Regulating the infrared by mode matching: A massless scalar in expanding spaces with constant deceleration

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(Received 12 June 2009; published 19 April 2011)

In this paper we consider a massless scalar field, with a possible coupling ξ to the Ricci scalar in a D dimensional Friedmann-Lemaître-Robertson-Walker space-time with a constant deceleration parameter $q = \epsilon - 1$, $\epsilon = -\dot{H}/H^2$. Correlation functions for the Bunch-Davies vacuum of such a theory have long been known to be infrared divergent for a wide range of values of ϵ. We resolve these divergences by explicitly matching the space-time under consideration to a space-time without infrared divergencies. Such a procedure ensures that all correlation functions with respect to the vacuum in the space-time of interest are infrared finite. In this newly defined vacuum we construct the coincidence limit of the propagator and as an example calculate the expectation value of the stress-energy tensor. We find that this approach gives both in the ultraviolet and in the infrared satisfactory results. Moreover, we find that, unless the effective mass due to the coupling to the Ricci scalar ξR is negative, quantum contributions to the energy density always dilute away faster, or just as fast, as the background energy density. Therefore, quantum backreaction is insignificant at the one-loop order, unless ξR is negative. Finally we compare this approach with known results where the infrared is regulated by placing the Universe in a finite box. In an accelerating universe, the results are qualitatively the same, provided one identifies the size of the Universe with the physical Hubble radius at the time of the matching. In a decelerating universe, however, the two schemes give different late time behavior for the quantum stress-energy tensor. This happens because in this case the length scale at which one regulates the infrared becomes sub-Hubble at late times.

DOI: 10.1103/PhysRevD.83.084035 PACS numbers: 04.62.+v, 98.80.Cq

I. INTRODUCTION

It is by now well established that on the largest scales, the Universe is well described by a homogeneous, isotropic, and spatially flat metric, given by [1]

$$g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a(t)^2 d\vec{x}\cdot d\vec{x}. \qquad (1)$$

Here a is the scale factor, and the derivatives on a define the Hubble parameter, H and the deceleration parameter q (in this paper we shall use the equivalent parameter ϵ)

$$H \equiv \frac{\dot{a}}{a}; \qquad q \equiv -1 - \frac{\dot{H}}{H^2} \equiv -1 + \epsilon. \qquad (2)$$

If $\epsilon > 1$ the expansion of the Universe is decelerating, if $\epsilon < 1$ it is accelerating, and if $\epsilon = 1$ it is curvature dominated. In a typical cosmological model, we couple this geometry to a certain perfect fluid, with energy density $\rho_b = \rho_b(t)$ and pressure $p_b = p_b(t)$. The dynamics of the scale factor are then governed by the Friedmann equations

$$H^2 = \frac{8\pi G_N}{3}\rho_b, \qquad \dot{H} = -4\pi G_N(\rho_b + p_b), \qquad (3)$$

where G_N indicates Newton's constant and for simplicity we assumed flat spatial sections. This is consistent with the current cosmological measurements [2], which give a very large lower limit on the radius of curvature R_c of the Universe, $R_c \geq 22h^{-1}$ Gpc ($R_c \geq 33h^{-1}$ Gpc) for a universe with positively (negatively) curved spatial sections, where $h = 0.705 \pm 0.013$ is the rescaled Hubble parameter (in units of 100 km/s/Mpc). If we now assume that the cosmological fluid obeys an equation of state

$$p_b = w_b\rho_b, \qquad (4)$$

with w constant, we immediately find that in such a case

$$\epsilon = \frac{3}{2}(1 + w_b) = \text{constant}. \qquad (5)$$

The equation of state (4) is in practice obeyed for many types of fluids. For example radiation corresponds to $w_b = 1/3$ and $\epsilon = 2$, nonrelativistic matter has $w_b = 0$ and $\epsilon = 3/2$, the cosmological constant has $w_b = -1$ and $\epsilon = 0$. If ϵ is a constant, we can find the Hubble parameter and the scale factor as a function of time

$$H(t) = \frac{H_0}{1 + \epsilon H_0 t}; \qquad a(t) = (1 + \epsilon H_0 t)^{1/\epsilon}, \qquad (6)$$

with H_0 a constant.

The behavior of a massless scalar field with a possible coupling to the Ricci scalar on such a background

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geometry has been extensively studied in the literature [3–6]. An interesting observation is that the kinetic operator of, for example, the graviton can be related to the kinetic operator of such a massless scalar field. Therefore the propagator of the graviton can, apart from some tensorial structure, be written in terms of propagators of massless scalar fields [7,8]. Similar observations can be made for vector fields [9] or antisymmetric tensor fields [10]. Therefore an understanding of the physics of the massless scalar field can be translated to vector and tensor fields as well. This presents us with the main motivation for our work: based on a proper understanding of the infrared (IR) sector of massless scalar fields one can learn about the infrared sector of the graviton and the corresponding quantum backreaction.

In particular many studies concern the special limit of $\epsilon \to 0$ (de Sitter) [11]. Because of the nature of the background space-time, the scalar field has nontrivial quantum properties. Because of the presence of a horizon, there is particle production. The particles are created mostly in the infrared, and it is this effect that leads, for example, to the creation of primordial density fluctuations. Also, it is the basis for many works on the quantum backreaction on the background space-time [12–20]. However, it was long ago realized that when one chooses the vacuum to contain only purely positive frequency modes, i.e., the so-called Bunch-Davies (BD) vacuum, the expectation of the two-point correlator for this vacuum diverges in the infrared, for a large—and physically relevant—range of ϵ [5]. In particular, if the massless scalar field is minimally coupled, there are infrared divergencies for all $\epsilon \leq 3/2$. The presence of these divergencies implies that in the infrared, the Bunch-Davies vacuum does not describe a physically sensible state. Although one might be tempted to use the concept of particle creation to explain the presence of the infrared divergence, one should be careful when doing this. The particle concept is strictly speaking not well defined on a curved background, since the vacuum, or zero particle state is not unambiguously defined. On the other hand, a two-point correlator is always well defined, but it necessarily depends on the vacuum state one chooses. Different choices of the vacuum lead to different infrared behavior and the infrared divergence should therefore be seen as arising from a choice which leads to more infrared correlations then that are physically acceptable.

One possible solution to this problem is to assume that the space-time manifold is spatially compact, for example, a torus T^{D-1}. This approach effectively introduces an infrared cutoff, when the sum over the modes is approximated by an integral [21]. In Ref. [22] the propagator is explicitly constructed using this regularization for any constant ϵ space, which is then used in Ref. [8] to study the quantum backreaction from tensor and scalar cosmological perturbations in constant ϵ spaces.

A different approach is to choose the mode functions such that the superhorizon modes are less singular than they would have been in the Bunch-Davies vacuum [4,6]. Because only the superhorizon modes change there would be no effect on the Hadamard short distance behavior of the propagator. The time dependence of the mode functions is determined by the scalar field equation but their initial values and the initial values of their first time derivatives can be freely specified. Thus, we can choose the initial values for the super-Hubble infrared modes to be in some infrared finite state. Such a choice would ensure then that there would be no infrared divergence, neither initially nor at any later time [23].

In this paper however, we propose a third regularization of the infrared. Instead of splitting the ultraviolet and the infrared sector, one could also consider the matching between the Bunch-Davies vacuum in an infrared safe space-time and the space-time one wishes to study for all modes. Also in this case the initial state does not lead to infrared divergences and thus also the final state will be safe. In this approach the ultraviolet mode functions will differ from the ultraviolet Bunch-Davies mode functions, but the ultraviolet divergent structure will not change. The advantage of this approach over the previous two is that one can quite easily envisage something like this to be realized in nature. For example if inflation was preceded by a radiation dominated epoch, we essentially have the scenario described above, if the field was in its Bunch-Davies vacuum during the radiation epoch and if the transition between the two geometries is fast enough. A disadvantage is that the sudden matching between the two geometries causes the Ricci scalar to change discontinuously. This causes a burst of particle creation, but we shall see that this does not significantly influence our main results.

There are other regularization schemes proposed in the literature. Here we mention a recent proposal by Parker [24], where it was argued that the infrared should be regulated by subtracting the adiabatic vacuum contribution. The proposal has been worked out in some detail in Refs. [25,26]. We feel that this scheme is not well motivated, since the infrared sector of constant ϵ spaces strongly breaks adiabaticity, which is the assumption made when constructing adiabatic vacua. Yet another possibility is to subtract the vacuum of comoving observers [27]. This procedure works well for de Sitter space [27], but it remains to be seen whether such a procedure can be successfully implemented in more general Friedmann-Lemaître-Robertson-Walker space-times. A final option to regulate the infrared is by giving the scalar field a small mass [28]. It is not clear however how to implement this approach to study the infrared properties and the corresponding backreaction of the graviton, which is the main motivation of the present work.

In all of these approaches one must realize what the regularization of an infrared divergence actually means.

The infrared divergence arises from a physically unacceptable choice for the vacuum. Thus, if one wants to solve the problem, one really has to change the physics of the system one is considering. The infrared regularization is thus not a mathematical tool that makes physical quantities calculable, like, for example, cutoff regularization for ultraviolet divergences. Instead, the infrared regularization is a change of the system under consideration, such that the system actually corresponds to a physically realizable situation in which various quantities are calculable. In all infrared regularizations mentioned above one is effectively introducing a new scale. This scale is thus not an arbitrary cutoff scale like in ultraviolet regularizations, but it is a new physical scale one has to introduce to the problem such that it physically makes sense.

In this paper we want to study the role of the quantum infrared fluctuations generated by the expansion of the Universe for a massless scalar field. In particular we would like to understand whether the energy density due to these fluctuations could ever become relevant for the evolution of the Universe. For a proper understanding of such a question, it is important to compare different regularization schemes for the infrared. In particular we expect that in an accelerating space-time, results will not depend that much on the regularization chosen, since the details of any such scheme will grow quickly to super-Hubble scales. The reason is that in accelerating space-times, physical length scales grow faster then the Hubble radius. In a decelerating space-time however, the opposite is true. Specific cases need to be studied to understand this issue in detail.

The paper is organized as follows. In Sec. II we discuss the conditions under which the infrared sector of the Bunch-Davies vacuum exhibits divergences, and show that they can be cured by matching onto an earlier radiation era. We then construct the corresponding coincident scalar propagator, which has a finite infrared, and which is used in Sec. III to study some properties of the stress-energy tensor. In Sec. IV we calculate the ultraviolet contribution to the stress-energy tensor and consider its renormalization. Section V, which is the central part of the paper, is devoted to the equation of state of the quantum fluid and to a study of the scaling of the quantum energy density. In Sec. VI we compare our results with the results presented in Refs. [8,22] by using a cutoff regularization procedure. Finally, in Sec. VII we summarize our results and conclude. Various technical details are relegated to the Appendices.

II. MATCHING ONTO AN EARLY RADIATION ERA

In this section we show how to regulate the infrared sector of a massless nonminimally coupled scalar field by matching onto an earlier epoch characterized by a regular infrared sector, which for definiteness we choose to be the radiation era. For simplicity we work with homogeneous cosmology, and assume that the Universe's expansion is driven by a perfect fluid whose equation of state $w_b = p_b/\rho_b$ changes suddenly[1] at the matching time from $w_b = 1/3$ (radiation) to the one with negative pressure $w_b < 0$ ("inflation"). In this section we demonstrate that, as a result of the matching conditions, the inflationary infrared inherits the good properties (finiteness) of the radiation era infrared sector, and finally we construct the corresponding coincident propagator.

Let us start with the action for a massless nonminimally coupled scalar ϕ,

$$S = \frac{1}{2}\int d^D x\sqrt{-g}\,\phi(\Box - \xi R)\phi, \tag{7}$$

where ξ is a constant coupling of the scalar field to the Ricci scalar R. The equation of motion for ϕ is

$$(\Box - \xi R)\phi = 0, \tag{8}$$

where in our background metric (1) the scalar d'Alembertian reads,

$$\Box = -\frac{\partial^2}{\partial t^2} + \frac{1}{a^2}\frac{\partial^2}{\partial \vec{x}^2} - (D-1)H\frac{\partial}{\partial t}, \tag{9}$$

and $R = (D-2)(D-2\epsilon)H^2$ denotes the Ricci scalar. The corresponding time-ordered (Feynman) propagator equation is then

$$(\Box - \xi R)i\Delta(x;x') = i\delta^D(x-x'). \tag{10}$$

In order to facilitate the dimensional regularization of the ultraviolet sector, we work in (complex) D space-time dimensions. The Feynman propagator (10) can be expressed in terms of the fields $\phi(x)$ as

$$\begin{aligned} i\Delta(x;x') &= \theta(t-t')\langle\Omega|\phi(x)\phi(x')|\Omega\rangle + \theta(t'-t) \\ &\quad\times\langle\Omega|\phi(x')\phi(x)|\Omega\rangle \\ &= \int\frac{d^{D-1}k}{(2\pi)^{D-1}}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}(\theta(t-t')\psi(t,k)\psi^*(t',k) \\ &\quad + \theta(t'-t)\psi(t',k)\psi^*(t,k)), \end{aligned} \tag{11}$$

where $|\Omega\rangle$ denotes the (vacuum) state of the system and $\psi(t,k)$ are the mode functions, defined by

[1]Even though we do not present here a complete model for the background, we parenthetically remark that similar conditions can be realized in a universe which is radiation dominated before inflation, and which is endowed with a scalar field that (in some regions of the Universe) sits in its false vacuum in a state containing a large potential energy. As the Universe expands the radiation energy density will dilute, leading eventually to the false vacuum energy dominance, triggering inflation. The onset of inflation corresponds to the matching time. Inflation ends when the scalar exits the false vacuum (e.g., via tunneling) and rolls down a potential hill towards its true vacuum with vanishing potential energy.

$$\phi(x) = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{i\vec{k}\cdot\vec{x}}[\psi(t,k)b(\vec{k}) + \psi^*(t,k)b^\dagger(-\vec{k})], \tag{12}$$

with (see Appendix A)

$$\psi(t,k) = a^{1-(D/2)}(\alpha(k)u(t,k) + \beta(k)u^*(t,k))$$
$$u(t,k) = \sqrt{\frac{\pi}{4(1-\epsilon)Ha}}H_\nu^{(1)}\left(\frac{k}{(1-\epsilon)Ha}\right) = \sqrt{\frac{\pi z}{4k}}H_\nu^{(1)}(z), \tag{13}$$

where $H_\nu^{(1)}$ denotes the Hankel function of the first kind, and

$$\nu^2 = \left(\frac{D-1-\epsilon}{2(1-\epsilon)}\right)^2 - \frac{(D-1)(D-2\epsilon)}{(1-\epsilon)^2}\xi = \frac{1}{4} - \frac{(D-2\epsilon)}{(1-\epsilon)^2}\left((D-1)\xi - \frac{D-2}{4}\right). \tag{14}$$

The vacuum $|\Omega\rangle$ is defined by $b(\vec{k})|\Omega\rangle = 0$, where $b(\vec{k})$ denotes the annihilation operator. The coefficients $\alpha(k)$ and $\beta(k)$ in (13) satisfy $|\alpha(k)|^2 - |\beta(k)|^2 = 1$. Since one phase in $\alpha(k)$ or $\beta(k)$ has no physical relevance, this condition still leaves us with a large number of possible vacua, characterized by the direct product, $|\Omega\rangle = \prod_{\vec{k}} \otimes |\{\beta(k)\}\rangle$. The standard choice of the vacuum made most frequently in the literature corresponds to the Bunch-Davies vacuum,

$$\alpha(k) = 1; \qquad \beta(k) = 0, \quad \forall\ \vec{k}. \tag{15}$$

This choice is very reasonable in the ultraviolet since it minimizes the field's ultraviolet energy, but it cannot correspond to a physical vacuum in the infrared, since it yields an infrared divergent state [4].[2] This divergence manifests itself, for example, as the infrared divergence in the two-point Green functions. This can be easily seen by considering the infrared of the propagator (11) and (13). Making use of the small argument expansion of the Hankel functions (A4) and (A3) one easily finds for the infrared behavior of the coincident propagator in the Bunch-Davies vacuum (15),

$$i\Delta(x;x)_{\rm BD} \propto \int_{\rm IR} k^{D-2-2\nu}(1 + \mathcal{O}(k^2))dk. \tag{16}$$

From this it immediately follows that the coincident propagator in the Bunch-Davies vacuum is infrared divergent for all [5,22]

$$\nu \geq \frac{D-1}{2}. \tag{17}$$

[2]Since it is based on adiabatic considerations, the energy minimization argument does not apply to the infrared, which is highly nonadiabatic.

When Eq. (14) is taken into account, this condition then implies for $D = 4$:

$$\xi \leq \frac{\epsilon(3-2\epsilon)}{6(2-\epsilon)} \qquad (0 \leq \epsilon < 2);$$
$$\xi \geq \frac{\epsilon(2\epsilon-3)}{6(\epsilon-2)} \qquad (\epsilon > 2). \tag{18}$$

These IR-divergent regions correspond to the shaded regions in Fig. 1. Let us first consider the minimally coupled case ($\xi = 0$). In this case an infrared divergence is present for all $\epsilon \leq 3/2$, which also means (5) that the pressure of the fluid driving the Universe's expansion is negative. The IR-divergent regions for the more general case when $\xi \neq 0$ in Fig. 1 are of course more complex. We can qualitatively understand this plot by realizing that the term $\xi R = 6\xi(2-\epsilon)H^2$ acts as an effective, time-dependent mass term for the scalar field. An IR-divergence is possible only when $\xi R < 0$, i.e., when $\xi < 0$ (for $0 \leq \epsilon < 2$) and when $\xi > 0$ (for $\epsilon > 2$).

We shall now show that an early radiation era regulates in a natural way the infrared divergence (16) of the Bunch-Davies vacuum in expanding spaces driven by a fluid of negative pressure. In our model at the matching time $t = \hat{t}$ the parameter ϵ changes suddenly from $\epsilon_0 = 2$ to an arbitrary ϵ. In order to match properly, we need not just to match the scale factor and the Hubble parameter

$$a(t \to \hat{t}) = \hat{a}; \qquad H(t \to \hat{t}) = \hat{H}, \tag{19}$$

but also the field's mode functions (13),

$$\psi_0(\hat{t},k) = \psi(\hat{t},k); \quad \frac{d}{dt}\psi_0(t,k)|_{t=\hat{t}} = \frac{d}{dt}\psi(t,k)|_{t=\hat{t}}, \tag{20}$$

where ψ_0 denotes the mode function of the early radiation era,

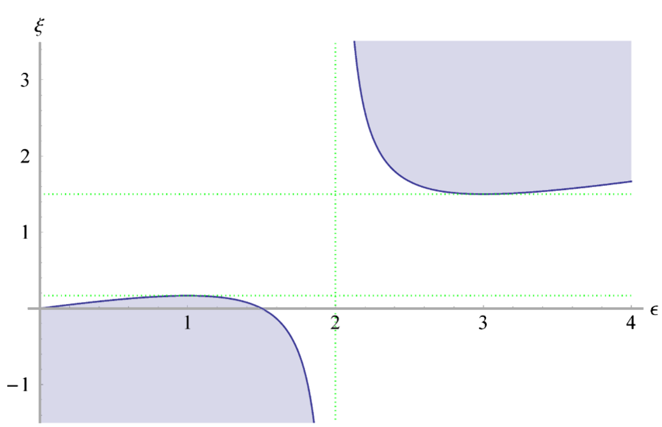

FIG. 1 (color online). The shaded regions indicate the regions where $\nu \geq 3/2$, with ν given in (14) and we choose $D = 4$. When this requirement is met, the coincident propagator (16) is infrared divergent. The regions are bounded by the curve $\xi_{\rm cr} = \epsilon(3-2\epsilon)/[6(2-\epsilon)]$ [see also (18)] and the dotted asymptotes are $\epsilon = 2$, $\xi = 1/6$ and $\xi = 3/2$.

$$\psi_0(t,k) \equiv \psi(t<\hat{t},k) = a^{1-D/2}\frac{-i}{\sqrt{2k}}e^{ik/(1-\epsilon_0)Ha}, \qquad (21)$$

which are just the mode functions of the conformal Minkowski vacuum.[3] We recall here that ϵ and ξ each have a very different physical origin: while ξ is a parameter in the Lagrangian, and thus fixed, ϵ is a dynamical quantity, whose value is set by the nature of the fluid causing background's expansion. The mode functions (21) are nonetheless independent of ξ, since in radiation era, the Ricci scalar is identically zero. We thus stress that our bare Lagrangian remains fixed throughout the history of the Universe.

The rest of the calculation needed to obtain the regulated propagator in the inflationary era are in principle straightforward, but in practice technically complex. When the matching coefficients $\alpha(k)$ and $\beta(k)$ of the mode functions (13) are evaluated by solving (20), the propagator (11) can be expressed in terms of an integral which depends not just on the distance function, but also on the matching time. The resulting propagator can be formally expressed in terms of the Appell functions, which are functions of two arguments and are a generalization of the Gauss' hypergeometric functions. Because of our insufficient understanding of the analytic structure of these functions, we were unable to use them to sensibly calculate the one-loop stress-energy tensor needed to study the backreaction. Instead, we evaluated the coincident propagator (which is sufficient to calculate the one-loop stress energy) as follows. We first matched the early radiation era modes onto the modes of a space-time corresponding to a $1/2$ integer ν, obtaining thus the answer in terms of finite sums, which we were able to explicitly evaluate. We then analytically extended the result to a general (complex) ν, and checked our answer numerically for some special values of ν. The details of this procedure are outlined in Appendix B. The main result is the following coincident propagator (B13):

$$\begin{aligned} i\Delta(x;x) &= \frac{H^2(1-\epsilon)^2}{16\pi^2}\Big(\nu^2-\frac{1}{4}\Big)\Big[-\frac{2\mu^{D-4}}{D-4}-c_\nu \\ &\quad + \ln\Big(\frac{4\pi\mu^2}{(1-\epsilon)^2H^2}\Big(1-\frac{1}{\zeta}\Big)^2\Big)+2\zeta-\zeta^2 \\ &\quad -\ln\Big(1-\frac{1}{\zeta^2}\Big)-\frac{\zeta^{1-2\nu}}{\nu-\frac{1}{2}}\times {}_2F_1\Big(1,\nu-\frac{1}{2};\nu+\frac{1}{2};\frac{1}{\zeta^2}\Big) \\ &\quad -\ln(1-\zeta^2)-\frac{\zeta^{3+2\nu}}{\nu+\frac{3}{2}}\times {}_2F_1\Big(1,\nu+\frac{3}{2};\nu+\frac{5}{2};\zeta^2\Big)\Big], \end{aligned} \qquad (22)$$

where $\zeta = \widehat{aH}/(aH)$ and ν is given in (14). Since in accelerating space-times $\dot{a} = aH$ (ζ) grows (decays) with time, while in decelerating space-times aH (ζ) decays (grows) with time, it is useful to rewrite (22) in a form more suitable for accelerating and decelerating space-times.

Thus, in the accelerating case ($\epsilon < 1$, $\zeta = \widehat{aH}/(aH) < 1$) from Eqs. (22) and (B14) we have

$$\begin{aligned} i\Delta(x;x) &= \frac{H^2(1-\epsilon)^2}{16\pi^2}\Big(\nu^2-\frac{1}{4}\Big)\Big[-\frac{2\mu^{D-4}}{D-4}-c_\nu+\pi\tan(\pi\nu) \\ &\quad +\ln\Big(\frac{4\pi\mu^2}{(1-\epsilon)^2H^2(1+\zeta)^2}\Big)+2\zeta-\zeta^2 \\ &\quad -\sum_{\pm}\frac{\zeta^{3\pm2\nu}}{\frac{3}{2}\pm\nu}\times {}_2F_1\Big(1,\frac{3}{2}\pm\nu;\frac{5}{2}\pm\nu;\zeta^2\Big)\Big]. \end{aligned} \qquad (23)$$

Notice the symmetry $\nu \to -\nu$ in all ζ dependent terms (although, one should keep in mind that we assumed $\nu > 0$). This symmetry has been observed when the infrared is regulated by placing the Universe in a comoving box [22]. On the other hand, in the decelerating case ($\epsilon > 1$, $\zeta > 1$) we get from Eqs. (22) and (B15),

$$\begin{aligned} i\Delta(x;x) &= \frac{H^2(1-\epsilon)^2}{16\pi^2}\Big(\nu^2-\frac{1}{4}\Big)\Big[-\frac{2\mu^{D-4}}{D-4}-c_\nu+\pi\tan(\pi\nu) \\ &\quad +\ln\Big(\frac{4\pi\mu^2}{(1-\epsilon)^2H^2(1+\zeta)^2}\Big)+2\zeta-\zeta^2 \\ &\quad -\sum_{\pm}\frac{\zeta^{1\mp2\nu}}{-\frac{1}{2}\pm\nu}\times {}_2F_1\Big(1,-\frac{1}{2}\pm\nu;\frac{1}{2}\pm\nu;\frac{1}{\zeta^2}\Big)\Big]. \end{aligned} \qquad (24)$$

Just like the accelerating case (23), this expression also exhibits a $\nu \to -\nu$ symmetry in all ζ dependent terms. It is worth noting that in the limit when ν is a half integer, $\nu \to N + 1/2$ ($N = 0, \pm1, \pm2, ..$), both coincident propagators (23) and (24) can be written in terms of finite sums. In the special case $N = 0$, both propagators are simply zero, while for all other cases the simple pole in the tangent cancels against the simple pole in the hypergeometric function.

From Eqs. (23) and (24) we can easily extract the leading order late time ($a \to \infty$) behavior of the coincident propagator in the accelerating case ($aH \to \infty$, such that $\zeta \to 0$) and decelerating case ($aH \to 0$, such that $\zeta \to \infty$):

$$\begin{aligned} i\Delta(x;x) &\propto H^2\zeta^{3-2\nu} \qquad (\epsilon<1,\ \nu>\tfrac{3}{2}); \\ i\Delta(x;x) &\propto H^2\zeta^{1+2\nu} \qquad (\epsilon>1,\ \nu>\tfrac{1}{2}). \end{aligned} \qquad (25)$$

In the special case when $\nu = 3/2$ the dependence on ζ in the accelerating case is logarithmic.

III. ENERGY MOMENTUM TENSOR

The energy momentum tensor is defined by

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}. \qquad (26)$$

[3] More generally, we could have defined the radiation era vacuum by thermal Green functions. That would complicate our treatment in that, in order to describe the early thermal state, we would have to use a density matrix description instead of the mode functions. Nevertheless, our final results for the backreaction would change in detail, but the qualitative features of our treatment would remain unchanged.

By varying the action (7) we obtain

$$T_{\mu\nu} = (\partial_\mu \phi)\partial_\nu \phi - \tfrac{1}{2} g_{\mu\nu} g^{\alpha\beta}(\partial_\alpha \phi)\partial_\beta \phi + \xi(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R)\phi^2 - \xi(\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box)\phi^2. \quad (27)$$

We take the trace to obtain

$$T^\mu{}_\mu = -\frac{D-2}{2}((\partial^\mu \phi)\partial_\mu \phi + R\xi\phi^2) + (D-1)\xi\Box\phi^2 \quad (28)$$

and, using the equation of motion (8) for ϕ, we find that the one-loop expectation value is given by

$$T_q \equiv \langle\Omega|T^\mu{}_\mu|\Omega\rangle = -\tfrac{1}{4}((D-2) - 4(D-1)\xi)\Box i\Delta(x;x). \quad (29)$$

Since any quantum correction will respect the symmetries of the underlying space-time, we know that we should be able to write (we use a subscript q to indicate we are considering the quantum corrections to the stress-energy tensor)

$$\langle\Omega|T^\mu{}_\nu|\Omega\rangle = \mathrm{diag}(-\rho_q, \underbrace{p_q, p_q, .., p_q}_{D-1}) \quad (30)$$

and then we can calculate ρ_q and p_q from T_q using the covariant stress-energy conservation, which we write as

$$\frac{1}{H}\dot{\rho}_q + D\rho_q = \rho_q - (D-1)p_q = -T_q. \quad (31)$$

Now an important question is whether the energy density in the quantum corrections can dominate over the energy density in the classical background. The background Friedmann equations (3) tell us the scaling of energy density

$$\rho_b = \frac{\hat{\rho}_b}{a^{3(1+w_b)}}, \quad (32)$$

where w_b denotes the equation of state parameter of the background and $\hat{\rho}_b = \rho_b(\hat{t})$. Since the background energy density is responsible for the expansion of the Universe, from (5) we have that

$$w_b \equiv \frac{p_b}{\rho_b} = -1 + \frac{2}{3}\epsilon. \quad (33)$$

Furthermore, we know from the conservation Eq. (31), which of course holds for any stress-energy tensor, that the energy density of the quantum contribution scales in general with the scale factor as

$$\rho_q = \frac{\hat{\rho}_q}{a^{3(1+w_q)}}, \quad (34)$$

where $w_q = p_q/\rho_q$ is the equation of state parameter for quantum matter contribution and $\hat{\rho}_q$ is the energy density at the matching, which is typically small when compared to the background density at the matching, $\hat{\rho}_q \sim \hat{H}^4 \sim (\hat{H}/M_P)^2\hat{\rho}_b$. Here $M_P = (8\pi G_N)^{-1/2} \simeq 2.4 \times 10^{18}$ GeV is the reduced Planck mass, $\hat{H} = H(\hat{t})$ is the Hubble parameter at the matching. For simplicity we took the scale factor at the matching to be equal to unity, $\hat{a} = a(\hat{t}) = 1$.

From Eqs. (33), (33), and (34) we see that

$$\frac{\rho_q}{\rho_b} = \frac{\hat{\rho}_q}{\hat{\rho}_b} a^{3(w_b - w_q)} \sim \left(\frac{\hat{H}}{M_p}\right)^2 a^{3(w_b - w_q)}, \quad (35)$$

which implies that the quantum contribution to the stress energy grows with respect to the background contribution whenever

$$w_q < w_b. \quad (36)$$

Hence, the principal task of our investigation is to find out whether this condition is ever met. If the answer is *affirmative*, then—in the light of Eq. (35)—we would be led to the conclusion that (after a sufficiently long time) the energy of quantum fluctuations would eventually dominate the energy of the background, leading to a strong quantum backreaction. Such a strong backreaction might then significantly change the evolution of the Universe. Of course this would then in principle ruin our *Ansatz* that ϵ is constant, and thus one should be careful in interpreting the result. If the backreaction is large, the critical time t_c after which quantum contributions start to dominate can be estimated to be

$$t_c = \hat{t}\left(\frac{\hat{\rho}_b}{\hat{\rho}_q}\right)^{(1+w_b)/2(w_b-w_q)} \sim \hat{t}\left(\frac{M_P}{\hat{H}}\right)^{(1+w_b)/(w_b-w_q)} \quad (w_q < w_b). \quad (37)$$

If however $w_q > w_b$, the energy density due to the quantum contributions will be negligible at all times. But before we address this question, we must renormalize the stress energy, which is what we do next.

IV. RENORMALIZATION

The only divergence arising in the one-loop stress-energy tensor is induced by the divergent part of the coincident propagator (B13), which we write in the form

$$i\Delta(x;x)_{\mathrm{div}} = -\frac{H^2[(D-2) - 4(D-1)\xi](D-2\epsilon)}{32\pi^2}\frac{\mu^{D-4}}{D-4}, \quad (38)$$

where we took into account Eq. (14). Now from the trace Eq. (29) we get for the divergent contribution to the stress-energy trace,

$$(T_q)_{\rm div} = \frac{[(D-2)-4(D-1)\xi]^2(D-2\epsilon)(D-1-3\epsilon)\epsilon}{64\pi^2} \times \frac{\mu^{D-4}H^4}{D-4} = \frac{3(1-6\xi)^2(2-\epsilon)(1-\epsilon)\epsilon}{8\pi^2}\frac{\mu^{D-4}H^4}{D-4} + \frac{(1-6\xi)\epsilon H^4}{16\pi^2}[(19-23\epsilon+6\epsilon^2) - 6\xi(15-17\epsilon+4\epsilon^2)], \tag{39}$$

where we took into account $\Box H^2 = -(\partial_t + (D-1)H)\partial_t H^2 = 2\epsilon(D-1-3\epsilon)H^4$.

It is known that this theory can be renormalized using only one counterterm, proportional to R^2. Indeed, taking a functional derivative with respect to $g^{\mu\nu}$ of the R^2 counterterm action results in [22]

$$({\tt ct})_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^Dx\sqrt{-g}\alpha R^2 = \alpha(4\nabla_\mu\nabla_\nu R - 4g_{\mu\nu}\Box R + g_{\mu\nu}R^2 - 4RR_{\mu\nu}). \tag{40}$$

Multiplying by $g^{\mu\nu}$ results in the trace

$$({\tt ct})^\mu_\mu = \alpha(-4(D-1)\Box R + (D-4)R^2) = \alpha(D-1)^2(D-2\epsilon)H^4(-8\epsilon(D-1-3\epsilon) + (D-4)(D-2\epsilon)) = -432\alpha(2-\epsilon)(1-\epsilon)\epsilon H^4 + 36\alpha(4-34\epsilon + 35\epsilon^2 - 8\epsilon^3)(D-4)H^4 + \mathcal{O}((D-4)^2). \tag{41}$$

Upon comparing this with Eq. (39), we can read off α which renormalizes the theory,

$$\alpha = \frac{(1-6\xi)^2}{1152\pi^2}\frac{\mu^{D-4}}{D-4} + \alpha_f, \tag{42}$$

where α_f is an arbitrary but finite constant. One can easily show that this choice of α renders all components of $(T^\nu_\mu)_q$ finite. Combining the two contributions (39) and (41), we get the following finite expression coming from the UV divergent contribution, which we indicate with a superscript (uv)

$$T_q^{(uv)} = (T_q)_{\rm div} + ({\tt ct})^\mu_\mu = \frac{(1-6\xi)(2-\epsilon)^2H^4}{32\pi^2}\Big((1-6\xi) + 4\frac{\epsilon(1-\epsilon)}{2-\epsilon}\Big) - 432\alpha_f(2-\epsilon)(1-\epsilon)\epsilon H^4. \tag{43}$$

In the following section we shall consider the contribution to the trace of the stress-energy tensor coming from the finite part of the coincident propagator (B13) to study the late time behavior of the quantum fluid. This contribution we shall indicate with a superscript (ir), $i\Delta^{(ir)} = i\Delta - i\Delta_{\rm div}$, where $i\Delta_{\rm div}$ is the divergent part of $i\Delta$ given in (38).

V. THE EQUATION OF STATE OF THE QUANTUM FLUID

In this section we calculate both the pressure and energy density of the quantum fluid that results from one-loop scalar fluctuations in our model. The general procedure is quite straightforward (albeit somewhat tedious): we first need to act with the d'Alembertian on the ultraviolet parts of the coincident propagator given in Eqs. (23) and (24) to get the trace of the stress-energy tensor $T_q^{(ir)}$ as defined in Eq. (29). To get the full T_q one has to add the finite contribution $T_q^{(uv)}$ in (43) that remained after renormalization. When we have obtained the full stress-energy tensor, we can solve the conservation Eq. (31) to obtain the quantum energy density ρ_q and quantum pressure p_q, from which we obtain the quantum equation of state parameter $w_q = p_q/\rho_q$. To facilitate this procedure we give a list of useful formulae in Appendices A and B.

To illustrate the procedure, we shall first calculate the contribution to ρ_q and p_q coming from the finite remainder from ultraviolet fluctuations (43). We denote these contributions by $\rho_q^{(uv)}$ and $p_q^{(uv)}$.

To calculate $\rho_q^{(uv)}$ we need to integrate Eq. (31), which we can do with the help of Eq. (D2) in Appendix D. The result is

$$\rho_q^{(uv)} = -\frac{(1-6\xi)(2-\epsilon)^2H^4}{128\pi^2(1-\epsilon)}((1-6\xi)+4\epsilon) + 108\alpha_f(2-\epsilon)\epsilon H^4. \tag{44}$$

The contribution to the pressure is then simply

$$p_q^{(uv)} = \frac{T_q^{(uv)} + \rho_q^{(uv)}}{3} = \Big(-\frac{(1-6\xi)(2-\epsilon)^2H^4}{128\pi^2(1-\epsilon)}((1-6\xi)+4\epsilon) + 108\alpha_f(2-\epsilon)\epsilon H^4\Big)\Big(-1+\frac{4}{3}\epsilon\Big) = \rho_q^{(uv)}\Big(-1+\frac{4}{3}\epsilon\Big). \tag{45}$$

Of course, the one-loop infrared contribution needs to be taken into account for a complete answer. We shall now consider the general infrared contributions to the quantum energy density and pressure. We first consider the accelerated and then the decelerated case.

A. Matching onto acceleration ($\epsilon < 1$) for a general ν

In this section we consider in some detail the general accelerating case ($\epsilon < 1$, $\zeta < 1$). The decelerating case we shall consider in Sec. V B. The contribution to the trace of the stress-energy tensor coming from the UV divergent terms has been calculated in (43). Our starting point for the remaining terms is the coincident propagator (23).

Now making use of Eq. (C1) in Appendix C the contribution to the trace of the quantum stress-energy tensor (29) from the remaining terms becomes

$$T_q^{(ir)} = -\frac{(1-6\xi)^2(1-\epsilon)(2-\epsilon)H^4}{16\pi^2}\bigg\{6\epsilon\bigg[\frac{1}{2}\ln\bigg(\frac{4\pi\mu^2}{(1-\epsilon)^2H^2(1+\zeta)^2}\bigg) - \frac{c_\nu}{2} - \sum_{\pm}\frac{\zeta^{3\pm2\nu}}{3\pm2\nu}\times {}_2F_1\bigg(1,\frac{3}{2}\pm\nu;\frac{5}{2}\pm\nu;\zeta^2\bigg) + \frac{\pi}{2}\tan(\pi\nu)\bigg] - \frac{3-5\epsilon}{1-\epsilon} + \frac{2(2-3\epsilon)}{1+\zeta} - \frac{1-\epsilon}{(1+\zeta)^2} + 2(1+\epsilon)\zeta - \zeta^2 - \sum_{\pm}[(3-5\epsilon)-(1-\epsilon)(3\pm2\nu)]\frac{\zeta^{3\pm2\nu}}{1-\zeta^2} + 2(1-\epsilon)\sum_{\pm}\frac{\zeta^{5\pm2\nu}}{(1-\zeta^2)^2}\bigg\}. \tag{46}$$

Recall that by assumption $\nu > 0$. The $\nu = 1/2$ pole of $\tan(\pi\nu)$ is harmless since in this case $\xi = 1/6$, such that the whole expression vanishes as the result of a vanishing prefactor. In Eq. (46) we made use the derivative of the hypergeometric function

$$\zeta\frac{d}{d\zeta}\frac{\zeta^{2a}}{2a}\times {}_2F_1(1,a;a+1;\zeta^2) = \frac{\zeta^{2a}}{1-\zeta^2}.$$

The total stress-energy trace is then obtained by adding Eqs. (43) and (46). A careful look at Eqs. (46) reveals a quadratic divergence in T_q at the matching time $t = \hat{t}$, at which $\zeta \to 1$. This divergence is also present in the energy density and pressure calculated below in Eqs. (50) and (51), and shows up as the logarithmic divergence in the coincident propagator (B13). This divergence occurs because of the sudden matching procedure assumed in this work, where ϵ jumps suddenly from $\epsilon = 2$ to an arbitrary ϵ, which also implies the sudden change in the Ricci scalar and the scalar field mass parameter. Of course, in any physically realistic situation any of those parameters will change smoothly. These sudden changes induce a change in the quantum contribution by an infinite amount at the matching. An attempt to renormalize it away by including it into the $1/(D-4)$ subtraction in Eq. (B4) would result in divergent contributions away from the matching, which would make matters only worse. But let us try to understand how bad the matching divergence actually is. From Eq. (46) we see that its contribution to T_q is of the order $\sim H^4/[1-\hat{a}\hat{H}/(aH)]^2$. By taking $t-\hat{t} = \delta t$ small, and requiring that $\rho_q < \rho_b = 3M_P^2H^2$, where $M_P = (8\pi G)^{-1/2}$ is the reduced Planck mass, we get that $\rho_q < \rho_b$ implies $\delta t > 1/[[1-\epsilon|M_P]$, which is of the order of the Planck time. Based on this estimate we expect that the matching divergence will be regulated whenever a sudden matching is replaced by a smooth matching, in which ϵ at the matching changes over a period of time that is much longer than the Planck time.

Before we calculate the quantum energy density, we take one more look at the form of Eq. (46). All terms containing ν dependent powers of ζ in (46) can be written as the following sum:

$$(T_q^{(ir)})_\nu = \frac{(1-6\xi)^2(1-\epsilon)(2-\epsilon)H^4}{16\pi^2}\sum_{\pm}\sum_{n=0}^{\infty}\bigg[\frac{6\epsilon}{3\pm2\nu+2n} + (3-5\epsilon)-(1-\epsilon)(3\pm2\nu+2n)\bigg]\zeta^{3\pm2\nu+2n}. \tag{47}$$

The contribution of this sum to the energy density can be, up to a ν dependent integration constant $\mathrm{cte}(\nu)\zeta^4H^4$, determined by integrating Eq. (31), which can be performed by making use of Eq. (D6). The result is

$$(\rho_q^{(ir)})_\nu = \frac{(1-6\xi)^2(1-\epsilon)(2-\epsilon)H^4}{16\pi^2}\sum_{\pm}\sum_{n=0}^{\infty}\bigg[\frac{6\epsilon}{3\pm2\nu+2n} + (3-5\epsilon)-(1-\epsilon)(3\pm2\nu+2n)\bigg]\times\frac{\zeta^{3\pm2\nu+2n}}{(1-\epsilon)(-1\pm2\nu+2n)} + \mathrm{cte}(\nu)\zeta^4H^4, \tag{48}$$

$$= \frac{(1-6\xi)^2(2-\epsilon)H^4}{64\pi^2}\bigg\{-6\epsilon\sum_{\pm}\frac{\zeta^{3\pm2\nu}}{3\pm2\nu}\times{}_2F_1\bigg(1,\frac{3}{2}\pm\nu;\frac{5}{2}\pm\nu;\zeta^2\bigg) - 4(1-\epsilon)\times\sum_{\pm}\frac{\zeta^{3\pm2\nu}}{1-\zeta^2} - 2(2-\epsilon)\sum_{\pm}\frac{\zeta^{3\pm2\nu}}{-1\pm2\nu}\times{}_2F_1\bigg(1,-\frac{1}{2}\pm\nu;\frac{1}{2}\pm\nu;\zeta^2\bigg) + ((2-\epsilon)\pi\tan(\pi\nu)+d_\nu)\zeta^4\bigg\},$$

where in the last step we chose the integration constant to be

$$\mathrm{cte}(\nu) = \frac{(1-6\xi)^2(2-\epsilon)}{64\pi^2}((2-\epsilon)\pi\tan(\pi\nu)+d_\nu), \tag{49}$$

where the tangent contribution is fixed by requiring that $\rho_q^{(ir)}$ be finite for all ν, and d_ν a ν dependent function, which is finite for all ν. Notice that the divergences at half integer ν coming from the first hypergeometric function are canceled by the contribution from the tangent in (46). The integration constant (49) scales as radiation, $\rho_q \propto 1/a^4$, and it can be uniquely fixed by calculating the full stress-energy tensor instead of just the trace. But in order to do that, we need the propagator off coincidence, which we do not have at this moment.

With this we are now ready to calculate the quantum energy density and pressure, which are obtained by integrating Eq. (32) with the help of Eqs. (D4) in Appendix D. The result for the energy density is

$$\rho_q^{(ir)} = \frac{(1-6\xi)^2(2-\epsilon)H^4}{64\pi^2}\Big\{6\epsilon\Big[\frac{1}{2}\ln\Big(\frac{4\pi\mu^2}{(1-\epsilon)^2H^2(1+\zeta)^2}\Big) - \frac{c_\nu}{2} - \frac{6-7\epsilon}{12(1-\epsilon)} - \sum_\pm \frac{\zeta^{3\pm2\nu}}{3\pm2\nu}\times {}_2F_1\Big(1,\frac{3}{2}\pm\nu;\frac{5}{2}\pm\nu;\zeta^2\Big) + \frac{\pi}{2}\tan(\pi\nu) + \zeta - \frac{1}{2}\zeta^2 + \frac{1}{3}\zeta^3\Big] - 2(2-\epsilon)\zeta^4\ln\Big(\frac{1+\zeta}{\zeta}\Big) + 4(1-\epsilon)\frac{\zeta^4}{1+\zeta} - 2(2-\epsilon) \times\sum_\pm \frac{\zeta^{3\pm2\nu}}{-1\pm2\nu}\times {}_2F_1\Big(1,-\frac{1}{2}\pm\nu;\frac{1}{2}\pm\nu;\zeta^2\Big) + ((2-\epsilon)\pi\tan(\pi\nu)+d_\nu)\zeta^4 - 4(1-\epsilon)\sum_\pm\frac{\zeta^{3\pm2\nu}}{1-\zeta^2}\Big\}, \tag{50}$$

with $\nu > 0$. The total energy density is obtained by adding this to the ultraviolet contribution (44). The quantum pressure p_q is then simply a sum of

$$p_q^{(ir)} = \tfrac{1}{3}(T_q^{(ir)} + \rho_q^{(ir)}) \tag{51}$$

and the ultraviolet contribution $p_q^{(uv)}$ in Eq. (45). The equation of state parameter w_q is

$$w_q = \frac{p_q}{\rho_q}. \tag{52}$$

In order to find out whether the quantum contribution can become important, we need to investigate whether the criterion (36) for the growth of quantum contribution with respect to the classical contribution is ever met [see also Eq. (37) above]. Rather then studying $w_q(t)$ in its full generality, we shall consider only the leading (late time) behavior of w_q. For this we need the leading order behavior of T_q, ρ_q, and p_q. Remember that in this section we consider an accelerating universe such that at late times $\zeta \to 0$.

1. *The case when* $\epsilon < 1$, $0 < \nu < 3/2$

From Eqs. (43), (46), (44), (45), (50), and (51) it follows that when $0 < \nu < 3/2$ the leading order late time contribution to the one-loop stress-energy trace, energy density, and pressure are

$$T_q \overset{\zeta\to0}{\longrightarrow} -\frac{(1-6\xi)^2(1-\epsilon)(2-\epsilon)H^4}{16\pi^2}\Big\{6\epsilon\Big[\ln\Big(\frac{H_0}{H}\Big)+\frac{\epsilon}{4(1-\epsilon)}\Big]\Big\}, \tag{53}$$

$$\rho_q \overset{\zeta\to0}{\longrightarrow} \frac{(1-6\xi)^2(2-\epsilon)H^4}{64\pi^2}\Big\{6\epsilon\ln\Big(\frac{H_0}{H}\Big)\Big\}, \tag{54}$$

$$p_q \overset{\zeta\to0}{\longrightarrow} \frac{(1-6\xi)^2(2-\epsilon)H^4}{64\pi^2}\Big\{6\epsilon\Big[\Big(-1+\frac{4}{3}\epsilon\Big)\ln\Big(\frac{H_0}{H}\Big) - \frac{\epsilon}{3}\Big]\Big\}, \tag{55}$$

where

$$\ln(H_0) = \frac{1}{2}\ln\Big(\frac{4\pi\mu^2}{(1-\epsilon)^2}\Big) - \frac{c_\nu}{2} - \frac{3-5\epsilon}{6(1-\epsilon)} - \frac{\epsilon}{4(1-\epsilon)} + \frac{\pi}{2}\tan(\pi\nu) - \frac{2-\epsilon}{3(1-\epsilon)} \times\Big(\frac{1}{4\epsilon}+\frac{1-\epsilon}{(2-\epsilon)(1-6\xi)}\Big) + \frac{1152\pi^2}{(1-6\xi)^2}\alpha_f. \tag{56}$$

These relations then imply for the equation of state parameter $w_q = p_q/\rho_q$:

$$w_q \overset{\zeta\to0}{\longrightarrow} \Big(-1+\frac{4}{3}\epsilon\Big) - \frac{\epsilon}{3\ln(H_0/H)} \to -1+\frac{4}{3}\epsilon. \tag{57}$$

The last implication follows from the observation that at late times $a\to\infty$, $H \propto a^{-\epsilon} \to 0$, such that formally $\ln(H_0/H) \to \infty$ as long as $\epsilon > 0$. When $\epsilon = 0$ one recovers the well-known result, $w_q = -1$. The results (53)–(57) also apply when ν is imaginary, or more generally whenever $\Re[\nu] < 3/2$.

2. *The case when* $\epsilon < 1$, $\nu > 3/2$

When $\nu > 3/2$ in general the lowest ν dependent power in (46) and (50) dominates the stress-energy tensor at late times. From the more convenient form (47) and (48) expressed as a series, we can read off the dominant contribution to T_q, ρ_q and p_q:

$$T_q \overset{\zeta\to0}{\longrightarrow} \frac{(1-6\xi)^2(1-\epsilon)(2-\epsilon)H^4}{16\pi^2}\Big[\frac{6\epsilon}{3-2\nu} + (3-5\epsilon) - (1-\epsilon)(3-2\nu)\Big]\zeta^{3-2\nu}, \tag{58}$$

$$\rho_q \overset{\zeta\to0}{\longrightarrow} -\frac{(1-6\xi)^2(2-\epsilon)H^4}{64\pi^2}\Big[\frac{6\epsilon}{3-2\nu} + (3-5\epsilon) - (1-\epsilon)(3-2\nu)\Big]\frac{4\zeta^{3-2\nu}}{1+2\nu}, \tag{59}$$

$$p_q \stackrel{\zeta\to 0}{\to} -\frac{(1-6\xi)^2(2-\epsilon)H^4}{64\pi^2}\left[\frac{6\epsilon}{3-2\nu}+(3-5\epsilon) - (1-\epsilon)(3-2\nu)\right]\frac{4[\epsilon-2(1-\epsilon)\nu]}{3[1+2\nu]}\zeta^{3-2\nu}. \quad (60)$$

From this we conclude,

$$w_q \stackrel{\zeta\to 0}{\to} -\frac{2}{3}(1-\epsilon)\nu+\frac{\epsilon}{3} \qquad (3/2<\nu)$$
$$\stackrel{\zeta\to 0}{\to} w_b+\frac{1}{3}\left((3-\epsilon)-\sqrt{(3-\epsilon)^2-24(2-\epsilon)\xi}\right) \quad (61)$$

from which we see that $w_q < w_b$ if ξ is negative and $w_q > w_b$ is ξ is positive, for all relevant ϵ (remember that this analysis holds for $\epsilon < 1$). Thus, we find that the quantum contribution to the energy density grows with respect to the background energy density for $\xi < 0$.

This result is correct however, provided the leading order contributions (58)–(60) do not vanish, which is the case when the expression in the square brackets does not vanish, i.e., when

$$\frac{6\epsilon}{3-2\nu}+(3-5\epsilon)-(1-\epsilon)(3-2\nu)\neq 0. \quad (62)$$

We shall now study in some detail the special case when (62) vanishes.

3. The special case when $\epsilon < 1$, $\xi = 0$

It is instructive to observe that (62) vanishes for the following two values of ν:

$$\nu \in \left\{0, \frac{3-\epsilon}{2(1-\epsilon)}\right\}, \quad (63)$$

or equivalently when

$$\xi \in \left\{\frac{(3-\epsilon)^2}{24(2-\epsilon)}, 0\right\}. \quad (64)$$

The first value in (63) is irrelevant since we consider $\nu > 3/2$. The second value however might be interesting. It corresponds to the minimal coupling of the scalar field to curvature, $\xi = 0$. If $3/2 < \nu < 5/2$ (or equivalently $0 < \epsilon < 1/2$), the leading order contribution in this case is logarithmic and the above analysis presented in Sec. VA 1 applies and we have [cf. Eq. (57)],

$$w_q \stackrel{\zeta\to 0}{\to} -1+\frac{4}{3}\epsilon, \quad (\xi=0, 3/2<\nu<5/2, 0<\epsilon<1/2). \quad (65)$$

When, on the other hand, $\nu > 5/2$ ($1/2 < \epsilon < 1$), the dominant contribution comes from the subdominant term in the ν dependent series (47) and (48). Since we study the case where $\xi = 0$ we have from (14) that $\nu = (3-\epsilon)/[2(1-\epsilon)]$ and we can write the stress-energy contributions as

$$T_q \stackrel{\zeta\to 0}{\to} \frac{(1-6\xi)^2(1-\epsilon)^2(2-\epsilon)(1+\epsilon)H^4}{16\pi^2(1-2\epsilon)}\zeta^{5-2\nu}, \quad (66)$$

$$\rho_q \stackrel{\zeta\to 0}{\to} -\frac{(1-6\xi)^2(1-\epsilon)^2(2-\epsilon)(1+\epsilon)H^4}{32\pi^2(1-2\epsilon)}\zeta^{5-2\nu}, \quad (5/2<\nu, 1/2<\epsilon<1), \quad (67)$$

$$p_q \stackrel{\zeta\to 0}{\to} \frac{(1-6\xi)^2(1-\epsilon)^2(2-\epsilon)(1+\epsilon)H^4}{96\pi^2(1-2\epsilon)}\zeta^{5-2\nu}. \quad (68)$$

Thus the equation of state parameter is in this case,

$$w_q \stackrel{\zeta\to 0}{\to} -\frac{1}{3}, \qquad (\xi=0, \nu>5/2, 1/2<\epsilon<1). \quad (69)$$

In Fig. 2 we illustrate the results of this section (these figures also already show the results for the decelerating case, $\epsilon > 1$ which we shall discuss in Sec. V B). From the previous discussion we found that in an accelerating

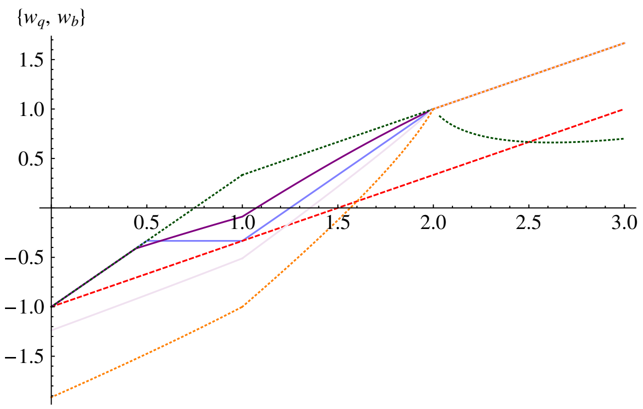

FIG. 2 (color online). The equation of state parameters for the background w_b and for the quantum fluid w_q *versus* $\epsilon = -\dot{H}/H^2$ for various cases. Quantum corrections become eventually strong if $w_q < w_b$ for any ϵ. The red dashed curve represents the background scaling, $w_b = -1 + \frac{2}{3}\epsilon$. The solid blue curve represents w_q for ξ on the boundary of stability: $\xi = 0$ for $\epsilon < 1$ and $\xi = |\epsilon - 1|/(3|2-\epsilon|)$ for $\epsilon > 1$). In this case $w_q = -1 + \frac{4}{3}\epsilon$ $(0 < \epsilon < 1/2)$, $w_q = -\frac{1}{3}$ $(1/2 < \epsilon < 1)$, $w_q = -\frac{5}{3} + \frac{4}{3}\epsilon$ $(1 < \epsilon < 2)$ and $w_q = -\frac{1}{3} + \frac{2}{3}\epsilon$ $(\epsilon > 2)$, such that the condition $w_q < w_b$ is not met for any choice of ϵ. Next, we show two curves for positive values of ξ: the solid dark purple ($\xi = 0.1$) and dotted dark green line ($\xi = 1$). For the case when $\xi = 0.1$ we have $w_q = -1 + \frac{4}{3}\epsilon$ $(0 < \epsilon < \epsilon_c \simeq 0.44)$, $w_q = -\frac{2}{3}\nu + \frac{\epsilon}{3}$ $(\epsilon_c < \epsilon < 1)$, $w_q = \epsilon - \frac{2}{3}(1+\nu)$ $(1 < \epsilon < 2)$ and $w_q = -\frac{1}{3} + \frac{2}{3}\epsilon$ $(\epsilon > 2)$. When $\xi = 1$ we have $w_q = -1 + \frac{4}{3}\epsilon$ $(0 < \epsilon < 1)$, $w_q = -\frac{1}{3} + \frac{2}{3}\epsilon$ $(1 < \epsilon < 2)$ and $w_q = \epsilon - \frac{2}{3} \times (1+\nu)$ $(\epsilon > 2)$. Finally, there are two curves for negative values of ξ: $\xi = -0.1$ (solid very light purple) and $\xi = -0.5$ (dotted light orange). Both curves show $w_q = -\frac{2}{3}\nu + \frac{\epsilon}{3}$ $(0 < \epsilon < 1)$, $w_q = \epsilon - \frac{2}{3}(1+\nu)$ $(1 < \epsilon < 2)$ and $w_q = -\frac{1}{3} + \frac{2}{3}\epsilon$ $(\epsilon > 2)$. In the negative ξ cases we see that when $w_q < w_b$ is generically met in accelerating universes $(0 < \epsilon < 1)$ and in addition in a part of the region where pressure $p_b < 1/3\rho_b$ $(1 \leq \epsilon < 2)$.

universe quantum effects can become important if $\xi < 0$ while they will always be subdominant for $\xi > 0$. The solid blue line in Fig. 2 shows the borderline behavior: $\xi = 0$. The equation of state parameter of quantum fluid $w_q = -1 + 4\epsilon/3$ [when $0 < \epsilon < 1/2$, corresponding to (65)] and $w_q = -1/3$[when $1/2 < \epsilon < 1$, corresponding to (69)] is always larger than the corresponding background parameter $w_b = -1 + 2\epsilon/3$, implying that the criterion (36) is never met and the quantum contribution can never become important. If, on the other hand, $\xi \neq 0$ the situation changes drastically, as can be seen from other plots in Fig. 2. For example, the dotted dark green ($\xi = 0.1$) and solid dark purple ($\xi = 1$) lines show $w_q = q_q(\epsilon)$ for positive values of ξ. The curves are given by $w_q = -1 + \frac{4}{3}\epsilon$ for $\nu < 3/2$ (corresponding to (57)) and $w_q = -\frac{2}{3}(1-\epsilon)\nu + \frac{\epsilon}{3}$ for $\nu > 3/2$ [corresponding to (61)]. We indeed see that we always have $w_q \geq w_b$, such that the energy density due to the quantum effects dilutes faster then (or equally fast to) the background energy density. Finally, the solid light purple ($\xi = -0.1$) and dotted light orange ($\xi = -0.5$) lines in Fig. 2 shows w_q for negative values of ξ. Since for negative ξ we always have that $\nu > 3/2$ we have [in accordance with (61)] that $w_q = -\frac{2}{3}(1-\epsilon)\nu + \frac{\epsilon}{3}$. We indeed see that $w_q < w_b$ in this case ($0 < \epsilon \leq 1$) and thus the energy density in the quantum contribution will become dominant at late enough times. The time it takes for the quantum contribution to become important can be estimated from Eq. (37) to be

$$t_c \sim \frac{1}{(1-\epsilon)\hat{H}}\left(\frac{M_P}{\hat{H}}\right)^{\epsilon(3-\epsilon)/(-6\xi(2-\epsilon))}, \qquad (\xi < 0, 0 < \epsilon < 1). \tag{70}$$

For $0 < -6\xi \ll 1$ this time can be very long, much longer than the age of the Universe. In fact, $-6\xi < 1$ can be tuned to make t_c of the order of the age of the Universe, in which case one could relate the contribution of quantum fluctuations to the dark energy of the Universe, and perhaps even address the question *"Why now?"* of the dark energy dominance. The above analysis works however only for matching the radiation era onto accelerating universes, in which the Universe spends most of the time, and that does not correspond to the observed Universe. Therefore, we also need to consider radiation matching onto the decelerated universes, which is what we hurriedly do next.

B. Matching onto deceleration for a general ν

The exact result for $T_q^{(ir)}$ in Eq. (46) is also correct for the decelerating universe. Since in a decelerating universe at late times $\zeta \to \infty$, it is more suitable to express $T_q^{(ir)}$ in terms of the hypergeometric function of $1/\zeta^2$. The result is

$$\begin{aligned} T_q^{(ir)} = &-\frac{(1-6\xi)^2(1-\epsilon)(2-\epsilon)H^4}{16\pi^2}\bigg\{6\epsilon\bigg[\frac{1}{2}\ln\bigg(\frac{4\pi\mu^2}{(1-\epsilon)^2H^2(1+\zeta)^2}\bigg) - \frac{c_\nu}{2} \\ &- \sum_\pm \frac{\zeta^{1\mp 2\nu}}{-1\pm 2\nu} \times {}_2F_1\bigg(1, -\frac{1}{2}\pm\nu; \frac{1}{2}\pm\nu; \frac{1}{\zeta^2}\bigg) + \frac{\pi}{2}\tan(\pi\nu)\bigg] - \frac{3-5\epsilon}{1-\epsilon} + \frac{2(2-3\epsilon)}{1+\zeta} - \frac{1-\epsilon}{(1+\zeta)^2} + 2(1+\epsilon)\zeta - \zeta^2 \\ &- \sum_\pm[(3-5\epsilon) - (1-\epsilon)(3\mp 2\nu)]\frac{\zeta^{3\mp 2\nu}}{1-\zeta^2} + 2(1-\epsilon)\sum_\pm\frac{\zeta^{5\mp 2\nu}}{(1-\zeta^2)^2}\bigg\}, \end{aligned} \tag{71}$$

where we made use of the identity

$$\frac{\zeta^{2\alpha}}{2\alpha} \times {}_2F_1(1, \alpha; \alpha+1; \zeta^2) = \frac{\zeta^{2(\alpha-1)}}{2(1-\alpha)} \times {}_2F_1\bigg(1, 1-\alpha; 2-\alpha; \frac{1}{\zeta^2}\bigg) + \frac{\pi}{2}\cot(\pi\alpha), \qquad (\zeta^2 > 1). \tag{72}$$

In the derivation of this identity, we took the central value along the cut of the hypergeometric function $\zeta^2 \geq 1$ (this prescription removes a possible imaginary contribution). Notice that for $\alpha = (3/2) \pm \nu$ the last bit in (72) does not contribute, since $\sum_\pm \cot[\pi(3/2) \pm \pi\nu] = 0$. The result (71) can be also obtained from Eq. (29) by acting with $\Box$ on Eq. (24), representing a check of (71). The identity (72) can be also used to derive $\rho_q^{(ir)}$ from Eq. (50),

$$\begin{aligned} \rho_q^{(ir)} = &\frac{(1-6\xi)^2(2-\epsilon)H^4}{64\pi^2}\bigg\{6\epsilon\bigg[\frac{1}{2}\ln\bigg(\frac{4\pi\mu^2}{(1-\epsilon)^2H^2(1+\zeta)^2}\bigg) - \frac{c_\nu}{2} - \frac{6-7\epsilon}{12(1-\epsilon)} - \sum_\pm\frac{\zeta^{1\mp 2\nu}}{-1\pm 2\nu} \times {}_2F_1\bigg(1, -\frac{1}{2}\pm\nu; \frac{1}{2}\pm\nu; \frac{1}{\zeta^2}\bigg) \\ &+ \frac{\pi}{2}\tan(\pi\nu) + \zeta - \frac{1}{2}\zeta^2 + \frac{1}{3}\zeta^3\bigg] - 2(2-\epsilon)\zeta^4\ln\bigg(1+\frac{1}{\zeta}\bigg) + 4(1-\epsilon)\frac{\zeta^4}{1+\zeta} \\ &- 2(2-\epsilon)\sum_\pm\frac{\zeta^{1\mp 2\nu}}{3\pm 2\nu} \times {}_2F_1\bigg(1, \frac{3}{2}\pm\nu; \frac{5}{2}\pm\nu; \frac{1}{\zeta^2}\bigg) + ((2-\epsilon)\pi\tan(\pi\nu) + d_\nu)\zeta^4 - 4(1-\epsilon)\sum_\pm\frac{\zeta^{3\mp 2\nu}}{1-\zeta^2}\bigg\}. \end{aligned} \tag{73}$$

The one-loop pressure $p_q^{(ir)}$ is, as before, obtained by inserting (71) and (73) into relation (51). We are now ready to consider the late time limit, $a \to \infty$ which implies for the decelerating case that $\zeta \to \infty$, of the stress-energy tensor and the corresponding equation of state parameter in a decelerating universe. Before we begin analyzing particular cases, let us observe the general structure of Eqs. (71) and (73). The curly brackets in (71) contain terms that grow $\propto \zeta^4$ and terms with ν-dependent powers of ζ. The ν-dependent powers can be summarized as

$$(T_q^{(ir)})_\nu = \frac{(1-6\xi)^2(1-\epsilon)(2-\epsilon)H^4}{16\pi^2}\sum_{\pm}\sum_{n=0}^{\infty}\bigg[\frac{6\epsilon}{-1\pm 2\nu+2n} - (3-5\epsilon)-(1-\epsilon)(-1\pm 2\nu+2n)\bigg]\zeta^{1\mp 2\nu-2n}. \quad (74)$$

Similarly, $\rho_q^{(ir)}$ in (73) contains terms that grow as $\propto \zeta^4$, and the terms that contain ν-dependent powers of ζ, which can be summarized as

$$(\rho_q^{(ir)})_\nu = \frac{(1-6\xi)^2(2-\epsilon)H^4}{64\pi^2}\sum_{\pm}\sum_{n=0}^{\infty}\bigg[\frac{6\epsilon}{-1\pm 2\nu+2n} - (3-5\epsilon)-(1-\epsilon)(-1\pm 2\nu+2n)\bigg] \times \frac{4\zeta^{1\mp 2\nu-2n}}{3\pm 2\nu+2n}. \quad (75)$$

This form could be also obtained by integrating (74), representing a nontrivial check of our result (73). It is now clear that the leading order late time behavior of ρ_q and T_q is dictated either by the term $\propto \zeta^4$ in ρ_q or by a term $\propto \zeta^{1+2\nu-2n}$ with $n = 0, 1, \ldots$ in Eqs. (74) and (75). In both cases, the ultraviolet contributions (43)–(45) are subdominant and can be neglected. Let us consider first in somewhat more detail the contribution $\propto \zeta^4$. This contribution is the dominant one, for all $\nu < 3/2$. It can be easily seen to lead to a $w_q = 1/3$ (since this contribution does is not influenced by the trace of the stress-energy tensor, its contribution must be traceless), independent of ϵ, and therefore this contribution will always lead to a strong backreaction, for larger enough values of ϵ (such that w_b is large). We do not believe however that this effect is due to infrared particle production. The reason is the following. A large backreaction, on physical grounds, is only expected when due to particle creation the infrared becomes highly correlated. This high amount of correlation leads to the infrared divergence and thus is present only for $\nu > 3/2$. Now not only does our procedure not fix this contribution uniquely, in Appendix F we show that this contribution *cannot* be determined uniquely in the present model. The reason is that it turns out that the contribution to the stress-energy tensor $\propto H^4\zeta^4$, is ultraviolet divergent. The physical origin for this divergence is probably the sudden matching between the two space-times we consider. Such a divergence then needs to be renormalized, and this renormalization inevitably introduces an arbitrariness in the coefficient in front of the $H^4\zeta^4$ term. Therefore we are free to choose the constant d_ν and for the purpose of present paper we choose the constant d_ν such that the term proportional to ζ^4 cancels. In this case we have that when $\nu < 1/2$ a term proportional to ζ^2 dominates, or when $\nu > 1/2$ the dominant term is proportional to $\zeta^{1+2\nu-2n}$.

1. *The case when* $\epsilon > 1$, $0 < \nu < 1/2$

In this case the dominant contribution to the one-loop stress-energy tensor comes from the term $\propto \zeta^2$ in (71) and

$$T_q \overset{\zeta\to\infty}{\longrightarrow} \frac{(1-6\xi)^2(1-\epsilon)(2-\epsilon)H^4}{16\pi^2}\zeta^2, \quad (76)$$

$$\rho_q \overset{\zeta\to\infty}{\longrightarrow} \frac{(1-6\xi)^2(2-\epsilon)H^4}{64\pi^2}[-2\zeta^2], \quad (77)$$

$$p_q \overset{\zeta\to\infty}{\longrightarrow} \frac{(1-6\xi)^2(2-\epsilon)H^4}{64\pi^2}\bigg[\frac{2}{3}(1-2\epsilon)\bigg]\zeta^2, \quad (78)$$

from which we conclude

$$w_q \overset{\zeta\to\infty}{\longrightarrow} -\frac{1}{3}+\frac{2}{3}\epsilon = w_b + \frac{2}{3} > w_b, \quad (79)$$

where w_b denotes the background equation of state parameter (33). Notice that the condition, $\nu < 1/2$ implies that $\xi < 1/6$, if $\epsilon > 2$ and $\xi > 1/6$, if $1 < \epsilon < 2$. In those regimes we therefore do not expect any significant backreaction.

2. *The case when* $\epsilon > 1$, $\nu > 1/2$

In this case the dominant contribution to T_q, ρ_q and p_q comes from a ν dependent power in Eqs. (71) and (73) [cf. also Eqs. (74) and (75)]:

$$T_q \overset{\zeta\to\infty}{\longrightarrow} -\frac{(1-6\xi)^2(1-\epsilon)(2-\epsilon)H^4}{16\pi^2}\bigg[\frac{6\epsilon}{1+2\nu}-(5\epsilon-3) + (\epsilon-1)(2\nu+1)\bigg]\zeta^{1+2\nu}, \quad (80)$$

$$\rho_q \overset{\zeta\to\infty}{\longrightarrow} \frac{(1-6\xi)^2(2-\epsilon)H^4}{64\pi^2}\bigg[\frac{6\epsilon}{1+2\nu}-(5\epsilon-3) + (\epsilon-1)(2\nu+1)\bigg]\frac{4\zeta^{1+2\nu}}{3-2\nu}, \quad (81)$$

$$p_q \overset{\zeta\to\infty}{\longrightarrow} \frac{(1-6\xi)^2(2-\epsilon)H^4}{64\pi^2}\bigg[\frac{6\epsilon}{1+2\nu}-(5\epsilon-3) + (\epsilon-1)(2\nu+1)\bigg]\frac{4}{3}\bigg[(\epsilon-1)+\frac{1}{3-2\nu}\bigg]\zeta^{1+2\nu}, \quad (82)$$

where we assumed that the term in square brackets does not vanish and we took $\nu \neq 3/2$.[4] From Eqs. (80)–(82) we conclude

$$w_q \overset{\zeta\to\infty}{\longrightarrow} \epsilon - \frac{2}{3} - \frac{2}{3}(\epsilon - 1)\nu = w_b + \frac{1}{3}(\epsilon + 1) - \frac{2}{3}(\epsilon - 1)\nu, \tag{83}$$

This equation implies that the quantum contribution will eventually dominate provided

$$\nu > \frac{\epsilon + 1}{2(\epsilon - 1)}, \tag{84}$$

or equivalently when

$$\begin{aligned} &\xi < -\frac{\epsilon - 1}{3(2 - \epsilon)}, \quad \text{when } 1 < \epsilon < 2 \\ &\xi > \frac{\epsilon - 1}{3(\epsilon - 2)}, \quad \text{when } \epsilon > 2. \end{aligned} \tag{85}$$

Together with the criterion $\xi < 0$ when $0 < \epsilon < 1$, these relations define the regions plotted in Fig. 3, for which one expects that the quantum one-loop contribution to the stress energy will eventually dominates over the classical stress-energy tensor that drives the Universe's expansion, and in which case the quantum backreaction can influence—and in fact change—the evolution of the Universe. We also indicated in the figure the boundary curves for an infrared divergence from Fig. 1. We come back to the difference between the boundary curves in the discussion.

Notice that the Ricci scalar is given by $R = 6\epsilon(2 - \epsilon)H^2$. Therefore if $\epsilon > 2$ the Ricci scalar is actually negative and thus the term $\xi R\phi^2$ acts in this case as a negative mass term. These conclusions are correct, provided the contributions (80)–(82) do not vanish, which will be the case when the sum of the terms in square brackets does not vanish, which is the case we consider next.

3. The special case when $\epsilon > 1$ and $\xi = -(\epsilon - 1)/[3(2 - \epsilon)]$

The case when the sum of the terms in square brackets (80)–(82) vanishes requires special attention, which is the case when

$$\nu \in \left\{1, \frac{\epsilon + 1}{2(\epsilon - 1)}\right\} \Longleftrightarrow \xi \in \left\{\frac{(5 - 3\epsilon)(1 + \epsilon)}{24(2 - \epsilon)}, -\frac{\epsilon - 1}{3(2 - \epsilon)}\right\}. \tag{86}$$

We shall only consider the second value, since we do not expect anything significant for $\nu < 3/2$. For this case the results of Sec. V B 1 apply for all $0 < \nu < 3/2$, and we have

[4] When $\nu = 3/2$, ρ_q and p_q acquire a logarithmic contribution of ζ, which yields the same w_q as is in Eq. (83).

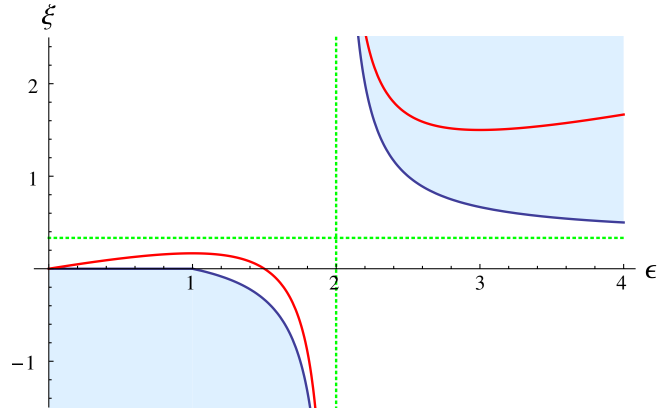

FIG. 3 (color online). Boundary values of ξ as a function of ϵ. The solid blue regions are the regions for which the quantum stress-energy scales slower that the classical stress energy, such that the quantum contribution will eventually dominate over the classical contribution. The regions are bounded by $\xi = 0$ $(0 < \epsilon < 1)$ and $\xi = -(\epsilon - 1)/[3(2 - \epsilon)]$ $(\epsilon > 1)$. The dotted asymptotes are given by $\epsilon = 2$ and $\xi = 1/3$. The red curves are the boundary curves indicating an infrared divergence, and are identical to those in Fig. 1.

$$\begin{aligned} &w_q \overset{\zeta\to\infty}{\longrightarrow} -\frac{1}{3} + \frac{2}{3}\epsilon = w_b + \frac{2}{3} > w_b, \\ &(0 < \nu < 3/2 \Leftrightarrow \epsilon > 2), \end{aligned} \tag{87}$$

When, on the other hand, $1 < \epsilon < 2$, such that $\nu > 3/2$ and $\xi < 0$, the subleading terms in the sums (74) and (75) dominate,

$$T_q \overset{\zeta\to\infty}{\longrightarrow} -\frac{(1 - 6\xi)^2(1 - \epsilon)^2(2 - \epsilon)(5 - 3\epsilon)H^4}{16\pi^2}\zeta^{2/(\epsilon-1)}, \tag{88}$$

$$\rho_q \overset{\zeta\to\infty}{\longrightarrow} \frac{(1 - 6\xi)^2(1 - \epsilon)(2 - \epsilon)(5 - 3\epsilon)H^4}{64\pi^2}\frac{2(\epsilon - 1)}{2\epsilon - 3}\zeta^{2/(\epsilon-1)}, \tag{89}$$

$$p_q \overset{\zeta\to\infty}{\longrightarrow} \frac{(1 - 6\xi)^2(1 - \epsilon)(2 - \epsilon)(5 - 3\epsilon)H^4}{64\pi^2}\frac{2(\epsilon - 1)}{3(2\epsilon - 3)} \times [2(2\epsilon - 3) + 1]\zeta^{2/(\epsilon-1)}. \tag{90}$$

This then implies for the equation of state parameter,

$$\begin{aligned} &w_q = -\tfrac{5}{3} + \tfrac{4}{3}\epsilon = w_b + \tfrac{2}{3}(\epsilon - 1), \\ &(\nu > \tfrac{3}{2} \Leftrightarrow 1 < \epsilon < 2). \end{aligned} \tag{91}$$

Together with Eq. (91) this implies that $w_q > w_b$, $\forall\ \epsilon > 1$.

Let us now go back to Fig. 2, already discussed at the end of Sec. VA. Now we can also understand the plots for $\epsilon > 1$. For the decelerating case we found that quantum contributions can dominate if $\xi < -\frac{\epsilon-1}{3(2-\epsilon)}$ for $1 < \epsilon < 2$ and $\xi > -\frac{\epsilon-1}{3(2-\epsilon)}$ for $\epsilon > 2$. Recall that the borderline case corresponds to the solid blue line in Fig. 2. Hence, for $\epsilon > 1$ we choose $\xi = -\frac{\epsilon-1}{3(2-\epsilon)}$. This implies that for $1 < \epsilon < 2$ we have that $w_q = -\frac{5}{3} + \frac{4}{3}\epsilon$, see (91), and for $\epsilon > 2$ we have $w_q = -\frac{1}{3} + \frac{2}{3}\epsilon$, see (87). The solid dark purple and dotted dark green lines show w_q for positive values of ξ. A positive value of ξ, less then 1/6 means that $\nu > 1/2$ for $\epsilon < 2$ and $\nu > 1/2$ for $\epsilon > 2$. Thus, in this case we show $w_q = \epsilon - \frac{2}{3} - \frac{2}{3}(\epsilon - 1)\nu$ [see (79)], for the region $1 < \epsilon < 2$ and $w_q = -\frac{1}{3} + \frac{2}{3}\epsilon$, see (83), for $\epsilon > 2$. On the other hand, if $\xi > 1/6$, we have that $\nu < 1/2$ for $1 < \epsilon < 2$ and that $\nu > 1/2$ for $\epsilon > 2$. Thus, in this case it is precisely the other way around. Moreover, since now $\xi > \frac{\epsilon-1}{3(\epsilon-2)}$ in the region where $\epsilon > 2$, we find from (87) that in this regime w_q can become smaller then w_b. Finally, the solid light purple ($\xi = -0.1$) and dotted light orange ($\xi = -0.5$) lines in Fig. 2 show w_q for negative values of ξ. If ξ is negative, we always have that $\nu > 1/2$ for $1 < \epsilon < 2$ and that $\nu < 1/2$ for $\epsilon > 2$. And thus we show $w_q = \epsilon - \frac{2}{3} - \frac{2}{3}(\epsilon - 1)\nu$, cf. (79), for the region $1 < \epsilon < 2$ and $w_q = -\frac{1}{3} + \frac{2}{3}\epsilon$, cf. Eq. (83), for $\epsilon > 2$. We find, cf. (85), that in the region $1 < \epsilon < 2$ we can have that $w_q < w_b$.

This completes our discussion for now. However there are special points, corresponding to $\nu = 3/2$ and $\nu = 5/2$, where the above discussion fails because of the logarithms appearing in ρ_q and p_q. However, it turns that the scaling of the quantum corrections, w_q does not deviate from the behavior described above. The details of this are presented in Appendix E.

VI. COMPARISON WITH THE CUTOFF REGULATED $T_{\mu\nu}$

In our earlier work [22,29], we calculated the one-loop stress energy due to scalar field quantum fluctuations [22] as well as due to graviton quantum fluctuations [29], where the infrared was regulated by working on a compact spatial manifold. In other words, we assumed that the Universe is a comoving box, with periodic boundary conditions (which to a good approximation can be described by a comoving infrared momentum cutoff). A detailed analysis for the graviton backreaction in Ref. [29] confirms our conjecture that a nonminimally coupled scalar can be used as a model to study qualitative features of the quantum backreaction for cosmological perturbations, at least for the case when the Universe is in a comoving box.

Generalizing the result from [22] to a nonminimally coupled scalar, we obtain for the renormalized stress-energy tensor

$$\begin{aligned}\langle\Omega|T_{\mu\nu}|\Omega\rangle = &-\frac{\epsilon(2-\epsilon)(1-6\xi)^2}{16\pi^2}H^4\left[\gamma_E + \ln\left(\frac{(1-\epsilon)^2H^2}{4\pi\mu^2}\right) + \psi\left(\frac{1}{2}-\nu\right) + \psi\left(\frac{1}{2}+\nu\right) + 2\frac{1-2\nu'}{1-2\nu} + 2\frac{1+2\nu'}{1+2\nu}\right]\\ &\times\left(\epsilon a^2\delta^0_\mu\delta^0_\nu + \left(\epsilon - \frac{3}{4}\right)g_{\mu\nu}\right) + \frac{H^4(1-6\xi)}{16\pi^2}\left[\left(-(1-4\xi)(2-\epsilon)\epsilon^2 - \frac{2}{3}(1-6\xi)(1-4\epsilon+\epsilon^2)\epsilon\right)a^2\delta^0_\mu\delta^0_\nu\right.\\ &\left.+\left(-\frac{1}{8}(-7+8\epsilon(1-4\xi)+30\xi) - \frac{1-6\xi}{6}(3-22\epsilon+22\epsilon^2-4\epsilon^3)\right)g_{\mu\nu}\right] + \sum_{N=0}^{\infty}(\langle\Omega|T_{\mu\nu}|\Omega\rangle_N + \langle\Omega|T_{\mu\nu}|\Omega\rangle^N),\end{aligned} \tag{92}$$

where $\nu' = d\nu/dD|_{D=4} =$ and the terms in the last line are the infrared corrections arising from the infrared cutoff, which can be written as [22]

$$\begin{aligned}\langle\Omega|T_{\mu\nu}|\Omega\rangle_N &= (\rho_N + p_N)a^2\delta^0_\mu\delta^0_\nu + p_N g_{\mu\nu}\\ &= \frac{1}{4\pi^{5/2}}\frac{(2(1-\epsilon)(N-\nu)+3-\epsilon)}{3+2N-2\nu}\frac{\Gamma(\nu-N)\Gamma(2\nu-N)}{\Gamma(\frac{1}{2}+\nu-N)\Gamma(N+1)}H^4(1-\epsilon)^2 z_0^{2N+3-2\nu}(1-6\xi)(N-\nu)\\ &\quad\times\left[\frac{1}{3}\left(\frac{4}{1-2N+2\nu} - (1-\epsilon)\right)a^2\delta^0_\mu\delta^0_\nu + \frac{1}{3}\left(\frac{1}{1-2N+2\nu} - (1-\epsilon)\right)g_{\mu\nu}\right]\end{aligned} \tag{93}$$

and

$$\begin{aligned}\langle\Omega|T_{\mu\nu}|\Omega\rangle^N &= \frac{1}{4\pi^{5/2}}\frac{(2(1-\epsilon)(N+\nu)+3-\epsilon)}{3+2N+2\nu}\frac{\Gamma(-\nu-N)\Gamma(-2\nu-N)}{\Gamma(\frac{1}{2}-\nu-N)\Gamma(N+1)}H^4(1-\epsilon)^2 z_0^{2N+3+2\nu}(1-6\xi)(N+\nu)\\ &\quad\times\left[\frac{1}{3}\left(\frac{4}{1-2N-2\nu} - (1-\epsilon)\right)a^2\delta^0_\mu\delta^0_\nu + \frac{1}{3}\left(\frac{1}{1-2N-2\nu} - (1-\epsilon)\right)g_{\mu\nu}\right],\end{aligned} \tag{94}$$

where $z_0 = k_0|\eta|$, k_0 is the (comoving) infrared cutoff scale ($k_0 = 2\pi/L$, where L is the comoving size of the Universe) and η is conformal time, defined by $a\eta = -1/[(1-\epsilon)H]$. We assume that the size of the Universe is at an initial time η_0 super-Hubble, such that $k_0|\eta_0| = k_0/[(1-\epsilon)a_0H_0] \ll 1$. Notice that in an accelerating universe ($0 < \epsilon < 1$) at late times z_0 decreases, such that $z_0 \to 0$ as $\eta \to 0-$ (or equivalently $a \to \infty$).

On the other hand, in a decelerating universe η increases, resulting in an increasing z_0. When $z_0 \sim 1$, the size of the Universe becomes comparable to the Hubble radius. Even later the comoving box shrinks to sub-Hubble sizes, and $z_0 \to \infty$ when η, $a \to \infty$. This case requires special attention, and it is treated below.

We shall now construct the leading order contribution from the corrections (93) and (94) to the stress-energy tensor in both accelerating and decelerating universes. If z_0 approaches zero, we see that the leading order contribution comes from the $N = 0$ term of (93). This term is growing if $\nu > 3/2$, as is expected, since this is the requirement for an infrared divergence (16). The case $\nu = 3/2$ leads to a logarithmic growth which, for brevity, we do not study here. The results turn out to be analogous to the case we consider here, described in Eqs. (E1)–(E3) for the accelerating case. If $\nu > 3/2$, the leading contribution in an accelerating universe is

$$\langle\Omega|T_{\mu\nu}|\Omega\rangle \overset{z_0\to 0}{\longrightarrow} \frac{4H^4(1-\epsilon)^4}{3\pi^2(1+2\nu)(2\nu-3)}\left(\frac{z_0}{2}\right)^{3-2\nu}(1-6\xi)\Gamma(\nu)\Gamma(\nu+1) \times\left(\frac{3-\epsilon}{2(1-\epsilon)}-\nu\right)\left[\left(\frac{3+\epsilon}{2(1-\epsilon)}-\nu\right)a^2\delta^0_\mu\delta^0_\nu + \left(\frac{\epsilon}{2(1-\epsilon)}-\nu\right)g_{\mu\nu}\right]+\mathcal{O}(z_0^{5-2\nu}). \tag{95}$$

This expression vanishes when $\xi = 0$, since then $\nu = (3-\epsilon)/[2(1-\epsilon)]$. In that specific case we need the next to leading order contribution, which is proportional to $z_0^{5-2\nu}$. The form of the leading order contribution (95) very similar to the corresponding matching case (58)–(61), with $z_0 \to \zeta$ (the precise numerical coefficients multiplying the leading order terms are, of course, different). Thus, in an accelerating universe, the two regularization procedures are qualitatively the same. The reasons for this is that any infrared regularization implies that we effectively suppress modes with wavelengths larger then some scale, be it given by z_0 or ζ. In an accelerating universe, this scale grows faster then the Hubble radius and therefore the precise details of this regularization become less and less visible as time goes on. Therefore, we indeed find that at late enough times the two regularization schemes give qualitatively the same result.

However, as mentioned above, in a decelerating spacetime we have a different situation: the sums over N run to infinity and, since z_0 grows, this leads in principle to fast growing terms. Moreover, we see that this happens when the Universe's size becomes sub-Hubble, independent of ν. Thus, also infrared perfectly finite space-times become dominated by the cutoff. The most reasonable approach to this case is to sum the sums over N when $z_0 \ll 1$ and then analytically extend to late times when $z_0 \gg 1$ and the Universe's size is sub-Hubble. We shall now illustrate how this procedure works by calculating the trace of the stress-energy tensor. After taking the trace of (93) we can perform the sum over N to obtain

$$\sum_{N=0}^{\infty}\langle\Omega|T^\mu{}_\mu|\Omega\rangle_N = \frac{2\Gamma(\nu)^2}{\pi^3(5-2\nu)(3-2\nu)}(1-6\xi)H^4(1-\epsilon)^4\left(\frac{z_0}{2}\right)^{3-2\nu} \times\left[\nu(5-2\nu)\left(\frac{3-\epsilon}{2(1-\epsilon)}-\nu\right){}_2F_3\left(\frac{1}{2}-\nu,\frac{3}{2}-\nu;1-2\nu,\frac{5}{2}-\nu,-\nu;-z_0^2\right)+\frac{z_0^2}{2}(3-2\nu) \times{}_2F_3\left(\frac{3}{2}-\nu,\frac{5}{2}-\nu;2-2\nu,\frac{7}{2}-\nu,1-\nu;-z_0^2\right)\right]. \tag{96}$$

The leading order can be studied by considering the asymptotic expansion of the ${}_2F_3$ hypergeometric functions. We have in general

$$\begin{aligned}{}_2F_3(a_1,a_2;b_1,b_2,b_3;-z) = \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{\Gamma(a_1)\Gamma(a_2)}\Bigg\{&\frac{\Gamma(a_1)\Gamma(a_2-a_1)}{\Gamma(b_1-a_1)\Gamma(b_2-a_1)\Gamma(b_3-a_1)}z^{-a_1}(1+\mathcal{O}(z^{-1}))\\ &+\frac{\Gamma(a_2)\Gamma(a_1-a_2)}{\Gamma(b_1-a_2)\Gamma(b_2-a_2)\Gamma(b_3-a_2)}z^{-a_2}(1+\mathcal{O}(z^{-1}))\\ &+\frac{z^\chi}{\sqrt{\pi}}\bigg[\cos(\pi\chi+2\sqrt{z})+\frac{1}{16\sqrt{z}}((3a_1+3a_2+b_1+b_2+b_3-2)(8\chi-2)\\ &+16(b_1b_2+b_1b_3+b_2b_3-a_1a_2)-3)\sin(\pi\chi+2\sqrt{z})\bigg](1+\mathcal{O}(z^{-1}))\Bigg\},\end{aligned} \tag{97}$$

with

$$\chi = \tfrac{1}{2}\left(a_1+a_2-b_1-b_2-b_3+\tfrac{1}{2}\right). \tag{98}$$

Using this we find that the leading order terms are

$$\sum_{N=0}^{\infty}\langle\Omega|T^{\mu}{}_{\mu}|\Omega\rangle_N = \frac{(1-6\xi)(1-\epsilon)^3H^4}{8\pi^2\sin^2(\pi\nu)}\bigg[\frac{z_0^2}{2}+z_0^2\Big(\frac{5\epsilon-1}{4}+(1-\epsilon)\nu^2\Big) \times\sin(2z_0+\pi\nu)-z_0^3(1-\epsilon)\cos(2z_0+\pi\nu)\bigg]+\mathcal{O}(z_0^0). \tag{99}$$

The second series $\sum_{N=0}^{\infty}\langle\Omega|T^{\mu}{}_{\mu}|\Omega\rangle^N$ in Eq. (92) can be obtained from (99) by interchanging ν with $-\nu$ and can be easily added, resulting in

$$T_q \simeq \sum_{N=0}^{\infty}(\langle\Omega|T^{\mu}{}_{\mu}|\Omega\rangle_N+\langle\Omega|T^{\mu}{}_{\mu}|\Omega\rangle^N) = \frac{(1-6\xi)(1-\epsilon)^3H^4}{8\pi^2\sin^2(\pi\nu)}\bigg[z_0^2+z_0^2\Big(\frac{5\epsilon-1}{2}+2(1-\epsilon)\nu^2\Big) \times\cos(\pi\nu)\sin(2z_0)-2z_0^3(1-\epsilon)\cos(\pi\nu)\cos(2z_0)\bigg] + \mathcal{O}(z_0^0). \tag{100}$$

Selecting the leading order term in (100), we can recast it as

$$T_q \simeq \frac{(1-6\xi)(\epsilon-1)H_0k_0^3}{4\pi^2}\frac{\cos(\pi\nu)}{\sin^2(\pi\nu)} \times\bigg[\cos\Big(\frac{2k_0}{(\epsilon-1)Ha}\Big)+\mathcal{O}(a^{1-\epsilon})\bigg]a^{-\epsilon-3}. \tag{101}$$

We shall not attempt to evaluate ρ_q and p_q in this case, since the asymptotic expansion would result in an expression that depends on the lowest value of z_0. Instead we shall compare the corresponding traces. When Eq. (101) is compared with the background contribution, $T_b=\rho_b-3p_b\propto 1/a^{3(1+w_b)}=a^{-2\epsilon}$, one finds that

$$\frac{T_q}{T_b}\sim\frac{k_0^3}{H_0M_P^2}a^{\epsilon-3}\cos\Big(\frac{2k_0}{(\epsilon-1)Ha}\Big), \tag{102}$$

which grows when $\epsilon>3$. Thus we find that the limiting case is kination, for which $\epsilon=3$ and $\rho_b\propto 1/a^6$, and the quantum contribution to the trace T_q given in (101) scales the same as the background contribution to the trace. For all $\epsilon>3$ we thus find that the quantum contribution will eventually dominate over the background energy density. There is one exception: as long as ν is not half integer, the scaling (102) is correct. However when ν is half integer, $\cos(\pi\nu)=0$ and the leading contribution (101) vanishes. The dominant contribution is then the term $\propto z_0^2$ in Eq. (100). We have seen in Sec. V B 1 that such a contribution will never lead to a strong backreaction. This is in contrast with what we found in the present work. Provided that we correctly disregarded the ζ^4 term, which arose as an integration constant, we have found in Sec. V B that, if the infrared is regulated using the mode matching, for all $\nu>1/2$, the scaling of the quantum energy is indeed governed by ν and not ϵ.

VII. SUMMARY AND DISCUSSION

In order to facilitate the reading of this rather technical paper, we shall now recap our main results. The main motivation for this paper is to study the role of the quantum infrared fluctuations generated by the expansion of the Universe for massless scalar fields, with a possible coupling to the Ricci scalar.

It is well known that the infrared sector of such a scalar field posses problems on a cosmological background: in space-times with a negative pressure, and constant acceleration/deceleration parameter $\epsilon=-\dot{H}/H^2$, the BD vacuum of massless minimally coupled scalars is infrared singular [4] (see Sec. II).

Of course, we know that our Universe must be infrared finite. That means that the infrared sector of the theory must be regulated. It is currently unknown what is the precise nature of the regularization scheme realized in our Universe. Hence, it is worth investigating different regularization schemes and compare the results. In the end, however, the presence of this infrared singularity is precisely what makes the present study interesting. The singularity indicates that there is a growth in long range correlations due to particle production. Even after regulating the infrared divergence, this growth of the correlations is still there, since it is simply a physical effect. These growing correlations can then in principle—after a sufficient amount of time—contribute significantly to the energy density in the Universe. If this is so, they might significantly alter the evolution of the Universe.

In this work we focus our attention on the regularization of the infrared which is executed by an epoch of the very early universe which has very little particle production and whose BD vacuum is therefore infrared finite. For definiteness we choose this early epoch to be radiation era. We then match it onto a constant ϵ space-time, and calculate the corresponding coincident scalar propagator. The exact solution for the mode functions can be expressed in terms of Hankel functions (13) with a rather complicated index ν (14). The BD vacuum corresponds to the choice (15), implying that one considers only positive frequency modes. As can be seen from Eq. (16) the propagator for the BD vacuum is IR singular whenever $\nu\geq(D-1)/2$, where D denotes the number of space-time dimensions. At the matching we require continuity of both the mode functions and their first derivatives. This implies that, after the matching, the mode functions become a mixture of positive and negative frequency modes, in such way that the (coincident) propagator (B2) is IR finite for all ϵ.

The propagator still suffers from the standard (logarithmic) ultraviolet divergence, which in dimensional regularization appears as a $1/(D-4)$ divergence (B6). This divergence induces an analogous divergence in the stress-energy tensor (39), and can be removed using standard techniques by an R^2 counterterm (41).

To construct the propagator, we have first calculated the coincident propagator for half-integer ν (B10) and (B11) and then analytically extended the result to all (complex) ν (B13). Since the coincident propagator for half integers (B12) is valid for all times later than the matching, we suspect that the analytic extension (B13) is unique. From the coincident propagator, one can derive the trace of the one-loop stress-energy tensor T_q, based on Eq. (29). The symmetries of the background space-time dictate the perfect fluid form of the quantum contribution to the stress-energy tensor and using the covariant conservation of the stress-energy tensor (31), one can derive the individual components (quantum energy density ρ_q and pressure p_q) of the one-loop $(T_{\mu\nu})_q$. The general results of this procedure are for accelerating universe presented in Eqs. (46) and in Eqs. (50) and (51) and for decelerating universes in Eqs. (71)–(73). Since we obtain ρ_q and p_q by an integration procedure, our results are unique up to an integration constant. Adding this integration constant corresponds to adding a component to ρ_q and p_q, which scales as a radiation fluid, $\propto 1/a^4$, and thus does not contribute to the trace of the stress-energy tensor T_q. To fix this component uniquely, we would need to know the propagator away from the coincident limit, which we have not derived in this paper. Since this undetermined radiation component yields a subdominant contribution at late times for accelerating universes, it is of no relevance for our discussion of the late time quantum stress-energy tensor in Sec. V A. It may be however relevant in decelerating universes. For the purpose of this work we have ignored this radiation contribution in our analysis of the late time quantum stress-energy tensor in decelerating universes presented in Sec. V B. It is also worth noting that the finite counterterms in Eqs. (42) and (43), that result from the (ultraviolet) renormalization, play no significant role in the quantum backreaction. This implies that the dominant backreaction is an infrared phenomenon, and can be explained by the particle creation in expanding space-times. This also means that conformally coupled scalars (whose infrared is not enhanced) will in general lead to a weaker backreaction, when compared to the one discussed in this work.

In order to study the significance of the quantum one-loop stress-energy tensor, in Secs. V A and V B we study in detail the equation of state parameter $w_q = p_q/\rho_q$ both in accelerating and in decelerating universes. If the following criterion is satisfied (36),

$$w_q < w_b,$$

(where $w_b = p_b/\rho_b$ is the equation of state parameter of the background fluid, driving the expansion of the universe), then the quantum contribution to the energy density dominates over the background contribution at late times. In this work we do not attempt to analyze what exactly happens in such a case (to start with, the assumptions underlying our calculations would become incorrect), but simply want to answer the question *if* and under what conditions this criterion is ever satisfied.

Our results are presented in detail in Secs. V A and V B. Since several cases require a separate discussion, it is difficult to get a quick grasp of the results. In order to facilitate a quicker understanding of our results we present in Fig. 2 the quantum equation of state parameter w_q vs the background equation of state parameter $w_b = -1 + (2/3)\epsilon$. The criterion (36) then tells us when the quantum contribution becomes dominant over the background contribution at late times. We shall now describe our results in some detail.

First, in Fig. 3 the shaded regions represent the regions in parameter space $\{\xi, \epsilon\}$ where the criterion $w_q < w_b$ is satisfied. For accelerated universes ($0 < \epsilon < 1$) this means simply that when $\xi < 0$, $w_q < w_b$. For decelerating universes ($\epsilon > 1$) we need to distinguish two cases. When $1 < \epsilon < 2$, then $\xi < -(\epsilon - 1)/[3(2-\epsilon)]$ (85) assures $w_q < w_b$. When on the other hand $\epsilon > 2$, $\xi > (\epsilon - 1)/[3(\epsilon - 2)]$ (85) is required in order that $w_q < w_b$. Keeping in mind that the Ricci scalar curvature $R = 6(2-\epsilon)H^2$ is positive when $0 < \epsilon < 2$ and negative when $\epsilon > 2$, we see that $w_q < w_b$ can be met only when the effective scalar "mass" parameter $m_{\rm eff}^2 = \xi R$ is (sufficiently) negative. A careful analysis presented in Secs. V A 3 and V B 2 shows that at the boundary of the shaded region in all cases $w_q \geq w_b$ is satisfied, such that the quantum contribution can never become important. This can also be seen from the solid blue line in Fig. 2.

Furthermore, from Figs. 1 and 3 we see that the criteria $\{\xi, \epsilon\}$ plane for quantum dominance and for infrared divergence are similar, but not the same. The reason for this is the nontrivial interplay between two effects. First there is the infrared divergence, second there is the question at what rate modes are entering or leaving the horizon. This second point essentially indicates that if modes are entering the horizon fast (large ϵ) the backreaction is enhanced, while for small ϵ (< 1), backreaction is reduced. The interplay between those effects explains why there can be both regions where there is an infrared divergence, but no enhanced backreaction, and regions where there is no divergence, but there is an enhanced backreaction.

Next, the solid dark purple ($\xi = 0.1$) and dotted dark green ($\xi = 1$) lines in Fig. 2 show w_q vs w_b for positive $\xi > 0$. When the coupling ξ is smaller than $1/3$, we have that $w_q > w_b$ such that the quantum stress energy can never be important. When however $\xi > 1/3$, in space-times with $\epsilon > 2$, w_q can become smaller than w_b, indicating the late time dominance of the one-loop contribution.

Finally, the solid light purple ($\xi = -0.1$) and dotted light orange ($\xi = -0.5$) lines in Fig. 2 show w_q when ξ is negative. In this case $w_q < w_b$ is always met for accelerating cases, but also for decelerating cases whenever $\xi < \xi_{\rm cr} = -(\epsilon - 1)/[3(2-\epsilon)]$. Notice that the critical ϵ (for which $\xi = \xi_{\rm cr}$) is always smaller than 2.

Apart from the regularization scheme presented in this work that involves matching from a nonsingular space-time, other infrared regularization schemes are possible. We do not know which regularization scheme was chosen by our Universe. We do know however that, since infrared regularization is physical, there are in principle physical observables that can distinguish between different regularization schemes, and thus we should be able to decide which one was picked by our Universe. But in order to find out the answer to that question, we need to investigate different plausible IR regularization schemes. Other regularization schemes include: placing the Universe in a large (comoving) box (this corresponds to an infrared momentum cutoff) which has been explored in Refs. [21,22,29], a tiny scalar mass [28] (it is not clear whether that is possible to implement for the graviton), a positive spatial curvature, subtracting an adiabatic [24] or a comoving vacuum [25–27], etc. While most of these schemes are physically well motivated, the implementation is often hindered by our lack of knowledge of the relevant propagators. In particular, we know the propagators in positively curved universes and for massive fields only in very special cases (de Sitter space, radiation era), which is not enough to conduct a sufficiently general analysis of the quantum backreaction regulated in this way.

Owing to the fact that the analysis of the Universe in a finite comoving box can be well approximated by an infrared momentum cutoff, in Refs. [8,22] we performed the one-loop analysis of the backreaction from a massless scalar field and the graviton in expanding universes with a constant ϵ parameter. In Sec. VI we present in some detail a comparison between the two regularization schemes. We find that in accelerated universes the leading order contributions to the corresponding late time stress-energy tensors are of a similar form in both regularization schemes in the sense that the leading order behavior with the scale factor is identical [cf. Eqs. (58)–(61) and (95)], albeit the coefficients of the leading order terms differ. In fact, the coefficients are comparable when the cutoff scale k_0 is chosen to be equal to the horizon scale $\hat{H}$ at the matching, i.e., when the initial comoving size of the Universe is of the order the Hubble radius. This is understandable, given the fact that the comoving box in accelerating universes grows with respect to the Hubble radius, i.e., as time elapses the Universe's size becomes more and more super-Hubble. The precise details about what happens on super-Hubble scales become less and less ''visible'' as time goes on. This is exactly what we find: at late times the two approaches give qualitatively the same answer.

On the other hand, in decelerating space-times the matching and cutoff regularization schemes yield very different results. The leading order contributions to the trace of the stress-energy tensor are in this case [cf. Eqs. (80) for $\nu > 1/2$, Eq. (76) for $\Re[\nu] < 1/2$ and (100) and (101)]

$$\langle\Omega|T^\mu{}_\mu|\Omega\rangle \propto H^4 z_0^3 \cos(2z_0) + \mathcal{O}(z_0^2); \quad \text{(cutoff)}, \qquad (103)$$

$$\langle\Omega|T^\mu{}_\mu|\Omega\rangle \propto \begin{cases} H^4\zeta^{2\nu+1} + \mathcal{O}(\zeta^{2\nu-1}); & (\text{matching}, \nu > 1/2) \\ H^4\zeta^2 + \mathcal{O}(\zeta^{2\nu+1}, \zeta); & (\text{matching}, \Re[\nu] < 1/2), \end{cases} \qquad (104)$$

where $\zeta = \hat{a}\hat{H}/(aH)$ and $z_0 = k_0/[(\epsilon-1)aH] = \{k_0/[(\epsilon-1)\hat{a}\hat{H}]\}\zeta$, such that z_0/ζ is a constant given by the ratio of the physical cutoff scale $k_0/\hat{a}$ and the (inverse) particle horizon $\sim(\epsilon-1)\hat{H}$ at the matching time. For the cutoff regulated case we find that the growth is independent of ν. The quantum contribution in this case becomes more and more dominant if ϵ becomes larger. In the mode-matching case we see that the growth is dependent on ν and moreover the effect is more profound the greater ν is. This is exactly what one would expect based on (38). What happens physically is that, in a decelerating space-time, physical scales grow slower then the Hubble radius, and eventually the comoving size of the Universe becomes sub-Hubble. This would eventually dominate all effects and thus what one is looking at then has nothing to do anymore with infrared particle production! The results (104), (100), and (101) are obtained by analytically extending to the limit $z_0 \to \infty$, which is precisely the limit in which the Universe's size becomes sub-Hubble. Then, not surprisingly, in this limit $T_q \propto 1/V_c$ becomes a function of the comoving volume V_c of space-time, representing an observable indicating the size of the Universe. In the infrared regularization presented in this work, all potentially relevant effects are however due to the infrared particle production, and since this is the physical effect we wish to study, we feel that this approach is therefore advantageous in decelerating space-times.

ACKNOWLEDGMENTS

We would like to thank R. P. Woodard for his contributions at the early stages of this project and useful suggestions later on and S. P. Miao for critically reading the manuscript.

APPENDIX A: NONMINIMALLY COUPLED MASSLESS SCALAR FIELD

In this appendix we recall some of the well-known details of the quantization of a free scalar field in cosmological background, which can be useful for the derivation of the formulae in the first part of Sec. II. From Eqs. (8), (9), and (13) it follows that the scalar field mode functions $\psi(t, k)$ obey

$$\left(\frac{d^2}{dt^2} + H\frac{d}{dt} + \frac{k^2}{a^2} + \frac{D-2\epsilon}{4}(2 - D + 4\xi(D-1))H^2\right) \times (a^{(D/2)-1}\psi) = 0, \tag{A1}$$

where we used that in the constant ϵ geometry under consideration

$$R = (D-1)(D-2\epsilon)H^2.$$

By assuming that ϵ in (5) is constant and writing $z = k/[(1-\epsilon)Ha]$, we obtain

$$\left(z^2\frac{d^2}{dz^2} + z^2 + \frac{1}{4} - \nu^2\right)(a^{(D/2)-1}\psi) = 0, \tag{A2}$$

where ν is given in Eq. (14). Notice that $\nu = 1/2$ for a conformally coupled scalar, $\xi = \xi_c = (D-2)/[4(D-1)]$, or when the universe is radiation dominated, in which case $\epsilon = D/2$. When $\nu =$ const. Eq. (A2) becomes the Bessel equation whose solution can be conveniently expressed in terms of the Hankel functions as shown in Eq. (13).

Next we note that the canonical commutation relations for the field and for the annihilation and creation operators,

$$[\phi(\vec{x}, t), a^{D-1}\dot{\phi}(\vec{x}', t)] = i\delta^{D-1}(\vec{x} - \vec{x}'),$$
$$[b(\vec{k}), b^\dagger(\vec{k}')] = (2\pi)^{D-1}\delta^{D-1}(\vec{k} - \vec{k}'),$$

imply for the mode function Wronskian

$$\psi(t, k)\dot{\psi}^*(t, k) - \psi^*(t, k)\dot{\psi}(t, k) = ia^{1-D}.$$

The properly normalized mode functions that solve (A2) are given in Eq. (13)

And finally, from Eqs. (8.403) and (8.405.1) of Ref. [30] we read off the definition for the Hankel function of the first kind,

$$H_\nu^{(1)}(z) = \frac{i}{\sin(\pi\nu)}(e^{-i\pi\nu}J_\nu(z) - J_{-\nu}(z)), \tag{A3}$$

such that the small k contribution to the integrand of (11) can be obtained from the power series for the Bessel function (see Eq. (8.402) of [30])

$$J_\nu(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(\nu+n+1)}\left(\frac{z}{2}\right)^{\nu+2n}. \tag{A4}$$

1. Matching

Here we show how to match an earlier radiation era with $\nu = 1/2$ to a later expanding space-time with an arbitrary ϵ. We shall work in the approximation of sudden matching, and we shall denote the matching time by $\hat{t}$. The matching conditions are given by Eqs. (19) and (20) and the relevant mode functions are given in Eqs. (13) and (21) for $t > \hat{t}$ and $t < \hat{t}$, respectively. We shall first do the matching for the case when the late time ν is a half integer. In this special case our algebra simplifies considerably because the mode functions (13) can be written as a finite sum,

$$u(t, k) = \frac{1}{\sqrt{2k}}(i)^{-\nu-(1/2)}e^{ik/(1-\epsilon)Ha} \times \sum_{n=0}^{\nu-(1/2)}\frac{\Gamma(\nu+\frac{1}{2}+n)}{n!\Gamma(\nu+\frac{1}{2}-n)}\left(\frac{-2ik}{(1-\epsilon)Ha}\right)^{-n}. \tag{A5}$$

The matching coefficients $\alpha(k)$ and $\beta(k)$ are completely fixed by the matching conditions (20):

$$\alpha = ia(u^*\dot{u}_0 - \dot{u}^*u_0)|_{t=\hat{t}} = i^{\nu+(1/2)}e^{-(ik/\hat{a}\hat{H})((\epsilon-\epsilon_0)/(1-\epsilon)(1-\epsilon_0))}\sum_{n=0}^{\nu-(1/2)}\left(-i - \frac{(1-\epsilon)n\hat{a}\hat{H}}{2k}\right)\frac{\Gamma(\nu+\frac{1}{2}+n)}{n!\Gamma(\nu+\frac{1}{2}-n)}\left(\frac{2ik}{(1-\epsilon)\hat{a}\hat{H}}\right)^{-n},$$
$$\beta = ia(\dot{u}u_0 - u\dot{u}_0)|_{t=\hat{t}} = i^{-(\nu+(1/2))}e^{-(ik/\hat{a}\hat{H})((\epsilon+\epsilon_0-2)/(1-\epsilon)(1-\epsilon_0))}\sum_{n=0}^{\nu-(1/2)}\left(\frac{(1-\epsilon)n\hat{a}\hat{H}}{2k}\right)\frac{\Gamma(\nu+\frac{1}{2}+n)}{n!\Gamma(\nu+\frac{1}{2}-n)}\left(\frac{-2ik}{(1-\epsilon)\hat{a}\hat{H}}\right)^{-n}, \tag{A6}$$

where $\hat{a} \equiv a(\hat{t})$ and $\hat{H} \equiv H(\hat{t})$. Notice that the series expansion used for the Hankel functions is also valid also for negative values of the argument, so we do not run into any problems due to the branch cut of the Hankel function for negative real arguments.

APPENDIX B: THE COINCIDENT PROPAGATOR

Now we shall calculate the coincident propagator, using the vacuum state $|\Omega\rangle$ defined by $b(\vec{k})|\Omega\rangle = 0$ and by (13) and (A6), which is sufficient to calculate the one-loop corrected stress-energy tensor. The coincident propagator is given by [22]

$$i\Delta(x;x) = \frac{1}{2^{D-2}\pi^{(D-1)/2}\Gamma\left(\frac{D-1}{2}\right)}\int dk k^{D-2}|\psi(t,k)|^2. \tag{B1}$$

Using Eqs. (13), (A5), and (A6) we obtain

$$\begin{aligned} i\Delta(x;x) = {}& \frac{a^{2-D}}{(4\pi)^{(D-1)/2}\Gamma\left(\frac{D-1}{2}\right)}\int dk k^{D-3}\sum_{q,r,m,n=0}^{\nu-(1/2)}\left\{\frac{\Gamma(\nu+\frac{1}{2}+n)\Gamma(\nu+\frac{1}{2}+m)\Gamma(\nu+\frac{1}{2}+q)\Gamma(\nu+\frac{1}{2}+r)}{\Gamma[\nu+\frac{1}{2}-n)\Gamma(\nu+\frac{1}{2}-m)\Gamma(\nu+\frac{1}{2}-q)\Gamma(\nu+\frac{1}{2}-r)}\right. \\ & \times\left[\left(1+\frac{i}{2}(1-\epsilon)(q-n)\frac{\hat{a}\hat{H}}{k}+\frac{1}{2}(1-\epsilon)^2 nq\frac{(\hat{a}\hat{H})^2}{k^2}\right)(-1)^{-q-m}+e^{2ik/(1-\epsilon)Ha(1-(aH/\hat{a}\hat{H}))}\right. \\ & \times\left(-\frac{i}{2}(1-\epsilon)q\frac{\hat{a}\hat{H}}{k}-\frac{1}{4}(1-\epsilon)^2 qn\frac{(\hat{a}\hat{H})^2}{k^2}\right)(-1)^{-r-m}+e^{(-2ik)/(1-\epsilon)Ha(1-(aH/\hat{a}\hat{H}))} \\ & \left.\left.\times\left(\frac{i}{2}(1-\epsilon)q\frac{\hat{a}\hat{H}}{k}-\frac{1}{4}(1-\epsilon)^2 qn\frac{(\hat{a}\hat{H})^2}{k^2}\right)(-1)^{-q-n}\right]\left(\frac{2ik}{(1-\epsilon)\hat{a}\hat{H}}\right)^{-n-q}\left(\frac{2ik}{(1-\epsilon)aH}\right)^{-m-r}\frac{1}{n!m!q!r!}\right\}. \end{aligned} \tag{B2}$$

One important property of this expression is that the integral converges in the infrared, or $k\to 0$. Of course, this was expected by construction, but it can be also shown explicitly. For example the quadruple sum in (B2) evaluates in the cases when $\nu = 3/2$, $5/2$ and $7/2$ to

$$\begin{aligned} \sum_{q,r,m,n=0}^{\nu-(1/2)}(\ldots)_{\nu=3/2} &= \left(\frac{aH}{\hat{a}\hat{H}}\right)^2\left[\frac{2}{3}+\frac{1}{3}\left(\frac{\hat{a}\hat{H}}{aH}\right)^3\right]^2+\mathcal{O}(k), \\ \sum_{q,r,m,n=0}^{\nu-(1/2)}(\ldots)_{\nu=5/2} &= \left(\frac{aH}{\hat{a}\hat{H}}\right)^4\left[\frac{3}{5}+\frac{2}{5}\left(\frac{\hat{a}\hat{H}}{aH}\right)^5\right]^2+\mathcal{O}(k), \\ \sum_{q,r,m,n=0}^{\nu-(1/2)}(\ldots)_{\nu=7/2} &= \left(\frac{aH}{\hat{a}\hat{H}}\right)^6\left[\frac{4}{7}+\frac{3}{7}\left(\frac{\hat{a}\hat{H}}{aH}\right)^7\right]^2+\mathcal{O}(k). \end{aligned} \tag{B3}$$

1. The ultraviolet

As long as $aH \neq \hat{a}\hat{H}$, all UV divergencies can be easily seen to come only from the first line in the square brackets in Eq. (B2). In fact, the second line has three types of contributions: UV-divergent terms, IR-divergent terms, and a term that is logarithmically divergent both in the UV and in the IR. Now we know that the IR-divergent terms will cancel against the IR-divergent terms coming from the second and third lines in the square brackets. So we shall simply drop all IR-divergent terms that we encounter (except for the logarithmic divergence). By explicitly calculating the first three terms of the sums, we find that the UV divergent contributions are

$$i\Delta(x;x)_{\rm UV} = \frac{a^{2-D}}{(4\pi)^{(D-1)/2}\Gamma(\frac{D-1}{2})}\int\left(k^{D-3}-\frac{a^2H^2}{8}k^{D-5}\times(1-\epsilon)^2(1-4\nu^2)\right)dk. \tag{B4}$$

The first (quadratically divergent) term is a scaleless integral, which we can automatically subtract in dimensional regularization [31,32]. The second contribution is logarithmically divergent. Since we know that the infrared divergent part should drop out in the final answer, we write the integral as

$$\int \to \int_0^{k_0}+\int_{k_0}^{\infty} \tag{B5}$$

and we drop the first, IR-divergent, integral. Now we obtain

$$i\Delta(x;x)_{UV} = \frac{H^2}{8\pi^2}(1-\epsilon)^2\left(\nu^2-\frac{1}{4}\right)\left[-\frac{\mu^{D-4}}{D-4}-\frac{\gamma_E}{2}+1+\frac{1}{2}\ln\left(\frac{\pi\mu^2a^2}{k_0^2}\right)+\mathcal{O}(D-4)\right], \tag{B6}$$

where one should note that the prefactor $(1-\epsilon)^2(\nu^2-\frac{1}{4})$ is still D dependent, as can be seen from (14), and we introduced a renormalization scale μ. The k_0 dependence of (B6) should—and does—drop out when we add the IR contribution. Notice that $\Delta(x;x)_{\rm UV}$ is exactly the same as the (divergent) UV contribution one would have obtained using the Bunch-Davies vacuum [22]. This was to be expected, since the ultraviolet of the theory is only sensitive to local physics and therefore does not "see" the matching.

2. The infrared

We now focus on the second line in square brackets of (B2) (the third line is simply the complex conjugate of the second line). As long as $aH \neq \widehat{aH}$, the contributions are UV finite, so we can put $D = 4$. We shall comment below on the special case $aH = \widehat{aH}$. The integrals are of the form

$$\int k^{-n}e^{i\alpha k}dk,$$

where n is a positive integer and $\alpha < 0$. We split the integral again in two ranges as in (B5) and take the limit $k_0 \to 0$. The lower integral is divergent, but these logarithmic divergences exactly cancel against the infrared divergences coming from the second line of (B2). The upper integral is given by ($\Im[\alpha] > 0$)

$$\int_{k_0}^{\infty} k^{-n} e^{i\alpha k} dk = k_0^{1-n} E_n(-i\alpha k_0), \tag{B7}$$

where n is an integer and

$$E_p(z) = \int_1^{\infty} dt \frac{e^{-zt}}{t^p}$$

denotes the exponential integral. For an integer index, the exponential integral has the following expansion around zero:

$$E_n(z) = \frac{(-z)^{n-1}}{\Gamma(n)}(\psi(n) - \ln(z)) - \sum_{m=0, m\neq n-1}^{\infty} \frac{(-z)^m}{(m-n+1)m!}, \tag{B8}$$

where $\psi(z) = (d/dz)\ln[\Gamma(z)]$ denotes the digamma function. Since we know that all negative powers of k_0 cancel, we obtain the following leading order result, which is obtained by expanding (B7) around $k_0 = 0$,

$$\int_{k_0}^{\infty} k^{-n} e^{\pm i\alpha k} dk = -\frac{(\pm i\alpha)^{n-1}}{\Gamma(n)}[\ln(\mp i\alpha k_0) - \psi(n)] + (\mathrm{IR} - \mathrm{div}) + \mathcal{O}(k_0). \tag{B9}$$

We use this expression to evaluate the third and fourth lines in (B2) and obtain

$$\begin{aligned} i\Delta(x;x)_{\mathrm{IR}} = {}& \frac{H^2(1-\epsilon)^2}{4\pi^2} \\ & \times \sum_{m,n,q,r=0}^{\nu-(1/2)} \frac{\Gamma(\nu+\frac{1}{2}+n)\Gamma(\nu+\frac{1}{2}+m)\Gamma(\nu+\frac{1}{2}+q)\Gamma(\nu+\frac{1}{2}+r)}{\Gamma(\nu+\frac{1}{2}-n)\Gamma(\nu+\frac{1}{2}-m)\Gamma(\nu+\frac{1}{2}-q)\Gamma(\nu+\frac{1}{2}-r)} \\ & \times \bigg[n(1-\zeta)\frac{\ln(|\alpha|k_0) - \psi(m+n+q+r+1)}{\Gamma(m+n+q+r+1)} \\ & - \frac{\ln(|\alpha|k_0) - \psi(m+n+q+r)}{\Gamma(m+n+q+r)}\bigg] \frac{q}{2}\frac{\zeta^2}{1-\zeta} \\ & \times (-(1-\zeta))^{n+q}\left(\frac{1}{\zeta} - 1\right)^{m+r} \frac{1}{n!m!q!r!}, \end{aligned} \tag{B10}$$

where $\alpha = \frac{2}{(1-\epsilon)aH}(1-\zeta^{-1})$ and $\zeta = (\hat{a}\hat{H})/(aH)$.

Quite remarkably, the sums in Eq. (B10) can be (almost completely) performed, resulting in

$$\begin{aligned} i\Delta(x;x)_{\mathrm{IR}} = {}& \frac{H^2(1-\epsilon)^2}{8\pi^2}\left(\nu^2 - \frac{1}{4}\right)\bigg[\sum_{n=1}^{\nu-3/2} \frac{1}{2n\zeta^{2n}} \\ & + \ln(|\alpha|k_0) - \frac{1}{2}c_\nu - 1 + \frac{1}{2}\gamma_E + \zeta + \sum_{n=2}^{\nu+1/2}\frac{\zeta^{2n}}{2n}\bigg], \end{aligned} \tag{B11}$$

where c_ν is a slowly varying function of ν.[5] Its precise value is not important, as it can be absorbed in a finite counterterm.

3. The complete coincident propagator

The full coincident propagator is given by adding the infrared (B11) and ultraviolet contributions (B6),

$$\begin{aligned} i\Delta(x;x) = {}& \frac{H^2(1-\epsilon)^2}{8\pi^2}\left(\nu^2 - \frac{1}{4}\right)\bigg[-\frac{\mu^{D-4}}{D-4} - \frac{c_\nu}{2} \\ & + \frac{1}{2}\ln\left(\frac{4\pi\mu^2}{(1-\epsilon)^2H^2}\left(1-\frac{1}{\zeta}\right)^2\right) + \zeta - \frac{1}{2}\zeta^2 \\ & + \sum_{n=1}^{\nu-3/2}\frac{\zeta^{-2n}}{2n} + \sum_{n=1}^{\nu+1/2}\frac{\zeta^{2n}}{2n}\bigg], \quad \left(\zeta = \frac{\hat{H}\hat{a}}{Ha}\right). \end{aligned} \tag{B12}$$

The two logarithms of k_0 coming from the infrared and the ultraviolet have indeed canceled. From the result (B12) we see immediately that when $|\zeta| \ll 1$ the leading order contribution to $i\Delta(x;x)$ is of a power law type, $\propto H^2\zeta^{3-2\nu}$, while when $|\zeta| \gg 1$ the leading order contribution goes as $\propto H^2\zeta^{2\nu+1}$. This observation is of crucial importance for the late time behavior of the one-loop stress-energy tensor investigated in Sec. V.

The two series in (B12) can be resummed, resulting in[6]

$$\begin{aligned} i\Delta(x;x) = {}& \frac{H^2(1-\epsilon)^2}{16\pi^2}\left(\nu^2 - \frac{1}{4}\right)\bigg[-\frac{2\mu^{D-4}}{D-4} - c_\nu \\ & + \ln\left(\frac{4\pi\mu^2}{(1-\epsilon)^2H^2}\left(1-\frac{1}{\zeta}\right)^2\right) + 2\zeta - \zeta^2 \\ & - \ln\left(1-\frac{1}{\zeta^2}\right) - \frac{\zeta^{1-2\nu}}{\nu-\frac{1}{2}} \times {}_2F_1\left(1, \nu-\frac{1}{2}; \nu+\frac{1}{2}; \frac{1}{\zeta^2}\right) \\ & - \ln(1-\zeta^2) - \frac{\zeta^{3+2\nu}}{\nu+\frac{3}{2}} \times {}_2F_1\left(1, \nu+\frac{3}{2}; \nu+\frac{5}{2}; \zeta^2\right)\bigg]. \end{aligned} \tag{B13}$$

We would like to interpret this result as the analytic extension of Eq. (B12) to arbitrary (complex) ν. In order to do this uniquely we need to specify the Riemann sheet of both the logarithm and of the hypergeometric function in the second line for $|\zeta| \leq 1$ and in the third line for $|\zeta| \geq 1$ (recall that the logarithm has a branch cut along the negative argument and that the hypergeometric function has a

[5]We were unable to evaluate the constant c_ν. The first few values are given by quite simple expressions. For $\nu = \{3/2, 5/2, 7/2, 9/2, 11/2\}$ we have $c_\nu = \psi(2\nu - 2) + \{1/2, 1/3, 1/2, 2/3, 137/168\}$.

[6]MATHEMATICA represents the answer in terms of the Lerch transcendent, `LerchΦ`$[z, s, \alpha]$, which is the following generalization of the Riemann ζ function, `LerchΦ`$[z, s, \alpha] = \sum_{n=0}^{\infty} z^n/(n+\alpha)^s$. For our purpose it is convenient to express the Lerch transcendent in terms of the Gauss' hypergeometric function,

$$\texttt{Lerch}\Phi[z, 1, \alpha] = \frac{{}_2F_1(1, \alpha; \alpha+1; z)}{\alpha}.$$

branch cut running along positive arguments $z \geq 1$). It turns out that the following cut prescription uniquely specifies the analytic extension to arbitrary ν:

$$\sum_{n=1}^{\nu-3/2} \frac{\zeta^{-2n}}{n} \overset{|\zeta|<1}{\longrightarrow} -\frac{1}{2}\sum_{\pm} \ln\left(1 - \frac{1}{\zeta^2 \pm i\varepsilon}\right) - \frac{1}{2}\sum_{\pm} \frac{\zeta^{1-2\nu}}{\nu - \frac{1}{2}} \times {}_2F_1\left(1, \nu - \frac{1}{2}; \nu + \frac{1}{2}; \frac{1}{\zeta^2 \pm i\varepsilon}\right) = -\ln\left(\frac{1-\zeta^2}{\zeta^2}\right) - \frac{\zeta^{3-2\nu}}{\frac{3}{2} - \nu} \times {}_2F_1\left(1, \frac{3}{2} - \nu; \frac{5}{2} - \nu; \zeta^2\right) + \pi \tan(\pi\nu), \tag{B14}$$

$$\sum_{n=1}^{\nu+1/2} \frac{\zeta^{2n}}{n} \overset{|\zeta|>1}{\longrightarrow} -\frac{1}{2}\sum_{\pm} \ln(1 - (\zeta^2 \pm i\varepsilon)) - \frac{1}{2}\sum_{\pm} \frac{\zeta^{3+2\nu}}{\nu + \frac{3}{2}} \times {}_2F_1\left(1, \nu + \frac{3}{2}; \nu + \frac{5}{2}; \zeta^2 \pm i\varepsilon\right) = -\ln\left(\frac{1 - 1/\zeta^2}{1/\zeta^2}\right) - \frac{\zeta^{2\nu+1}}{-\frac{1}{2} - \nu} \times {}_2F_1\left(1, -\frac{1}{2} - \nu; \frac{1}{2} - \nu; \frac{1}{\zeta^2}\right) + \pi \tan(\pi\nu), \tag{B15}$$

where $\varepsilon > 0$ is an infinitesimal parameter. Notice that the results in (B14) and (B15) are independent on the cut prescription. When Eqs. (B14) and (B15) are used in (B13), one can obtain the coincident propagators in the form suitable for accelerating (23) and decelerating (24) space-times, respectively.

4. Numerical check of the analytic extension

In order to test the analytic extension procedure described above, we have performed some numerical checks. To get a numerical result for the propagator we use Eq. (13) for the mode functions. From those we can easily get the general expression for the matching coefficients $\alpha(k)$ and $\beta(k)$ from the first lines of (A6). When evaluated numerically Eq. (11) then gives an integral which, after the appropriate regularization, corresponds to (23). This is, of course, true up to the $1/(D-4)$ term, which is removed by renormalization. There are two ultraviolet divergences: a quadratic one and a logarithmic one. We can easily read off both divergences from (B4). We know those expressions for the divergences are also valid for non half integer ν. The quadratic divergence we can simply subtract from our numerical result. In order to correctly regulate the logarithmic divergence, we choose an ultraviolet cutoff Λ for our numerical integrals and realize from (B4) that we must identify

$$\ln(\Lambda) = \ln(a\mu\sqrt{\pi}) + 1 - \tfrac{1}{2}\gamma_E. \tag{B16}$$

To smoothen the numerical procedure, we also choose an infrared cutoff k_0. Apart from the $1/(D-4)$ term, the numerically integrated result should now correspond to (23) for all values of ν, up to the undetermined constant c_ν in the analytic result (23). We fix this constant for a certain fixed, but arbitrary set of parameters. As an example we show in Fig. 4 the result for $\nu = 1.7$. The main plot shows the behavior of the coincident propagator divided by the Hubble scale $\Delta(x;x)/H^2$ versus conformal time defined by

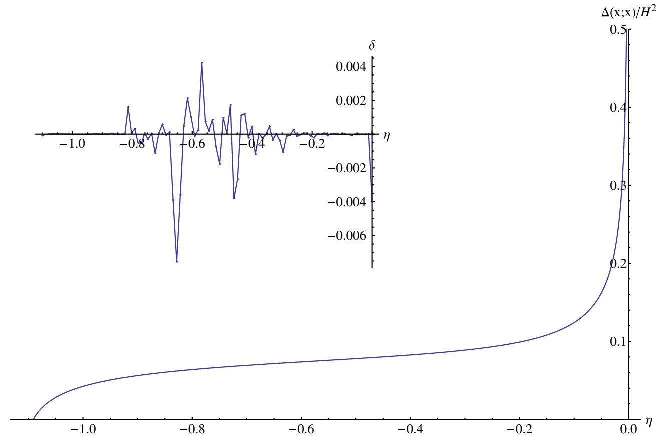

FIG. 4 (color online). The main plot shows the coincident propagator (23), rescaled by H^2 versus conformal time (B17). The inset shows the difference between the analytically calculated result and a direct numerical integration of (11), using the mode functions and the α and β coefficients in terms of Hankel functions (13). The parameters chosen are $a = 4$, $\nu = 1.7$, $\epsilon = 0.1$, $\Lambda = 10^4(\hat{a}\hat{H})$, $k_0 = 10^{-6}(\hat{a}\hat{H})$. We find for this case that $c_\nu \approx \psi(2\nu - 2) + 0.348$.

$$\eta = -\frac{1}{(1-\epsilon)Ha}. \tag{B17}$$

For the matching point we choose $aH = \widehat{aH}$, such that for $\epsilon = 0.1$ conformal time at the matching is $\hat{\eta} = -1.1(\widehat{aH})^{-1}$. The inset in the figure shows the difference between the numerically calculated propagator and the analytic solution (23). We thus find that, up to a numerical noise of the order of 1% or less, the two results agree. Such a noise level is not unreasonable given the complexity of the numerical integrations. We have also checked several other values of ν, which give similar results. Albeit we do not have a rigorous mathematical proof, these numerical results strongly suggest that the analytic extension to all ν made in (23) and (24) is correct.

APPENDIX C: ACTING WITH THE D'ALEMBERTIAN

In order to calculate the contribution to the trace of the stress-energy tensor (29), the following identities are useful:

$$\square[H^2 f(\zeta)] = (1-\epsilon)H^4\left[6\epsilon + (3 - 5\epsilon)\zeta\frac{d}{d\zeta} - (1-\epsilon)\zeta\frac{d}{d\zeta}\zeta\frac{d}{d\zeta}\right]f(\zeta), \tag{C1}$$

$$\Box[H^2\ln(H^2)] = 2\epsilon H^4[3(1-\epsilon)\ln(H^2) + (3-5\epsilon)], \quad \text{(C2)}$$

where we made use of $\Box = -(\partial_t + 3H)\partial_t$ and $\partial_t = -(1-\epsilon)H\zeta\partial_\zeta$. From these we easily get the following useful identities:

$$\begin{aligned}\Box[H^2\ln(1\pm\zeta)] &= (1-\epsilon)H^4\bigg[6\epsilon\ln(1\pm\zeta) + (3-5\epsilon) \\ &\quad - \frac{2(2-3\epsilon)}{1\pm\zeta} + \frac{1-\epsilon}{(1\pm\zeta)^2}\bigg], \\ \Box[H^2] &= (1-\epsilon)H^4(6\epsilon), \\ \Box[H^2\ln(\zeta)] &= (1-\epsilon)H^4[6\epsilon\ln(\zeta) + (3-5\epsilon)], \\ \Box[H^2\zeta^\omega] &= (1-\epsilon)H^4[6\epsilon + (3-5\epsilon)\omega - (1-\epsilon)\omega^2].\end{aligned} \quad \text{(C3)}$$

From these identities we can also obtain how the d'Alembertian acts on the logarithms in (23) and (24)

$$\begin{aligned}&\Box\bigg[H^2\ln\bigg(\frac{\mu^2}{H^2}\Big(1-\frac{1}{\zeta}\Big)^2\bigg)\bigg] \\ &= (1-\epsilon)H^4\bigg\{6\epsilon\ln\bigg[\frac{\mu^2}{H^2}\Big(1-\frac{1}{\zeta}\Big)^2\bigg] - \frac{2\epsilon(3-5\epsilon)}{1-\epsilon} \\ &\quad - \frac{4(2-3\epsilon)}{1-\zeta} + \frac{2(1-\epsilon)}{(1-\zeta)^2}\bigg\}\bigg[H^2\ln\bigg(\frac{\mu^2}{H^2(1+\zeta)^2}\bigg)\bigg] \\ &= (1-\epsilon)H^4\bigg\{6\epsilon\ln\bigg[\frac{\mu^2}{H^2(1+\zeta)^2}\bigg] - \frac{2(3-5\epsilon)}{1-\epsilon} \\ &\quad + \frac{4(2-3\epsilon)}{1+\zeta} - \frac{2(1-\epsilon)}{(1+\zeta)^2}\bigg\}.\end{aligned} \quad \text{(C4)}$$

These expressions are used in Sec. V to calculate the trace of the stress-energy tensor from the coincident propagator.

APPENDIX D: INTEGRATING THE STRESS-ENERGY CONSERVATION EQUATION

The covariant conservation of the stress-energy tensor for quantum fluctuations in Friedmann-Lemaître-Robertson-Walker space-times acquires in $D = 4$ the form (31):

$$\frac{1}{H}\dot{\rho}_q + 4\rho_q = \rho_q - 3p_q = -T_q. \quad \text{(D1)}$$

When the stress-energy trace T_q is a function of the Hubble parameter H only, we have $d/dt = -\epsilon H d/dH$, such that Eq. (D1) can be integrated,

$$\rho_q = \frac{H^{4/\epsilon}}{\epsilon}\int^H \frac{d\tilde{H}}{H^{1+4/\epsilon}}T_q(\tilde{H}). \quad \text{(D2)}$$

If, on the other hand, $T_q = H^4\tau_q(\zeta)$ with $\rho_q = H^4 r_q(\zeta)$, Eq. (D1) can be recast as

$$\Big(4 - \zeta\frac{d}{d\zeta}\Big)r_q = -\frac{\tau_q}{1-\epsilon}. \quad \text{(D3)}$$

This can be easily integrated to yield

$$\rho_q = H^4\frac{\zeta^4}{1-\epsilon}\int^\zeta \frac{d\tilde{\zeta}}{\tilde{\zeta}^5}\tau_q(\tilde{\zeta}), \qquad p_q = \frac{1}{3}(\rho_q + T_q). \quad \text{(D4)}$$

The simple useful examples that make use of Eq. (D3) are

$$\begin{aligned}T_q &= t_0H^4 \Rightarrow \rho_q = t_0H^4\Big(-\frac{1}{4(1-\epsilon)}\Big), \\ T_q &= t_1H^4\ln(H^2) \Rightarrow \rho_q \\ &= t_1H^4\Big(-\frac{1}{4(1-\epsilon)}\Big)\Big(\ln(H^2) + \frac{\epsilon}{2(1-\epsilon)}\Big).\end{aligned} \quad \text{(D5)}$$

Useful examples which make use of the integral (D4) are

$$\begin{aligned}T_q &= t_2H^4\zeta^\omega \Rightarrow \rho_q = t_2\frac{H^4\zeta^\omega}{(1-\epsilon)(\omega-4)}, \qquad (\omega \neq 4) \\ T_q &= t_3H^4\zeta^4 \Rightarrow \rho_q = -t_3\frac{H^4}{1-\epsilon}\zeta^4\ln(\zeta), \\ T_q &= t_4H^4\ln(\zeta) \Rightarrow \rho_q = t_4H^4\Big(-\frac{1}{4(1-\epsilon)}\Big)\Big(\ln(\zeta) + \frac{1}{4}\Big).\end{aligned} \quad \text{(D6)}$$

Here t_n ($n = 0, 1, 2, 3, 4$) are constants, which in general depend on ϵ and ξ but not on time.

APPENDIX E: THE SPECIAL CASES WHEN $\nu = 3/2$ AND $\nu = 5/2$

The expressions (58)–(60), (81), and (82) are singular in the limit when $\nu = 3/2$, indicating that the $\nu = 3/2$ case requires a special attention. We shall first consider the accelerating case, and then the decelerating case. Furthermore, Eqs. (89) and (90) are singular in the limit when $\epsilon = 3/2$. This singular behavior can be traced back to the $\nu = 5/2$ divergence in Eqs. (73) [see the $n = 1$ member of the sum (75)]. This special case is also considered below.

1. The special case when $\epsilon < 1$, $\nu = 3/2$

In order to get the correct stress-energy tensor for $\nu = 3/2$ we need to take the $\nu = 3/2$ limit of Eqs. (46) and (50). This amounts to adding the corresponding contributions from (53)–(55) and (58)–(60) and the $(\pi/2)\tan(\pi\nu)$ from (46) and (50). The result is

$$\begin{aligned}T_q &\overset{\zeta\to0}{\to} -\frac{(1-6\xi)^2(1-\epsilon)(2-\epsilon)H^4}{16\pi^2} \\ &\quad \times\bigg\{6\epsilon\bigg[\ln\Big(\frac{H_0a}{\hat{H}\hat{a}}\Big) + \frac{\epsilon}{4(1-\epsilon)}\bigg] - (3-5\epsilon)\bigg\},\end{aligned} \quad \text{(E1)}$$

$$\rho_q \overset{\zeta\to0}{\to} \frac{(1-6\xi)^2(2-\epsilon)H^4}{64\pi^2}\bigg\{6\epsilon\ln\Big(\frac{H_0a}{\hat{H}\hat{a}}\Big) - (3-5\epsilon)\bigg\}, \quad \text{(E2)}$$

$$p_q \overset{\zeta\to 0}{\longrightarrow} \frac{(1-6\xi)^2(2-\epsilon)H^4}{64\pi^2}\bigg\{\Big(-1+\frac{4}{3}\epsilon\Big)\bigg[6\epsilon\ln\Big(\frac{H_0 a}{\hat{H}\,\hat{a}}\Big) - (3-5\epsilon)\bigg] - 2\epsilon^2\bigg\}, \tag{E3}$$

where H_0 is defined in (56). Taking a ratio of (E2) and (E3) gives the equation of state parameter for this case,

$$w_q \overset{\zeta\to 0}{\longrightarrow} -1+\frac{4}{3}\epsilon - \frac{2\epsilon^2}{6\epsilon\ln[H_0 a/(\hat{H}\,\hat{a})]-(3-5\epsilon)} \to -1+\frac{4}{3}\epsilon = w_b + \frac{2}{3}\epsilon. \tag{E4}$$

Notice that this result agrees (in the limit when $a\to\infty$) with both w_q in Eq. (57) and with $\nu\to 3/2$ limit—or equivalently the $\xi\to\epsilon(3-2\epsilon)/[6(2-\epsilon)]$ limit—of relation (61). Hence, when it comes to the equation of state parameter w_q at late times, there is nothing special about the point $\nu = 3/2$: the curves shown in Fig. 2 are continuous at $\nu = 3/2$. The only special point is the logarithmic form of the stress-energy tensor, as can be seen in Eqs. (E1)–(E3). One can check that the results identical to (E1)–(E4) can be obtained by calculating the stress-energy tensor from the $\nu = 3/2$ case of the half-integer coincident propagator (B12), representing a check of Eqs. (E1)–(E3), as well as a check of our procedure based on analytic continuation. Notice that the logarithms drop out in the de Sitter limit when $\epsilon\to 0$, which is a well-known one-loop result. The logarithms are, however, expected to reappear at two or higher loop order also in de Sitter space both in massless scalar theories [33] as well as in quantum gravity [12].

2. The special case when $\epsilon > 1$, $\nu = 3/2$

Just as in the accelerating case, when matching onto deceleration the limit $\nu\to 3/2$ appears singular, as can be seen from Eqs. (81) and (82). The full expressions (71) and (73) are of course regular (thanks to the $\tan(\pi\nu)$ terms). The get the correct late time limit in this case, it suffices to add the $\tan(\pi\nu)$ terms from (71) and (73) to Eqs. (81) and (82) and take the $\nu\to 3/2$ limit. The results are finite and—just as in the accelerating case—they acquire logarithms

$$T_q \overset{\zeta\to\infty}{\longrightarrow} \frac{(1-6\xi)^2(1-\epsilon)(2-\epsilon)^2H^4}{32\pi^2}\zeta^4, \qquad (\nu = 3/2), \tag{E5}$$

$$\rho_q \overset{\zeta\to\infty}{\longrightarrow} \frac{(1-6\xi)^2(2-\epsilon)^2H^4}{32\pi^2}\zeta^4\ln(\zeta), \tag{E6}$$

$$p_q \overset{\zeta\to\infty}{\longrightarrow} \frac{(1-6\xi)^2(2-\epsilon)^2H^4}{32\pi^2}\frac{\zeta^4}{3}(\ln(\zeta)+(1-\epsilon)). \tag{E7}$$

The equation of state parameter w_q follows trivially from these relations,

$$w_q \overset{\zeta\to\infty}{\longrightarrow} \frac{1}{3} - \frac{\epsilon-1}{\ln(\zeta)} \to \frac{1}{3}. \tag{E8}$$

This relativistic fluid scaling is in accordance with the $\nu = 3/2$ limit of Eq. (83). Curiously w_q in (E8) does not depend on ϵ, implying that as long as the relation $\xi = (3\epsilon-2)/[6(2-\epsilon)]$ holds, $w_q = 1/3$ will not depend on ϵ. Just as above, it is worth commenting that results identical to (E5) can be obtained by calculating the stress-energy tensor from the half-integer coincident propagator (B12) by taking $\nu = 3/2$ and assuming $\epsilon > 1$. In the minimally coupled scalar case when $\xi = 0$, the $\nu = 3/2$ case corresponds to the matter era, $\epsilon = 3/2$. In this case the logarithmic one-loop structure exhibited in Eqs. (E5)–(E7) is also well known in the literature.

3. The special case when $\nu = 5/2$, $\epsilon = 3/2$

The last special case that requires attention is the $\nu\to 5/2$ limit on the boundary of stability, where $\xi = -(\epsilon-1)/[3(2-\epsilon)]$ as in Eq. (86) and $\epsilon = 3/2$, such that Eqs. (89) and (90) appear singular. This limit corresponds to the $\nu = 3/2$, $n = 1$ term in Eq. (75). Similarly as above, in this limit one gets a finite result when the term $\propto \zeta^4\pi\tan(\pi\nu)$ from Eq. (73) is included. In this case one reproduces the $\nu = 3/2$ case discussed above in Eqs. (E5) with $\epsilon\to 3/2$:

$$T_q \overset{\zeta\to\infty}{\longrightarrow} -\frac{(1-6\xi)^2H^4}{256\pi^2}\zeta^4, \qquad (\nu = 5/2, \epsilon = 3/2), \tag{E9}$$

$$\rho_q \overset{\zeta\to\infty}{\longrightarrow} \frac{(1-6\xi)^2H^4}{128\pi^2}\zeta^4\ln(\zeta), \tag{E10}$$

$$p_q \overset{\zeta\to\infty}{\longrightarrow} \frac{(1-6\xi)^2(2-\epsilon)^2H^4}{128\pi^2}\frac{\zeta^4}{3}\Big(\ln(\zeta)-\frac{1}{2}\Big). \tag{E11}$$

The equation of state parameter is then

$$w_q \overset{\zeta\to\infty}{\longrightarrow} \frac{1}{3} - \frac{1}{2\ln(\zeta)} \to \frac{1}{3}. \tag{E12}$$

We have thus shown that a finite answer is obtained for all values of ν.

APPENDIX F: EXPLICITLY CALCULATING THE $H^4\zeta^4$ CONTRIBUTION

In this appendix we shall show that the contribution to the stress-energy tensor proportional to $H^4\zeta^4$ is actually ultraviolet divergent. The finite contribution proportional to $H^4\zeta^4$ will therefore—after renormalization—always be undetermined, until it is fixed by a measurement. Thus, not only does our procedure to calculate ρ_q and p_q from the trace T_q in Sec. V not determine this constant uniquely, it turns out that in the given model the sudden matching at $t = \hat{t}$ generates an infinite amount of conformal fluctuations and therefore *cannot* be determined uniquely. In order to show this, we use (27) to obtain

$$\langle\Omega|T_{00}|\Omega\rangle + \frac{1}{D-2}\langle\Omega|T|\Omega\rangle = \left(\partial_t\partial_{\tilde{t}} + \xi(R_{00} - \nabla_t\partial_t) + \frac{1}{D-2}\xi\Box\right)i\Delta(x;\tilde{x})\Big|_{x=\tilde{x}}, \tag{F1}$$

where $T = T^{\mu}{}_{\mu}$. Using (29) we then find

$$\langle\Omega|T_{00}|\Omega\rangle + \frac{1-4\xi}{D-2-4\xi(D-1)}\langle\Omega|T|\Omega\rangle = (\partial_t\partial_{\tilde{t}} + \xi(R_{00} - \nabla_t\partial_t))i\Delta(x;\tilde{x})|_{x=\tilde{x}}. \tag{F2}$$

Now this equation depends also on T_{00} and therefore has a well-defined contribution proportional to $H^4\zeta^4$, which we could not determine before, when we only calculated the contribution proportional to T. We shall first consider the first term of the right-hand side (RHS)

$$\partial_t\partial_{\tilde{t}}i\Delta(x;\tilde{x})|_{x=\tilde{x}} = \frac{1}{2^{D-2}\pi^{(D-1)/2}\Gamma(\frac{D-1}{2})}\times\int dk k^{D-2}|\partial_t\psi(t,k)|^2. \tag{F3}$$

Now for the purpose of this section, we shall only consider the ultraviolet divergent contribution to (F3). We saw in Appendix B that away from the matching point, only those terms that are polynomial in k contribute to the UV divergence. Thus, using (13), we see that we can write

$$|\partial_t\psi(t,k)|^2 \overset{\text{UV-div}}{\longrightarrow} (|\alpha|^2 + |\beta|^2)|\partial_t(a(t)^{1-(D/2)}u(t,k))|^2. \tag{F4}$$

Using similar techniques that led to Eq. (B2) we find in this case

$$\begin{aligned}\partial_t\partial_{\tilde{t}}i\Delta(x;\tilde{x})|_{x=\tilde{x},\mathrm{UV}} ={}& \frac{a^{-D}}{2^{D-2}\pi^{(D-1)/2}\Gamma(\frac{D-1}{2})}\int dk k^{D-1}\sum_{q,r,m,n=0}^{\nu-(1/2)}\bigg\{\frac{\Gamma(\nu-\frac{1}{2}+n)\Gamma(\nu+\frac{1}{2}+m)\Gamma(\nu+\frac{1}{2}+q)\Gamma(\nu-\frac{1}{2}+r)}{\Gamma[\nu+\frac{3}{2}-n)\Gamma(\nu+\frac{1}{2}-m)\Gamma(\nu+\frac{1}{2}-q)\Gamma(\nu+\frac{3}{2}-r)}\\ &\times\bigg\{\frac{1}{256}\bigg(2 - i(m-q)(1-\epsilon)\frac{\hat{a}\hat{H}}{k} + mq(1-\epsilon)^2\frac{(\hat{a}\hat{H})^2}{q^2}\bigg)\bigg(4(1-4n^2-4\nu^2)(1-4r^2-4\nu^2)\\ &+ 8i(n-r)(D-1-\epsilon)((1-2n)(1-2r) - 4(1-2n-2r)\nu^2)\frac{aH}{k} + (D-1-\epsilon)^2((1-2n)^2 - 4\nu^2)\\ &\times((1-2r)^2 - 4\nu^2)\frac{a^2H^2}{k^2}\bigg)\bigg\}(-1)^{-q-n}\bigg(\frac{2ik}{(1-\epsilon)\hat{a}\hat{H}}\bigg)^{-m-q}\bigg(\frac{2ik}{(1-\epsilon)aH}\bigg)^{-n-r}\frac{1}{n!m!q!r!}\bigg\},\end{aligned} \tag{F5}$$

where as before $\hat{a} = a(\hat{t})$, $\hat{H} = H(\hat{t})$. To obtain the UV divergent terms, we sum the first four terms of (F5) to obtain

$$\begin{aligned}\partial_t\partial_{\tilde{t}}i\Delta(x;\tilde{x})|_{x=\tilde{x},\mathrm{UV}} ={}& \frac{a^{-D}}{2^{D-2}\pi^{(D-1)/2}\Gamma(\frac{D-1}{2})}\int dk k^{D-1}\bigg\{\frac{1}{2} + \frac{1}{16}(9 - 2\epsilon + \epsilon^2\\ &+ 2D(D-4) - 4(1-\epsilon)^2\nu^2)\frac{a^2H^2}{k^2} + \frac{1}{64}(1-\epsilon)^2\\ &\times\bigg(\nu^2 - \frac{1}{4}\bigg)\bigg[4(1-\epsilon)^2\bigg(\nu^2 - \frac{1}{4}\bigg)\zeta^4 + (73 + 4D^2\\ &- 16D(2-\epsilon) - 82\epsilon + 25\epsilon^2 - 4(1-\epsilon)^2\nu^2)\bigg]\frac{a^4H^4}{k^4}\bigg\},\end{aligned} \tag{F6}$$

where $\zeta = \frac{\hat{a}\hat{H}}{aH}$. The power law divergences (k^{D-1} and k^{D-3}) are automatically subtracted in dimensional regularization, and we are thus left with the logarithmic divergence. We evaluate this divergence as in Sec. VII and obtain for the $1/(D-4)$ part

$$\frac{(2-\epsilon)(1-6\xi)}{32\pi^2(D-4)}(3(\epsilon(1-2\epsilon) - 2\xi(2-\epsilon)) - (2-\epsilon)(1-6\xi)\zeta^4)\mu^{D-4}H^4, \tag{F7}$$

where we used the expression for ν from (14). The $1/(D-4)$ part from the other two terms from the RHS of (F2) are easily evaluated, using (B6) and

$$R_{00} = -3(1-\epsilon)H^2, \qquad \nabla_t\partial_t H^2 = 6\epsilon^2H^4. \tag{F8}$$

We can put all terms together and obtain for the $1/(D-4)$ contribution to the RHS of (F2)

$$\frac{(2-\epsilon)(1-6\xi)}{32\pi^2(D-4)}(3\epsilon(1-2\epsilon-2\xi(1-4\epsilon)) - (2-\epsilon)(1-6\xi)\zeta^4)H^4. \tag{F9}$$

Thus we indeed find that the contribution comes in two parts: one proportional to H^4, and one proportional to $H^4\zeta^4$. As a check, we can calculate the left-hand side of (F1), using the ultraviolet contributions we obtained for the trace in (39). Using the conservation equation we can then find, using (D2) also the divergent contribution to the energy density, since if $T \propto H^4$ we have, apart from a possible $H^4\zeta^4$ contribution, that

$$T_{00} = -\frac{1}{4(1-\epsilon)}T. \tag{F10}$$

Using (39) we then thus find for the $1/(D-4)$ part of the left-hand side

$$\frac{(2-\epsilon)(1-6\xi)}{32\pi^2(D-4)}(3\epsilon(1-2\epsilon-2\xi(1-4\epsilon)))H^4. \quad \text{(F11)}$$

In other words, the calculation leading to (F9) is consistent with the calculation in the rest of this paper, apart from the $H^4\zeta^4$ term. From (F9) we thus see that there is a divergence $H^4\zeta^4/(D-4)$ contributing to the stress-energy tensor. This divergence could be subtracted by a counterterm of the form

$$\alpha \int d^D x\sqrt{-g}p_r, \quad \text{(F12)}$$

where α is a constant (which will depend on the matching time $\hat{t}$) and p_r is the pressure of some radiation fluid, obeying $p_r = 1/3\rho_r$. This fluid could, for example, be a photon fluid, or a scalar field. This renormalization is not completely satisfactory, since it requires the addition of a new field, not present in the original model. However, one needs to keep in mind here that our model is incomplete anyway. The sudden matching is put in by hand, where in a realistic model, it should arise from some dynamics. Moreover, in a realistic model, the matching is never instantaneous, which would probably remove this UV divergence anyway. Therefore we do not feel that these pathologies, arising from the sudden matching are very problematic. However, given the fact that the present model needs a counterterm like (F12), the undetermined finite part contributing to the counterterm makes any $H^4\zeta^4$ contribution to the stress-energy tensor arbitrary.

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