Lie algebras and algebraic varieties associated with PDEs and Bäcklund transformations

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Analogy between geometry of manifolds and geometry of PDEs

Geometry of manifolds:	**Geometry of PDEs:**
Manifold	Infinite prolongation of a PDE in a jet space
de Rham cohomology of a manifold	Horizontal cohomology of a PDE (conservation laws)
Coverings	Differential coverings (Bäcklund transformations)
Fundamental group of a manifold	Fundamental Lie algebra of a PDE

Differential coverings (A. Vinogradov, I. Krasilshchik)

Example: Miura transformation

$$\mathrm{KdV} = \left\{u_t = u_{xxx} + 6uu_x\right\} \quad \xleftarrow{u=v_x-v^2} \quad \mathrm{mKdV} = \left\{v_t = v_{xxx} - 6v^2 v_x\right\}$$

This is a map from solutions $v(x,t)$ of mKdV to solutions $u(x,t)$ of KdV. The preimage of each solution $u(x,t)$ of KdV is a one-parameter family of solutions $v(x,t)$ of mKdV.

General definition of coverings in coordinates:

$$\mathcal{E}_1 = \left\{F_\alpha\left(x_i, u^j, \frac{\partial u^p}{\partial x_s}, \ldots\right) = 0\right\} \quad \longleftarrow \quad \mathcal{E}_2 = \left\{G_\beta\left(y_i, v^k, \frac{\partial v^l}{\partial y_s}, \ldots\right) = 0\right\}$$

$$u^j = \varphi\left(y_s, v^a, \frac{\partial v^b}{\partial y_s}, \ldots\right), \qquad x_i = \psi\left(y_s, v^a, \frac{\partial v^b}{\partial y_s}, \ldots\right)$$

The preimage of each solution $u^j(x_i)$ of $\mathcal{E}_1$ is a family of $\mathcal{E}_2$ solutions $v^k(y_i)$ dependent on a finite number D of parameters.
D is the *dimension of fibers* of the covering.

$\mathcal{E}_1$ and $\mathcal{E}_2$ are connected by a **Bäcklund transformation** if there is $\mathcal{E}_3$ and a pair of coverings

$$\begin{array}{ccc} & \mathcal{E}_3 & \\ \swarrow & & \searrow \\ \mathcal{E}_1 & & \mathcal{E}_2 \end{array}$$

This allows to obtain solutions of $\mathcal{E}_2$ from solutions of $\mathcal{E}_1$:
take a solution of $\mathcal{E}_1$, find its preimage in $\mathcal{E}_3$, and project it to $\mathcal{E}_2$.

Example:

$$\begin{array}{ccc} & v_t = v_{xxx} - 6v^2 v_x + 6\lambda v_x & \\ {\scriptstyle u = v_x - v^2 + \lambda}\swarrow & & \searrow{\scriptstyle u = -v_x - v^2 + \lambda} \\ u_t = u_{xxx} + 6uu_x & & u_t = u_{xxx} + 6uu_x \end{array}$$

Trivial solution $u(x,t) = \mathrm{const}$ $\mapsto$ 1-soliton solution $\mapsto$ 2-soliton solution $\mapsto \dots$

Example: the infinite prolongation of KdV.

Infinite jet space $J^\infty = (x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots)$.

Total derivative operators

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + u_{xxx} \frac{\partial}{\partial u_{xx}} + \dots$$

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + u_{xxt} \frac{\partial}{\partial u_{xx}} + \dots$$

are commuting vector fields on J^∞.

Consider the submanifold $\mathcal{E} \subset J^\infty$ determined by KdV and all its differential consequences

$$u_t = u_{xxx} + 6uu_x, \quad u_{tt} = u_{xxxt} + 6u_t u_x + 6uu_{xt}, \quad u_{tx} = u_{xxxx} + 6u_x^2 + 6uu_{xx}, \dots$$

D_x, D_t are tangent to $\mathcal{E}$ and span a 2-dimensional distribution on $\mathcal{E}$.

Solutions of KdV correspond to integral submanifolds of this distribution.

$$\sigma = i_1 \dots i_k, \qquad u^j_\sigma = \frac{\partial^k u^j}{\partial x_{i_1} \dots \partial x_{i_k}}.$$

The **infinite jet space** $J^\infty = (x_i, u^j, u^j_\sigma, \dots)$.

Total derivative operators $D_{x_i} = \dfrac{\partial}{\partial x_i} + \sum_{\sigma,j} u^j_{\sigma i} \dfrac{\partial}{\partial u^j_\sigma}$ are vector fields on J^∞.

$$\textbf{PDE:} \quad F_r(x_i, u^j, u^j_\sigma, \dots) = 0, \quad r = 1, \dots, s.$$

$$\textbf{Infinite prolongation of the PDE:} \quad \mathcal{E} = \left\{ D_{x_{i_1}} \dots D_{x_{i_p}}(F_r) = 0 \right\} \subset J^\infty$$

Vector fields D_{x_i} are tangent to $\mathcal{E}$ and span the **Cartan distribution** $\mathcal{C}(\mathcal{E})$ on $\mathcal{E}$.

Solutions of the PDE correspond to integral submanifolds of this distribution.

An **object** of the **category of PDEs** is a pair $(\mathcal{E}, \mathcal{C}(\mathcal{E}))$, where $\mathcal{E}$ is a manifold and $\mathcal{C}(\mathcal{E})$ is a distribution on $\mathcal{E}$, such that $(\mathcal{E}, \mathcal{C}(\mathcal{E}))$ is locally isomorphic to the infinite prolongation of a PDE.

A **morphism** $\tau\colon (\mathcal{E}_2, \mathcal{C}(\mathcal{E}_2)) \to (\mathcal{E}_1, \mathcal{C}(\mathcal{E}_1))$ is a smooth map $\tau\colon \mathcal{E}_2 \to \mathcal{E}_1$

$$\forall\, a \in \mathcal{E}_2 \qquad \tau_*\colon\, T_a\mathcal{E}_2 \to\, T_{\tau(a)}\mathcal{E}_1 \qquad \tau_*\big(\mathcal{C}(\mathcal{E}_2)_a\big) \subset \mathcal{C}(\mathcal{E}_1)_{\tau(a)}$$

A morphism τ is a **differential covering** if $\tau\colon \mathcal{E}_2 \to \mathcal{E}_1$ is a bundle with finite-dimensional fibers and

$$\forall\, a \in \mathcal{E}_2 \qquad \tau_*\colon \mathcal{C}(\mathcal{E}_2)_a \longrightarrow \mathcal{C}(\mathcal{E}_1)_{\tau(a)} \ \text{ is an isomorphism.}$$

If $\mathcal{C}(\mathcal{E})_a = T_a\mathcal{E}$ then differential coverings are topological coverings.

Topological coverings of a manifold M are determined by actions of the fundamental group $\pi_1(M, a)$ for $a \in M$.

We need an analog of $\pi_1(M, a)$ for differential coverings. This analog will be a Lie algebra, because differential coverings are studied locally.

For any analytic PDE $\mathcal{E}$, we naturally define a Lie algebra $\pi_1(\mathcal{E}, a)$ for every point $a \in \mathcal{E}$.
$\pi_1(\mathcal{E}, a)$ is called the **fundamental Lie algebra** of $\mathcal{E}$ at $a \in \mathcal{E}$.

The correspondence $(\mathcal{E}, a) \mapsto \pi_1(\mathcal{E}, a)$ is a functor from the category of PDEs to the category of Lie algebras.

Coverings over $\mathcal{E}$ with fibers W are determined by actions of $\pi_1(\mathcal{E}, a)$ on W (homomorphisms from $\pi_1(\mathcal{E}, a)$ to the Lie algebra of vector fields on W).

For any covering $\tau\colon \mathcal{E}' \to \mathcal{E}$, the algebra $\pi_1(\mathcal{E}, a)$ acts on the fiber $\tau^{-1}(a)$.
Morphisms of coverings preserve the action of $\pi_1(\mathcal{E}, a)$.

If the PDE satisfies some non-degeneracy conditions, any action of $\pi_1(\mathcal{E}, a)$ on W gives a covering with fiber W on the level of formal power series. Usually these formal power series converge, so one gets locally an analytic covering.

There is an algorithm to compute the algebra $\pi_1(\mathcal{E}, a)$ in terms of generators and relations. (The number of generators and relations may be infinite.)

For a topological covering $\tau\colon M' \to M$,
$a' \in M', \qquad a = \tau(a') \in M, \qquad \pi_1(M', a') \hookrightarrow \pi_1(M, a).$

For a differential covering $\tau\colon \mathcal{E}' \to \mathcal{E}, \qquad a' \in \mathcal{E}', \qquad a = \tau(a') \in \mathcal{E}$,
$\pi_1(\mathcal{E}', a')$ is isomorphic to a subalgebra of $\pi_1(\mathcal{E}, a)$ of finite codimension.

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be connected by a Bäcklund transformation

$$\mathcal{E}_1 \xleftarrow{\;\tau_1\;} \mathcal{E}_3 \xrightarrow{\;\tau_2\;} \mathcal{E}_2$$

$$a_3 \in \mathcal{E}_3, \qquad a_1 = \tau_1(a_3) \in \mathcal{E}_1, \qquad a_2 = \tau_2(a_3) \in \mathcal{E}_2,$$

$$\pi_1(\mathcal{E}_3, a_3) \hookrightarrow \pi_1(\mathcal{E}_1, a_1), \qquad \pi_1(\mathcal{E}_3, a_3) \hookrightarrow \pi_1(\mathcal{E}_2, a_2)$$

Therefore, $\pi_1(\mathcal{E}_1, a_1)$ and $\pi_1(\mathcal{E}_2, a_2)$ have a common subalgebra of finite codimension. This is a powerful necessary condition for existence of a Bäcklund transformation between $\mathcal{E}_1$ and $\mathcal{E}_2$.

If $\mathcal{E}$ is integrable by zero-curvature representations (like KdV, sine-Gordon, WDVV), then $\dim \pi_1(\mathcal{E}, a) = \infty$.

For a wide class of PDEs, $\pi_1(\mathcal{E}, a) \cong \pi_1(\mathcal{E}, b) \quad \forall\, a,\ b \in \mathcal{E}$.

In computations, $\pi_1(\mathcal{E}, a)$ is the inverse limit of a sequence of surjective homomorphisms of Lie algebras

$$\cdots \to F^{k+1}(\mathcal{E}, a) \to F^k(\mathcal{E}, a) \to \cdots \to F^1(\mathcal{E}, a) \to F^0(\mathcal{E}, a)$$

Actions of $F^k(\mathcal{E}, a)$ classify (with respect to gauge equivalence) coverings dependent on jets of order $k + p - 1$, where p is the order of the PDE $\mathcal{E}$.

In coordinate computations, an algebra similar to $F^0(\mathcal{E}, a)$ was introduced for some PDEs by H. Wahlquist and F. Estabrook. A. Vinogradov noticed (1986) that this Lie algebra plays a role similar to the fundamental group.
But $F^0(\mathcal{E}, a)$ does not have any coordinate-independent meaning.

The explicit structure of $F^0(\mathcal{E}, a)$ was computed for many PDEs by H. van Eck, G. Roelofs, R. Martini.

Examples: for the KdV, NLS, Krichever-Novikov, Landau-Lifshitz equations,

$$F^k(\mathcal{E}, a) = \mathcal{L} \oplus N_k$$

$\mathcal{L}$ is some infinite-dimensional Lie algebra of certain matrix-valued functions on an algebraic curve of genus 1 or 0,
N_k is finite-dimensional and nilpotent.

For the Krichever-Novikov equation, $F^0(\mathcal{E}, a) = 0$.

How to extract algebraic curves from $\pi_1(\mathcal{E}, a)$

Let $S(\mathcal{E}, a)$ be the Lie algebra obtained from $\pi_1(\mathcal{E}, a)$ by 'killing' all solvable ideals.

$$A(\mathcal{E}, a) = \left\{ f\colon S(\mathcal{E}, a) \to S(\mathcal{E}, a) \;\middle|\; f([p_1, p_2]) = [f(p_1), p_2] \right\}$$

In the above examples, $A(\mathcal{E}, a)$ is isomorphic to the algebra of polynomial functions on an algebraic curve.

Rational curve (genus = 0) for KdV and nonlinear-Schrödinger.
Elliptic curve for Krichever-Novikov and Landau-Lifshitz.
(In the computation, we use some results of D. Demskoi, V. Sokolov.)

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be some PDEs from these examples, $\quad a_1 \in \mathcal{E}_1, \quad a_2 \in \mathcal{E}_2$. If the curves $A(\mathcal{E}_1, a_1)$ and $A(\mathcal{E}_2, a_2)$ are not birationally equivalent, then there is no Bäcklund transformation between $\mathcal{E}_1$ and $\mathcal{E}_2$.

This solves a classical problem about the classification of some classes of PDEs with respect to Bäcklund transformations.

$A(\mathcal{E}, a)$ provides an invariant meaning for algebraic curves related to PDEs.

An m-component generalization of Landau-Lifshitz was introduced by I. Golubchik and V. Sokolov.

For this PDE, the Lie algebras $F^k(\mathcal{E}, a)$ have the following structure (S. Ig., J. van de Leur, G. Manno, V. Trushkov):

$F^0(\mathcal{E}, a)$ is isomorphic to the infinite-dimensional Lie algebra $\mathbf{L}$ of certain matrix-valued functions on an algebraic curve of genus $1+(m-3)2^{m-2}$.
For any $k \geq 1$, there is a surjective homomorphism
$F^k(\mathcal{E}, a) \to \mathbf{L} \oplus \mathfrak{so}_{m-1}(\mathbb{C})$ with solvable kernel.

For the Darboux–Egoroff system (which is used in topological field theory and the theory of Frobenius manifolds),
there is a surjective homomorphism from $\pi_1(\mathcal{E}, a)$ to the $\mathbb{Z}_+$-graded part of a twisted affine Kac-Moody algebra.

The fundamental group $\pi_1(M, a)$ can be defined using only topological coverings of M (without using loops in M).

$g \in \pi_1(M, a)$ gives a transformation $g_\tau \colon \tau^{-1}(a) \to \tau^{-1}(a)$ for each $\tau \colon \tilde{M} \to M$

$$\text{For any} \quad \begin{array}{ccc} M_1 & \xrightarrow{\ \varphi\ } & M_2 \\ & {\scriptstyle \tau_1} \searrow \quad \swarrow {\scriptstyle \tau_2} & \\ & M & \end{array}, \qquad \text{one has} \quad g_{\tau_2} \circ \varphi = \varphi \circ g_{\tau_1} \qquad (1)$$

$g \in \pi_1(M, a)$ is uniquely determined by the collection of transformations $\left\{ g_\tau \colon \tau^{-1}(a) \to \tau^{-1}(a) \;\middle|\; \tau \text{ is a covering} \right\}$.
One can define an element of $\pi_1(M, a)$ as a collection of such transformations satisfying (1).

To define $\pi_1(\mathcal{E}, a)$, replace M by $\mathcal{E}$, topological coverings by differential coverings, transformations on fibers by vector fields on fibers.

To obtain the correct definition of $\pi_1(\mathcal{E}, a)$, consider 'generalized coverings': **formal zero-curvature representations with coefficients in Lie algebras**.

Analytic functions are replaced by formal power series,
Lie algebras of vector fields are replaced by arbitrary Lie algebras.

Let L be a Lie algebra. Let ω be a formal L-valued differential 1-form on $\mathcal{C}(\mathcal{E})$.
ω is called a **zero-curvature representation (ZCR)** if it satisfies the
$\bar{d}$-Maurer-Cartan equation $\bar{d}(\omega) + \dfrac{1}{2}[\omega, \omega] = 0$.
$g \in \pi_1(\mathcal{E}, a)$ determines $g_\omega \in L$ for each L-valued ZCR ω.
$g \in \pi_1(\mathcal{E}, a)$ is determined uniquely by the collection of elements
$\left\{g_\omega \in L \,\middle|\, \forall\, L\ \forall\, \omega\right\}$, which are in agreement with respect to morphisms of ZCRs.
We define an element of $\pi_1(\mathcal{E}, a)$ as such a collection.

To compute $\pi_1(\mathcal{E}, a)$ in terms of generators and relations, we find a 'normal' form for ZCRs with respect to the action of the group of gauge transformations. (This involves a step similar to finding a Gröbner basis.)
Then we take a 'general' ZCR $\tilde{\omega}$ in this normal form, the coefficients of $\tilde{\omega}$ are regarded as generators of a Lie algebra, and the equation $\bar{d}(\tilde{\omega}) + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}] = 0$ provides some Lie algebraic relations for these generators.