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A NOTE ON THE GREEDY β-TRANSFORMATION WITH ARBITRARY DIGITS

by

Karma Dajani & Charlene Kalle

Abstract. — We consider a generalization of the greedy and lazy β-expansions with digit set $A = \{a_0 < a_1 < \cdots < a_m\}$. We prove that the transformation generating such expansions admits a unique absolutely continuous invariant ergodic measure. Furthermore, the support of this measure is an interval.

***Résumé* (Note sur la β-transformation avec des chiffres arbitraires).** — Nous considérons une généralisation des « greedy » et « lazy » β-développements avec chiffres dans un alphabet $A = \{a_0 < a_1 < \cdots < a_m\}$. Nous montrons que la transformation qui donne ces développements possède une unique mesure qui soit invariante, ergodique et absolument continue par rapport à la mesure de Lebesgue. En outre, le support de cette mesure est un intervalle.

1. Introduction

Let $\beta > 1$ be a real number, and let T_β be the transformation of the unit interval [0,1) given by $T_\beta x = \beta x (\text{mod} 1)$. This transformation gives rise to the β-expansion introduced by Rényi [19]: for any $0 \le x < 1$,

$$x = \sum_{k=1}^{\infty} d_k \beta^{-k} \,, \tag{1}$$

where $d_k = d_k(x, \beta) = \lfloor \beta T_\beta^{k-1} x \rfloor$, $k \ge 1$ (here $\lfloor \xi \rfloor$ denotes the greatest integer not exceeding ξ). Rényi showed that for each $\beta > 1$ the β-transformation T_β is ergodic, and that there exists a unique probability measure ν_β, equivalent to Lebesgue measure and invariant under T_β, such that for each Borel measurable set $B \in \mathcal{B}$ one has

$$\nu_\beta(B) = \int_B h_\beta(x)\, dx \,,$$

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where $h_\beta(x)$ is a measurable function satisfying

$$1 - \frac{1}{\beta} \leq h_\beta(x) \leq \frac{1}{1 - \frac{1}{\beta}} .$$

Shortly afterwards, Parry [15] gave an explicit formula for the density of the invariant measure, namely,

$$h_\beta(x) = \frac{1}{F(\beta)} \sum_{x < T^n(1)} \frac{1}{\beta^n} \; x \in [0,1) ,$$

where $F(\beta) = \int_0^1 \left(\sum_{x<T^n(1)} \frac{1}{\beta^n} \right) dx$ is a normalizing constant.

The β-transformation given above can be defined geometrically in the following way. There exists a subdivision $\alpha_0 = 0 < \alpha_1 < \cdots < \alpha_m < \alpha_{m+1} = 1$ of $[0,1)$, such that T_β is linear with slope β on each subinterval $I_j = [\alpha_j, \alpha_{j+1})$. Further, $TI_j = [0,1)$, $T\alpha_j = 0$ for $j = 0, 1, \cdots, m$, and $T1 = \lim_{x \to 1} Tx \leq 1$. This implies that on I_j, T is given by $Tx = \beta x - j$. Iterations of T give expansions of the form (1) with digit $d_k \in A = \{0, 1, \cdots, m\}$ (notice that $m = \lfloor \beta \rfloor$).

Expansion (1) is also sometimes known in the literature by the *greedy expansion*. The reason is that, for each k the digit d_k in (1) is the largest element of $A = \{0, 1, \cdots, \lfloor \beta \rfloor\}$ such that $\sum_{j=1}^k d_j \beta^{-j} \leq x$. So at each step, the greedy expansion chooses the largest possible digit. Taking this point of view, Pedicini [17] generalized the above notion to greedy expansions with digits in some set $A = \{a_0, \ldots, a_m\}$, with $a_0 < \cdots < a_m$. More precisely, he studied the combinatorial and arithmetic properties of expansions of the form $x = \sum_{k=1}^\infty d_k \beta^{-k}$ such that for each k, d_k is the largest element of $A = \{a_0, a_1, \ldots a_m\}$ with $\sum_{j=1}^k d_j \beta^{-j} \leq x$ (see [4] for more detail). He showed that every point in the interval $\left[\frac{a_0}{\beta-1}, \frac{a_m}{\beta-1}\right]$ has a greedy expansion with digits in the set A, if and only if

$$\max_{0 \leq j \leq m-1} (a_{j+1} - a_j) \leq \frac{a_m - a_0}{\beta - 1}. \tag{2}$$

He called such expansions *greedy expansions with deleted digits*. This terminology has already been used by several authors (see [11], [18], [12], [17], [4], [21]), but in most cases the digit sets under consideration contain only non-negative integers. Since our digit sets will contain arbitrary real numbers, we choose to adopt *greedy expansions with arbitrary digits* instead or we will refer explicitly to the digit set that we use. We will now give a geometrical description of the underlying map generating these expansions, which shows why condition (2) imposed by Pedicini is a natural one. Furthermore, it allows us to put these transformations in the general framework of piecewise linear maps, for which a rich theory has been developed, and which we use to analyze and understand the ergodic properties of the β-transformations with arbitrary digits. The description is similar to the one given for the classical greedy expansion defined above. We consider transformations T whose domain is some interval $[0,\alpha]$ with the following properties

(i) there exists a subdivision $0=\alpha_0<\alpha_1<\cdots<\alpha_m<\alpha_{m+1}=\alpha$ with corresponding interval partition $J_j=[\alpha_j,\alpha_{j+1})$, $j=0,1,\dots m-1$, $J_m=[\alpha_m,\alpha]$ such that T on each J_j is linear with slope β,

(ii) $T\alpha_j=0$, $j=0,1,\dots,m$, and $T\alpha=\alpha$,

(iii) $TJ_j\subset[0,\alpha]$, $j=0,1,\dots m-1$.

Note that from (ii) and the linearity of T, we have that $TJ_m=[0,\alpha]$ so that T is surjective. Setting $a_j=\beta\alpha_j$ for $j=0,1,\dots m$, one sees from (i) and (ii) that on each J_j, T has the form $Tx=\beta x-a_j$, $j=0,1,\dots m$. From the second equation in (ii), one has that $\alpha=\frac{a_m}{\beta-1}$. From (iii) one gets $\max_{0\le j\le m-1}(a_{j+1}-a_j)\le\alpha=\frac{a_m}{\beta-1}$, which is condition (2) with $a_0=0$. See Figure 1 for the graph of T. Iterations of T generate greedy expansions with digits in A as described by Pedicini.

The above geometrical description can be slightly modified in order to capture the case $a_0\neq 0$. The interval $[0,\alpha]$ is replaced by $[\alpha_0,\alpha]$ $(0\neq\alpha_0<\alpha)$, and in condition (ii) we replace the first equality by $T\alpha_j=\alpha_0$ for $i=0,1,\dots,m$. However, a simple translation by α_0 conjugates a transformation T in this class with domain $[\alpha_0,\alpha]$ and subdivision $\alpha_0<\alpha_1<\cdots<\alpha_m<\alpha_{m+1}=\alpha$ to a transformation S of the previous class on $[0,\alpha-\alpha_0]$ with subdivision $0=\gamma_0<\gamma_1<\cdots<\gamma_m<\gamma_{m+1}=\alpha-\alpha_0$, $\gamma_i=\alpha_i-\alpha_0$, $i=0,1,\dots,m+1$. Setting now $a_j=\beta\alpha_j-\alpha_0$, and using the conjugation with S or the defining properties of T, we see that

$$Tx=\begin{cases}\beta x-a_j, & \text{if } x\in\left[\dfrac{a_0}{\beta-1}+\dfrac{a_j-a_0}{\beta},\dfrac{a_0}{\beta-1}+\dfrac{a_{j+1}-a_0}{\beta}\right),\\ & \text{for } j=0,\dots,m-1,\\ \\ \beta x-a_m, & \text{if } x\in\left[\dfrac{a_0}{\beta-1}+\dfrac{a_m-a_0}{\beta},\dfrac{a_m}{\beta-1}\right].\end{cases}$$

We call $T=T_{\beta,A}$ the *greedy transformations with digits in the set* $A=\{a_0,\dots,a_m\}$ with $a_0<\cdots<a_m$. Clearly, A satisfies condition (2), and a set with this property is called an *allowable digit set for* β. From the above, and to keep the exposition simple, we will assume with no loss of generality that $a_0=0=\alpha_0$.

As explained above, the greedy expansion chooses at each stage the largest possible digit. One can look at the other extreme case, namely β-expansions $\sum_{k=1}^{\infty}d_k\beta^{-k}$ such that for each k, d_k is the smallest member of A satisfying

$$x\le\sum_{i=1}^{k}\frac{d_i}{\beta^i}+\sum_{i=k+1}^{\infty}\frac{a_m}{\beta^i}. \tag{3}$$

These expansions are known as *lazy expansions*, and were studied in the classical case, *i.e.*, $A = \{0, 1, \dots \lfloor\beta\rfloor\}$ by many authors (see for example [5], [6], [8] and [7]), and for the general case $A = \{a_0, \dots, a_m\}$ by Pedicini [17] (see also [4]). Pedicini showed that under condition (2), every $x \in \left[\frac{a_0}{\beta-1}, \frac{a_m}{\beta-1}\right]$ has a lazy expansion. In [4] the underlying transformation generating lazy expansions with digit set A was given, and was shown to be conjugate to the greedy expansion $T = T_{\beta,\tilde{A}}$ with $\tilde{A} = \{\tilde{a}_0, \dots, \tilde{a}_m\}$, where $\tilde{a}_i = a_0 + a_m - a_{m-i}$, $i \in \{0, \dots, m\}$. The isomorphism ϕ is given by

$$\begin{aligned} \phi : \quad \left[\frac{a_0}{\beta-1}, \frac{a_m}{\beta-1}\right] &\longrightarrow \left[\frac{a_0}{\beta-1}, \frac{a_m}{\beta-1}\right] \\ x &\longmapsto \frac{a_0+a_m}{\beta-1} - x. \end{aligned}$$

So we have $L \circ \phi = \phi \circ T$. The explicit definition of $L = L_{\beta,A}$ is given by

$$Lx = \begin{cases} \beta x - a_0, & \text{if } x \in \left[\dfrac{a_0}{\beta-1}, \dfrac{a_m}{\beta-1} - \dfrac{a_m - a_0}{\beta}\right], \\[2ex] \beta x - a_j, & \text{if } x \in \left(\dfrac{a_m}{\beta-1} - \dfrac{a_m - a_{j-1}}{\beta}, \dfrac{a_m}{\beta-1} - \dfrac{a_m - a_j}{\beta}\right], \\ & \text{for } j = 1, \dots, m. \end{cases}$$

As mentioned, throughout we will assume that $a_0 = 0$.

A number of articles have been published on invariant measures of piecewise monotonic transformations. Among others, the articles [1] by Buzzi and Sarig, [2] by Byers and Boyarsky, [3] by Byers, Góra and Boyarsky, [9] and [10] by Hofbauer, [13] by Lasota and Yorke, [14] by Li and Yorke, [20] by Schweiger and [22] by Wilkinson state a variety of results regarding invariant measures of this kind of transformations and their ergodicity.

In the first section of this article we will prove the existence of a unique absolutely continuous invariant ergodic measure μ for the greedy transformation with arbitrary digits using the results found by Li and Yorke in [14]. We will show that the support of this measure is the smallest interval of the form $[0, t)$ such $T([0, t)) \subset [0, t)$, and we identify t explicitly. This leads to the following theorem

Theorem 1.1. — *The restriction of T to the interval $[0, t)$ admits a unique invariant ergodic measure that is equivalent to Lebesgue measure on this interval.*

We give similar results for the lazy transformation with arbitrary digits. In the last section we consider in more detail two classes of greedy transformations with arbitrary digits, and we give an explicit formula for the density of their absolutely continuous invariant measures. For the first class an article by Wilkinson ([22]) has been an important source. For the second class we use an article by Byers and Boyarski ([2]), which is based on [16] by Parry.

2. Ergodic absolutely continuous measures

Let $\beta > 1$ and $A = \{a_0, a_1, \ldots, a_m\}$ be an allowable digit set with $a_0 = 0$. Let $T = T_{\beta,A} : \big[0, \frac{a_m}{\beta-1}\big] \to \big[0, \frac{a_m}{\beta-1}\big]$ be the greedy transformation with digits in A. This is a piecewise linear, strictly increasing transformation, which has its discontinuities in the points $\frac{a_i}{\beta}$ for $i = 1, \ldots, m$. Let J denote the set containing these points. Then J is finite and for each $x \in \big[0, \frac{a_m}{\beta-1}\big] \setminus J$ we have $T'(x) = \beta > 1$. The points in J give a partition $\Delta = \{\Delta_i\}_{i=0}^m$ of the interval $\big[0, \frac{a_m}{\beta-1}\big]$, where $\Delta_m = \big[\frac{a_m}{\beta}, \frac{a_m}{\beta-1}\big]$ and for $i \in \{0, \ldots, m-1\}$, $\Delta_i = \big[\frac{a_i}{\beta}, \frac{a_{i+1}}{\beta}\big)$. Define for $i \in \{1, \ldots, m\}$ the values y_i to be the values obtained from T by taking the limit from the left to the points $\frac{a_i}{\beta}$, *i.e.*, $y_i = a_i - a_{i-1}$. (See Figure 1)

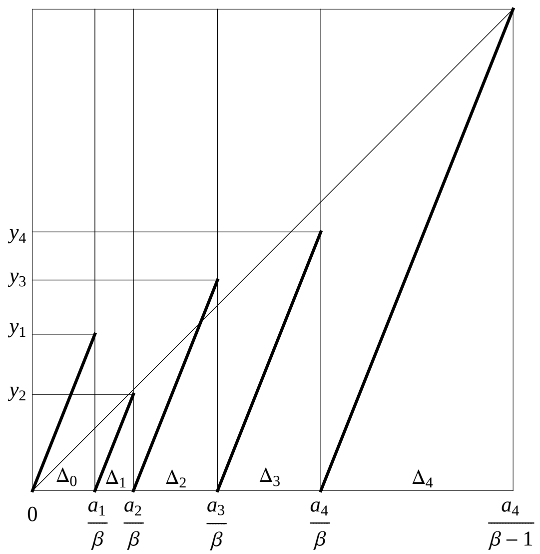

FIGURE 1. The greedy β-transformation with $\beta = 2.5$ and $A = \{0, 1.35, 1.75, 3.3, 6\}$.

Let λ denote the Lebesgue measure. We begin by defining different notions of invariance under the transformation T and by stating the results found by Lasota and Yorke in [13] and by Li and Yorke in [14]. Let μ be a Borel measure on $\big[0, \frac{a_m}{\beta-1}\big]$. A μ-integrable function f is called *an invariant function under* T if for all measurable sets E, $\int_E f\, d\mu = \int_{T^{-1}E} f\, d\mu$. We call a Lebesgue measurable set E *forward invariant under* T if $TE = E$ modulo sets of λ-measure zero. It was shown in [13] that there exists an invariant measure, absolutely continuous with respect to the Lebesgue measure, for transformations τ that are piecewise continuous with a finite set of points of discontinuity and that have a derivative bigger than 1 for points outside this finite set. In [14], Li and Yorke studied these invariant measures in more detail. Their results translate in the following way to our particular greedy transformation T. For the transformation T, there exist sets $B_1, \ldots, B_n$ and functions $f_1, \ldots, f_n$, where $n \leq m$, such that all the following hold.

(c1) For each $i \in \{1, \ldots n\}$, B_i is a finite union of closed subintervals of $\left[0, \frac{a_m}{\beta-1}\right]$. Each B_i contains at least one of the elements of J in its interior. Moreover, each B_i is forward invariant under T.

(c2) $B_i \cap B_j$ contains at most a finite number of points, when $i \neq j$, $i, j \in \{1, \ldots, n\}$.

(c3) For almost all $x \in \left[0, \frac{a_m}{\beta-1}\right] \setminus J$, there is an $i \in \{1, \ldots, n\}$ such that the closure of the forward orbit of x under T equals the set B_i, *i.e.*,

$$\Lambda(x) := \bigcap_{N=1}^{\infty} \overline{\{T^n(x)\}_{n=N}^{\infty}} = B_i.$$

(c4) For each $i \in \{1, \ldots n\}$, B_i is the support of the function f_i, *i.e.*, $f_i > 0$ λ a.e. on B_i and $f_i = 0$ on B_i^c. Moreover, $\int_{B_i} f_i \, d\lambda = 1$.

(c5) For each $i \in \{1, \ldots n\}$, f_i is invariant under T and if g is invariant under T and satisfies (c4) for some i, then $g = f_i$ λ almost everywhere.

(c6) Each function f that is invariant under T can be written as $f = \sum_{i=1}^{n} b_i f_i$ with a suitably chosen set of constants $\{b_i\}_{i=1}^{n}$.

(c7) If f is an invariant function and E is a measurable set, such that TE is measurable and $TE \subseteq E$ λ a.e., then $f \cdot 1_E$ is an invariant function, where 1_E denotes the indicator function of the set E.

Remark 2.1. — The last result was proven in [14] for sets E such that $TE = E$ for λ a.e. $x \in E$, however the proof of Li and Yorke still holds under the weaker assumption that $TE \subseteq E$ λ a.e.

We will first make some observations about the sets B_i.

Lemma 2.2. — *Let $I \subseteq \left[0, \frac{a_m}{\beta-1}\right]$ be a closed interval.*

(i) *If I is forward invariant under T and contains at least one element of J in its interior, then $0 \in I$.*

(ii) *If I does not contain an element of J in its interior, then I is not forward invariant under T.*

Proof. — The first part of the lemma follows immediately from the fact that for each $i \in \{1, \ldots, m\}$, $T\left(\frac{a_i}{\beta}\right) = 0$.

For the second part it is enough to notice that if I does not contain an element of J in its interior, then $\lambda(TI) = \beta\lambda(I)$. □

Remark 2.3. — As an immediate consequence of this lemma, we have that there cannot exist two or more sets B_i satisfying (c1) and (c2). To see this, suppose that the sets B_i and B_j both satisfy (c1). Then they are both forward invariant under T, so by the previous lemma there should exist numbers $0 < x_i, x_j \leq \frac{a_m}{\beta-1}$ such that $[0, x_i] \subseteq B_i$ and $[0, x_j] \subseteq B_j$, but this contradicts (c2). So by the previously stated results from [**14**], there exists a number $0 < x \leq \frac{a_m}{\beta-1}$ and a finite number of closed intervals $I_1, \ldots, I_k \subseteq \left[0, \frac{a_m}{\beta-1}\right]$ with $T(I_j)$ containing a closed interval of positive Lebesgue measure, such that the set

$$B := [0, x] \cup \bigcup_{i=1}^{k} I_i \tag{4}$$

satisfies (c1) to (c7) for an invariant probability density function f. This implies that there exists a unique invariant measure for T that is equivalent to the Lebesgue measure on B. Notice that without loss of generality, we can assume that B is the finite union of disjoint closed intervals. The fact that B is forward invariant implies that $T[0, x] \subseteq [0, x]$.

The next lemma states that if B contains a closed interval whose image under T is contained in itself, then B is exactly this interval.

Lemma 2.4. — *Let f be the density function of an invariant absolutely continuous probability measure as in Remark 2.3 and let B be its support. Suppose that $[\alpha_1, \alpha_2] \subseteq B$ is a closed interval. If $T[\alpha_1, \alpha_2] \subseteq [\alpha_1, \alpha_2]$ λ a.e., then $[\alpha_1, \alpha_2] = B$. Consequently, $[\alpha_1, \alpha_2]$ is a forward invariant set.*

Proof. — Consider the function $g = f \cdot 1_{[\alpha_1, \alpha_2]}$. Since f is an invariant function and $[\alpha_1, \alpha_2]$ satisfies $T[\alpha_1, \alpha_2] \subseteq [\alpha_1, \alpha_2]$ λ a.e., by (c7) we know that also the function g is invariant, with its support contained in the support of f. By (c6) there exists a constant c, such that $g = c \cdot f$. Now define the function

$$h = \frac{g}{\int g \, d\lambda}.$$

Then h is an invariant probability density function and $h = c' \cdot f$, with $c' = c / \int g \, d\lambda$. This means that $h = f$ λ a.e., so that $1_{[\alpha_1, \alpha_2]}(x) = 1$ for λ almost all $x \in B$. Since B is a finite union of closed intervals, it follows that $B = [\alpha_1, \alpha_2]$. By (c1), $[\alpha_1, \alpha_2]$ is forward invariant. □

Remark 2.5. — By the same reasoning as in the proof of the previous lemma, it can be shown that T is ergodic with respect to the invariant measure. To see this, let μ be the measure given by $\mu(E) = \int_E f \, d\lambda$ for each measurable set E and suppose that A is a measurable set such that $T^{-1}A = A$ λ a.e. and $\mu(A) > 0$. Then $TA \subseteq A$ λ

a.e., so by (c7) the function $g = f \cdot 1_A$ is invariant. Following the idea of the proof of Lemma 2.4 gives that $1_A = 1$ λ a.e., so $\mu(A) = 1$.

By Remark 2.3, there exists an element $x \in \left[0, \frac{a_m}{\beta-1}\right]$ such that the support B of f contains the interval $[0,x]$ with $T[0,x] \subseteq [0,x]$. By Lemma 2.4, we see that $B = [0,x] = T[0,x]$ λ a.e. The next two lemmas specify the value of x. First we define the following value. Let $y_{i_0} = \max\left\{y_i : \frac{a_i}{\beta} \leq x\right\}$ and if there are two such values, then let y_{i_0} be the one with the smallest index.

Lemma 2.6. — *Let $B = [0,x]$ be the support of the probability density function f as described above. Then $B = [0, y_{i_0}]$.*

Proof. — Since $T[0,x] = [0,x]$ λ a.e., we have that $y_i \leq x$ for any i such that $\dfrac{a_i}{\beta} \leq x$. Hence $y_{i_0} \leq x$. Also $Tx \leq x$.

Suppose $x \in \Delta_k$ for some $k \in \{0, \ldots, m\}$. Then by the definition of y_{i_0},

$$T\left[0, \frac{a_k}{\beta}\right) \subseteq [0, y_{i_0}] \quad \lambda \text{ a.e.} \tag{5}$$

If $y_{i_0} \in \left[0, \frac{a_k}{\beta}\right]$, then $T[0,y_{i_0}] \subseteq [0,y_{i_0}] \subseteq [0,x]$ λ a.e. and thus by Lemma 2.4, $[0, y_{i_0}] = [0,x]$ λ a.e. If on the other hand $y_{i_0} \in \left[\frac{a_k}{\beta}, x\right]$, then since $Tx \leq x$, we also have $Ty_{i_0} \leq y_{i_0}$ and this means that

$$T\left[\frac{a_k}{\beta}, y_{i_0}\right] \subseteq [0, y_{i_0}] \quad \lambda \text{ a.e.}$$

Combining this with equation (5) gives that $T[0,y_{i_0}] \subseteq [0,y_{i_0}] \subseteq [0,x]$ λ a.e., so again by Lemma 2.4 we have that $[0,y_{i_0}] = [0,x]$. □

From the previous lemma we know that x is one of the values y_i, $i \in \{1, \ldots, m\}$. The next lemma states explicitly which of these values it is.

Lemma 2.7. — *Let y_{i_0} be defined as above. Then*

$$i_0 = \min\{i : T[0,y_i] \subseteq [0,y_i] \quad \lambda \text{ a.e.}\}. \tag{6}$$

Proof. — Since $[0,x] = [0,y_{i_0}]$ is the support of the invariant probability density function f, we must have by Lemma 2.4 that $T[0,y_i] \not\subseteq [0,y_i]$ λ a.e. for any $y_i < y_{i_0}$. In particular, by the definition of y_{i_0} we have that if $i < i_0$, then

$$\frac{a_i}{\beta} < \frac{a_{i_0}}{\beta} \leq y_{i_0}.$$

This implies that $y_i < y_{i_0}$ and thus that $T[0,y_i] \not\subseteq [0,y_i]$ λ a.e. Hence $i_0 = \min\{i : T[0,y_i] \subseteq [0,y_i]$ λ a.e.$\}$. □

In the previous lemmas and remarks we have established the existence of a unique absolutely continuous invariant measure for the greedy transformation with arbitrary digits, that is ergodic, and we have given its support. These results are summarized in the following theorem.

Theorem 2.8. — *Let $\beta > 1$ and let $A = \{0, a_1, \ldots, a_m\}$ be an allowable digit set for β. If $T : \left[0, \frac{a_m}{\beta-1}\right] \to \left[0, \frac{a_m}{\beta-1}\right]$ is the greedy β-transformation with digits in A, then there exists a unique absolutely continuous invariant measure, that is ergodic. Furthermore, the support of the probability density function f is the interval $[0, y_{i_0}]$, where $i_0 = \min\{i : T[0, y_i] \subseteq [0, y_i]\ \lambda\ a.e.\}$.*

As a corollary we get Theorem 1.1, stated in the introduction.

Now consider the lazy transformation with arbitrary digits for $\beta 1$ and allowable digit set $\tilde{A} = \{\tilde{a}_0, \ldots, \tilde{a}_m\}$, where $\tilde{a}_0 = 0$. Indicate the points of discontinuity of L in the following way. For $i \in \{0, \ldots, m-1\}$, let $\tilde{\ell}_i = \frac{\tilde{a}_m}{\beta-1} - \frac{\tilde{a}_m - \tilde{a}_i}{\beta}$. In the same way as was done for the greedy transformation, we can make a partition $\{\tilde{\Delta}_i\}_{i=0}^m$, using these points of discontinuity. Let $\tilde{\Delta}_0 = [0, \tilde{\ell}_0]$ and $\tilde{\Delta}_m = \left(\tilde{\ell}_{m-1}, \frac{\tilde{a}_m}{\beta-1}\right]$ and for all $i \in \{1, \ldots, m-1\}$, define

$$\tilde{\Delta}_i = (\tilde{\ell}_{i-1}, \tilde{\ell}_i].$$

For each $i \in \{1, \ldots, m\}$, let $\tilde{y}_i$ denote the value of L when taking the limit from the right to the point $\tilde{\ell}_{i-1}$, *i.e.*, $\tilde{y}_i = \frac{\tilde{a}_m}{\beta-1} - (\tilde{a}_i - \tilde{a}_{i-1})$. (See Figure 2)

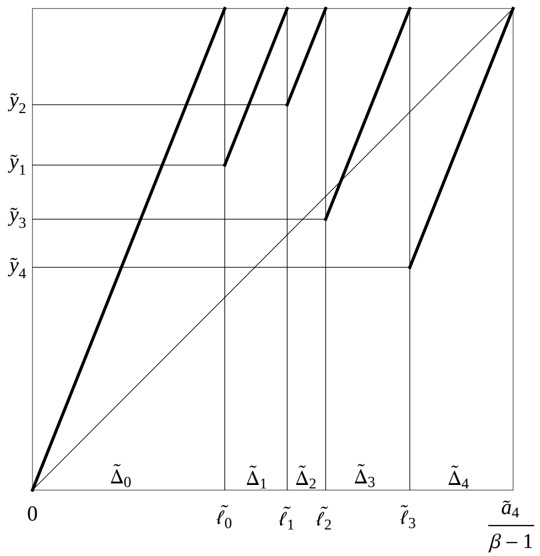

FIGURE 2. The lazy β-transformation with $\beta = 2.5$ and $A = \{0, 1.35, 1.75, 3.3, 6\}$.

The following corollary follows directly from Theorem 2.8.

Corollary 2.9. — *Let L be the lazy transformation for $\beta 1$ and allowable digit set $\tilde{A} = \{\tilde{a}_0, \dots, \tilde{a}_m\}$, for which $\tilde{a}_0 = 0$. Let T be the greedy transformation for the same $\beta > 1$ and with allowable digit set $A = \{a_0, \dots, a_m\}$, such that $a_i = \tilde{a}_m - \tilde{a}_{m-i}$. Then there exists a unique absolutely continuous invariant measure ν for L, that is ergodic. Let i_0 be defined for the greedy transformation as in equation (6). Then the support of the measure ν is given by the interval $\left[\tilde{y}_{m-i_0+1}, \frac{\tilde{a}_m}{\beta-1}\right]$.*

Proof. — By Theorem 2.8 we have that the interval $[0, y_{i_0}]$ is the support of the density function of the invariant measure for the greedy transformation. Let μ be the absolutely continuous invariant measure for T. Since T and L are isomorphic with isomorphism ϕ, given by $\phi(x) = \frac{a_m}{\beta-1} - x$, L also has a unique absolutely continuous invariant measure, given by $\nu = \mu \circ \phi^{-1}$. This is an ergodic measure and its support is given by

$$\phi([0, y_{i_0}]) = \left[\tilde{y}_{m-i_0+1}, \frac{\tilde{a}_m}{\beta - 1}\right]. \quad \square$$

3. Examples of explicitly calculable invariant measures

In the previous section it is shown that by the results of Li and Yorke in [14], on $[0, y_{i_0}]$ T has a unique invariant measure that is equivalent to the normalized Lebesgue measure. The same result holds for the lazy transformation. In general we can not give an explicit expression of this invariant measure, but in the next section we discuss two cases of which we do know what the invariant measure is.

3.1. Imposing a condition on the number of digits. — The first example is a particular case of the transformations studied by Wilkinson in [22]. In this paper, Wilkinson derives a formula for the density of an absolutely continuous invariant measure for certain piecewise linear transformations. Before we go into more detail, let us give some definitions. We consider our greedy transformation T with $\beta > 1$ and $A = \{0, a_1, \dots, a_m\}$ an allowable digit set for β, but with the extra restriction that $m < \beta \leq m + 1$. In [4] it was shown that condition (2) implies that $\lceil \beta \rceil \leq m + 1$, so this example is the case in which the smallest number of digits is considered.

By the previous section, we have that the support of the absolutely continuous invariant measure is given by $[0, y_{i_0}]$. Suppose that N is the largest index such that $\frac{a_N}{\beta} < y_{i_0}$. Then the points $\frac{a_i}{\beta}$, $i = 1, \dots, N$, give an interval partition of the interval $[0, y_{i_0}]$ as before. Let $\Delta = \{\Delta_0, \dots, \Delta_N\}$ be the partition of $[0, y_{i_0}]$, such that

$$\Delta_0 = \left[0, \frac{a_1}{\beta}\right), \quad \Delta_N = \left[\frac{a_N}{\beta}, y_{i_0}\right]$$

and for $i = 1, \ldots, N-1$,

$$\Delta_i = \left[\frac{a_i}{\beta}, \frac{a_{i+1}}{\beta}\right).$$

An element $\Delta_i \in \Delta$ is called a *full interval of rank 1* if $\lambda(T\Delta_i) = 1$, where λ is the normalized Lebesgue measure on the interval $[0, y_{i_0}]$. Otherwise we call it *non-full*. Using Δ and T, we can make the sequence of partitions $\{\Delta^{(n)}\}$ in the usual way. For $n \geq 1$, $\Delta^{(n)} = \bigvee_{k=0}^{n-1} T^{-k}\Delta$. The elements of $\Delta^{(n)}$ are intervals and are called the *fundamental intervals of rank n*. An element $E^{(n)} \in \Delta^{(n)}$ is called *full of rank n* if $\lambda(T^n E^{(n)}) = 1$ and *non-full* otherwise. Now let $I(E^{(n)})$ be the number of non-full fundamentals interval of rank $n+1$, that are contained in $E^{(n)}$ and let

$$I_n = \sup_{E^{(n)} \in \Delta^{(n)}} I(E^{(n)}).$$

So for each fundamental interval of rank n, we take the number of non-full fundamental intervals of rank $n+1$ and I_n indicates the supremum of these numbers over all the fundamental intervals of rank n. If we then take the supremum over all ranks, we get a number I, *i.e.*,

$$I = \sup_{n \geq 0} I_n,$$

where I_0 is the number of non-full intervals of rank 1. In [22] Wilkinson derived a formula for the absolutely continuous invariant measure under the condition that $\beta > I$. We will adapt his result to our case and generalize it to our setting. For each $K \geq 0$, let $\bar{I}_K = \sup_{n \geq K} I_n$ and let B_n denote the union of those fundamental intervals of rank n which are full, but which are not a subset of any full fundamental interval of lower rank. Notice that $I = \bar{I}_0$. We have the following lemma, which is a generalization of Corollary 4.5 in [22].

Lemma 3.1. — *Let $\bar{I}_K$ and B_n be as above and suppose that $\beta > \bar{I}_K$ for some $K \geq 0$. Then*

$$\sum_{n=1}^{\infty} \lambda(B_n) = 1.$$

Proof. — Consider the support $[0, y_{i_0}]$ and fill it as far as possible with full fundamental intervals of rank 1. Since every non-full fundamental interval of rank 1 has Lebesgue measure smaller than $\frac{1}{\beta}$, the remaining part has Lebesgue measure smaller than $\frac{I_0}{\beta}$. Now fill this part as far as possible with full fundamental intervals of rank 2. The remaining part has Lebesgue measure smaller than $\frac{I_0 \cdot I_1}{\beta^2}$. If we continue in this manner, after $n+1$ steps the remaining part will have Lebesgue measure smaller than

$$I_0 \cdot I_1 \cdot \cdots \cdot I_n \cdot \frac{1}{\beta^n}.$$

And by hypothesis we have

$$\lim_{n\to\infty} I_0 \cdot I_1 \cdot \dots \cdot I_n \cdot \frac{1}{\beta^n} \leq I_0 \cdot I_1 \cdot \dots \cdot I_{K-1} \lim_{n\to\infty} \left(\frac{\bar{I}_K}{\beta}\right)^n = 0,$$

which completes the proof. □

The next theorem is an adaptation of the formula by Wilkinson and a generalization of Theorem 5.12 of [22]. It gives an explicit expression of the absolutely continuous invariant measure of the greedy transformation with arbitrary digits under the assumption that $\beta > \bar{I}_K$ for some $K \geq 0$. Before we state the theorem, we need the following notation. Let D_n be the collection of all non-full fundamental intervals of rank n, that are not subsets of any full fundamental interval of lower rank. Let $x \in [0, y_{i_0}]$. Define $\phi_0(x) = 1$ and for $n \geq 1$, let

$$\phi_n(x) = \sum_{E^{(n)} \in D_n} \frac{1}{\beta^n} 1_{T^n E^{(n)}}(x). \tag{7}$$

Theorem 3.2. — *If $\beta > \bar{I}_K$ for some $K \geq 0$, then the functions ϕ_n, $n \geq 0$ and ϕ, given by*

$$\begin{aligned} \phi : \quad [0, y_{i_0}] &\longrightarrow [0, y_{i_0}] \\ x &\longmapsto \sum_{n=0}^{\infty} \phi_n(x) \end{aligned}$$

are Lebesgue integrable and the function h given by

$$\begin{aligned} h : \quad [0, y_{i_0}] &\longrightarrow [0, y_{i_0}] \\ x &\longmapsto \frac{\phi(x)}{\int \phi(x)\, d\lambda(x)} \end{aligned}$$

is the density of the absolutely continuous invariant measure of T.

Proof. — The proof follows from Lemma 3.1 and a slight adaptation of the corresponding proof in [22]. □

Remark 3.3. — In the case $K = 0$ Theorem 3.2 reduces to the theorem proved by K. Wilkinson.

The next theorem states that in the case $m < \beta \leq m + 1$, we have that $\beta > I$, so we can immediately apply Theorem 3.2.

Theorem 3.4. — *Let $\beta > 1$ and $A = \{0, a_1, \dots, a_m\}$ be an allowable digit set, such that $m < \beta \leq m + 1$. Let T be the greedy transformation for this β and A. Then the unique absolutely continuous invariant density is given by Theorem 3.2.*

Proof. — It is enough to show that $\beta > I$. First notice that Δ_{i_0-1} is a full fundamental interval, so that we have $I_0 \leq N \leq m < \beta$. By the definition of y_{i_0} we have that $\frac{a_{i_0}}{\beta} < y_{i_0}$, which means that $\Delta_{i_0-1} \neq \Delta_N$. Since for each fundamental interval of rank n, $E^{(n)}$, we have that $T^n E^{(n)}$ is an interval of the form $[0, y] \subseteq [0, y_{i_0}]$, we know that $E^{(n)}$ can contain at most N non-full fundamental intervals of rank n. So $I_n \leq N$ for each $n \geq 1$, which means that $I < \beta$, as we wanted. □

We consider two examples. The first one satisfies the condition of Theorem 3.4. The second one does not satisfy the condition of this theorem, but in this case we can apply Theorem 3.2.

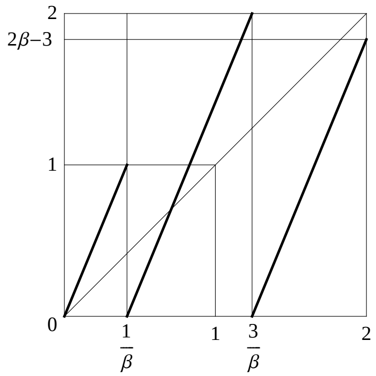

FIGURE 3. The greedy β-transformation with $\beta = 1+\sqrt{2}$ and $A = \{0, 1, 3\}$ on the interval $[0, 2]$.

Example 3.5. — First, let $\beta = 1 + \sqrt{2}$ be the positive solution of the equation $\beta^2 - 2\beta - 1 = 0$ and consider the allowable digit set $A = \{0, 1, 3\}$. The interval $[0, 2]$ is the support of the invariant measure, see Figure 3. The orbits of the points 1 and 2 are as follows.

$$
\begin{aligned}
&T1 = \beta - 1, && T^2 1 = T(\beta - 1) = \tfrac{1}{\beta}, && T^3 1 = T(\tfrac{1}{\beta}) = 0,\\
&T2 = 2\beta - 3, && T^2 2 = T(2\beta - 3) = \beta - 1, && T^3 2 = T(\beta - 1) = \tfrac{1}{\beta}.
\end{aligned}
$$

Notice that the condition of Theorem 3.4 is satisfied, so equation (7) gives for $x \in [0, 2]$,

$$
\phi(x) = 1 + \frac{1}{\beta} 1_{[0,1)}(x) + \frac{1}{\beta} 1_{[0,2\beta-3)}(x) + \sum_{k=0}^{\infty} \frac{c_{k+2}}{\beta^{k+2}} 1_{[0,1)}(x)
$$
$$
+ \sum_{k=0}^{\infty} \frac{c_{k+1}}{\beta^{k+2}} 1_{[0,\beta-1)}(x) + \sum_{k=0}^{\infty} \frac{c_k}{\beta^{k+2}} 1_{[0,\frac{1}{\beta})}(x),
$$

where c_k is the k-th term of the tribonacci sequence, *i.e.*, $c_k = c_{k-1} + c_{k-2} + c_{k-3}$, starting with $c_0 = 0$, $c_1 = c_2 = 2$. From the recurrence relation one can easily derive the following identity,

$$\sum_{k=0}^{\infty} c_k x^k = \frac{2x}{1-x-x^2-x^3}.$$

Using this formula, we get that

$$\sum_{k=0}^{\infty} \frac{c_{k+2}}{\beta^{k+2}} = \frac{\frac{2}{\beta}}{1-\frac{1}{\beta}-\frac{1}{\beta^2}-\frac{1}{\beta^3}} - \frac{2}{\beta} = 2 - \frac{1}{\beta},$$

$$\sum_{k=0}^{\infty} \frac{c_{k+1}}{\beta^{k+2}} = \frac{1}{\beta}\left[\frac{\frac{2}{\beta}}{1-\frac{1}{\beta}-\frac{1}{\beta^2}-\frac{1}{\beta^3}}\right] = 1,$$

$$\sum_{k=0}^{\infty} \frac{c_{k}}{\beta^{k+2}} = \frac{1}{\beta^2}\left[\frac{\frac{2}{\beta}}{1-\frac{1}{\beta}-\frac{1}{\beta^2}-\frac{1}{\beta^3}}\right] = \frac{1}{\beta}.$$

So, now

$$\phi(x) = 1 + \frac{1}{\beta}\cdot 1_{[0,2\beta-3)}(x) + 2\cdot 1_{[0,1)}(x) + 1_{[0,\beta-1)}(x) + \frac{1}{\beta}\cdot 1_{[0,\frac{1}{\beta})}(x)$$

and rewriting this, leads to

$$\begin{aligned}\phi(x) = 2\beta\cdot 1_{[0,\frac{1}{\beta})}(x) + (\beta+2)\cdot 1_{[\frac{1}{\beta},1)}(x) + \beta\cdot 1_{[1,\beta-1)}(x) \\ + (\beta-1)\cdot 1_{[\beta-1,2\beta-3)}(x) + 1_{[2\beta-3,2]}(x).\end{aligned}$$

Since

$$\int_{[0,2]} \phi(x)\, d\lambda(x) = \frac{8\beta-4}{\beta},$$

it is easily seen that the measure μ given by

$$\mu(E) = \int_E \frac{\beta}{8\beta-4}\phi(x)\, d\lambda(x),$$

for every measurable set E, is the unique invariant measure that is absolutely continuous with respect to Lebesgue measure.

Example 3.6. — For our second example we let $\beta = \frac{1+\sqrt{5}}{2}$ be the golden mean, *i.e.*, the positive solution of the equation $\beta^2-\beta-1=0$, and we consider the allowable digit set $A=\{0,2\beta,5\}$. Notice that with this combination of β and A the condition of Theorem 3.4 is not satisfied. The support of the invariant measure is given by the interval $[0,2\beta]$, see Figure 4. We now look at the orbits of the points 2β and $5-2\beta$.

$T(2\beta) = 2\beta-3,\quad T^2(2\beta) = 2-\beta,\quad T^3(2\beta) = \beta-1,\quad T^4(2\beta) = 1,$
$T^5(2\beta) = \beta+1,\quad T^6(2\beta) = 1,$
$T(5-2\beta) = 3\beta-2,\quad T^2(5-2\beta) = 3-\beta,\quad T^3(5-2\beta) = 2\beta-1,$
$T^4(5-2\beta) = 2-\beta.$

And we see that after the first iteration both orbits never hit Δ_2 again. This means that the number of non-full fundamental intervals of rank $n \geq 2$ that are not a subset

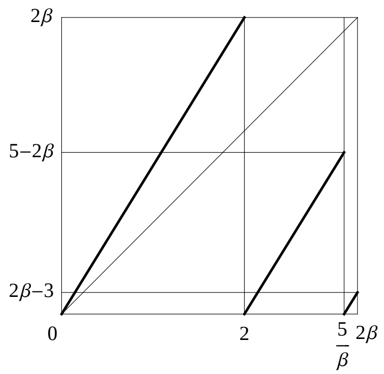

FIGURE 4. The greedy β-transformation with $\beta = \frac{1+\sqrt{5}}{2}$ and $A = \{0, 2\beta, 5\}$ on the interval $[0, 2\beta]$.

of any full interval of lower rank is at most 1, hence $\bar{I}_2 = 1 < \beta$. By Theorem 3.2 we can apply formula (7) to get

$$\begin{aligned}\phi(x) &= 1 + \frac{1}{\beta}\cdot 1_{[0,5-2\beta)}(x) + \frac{1}{\beta}\cdot 1_{[0,2\beta-3)}(x) + \frac{1}{\beta^2}\cdot 1_{[0,3\beta-2)}(x) + \frac{2}{\beta^3}\cdot 1_{[0,2-\beta)}(x)\\ &\quad + \frac{1}{\beta^3}\cdot 1_{[0,3-\beta)}(x) + \frac{2}{\beta^4}\cdot 1_{[0,\beta-1)}(x) + \frac{1}{\beta^4}\cdot 1_{[0,2\beta-1)}(x)\\ &\quad + \left[\frac{1}{\beta^4} + \frac{2}{\beta^7}\sum_{k=0}^{\infty}\frac{1}{\beta^{3k}}\right] 1_{[0,1)}(x) + \left[\frac{1}{\beta^5} + \frac{2}{\beta^8}\sum_{k=0}^{\infty}\frac{1}{\beta^{3k}}\right] 1_{[0,\beta)}(x)\\ &\quad + \left[\frac{1}{\beta^6} + \frac{2}{\beta^9}\sum_{k=0}^{\infty}\frac{1}{\beta^{3k}}\right] 1_{[0,\beta+1)}(x).\end{aligned}$$

Rewriting this will get you

$$\begin{aligned}\phi(x) &= (2\beta+1)\cdot 1_{[0,2\beta-3)}(x) + (\beta+2)\cdot 1_{[2\beta-1,2-\beta)}(x) + (8-3\beta)\cdot 1_{[2-\beta,\beta-1)}(x)\\ &\quad + (3\beta-2)\cdot 1_{[\beta-1,1)}(x) + (\beta+1)\cdot 1_{[1,3-\beta)}(x) + (4-\beta)\cdot 1_{[3-\beta,\beta)}(x)\\ &\quad + (2\beta-1)\cdot 1_{[\beta,5-2\beta)}(x) + \beta\cdot 1_{[5-2\beta,2\beta-1)}(x) + (4\beta-5)\cdot 1_{[2\beta-1,\beta+1)}(x)\\ &\quad + (3-\beta)\cdot 1_{[\beta+1,3\beta-2)}(x) + 1_{[3\beta-2,2\beta]}(x).\end{aligned}$$

Furthermore, we have

$$\int_{[0,2\beta]} \phi(x)\, d\lambda(x) = \frac{49-23\beta}{\beta},$$

so the density h of the unique absolutely continuous invariant measure is given by

$$h(x) = \frac{\beta\phi(x)}{49-23\beta}.$$

Remark 3.7. — In [22] more is said about piecewise linear transformations with maximal slope β for which $I < \beta$. For example, Wilkinson proves that these transformations are exact and weak Bernoulli. We can remark that greedy β-transformations with arbitrary digit for which the number of digits $m+1$ satisfies $m < \beta \leq m+1$ have these same properties, *i.e.*, they are exact and weak Bernoulli.

3.2. Ultimately periodic endpoints. — The condition we impose on the system in this second example is that the endpoints of the transformation must have ultimately periodic orbits. What we mean by this is clarified in the following definition.
Let T be the greedy transformation for $\beta > 1$ and allowable digit set $A = \{0, a_1, \ldots, a_m\}$, restricted to $[0, y_{i_0}]$ as given in Theorem 2.8. Let $N \geq 1$ be the largest index such that $\frac{a_N}{\beta} < y_{i_0}$. We say that *the endpoints of* T *have ultimately periodic orbits* if for each $i \in \{1, \ldots, N\}$ there exist numbers $u(i)$, $p(i)$, such that $T^{u(i)+p(i)} y_i = T^{u(i)} y_i$. In this case we say that the points y_i have ultimately periodic orbits of period $p(i)$.

In [2] Byers and Boyarsky proved some nice results about the absolutely continuous invariant measure of a certain class of piecewise linear functions, namely the piecewise linear Markov maps, which they defined as follows. Let $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n = 1$ be a partition of the interval $[0,1]$, denoted by $\mathcal{P}$. A map $\tau : [0,1] \to [0,1]$ is called a *piecewise linear Markov map* if $\tau|_{(\alpha_i, \alpha_{i+1})}$ is a linear homeomorphism onto some interval $(\alpha_{j(i)}, \alpha_{k(i)})$ for all $i \in \{0, \ldots, n-1\}$. If τ is such a map, it induces an $n \times n$ 0-1 matrix $M = M_\tau$ in the following way. The entry m_{ij} equals 1 if $[\alpha_j, \alpha_{j+1}) \subseteq \tau[\alpha_i, \alpha_{i+1})$ and 0 otherwise. The fact that τ is a piecewise linear Markov map guarantees that the nonzero entries in each row are contiguous. In [2], Byers and Boyarsky proved the following results.
Let τ be a piecewise linear Markov map, that is expanding and of constant slope and suppose that the 0-1 matrix M it induces is irreducible. Then

(d1) There exists a unique invariant probability measure μ, that is equivalent to Lebesgue measure and that maximizes entropy.
(d2) The entropy of τ with respect to this measure μ equals $\log \beta$, where β is the spectral radius of M and is also equal to the slope of τ.

A combination of these results, with those found by Parry in [16], gives that the invariant measure μ of τ can be found in the following way. Let β be as given by (d2). Let $\mathbf{v} = (v_0, \ldots, v_n)$ be the right eigenvector of M, belonging to the eigenvalue β and such that $\sum_{i=0}^{n} v_i = 1$ and suppose that $\mathbf{u} = (u_0, \ldots, u_n)$ is the left eigenvector of M

belonging to eigenvalue β and such that $\sum_{i=0}^{n} u_i v_i = 1$. Then the function

$$\begin{aligned} \phi: \quad [0,1] &\longrightarrow [0,1] \\ x &\longmapsto \sum_{i=0}^{n} u_i \cdot 1_{[\alpha_i,\alpha_{i+1})}(x) \end{aligned} \tag{8}$$

is the density of the unique absolutely continuous invariant measure for τ that we are looking for.

Now, consider the greedy transformation with arbitrary digits T that has ultimately periodic endpoints. Using the orbits of these endpoints, we make a partition $\mathcal{P}$ of the interval $[0, y_{i_0}]$. Consider the set

$$I = \{T^k y_i : 1 \le i \le N, k \ge 0\} \cup \left\{ \frac{a_i}{\beta} : 1 \le i \le N\} \cup \{0\right\}.$$

Since all the orbits of the points y_i are periodic, this set only contains a finite number of elements, say $n+1$ elements, so we can put them in increasing order to get a sequence $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n = y_{i_0}$. This gives us the partition $\mathcal{P}$, *i.e.*, $\mathcal{P} = \{P_i\}_{i=0}^{n}$ with $P_i = [\alpha_i, \alpha_{i+1})$. The next lemma states that in this case T is a piecewise linear Markov map for this partition.

Lemma 3.8. — *Let T and $\mathcal{P}$ be as above. Then T is linear on each of the elements of $\mathcal{P}$ and for each $i \in \{0, \ldots, n\}$ we have $TP_i = [\alpha_{j(i)}, \alpha_{k(i)})$, for some $\alpha_{j(i)}, \alpha_{k(i)} \in I$.*

Proof. — Since all the points $\frac{a_i}{\beta}$ are in I, it is easy to see that T is linear on each element of $\mathcal{P}$. Now fix an element $P_i \in \mathcal{P}$. If $\alpha_i = \frac{a_j}{\beta}$ for some $j \in \{1, \ldots N\}$, then $TP_i = [0, T^k y_\ell)$ for some $\ell \in \{1, \ldots, N\}$ and $k \ge 0$. Suppose $\alpha_i = T^k y_\ell$. If $\alpha_{i+1} = \frac{a_j}{\beta}$, then $T\alpha_i < y_j$, so $TP_i = [T^{k+1} y_\ell, y_j)$. If, on the other hand $\alpha_{i+1} = T^m y_j$, then $T\alpha_i < T\alpha_{i+1}$ and $TP_i = [T^{k+1} y_\ell, T^{m+1} y_j)$. In all cases TP_i is of the desired form. □

The following lemma states that each element of $\mathcal{P}$ is eventually mapped in each other element of $\mathcal{P}$. This implies that the matrix that T induces is irreducible.

Lemma 3.9. — *For each $i, j \in \{0, \ldots, n\}$ there exists an $k \ge 0$, such that $P_j \subseteq T^k P_i$.*

Proof. — Let P_i and P_j be two elements of $\mathcal{P}$. Let μ be the measure given by Theorem 2.8 and let ϕ be its density. By (c4) we know that $\phi > 0$ almost everywhere on P_i and P_j, so P_i and P_j both have strictly positive measure. Moreover, μ is ergodic, so there exists a $k \ge 0$, such that

$$\mu(P_i \cap T^{-k} P_j) > 0. \tag{9}$$

By Lemma 3.8, we have that $T^k P_i = \bigcup_{\ell \in \alpha} P_\ell$ for some index set $\alpha \subseteq \{0, \ldots, n\}$. Then (9) gives that $\mu(P_j \cap \bigcup_{\ell \in \alpha} P_\ell) > 0$, so that $P_j \subseteq T^k P_i$. □

It is easy to see that the transformation T, restricted to $[0, y_{i_0}]$ is isomorphic to a greedy transformation with arbitrary digits $\bar{T}$, defined on the interval $[0,1]$. The isomorphism ψ is given by $\psi : [0,1] \to [0, y_{i_0}] : x \mapsto y_{i_0} x$ and we have $T \circ \psi = \psi \circ \bar{T}$. The transformation T induces an $n \times n$ 0-1 matrix M in the following way. The entry $m_{ij} = 1$ if $P_j \subseteq TP_i$ and $m_{ij} = 0$ otherwise. By Lemma 3.8 the ones in each row of M are consecutive and by Lemma 3.9, the matrix M is irreducible. Let $\mathbf{v} = (v_0, \ldots, v_n)$ be the right eigenvector of M, belonging to the eigenvalue β and such that $\sum_{i=0}^{n} v_i = 1$ and suppose that $\mathbf{u} = (u_0, \ldots, u_n)$ is the left eigenvector of M belonging to eigenvalue β and such that $\sum_{i=0}^{n} u_i v_i = 1$. The transformation $\bar{T}$ induces the same matrix and therefore, we have the following theorem.

Theorem 3.10. — *Let $\beta > 1$ and $A = \{0, a_1, \ldots, a_m\}$ be an allowable digit set. Consider T on $[0, y_{i_0}]$ and suppose that all the endpoints of T have periodic orbits. Let $\bar{T}$ be the greedy transformation with arbitrary digits, defined on $[0,1]$, which is isomorphic to T by the isomorphism ψ defined above. Then the unique absolutely continuous invariant measure μ_T is the unique measure that maximizes entropy. This entropy is given by $\log \beta$. The measure μ_T is defined for all measurable sets B by*

$$\mu_T(B) = \mu_{\bar{T}}(\psi^{-1}(B)),$$

where $\mu_{\bar{T}}$ is the absolutely continuous invariant measure for $\bar{T}$, of which the probability density function is given by equation (8).

We consider the same two examples as in the previous paragraph.

Example 3.11. — In the first example, $\beta = 1 + \sqrt{2}$ was the positive solution of the equation $\beta^2 - 2\beta - 1 = 0$, and the allowable digit set considered was $A = \{0, 1, 3\}$. The support of the absolutely continuous measure was the interval $[0, 2]$. We already saw that both endpoints have finite orbits and the partition $\mathcal{P} = \{P_i\}_{i=0}^{5}$, given by these orbits is as follows:

$$P_0 = \left[0, \frac{1}{\beta}\right), \qquad P_1 = \left[\frac{1}{\beta}, 1\right), \qquad P_2 = \left[1, \frac{3}{\beta}\right),$$
$$P_3 = \left[\frac{3}{\beta}, \beta - 1\right), \quad P_4 = [\beta - 1, 2\beta - 3), \quad P_5 = [2\beta - 3, 2].$$

This gives us the following 0-1 matrix:

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $\mathbf{v} = (v_0, \ldots, v_5)$ be the right eigenvector of M with eigenvalue β and such that $\sum_{i=0}^{5} v_i = 1$ and let $\mathbf{u} = (u_0, \ldots, u_5)$ be the left eigenvector of M for the eigenvalue β and such that $\sum_{i=0}^{5} u_i v_i = 1$. Then

$$\mathbf{v} = \frac{1}{2\beta^2}(\beta, \beta+1, \beta-1, 1, \beta, 1)$$

and

$$\mathbf{u} = \frac{\beta}{4\beta-2}(2\beta, \beta+2, \beta, \beta, \beta-1, 1).$$

This means that the invariant probability density for our transformation T is given by

$$\begin{aligned} h(x) = \frac{1}{2} \cdot \frac{\beta}{4\beta-2} & \left[2\beta \cdot 1_{[0,\frac{1}{\beta})}(x) + (\beta+2) \cdot 1_{[\frac{1}{\beta},1)}(x) + \beta \cdot 1_{[1,\beta-1)}(x) \right. \\ & \left. + (\beta-1) \cdot 1_{[\beta-1,2\beta-3)}(x) + 1_{[2\beta-3,2]}(x)\right], \end{aligned}$$

just as we obtained before.

Example 3.12. — In the second example, $\beta = \frac{1+\sqrt{5}}{2}$ was the golden mean and $A = \{0, 2\beta, 5\}$ was the allowable digit set. The support of the absolutely continuous measure was $[0, 2\beta]$. In this case, both the orbit of 2β and that of $5 - 2\beta$ were eventually periodic and the partition $\mathcal{P} = \{P_i\}_{i=0}^{12}$ and 0-1 matrix M they give are the following. For the partition we have

$$\begin{aligned} & P_0 = [0, 2\beta-3), && P_1 = [2\beta-3, 2-\beta), && P_2 = \left[2-\beta, \frac{1}{\beta}\right), \\ & P_3 = \left[\frac{1}{\beta}, 1\right), && P_4 = [1, 3-\beta), && P_5 = [3-\beta, \beta), \\ & P_6 = [\beta, 5-2\beta), && P_7 = [5-2\beta, 2), && P_8 = [2, 2\beta-1), \\ & P_9 = [2\beta-1, \beta+1), && P_{10} = [\beta+1, 3\beta-2), && P_{11} = \left[3\beta-2, \frac{5}{\beta}\right), \\ & P_{12} = \left[\frac{5}{\beta}, 2\beta\right]. \end{aligned}$$

And for the matrix,

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The right eigenvector, $\mathbf{v}$, with eigenvalue β and such that the sum of its elements equals 1 is

$$\mathbf{v} = \frac{1}{10\beta+6}(\beta, 1, \beta, \beta+1, \beta+1, \beta, 1, \beta, \beta, \beta+1, \beta, \beta, 1)$$

and the left eigenvector, $\mathbf{u}$, belonging to the eigenvalue β and such that the dot product with $\mathbf{v}$ is 1 is

$$\mathbf{u} = \frac{10\beta+6}{29\beta+3}(2\beta+1, 2+\beta, 8-3\beta, 3\beta-2, \beta+1, 4-\beta, 2\beta-1, \beta, \beta, 4\beta-5, 3-\beta, 1, 1).$$

The invariant probability density then is

$$\begin{aligned} h(x) = \frac{1}{2\beta} \cdot \frac{10\beta+6}{29\beta+3} & [(2\beta+1) \cdot 1_{[0,2\beta-3)}(x) + (\beta+2) \cdot 1_{[2\beta-1,2-\beta)}(x) \\ & + (8-3\beta) \cdot 1_{[2-\beta,\beta-1)}(x) + (3\beta-2) \cdot 1_{[\beta-1,1)}(x) + (\beta+1) \cdot 1_{[1,3-\beta)}(x) \\ & + (4-\beta) \cdot 1_{[3-\beta,\beta)}(x) + (2\beta-1) \cdot 1_{[\beta,5-2\beta)}(x) + \beta \cdot 1_{[5-2\beta,2\beta-1)}(x) \\ & + (4\beta-5) \cdot 1_{[2\beta-1,\beta+1)}(x) + (3-\beta) \cdot 1_{[\beta+1,3\beta-2)}(x) + 1_{[3\beta-2,2\beta]}(x)]. \end{aligned}$$

And again, this is equal to the result from the previous paragraph.

4. Conclusions

In the second section of this article we have established that the greedy β-transformation with arbitrary digits has a unique invariant measure that is absolutely continuous with respect to the Lebesgue measure. We saw that this measure is ergodic and gave the interval on which the density function is strictly positive.

In the last section we have studied two specific examples of the greedy transformation in which this absolutely continuous invariant measure can be explicitly calculated. The first example put a restraint on the number of digits that can be chosen. If this number $m+1$, satisfies $m < \beta \leq m+1$, then the density of the absolutely continuous invariant measure is given by Wilkinson's formula. We remarked that in this case the absolutely continuous, invariant measure is exact and weak Bernoulli. The second example we studied is the case in which the endpoints of the transformation have ultimately periodic orbits. In that case the absolutely continuous invariant measure is also the measure of maximal entropy (with entropy equal to $\log\beta$) and its density is given by Parry's formula.

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