

Lie algebras and algebraic varieties associated
with PDEs and Bäcklund transformations

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Analogy between geometry of manifolds and geometry of PDEs

Geometry of manifolds:

Manifold

de Rham cohomology
of a manifold

Coverings

Fundamental group
of a manifold

Geometry of PDEs:

Infinite prolongation
of a PDE in a jet space

Horizontal cohomology of a PDE
(conservation laws)

Differential coverings
(Bäcklund transformations)

Fundamental Lie algebra
of a PDE

Example: Miura transformation

$$\text{KdV} = \{u_t = u_{xxx} + 6uu_x\} \xleftarrow{u=v_x-v^2} \text{mKdV} = \{v_t = v_{xxx} - 6v^2v_x\}$$

This is a map from solutions $v(x, t)$ of mKdV to solutions $u(x, t)$ of KdV. The preimage of each solution $u(x, t)$ of KdV is a one-parameter family of solutions $v(x, t)$ of mKdV.

General definition of coverings in coordinates:

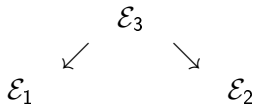
$$\mathcal{E}_1 = \left\{ F_\alpha(x_i, u^j, \frac{\partial u^p}{\partial x_s}, \dots) = 0 \right\} \longleftarrow \mathcal{E}_2 = \left\{ G_\beta(y_i, v^k, \frac{\partial v^l}{\partial y_s}, \dots) = 0 \right\}$$

$$u^j = \varphi(y_s, v^a, \frac{\partial v^b}{\partial y_s}, \dots), \quad x_i = \psi(y_s, v^a, \frac{\partial v^b}{\partial y_s}, \dots)$$

The preimage of each solution $u^j(x_i)$ of \mathcal{E}_1 is a family of \mathcal{E}_2 solutions $v^k(y_i)$ dependent on a finite number D of parameters.

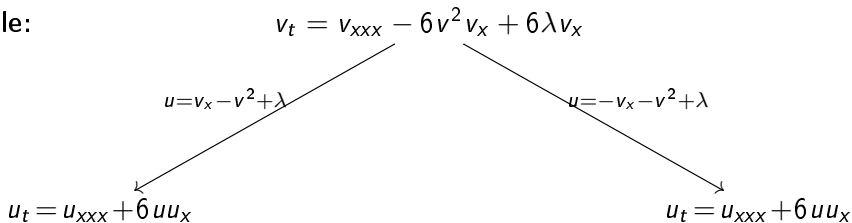
D is the *dimension of fibers* of the covering.

\mathcal{E}_1 and \mathcal{E}_2 are connected by a **Bäcklund transformation** if there is \mathcal{E}_3 and a pair of coverings



This allows to obtain solutions of \mathcal{E}_2 from solutions of \mathcal{E}_1 :
take a solution of \mathcal{E}_1 , find its preimage in \mathcal{E}_3 , and project it to \mathcal{E}_2 .

Example:



Trivial solution $u(x, t) = \text{const}$ \mapsto 1-soliton solution \mapsto 2-soliton solution $\mapsto \dots$

Example: the infinite prolongation of KdV.

Infinite jet space $J^\infty = (x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots)$.

Total derivative operators

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + u_{xxx} \frac{\partial}{\partial u_{xx}} + \dots$$

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + u_{xxt} \frac{\partial}{\partial u_{xx}} + \dots$$

are commuting vector fields on J^∞ .

Consider the submanifold $\mathcal{E} \subset J^\infty$ determined by KdV and all its differential consequences

$$u_t = u_{xxx} + 6uu_x, \quad u_{tt} = u_{xxx}t + 6u_t u_x + 6uu_{xt}, \quad u_{tx} = u_{xxxx} + 6u_x^2 + 6uu_{xx}, \dots$$

D_x, D_t are tangent to \mathcal{E} and span a 2-dimensional distribution on \mathcal{E} .

Solutions of KdV correspond to integral submanifolds of this distribution.

$$\sigma = i_1 \dots i_k, \quad u_\sigma^j = \frac{\partial^k u^j}{\partial x_{i_1} \dots \partial x_{i_k}}.$$

The **infinite jet space** $J^\infty = (x_i, u^j, u_\sigma^j, \dots)$.

Total derivative operators $D_{x_i} = \frac{\partial}{\partial x_i} + \sum_{\sigma^j} u_{\sigma^j}^j \frac{\partial}{\partial u_\sigma^j}$ are vector fields on J^∞ .

$$\text{PDE: } F_r(x_i, u^j, u_\sigma^j, \dots) = 0, \quad r = 1, \dots, s.$$

Infinite prolongation of the PDE: $\mathcal{E} = \{D_{x_{i_1}} \dots D_{x_{i_p}}(F_r) = 0\} \subset J^\infty$

Vector fields D_{x_i} are tangent to \mathcal{E} and span the **Cartan distribution** $\mathcal{C}(\mathcal{E})$ on \mathcal{E} .

Solutions of the PDE correspond to integral submanifolds of this distribution.

An **object** of the **category of PDEs** is a pair $(\mathcal{E}, \mathcal{C}(\mathcal{E}))$, where \mathcal{E} is a manifold and $\mathcal{C}(\mathcal{E})$ is a distribution on \mathcal{E} , such that $(\mathcal{E}, \mathcal{C}(\mathcal{E}))$ is locally isomorphic to the infinite prolongation of a PDE.

A **morphism** $\tau: (\mathcal{E}_2, \mathcal{C}(\mathcal{E}_2)) \rightarrow (\mathcal{E}_1, \mathcal{C}(\mathcal{E}_1))$ is a smooth map $\tau: \mathcal{E}_2 \rightarrow \mathcal{E}_1$

$$\forall a \in \mathcal{E}_2 \quad \tau_*: T_a \mathcal{E}_2 \rightarrow T_{\tau(a)} \mathcal{E}_1 \quad \tau_*(\mathcal{C}(\mathcal{E}_2)_a) \subset \mathcal{C}(\mathcal{E}_1)_{\tau(a)}$$

A morphism τ is a **differential covering** if $\tau: \mathcal{E}_2 \rightarrow \mathcal{E}_1$ is a bundle with finite-dimensional fibers and

$$\forall a \in \mathcal{E}_2 \quad \tau_*: \mathcal{C}(\mathcal{E}_2)_a \longrightarrow \mathcal{C}(\mathcal{E}_1)_{\tau(a)} \text{ is an isomorphism.}$$

If $\mathcal{C}(\mathcal{E})_a = T_a \mathcal{E}$ then differential coverings are topological coverings.

Topological coverings of a manifold M are determined by actions of the fundamental group $\pi_1(M, a)$ for $a \in M$.

We need an analog of $\pi_1(M, a)$ for differential coverings. This analog will be a Lie algebra, because differential coverings are studied locally.

For any analytic PDE \mathcal{E} , we naturally define a Lie algebra $\pi_1(\mathcal{E}, a)$ for every point $a \in \mathcal{E}$.

$\pi_1(\mathcal{E}, a)$ is called the **fundamental Lie algebra** of \mathcal{E} at $a \in \mathcal{E}$.

The correspondence $(\mathcal{E}, a) \mapsto \pi_1(\mathcal{E}, a)$ is a functor from the category of PDEs to the category of Lie algebras.

Coverings over \mathcal{E} with fibers W are determined by actions of $\pi_1(\mathcal{E}, a)$ on W (homomorphisms from $\pi_1(\mathcal{E}, a)$ to the Lie algebra of vector fields on W).

For any covering $\tau: \mathcal{E}' \rightarrow \mathcal{E}$, the algebra $\pi_1(\mathcal{E}, a)$ acts on the fiber $\tau^{-1}(a)$. Morphisms of coverings preserve the action of $\pi_1(\mathcal{E}, a)$.

If the PDE satisfies some non-degeneracy conditions, any action of $\pi_1(\mathcal{E}, a)$ on W gives a covering with fiber W on the level of formal power series. Usually these formal power series converge, so one gets locally an analytic covering.

There is an algorithm to compute the algebra $\pi_1(\mathcal{E}, a)$ in terms of generators and relations. (The number of generators and relations may be infinite.)

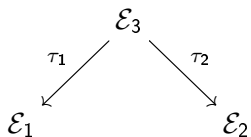
For a topological covering $\tau: M' \rightarrow M$,

$$a' \in M', \quad a = \tau(a') \in M, \quad \pi_1(M', a') \hookrightarrow \pi_1(M, a).$$

For a differential covering $\tau: \mathcal{E}' \rightarrow \mathcal{E}$, $a' \in \mathcal{E}'$, $a = \tau(a') \in \mathcal{E}$,

$\pi_1(\mathcal{E}', a')$ is isomorphic to a subalgebra of $\pi_1(\mathcal{E}, a)$ of finite codimension.

Let \mathcal{E}_1 and \mathcal{E}_2 be connected by a Bäcklund transformation



$$a_3 \in \mathcal{E}_3, \quad a_1 = \tau_1(a_3) \in \mathcal{E}_1, \quad a_2 = \tau_2(a_3) \in \mathcal{E}_2,$$

$$\pi_1(\mathcal{E}_3, a_3) \hookrightarrow \pi_1(\mathcal{E}_1, a_1), \quad \pi_1(\mathcal{E}_3, a_3) \hookrightarrow \pi_1(\mathcal{E}_2, a_2)$$

Therefore, $\pi_1(\mathcal{E}_1, a_1)$ and $\pi_1(\mathcal{E}_2, a_2)$ have a common subalgebra of finite codimension. This is a powerful necessary condition for existence of a Bäcklund transformation between \mathcal{E}_1 and \mathcal{E}_2 .

If \mathcal{E} is integrable by zero-curvature representations (like KdV, sine-Gordon, WDVV), then $\dim \pi_1(\mathcal{E}, a) = \infty$.

For a wide class of PDEs, $\pi_1(\mathcal{E}, a) \cong \pi_1(\mathcal{E}, b) \quad \forall a, b \in \mathcal{E}$.

In computations, $\pi_1(\mathcal{E}, a)$ is the inverse limit of a sequence of surjective homomorphisms of Lie algebras

$$\cdots \rightarrow F^{k+1}(\mathcal{E}, a) \rightarrow F^k(\mathcal{E}, a) \rightarrow \cdots \rightarrow F^1(\mathcal{E}, a) \rightarrow F^0(\mathcal{E}, a)$$

Actions of $F^k(\mathcal{E}, a)$ classify (with respect to gauge equivalence) coverings dependent on jets of order $k + p - 1$, where p is the order of the PDE \mathcal{E} .

In coordinate computations, an algebra similar to $F^0(\mathcal{E}, a)$ was introduced for some PDEs by H. Wahlquist and F. Estabrook. A. Vinogradov noticed (1986) that this Lie algebra plays a role similar to the fundamental group. But $F^0(\mathcal{E}, a)$ does not have any coordinate-independent meaning.

The explicit structure of $F^0(\mathcal{E}, a)$ was computed for many PDEs by H. van Eck, G. Roelofs, R. Martini.

Examples: for the KdV, NLS, Krichever-Novikov, Landau-Lifshitz equations,

$$F^k(\mathcal{E}, a) = \mathcal{L} \oplus N_k$$

\mathcal{L} is some infinite-dimensional Lie algebra of certain matrix-valued functions on an algebraic curve of genus 1 or 0,

N_k is finite-dimensional and nilpotent.

For the Krichever-Novikov equation, $F^0(\mathcal{E}, a) = 0$.

How to extract algebraic curves from $\pi_1(\mathcal{E}, a)$

Let $S(\mathcal{E}, a)$ be the Lie algebra obtained from $\pi_1(\mathcal{E}, a)$ by 'killing' all solvable ideals.

$$A(\mathcal{E}, a) = \{ f: S(\mathcal{E}, a) \rightarrow S(\mathcal{E}, a) \mid f([p_1, p_2]) = [f(p_1), p_2] \}$$

In the above examples, $A(\mathcal{E}, a)$ is isomorphic to the algebra of polynomial functions on an algebraic curve.

Rational curve (genus = 0) for KdV and nonlinear-Schrödinger.

Elliptic curve for Krichever-Novikov and Landau-Lifshitz.

(In the computation, we use some results of D. Demskoi, V. Sokolov.)

Let \mathcal{E}_1 and \mathcal{E}_2 be some PDEs from these examples, $a_1 \in \mathcal{E}_1$, $a_2 \in \mathcal{E}_2$. If the curves $A(\mathcal{E}_1, a_1)$ and $A(\mathcal{E}_2, a_2)$ are not birationally equivalent, then there is no Bäcklund transformation between \mathcal{E}_1 and \mathcal{E}_2 .

This solves a classical problem about the classification of some classes of PDEs with respect to Bäcklund transformations.

$A(\mathcal{E}, a)$ provides an invariant meaning for algebraic curves related to PDEs.

An m -component generalization of Landau-Lifshitz was introduced by I. Golubchik and V. Sokolov.

For this PDE, the Lie algebras $F^k(\mathcal{E}, a)$ have the following structure (S. Ig., J. van de Leur, G. Manno, V. Trushkov):

$F^0(\mathcal{E}, a)$ is isomorphic to the infinite-dimensional Lie algebra \mathbf{L} of certain matrix-valued functions on an algebraic curve of genus $1 + (m-3)2^{m-2}$.

For any $k \geq 1$, there is a surjective homomorphism $F^k(\mathcal{E}, a) \rightarrow \mathbf{L} \oplus \mathfrak{so}_{m-1}(\mathbb{C})$ with solvable kernel.

For the Darboux–Egoroff system (which is used in topological field theory and the theory of Frobenius manifolds), there is a surjective homomorphism from $\pi_1(\mathcal{E}, a)$ to the \mathbb{Z}_+ -graded part of a twisted affine Kac-Moody algebra.

The fundamental group $\pi_1(M, a)$ can be defined using only topological coverings of M (without using loops in M).

$g \in \pi_1(M, a)$ gives a transformation $g_\tau: \tau^{-1}(a) \rightarrow \tau^{-1}(a)$ for each $\tau: \tilde{M} \rightarrow M$

For any $M_1 \xrightarrow{\varphi} M_2$, one has $g_{\tau_2} \circ \varphi = \varphi \circ g_{\tau_1}$ (1)

$g \in \pi_1(M, a)$ is uniquely determined by the collection of transformations $\{g_\tau: \tau^{-1}(a) \rightarrow \tau^{-1}(a) \mid \tau \text{ is a covering}\}$.

One can define an element of $\pi_1(M, a)$ as a collection of such transformations satisfying (1).

To define $\pi_1(\mathcal{E}, a)$, replace M by \mathcal{E} , topological coverings by differential coverings, transformations on fibers by vector fields on fibers.

To obtain the correct definition of $\pi_1(\mathcal{E}, a)$, consider 'generalized coverings': **formal zero-curvature representations with coefficients in Lie algebras.**

Analytic functions are replaced by formal power series,

Lie algebras of vector fields are replaced by arbitrary Lie algebras.

Let L be a Lie algebra. Let ω be a formal L -valued differential 1-form on $\mathcal{C}(\mathcal{E})$.

ω is called a **zero-curvature representation (ZCR)** if it satisfies the

$$\bar{d}\text{-Maurer-Cartan equation } \bar{d}(\omega) + \frac{1}{2}[\omega, \omega] = 0.$$

$g \in \pi_1(\mathcal{E}, a)$ determines $g_\omega \in L$ for each L -valued ZCR ω .

$g \in \pi_1(\mathcal{E}, a)$ is determined uniquely by the collection of elements

$\{g_\omega \in L \mid \forall L \forall \omega\}$, which are in agreement with respect to morphisms of ZCRs.

We define an element of $\pi_1(\mathcal{E}, a)$ as such a collection.

To compute $\pi_1(\mathcal{E}, a)$ in terms of generators and relations, we find a 'normal' form for ZCRs with respect to the action of the group of gauge transformations. (This involves a step similar to finding a Gröbner basis.)

Then we take a 'general' ZCR $\tilde{\omega}$ in this normal form, the coefficients of $\tilde{\omega}$ are regarded as generators of a Lie algebra, and the equation $\bar{d}(\tilde{\omega}) + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}] = 0$ provides some Lie algebraic relations for these generators.