Hodge classes on abelian varieties of low dimension

B. J. J. Moonen* and Yu. G. Zarhin**

Introduction.

In this paper we study Hodge classes on complex abelian varieties X. If $\dim(X) \leq 3$ then it is well-known that every Hodge class on X is a linear combination of products of divisor classes. In [10] the authors showed that if X is simple of dimension 4 then every Hodge class is a linear combination of products of divisor classes and Weil classes—if there are any. (The notion of a Weil class shall be briefly reviewed in (1.8).)

The aim of this note is to extend this to arbitrary abelian varieties of dimension ≤ 5. In order to state our main results, let us describe some special cases.

(a) The abelian variety X is isogenous to a product $X_1 \times X_2$ where X_1 is an elliptic curve with complex multiplication by an imaginary quadratic field k and where X_2 is a simple abelian threefold such that there exists an embedding $k \hookrightarrow \mathrm{End}^0(X_2)$.

(b) The abelian variety X is simple of dimension 4 such that $\mathrm{End}^0(X)$ is a field containing an imaginary quadratic field k which acts on the tangent space $T_{X,0}$ with multiplicities $(2,2)$. (See §1 for further explanation.)

(c) The abelian variety X is simple of dimension 4 with $D = \mathrm{End}^0(X)$ a definite quaternion algebra over $\mathbb{Q}$. (Type III in the Albert classification.) Note that for every $\alpha \in D \setminus \mathbb{Q}$ the subalgebra $\mathbb{Q}(\alpha) \subset D$ is an imaginary quadratic field.

(d) The abelian variety X is simple of dimension 4 with $\mathrm{End}^0(X) = \mathbb{Q}$.

(0.1) Theorem. *Let X be a complex abelian variety with $\dim(X) \leq 4$. Write $V = H_1(X(\mathbb{C}), \mathbb{Q})$ and let $\varphi\colon V \times V \to \mathbb{Q}$ be the Riemann form associated to a polarization of X. Write $D = \mathrm{End}^0(X)$ and let $\mathrm{Sp}_D(V, \varphi)$ denote the centralizer of D inside the symplectic group $\mathrm{Sp}(V, \varphi)$.*

(i) *Suppose we are in case* (a) *or* (b). *Then the Hodge ring $\mathcal{B}^\bullet(X)$ is generated by the subalgebra $\mathcal{D}^\bullet(X)$ of divisor classes together with the space of Weil classes $W_k \subset \mathcal{B}^2(X)$. The Hodge group $\mathrm{Hg}(X)$ is strictly contained in $\mathrm{Sp}_D(V, \varphi)$.*

(ii) *Suppose we are in case* (c). *Then $\mathrm{Hg}(X) = \mathrm{Sp}_D(V, \varphi)$. The Hodge ring $\mathcal{B}^\bullet(X)$ is generated by the divisor classes together with the spaces of Weil classes $W_k \subset \mathcal{B}^2(X)$, where k runs through the set of imaginary quadratic fields contained in D.*

(iii) *Suppose we are in case* (d). *Then the Hodge ring $\mathcal{B}^\bullet(X)$ is generated by divisor classes, i.e., $\mathcal{B}^\bullet(X) = \mathcal{D}^\bullet(X)$. Either $\mathrm{Hg}(X) = \mathrm{Sp}(V, \varphi)$, in which case $\mathcal{B}^\bullet(X^n) = \mathcal{B}^\bullet(X^n)$ for all n, or $\mathrm{Hg}(X)$ is isogenous to a $\mathbb{Q}$-form of $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$, in which case there are exceptional Hodge classes in $\mathcal{B}^2(X^2)$. In the latter case these exceptional Hodge classes are not of Weil type.*

(iv) *Suppose we are not in one of the cases* (a), (b), (c) *or* (d). *Then $\mathrm{Hg}(X) = \mathrm{Sp}_D(V, \varphi)$ and $\mathcal{B}^\bullet(X^n) = \mathcal{D}^\bullet(X^n)$ for all n.*

Let us note that in the cases (a), (b) and (c) the Weil classes are really needed to generate the Hodge ring; in these cases we have $\mathcal{D}^2(X) \neq \mathcal{B}^2(X)$. See [11], especially Example 8 and Criterion 13.

---

* Research made possible by a fellowship of the Royal Netherlands Academy of Arts and Sciences.
** Supported by the National Science Foundation.

We have a similar result for abelian varieties of dimension 5. Again we have to single out some special cases.

(e) The abelian variety X is isogenous to a product $X_1^2 \times X_2$, where X_1 and X_2 are as in (a).

(f) The abelian variety X is isogenous to a product $X_0 \times X_1 \times X_2$, where X_0 is an elliptic curve, where X_1 and X_2 are as in (a), and such that X_0 and X_1 are not isogenous.

(g) The abelian variety X is isogenous to a product $X_1 \times X_2$ where X_1 is an elliptic curve with complex multiplication by an imaginary quadratic field k and where X_2 is a simple abelian fourfold such that there exists an embedding $k \hookrightarrow \mathrm{End}^0(X_2)$ via which k acts on $T_{X_2,0}$ with multiplicities $(1,3)$.

(0.2) Theorem. *Let X be a complex abelian variety of dimension 5. Let V, φ, D and $\mathrm{Sp}_D(V,\varphi)$ have the same meaning as in* (0.1).

(i) *Suppose we are in case* (e). *Consider the space of Weil classes $W_k \subset \mathcal{B}^2(X_1 \times X_2)$ and write $W_{k,\alpha} \subset \mathcal{B}^2(X)$ for its image under the map $\mathcal{B}^2(X_1 \times X_2) \to \mathcal{B}^2(X)$ induced by a surjective homomorphism $\alpha\colon X \to X_1 \times X_2$. Then the Hodge ring $\mathcal{B}^\bullet(X)$ is generated by the subalgebra $\mathcal{D}^\bullet(X)$ of divisor classes together with the subspaces $W_{k,\alpha}$. The Hodge group $\mathrm{Hg}(X)$ is strictly contained in $\mathrm{Sp}_D(V,\varphi)$.*

(ii) *Suppose we are in case* (f). *Then $\mathrm{Hg}(X) = \mathrm{Hg}(X_0) \times \mathrm{Hg}(X_1 \times X_2)$. For every $n \geq 1$ the Hodge ring $\mathcal{B}^\bullet(X^n)$ is generated by the images of $\mathcal{B}^\bullet(X_0^n)$ and $\mathcal{B}^\bullet(X_1^n \times X_2^n)$. In particular, $\mathcal{B}^\bullet(X)$ is generated by the divisor classes $\mathcal{D}^\bullet(X)$ together with the pull-backs of the Weil classes in $W_k \subset \mathcal{B}^2(X_1 \times X_2)$.*

(iii) *Suppose we are in case* (g). *Then the Hodge ring $\mathcal{B}^\bullet(X)$ is generated by divisor classes, i.e., $\mathcal{B}^\bullet(X) = \mathcal{D}^\bullet(X)$. The Hodge group $\mathrm{Hg}(X)$ is strictly contained in $\mathrm{Sp}_D(V,\varphi)$.*

(iv) *Suppose we are not in one of the cases* (e), (f) *or* (g). *Decompose X, up to isogeny, as a product of elementary abelian varieties, say $X \sim Y_1^{m_1} \times \cdots \times Y_r^{m_r}$. Then $\mathrm{Hg}(X) = \mathrm{Hg}(Y_1^{m_1}) \times \cdots \mathrm{Hg}(Y_r^{m_r})$. For every $n \geq 1$ the Hodge ring $\mathcal{B}^\bullet(X^n)$ is generated by the images of the Hodge rings $\mathcal{B}^\bullet(Y_j^{m_j})$. In particular, if X has no simple factor of dimension 4 then $\mathrm{Hg}(X) = \mathrm{Sp}_D(V,\varphi)$ and $\mathcal{B}^\bullet(X^n) = \mathcal{D}^\bullet(X^n)$ for every $n \geq 1$.*

In the cases (e) and (f) the pull-backs of the Weil classes are needed to generate the Hodge ring of X; we have $\mathcal{D}^2(X) \neq \mathcal{B}^2(X)$ and $\mathcal{D}^3(X) \neq \mathcal{B}^3(X)$. In the decomposition (up to isogeny) $X \sim Y_1^{m_1} \times \cdots \times Y_r^{m_r}$ in (iv) we require the Y_j to be simple, pairwise non-isogenous, and the m_j are positive integers.

For *simple* abelian varieties the above results were already known: the case $\dim(X) = 4$ was treated in [10]; for simple X with $\dim(X)$ a prime number one uses a result of Tankeev together with a theorem of Hazama and Murty. (The relevant statements are recalled in (1.7) and (2.7) below.) In the present paper we are therefore mainly concerned with non-simple abelian varieties. We prove some lemmas which in certain cases allow us to determine the Hodge group of a product $X_1 \times X_2$, knowing the Hodge groups $\mathrm{Hg}(X_i)$ of the factors. Using these results we shall determine the Hodge groups of all complex abelian varieties X with $\dim(X) \leq 5$ (again referring to [10], [22], [17] for the results in case X is simple).

The paper is organised as follows. In the first section we review the notion of a Hodge group and we recall a number of properties that we shall use. In §2 we give an overview of the situation for simple abelian varieties of low dimension. In §3 we prove a couple of general lemmas which allow us to analyse certain product situations. In §4 we analyse Hodge groups of simple abelian surfaces of CM-type. Putting everything together the main theorems are proven in §5.

§1. Hodge groups of abelian varieties.

(1.1) Let X be an abelian variety over an algebraically closed field k. Set $D = \mathrm{End}^0(X) := \mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. A polarization of X induces a positive (Rosati-) involution, say $d \mapsto d^\dagger$, of D.

Now assume that X is simple. Then D is a division algebra and we have $D \supset F \supset F_0 \supset \mathbb{Q}$ with

$$F = \mathrm{Cent}(D)\,, \qquad F_0 = \{a \in F \mid a^\dagger = a\}\,.$$

We write

$$e_0 = [F_0 : \mathbb{Q}]\,, \qquad e = [F : \mathbb{Q}]\,, \qquad d^2 = [D : F]\,.$$

By the classification due to Albert (see [13], § 21) the division algebra D is of one of the following types.

Type I(e_0): $e = e_0$, $d = 1$; $D = F = F_0$ is a totally real field.

Type II(e_0): $e = e_0$, $d = 2$; D is a quaternion algebra over a totally real field $F = F_0$; D splits at all infinite places.

Type III(e_0): $e = e_0$, $d = 2$; D is a quaternion algebra over a totally real field $F = F_0$; D is inert at all infinite places.

Type IV(e_0, d): $e = 2e_0$; F is a CM-field with totally real subfield F_0; D is a division algebra of rank d^2 over F.

We say that a (simple) abelian variety X is of Type $A(?)$ (with $A \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$) if $\mathrm{End}^0(X)$ is an algebra of the corresponding type.

We refer to [15] for results about which algebras in the Albert classification occur as the endomorphism algebra of an abelian variety. (Note that there is a misprint in Table 8.1 of [15]; the author informs us that in the last line of this table it should read: "occurs if and only if $2g/ed^2 \geq 1$ but *excluded* IV$(1,1)$, $g = 2$ and IV$(1,1)$, $g = 4$.")

(1.2) Let X be a complex abelian variety, $X \neq 0$. We write $V = V_X = H_1(X(\mathbb{C}), \mathbb{Q})$, which is a polarizable $\mathbb{Q}$-Hodge structure of type $(-1,0) + (0,-1)$. This Hodge structure can be described by giving a homomorphism of algebraic groups over $\mathbb{R}$

$$h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}\,,$$

where $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$.

The Mumford-Tate group $\mathrm{MT}(X)$ of X is defined to be the smallest algebraic subgroup $M \subset \mathrm{GL}(V)$ (over $\mathbb{Q}$) such that h factors through $M_{\mathbb{R}}$. In practice it is often more convenient to work with the *Hodge group* $\mathrm{Hg}(X)$. We can define it by $\mathrm{Hg}(X) = \mathrm{MT}(X) \cap \mathrm{SL}(V)$. For a more direct definition, consider the $\mathbb{R}$-subtorus $U^1 \subset \mathbb{S}$ given on points by

$$U^1(\mathbb{R}) = \{z \in \mathbb{C}^* \mid z\bar{z} = 1\} \subset \mathbb{C}^* = \mathbb{S}(\mathbb{R})\,.$$

Then $\mathrm{Hg}(X)$ is the smallest algebraic subgroup $H \subset \mathrm{GL}(V)$ such that the restriction of h to U^1 factors through $H_{\mathbb{R}}$.

The Mumford-Tate group $\mathrm{MT}(X)$ contains the torus $\mathbb{G}_{m,\mathbb{Q}} \subset \mathrm{GL}(V)$ of homotheties. The group $\mathrm{MT}(X)$ is the almost direct product of $\mathbb{G}_{m,\mathbb{Q}}$ and $\mathrm{Hg}(X)$.

The Hodge group $\mathrm{Hg}(X)$ is a connected reductive algebraic group. Viewing $D = \mathrm{End}^0(X)$ as a subalgebra of $\mathrm{End}_{\mathbb{Q}}(V)$ we have $D = \mathrm{End}_{\mathbb{Q}}(V)^{\mathrm{Hg}(X)}$. If $\varphi\colon V \times V \to \mathbb{Q}$ is the Riemann form associated to a polarization of X (so φ is a symplectic form) then

$$\mathrm{Hg}(X) \subset \mathrm{Sp}_D(V, \varphi)\,,$$

the centralizer of D in the symplectic group $\mathrm{Sp}(V, \varphi)$.

The Hodge group $\mathrm{Hg}(X)$ is a torus if and only if X is of CM-type. If X has no factors of Type IV then $\mathrm{Hg}(X)$ is semi-simple. (See [12], §2 and [21], Lemma 1.4.)

For $n \geq 1$ we can identify $\mathrm{Hg}(X^n)$ with $\mathrm{Hg}(X)$, acting diagonally on $V_{X^n} = (V_X)^n$. More generally, if $n_1, \ldots, n_r \in \mathbb{Z}_{\geq 1}$ then we can identify $\mathrm{Hg}(X_1^{n_1} \times \cdots \times X_r^{n_r})$ with $\mathrm{Hg}(X_1 \times \cdots \times X_r)$.

(1.3) Write $\mathfrak{hg}(X)$ for the Lie algebra of $\mathrm{Hg}(X)$. If W is a $\mathrm{Hg}(X)$-module then $W^{\mathrm{Hg}(X)} = W^{\mathfrak{hg}(X)}$, since $\mathrm{Hg}(X)$ is connected. Thus, for instance, $\mathrm{End}^0(X)$ can be computed as the $\mathfrak{hg}(X)$-invariants in $\mathrm{End}_{\mathbb{Q}}(V)$.

The following description of $\mathfrak{hg}(X)$ proves to be very useful. We have a Hodge decomposition $V_{\mathbb{C}} = V_{\mathbb{C}}^{-1,0} \oplus V_{\mathbb{C}}^{0,-1}$. Let the endomorphism $J = J_X \in \mathrm{End}(V_{\mathbb{C}})$ be given by

$$J_X(v) = \begin{cases} iv, & \text{if } v \in V_{\mathbb{C}}^{-1,0}, \\ -iv, & \text{if } v \in V_{\mathbb{C}}^{0,-1}. \end{cases}$$

Note that $J_X^2 = -\mathrm{id}$. Then $\mathfrak{hg}(X) \subset \mathrm{End}(V)$ is the smallest $\mathbb{Q}$-Lie subalgebra $\mathfrak{h} \subset \mathrm{End}(V)$ such that $\mathfrak{h}_\mathbb{C}$ contains J_X; see [26]. In fact, since $V_\mathbb{C}^{-1,0}$ and $V_\mathbb{C}^{0,-1}$ are complex conjugate we even have $J_X \in \mathfrak{hg}(X)_\mathbb{R}$.

(1.4) The cohomology ring $H^\bullet(X,\mathbb{Q})$ is naturally isomorphic to the exterior algebra on $V^\vee$. The Hodge group $\mathrm{Hg}(X)$ acts on this ring. The $\mathrm{Hg}(X)$-invariants in $H^\bullet(X,\mathbb{Q})$ are precisely the Hodge classes. Writing $\mathcal{B}^i(X) \subset H^{2i}(X,\mathbb{Q})$ for the subspace of Hodge classes we obtain a graded $\mathbb{Q}$-algebra $\mathcal{B}^\bullet(X) = \oplus_i \mathcal{B}^i(X)$, called the Hodge ring of X.

The Hodge classes in $H^2(X,\mathbb{Q})$ (i.e., the elements of $\mathcal{B}^1(X) = H^2(X,\mathbb{Q})^{\mathrm{Hg}(X)}$) are called the divisor classes. We write $\mathcal{D}^\bullet(X) \subset \mathcal{B}^\bullet(X)$ for the $\mathbb{Q}$-subalgebra generated by the divisor classes. The Hodge classes in $\mathcal{D}^\bullet(X)$ are called the *decomposable* Hodge classes. The elements of $\mathcal{B}^\bullet(X)$ not in $\mathcal{D}^\bullet(X)$ are called *exceptional* Hodge classes.

(1.5) Let $\mathfrak{h}$ be a reductive Lie algebra over $\mathbb{Q}$. Consider a semi-simple (finite dimensional) representation $\rho\colon \mathfrak{h} \to \mathrm{End}(V)$. The Lie algebra $\mathfrak{h} \otimes \mathbb{C}$ decomposes as a direct sum $\mathfrak{h}_\mathbb{C} = \mathfrak{c} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$, where $\mathfrak{c}$ is its center and where the $\mathfrak{g}_i$ are its simple factors. Let W be an irreducible $\mathfrak{h}_\mathbb{C}$-submodule of $V_\mathbb{C}$. It decomposes as an external tensor product $W = \chi_0 \boxtimes W_1 \boxtimes \cdots \boxtimes W_r$, where χ_0 is a character of $\mathfrak{c}$ and where W_i is an irreducible representation of $\mathfrak{g}_i$. We shall say that ρ is *of length 1* if

(i) all simple factors $\mathfrak{g}_i$ are of classical type A_ℓ, B_ℓ, C_ℓ or D_ℓ,

and if for every irreducible $\mathfrak{h}_\mathbb{C}$-submodule W of $V_\mathbb{C}$ we have

(ii) precisely one of the representations W_i is non-trivial,

(iii) if W_i is a non-trivial $\mathfrak{g}_i$-module then its highest weight (w.r.t. a chosen Cartan subalgebra of $\mathfrak{g}_i$ and a choice of a basis for the root system) is miniscule in the sense of [2], Chap. 8, §7, n° 3.

(This is to be compared with [25], where the length of an irreducible representation of a simple Lie algebra is defined.)

Now let X be a complex abelian variety, and consider the tautological representation $\rho\colon \mathfrak{hg}(X) \to \mathrm{End}(V_X)$. The fact that V_X is a Hodge structure of level 1 implies that ρ is a symplectic representation of length 1; see [3]. See also [16], §4 and [25].

Let us note that if $\rho\colon \mathfrak{h} \to \mathrm{End}(V)$ is a representation of length 1 and if $\mathfrak{g}$ is an ideal of $\mathfrak{h}$ then $V = V^\mathfrak{g} \oplus V'$ where $V' \subseteq V$ is a $\mathfrak{g}$-submodule such that the representation $\rho'\colon \mathfrak{g} \to \mathrm{End}(V')$ is again of length 1.

(1.6) Remark. Later in the paper we shall consider $\mathbb{Q}$-Lie algebras $\mathfrak{h}$ for which there is a unique faithful irreducible representation (up to isomorphism) which is of length 1. For instance, let $\mathfrak{h}$ be a simple $\mathbb{Q}$-Lie algebra. Then there exists a number field K and an absolutely simple K-Lie algebra $\mathfrak{g}$ such that $\mathfrak{h} \cong \mathrm{Res}_{K/\mathbb{Q}}\, \mathfrak{g}$. Writing Σ_K for the set of embeddings of K into $\mathbb{C}$ we have $\mathfrak{h}_\mathbb{C} = \oplus_{\sigma\in\Sigma_K} \mathfrak{g}_{(\sigma)}$, where $\mathfrak{g}_{(\sigma)} = \mathfrak{g} \otimes_{K,\sigma} \mathbb{C}$. We claim that if the (absolute) root system of $\mathfrak{g}$ is of type C_ℓ ($\ell \geq 1$) then $\mathfrak{h}$ has a unique irreducible representation of length 1.

To see this, let us first remark that a simple Lie algebra of type C_ℓ ($\ell \geq 1$) over $\mathbb{C}$ has a unique irreducible representation with miniscule highest weight, see [2], Chap. 8, §7, n° 3. Now write $\Sigma_K = \{\sigma_1, \ldots, \sigma_r\}$ and let $V_{(i)}$ ($1 \leq i \leq r$) be the irreducible $\mathfrak{h}_\mathbb{C}$-module which is irreducible as a $\mathfrak{g}_{(\sigma_i)}$-module with miniscule highest weight, and on which the factors $\mathfrak{g}_{(\sigma_j)}$ with $i \neq j$ act trivially. If $\rho\colon \mathfrak{h} \to \mathrm{End}(V)$ is an irreducible representation of length 1 then

$$V_\mathbb{C} \cong V_{(1)}^{m_1} \oplus \cdots \oplus V_{(r)}^{m_r}$$

for certain multiplicities m_i. But if L is the normal closure of K inside $\mathbb{C}$ then $\mathrm{Gal}(L/\mathbb{Q})$ permutes the factors $\mathfrak{g}_{(\sigma_i)}$ transitively, and it follows from the fact that $V_\mathbb{C}$ is defined over $\mathbb{Q}$ that we must have $m_1 = m_2 = \cdots = m_r$. Therefore, if $\rho'\colon \mathfrak{h} \to \mathrm{End}(V')$ is another irreducible representation of length 1 then there is a relation $(\rho_\mathbb{C})^M \cong (\rho'_\mathbb{C})^N$ for certain integers M and N. But this is possible only if $\rho \cong \rho'$.

That, conversely, every $\mathfrak{h}$ as above has an irreducible (symplectic) representation of length 1 can be seen from the description of such $\mathfrak{h}$'s in terms of algebras with involution, as in [8], Chap. X.

(1.7) Consider the following condition on the complex abelian variety X:

(D) $$\mathcal{B}^\bullet(X^n) = \mathcal{D}^\bullet(X^n) \qquad \text{for all } n\,.$$

If this condition is satisfied then the Hodge conjecture is "trivially" true for all X^n.

As recalled above, the Hodge group $\mathrm{Hg}(X)$ is contained in the algebraic group $\mathrm{Sp}_D(V,\varphi)$. It was shown by Hazama [5] and Murty [14] (independently) that

$$\mathrm{Hg}(X) = \mathrm{Sp}_D(V,\varphi) \quad \Longleftrightarrow \quad \begin{pmatrix} X \text{ has no factors of type III} \\ \textit{and} \\ \mathcal{D}^\bullet(X^n) = \mathcal{B}^\bullet(X^n) \text{ for all } n \end{pmatrix}.$$

(1.8) Let K be a subfield of $\mathrm{End}^0(X)$, with $1 \in K$ acting as the identity on X. Write Σ_K for the set of embeddings of K into $\mathbb{C}$. Let $T_{X,0}$ be the tangent space of X at the origin. The action of (an order of) K on X makes $T_{X,0}$ into a module under $K \otimes_{\mathbb{Q}} \mathbb{C} = \prod_{\sigma \in \Sigma_K} \mathbb{C}$. This gives a decomposition

$$T_{X,0} = \bigoplus_{\sigma \in \Sigma_K} T^{(\sigma)}.$$

Let $n_\sigma = \dim_{\mathbb{C}} T^{(\sigma)}$. If $\overline{\sigma}\colon K \to \mathbb{C}$ is the complex conjugate of σ then $n_\sigma + n_{\overline{\sigma}} = r := 2\dim(X)/[K:\mathbb{Q}]$.

If K is imaginary quadratic then we say that it acts on $T_{X,0}$ with multiplicities (a,b) if $n_\sigma = a$, $n_{\overline{\sigma}} = b$ for some ordering $\Sigma_K = \{\sigma, \overline{\sigma}\}$.

The inclusion $K \subset \mathrm{End}^0(X)$ induces on V_X the structure of an r-dimensional K-vector space. The 1-dimensional K-vector space $W_K = W_K(X) := \wedge_K^r V_X^\vee$ can be identified in a natural way with a subspace of $H^r(X,\mathbb{Q})$; we call W_K the space of Weil classes w.r.t. K. It is known that either W_K consists entirely of Hodge classes or $0 \in W_K$ is the only Hodge class in W_K. Whether W_K consists of Hodge classes and, if so, whether these classes are exceptional or not, can be answered purely in terms of the data $K \subset \mathrm{End}^0(X)$ and the action of K on $T_{X,0}$, see [11]. For instance, it is shown there that W_K consists of Hodge classes if and only if $n_\sigma = n_{\overline{\sigma}}$ for all $\sigma \in \Sigma_K$.

§2. Simple abelian varieties of dimension ≤ 5.

We shall give a short overview of the situation for simple complex abelian varieties of low dimension. Thus, in this section we shall assume X to be *simple*.

For $g := \dim(X) \leq 3$ and $g = 5$ we always find that $\mathrm{Hg}(X) = \mathrm{Sp}_D(V,\varphi)$. Since type III does not occur for $g \leq 3$ and $g = 5$ (X simple!), it follows that $\mathcal{B}^\bullet(X^n) = \mathcal{D}^\bullet(X^n)$ for all n. (See (1.7).) In particular the Hodge conjecture is true for all such X^n. A useful references for the results stated below is [17].

We shall give an overview of the cases that occur. If F is a CM-field with totally real subfield F_0 and complex conjugation $x \mapsto \bar{x}$ then we shall write U_F for the algebraic torus over $\mathbb{Q}$ given on points by

$$\mathrm{U}_F(R) = \{x \in (F \otimes_{\mathbb{Q}} R)^* \mid x\bar{x} = 1\}.$$

(2.1) g=1. There are two cases to distinguish.

Type I(1): X is an elliptic curve with $\mathrm{End}^0(X) = \mathbb{Q}$. Then $\mathrm{Hg}(X) = \mathrm{Sp}(V,\varphi) \cong \mathrm{SL}_{2,\mathbb{Q}}$.

Type IV(1,1): X is an elliptic curve with CM by an imaginary quadratic field F. Then $\mathrm{Hg}(X) = \mathrm{U}_F$.

(2.2) g=2. There are four cases.

Type I(1): X is an abelian surface with $\mathrm{End}^0(X) = \mathbb{Q}$. Then $\mathrm{Hg}(X) = \mathrm{Sp}(V,\varphi) \cong \mathrm{Sp}_{4,\mathbb{Q}}$.

Type I(2): $\mathrm{End}^0(X) = F$ is a real quadratic field. Then there is a unique F-symplectic form $\psi\colon V \times V \to F$ such that $\varphi = \mathrm{tr}_{F/\mathbb{Q}}\psi$. The Hodge group is given by $\mathrm{Hg}(X) = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{Sp}_F(V,\psi)$.

Type II(1): $D = \mathrm{End}^0(X)$ is a quaternion algebra over $\mathbb{Q}$, split at ∞. Write D^{opp} for the opposite algebra, and let $x \mapsto x^*$ be the canonical involution. Then $\mathrm{Hg}(X)$ is the algebraic group $\mathrm{U}_{D^{\mathrm{opp}}}$ given on points by $\mathrm{U}_{D^{\mathrm{opp}}}(\mathbb{Q}) = \{x \in (D^{\mathrm{opp}})^* \mid xx^* = 1\}$.

Type IV(2,1): $\mathrm{End}^0(X) = F$ is a quartic CM-field not containing an imaginary quadratic subfield. We have $\mathrm{Hg}(X) = \mathrm{U}_F$.

(2.3) *g*=3. There are four cases.

Type I(1): X is an abelian 3-fold with $\mathrm{End}^0(X) = \mathbb{Q}$. Then $\mathrm{Hg}(X) = \mathrm{Sp}(V, \varphi) \cong \mathrm{Sp}_{6,\mathbb{Q}}$.

Type I(3): $\mathrm{End}^0(X) = F$ is a totally real cubic field. There is a unique F-symplectic form $\psi: V \times V \to F$ such that $\varphi = \mathrm{tr}_{F/\mathbb{Q}}\psi$. The Hodge group is given by $\mathrm{Hg}(X) = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{Sp}_F(V, \psi)$.

Type IV(1,1): $\mathrm{End}^0(X) = F$ is an imaginary quadratic field; given $a \in F$ with $\bar{a} = -a$ there is a unique F-hermitian form $\psi: V \times V \to F$ such that $\varphi = \mathrm{tr}_{F/\mathbb{Q}}(a \cdot \psi)$ and $\mathrm{Hg}(X) = \mathrm{U}_F(V, \psi)$.

Type IV(3,1): $\mathrm{End}^0(X) = F$ is a CM-field of degree 6 over $\mathbb{Q}$. Then $\mathrm{Hg}(X) = \mathrm{U}_F$.

The following proposition is easily read off from the above, using Remark (1.6) and Lemma (3.7) below.

(2.4) Proposition. *Let X be a simple complex abelian variety with $g = \dim(X) \leq 3$.*

(i) *Suppose X is of CM-type. Then $\mathrm{Hg}(X)$ is a g-dimensional algebraic torus. It is $\mathbb{Q}$-simple, except when $\dim(X) = 3$ and the sextic CM-field $\mathrm{End}^0(X)$ contains an imaginary quadratic field.*

(ii) *Suppose X is not of CM-type. Then $\mathrm{Hg}(X)$ is a $\mathbb{Q}$-simple algebraic group, except when $\dim(X) = 3$ and $\mathrm{End}^0(X)$ is an imaginary quadratic field. If $\mathrm{Hg}(X)$ is $\mathbb{Q}$-simple then (up to isomorphism) there is exactly one faithful irreducible representation of $\mathfrak{hg}(X)$ over $\mathbb{Q}$ which is of length 1.*

(2.5) *g*=4. The case $g = 4$ is more involved and was studied in [10]. In particular, in op. cit. we already proved Thm. (0.1) for simple abelian fourfolds. (This covers the cases (b), (c) and (d) of the introduction.) We here only recall some of the most interesting cases.

(i) For $g = 4$ it is no longer true that $\mathrm{Hg}(X)$ is determined by $\mathrm{End}^0(X)$ together with its action on the tangent space at the origin. Namely, if $g = 4$ and $\mathrm{End}^0(X) = \mathbb{Q}$ then either $\mathrm{Hg}(X) = \mathrm{Sp}(V, \varphi) \cong \mathrm{Sp}_{8,\mathbb{Q}}$, or $\mathrm{Hg}(X)$ is a $\mathbb{Q}$-form of an almost direct product of three copies of SL_2. (See [12].) In both cases the Hodge ring of X is generated by divisor classes, but if $\mathrm{Hg}(X)$ is isogenous to a $\mathbb{Q}$-form of SL_2^3 then there are exceptional Hodge classes in $H^4(X^2, \mathbb{Q})$.

(ii) For $g = 4$ we find cases where in addition to divisor classes we also need Weil classes to generate the Hodge ring. This happens if $\mathrm{End}^0(X)$ contains an imaginary quadratic field k which acts on the tangent space with multiplicities $(2, 2)$. If X is of Type III then this is the case (e.g., see [14], [11]); further it can occur only for X of Type IV(1,1) or of Type IV(4,1). Only in very special cases these Weil classes are known to be algebraic, see [18] and [23].

(2.6) *g*=5. As already stated above, $\mathrm{Hg}(X) = \mathrm{Sp}_D(V, \varphi)$ for all simple abelian 5-folds. The point here is that 5 is a prime number, since in fact we have the following result, due to Tankeev [22]. (See also Ribet's paper [17].)

(2.7) Theorem. *Let X be a simple complex abelian variety such that $\dim(X)$ is a prime number. Then $\mathrm{Hg}(X) = \mathrm{Sp}_D(V, \varphi)$ and $\mathcal{B}^\bullet(X^n) = \mathcal{D}^\bullet(X^n)$ for every $n \geq 1$.*

In connection with this result let us note that a simple X of prime dimension cannot be of Type III, so that the result of Hazama and Murty in (1.7) applies.

§3. The Hodge group of a product of abelian varieties.

(3.1) Let X_1 and X_2 be complex abelian varieties. Write $X = X_1 \times X_2$. Then $\mathrm{Hg}(X)$ is an algebraic subgroup of $\mathrm{Hg}(X_1) \times \mathrm{Hg}(X_2)$. The two projections $\mathrm{pr}_i \colon \mathrm{Hg}(X) \to \mathrm{Hg}(X_i)$ are surjective. From this one easily shows that there exist Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{g}_3$ and an automorphism φ of $\mathfrak{g}_3$ such that

$$\mathfrak{hg}(X_1) \cong \mathfrak{g}_1 \oplus \mathfrak{g}_3 , \qquad \mathfrak{hg}(X_2) \cong \mathfrak{g}_2 \oplus \mathfrak{g}_3 ,$$

and

$$\begin{array}{ccccc} \mathfrak{hg}(X_1 \times X_2) & & \subseteq & & \mathfrak{hg}(X_1) \oplus \mathfrak{hg}(X_2) \\ \wr\Vert & & & & \Vert\wr \\ \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \Gamma_\varphi & \subseteq & \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \oplus \mathfrak{g}_3 & = & (\mathfrak{g}_1 \oplus \mathfrak{g}_3) \oplus (\mathfrak{g}_2 \oplus \mathfrak{g}_3) \end{array} ,$$

where $\Gamma_\varphi \subseteq (\mathfrak{g}_3 \oplus \mathfrak{g}_3)$ is the graph of the automorphism φ.

We may have that

$$\mathrm{Hg}(X_1 \times X_2) \neq \mathrm{Hg}(X_1) \times \mathrm{Hg}(X_2) . \tag{1}$$

(I.e., $\mathfrak{g}_3 \neq 0$ in the above.) This holds if and only if for some m and n the Hodge ring $\mathcal{B}^\bullet(X_1^m \times X_2^n)$ is not generated by the elements coming from $\mathcal{B}^\bullet(X_1^m)$ and $\mathcal{B}^\bullet(X_2^n)$.

In certain cases one can show that an inequality (1) can only hold if $\mathrm{Hom}(X_1, X_2) \neq 0$. For instance, we have the following result of Hazama [6].

(3.2) Theorem. *Let X_1 and X_2 be complex abelian varieties which both satisfy condition* (D) *in* (1.7).

(i) *Suppose X_1 and X_2 contain no factors of Type IV. Then $X_1 \times X_2$ again satisfies* (D)*, and either* $\mathrm{Hom}(X_1, X_2) \neq 0$ *or* $\mathrm{Hg}(X_1 \times X_2) = \mathrm{Hg}(X_1) \times \mathrm{Hg}(X_2)$.

(ii) *Suppose X_1 has no factors of Type IV and X_2 is of CM-type. Then $X_1 \times X_2$ again satisfies* (D) *and* $\mathrm{Hg}(X_1 \times X_2) = \mathrm{Hg}(X_1) \times \mathrm{Hg}(X_2)$.

The next lemmas are aimed at proving similar conclusions in other cases.

(3.3) Lemma. *Let X be a complex abelian variety. Suppose that $\mathfrak{hg}(X)$ is semi-simple and that V_X is the only irreducible $\mathfrak{hg}(X)$-representation of length 1 (up to isomorphism). Let Y be a simple complex abelian variety such that $\mathfrak{hg}(Y)$ splits as $\mathfrak{hg}(Y) = \mathfrak{g} \oplus \mathfrak{h}$; correspondingly we can write $J_Y = J_1 + J_2$ with $J_1 \in \mathfrak{g}_{\mathbb{C}}$ and $J_2 \in \mathfrak{h}_{\mathbb{C}}$. Suppose there exists an isomorphism $\mathfrak{hg}(X) \xrightarrow{\sim} \mathfrak{g}$ with $J_X \mapsto J_1$. Then $\mathfrak{h} = 0$ and Y is isogenous to X.*

Proof. Write $D = \mathrm{End}^0(X)$ and $F = \mathrm{Cent}(D)$; set $e = [F : \mathbb{Q}]$ and $d^2 = \dim_F(D)$. We have $D \otimes_{\mathbb{Q}} \mathbb{C} \cong M_d(\mathbb{C})_{(1)} \times \cdots \times M_d(\mathbb{C})_{(e)}$. There are irreducible $\mathfrak{hg}(X)_{\mathbb{C}}$-modules $U_1, \ldots, U_e$, pairwise non-isomorphic, such that $V_X \otimes_{\mathbb{Q}} \mathbb{C} \cong U_1^d \oplus \cdots \oplus U_e^d$ as $\mathfrak{hg}(X)_{\mathbb{C}}$-modules.

As $\mathfrak{hg}(X)$ is semi-simple, the F-linear trace map $\mathrm{tr}_F \colon \mathfrak{hg}(X) \subset \mathrm{End}_F(V_X) \to F$ is zero. It follows that $\mathfrak{hg}(X)_{\mathbb{C}}$ acts on each of the summands U_j^d through $\mathfrak{sl}(U_j^d)$. In particular, on each of the summands U_j^d the operator J_X has $+i$ and $-i$ as its eigenvalues (as it has zero trace and satisfies $J_X^2 = -\mathrm{id}$).

Fix an isomorphism $\varphi \colon \mathfrak{hg}(X) \xrightarrow{\sim} \mathfrak{g}$ with $J_X \mapsto J_1$. Note that there are no non-trivial $\mathfrak{g}$-invariants in V_Y, as $(V_Y)^{\mathfrak{g}}$ is a $\mathfrak{hg}(Y)$-submodule of V_Y and Y is simple. The assumption that V_X is the only irreducible $\mathfrak{g}$-module of length 1 therefore implies that $V_Y \cong V_X^q$ as $\mathfrak{g}$-modules, for some $q \geq 1$. Then $\mathfrak{h}$ acts on V_Y through an embedding $\mathfrak{h} \hookrightarrow \mathrm{End}_{\mathfrak{g}}(V_Y) = M_q(D)$. Thus

$$V_{Y,\mathbb{C}} \cong U_1^{dq} \oplus \cdots \oplus U_e^{dq} ,$$

as $\mathfrak{g}_{\mathbb{C}}$-modules and each of the factors U_j^{dq} is stable under $\mathfrak{h}_{\mathbb{C}}$. If λ is an eigenvalue of J_2 on U_j^{dq} then we find that both $i + \lambda$ and $-i + \lambda$ occur as eigenvalues of J_Y on $U_j^{dq} \subseteq V_{Y,\mathbb{C}}$. By definition of J_Y this is possible only if $\lambda = 0$. We conclude that J_2 acts trivially on each factor U_j^{dq}. Hence $\mathfrak{h} = 0$.

The graph $\Gamma_\varphi \subset \mathfrak{hg}(X) \times \mathfrak{hg}(Y)$ is a $\mathbb{Q}$-Lie subalgebra such that $\Gamma_{\varphi,\mathbb{C}} \ni J_{X \times Y} = (J_X, J_Y)$. Therefore, $\mathfrak{hg}(X \times Y) = \Gamma_\varphi$ and some multiple of φ corresponds to an isogeny from X to Y. $\square$

(3.4) Lemma. *Let X_1 and X_2 be nonzero complex abelian varieties. Write $X = X_1 \times X_2$. Assume that $\mathfrak{hg}(X_2)$ is a $\mathbb{Q}$-simple Lie algebra and that V_{X_2} is the only irreducible $\mathfrak{hg}(X_2)$-module of length 1 (up to isomorphism). Then either* $\mathrm{Hg}(X) = \mathrm{Hg}(X_1) \times \mathrm{Hg}(X_2)$ *or* $\mathrm{Hom}(X_2, X_1) \neq 0$.

Proof. Assume that $\mathrm{Hg}(X) \neq \mathrm{Hg}(X_1) \times \mathrm{Hg}(X_2)$. Using the notations of (3.1) the assumption that $\mathfrak{hg}(X_2)$ is $\mathbb{Q}$-simple implies that $\mathfrak{hg}(X) = \mathfrak{g}_1 \oplus \mathfrak{g}_3 \xrightarrow{\sim} \mathfrak{hg}(X_1)$ and $\mathfrak{hg}(X_2) \cong \mathfrak{g}_3$.

There exists a *simple* abelian subvariety $Y \subset X_1$ such that the ideal $\mathfrak{g}_3 \subset \mathfrak{hg}(X_1)$ acts non-trivially on $V_Y \subset V_{X_1}$. There is a quotient $\mathfrak{g}_1'$ of $\mathfrak{g}_1$ such that $\mathfrak{hg}(Y) = \mathfrak{g}_1' \oplus \mathfrak{g}_3$. Notice that via $\mathfrak{hg}(Y) \xleftarrow{\sim} \mathfrak{hg}(Y \times X_2) = \mathfrak{g}_1' \oplus \mathfrak{g}_3 \twoheadrightarrow \mathfrak{hg}(X_2)$ we obtain an isomorphism $\mathfrak{hg}(X_2) \xrightarrow{\sim} \mathfrak{g}_3$ mapping J_{X_2} to the $\mathfrak{g}_3$-component of J_Y. Lemma (3.3) then gives $\mathrm{Hom}(X_2, Y) \neq 0$. □

(3.5) Remark. It was shown by Borovoi [1] that $\mathfrak{hg}(X)$ is $\mathbb{Q}$-simple if $\mathrm{End}^0(X) = \mathbb{Q}$. For a generalization of this result to absolutely irreducible Hodge structures of arbitrary level see [26].

(3.6) Lemma. *Let X_1 and X_2 be nonzero complex abelian varieties. Assume that the Hodge group* $\mathrm{Hg}(X_2)$ *is a $\mathbb{Q}$-simple algebraic torus. (In particular X_2 is a simple abelian variety of CM-type.) Write $X = X_1 \times X_2$. If* $\mathrm{Hg}(X) \neq \mathrm{Hg}(X_1) \times \mathrm{Hg}(X_2)$ *then the center of* $\mathrm{Hg}(X_1)$ *contains an algebraic torus which is $\mathbb{Q}$-isogenous to* $\mathrm{Hg}(X_2)$.

Proof. Suppose that $\mathrm{Hg}(X) \neq \mathrm{Hg}(X_1) \times \mathrm{Hg}(X_2)$. The assumption that $\mathrm{Hg}(X_2)$ is $\mathbb{Q}$-simple implies that $\mathfrak{hg}(X_2)$ does not contain a proper algebraic Lie subalgebra. Using the notations of (3.1) we then have that $\mathfrak{hg}(X) = \mathfrak{g}_1 \oplus \mathfrak{g}_3 \xrightarrow{\sim} \mathfrak{hg}(X_1)$ and $\mathfrak{hg}(X_2) \cong \mathfrak{g}_3$. This readily implies the lemma, noting that $\mathfrak{g}_1$ and $\mathfrak{g}_3$ are algebraic Lie subalgebras of $\mathfrak{hg}(X)$. □

Next let us recall a lemma from [9] that was also used in [10]. This lemma was also stated in [4], where it is attributed to Ribet. To formulate it, we need the following notation. Suppose F is a CM-field containing an imaginary quadratic field k. In §2 above we defined the algebraic torus U_F over $\mathbb{Q}$. The subfield $k \subset F$ gives rise to a subtorus $\mathrm{SU}_{F/k} \subset \mathrm{U}_F$ of codimension 1, by

$$\mathrm{SU}_{F/k} = \mathrm{Ker}(\mathrm{Nm}_{F/k}\colon \mathrm{U}_F \to \mathrm{U}_k)\,.$$

With this notation, we have the following lemma. For a proof we refer to [10].

(3.7) Lemma. *Let F be a CM-field. Suppose H is an algebraic subtorus of U_F of codimension 1. Then there exists an imaginary quadratic subfield $k \subset F$ such that $H = \mathrm{SU}_{F/k}$.*

Combining the above lemmas with the facts in (2.1) gives the following result.

(3.8) Proposition. *Let X be an abelian variety and let E be an elliptic curve, both over $\mathbb{C}$. Suppose* $\mathrm{Hom}(E, X) = 0$. *Then either* $\mathrm{Hg}(X \times E) = \mathrm{Hg}(X) \times \mathrm{Hg}(E)$ *or* $\mathrm{End}^0(E) = k$ *is an imaginary quadratic field such that there exists an embedding of k into the center of* $\mathrm{End}^0(X)$.

Proof. If $\mathrm{End}^0(E) = \mathbb{Q}$ then we apply Lemma (3.4). Hence we may assume that $\mathrm{End}^0(E) = k$ is an imaginary quadratic field, so that $\mathrm{Hg}(E)$ is the rank 1 torus U_k.

Write C for the center of $\mathrm{End}^0(X)$. Then C has the form $C = K_1 \times \cdots \times K_m \times F_1 \times \cdots \times F_n$, where $K_1, \ldots, K_m$ are totally real fields and $F_1, \cdots, F_n$ are CM-fields. The center Z of $\mathrm{Hg}(X)$ is contained in $\mathrm{U}_{F_1} \times \cdots \times \mathrm{U}_{F_n}$. By Lemma (3.6), if $\mathrm{Hg}(X \times E) \neq \mathrm{Hg}(X) \times \mathrm{Hg}(E)$ then there is a homomorphism $\mathrm{U}_k \to \mathrm{U}_{F_1} \times \cdots \times \mathrm{U}_{F_n}$ with finite kernel. If U_{F_i} is a factor such that the projection of U_k to U_{F_i} has rank 1 then it easily follows from Lemma (3.7) that there exists an embedding $k \to F_i$. This proves the claim. □

As an easy corollary we obtain a result first proved by Imai [7].

(3.9) Corollary. *Let $X_1, \ldots, X_n$ be elliptic curves over $\mathbb{C}$, no two of which are isogenous. Write $X = X_1 \times \cdots \times X_n$. Then* $\mathrm{Hg}(X) = \mathrm{Hg}(X_1) \times \cdots \times \mathrm{Hg}(X_n)$. *In particular, every product of elliptic curves satisfies condition* (D) *in* (1.7).

Proof. Immediate from the proposition, by induction on the number of factors. □

§4. Hodge groups of simple abelian surfaces of CM-type.

In this section we study Hodge groups of simple abelian surfaces of CM-type. We use this to prove Thm. (0.1) for the product of two such surfaces.

(4.1) Let F be a CM-field. Write Σ_F for the set of embeddings $F \to \mathbb{C}$. Let $\iota\colon x \mapsto \bar{x}$ denote the complex conjugation on F. (Recall that ι is independent of the choice of an embedding of F into $\mathbb{C}$.) By a CM-type for F we mean a subset $\Phi \subset \Sigma_F$ such that, writing $\overline{\Phi} = \{\bar{\varphi} \mid \varphi \in \Phi\}$, we have $\Sigma_F = \Phi \amalg \overline{\Phi}$.

Write $F_0 \subset F$ for the totally real subfield. The choice of a CM-type Φ for F is equivalent to giving an identification $F \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} \mathbb{C}^{\Sigma_{F_0}}$. Writing $J = J_\Phi \in F \otimes_{\mathbb{Q}} \mathbb{R}$ for the element which maps to $(i, i, \ldots, i)$ we obtain a bijection

$$\{\text{CM-types for } F\} \xrightarrow{\sim} \Gamma_F := \{J \in F \otimes_{\mathbb{Q}} \mathbb{R} \mid J^2 = -1\}$$

which is equivariant for the natural $\mathrm{Aut}(F)$-action on both sides.

To the CM-type (F, Φ) we can associate an isogeny class of complex abelian varieties by taking F as a $\mathbb{Q}$-lattice and J_Φ as a complex structure. Two CM-types (F, Φ) and (F, Ψ) give rise to the same isogeny class if and only if there exists an automorphism $\alpha \in \mathrm{Aut}(F)$ with $\Psi = {}^\alpha\Phi$. Note that if X is an abelian variety in the isogeny class associated to (F, Φ) then J_Φ is just the operator J_X as in (1.3). We have $J_{\overline{\Phi}} = -J_\Phi$.

Now let F be a quartic CM-field which does not contain an imaginary quadratic subfield. Then either (i) F is Galois over $\mathbb{Q}$, in which case $\mathrm{Aut}(F)$ is cyclic of order 4 acting transitively on Γ_F, or (ii) F is not Galois over $\mathbb{Q}$, its normal closure L has degree 8 over $\mathbb{Q}$, and $\mathrm{Aut}(F) = \{\mathrm{id}, \iota\}$. In case (i) there is only one isogeny class of abelian surfaces with CM by F, in case (ii) there are two such isogeny classes.

(4.2) Proposition. *Let X_1 and X_2 be two simple abelian surfaces with CM by the same quartic CM-field F. Suppose X_1 and X_2 are not isogenous. Write $X = X_1 \times X_2$. Then* $\mathrm{Hg}(X) = \mathrm{Hg}(X_1) \times \mathrm{Hg}(X_2)$.

Proof. Fix isomorphisms $F \cong \mathrm{End}^0(X_i)$; this gives identifications $\mathrm{Hg}(X_i) = \mathrm{U}_F$. As just explained, the assumption that $X_1 \not\sim X_2$ implies that F is not Galois over $\mathbb{Q}$. *A priori* the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ could be isomorphic to either the dihedral group D_4 or the quaternion group Q. By [20], Prop. 14 and Prop. 18, the CM-field F does not contain an imaginary quadratic field. Lemma (3.7) then shows that the torus U_F is $\mathbb{Q}$-simple. Writing $X^* = X^*(\mathrm{U}_F)$ for its character group this means that $X^*_{\mathbb{Q}}$ is an irreducible 2-dimensional $\mathbb{Q}$-representation of $\mathrm{Gal}(L/\mathbb{Q})$. But the group Q does not admit such a representation (cf. [19], Sect. 12.2, p. 108), hence $\mathrm{Gal}(L/\mathbb{Q}) \cong D_4$. Now consider the "standard" representation $\rho\colon D_4 \to \mathrm{GL}_2(\mathbb{Q})$, realizing D_4 as the subgroup of $\mathrm{GL}_2(\mathbb{Z})$ generated by the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

We remark that this ρ is the *only* irreducible 2-dimensional $\mathbb{Q}$-representation of D_4 (up to isomorphism), and that ρ is absolutely irreducible. Furthermore, one easily shows that the only D_4-stable lattices in $\mathbb{Q}^2$ are the lattices $q \cdot \mathbb{Z}^2$, for $q \in \mathbb{Q}^*$.

Assume that $\mathrm{Hg}(X) \neq \mathrm{Hg}(X_1) \times \mathrm{Hg}(X_2)$. The fact that U_F is $\mathbb{Q}$-simple then implies that $\mathrm{Hg}(X)$ is isogenous to U_F, and by the previous considerations we see that it is even isomorphic to U_F. Fix an isomorphism $\mathrm{Hg}(X) \xrightarrow{\sim} \mathrm{U}_F$ and identify $X^*(\mathrm{U}_F)$ with the "standard" D_4-module $\mathbb{Z}^2$ (i.e., $\mathbb{Z}^2$ with D_4-action given by ρ). Recall that we also have fixed identifications $\mathrm{Hg}(X_i) = \mathrm{U}_F$.

Each of the two projections $\mathrm{pr}_i\colon \mathrm{Hg}(X) \to \mathrm{Hg}(X_i)$ is surjective. The above considerations then show that there exist non-zero integers m_i $(i = 1, 2)$ such that pr_i is given by "raising to the m_ith power" on U_F. The induced map on Lie algebras $\mathrm{pr}_i\colon \mathfrak{hg}(X) = \mathfrak{u}_F \to \mathfrak{hg}(X_i) = \mathfrak{u}_F$ is therefore given by multiplication by m_i.

Under pr_i, the element $J_X \in \mathfrak{hg}(X) \otimes \mathbb{C}$ is mapped to $J_i = J_{X_i}$. Under the given identifications $\mathfrak{hg}(X_1) = \mathfrak{u}_F = \mathfrak{hg}(X_2)$ we thus find that $J_2 = (m_2/m_1) \cdot J_1$. Since both J_1 and J_2, viewed as elements of F, satisfy $J_i^2 = -1$ it follows that $m_1 = \pm m_2$. On the other hand, the homomorphism $\mathrm{Hg}(X) \to \mathrm{Hg}(X_1) \times \mathrm{Hg}(X_2)$ is injective, which means that m_1 and m_2 are relatively prime. It follows that $m_1, m_2 \in \{\pm 1\}$. In particular,

V_{X_1} and V_{X_2} are isomorphic as representations of $\mathrm{Hg}(X)$. This contradicts the assumption that $X_1 \not\sim X_2$. $\square$

§5. Proof of the main result.

(5.1) Let X be a complex abelian variety with $g = \dim(X) \leq 4$. Our first goal is to prove (0.1). As recalled above we already know this in case X is simple. In the rest of this section we may, and will, therefore assume that X is *not simple*.

Up to isogeny we can decompose X as $X \sim Y_1^{m_1} \times \cdots \times Y_r^{m_r}$ where $Y_1, \ldots, Y_r$ ($r \in \mathbb{Z}_{\geq 1}$) are simple, pairwise non-isogenous abelian varieties and $m_1, \ldots, m_r \in \mathbb{Z}_{\geq 1}$. Correspondingly, the endomorphism algebra D decomposes as $D = D_1 \times \cdots \times D_r$ where $D_i = \mathrm{End}^0(Y_i^{m_i}) \cong M_{m_i}(\mathrm{End}^0(Y_i))$. Write $V = H_1(X, \mathbb{Q})$ and $V_i = H_1(Y_i^{m_i}, \mathbb{Q})$. Choose polarizations λ_i of $Y_i^{m_i}$, let λ be the "product" polarization $\lambda = (\lambda_1, \ldots, \lambda_r)$ of X, and let $\varphi_i\colon V_i \times V_i \to \mathbb{Q}$ resp. $\varphi\colon V \times V \to \mathbb{Q}$ be the associated Riemann forms. With these notations we have the obvious remark that $\mathrm{Hg}(X) = \mathrm{Sp}_D(V, \varphi)$ if and only if $\mathrm{Hg}(X) = \mathrm{Hg}(Y_1^{m_1}) \times \cdots \times \mathrm{Hg}(Y_r^{m_r})$ and $\mathrm{Hg}(Y_i^{m_i}) = \mathrm{Sp}_{D_i}(V_i, \varphi_i)$ for all i.

Now assume that X is not simple, $\dim(X) = 4$ and that we are not in case (a) of the introduction. Note that X has no factors of Type III (since Type III does not occur in dimension ≤ 3). By the Theorem (1.7) of Hazama and Murty and the results discussed in §2, we then see that in order to prove (0.1) for X it suffices to show that $\mathrm{Hg}(X) = \mathrm{Hg}(Y_1^{m_1}) \times \cdots \times \mathrm{Hg}(Y_r^{m_r})$.

(5.2) Suppose $g = 3$. Suppose also that X decomposes, up to isogeny, as a product $X \sim X_1 \times X_2$ of an elliptic curve X_1 and a simple abelian surface X_2. Then the center of $\mathrm{End}^0(X_2)$ does not contain an imaginary quadratic field. By Prop. (3.8) it follows that $\mathrm{Hg}(X) = \mathrm{Hg}(X_1) \times \mathrm{Hg}(X_2)$.

Combining this with Cor. (3.9), we have proven (0.1) in case $\dim(X) \leq 3$. In particular, for every complex abelian variety X of dimension ≤ 3 we have $\mathrm{Hg}(X) = \mathrm{Sp}_D(V, \varphi)$ and condition (D) in (1.7) is satisfied.

(5.3) Let X be a complex abelian variety which is isogenous to a product, say $X \sim X_1 \times X_2$, where X_1 is an elliptic curve and X_2 is a simple abelian threefold. Suppose furthermore that $k := \mathrm{End}^0(X_1)$ is an imaginary quadratic field and that there exists an embedding $k \hookrightarrow F := \mathrm{End}^0(X_2)$. This means we are in case (a) of the introduction. Either (a1) $F = k$, or (a2) F is a sextic CM-field.

Embed k as a subfield of $\mathrm{End}^0(X)$ such that it acts with multiplicities $(2, 2)$ on the tangent space $T_{X,0}$. (Our assumption that X_2 is simple implies that k acts on $T_{X_2,0}$ with multiplicities $(1, 2)$, see [20], Prop. 14. Therefore, if we fix $\mathrm{End}^0(X_1) = k \hookrightarrow \mathrm{End}^0(X_2)$ then either $\alpha \mapsto (\alpha, \alpha) \in \mathrm{End}^0(X_1) \times \mathrm{End}^0(X_2)$ or $\alpha \mapsto (\bar{\alpha}, \alpha)$ gives an embedding as required.) Then the space $W_k \subset H^4(X, \mathbb{Q})$ consists of Hodge classes. The Hodge group $\mathrm{Hg}(X)$ is contained in $\mathrm{Hg}(X_1) \times \mathrm{Hg}(X_2) = \mathrm{U}_k \times \mathrm{U}_k(V_{X_2}, \psi_{X_2})$ (case (a1)) resp. $\mathrm{Hg}(X_1) \times \mathrm{Hg}(X_2) = \mathrm{U}_k \times \mathrm{U}_F$ (case (a2)). (See §2 for notations.) The Hodge group acts trivially on W_k, i.e., its elements have trivial k-linear determinant. We then easily find that we must have

$$\mathrm{Hg}(X) = \{(u_1, u_2) \in \mathrm{U}_k \times \mathrm{U}_k(V_{X_2}, \psi_{X_2}) \bigm| u_1 \cdot \det_k(u_2) = 1\} \quad (\text{case (a1)}),$$

respectively

$$\mathrm{Hg}(X) = \{(u_1, u_2) \in \mathrm{U}_k \times \mathrm{U}_F \bigm| u_1 \cdot \det_k(u_2) = 1\} \quad (\text{case (a2)}),$$

where $\det_k\colon \mathrm{U}_k(V_{X_2}, \psi_{X_2}) \to \mathrm{U}_k$ is the k-linear determinant map, resp. $\det_k = \mathrm{Nm}_{F/k}\colon \mathrm{U}_F \to \mathrm{U}_k$. (To see our claim, note that U_k has rank 1.)

The Künneth decomposition gives

$$H^4(X, \mathbb{Q}) = [H^2(X_1, \mathbb{Q}) \otimes H^2(X_2, \mathbb{Q})] \oplus [H^1(X_1, \mathbb{Q}) \otimes H^3(X_2, \mathbb{Q})] \oplus [H^0(X_1, \mathbb{Q}) \otimes H^4(X_2, \mathbb{Q})].$$

The Hodge classes in $H^2(X_1, \mathbb{Q}) \otimes H^2(X_2, \mathbb{Q}) \cong H^2(X_2, \mathbb{Q})(-1)$ and those in $H^0(X_1, \mathbb{Q}) \otimes H^4(X_2, \mathbb{Q}) \cong H^4(X_2, \mathbb{Q})$ are linear combination of products of divisor classes. The space of Weil classes W_k is a subspace of $H^1(X_1,\mathbb{Q})\otimes H^3(X_2,\mathbb{Q})$. (Since we are viewing W_k as a *subspace* of $H^4(X,\mathbb{Q})$, rather than a *quotient*, some of our identifications may seem a little unnatural, cf. [11], Sect. 7.)

We have $H^1(X_1,\mathbb{Q})\otimes H^3(X_2,\mathbb{Q})\cong \mathrm{Hom}_{\mathrm{HS}}(H^1(X_1,\mathbb{Q}),H^3(X_2,\mathbb{Q}))(-1)$, where "$\mathrm{Hom}_{\mathrm{HS}}$" denotes the space of homomorphisms of $\mathbb{Q}$-Hodge structures. The Hodge structure $H^1(X_1,\mathbb{Q})$ is irreducible and has endomorphism ring k. Therefore, our assertion that W_k is the space of Hodge classes in $H^1(X_1,\mathbb{Q})\otimes H^3(X_2,\mathbb{Q})$ is equivalent to saying that $H^3(X_2,\mathbb{Q})$ contains only one copy of $H^1(X_1,\mathbb{Q})(1)$ as a rational sub-Hodge structure. It suffices to prove this in case (a2), since the group $\mathrm{U}_k(V_{X_2},\psi_{X_2})$ contains tori of the form U_F where F is a sextic CM-field containing k. (Put differently: we can specialize from case (a1) to case (a2).)

Suppose then that $W\subset H^3(X_2,\mathbb{Q})$ is a sub-Hodge structure isomorphic to $H^1(X_1,\mathbb{Q})(1)$. Then $\mathrm{Hg}(X_2)=\mathrm{U}_F$ acts on W through the torus $\mathrm{Hg}(W)=\mathrm{Hg}(X_1)=\mathrm{U}_k$. The kernel of the corresponding homomorphism $\mathrm{U}_F\to\mathrm{U}_k$ is necessarily the subtorus $\mathrm{SU}_{F/k}\subset\mathrm{U}_F$. (Cf. Lemma (3.7).) But now we remark that the space of $\mathrm{SU}_{F/k}$-invariants in $H^3(X_2,\mathbb{Q})$ has $\mathbb{Q}$-dimension 2, which proves our claim. (In fact, the space of $\mathrm{SU}_{F/k}$-invariants in $H^3(X_2,\mathbb{Q})$ is precisely the subspace $W_k(X_2)\subset H^3(X_2,\mathbb{Q})$, which is naturally a 1-dimensional k-vector space.)

In sum, the previous arguments prove (0.1) for case (a).

(5.4) Let X be a non-simple complex abelian fourfold. Suppose X is not of CM-type. Then X contains a simple abelian subvariety X_2 which is not of CM-type. We can write $X\sim X_1\times X_2^r$ with $r\geq 1$ and $\mathrm{Hom}(X_2,X_1)=0$.

Suppose that we are not in case (a). We want to show that $\mathrm{Hg}(X)=\mathrm{Hg}(X_1)\times\mathrm{Hg}(X_2^r)$. If $r>1$ then we are reduced to the case $g\leq 3$, since $\mathrm{Hg}(X_1\times X_2^r)\cong\mathrm{Hg}(X_1\times X_2)$. Assume then that $r=1$. We distinguish two cases. If $\dim(X_2)=3$ then X_1 is an elliptic curve and we can apply (3.8), which works since we not in case (a). If $\dim(X_2)<3$ then the desired equality $\mathrm{Hg}(X)=\mathrm{Hg}(X_1)\times\mathrm{Hg}(X_2)$ follows from (2.4) and (3.4).

(5.5) Let X be a non-simple complex abelian fourfold of CM-type. Suppose that X is isogenous to $X_1\times X_2^r$ with $\dim(X_1)=1$ and $\mathrm{Hom}(X_1,X_2)=0$. If we are not in case (a) then there is no embedding of $\mathrm{End}^0(X_1)$ into the center of $\mathrm{End}^0(X_2)$. It thus follows from Prop. (3.8) that $\mathrm{Hg}(X)=\mathrm{Hg}(X_1)\times\mathrm{Hg}(X_2^r)$.

This only leaves us with the case where $X\sim X_1\times X_2$, with X_1 and X_2 simple abelian surfaces. If X_1 and X_2 are isogenous then we are done. If X_1 and X_2 are not isogenous then Proposition (4.2) shows that $\mathrm{Hg}(X)=\mathrm{Hg}(X_1)\times\mathrm{Hg}(X_2)$.

This completes the proof of Theorem (0.1).

We now turn to the proof of Thm. (0.2). As we have seen in §2, the statement is known if X is simple. So again we may, and shall, assume X to be non-simple. Furthermore we can assume that every simple factor of X occurs with multiplicity 1.

Write $X_1\subseteq X$ for the maximal abelian subvariety which has no factors of Type IV, and $X_2\subseteq X$ for the maximal abelian subvariety of which all factors are of Type IV. Write $d_i=\dim(X_i)$. We shall treat the possibilities case by case.

(5.6) Suppose $(d_1,d_2)=(5,0)$, so that X has no factors of Type IV. If X contains an elliptic curve E then $\mathrm{End}^0(E)=\mathbb{Q}$ (since E is not of Type IV) and Thm. (0.2) follows by Prop. (3.8). If X does not contain an elliptic curve then all its simple factors satisfy condition (D) in (1.7) and we conclude using Thm. (3.2).

(5.7) Suppose $(d_1,d_2)=(4,1)$ or $(d_1,d_2)=(3,2)$. Then X_2 is of CM-type and Thm. (3.2) gives $\mathrm{Hg}(X)=\mathrm{Hg}(X_1)\times\mathrm{Hg}(X_2)$.

(5.8) Suppose $(d_1,d_2)=(2,3)$. If X_1 is simple then Lemma (3.4) gives $\mathrm{Hg}(X)=\mathrm{Hg}(X_1)\times\mathrm{Hg}(X_2)$. If X_1 is not simple then it is isogenous to a product of two elliptic curves, $X_1\sim E_1\times E_2$ with $\mathrm{End}^0(E_1)=\mathrm{End}^0(E_2)=\mathbb{Q}$ and where we may assume E_1 and E_2 to be non-isogenous. Prop. (3.8) then gives $\mathrm{Hg}(X)=\mathrm{Hg}(E_1)\times\mathrm{Hg}(E_2)\times\mathrm{Hg}(X_2)$.

(5.9) Suppose $(d_1, d_2) = (1, 4)$. Then $\mathrm{Hg}(X) = \mathrm{Hg}(X_1) \times \mathrm{Hg}(X_2)$ by Prop. (3.8).

(5.10) From now on, let us assume that $(d_1, d_2) = (0, 5)$, meaning that all simple factors of X are of Type IV. Let $d_{\min}$ be the minimal dimension of a simple factor of X. Since we assume X to be non-simple we have $d_{\min} = 1$ or $d_{\min} = 2$.

First suppose that $d_{\min} = 2$. Then $X \sim Y_1 \times Y_2$ where Y_1 is a simple abelian surface and Y_2 is a simple abelian threefold. Note that Y_1 is of CM-type with $\mathrm{Hg}(Y_1) = \mathrm{U}_{F_1}$, where $F_1 = \mathrm{End}^0(Y_1)$.

If Y_2 is not of CM-type then Lemma (3.6) readily gives $\mathrm{Hg}(X) = \mathrm{Hg}(Y_1) \times \mathrm{Hg}(Y_2)$. If Y_2 is of CM-type then $F_2 = \mathrm{End}^0(Y_2)$ is a sextic CM-field. By Lemma (3.6) and Lemma (3.7), we can have $\mathrm{Hg}(X) \neq \mathrm{Hg}(Y_1) \times \mathrm{Hg}(Y_2)$ only if F_2 contains an imaginary quadratic field k such that U_{F_1} is isogenous to $\mathrm{SU}_{F_2/k}$. Suppose this is the case. Write Ω_1 for the normal closure of F_1 over $\mathbb{Q}$. Either $\Omega_1 = F_1$ and $\mathrm{Gal}(\Omega_1/\mathbb{Q}) = \mathbb{Z}/4\mathbb{Z}$ (as F_1 does not contain an imaginary quadratic field), or Ω_1 has degree 8 over $\mathbb{Q}$. Next write K_2 for the totally real subfield of F_2 and let Ω_2 be the normal closure of K_2 over $\mathbb{Q}$. As F_2 contains the imaginary quadratic field k, the normal closure of F_2 over $\mathbb{Q}$ is the compositum $k \cdot \Omega_2$. The Galois group $\mathrm{Gal}(k \cdot \Omega_2/\mathbb{Q})$ is either $\mathbb{Z}/2\mathbb{Z} \times \mathfrak{S}_3$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Now Ω_1 is the splitting field of the $\mathbb{Q}$-torus U_{F_1} and $k \cdot \Omega_2$ contains the splitting field of U_{F_2}. The assumption that U_{F_1} is isogenous to $\mathrm{SU}_{F_2/k}$ thus implies that $\Omega_1 \subseteq k \cdot \Omega_2$. Looking at Galois groups we obtain a contradiction. Hence again $\mathrm{Hg}(X) = \mathrm{Hg}(Y_1) \times \mathrm{Hg}(Y_2)$.

(5.11) From now on, let us assume that $(d_1, d_2) = (0, 5)$ and that $d_{\min} = 1$. Write $X \sim E \times Y$, where E is an elliptic curve and $\dim(Y) = 4$. Without loss of generality we may assume that $\mathrm{Hom}(E, Y) = 0$. (If not then we are reduced to the case $\dim(X) \leq 4$.) Let $d_{\max}$ be the maximal dimension of a simple factor of X.

If $d_{\max} \leq 2$ then all simple factors of Y are of CM-type and there does not exist an embedding of $\mathrm{End}^0(E)$ into the center of $\mathrm{End}^0(Y)$. Then Prop. (3.8) gives $\mathrm{Hg}(X) = \mathrm{Hg}(E) \times \mathrm{Hg}(Y)$.

If $d_{\max} = 3$ then Y is isogenous to a product of an elliptic curve Y_1 and a simple abelian threefold Y_2. If $\mathrm{End}^0(Y_2)$ contains an imaginary quadratic field then this subfield is unique. Therefore, possibly after interchanging the roles of E and Y_1 we find that there does not exist an embedding of $\mathrm{End}^0(E)$ into the center of $\mathrm{End}^0(Y)$. (Note that $\mathrm{End}^0(E) = \mathrm{End}^0(Y_1)$ implies that $E \sim Y_1$, which we excluded.) Again by Prop. (3.8) we then find $\mathrm{Hg}(X) = \mathrm{Hg}(E) \times \mathrm{Hg}(Y)$.

Finally, let us assume that $d_{\max} = 4$, i.e., that Y is simple (of Type IV). Write $k = \mathrm{End}^0(E)$ and $F = \mathrm{End}^0(Y)$. If there is no embedding $j\colon k \hookrightarrow F$ then Prop. (3.8) gives $\mathrm{Hg}(X) = \mathrm{Hg}(E) \times \mathrm{Hg}(Y)$. Suppose then that there exists an embedding j. We distinguish 2 cases.

Case 1: Suppose that k acts on $T_{Y,0}$ with multiplicities $(2, 2)$. Then $\mathrm{Hg}(Y) = \mathrm{SU}_{F/k}$. By rank considerations, $\mathrm{Hg}(X) \neq \mathrm{Hg}(E) \times \mathrm{Hg}(Y)$ is possible only if there exists a non-trivial homomorphism $\alpha\colon \mathrm{SU}_{F/k} \to \mathrm{Hg}(E) = \mathrm{U}_k$. Choose a homomorphism $j\colon \mathrm{U}_F \to \mathrm{SU}_{F/k}$ such that the composition $\mathrm{SU}_{F/k} \hookrightarrow \mathrm{U}_F \xrightarrow{j} \mathrm{SU}_{F/k}$ is an isogeny. Then the identity component of $\mathrm{Ker}(\alpha \circ j\colon \mathrm{U}_F \to \mathrm{U}_k)$ is a codimension 1 subtorus of U_F other than $\mathrm{SU}_{F/k}$. Using (3.7) we now easily obtain a contradiction. Hence $\mathrm{Hg}(X) = \mathrm{Hg}(E) \times \mathrm{Hg}(Y)$.

Case 2: Suppose that k acts on $T_{Y,0}$ with multiplicities $(1, 3)$. (By [20], Prop. 14 this is the only other case that occurs.) Rather than looking at $E \times Y$, let us look at $Z := E^2 \times Y$. There is an embedding $k \hookrightarrow \mathrm{End}^0(Z)$ such that k acts on $T_{Z,0}$ with multiplicities $(3, 3)$. This implies that the corresponding space of Weil classes $W_k \subset H^6(Z, \mathbb{Q})$ consists of Hodge classes and that $\mathrm{Hg}(Z) \subseteq \mathrm{SU}_k(V_Z, \psi)$. (For this last conclusion, see [10], Lemma 2.8.) Returning to our original abelian variety $X \sim E \times Y$ we find that $\mathrm{Hg}(X)$ is contained in the subgroup $H \subset \mathrm{Hg}(E) \times \mathrm{Hg}(Y) = \mathrm{U}_k \times \mathrm{U}_F(V_Y, \psi)$ given by

$$H = \{(u_1, u_2) \in \mathrm{U}_k \times \mathrm{U}_F(V_Y, \psi) \mid u_1^2 \cdot \det_k(u_2) = 1\},$$

where $\det_k\colon \mathrm{Hg}(Y) = \mathrm{U}_F(V_Y, \psi) \to \mathrm{U}_k$ is the k-linear determinant. By dimension considerations, noting that H is connected and that $H \to \mathrm{Hg}(Y)$ is an isogeny, we then find that $\mathrm{Hg}(X) = H$.

(5.12) We have now computed the Hodge groups of all complex abelian 5-folds. It remains to be shown that this indeed gives the conclusions as stated in Thm. (0.2). Part (iv) of the theorem follows by going through the above and using (0.1) and (2.7). All that remains to be done is the computation of the Hodge rings in the cases (e), (f) and (g).

Case (f) is easy. It was established in (5.9) and (5.10) that $\mathrm{Hg}(X) = \mathrm{Hg}(X_0) \times \mathrm{Hg}(X_1 \times X_2)$. (Notations as in the introduction.) The rest of statement (ii) of (0.2) readily follows.

Next suppose we are in case (e). By the duality $H^j(X,\mathbb{Q})(5) \cong H^{10-j}(X,\mathbb{Q})^\vee$ we only have to show that $\mathcal{B}^2(X) \subset H^4(X,\mathbb{Q})$ is generated by $\mathcal{D}^2(X)$ and the spaces $W_{k,\alpha}$. The Künneth formula gives

$$\begin{aligned} H^4(X,\mathbb{Q}) = & [H^4(X_1^2,\mathbb{Q}) \otimes H^0(X_2,\mathbb{Q})] \oplus [H^3(X_1^2,\mathbb{Q}) \otimes H^1(X_2,\mathbb{Q})] \oplus [H^2(X_1^2,\mathbb{Q}) \otimes H^2(X_2,\mathbb{Q})] \\ & \oplus [H^1(X_1^2,\mathbb{Q}) \otimes H^3(X_2,\mathbb{Q})] \oplus [H^0(X_1^2,\mathbb{Q}) \otimes H^4(X_2,\mathbb{Q})]\,. \end{aligned}$$

In $H^4 \otimes H^0$ and $H^0 \otimes H^4$ we only have divisor classes. In

$$H^3(X_1^2,\mathbb{Q}) \otimes H^1(X_2,\mathbb{Q}) \cong \mathrm{Hom}(H^1(X_1^2,\mathbb{Q}), H^1(X_2,\mathbb{Q}))(-2)$$

there are no non-zero Hodge classes, as there are no non-zero homomorphisms from X_1^2 to X_2. Next we have $H^1(X_1^2,\mathbb{Q}) \otimes H^3(X_2,\mathbb{Q}) \cong [H^1(X_1,\mathbb{Q}) \otimes H^3(X_2,\mathbb{Q})]^{\oplus 2}$, so that the Hodge classes in $H^1 \otimes H^3$ are just the elements of the spaces $W_{k,\alpha}$. (See (5.3). Also note that in fact we only need two spaces W_{k,α_1} and W_{k,α_2} for "linear independent" choices α_1 and α_2.)

To settle case (e) it thus remains to compute the Hodge classes in $H^2 \otimes H^2$. Write $V_2 = H_1(X_2,\mathbb{Q})$ and $F = \mathrm{End}^0(X_2)$. Either $F = k$ or F is a sextic CM-field. Fix an element $a \in F$ with $\bar{a} = -a$. The Hodge group $\mathrm{Hg}(X_2)$ is the unitary group $\mathrm{U}_F(V_2,\psi)$, where $\psi\colon V_2 \times V_2 \to F$ is an F-hermitian form such that $\mathrm{tr}_{F/\mathbb{Q}}(a \cdot \psi)$ is the Riemann form of a polarization. (See (2.3) and notice that this description also applies if F is a sextic CM-field.) Consider the algebraic subgroup $\mathrm{SU}_{F/k}(V_2,\psi) = \mathrm{Ker}(\det_k\colon \mathrm{U}_F(V_2,\psi) \to \mathrm{U}_k)$. We claim that $\mathrm{SU}_{F/k}(V_2,\psi)$ and $\mathrm{U}_F(V_2,\psi)$ have the same centralizer in $\mathrm{End}(V_2)$. To see this we can extend scalars to $\mathbb{C}$, where, treating the cases $F = k$ and $[F:\mathbb{Q}] = 6$ separately, the claim is easily verified. As $H^2(X_2,\mathbb{Q})$ is isomorphic to a sub-Hodge structure of $\mathrm{End}(V_2)(-1)$ it follows that the space of $\mathrm{SU}_{F/k}(V_2,\psi)$-invariants in $H^2(X_2,\mathbb{Q})$ is equal to the space $\mathcal{B}^1(X_2)$ of $\mathrm{Hg}(X_2)$-invariants. Now our description of $\mathrm{Hg}(X) \cong \mathrm{Hg}(X_1 \times X_2)$ in (5.3) above shows that $\mathrm{Hg}(X) \supset \{1\} \times \mathrm{SU}_{F/k}(V_2,\psi)$, so that the Hodge classes in $H^2 \otimes H^2$ are contained in $H^2(X_1^2,\mathbb{Q}) \otimes \mathcal{B}^1(X_2)$. It readily follows that the Hodge classes in $H^2 \otimes H^2$ must lie in $\mathcal{B}^1(X_1^2) \otimes \mathcal{B}^1(X_2)$ and are therefore decomposable. This finishes the proof of (i) of Thm. (0.2).

Finally, suppose we are in case (g). Again we only have to look at $H^4(X,\mathbb{Q})$, and the only interesting Künneth component here is $H^1(X_1,\mathbb{Q}) \otimes H^3(X_2,\mathbb{Q})$. As we have shown, $\mathrm{Hg}(X) = \{(u_1,u_2) \in \mathrm{U}_k \times \mathrm{Hg}(Y) \mid u_1^2 \cdot \det_k(u_2) = 1\}$. In particular we have an element $(-1,1) \in \mathrm{Hg}(X)$ which acts on $H^1(X_1,\mathbb{Q}) \otimes H^3(X_2,\mathbb{Q})$ as -1. This shows there are no Hodge classes in $H^1(X_1,\mathbb{Q}) \otimes H^3(X_2,\mathbb{Q})$ and that $\mathcal{B}^\bullet(X)$ is generated by divisor classes.

References.

[1] M. V. Borovoi, *The Hodge group and the algebra of endomorphisms of an abelian variety*, In: A. L. Onishchik (ed.), Problems in group theory and homological algebra, Yaroslav. Gos. Univ., Yaroslavl (1981), 124–126. (MR 84m:14047).

[2] N. Bourbaki, *Groupes et Algèbres de Lie*, Chap. 1, Hermann, Paris (1960); Chapters 7 et 8, Hermann, Paris (1975).

[3] P. Deligne, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, In: A. Borel and W. Casselman (eds.), Automorphic forms, representations, and L-functions, Proc. Symp. in pure Math. **33**(2), AMS, Providence (1979), 247–289.

[4] F. Hazama, *Hodge cycles on abelian varieties of CM-type*, Res. Act. Fac. Sci. Engrg. Tokyo Denki Univ. **5** (1983), 31–33. (MR 86g:14024).

[5] F. Hazama, *Algebraic cycles on certain abelian varieties and powers of special surfaces*, J. Fac. Sci. Univ. Tokyo, Sect. Ia, **31** (1984), 487–520.

[6] F. Hazama, *Algebraic cycles on nonsimple abelian varieties*, Duke Math. J. **58** (1989), 31–37.

[7] H. Imai, *On the Hodge groups of some abelian varieties*, Kodai Math. Sem. Rep. **27** (1976), 367–372.

[8] N. Jacobson, *Lie Algebras*, Interscience Tracts **10**, John Wiley and Sons, New York (1962). Republished by Dover Publications, Inc., New York (1979).

[9] H. W. Lenstra and Yu. G. Zarhin, *The Tate conjecture for almost ordinary abelian varieties over finite fields*, In: F. Gouvêa and N. Yui (eds.), Advances in Number Theory: Proceedings of the third conference of the CNTA, 1991, Clarendon Press, Oxford (1993), 179–194.

[10] B. J. J. Moonen and Yu. G. Zarhin, *Hodge classes and Tate classes on simple abelian fourfolds*, Duke Math. J. **77** (1995), 553–581.

[11] B. J. J. Moonen and Yu. G. Zarhin, *Weil classes on abelian varieties*, J. reine angew. Math. **496** (1998), 83–92.

[12] D. Mumford, *A Note of Shimura's paper "Discontinuous groups and abelian varieties"*, Math. Ann. **181** (1969), 345–351.

[13] D. Mumford, *Abelian varieties*, 2nd Edition, Oxford Univ. Press, Oxford (1974).

[14] V. K. Murty, *Exceptional Hodge classes on certain abelian varieties*, Math. Ann. **268** (1984), 197–206.

[15] F. Oort, *Endomorphism algebras of abelian varieties*, In: H. Hijikata et al. (eds.), Algebraic geometry and commutative algebra in honor of Masayoshi Nagata, Part II, Kinokuniya Company, Ltd., Tokyo (1988), 469–502.

[16] R. Pink, *ℓ-adic algebraic monodromy groups, cocharacters, and the Mumford-Tate conjecture*, J. reine angew. Math. **495** (1998), 187–237.

[17] K. Ribet, *Hodge classes on certain types of abelian varieties*, Amer. J. Math. **105** (1983), 523–538.

[18] C. Schoen, *Hodge classes on self-products of a variety with an automorphism*, Compositio Math. **65** (1988), 3–32; *Addendum*, Compositio Math. **114** (1998), 329–336.

[19] J-P. Serre, *Représentations linéaires des groupes finis (troisième éd.)*, Hermann, Paris (1978).

[20] G. Shimura, *On analytic families of polarized abelian varieties and automorphic functions*, Ann. of Math. **78** (1963), 149–192.

[21] S. G. Tankeev, *On Algebraic cycles on abelian varieties. II*, Math. USSR Izv. **14** (1980), 383–394.

[22] S. G. Tankeev, *Cycles on simple abelian varieties of prime dimension*, Math. USSR Izv. **20** (1983), 157–171.

[23] B. van Geemen, *Theta functions and cycles on some abelian fourfolds*, Math. Z. **221** (1996), 617–631.

[24] A. Weil, *Abelian varieties and the Hodge ring*, Collected papers, Vol. III, [1977c], 421–429.

[25] Yu. G. Zarhin, *Weights of simple Lie algebras in the cohomology of algebraic varieties*, Math. USSR Izv. **24** (1985), 245–282.

[26] Yu. G. Zarhin, *Linear irreducible Lie algebras and Hodge structures*, In: S. Bloch, I. Dolgachev, W. Fulton (eds.), Proceedings of the USA-USSR Symposium on Algebraic Geometry, Chicago 1989, Lecture Notes in Math. **1479**, Springer-Verlag, Berlin (1991), 281–297.

B. J. J. Moonen, Department of Mathematics, University of Utrecht, P.O. Box 80.010, NL-3508 TA Utrecht, The Netherlands. E-mail: `moonen@math.uu.nl`

Yu. G. Zarhin, Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA. E-mail: `zarhin@math.psu.edu`