity map. The set of all H -cocycles is denoted by $Z_{f,c}^1(X \times \Gamma, A)$ (or $Z_{f,c}^1(A)$). If for an H -cocycle $\delta(x, \gamma)$ there exist a normalized Borel map $p : G \rightarrow A$ and a measurable map $a : X \rightarrow A$ such that

$$\delta(x, \gamma) = p(c(x, \gamma)) + c(x, \gamma)^{-1} \cdot a(\gamma x) - a(x)$$

then δ is called an H -coboundary.

H -cocycles arise naturally in the following way. Let π be a cocycle over $X \times \Gamma$ with values in E_f . Then $\pi = (c, \alpha)$ where c and α are the projections of π onto G and A respectively. It is easily seen that $c \in Z^1(X \times \Gamma, G)$ and $\alpha \in Z_{f,c}^1(X \times \Gamma, A)$. The converse is also true [B, Da2].

Let K be a l.c.s.c. group with the Haar measure m_K . Let $\Gamma \subset \text{Aut}(X, \mathcal{B}, \mu)$, and $c \in Z^1(X \times \Gamma, K)$. Define the group of automorphisms $\Gamma(c) \subset \text{Aut}(X \times K, \mu \times m_K)$ whose elements act by the formula:

$$(1.7) \quad \gamma(c)(x, k) = (\gamma x, c(x, \gamma)k), \quad (x, k) \in X \times K, \gamma \in \Gamma.$$

The group $\Gamma(c)$ is called the *skew product*. If $\Gamma(c)$ is ergodic on $(X \times K, \mu \times m_K)$, then the cocycle c is said to be of *dense range* in K [Sch].

Let us consider the action V of K on $(X \times K, \mu \times m_K)$:

$$V(h)(x, k) = (x, kh^{-1}), \quad h \in K.$$

Denote by ξ the measurable partition of $X \times K$ into $\Gamma(c)$ -ergodic components. The groups $\Gamma(c)$ and $V(K)$ pairwise commute. Therefore, V generates on $((X \times K)/\xi, (\mu \times m_K)/\xi)$ a new action $W_{(\Gamma, c)}$ of K which is called the *Mackey action* (or the *action associated to the pair* (Γ, c)). Note that $W_{(\Gamma, c)}(K)$ is ergodic if and only if Γ is ergodic and $W_{(\Gamma, c)}(K)$ is isomorphic to the translation on K if and only if c is a coboundary.

Remark 1.4 Recall some results from [AHS, BG1, GS1, GS2] about Mackey actions that will be used later on.

(1) It was proved that if $U(K)$ is an amenable ergodic nonsingular action of K on a measure space, then there exists a pair (Γ, d) , where Γ is a countable ergodic approximately finite (a.f.) group of measure preserving automorphisms and d is a recurrent cocycle from $Z^1(X \times \Gamma, K)$, such that $U(K)$ and $W_{(\Gamma, d)}(K)$ are isomorphic. In particular, Γ may be taken to be of the form $\Gamma(c)$ where c is a cocycle with dense range in some amenable l.c.s.c. group G .

(2) Let Γ_i be an ergodic a.f. measure preserving group of automorphisms of $(X_i, \mathcal{B}_i, \mu_i)$ and let $d_i \in Z^1(X \times \Gamma_i, K)$ be a recurrent cocycle, $i = 1, 2$. Then, the Mackey actions $W_{(\Gamma_1, d_1)}(K)$ and $W_{(\Gamma_2, d_2)}(K)$ are isomorphic if and only if there is an isomorphism $R: X_1 \rightarrow X_2$ such that $R[\Gamma_1]R^{-1} = [\Gamma_2]$ and cocycles $d_1(x, \gamma_1)$ and $d_2 \circ R(x, \gamma_1) := d_2(Rx, R\gamma_1 R^{-1})$, $(x, \gamma_1) \in X_1 \times \Gamma_1$, are cohomologous, i.e. there exists a measurable map $\varphi: X_1 \rightarrow K$ such that $d_2 \circ R(x, \gamma_1) = \varphi(\gamma_1 x) d_1(x, \gamma_1) \varphi(x)^{-1}$. Such cocycles (or, more generally, the pairs (Γ_1, d_1) and (Γ_2, d_2)) are called *weakly equivalent*. We will use the fact that if c and c_1 are cocycles with dense range over Γ , then they are weakly equivalent.

(3) Let $\{U(K)\}$ be the class of K -actions isomorphic to an action U of K . Let $\Gamma(c)$ be an ergodic countable a.f. measure preserving group where c is a cocycle with dense range in G . It follows from the above facts that $\{U(K)\}$ contains the Mackey action $W_{(\Gamma(c), d)}(K)$ where d is a recurrent cocycle over $\Gamma(c)$ with values in K . Conversely, if $\Gamma(c)$ is fixed, then every cocycle d over $\Gamma(c)$ determines a class of isomorphic K -actions. Furthermore, two cocycles d

and d_1 over $\Gamma(c)$ determine the same class if and only if they are weakly equivalent.

We will also need the following technical statement (the proof is standard).

Remark 1.5 Let (X, μ) be a Lebesgue space and let K and H be l.c.s.c. groups with the Haar measures m_K and m_H respectively. Suppose that F is a measurable map from $(X \times K, \mu \times m_K)$ into (H, m_H) such that $F(x, k) = h_0$ for a.e. $(x, k) \in X \times K$ where $h_0 \in H$. Assume that F is continuous in k for μ -a.e. $x \in X$. Then there exists a measurable set $D \subset X$, $\mu(X - D) = 0$, such that $F(x, k) = h_0$ for all $(x, k) \in D \times K$.

2. EXTENDIBLE AND NON-EXTENDIBLE ACTIONS

We begin with the definition of extendible actions on a measure space.

Definition 2.1 Let V be an ergodic action of an abelian l.c.s.c. group A by non-singular automorphisms on a measure space (X, \mathcal{B}, μ) . Let $0 \rightarrow A \xrightarrow{i} E \xrightarrow{j} G \rightarrow 1$ be a group extension of A by an amenable l.c.s.c. group G . We say that V is extendible to an action of E if there exists an action W of E on (X, \mathcal{B}, μ) such that $V(A)$ is isomorphic to $W(i(A))$.

Remark 2.2 (1) Let W be an ergodic action of E (or $E_f(G, A)$) on a measure space and let θ be a group automorphism of E that acts identically on A . Denote the group of all such automorphisms by $\text{Aut}(E; A)$. Elements from $\text{Aut}(E_f; A)$ are described explicitly in Proposition 1.2. Given $\theta \in \text{Aut}(E; A)$, define a new action $\theta^*(W)$ of E by setting up $\theta^*(W)(k) = W(\theta(k))$, $k \in E$. Clearly, W and $\theta^*(W)$ have the same action of A . Therefore, if an action V of A is extendible to an action W of E , then V is also extendible to the action $\theta^*(W)$ of E for any $\theta \in \text{Aut}(E; A)$.

(2) We note that if $V(A)$ is extendible to an action of $E_f(G, A)$ then it is also extendible to an action of $E_{f_1}(G, A)$ where $f_1 = f + f_p$ is cohomologous to f .

Theorem 2.3 Let T be the translation on A : $T(a)(b) = ab$, $a, b \in A$. Then:

- (1) T is extendible to an action of E if and only if E splits.
- (2) If $E = E_f(G, A)$, then T is extendible to an action of E_f if and only if f is a 2-coboundary.

Proof. (1) Suppose that $q : G \rightarrow E$ is a group homomorphism such that $j \circ q = \text{id}$. We can easily find an action of E that extends T . Given $k \in E$ take $g \in G$, $a \in A$, such that $k = q(g)a$. Define $g \cdot a = q(g)aq(g)^{-1}$. Then set

$$W(k)(b) = W(q(g)a)(b) = g \cdot (ab), \quad k \in E.$$

Clearly, W extends T . Next, if $k_1 = q(g_1)a_1$, $k_2 = q(g_2)a_2$, then

$$\begin{aligned}
W(k_1 k_2)(b) &= W(q(g_1 g_2)(g_2^{-1} \cdot a_1 a_2))(b) \\
&= (g_1 g_2) \cdot [(g_2^{-1} \cdot a_1) a_2 b] \\
&= g_1 \cdot [a_1 g_2 \cdot (a_2 b)] \\
&= W(q(g_1) a_1) W(q(g_2) a_2)(b) \\
&= W(k_1) W(k_2)(b).
\end{aligned}$$

Note that if $k = q(g)a$, then $k^{-1} = q(g^{-1})(g \cdot a^{-1})$ and $W(k^{-1})(b) = W(k)^{-1}(b)$. Observe that in this proof we have not used that A is abelian.

To prove the converse, we have to assume that A is abelian (the group operation in E and A will be again written multiplicatively). Let T be extendible to an action W of E . Let $q : G \rightarrow E$ be a Borel normalized section. Denote $W(q(g)) = \tau(g)$, then $(g, b) \mapsto \tau(g)(b)$ is a Borel map from $G \times A$ into A that leaves the Haar measure m_A quasi-invariant for every $g \in G$. Then $g \in G$ defines a group homomorphism $a \mapsto g \cdot a$ where, by definition, $g \cdot a = q(g) a q(g)^{-1}$. Let $f : G \times G \rightarrow A$ be a 2-cocycle such that $q(g_1 g_2) f(g_1, g_2) = q(g_1) q(g_2)$. Then we get the following relations:

$$(2.1) \quad \tau(g)T(a) = T(g \cdot a)\tau(g),$$

$$(2.2) \quad \tau(g_1)\tau(g_2) = \tau(g_1 g_2)T(f(g_1, g_2)).$$

Relation (2.2) implies that $\tau(g)(ab) = (g \cdot a)\tau(g)(b)$ for all $b \in A$. Then, for $b = 1$, we have

$$(2.3) \quad \tau(g)(a) = (g \cdot a)s_g$$

where $s_g = \tau(g)(1)$ is a Borel map from G into A . In such a way, the W -‘action’ of G (i.e. the maps $\tau(g)$) can be found by (2.3). Note that (2.1) holds automatically if (2.3) is true. Furthermore, it follows from (2.2) and (2.3) that there is a relation between f and s_g . Indeed,

$$\begin{aligned}
\tau(g_1)\tau(g_2)(b) &= [(g_1 g_2) \cdot b](g_1 \cdot s_{g_2})s_{g_1} \\
\tau(g_1 g_2)T(f(g_1, g_2))(b) &= [(g_1 g_2) \cdot b][(g_1 g_2) \cdot f(g_1, g_2)]s_{g_1 g_2}.
\end{aligned}$$

Thus,

$$(g_1 g_2) \cdot f(g_1, g_2) = (g_1 \cdot s_{g_2})(s_{g_1 g_2})^{-1} s_{g_1}$$

or

$$f(g_1, g_2) = (g_2^{-1} \cdot s_{g_2})[(g_1 g_2)^{-1} \cdot s_{g_1 g_2}]^{-1}[(g_2^{-1} g_1^{-1}) \cdot s_{g_1}].$$

Denote $p(g) = (g^{-1} \cdot s_g)^{-1}$. Then

$$f(g_1, g_2) = [g_2^{-1} \cdot p(g_1)]^{-1} p(g_1 g_2) (p(g_2))^{-1}.$$

Clearly, $p(e) = 0$. Therefore, E splits since f is a 2-coboundary.

(2) In the case $E = E_f(G, A)$ the proof is the same. We should only note that because f is continuous and p is Borel, then, by Lemma 1.1, we get that p is a continuous map, and therefore, f is a 2-coboundary (see (1.5)). \square

Now we recall some notations and results from [B] that will be used later on. Let $\pi : X \times \Gamma \rightarrow E_f(G, A)$ be a cocycle over a countable ergodic measure preserving group of automorphisms Γ acting on (X, \mathcal{B}, μ) . Then $\pi = (c, \alpha)$ where $c \in Z^1(X \times \Gamma, G)$ and $\alpha \in Z_{f,c}^1(X \times \Gamma, A)$.

It can be easily verified that every $\alpha \in Z_{f,c}^1(X \times \Gamma, A)$ generates a cocycle b_α from $Z^1(X \times G \times \Gamma(c), A)$ where

$$(2.4) \quad b_\alpha(x, h, \gamma(c)) = h^{-1} \cdot \alpha(x, \gamma) + f(c(x, \gamma), h), \quad \gamma \in \Gamma.$$

We obtain two simple consequences of this fact. Firstly, $\alpha \mapsto b_\alpha$ defines a map S from $Z_{f,c}^1(X \times \Gamma, A)$ into $Z^1(X \times G \times \Gamma(c), A)$ where $f \in Z^2(G, A)$, $c \in Z^1(X \times \Gamma, G)$. Denote by $I(f, c)$ the image of $Z_{f,c}^1(X \times \Gamma, A)$ under the map S . Then $I(f, c) \subset Z^1(X \times G \times \Gamma(c), A)$ and we will see below that cocycles from $I(f, c)$ produce extendible actions of A via the Mackey construction. One can show that a cocycle $d : X \times G \times \Gamma(c) \rightarrow A$ belongs to $I(f, c)$ if and only if

$$(2.5) \quad d(x, h, \gamma(c)) = h^{-1} \cdot d(x, e, \gamma(c)) + f(c(x, \gamma), h)$$

for a.e. $(x, h) \in X \times G$. Indeed, (2.5) holds for a.e. $x \in X$ and all $h \in G$ (Remark 1.5). Define $\alpha(x, \gamma) = d(x, e, \gamma(c))$, then $d = b_\alpha$. Secondly, we can consider the Mackey action $W_{(\Gamma(c), b_\alpha)}$ of A associated with $(\Gamma(c), b_\alpha)$ as well as the Mackey action $W_{(\Gamma, \pi)}$ of (e, A) associated with (Γ, π) . It turns out that these two actions are isomorphic.

Theorem 2.4 [B] *Given a cocycle $\pi = (c, \alpha) : X \times \Gamma \rightarrow E_f(G, A)$, let b_α be defined by (2.4). Then, $W_{(\Gamma(c), b_\alpha)}(A)$ is isomorphic to $W_{(\Gamma, \pi)}(e, A)$.*

We will study only *ergodic actions* of E_f and (e, A) . Theorem 2.4 shows that, in this case, cocycles $c \in Z^1(X \times \Gamma, G)$ must necessarily be of *dense range*.

The next theorem answers the question when an ergodic action of A can be extended to an action of $E_f(G, A)$. The key point here is that, without loss of generality, we may deal only with Mackey actions of A and E_f and therefore use the results mentioned in Remark 1.4.

Theorem 2.5 [B] *Let V be an ergodic nonsingular action of A on a measure space (Ω, m) and let f be a 2-cocycle from $Z^2(G, A)$. Then the following statements are equivalent:*

- (i) V is extendible to an action of $E_f(G, A)$;
- (ii) for some cocycle $c \in Z^1(X \times \Gamma, G)$ with dense range, there exists a cocycle $d \in I(f, c)$ such that $V(A)$ is isomorphic to the Mackey action $W_{(\Gamma(c), d)}(A)$;
- (iii) for every cocycle $c \in Z^1(X \times \Gamma, G)$ with dense range, there exists a cocycle $d \in I(f, c)$ such that $V(A)$ is isomorphic to the Mackey action $W_{(\Gamma(c), d)}(A)$.

The last statement of Theorem 2.5 asserts that extendibility of $V(A)$ does not depend on a choice of c . In other words, this property does not depend on a realization of $V(A)$ as an associated action.

To clarify Theorems 2.4 and 2.5, we note that if $d \in I(f, c)$, then $W_{(\Gamma(c), d)}(A)$

is obviously extendible to the Mackey action of $E_f(G, A)$ built by (Γ, π) . Indeed, since $d = b_a$, we get that

$$W_{(\Gamma(c), d)}(A) \simeq W_{(\Gamma(c), b_a)}(A) \simeq W_{(\Gamma, \pi)}(e, A).$$

The last Mackey action is extendible to the action $W_{(\Gamma, \pi)}(E_f)$. Based on this observation, we will call actions of A of the form $W_{(\Gamma(c), d)}(A)$, where $d \in I(f, c)$, *explicitly extendible*. This term emphasizes the fact that $W_{(\Gamma(c), d)}(A)$ can be considered as a subgroup in $W_{(\Gamma, \pi)}(E_f)$. On the other hand, if we know that an action $V(A)$ is extendible, then it this means that $V(A)$ is *isomorphic* to an explicitly extendible action of A .

Remark that to show that (ii) and (iii) are equivalent, we use the following statement proved in [B]: If c and c_1 are two cocycles from $Z^1(X \times \Gamma, G)$ with dense ranges in G , then for any cocycle $d \in I(f, c) \subset Z^1(X \times G \times \Gamma(c), A)$ there exists a cocycle $d_1 \in I(f, c_1) \subset Z^1(X \times G \times \Gamma(c_1), A)$ such that the Mackey actions $W_{(\Gamma(c), d)}(A)$ and $W_{(\Gamma(c_1), d_1)}(A)$ are isomorphic.

The next proposition gives another approach to the extendibility problem. We will work here with a group extension E of an abelian group A by G as in Theorem 2.3. Let V be an ergodic action of the group A on a measure space (X, \mathcal{B}, μ) , and suppose that $\tau : G \rightarrow \text{Aut}(X, \mathcal{B}, \mu)$ is a map satisfying the conditions: $\tau(g_1)\tau(g_2) = \tau(g_1g_2)V(f(g_1, g_2))$ and $\tau(g^{-1}) = \tau(g)^{-1}V(f(g, g^{-1}))$ (the latter is equivalent to $\tau(e) = \text{id}$) where f is a 2-cocycle defined by a normalized section $g : G \rightarrow E$ as in Section 1 and $g, g_1, g_2 \in G$. We call such a τ an f -action of G with respect to V .

Proposition 2.6 *An ergodic action V of A is extendible to an action W of E if and only if there exists an f -action τ of G such that*

$$(2.6) \quad \tau(g)V(a)\tau(g)^{-1} = V(g \cdot a), \quad g \in G, \quad a \in A,$$

where $g \cdot a = q(g)aq(g)^{-1}$.

Proof. The proposition can be easily proved by utilizing the same argument as in the proof of Theorem 2.3. \square

Based on Theorem 2.3, we can deduce some results about extendibility of Mackey actions associated to $\Gamma(c)$ -coboundaries. From now, we work with *topologically trivial group extensions*.

Lemma 2.7 *Let $c \in Z^1(X \times \Gamma, G)$ be a cocycle with dense range and let $d \in B^1(X \times G \times \Gamma(c), A)$ be a $\Gamma(c)$ -coboundary. Then $d \in I(f_p, c)$ if and only if there exists a measurable map $\xi : X \rightarrow A$ such that for a.e. $x \in X$ and all $h \in G$*

$$(2.7) \quad d(x, h, \gamma(c)) = (h^{-1}c(x, \gamma)^{-1}) \cdot \xi(\gamma x) + p(c(x, \gamma)h) - (h^{-1} \cdot \xi(x) + p(h)).$$

Proof. It is straightforward to check that if d satisfies (2.7), then $d \in I(f_p, c)$.

Conversely, let $s : X \times G \rightarrow A$ be a measurable map and assume that

$d(x, h, \gamma(c)) := s(\gamma(c)(x, h)) - s(x, h)$ belongs to $I(f_p, c)$. It follows from (2.5) and (1.5) that

$$\begin{aligned} s(\gamma(c)(x, h)) - h^{-1} \cdot s(\gamma(c)(x, e)) - p(c(x, \gamma)h) - h^{-1} \cdot p(c(x, \gamma)) \\ = s(x, h) - h^{-1} \cdot s(x, e) - p(h) - h^{-1} \cdot p(e). \end{aligned}$$

Let $\tilde{p}(x, h) = p(h)$, $(x, h) \in X \times G$. Then the last expression has the form

$$\begin{aligned} s(\gamma(c)(x, h)) - h^{-1} \cdot s(\gamma(c)(x, e)) - \tilde{p}(\gamma(c)(x, h)) + h^{-1} \cdot \tilde{p}(\gamma(c)(x, e)) \\ = s(x, h) - h^{-1} \cdot s(x, e) - \tilde{p}(x, h) + h^{-1} \cdot \tilde{p}(x, e). \end{aligned}$$

In view of ergodicity of $\Gamma(c)$, there exists some $a \in A$ such that $s(x, h) = h^{-1} \cdot s(x, e) + p(h) + a$ for a.e. $x \in X$ and all $h \in G$ (see Remark 1.5). Therefore (2.7) holds where $\xi(x) = s(x, e)$. \square

Proposition 2.8 *Let $d \in B^1(X \times G \times \Gamma(c), A)$ where c has dense range. Then $W_{(\Gamma(c), d)}(A)$ is extendible to an action $E_f(G, A)$ if and only if f is a 2-coboundary.*

Proof. The assertion is a direct consequence of Theorem 2.3 because $W_{(\Gamma(c), d)}(A)$ is isomorphic to the translation T on A . \square

Corollary 2.9 *Let c be a cocycle with dense range. Then $B^1(X \times G \times \Gamma(c), A) \cap I(f, c) \neq \emptyset$ if and only if f is a 2-coboundary.*

Proof. The statement follows immediately from Lemma 2.7 and Proposition 2.8. Here we give a short direct proof of the corollary assuming, for simplicity, that G is countable.

Let $d \in I(f, c)$ be a $\Gamma(c)$ -coboundary. Then there exists a measurable function $\xi : X \times \Gamma \rightarrow A$ such that $d(x, h, \gamma(c)) = \xi(\gamma(c)(x, h)) - \xi(x, h)$ and d satisfies (2.5). Then

$$(2.8) \quad \xi(\gamma(c)(x, h)) - \xi(x, h) = h^{-1} \cdot \xi(\gamma(c)(x, e)) - h^{-1} \cdot \xi(x, e) + f(c(x, \gamma), h).$$

Let $\Gamma_0 = \{\gamma \in [\Gamma] : c(x, \gamma) = e \text{ a.e.}\}$, then Γ_0 is ergodic (recall that G is countable and $\Gamma(c)$ is ergodic). It follows from (2.8) that for each $\gamma_0 \in \Gamma_0$,

$$\xi(\gamma_0 x, h) - \xi(x, h) = h^{-1} \cdot \xi(\gamma_0 x, e) - h^{-1} \cdot \xi(x, e).$$

By ergodicity of Γ_0 we get that

$$(2.9) \quad \xi(x, h) = h^{-1} \cdot \zeta(x) + r(h)$$

where $\zeta(x) = \xi(x, e)$, and r is a normalized Borel map from G into A . It follows from (2.8) into (2.9) that

$$(2.10) \quad f(c(x, \gamma), h) = -h^{-1} \cdot r(c(x, \gamma)) + r(c(x, \gamma)h) - r(h).$$

Let $G = \{g_i : i \in \mathbb{N}\}$ and let $\gamma_i \in [\Gamma]$ be such that $c(x, \gamma_i) = g_i$ a.e. Taking $\gamma = \gamma_i$ in (2.10), we get that f is a 2-coboundary. \square

Recall that $Z^1(X \times G \times \Gamma(c), A)$ is a Polish space with respect to the topology τ

of convergence in measure. If Γ is a.f., then $B^1(X \times G \times \Gamma(c), A)$ is dense in $Z^1(X \times G \times \Gamma(c), A)$ [Sch]. Let $[d]$ denote the set of cocycles from $Z^1(X \times G \times \Gamma(c), A)$ weakly equivalent to a cocycle d (in particular, cohomologous to d). Then, $[d]$ is dense in $Z^1(X \times G \times \Gamma(c), A)$ for every d .

As mentioned in Remark 1.4, every ergodic nonsingular A -action is isomorphic to the associated action $W_{(\Gamma(c), d)}(A)$ where an ergodic group $\Gamma(c)$ may be chosen a priori and d is a cocycle from $Z^1(X \times G \times \Gamma(c), A)$. Thus, roughly speaking, one can state that the variety of associated actions of A is determined by cocycles from $Z^1(X \times G \times \Gamma(c), A)$. Note that if such a cocycle d is of the form (2.5), then the corresponding action is *explicitly* extendible to an action of E_f . Our goal now is to answer the question: How typical are cocycles d which determine explicitly extendible actions of A ?

Consider the set

$$I(c) = \bigcup_{f \in Z^2(G, A)} I(f, c)$$

formed by explicitly extendible cocycles. Indeed, if $d \in I(c)$, then there exists $f \in Z^2(G, A)$ such that $d = b_\alpha$ for some $\alpha \in Z_{f, c}^1$. Then $W_{(\Gamma(c), d)}(A) = W_{(\Gamma(c), b_\alpha)}(A)$ is isomorphic to $W_{(\Gamma, \pi)}(e, A)$ by Theorem 2.4, and hence is extended to E_f .

Next, we note that $I(c)$ is a subgroup in $Z^1(X \times G \times \Gamma(c), A)$ since $I(f, c) + I(f_1, c) = I(f + f_1, c)$. Assuming that c has a dense range in G , we see that $I(f, c) \cap I(f_1, c) = \emptyset$ when $f \neq f_1$. It is known that $I(c)$ does not depend on c up to isomorphism [B].

To clarify the structure of $I(c)$, we first observe that $I(0, c)$ is a closed subgroup of $I(c)$. For this, take $d_k \in I(0, c)$ such that $d_k \rightarrow d$ (in measure), $k \in \mathbb{N}$. It follows from Remark 1.5. that the relation $d_k(x, h, \gamma(c)) = h^{-1} \cdot d_k(x, e, \gamma(c))$ holds for all $k \in \mathbb{N}$ and all $(x, h) \in D \times G$ where $\mu(X - D) = 0$. Taking the pointwise limit in the above relation, we obtain that $d \in I(0, c)$. Let d_f be a fixed cocycle from $I(f, c)$, $f \in Z^2(G, A)$. Then, the relation

$$(2.11) \quad I(f, c) = d_f + I(0, c)$$

shows that $I(f, c)$ is also closed. Therefore, $I(c)$ is a disjoint union of closed subsets.

Given $f \in Z^2(G, A)$ and $c \in Z^1(X \times \Gamma, G)$, we point out how one can construct a cocycle from $I(f, c)$. Since Γ is a.f., one may assume that $\Gamma = \{T^n : n \in \mathbb{N}\}$ where T is an ergodic automorphism [CFW]. Let $u : X \rightarrow A$ be a measurable function. Given $f \in Z^2(G, A)$, set

$$\begin{aligned} \alpha_f(x, T) &= u(x), \\ \alpha_f(x, T^2) &= f(c(Tx, T), c(x, T)) + c(x, T)^{-1} \cdot u(Tx) + u(x) \end{aligned}$$

and so on. In such a way, we define an H -cocycle α^f (see (1.6)). Denote

$$d_f(x, h, T^n(c)) = h^{-1} \cdot \alpha_f(x, T^n) + f(c(x, T^n), h).$$

Clearly, $d_f \in I(f, c)$. It is easily seen that if, in particular, $u(x) = 0$, then this construction defines an injective group homomorphism $f \mapsto d_f$ from $Z^2(G, A)$ into $Z^1(X \times G \times \Gamma(c), A)$. It follows from (2.11) that $I(c)$ is isomorphic to the direct product $Z^2(G, A) \times I(0, c)$.

Let us consider the case of a countable group G .

Theorem 2.10 *Let G be a countable amenable group and let Γ be an ergodic a.f. group of measure preserving automorphisms and let $c \in Z^1(X \times \Gamma, G)$ be a cocycle with dense range in G . Then $I(c)$ is nowhere dense in $Z^1(X \times G \times \Gamma(c), A)$ with respect to the topology τ of convergence in measure.*

Proof. We first show that $I(c)$ is a closed subgroup of $Z^1(X \times G \times \Gamma(c), A)$. Let (d_n) be a sequence from $I(c)$ converging in measure to some cocycle d . Then every d_n defines uniquely a 2-cocycle f_n such that

$$d_n(x, h, \gamma(c)) - h^{-1} \cdot d_n(x, e, \gamma(c)) = f_n(c(x, \gamma), h).$$

Note that the right hand side depends neither on $x \in X$ nor on $\gamma \in \Gamma$ but only on the value $c(x, \gamma)$ and $h \in G$. Passing to a subsequence, we see that the left-hand side has a limit in the sense of almost sure convergence. By countability, the exceptional null set N for this convergence may be chosen to be independent of all $\gamma \in \Gamma$ and $h \in G$ (see Remark 1.5). Since the cocycle c has dense range, the set N can be also chosen so that for any $g \in G$ and any $x \notin N$ there exists a $\gamma = \gamma(x, g) \in \Gamma$ such that $c(x, \gamma) = g$. Then define the function f on $G \times G$ by choosing for $g \in G$ any $x \notin N$ and by putting for any $h \in G$,

$$f(g, h) = \lim_{n \rightarrow \infty} [d_n(x, h, \gamma(c)) - h^{-1} \cdot d_n(x, e, \gamma(c))]$$

where $\gamma = \gamma(x, g)$. One then sees that $f \in Z^2(G, A)$.

Since Γ (and hence $\Gamma(c)$) is a.f., the set $B^1 = B^1(X \times G \times \Gamma(c), A)$ is dense in $Z^1(X \times G \times \Gamma(c), A)$ in τ . Denote by $M = B^1 \cap Z^1(X \times G \times \Gamma(c), A)$. We prove that $B^1 \setminus M$ is still dense in $Z^1(X \times G \times \Gamma(c), A)$. To see this, we will show that every coboundary d from M can be approximated in τ by a coboundary $d' \in B^1 \setminus M$. Indeed, Lemma 2.7 says that d belongs to M if and only if $d(x, h, \gamma(c)) = s(\gamma(c)(x, h)) - s(x, h)$ where $s(x, h) = h^{-1} \cdot \xi(x) + p(h)$. Since G is countable, the full group $[I]$ contains the ergodic subgroup $\Gamma_0 = \{\gamma \in [I] : c(x, \gamma) = e \text{ for a.e. } x \in X\}$. Let $d \in M$ and let s, ξ, p be as above and $\epsilon > 0$. Then

$$d(x, h, \gamma_0(c)) = h^{-1} \cdot (\xi(\gamma_0 x) - \xi(x)), \quad \gamma_0 \in \Gamma_0.$$

We show that there exists a $\Gamma(c)$ -coboundary $d' \notin M$ which satisfies the relation

$$\mu\{x \in X : d'(x, h, \gamma(c)) \neq d(x, h, \gamma(c))\} < \epsilon, \quad h \in G, \gamma \in \Gamma.$$

Choose and fix $a_0 \in A$ which is not zero and take disjoint subsets $D(h)$, $h \in G$,

of X such that $\mu(\bigcup_{h \in G} D(h)) < \varepsilon/2$. For every $h \in G$, define the measurable function $I(D(h)) : X \rightarrow A$ by

$$I(D(h))(x) = \begin{cases} 0, & \text{if } x \notin D(h) \\ a_0, & \text{if } x \in D(h). \end{cases}$$

Set $k(x, h) = s(x, h) + I(D(h))(x)$. Now define the $\Gamma(c)$ -coboundary d' by setting

$$d'(x, h, \gamma(c)) = k(\gamma(c)(x, h)) - k(x, h).$$

Note that

$$d'(x, h, \gamma(c)) = d(x, h, \gamma(c)) + I(D(c(x, \gamma)h))(\gamma x) - I(D(h))(x).$$

Then, d' is ε -close to d in τ -topology, because

$$\begin{aligned} & \mu\{x \in X : d'(x, h, \gamma(c)) \neq d(x, h, \gamma(c))\} \\ &= \mu\{x \in X : I(D(c(x, \gamma)h))(\gamma x) - I(D(h))(x) \neq 0\} \\ &\leq \mu\{x \in X : \gamma x \in D(c(x, \gamma)h)\} + \mu(D(h)) < \varepsilon. \end{aligned}$$

Suppose that $d' \in M$. Then d' must be of the form $d'(x, h, \gamma(c)) = s'(\gamma(c)(x, h)) - s'(x, h)$, where $s'(x, h) = h^{-1} \cdot \xi'(x) + p'(h)$. It follows from the above relations for d' that if $\gamma_0 \in \Gamma_0$, then

$$h^{-1} \cdot (\xi'(\gamma_0 x) - \xi'(x)) = h^{-1} \cdot (\xi(\gamma_0 x) - \xi(x)) + I(D(h))(\gamma_0 x) - I(D(h))(x).$$

Therefore, for every $h \in G$, the function

$$c_h(x) = h^{-1} \cdot (\xi'(x) - \xi(x)) - I(D(h))(x)$$

is Γ_0 -invariant and hence $c_h(x)$ is a constant $c(h) \in A$ a.e. Taking x from $D(h)$, $h \neq e$, we see that the relation $c(h) = h^{-1} \cdot c(e) - a_0$ holds. Thus, for any $h \neq e$ and a.e. $x \in X$, we get that

$$h \cdot a_0 + I(D(e))(x) = h \cdot I(D(h))(x).$$

If we choose $x \notin D(h) \cup D(e)$, then we have $h \cdot a_0 = 0$, which implies $a_0 = 0$, a contradiction.

To complete the proof, we observe that the interior of $I(c)$ is empty since $Z^1(X \times G \times \Gamma(c), A) \setminus I(c)$ is open and contains the dense subset $B^1 \setminus M$. \square

The next theorem gives a sufficient condition for a cocycle $d \in Z^1(X \times G \times \Gamma(c), A)$ to belong to $I(c)$. In other words, for d satisfying the condition of the theorem, there exists $f \in Z^2(G, A)$ such that $W_{(\Gamma(c), d)}(A)$ may be extended to an action of E_f .

Theorem 2.11 *Let Γ be an ergodic a.f. countable group of automorphisms of (X, \mathcal{B}, μ) and let c be a cocycle over Γ with dense range in a countable amenable group G . Assume that for given $d \in Z^1(X \times G \times \Gamma(c), A)$ there exists a subset*

$N \subset X$, $\mu(N) = 0$, such that $d(x, h, \gamma(c)) - h^{-1} \cdot d(x, e, \gamma(c))$ does not depend on $x \in X - N$. Then $d \in I(c)$.

Proof. It follows from [BG2] that Γ is generated by ergodic subgroup $\Gamma_0 = \{\gamma \in [\Gamma] : c(x, \gamma) = e \text{ for a.e. } x \in X\}$ and the automorphisms $\gamma_g \in [\Gamma] \cap N[\Gamma_0]$ such that $c(x, \gamma_g) = g$ for a.e. $x \in X$ and $g \in G$. Set $\gamma_e = \mathbb{I}$. Then, for each $\gamma \in \Gamma$ and a.e. $x \in X$, there are $g = g(x) \in G$ and $\gamma_0 \in \Gamma_0$ such that $\gamma x = \gamma_g \gamma_0 x$. Without loss of generality, Γ may be taken as a free a.f. group of automorphisms. It allows one to extend d to the full group $[\Gamma]$. By the assumption of the theorem, we can define

$$(2.12) \quad F(\gamma, h) = d(x, h, \gamma(c)) - h^{-1} \cdot d(x, e, \gamma(c)),$$

where $\gamma \in \Gamma$ and $x \in X - N$, $\mu(N) = 0$. If h is fixed and γ runs over the full subgroup $\Gamma_0 \subset [\Gamma]$, then $F(\gamma, h)$ defines a group homomorphism F_h from Γ_0 into A . Furthermore, F_h is continuous with respect to the uniform topology (see e.g. [HO]). Thus, the kernel of F_h , $\ker(F_h)$, must be a normal closed subgroup in Γ_0 . It follows from [Dye] that $\ker(F_h)$ is either $\{\mathbb{I}\}$ or Γ_0 . But if for $\gamma_1, \gamma_2 \in \Gamma_0$, $\mu(\{x \in X : \gamma_1 x = \gamma_2 x\}) > 0$, then $F_h(\gamma_1) = F_h(\gamma_2)$. This shows that $\ker(F_h) = \Gamma_0$. In other words, we proved that for all $\gamma_0 \in \Gamma_0$

$$(2.13) \quad d(x, h, \gamma_0(c)) - h^{-1} \cdot d(x, e, \gamma_0(c)) = 0.$$

Define

$$(2.14) \quad f(g, h) = F(\gamma_g, h), \quad g, h \in G.$$

Note that $f(e, g) = f(g, e) = 0$, $g \in G$. Next, take $\gamma \in G$, then $\gamma x = \gamma_g \gamma_0 x$ where g depends on x and $\gamma_0 \in \Gamma_0$. We get for a.e. $x \in X$ that

$$\begin{aligned} & d(x, h, \gamma(c)) - h^{-1} \cdot d(x, e, \gamma(c)) \\ &= d(\gamma_0 x, h, \gamma_g(c)) - h^{-1} \cdot d(\gamma_0 x, e, \gamma_g(c)) \\ (2.15) \quad & + d(x, h, \gamma_0(c)) - h^{-1} d(x, e, \gamma_0(c)) \\ &= f(g, h) \quad (\text{by (2.13), (2.14)}) \\ &= f(c(\gamma_0 x, \gamma_g) c(x, \gamma_0), h) \\ &= f(c(x, \gamma), h). \end{aligned}$$

If we prove that f is a 2-cocycle, then (2.15) would imply that $d \in I(f, c) \subset I(c)$. Thus, it remains to show that $f \in Z^2(G, A)$. By definition of f , this is equivalent to showing that for all $g_1, g_2, g_3 \in G$,

$$\begin{aligned} (2.16) \quad & g_3^{-1} \cdot d(x, g_2, \gamma_{g_1}(c)) + d(x, g_3, \gamma_{g_1 g_2}(c)) - g_3^{-1} \cdot d(x, e, \gamma_{g_1 g_2}(c)) \\ &= d(x, g_3, \gamma_{g_2}(c)) - g_3^{-1} \cdot d(x, e, \gamma_{g_2}(c)) + d(x, g_2 g_3, \gamma_{g_1}(c)). \end{aligned}$$

It is easily seen that $\gamma_{g_1 g_2}(c) = \gamma_{g_1}(c) \gamma_{g_2}(c) \gamma_0(c)$ for some γ_0 from Γ_0 . Therefore, we can get from (2.14) and the assumption of the theorem that

$$\begin{aligned}
(2.17) \quad & d(x, g_3, \gamma_{g_1}(c)\gamma_{g_2}(c)\gamma_0(c)) - g_3^{-1} \cdot d(x, e, \gamma_{g_1}(c)\gamma_{g_2}(c)\gamma_0(c)) \\
& = d(\gamma_{g_2}\gamma_0x, g_2g_3, \gamma_{g_1}(c)) - g_3^{-1} \cdot d(\gamma_{g_2}\gamma_0x, g_2, \gamma_{g_1}(c)) \\
& \quad + d(x, g_3, \gamma_{g_2}(c)) - g_3^{-1} \cdot d(x, e, \gamma_{g_2}(c)).
\end{aligned}$$

It follows from (2.17) that (2.16) may be transformed into the following relation:

$$\begin{aligned}
(2.18) \quad & d(\gamma_{g_2}\gamma_0x, g_2g_3, \gamma_{g_1}(c)) - g_3^{-1} \cdot d(\gamma_{g_2}\gamma_0x, g_2, \gamma_{g_1}(c)) \\
& = d(x, g_2g_3, \gamma_{g_1}(c)) - g_3^{-1} \cdot d(x, g_2, \gamma_{g_1}(c)).
\end{aligned}$$

To see that (2.18) is true, let us add and subtract $g_3^{-1}g_2^{-1} \cdot d(\gamma_{g_2}\gamma_0x, e, \gamma_{g_1}(c))$ from the left-hand side of (2.18), and $g_3^{-1}g_2^{-1} \cdot d(x, e, \gamma_{g_1}(c))$ from the right-hand side. Then

$$\begin{aligned}
& d(\gamma_{g_2}\gamma_0x, g_2g_3, \gamma_{g_1}(c)) - g_3^{-1}g_2^{-1} \cdot d(\gamma_{g_2}\gamma_0x, e, \gamma_{g_1}(c)) \\
& \quad - g_3^{-1} \cdot [d(\gamma_{g_2}\gamma_0x, g_2, \gamma_{g_1}(c)) - g_2^{-1} \cdot d(\gamma_{g_2}\gamma_0x, e, \gamma_{g_1}(c))] \\
& = d(x, g_2g_3, \gamma_{g_1}(c)) - g_3^{-1}g_2^{-1} \cdot d(x, e, \gamma_{g_1}(c)) \\
& \quad - g_3^{-1} \cdot [d(x, g_2, \gamma_{g_1}(c)) - g_2^{-1} \cdot d(x, e, \gamma_{g_1}(c))].
\end{aligned}$$

Clearly, the last relation (and therefore (2.16)) holds. \square

We conclude this section with several examples of extendible and non-extendible actions.

Examples 2.12 (1) It is a simple exercise to construct an extendible (even explicitly) action of A taking into account Theorem 2.5 and relation (2.5). To find a non-extendible action is a more delicate problem. It is not sufficient to take a cocycle d which is not in $I(f, c)$ because d may be weakly equivalent to some $d_1 \in I(f, c)$ and therefore the action $W_{(\Gamma(c), d)}(A)$ turns out to be extendible. Moreover, note that the class $[d]$ of cocycles weakly equivalent to d is dense in $Z^1(X \times G \times \Gamma(c), A)$.

In the examples given below, we take $A = \mathbb{R}$, $G = \mathbb{R}_+^*$, and $E_0 = \mathbb{R}_+^* \ltimes \mathbb{R}$.

(2) We first remark that one can find a countable ergodic group of measure preserving automorphisms Γ and cocycles c and d such that the Mackey action of \mathbb{R} generated by $(\Gamma(c), d)$ is isomorphic to a special flow $W_{(Q, \varphi)}(\mathbb{R})$ built by a base automorphism Q acting on the unit circle \mathbb{S} and a ceiling function $\varphi = 1$. Moreover, Q is isomorphic to the circle rotation with an irrational rotation number. Then we claim that $W_{(Q, \varphi)}(\mathbb{R})$ cannot be extended to an action of $E_0 = \mathbb{R}_+^* \ltimes \mathbb{R}$. For this, it suffices to show that Proposition 2.6 fails. Let us assume that there exists an action τ of \mathbb{R}_+^* on $Y = \mathbb{S} \times [0, 1)$ such that for $(x, u) \in Y$, $p \in \mathbb{R}_+^*$, $t \in \mathbb{R}$, one has

$$(2.19) \quad \tau(p)W(t)(x, u) = W(pt)\tau(p)(x, u)$$

where $W(t) = W_{(Q, 1)}(t)$. Note that $W(1)(x, u) = (Qx, u)$, $(x, u) \in Y$. Then, it

follows from (2.19) that $W(1)$ is isomorphic to $W(2)$ where $W(2)(x, u) = (Q^2x, u)$. We get a contradiction since their spectra are different.

(3) In (2), we have found an example of \mathbb{R} -action which is not extendible to an action of $\mathbb{R}_+^* \ltimes \mathbb{R}$ and has zero entropy. Relation (2.12) (or, more generally, (2.6)) shows that, for every $p \in \mathbb{R}_+^*$ and $t \in \mathbb{R}$, the automorphisms $W(t)$ and $W(pt)$ must be isomorphic. This simple observation allows us to produce a family of \mathbb{R} -actions with positive entropy that cannot be extended to actions of $\mathbb{R}_+^* \ltimes \mathbb{R}$. Let $W_{(Q, \varphi)}(\mathbb{R})$ be an ergodic special flow of measure preserving automorphisms such that the entropy $h(Q) > 0$ and finite. Then $W_{(Q, \varphi)}(\mathbb{R})$ is not extendible to an action of $\mathbb{R}_+^* \ltimes \mathbb{R}$ since $W(t)$ and $W(pt)$ have different entropies. On the other hand, suppose we are given an ergodic measure preserving action U of $\mathbb{R}_+^* \ltimes \mathbb{R}$. Then the automorphism group $U(\{1\} \times \mathbb{R})$ has the property: the entropy of every $U(\{1\} \times t)$, $t \in \mathbb{R}$, is either 0 or ∞ .

(4) (This example was given by I. Kornfeld). Let $U(t)$ be the horocycle flow and let $\psi(s)$ be the geodesic flow in the Poincaré half plane $\{z \in \mathbb{C} : \text{Im}(z) > 0\}$, $s, t \in \mathbb{R}$. It is well known that they are related in the following way:

$$(2.20) \quad \psi(s)U(t)\psi(-s) = U(te^s).$$

Define $\tau(p) = \psi(\log p)$. It follows from (2.20) and Proposition 2.6 that the horocycle flow can be extended to the action of $\mathbb{R}_+^* \ltimes \mathbb{R}$ defined by τ and U .

3. ISOMORPHIC ACTIONS OF GROUP EXTENSIONS

This section is devoted to the solution of the following problem: Suppose we are given by two isomorphic actions of A , V_1 and V_2 . Assume they are extendible to actions U_1 and U_2 of $E_f(G, A)$. The question is: *Under what conditions are U_1 and U_2 isomorphic?*

As above we represent any ergodic nonsingular action of E_f as the Mackey action $W_{(\Gamma, \pi)}(E_f)$ where Γ is an ergodic measure preserving a.f. countable group of automorphisms of (X, \mathcal{B}, μ) and $\pi : X \times \Gamma \rightarrow E_f$ is a cocycle. The following proposition is an immediate consequence of the results from [GS1, GS2] discussed in Section 1.

Proposition 3.1 *Let U_1 and U_2 be two isomorphic ergodic actions of E_f . Then there exist pairs (Γ, π_1) and (Γ, π_2) such that U_i is isomorphic to $W_{(\Gamma, \pi_i)}$, $i = 1, 2$, and the cocycles π_1, π_2 are weakly equivalent, i.e. there is $R \in N[\Gamma]$ such that $\pi_1 \circ R$ is cohomologous to π_2 where Γ is an ergodic countable a.f. group of measure preserving automorphisms.*

Consider first the converse problem to the question formulated above.

Theorem 3.2 *Let $W_{(\Gamma, \pi)}$ and $W_{(\Gamma, \pi_1)}$ be isomorphic ergodic actions of E_f where $\pi = (c, \alpha)$ and $\pi_1 = (c_1, \alpha_1)$ are cocycles with values in $E_f(G, A)$. Then there exists an automorphism $R'(x, h) = (Rx, k(x)h)$ of $(X \times G, \mu \times m_G)$, $R \in N[\Gamma]$, such that the cocycles $b_\alpha \circ R'$ and b_{α_1} are $\Gamma(c_1)$ -cohomologous.*

Proof. It follows from Proposition 3.1 that there exist a measurable function $\Phi(x) = (k(x), u(x)) : X \rightarrow E_f$ and an automorphism $R \in N[\Gamma]$ such that for any $\gamma \in \Gamma$ and a.e. $x \in X$, $\pi \circ R(x, \gamma) = \Phi(\gamma x) \pi_1(x, \gamma) \Phi(x)^{-1}$. By (1.4), $\Phi(x)^{-1} = (k(x)^{-1}, -f(k(x), k(x)^{-1}) - k(x) \cdot u(x))$. Then

$$\begin{aligned} \pi \circ R(x, \gamma) &= (c \circ R(x, \gamma), \alpha \circ R(x, \gamma)) \\ &= (k(\gamma x), u(\gamma x))(c_1(x, \gamma), \alpha_1(x, \gamma))(k(x)^{-1}, -f(k(x), k(x)^{-1}) - k(x) \cdot u(x)), \end{aligned}$$

and therefore

$$\begin{aligned} (3.1) \quad c \circ R(x, \gamma) &= k(\gamma x) c_1(x, \gamma) k(x)^{-1}, \\ (3.2) \quad \alpha \circ R(x, \gamma) &= f(k(\gamma x) c_1(x, \gamma), k(x)^{-1}) + k(x) \cdot f(k(\gamma x), c_1(x, \gamma)) \\ &\quad + k(x) c_1(x, \gamma)^{-1} \cdot u(\gamma x) + k(x) \cdot \alpha_1(x, \gamma) - f(k(x), k(x)^{-1}) - k(x) \cdot u(x). \end{aligned}$$

Applying (1.2), we get

$$\begin{aligned} &k(x) \cdot f(k(\gamma x), c_1(x, \gamma)) + f(k(\gamma x) c_1(x, \gamma), k(x)^{-1}) \\ &= f(k(\gamma x), c_1(x, \gamma) k(x)^{-1}) + f(c_1(x, \gamma), k(x)^{-1}) \end{aligned}$$

and then (3.2) is transformed into

$$\begin{aligned} \alpha \circ R(x, \gamma) &= f(k(\gamma x), c_1(x, \gamma) k(x)^{-1}) + f(c_1(x, \gamma), k(x)^{-1}) - f(k(x), k(x)^{-1}) \\ &\quad + k(x) \cdot \alpha_1(x, \gamma) + k(x) c_1(x, \gamma)^{-1} \cdot u(\gamma x) - k(x) \cdot u(x). \end{aligned}$$

It is easy to check that if $\alpha \in Z_{f,c}^1(A)$, then $\alpha \circ R \in Z_{f,coR}^1(A)$.

Define the automorphism $R' \in \text{Aut}(X \times G, \mu \times m_G)$ by setting $R'(x, h) = (Rx, k(x)h)$. It is a simple exercise to verify (using (3.1)) that

$$(3.4) \quad R' \gamma(c_1)(R')^{-1} = R \gamma R^{-1}(c).$$

To compute $b_\alpha \circ R'$ and b_{α_1} we use (2.4), (3.3), and (3.4):

$$b_{\alpha_1}(x, h, \gamma(c_1)) = h^{-1} \cdot \alpha_1(x, \gamma) + f(c_1(x, \gamma), h)$$

and

$$\begin{aligned} &b_\alpha \circ R'(x, h, \gamma(c_1)) \\ &= (h^{-1} k(x)^{-1}) \cdot \alpha(Rx, R \gamma R^{-1}) + f(c(Rx, R \gamma R^{-1}), k(x)h) \\ (3.5) \quad &= h^{-1} k(x)^{-1} \cdot f(k(\gamma x), c_1(x, \gamma) k(x)^{-1}) + h^{-1} k(x)^{-1} \cdot f(c_1(x, \gamma), k(x)^{-1}) \\ &\quad - h^{-1} k(x)^{-1} \cdot f(k(x), k(x)^{-1}) + h^{-1} \cdot \alpha_1(x, \gamma) + h^{-1} c_1(x, \gamma)^{-1} \cdot u(\gamma x) \\ &\quad - h^{-1} \cdot u(x) + f(c \circ R(x, \gamma), k(x)h). \end{aligned}$$

Applying (1.2) to the first 3 terms of (3.5), we can rewrite

$$\begin{aligned} b_\alpha \circ R'(x, h, \gamma(c_1)) &= f(k(\gamma x), c_1(x, \gamma)h) + f(c_1(x, \gamma), h) \\ &\quad - f(k(x), h) + h^{-1} \cdot \alpha_1(x, \gamma) + h^{-1} c_1(x, \gamma)^{-1} \cdot u(\gamma x) - h^{-1} \cdot u(x). \end{aligned}$$

Thus,

$$\begin{aligned}
& b_\alpha \circ R'(x, h, \gamma(c_1)) - b_{\alpha_1}(x, h, \gamma(c_1)) \\
& = f(k(\gamma x), c_1(x, \gamma)h) - f(k(x), h) + h^{-1}c_1(x, \gamma)^{-1} \cdot u(\gamma x) - h^{-1} \cdot u(x).
\end{aligned}$$

Denote by $\xi(x, h) = f(k(x), h) + h^{-1} \cdot u(x)$. To complete the proof, we note that

$$b_\alpha \circ R'(x, h, \gamma(c_1)) - b_{\alpha_1}(x, h, \gamma(c_1)) = \xi(\gamma(c_1)(x, h)) - \xi(x, h). \quad \square$$

Remark 3.3 It is not surprising that cocycles b_α and b_{α_1} are weakly equivalent since the Mackey actions $W_{(\Gamma(c), b_\alpha)}(A)$ and $W_{(\Gamma(c_1), b_{\alpha_1})}(A)$ must be isomorphic by Theorem 2.4. The non-trivial part of Theorem 3.2 consists of the explicit description of the automorphism R' that implements the isomorphism of these Mackey actions.

Now our goal is to prove the converse statement. To do this, we will have to assume that G is countable.

It is known [GS2] that if c and c_1 are cocycles over an ergodic a.f. automorphism group Γ both with dense ranges in G , then they are weakly equivalent, i.e. there exist an $R \in N[\Gamma]$ and a measurable map $k : X \rightarrow G$ such that $c \circ R(x, \gamma) = k(\gamma x)c_1(x, \gamma)k(x)^{-1}$, $(x, \gamma) \in X \times \Gamma$.

Theorem 3.4 *Let G be a countable amenable group and let c, c_1, R , and $k(x)$ be as above. Define $R'(x, h) = (Rx, k(x)h)$, $(x, h) \in X \times G$. Given $\alpha \in Z_{f, c}^1(A)$ and $\alpha_1 \in Z_{f, c_1}^1(A)$, assume that the cocycles $b_\alpha \circ R'$ and b_{α_1} are $\Gamma(c_1)$ -cohomologous, that is $W_{(\Gamma(c), \alpha)}(A)$ and $W_{(\Gamma(c_1), \alpha_1)}(A)$ are isomorphic. Then there exists a group automorphism θ of E_f such that the Mackey actions $W_{(\Gamma, \pi_1)}(E_f)$ and $\theta^*(W)_{(\Gamma, \pi)}(E_f)$ are isomorphic where $\pi = (c, \alpha)$ and $\pi_1 = (c_1, \alpha_1)$.*

Proof. It follows from the assumption that there exists a measurable map $q : X \times G \rightarrow A$ such that for any $\gamma \in \Gamma$ and a.e. $(x, h) \in X \times G$

$$b_\alpha \circ R'(x, h, \gamma(c_1)) - b_{\alpha_1}(x, h, \gamma(c_1)) = q(\gamma(c_1)(x, h)) - q(x, h).$$

By (2.4), we have

$$\begin{aligned}
(3.6) \quad & k(x)^{-1} \cdot \alpha \circ R(x, \gamma) - \alpha_1(x, \gamma) \\
& = h \cdot f(c_1(x, \gamma), h) - h \cdot f(c \circ R(x, \gamma), k(x)h) + h \cdot q(\gamma(c_1)(x, h)) - h \cdot q(x, h).
\end{aligned}$$

The left-hand side in (3.6) does not depend on $h \in G$. Therefore, we can put $h = e$ in (3.6). Then we have

$$\begin{aligned}
(3.7) \quad & h \cdot f(c_1(x, \gamma), h) - h \cdot f(c \circ R(x, \gamma), k(x)h) + h \cdot q(\gamma(c_1)(x, h)) - h \cdot q(x, h) \\
& = -f(k(\gamma x)c_1(x, \gamma)k(x)^{-1}, k(x)) + q(\gamma(c_1)(x, e)) - q(x, e).
\end{aligned}$$

Since

$$h \cdot f(c_1(x, \gamma), h) = -f(c_1(x, \gamma)h, h^{-1}) + f(h, h^{-1}),$$

and

$$\begin{aligned}
& -h \cdot f(c \circ R(x, \gamma), k(x)h) \\
& = f(k(\gamma x)c_1(x, \gamma)h, h^{-1}) - f(k(x)h, h^{-1}) - f(k(\gamma x)c_1(x, \gamma)k(x)^{-1}, k(x)),
\end{aligned}$$

relation (3.7) can be written in the following form:

$$\begin{aligned}
(3.8) \quad & q(\gamma(c_1)(x, e)) - q(x, e) \\
& = -f(c_1(x, \gamma)h, h^{-1}) + f(h, h^{-1}) + f(k(\gamma x)c_1(x, \gamma)h, h^{-1}) \\
& \quad - f(k(x)h, h^{-1}) + h \cdot q(\gamma(c_1)(x, h)) - h \cdot q(x, h).
\end{aligned}$$

We use (1.2):

$$\begin{aligned}
& f(k(\gamma x)c_1(x, \gamma)h, h^{-1}) - f(c_1(x, \gamma)h, h^{-1}) \\
& = -h \cdot f(k(\gamma x), c_1(x, \gamma)h) + f(k(\gamma x), c_1(x, \gamma)), \\
& f(k(x)h, h^{-1}) - f(h, h^{-1}) = -h \cdot f(k(x), h).
\end{aligned}$$

Then it follows from (3.8) that

$$\begin{aligned}
(3.9) \quad & -h \cdot f(k(\gamma x), c_1(x, \gamma)h) + f(k(\gamma x), c_1(x, \gamma)) \\
& + h \cdot q(\gamma(c_1)(x, h)) - q(\gamma(c_1)(x, e)) \\
& = -h \cdot f(k(x), h) + h \cdot q(x, h) - q(x, e).
\end{aligned}$$

Consider the measurable function $F : X \times G \rightarrow A$

$$(3.10) \quad F(x, h) = -h \cdot f(k(x), h) + h \cdot q(x, h) - q(x, e).$$

Note that for every fixed $h \in G$, F is constant a.e. on X . To see this, define the ergodic subgroup $\Gamma_0 = \{\gamma \in [\Gamma] : c_1(x, \gamma) = 0 \text{ for a.e. } x \in X\}$. For $h \in G$, $\gamma \in \Gamma_0$, we get from (3.9) that $F(\gamma x, h) = F(x, h)$, i.e. $F(x, h) = \varphi(h)$ a.e.

We show that $\varphi(h)$ satisfies the relation

$$(3.11) \quad \varphi(gh) = \varphi(g) + g \cdot \varphi(h).$$

Indeed, let $\gamma_g \in [\Gamma]$ be chosen such that $c_1(x, \gamma_g) = g$, $g \in G$ where $x \in X - N$, $\mu(N) = 0$. Denote $y = \gamma_g x$. Then we get from (3.9) and (3.10) that

$$\begin{aligned}
\varphi(h) & = -h \cdot f(k(y), gh) + f(k(y), g) + h \cdot q(y, gh) - q(y, g) \\
& = g^{-1}[-(gh) \cdot f(k(y), gh) + g \cdot f(k(y), g) + gh \cdot q(y, gh) - g \cdot q(y, g)] \\
& = g^{-1} \cdot (\varphi(gh) - \varphi(g)).
\end{aligned}$$

It follows from (3.11) that $\varphi(h)$ generates a group automorphism $\theta \in \text{Aut}(E_f; A)$ defined by $\theta(h, a) = (h, a - \varphi(h^{-1}))$ (see Proposition 1.2).

Finally, let us define the measure space isomorphism $\Phi : X \times E_f \rightarrow X \times E_f$ as follows:

$$\Phi(x, h, a) = (Rx, k(x)h, a + q(x, h)).$$

Note that, due to (3.10), Φ can be written down as follows:

$$\Phi(x, h, a) = (Rx, k(x)h, a + h^{-1} \cdot q(x, e) - \varphi(h^{-1}) + f(k(x), h))$$

Recall that the translation on $X \times E_f$ by the group E_f is defined by

$$\begin{aligned} T_{(h,a)}(x, g, b) &= (x, (g, b)(h, a)^{-1}) \\ &= (x, gh^{-1}, h \cdot b + f(g, h^{-1}) - f(h, h^{-1}) - h \cdot a) \end{aligned}$$

Claim 1. $\Phi \cdot T_{(h,a)} = T_{\theta(h,a)} \cdot \Phi$

We compute

$$\begin{aligned} \Phi \cdot T_{(h,a)}(x, g, b) &= (Rx, k(x)gh^{-1}, f(g, h^{-1}) + h \cdot b - f(h, h^{-1}) - h \cdot a \\ &\quad + hg^{-1} \cdot q(x, e) - \varphi(hg^{-1}) + f(k(x), gh^{-1})) \end{aligned}$$

and

$$\begin{aligned} T_{\theta(h,a)} \cdot \Phi(x, g, b) &= (Rx, k(x)gh^{-1}, h \cdot b + hg^{-1} \cdot q(x, e) - h \cdot \varphi(g^{-1}) + h \cdot f(k(x), g) \\ &\quad + f(k(x)g, h^{-1}) - f(h, h^{-1}) - h \cdot a + h \cdot \varphi(h^{-1})) \end{aligned}$$

Note that $f(k(x), gh^{-1}) = -f(g, h^{-1}) + h \cdot f(k(x), g) + f(k(x)g, h^{-1})$. To compare the third coordinates in $\varphi \cdot T_{(h,a)}(x, g, b)$ and $T_{\theta(h,a)} \cdot \Phi(x, g, b)$, we notice that their difference is equal to $-\varphi(hg^{-1}) + h \cdot \varphi(g^{-1}) - h \cdot \varphi(h^{-1}) = 0$. The claim is proved.

Claim 2. Let $x \in X$ and let γ and $\gamma' \in \Gamma$, be such that $\gamma'Rx = R\gamma x$. Then $\Phi(\gamma(\pi_1)(x, h, a)) = \gamma'(\pi)\Phi(x, h, a)$.

To show this, set $y = \gamma x$. Then

$$\begin{aligned} \Phi(\gamma(\pi_1)(x, h, a)) &= \Phi(y, c_1(x, \gamma)h, b_{\alpha_1}(x, h, \gamma(c_1)) + a) \\ &= (Ry, k(y)c_1(x, \gamma)h, b_{\alpha_1}(x, h, \gamma(c_1)) + a + q(y, c_1(x, \gamma)h)) \end{aligned}$$

and

$$\begin{aligned} \gamma'(\pi)\Phi(x, h, a) &= \gamma'(\pi)(Rx, k(x)h, a + q(x, h)) \\ &= (\gamma'Rx, c(Rx, \gamma')k(x)h, b_{\alpha}(Rx, k(x)h, \gamma'(c)) + a + q(x, h)). \end{aligned}$$

Since $\gamma'(c) = R'\gamma(c_1)(R')^{-1}$, we have that

$$b_{\alpha}(Rx, k(x)h, \gamma'(c)) - b_{\alpha_1}(x, h, \gamma(c_1)) = q(\gamma x, c_1(x, \gamma)h) - q(x, h).$$

Thus the proof of the claim is complete.

Now let us return to the theorem proof. Claims 1 and 2 imply that the map Φ sends every $\Gamma(\pi_1)$ -ergodic component to a $\Gamma(\pi)$ -ergodic component. We denote by $\tilde{\Phi}$ the map, induced by Φ , from the quotient space $(X \times E_f)/\xi(\Gamma(\pi))$ onto $(X \times E_f)/\xi(\Gamma(\pi_1))$ where $\xi(\Gamma(\pi))$ and $\xi(\Gamma(\pi_1))$ are partitions into ergodic components of $\Gamma(\pi)$ and $\Gamma(\pi_1)$ respectively. Then the map $\tilde{\Phi}$ implements the conjugacy between the Mackey actions, that is,

$$\tilde{\Phi}W_{(\Gamma, \pi_1)}(h, a) = \theta^*(W)_{(\Gamma, \pi)}(\tilde{\Phi}(h, a)), \quad \text{for all } (h, a) \in E_f. \quad \square$$

The following statement is an immediate consequence of Theorems 3.2 and 3.4. Recall that any ergodic nonsingular action V of A can be represented as the Mackey action $W_{(\Gamma(c), d)}(A)$ where $\Gamma(c)$ is an ergodic a.f. measure preserving group of automorphisms, $c : X \times \Gamma \rightarrow G$ is a cocycle with dense range and $d \in Z^1(X \times G \times \Gamma(c), A)$. We consider the case of a countable group G .

Theorem 3.5 *Assume that two ergodic nonsingular actions V and V_1 of A are isomorphic, and they are extendible to actions of $E_f(G, A)$. This means that for the corresponding Mackey actions $W_{(\Gamma(c), d)}(A)$ and $W_{(\Gamma(c_1), d_1)}(A)$, there exists an automorphism R' on $X \times G$ such that $R'[\Gamma(c_1)](R')^{-1} = [\Gamma(c)]$ and $d \circ R'$ is $\Gamma(c_1)$ -cohomologous to d_1 . Then the actions of E_f extended from $W_{(\Gamma(c), d)}(A)$ and $W_{(\Gamma(c_1), d_1)}(A)$ are isomorphic if and only if R' is of the skew product form, i.e. $R'(x, h) = (Rx, k(x)h)$.*

Proof. It follows from Theorem 2.5 that the Mackey actions $W_{(\Gamma(c), d)}(A)$ and $W_{(\Gamma(c_1), d_1)}(A)$ can be chosen such that $d \in I(f, c)$ and $d_1 \in I(f, c_1)$, i.e. $d = b_\alpha$ and $d_1 = b_{\alpha_1}$. Therefore, the automorphism R' satisfies the conditions of Theorem 3.4, and we get that the theorem holds. \square

Remark 3.6 It is not difficult to point out nonisomorphic actions of $E_f(G, A)$ that have isomorphic actions of A . Assume, for simplicity, that G and A are countable groups. For every $f \in Z^2(G, A)$ consider a Bernoulli action U of E_f with infinite entropy. Then the subgroup $U(e, A)$ also has infinite entropy. Therefore if f_1 and f_2 non-cohomologous (i.e. E_{f_1} and E_{f_2} are nonisomorphic) the corresponding Bernoulli actions $U_1(e, A)$ and $U_2(e, A)$ of A are isomorphic.

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