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dear experimental signature of the superfluid phase.

supernovae degrees of freedom. The observation of these additional modes in a
that in superfluid helium there are two additional modes associated with the
a quantum crystal and observed in recent experiments. Furthermore, we find
propagating instead of a diffuse defect mode, the former is appropriate for
hydrodynamics is modified in such a way that it leads to the presence of a
that describe the collective modes of these phases. In particular, the usual
We derive the hydrodynamic equations of motion of solid and superfluid, 1He,

Abstract

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Collective Modes in Superfluid 4He
The temperature dependence of the quantum crystal, in terms of the distance of the atoms from each other, was measured by the famous Lenz-Overhauser effect. The phenomenon of superfluidity, where the atoms form a single quantum fluid, was first observed in this point of view. Using the famous Lenz-Overhauser effect, it was observed that the atoms form a single quantum fluid, which is known as superfluid helium. The phenomenon of superfluidity was first observed in helium by Kamerlingh Onnes in 1911 and has been a subject of experimental and theoretical research for decades. In 1966, the low temperature behavior of the strongly interacting quantum liquid, He, has been

I. INTRODUCTION
which did include propagation behavior of the central defect mode. Moreover, the long-range interstitials, derived the hydrodynamic equations for an isotropic supersolid in two dimensions. Recently, Shook et al. in response to experiments with supersolid or supersolid-like behavior, a propagation mode, as is confirmed by the experiments of Lefever and Goodkind, for a classical but not for a quantum crystal, where the defect mode is expected to be a Martin C. A. assume diffusion dynamics for this defect mode. This seems to be appropriate does not lead to the required defect mode in the normal state of the crystal. In addition, it is also incomplete, because it does not include the non-condensed defects and as a result a mode in the defect density. This implies that the hydrodynamics of Andrew and Hirst's incomplete since it yields the wrong number of modes. They identified the missing mode as a hydrodynamic equation for a classical crystal, which does not include defects, is necessarily necessarily proliferation in the discussion of the Andrew and Hirst's hydrodynamics.

However, it was pointed out by Martin C. A. that the conditional treatment of the flow of an ideal crystal, in addition to this pioneering work, this has now recently presented a by including the effect of Bose-Einstein condensation of the defects on the hydrodynamics. As mentioned above, the flow from the supersolid to the supersolid phase at a temperature \( T \). After mentioning above, the flow from the normal phase to the supersolid phase at some temperature \( T \) and subsequently

\[ p \rightarrow p + \Delta p, \quad q \rightarrow q + \Delta q, \quad \text{where } \Delta p \text{ and } \Delta q \text{ are the changes in the density and velocity, respectively.} \]

Following a certain trajectory in this phase diagram, \( H \) may undergo a transition defects. Thus the phase diagram of \( H \) in these dimensions would be qualitatively given by the zero momentum state of the point defects, i.e., a Bose-Einstein condensation of the point defects. To consistently interpret their data they have had to assume a microscopic population of defects on the density of defects in the same way as in a dense Bose gas. They found a relation between the temperature dependence of the phase velocity and that of the defect density of defects in a quantum crystal. Furthermore, assuming the speed of sound of the defect mode to be the attenuation needed a coupling to thermally activated excitations, consistent with the existence of a propagation mode in the gas of point defects that is expected to be present in the crystal.

\[ p \rightarrow p + \Delta p, \quad q \rightarrow q + \Delta q, \quad \text{where } \Delta p \text{ and } \Delta q \text{ are the changes in the density and velocity, respectively.} \]
II. HYDRODYNAMICS OF SOLIDS WITH POINT DEFECTS

In this section we derive the hydrodynamic equations of a solid with point defects. This will be

Discussion and outlook in Sec. 

We conclude with a

understanding that experimental is not necessary to include temperature fluctuations into our

by Leung and Gokhale in the light of our results. It should be noted that in order to

our hydrodynamic equations in the usual way and in Sec. we discuss the experiment

and dissipation is included. In Sec. we then add a supplementary degree of freedom to

as a limiting case of a dissociation. Also, a more microscopic point of view is presented

we obtain the interaction between phonons and a point defect, by solving the point defect

describing a solid with dislocations, using methods developed by Kleinert. From this action

a normal solid with point defects will be derived. This is achieved by deriving an action

The paper is organized as follows. In Sec. we discuss the hydrodynamic equations describing

sytem.

Properties of sound in solid H and lead to the first claim of a superfluid phase in this

a microscopic point of view the phenomenological equations that successfully explained the

H, which is a hexagonally close packed (HCP) crystal. Thus we hope to justify from

renormalized system and second of all we have to take into account the anisotropy of solid

helium, we have to extend them in two ways. First of all we have to consider a three

model their data. However, to apply these promising results to the experiments with solid

nucleus part of the solid hydrodynamics derived by these authors turns out to be identical.
A. Gauge theory of phonons and dislocations

We start with deriving a gauge theory that describes the solid phase at long wavelengths. This section is closely related to previous work done by *.* However, the differences can involve higher gradient elasticity, which would be needed if we also wanted to describe point defects. The dislocation theory presented in this section is based on the analogy of the action describing a lattice interaction with electromagnetic fields. An intuitive microscopic picture of point defects is also presented with leads to an alternative derivation of the action describing a
\[ (2) \quad \int \left( \frac{(\vec{a} \cdot \vec{n}) (\vec{x} - \vec{x})}{|\vec{x}|^3} \right) \cdot \vec{p} \cdot \left( \sum_{\gamma} \vec{q} \right) = \sum_{\gamma} \left( \frac{\alpha_{\gamma}}{\gamma} \right) \]

The gradient of the displacement field which is called the plastic distortion and is given by the Burgers vector \( \vec{b} \). This displacement gives a delta function contribution to the stress, and the stress at a point \( \vec{x} \) is given by the integral of the delta function over the volume. The displacement is discontinuous across the surface \( \partial \), with a jump in the displacement field that is discontinuous everywhere except at the boundary \( \partial \). The displacement field which describes a discontinuity created by this can be obtained from the Volterra construction, which we have previously explained. Given the discontinuity, we can use the Volterra construction to determine the displacement field which describes the discontinuity in a solid formed by subtracting from the total displacement field which describes the discontinuity on the real axis. If we consider the contributions from the boundaries, it is much more convenient to use a single-valued displacement field which takes values on the real axis. If we would like to perform a path integral over the total displacement field, the action

\[ (3) \quad \int \left( \frac{(\vec{s} \cdot \vec{x})}{|\vec{x}|^3} \right) \cdot \vec{p} \cdot \left( \sum_{\gamma} \vec{q} \right) = \sum_{\gamma} \left( \frac{\alpha_{\gamma}}{\gamma} \right) \]

is given by the equation

\[ (4) \quad \int \left( \frac{(\vec{s} \cdot \vec{x})}{|\vec{x}|^3} \right) \cdot \vec{p} \cdot \left( \sum_{\gamma} \vec{q} \right) = \sum_{\gamma} \left( \frac{\alpha_{\gamma}}{\gamma} \right) \]

where \( \vec{s} \) is the dislocation density. If \( \vec{s} \) is an integer vector, this is the so-called non-differential form of the dislocation density. The dislocation can be written as a differential form as follows:

\[ (2) \quad \int \left( \frac{(\vec{s} \cdot \vec{x})}{|\vec{x}|^3} \right) \cdot \vec{p} \cdot \left( \sum_{\gamma} \vec{q} \right) = \sum_{\gamma} \left( \frac{\alpha_{\gamma}}{\gamma} \right) \]
\[
\mathcal{Z} = \frac{1}{\mathcal{Z}} \left( \frac{d}{\mathcal{Z}} \right)
\]

is symmetric under the exchange of \( \gamma \leftrightarrow h \) and is defined by

\[
\gamma \leftrightarrow h \text{ if and only if }
\]

Note also that the action contains only the symmetric part of the stress tensor, and we should have the functional integrals over the momenta density \( \nu \) and the stress tensor \( \sigma \).

Furthermore, to obtain the path integral representation of the partition function we

\[
\mathcal{Z} = \frac{1}{\mathcal{Z}} \left( \frac{d}{\mathcal{Z}} \right)
\]

and

\[
\mathcal{Z} = \frac{1}{\mathcal{Z}} \left( \frac{d}{\mathcal{Z}} \right)
\]

respectively. It amounts to adding to the Lagrangian density in Eq. (1) the quartic terms \( \mathcal{L}^{(4)} \) and \( \frac{1}{2} \mathcal{L}^{(2)} n = \mathcal{L}^{(2)} n \) which are connected to the stress tensor and the stress tensor density, respectively. Physically these fields are the stress tensor and the stress tensor density.

We do this by introducing two new fields by means of a Hubbard-Stratonovich transformation. Physically these fields are the stress tensor and the stress tensor density, respectively. Physically these fields are the stress tensor and the stress tensor density.

To be able to actually calculate the interaction between the phonons and the defects we need to able to actually calculate the interaction between the phonons and the defects.

For a perfect crystal such transformations are generalized to include dislocations. Then the action introduced in the beginning of this section is indeed just the classical action of the displacement field that is used to calculate the spatial and time derivatives of the displacement field. The value of these physical quantities equals what one would get by using the multivalued

\[
\gamma \leftrightarrow h \text{ if and only if }
\]

field should be continuous and therefore given by

\[
\gamma \leftrightarrow h \text{ if and only if }
\]

Furthermore, the interaction between the physical fields of the spatial and time derivatives of the displacement field gives a delta function contribution of the displacement field. Furthermore, if the displacement field is moving with a speed \( v \), the time derivative of the integral measure is defined by

\[
\int_0^t d\tau \int_0^\tau d\tau' \int_0^\tau' d\tau'' \int_0^\tau'' d\tau''' = \int_0^t d\tau
\]
Gauge transformations are invariant under a group of transformations which reduces the number of gauge degrees of freedom. Indeed, the number of gauge transformations are the same as the number of free degrees of freedom. However, we note that these gauge transformations are the same as the number of free degrees of freedom.

At first sight, one might therefore think that the gauge freedom removes 1 free degree of freedom.

\[ \cdot \left( \tilde{F}^\mu \partial_\mu \phi - \tilde{F}^\nu \partial_\nu \phi \right) + \tilde{F}^\mu \phi \rightleftharpoons \tilde{F}^\mu \]

\[ \tilde{F}^\mu \phi \rightleftharpoons \tilde{F}^\nu \phi \]

These imply that the action is not invariant under the gauge transformations and are therefore Held. Indeed, the expressions for \( \tilde{F}^\mu \phi \) and \( \tilde{F}^\nu \phi \) are invariant under the gauge transformation. In the process of rewriting the action we have ended up with the new degrees of freedom.

\[ \cdot \left\{ \tilde{F}^\mu \tilde{F}^\nu \tilde{F}^\sigma \tilde{F}^\rho \right\} - \tilde{F}^\mu \tilde{F}^\nu \tilde{F}^\sigma \tilde{F}^\rho \int \int \frac{p}{\phi} \frac{\partial^\phi}{\phi^2} = \left[ \tilde{F}^\mu , \tilde{F}^\nu \right] \tilde{x} S \]

Written in terms of the displacement density \( \tilde{F}^\mu \phi \) and displacement current density \( \tilde{F}^\mu \phi \), and performing some partial integrations, we find that this part of the action can be written in terms of the interaction, i.e., the last two terms in the right-hand side of Eq. (11).

\[ \cdot \tilde{F}^\mu \phi = \tilde{d} \phi \]

\[ \cdot \tilde{F}^\mu \phi \rightleftharpoons \tilde{F}^\nu \phi \]

This is Newton's law. In order to automatically satisfy these constraints we rewrite the action as

\[ \cdot \left\{ \tilde{m} \tilde{F}^\mu \tilde{F}^\nu \tilde{F}^\sigma \tilde{F}^\rho \right\} - \tilde{m} \tilde{F}^\mu \tilde{F}^\nu \tilde{F}^\sigma \tilde{F}^\rho \int \int \frac{p}{\phi} \frac{\partial^\phi}{\phi^2} = \left[ \tilde{m} \tilde{F}^\mu , \tilde{F}^\nu \right] \tilde{x} S \]

We now integrate out the displacement field, which leads to the constraints.
\[
(\text{11}) \cdot \left\{ \phi_y(\phi_{\phi_y} + \phi_{\phi_y}^T) \right\} - \frac{\partial \phi}{\partial y} - \frac{\partial \phi}{\partial y} + \left( \frac{\partial \phi}{\partial y} + \phi_{\phi_y}(n^2 \theta) \right) \frac{\partial \phi}{\partial y} - \frac{\partial \phi}{\partial y} + \phi_{\phi_y} \frac{\partial \phi}{\partial y} \right\} \exp \int_{\Omega} \frac{1}{2} \right\} \quad \text{as} \quad [1, \text{SN}]
\]

\[
\text{performing a Hubbard-Stratonovich transformation we get}
\]

\[
\text{where denotes the partition function with non-vanishing source terms. Again}
\]

\[
(\text{21}) \quad : \quad \phi = \phi_y(\phi_{\phi_y} + \phi_{\phi_y}^T) \quad \exp \int_{\Omega} \frac{1}{2} \right\} \quad \text{as} \quad [1, \text{SN}]
\]

\[
\text{Expectation values of } \phi_{\phi_y} \text{ and } \phi_{\phi_y}^T \text{ are now easily calculated as follows}
\]

\[
\exp \int_{\Omega} \frac{1}{2} \right\} \quad \text{as} \quad [1, \text{SN}]
\]

The proof is given by addressing the action in Eq. 1 to source terms proportional to the

\[
(\text{41}) \quad \phi = \phi_y(\phi_{\phi_y} + \phi_{\phi_y}^T) \quad \exp \int_{\Omega} \frac{1}{2} \right\} \quad \text{as} \quad [1, \text{SN}]
\]

\[
\text{The equations of motion for a solid with point defects. They are}
\]

\[
\text{prove the following equations which we will need later on when deriving the hydrodynamic}
\]

\[
\text{degrees of freedom, i.e. fix the gauge. Before we embark on this problem however, we will}
\]

\[
\text{In order to extract physically relevant information we will have to remove the gauge}
\]

\[
\text{6 phonon modes.}
\]

\[
\text{This means that we end up with 6 physical degrees of freedom, corresponding to the usual}
\]

\[
\text{we should also demand to be symmetric, which will remove another 3 degrees of freedom.}
\]

\[
\text{Note that the constraints in Eq. 6 in \phi_{\phi_y} \text{ and the constraints in Eq. 7 remove 3 of these. Note that}
\]

\[
\text{in the fields } \phi \text{ and } \phi_{\phi_y} \text{ which is exactly what we expect because there are 3 degrees of}
\]

\[
\text{the equations for } \phi \text{ and } \phi_{\phi_y} \text{ which is exactly what we expect because there are 3 degrees of}
\]

\[
\text{As a result the gauge freedom in Eq. 6 only removes 1 degree of freedom in}
\]

\[
\phi(\phi_{\phi_y} + \phi_{\phi_y}^T) \left\{ \frac{1}{2} \right\} \exp \int_{\Omega} \frac{1}{2} \right\} \quad \text{as} \quad [1, \text{SN}]
\]

\[
\phi(\phi_{\phi_y} + \phi_{\phi_y}^T) \left\{ \frac{1}{2} \right\} \exp \int_{\Omega} \frac{1}{2} \right\} \quad \text{as} \quad [1, \text{SN}]
\]
form a complete set of the irreducible representations of the rotation group in this space. From group theory we know that the
momentum space with itself, i.e., \( d \otimes d \). If we then take the direct product of
form of the fields and add \( \frac{d \theta}{2 \pi} \gamma^{\alpha} \) in the helicity-basis, we expand the Fourier trans-
of the remaining unpolarised degrees of freedom, we obtain the following equations:

\[
\begin{align*}
0 & = \int_0^\beta \frac{d \theta}{2 \pi} \gamma^{\alpha} \left( e^{\frac{i \theta}{2} J^\alpha} - e^{-\frac{i \theta}{2} J^\alpha} \right) f_{\alpha} J^\alpha
\end{align*}
\]

Free action \( S_0 \) becomes

\[
\begin{align*}
0 & = \int_0^\beta d \theta \gamma^{\alpha} \left( e^{\frac{i \theta}{2} J^\alpha} - e^{-\frac{i \theta}{2} J^\alpha} \right) f_{\alpha} J^\alpha
\end{align*}
\]

If we now take the form also be symmetric, in terms of the fields the

\[
\begin{align*}
\gamma^{\alpha} \gamma^{\beta} & = 0
\end{align*}
\]

\[
\begin{align*}
\gamma^{\alpha} \gamma^{\beta} & = 0, \quad \gamma^{\alpha} \gamma^{\beta} = 0
\end{align*}
\]

In this gauge, our original gauge transformation reduces to

\[
\begin{align*}
\gamma^{\alpha} \gamma^{\beta} & = 0
\end{align*}
\]

To choose a gauge in which is always possible by letting \( f_{\alpha} \gamma^{\alpha} = 0 \), which is invariant under the transformations of original form. We use the latter invariance
present in the action \( S = S_0 + S_{\text{inv}} \) mentioned in Appendix B, and examine Fig.

After this discussion, we return to the elimination of the non-physical degrees of freedom

\[
\begin{align*}
\text{Fig. } (\text{a}) \text{ now follow by differentiation.}
\end{align*}
\]
\( (\varepsilon) \quad 0 = (e^{-\varepsilon})^\gamma \theta = (e^{-\varepsilon})^\gamma = (e^{-\varepsilon})^\gamma \)

\[ \{ J, (\varepsilon^{-1})'(\varepsilon' \varepsilon') \} \]

The elements in the space of symmetric second rank tensors, in addition the elements from the old to the new basis is unitary and hence the new basis is also orthonormal. The components of the corresponding to \( u \chi \) are unphysical and disappear.

\[ (\varepsilon) \quad \delta_{\varepsilon \varepsilon}(\theta + \varepsilon)^\gamma = \frac{E^\Lambda}{1} = \frac{\theta}{1} \]

\[ (\varepsilon) \quad (\theta - \varepsilon)^\gamma = \frac{E^\Lambda}{1} = \frac{\theta}{1} \]

Where

\[ (\varepsilon) \quad \{ (\theta^{-1})', (\varepsilon^{-1})', (\varepsilon^{-1})', (\varepsilon^{-1})' \} \]

From this expression we see that if we choose a new basis in which to develop \( u \chi \) given by

\[ (\varepsilon) \quad (\theta^{-1})'(1 - \varepsilon) + (\varepsilon^{-1})'(1 - \varepsilon) = (\theta + \varepsilon)^\gamma + (\theta - \varepsilon)^\gamma \]

If we choose as a basis in Fourier space \( u \gamma \) in terms of \( \{ (\theta^{-1})', (\varepsilon^{-1})', (\varepsilon^{-1})' \} \) we develop \( u \gamma \) in that basis, this transformation reads up to a factor \( \chi \)

\[ (\varepsilon) \quad \zeta + \varepsilon_i \theta + \varepsilon \chi \leftarrow u \chi \]

Following transformation note that the expression \( u \chi \theta \theta^\varepsilon \gamma \varepsilon \) is symmetric, traceless and invariant under the exchange of \( (0 \varepsilon) \), \( (0 \varepsilon) \), \( (0 \varepsilon) \), \( (0 \varepsilon) \), \( (0 \varepsilon) \), \( (0 \varepsilon) \)

\[ (\varepsilon) \quad \{ (\theta^{-1})', (\varepsilon^{-1})', (\varepsilon^{-1})' \} \]
If furthermore realize that

\[
\tau \frac{\varepsilon}{\tau J^+ \Theta} - \chi = \mu^\prime \chi
\]

(63)

\[
\frac{\varepsilon}{(r - \varepsilon)\tau I^\prime \Theta} + (r - \varepsilon)\chi = (r - \varepsilon)\chi
\]

(63)

\[
\frac{\varepsilon}{(r + \varepsilon)\tau I^\prime \Theta} + (r + \varepsilon)\chi = (r + \varepsilon)\chi
\]

\[
\text{To remove this remaining freedom we introduce three new, invariant, fields}
\]

\[
(q^\prime \cdot \lambda^\prime)\{T^\prime (1 - \varepsilon) \varepsilon (1 + \varepsilon)\}
\]

(82)

\[
(q^\prime \cdot \lambda^\prime)\{T^\prime (1 - \varepsilon) \varepsilon (1 + \varepsilon)\}
\]

(82)

\[
(q^\prime \cdot \lambda^\prime)\{T^\prime (1 - \varepsilon) \varepsilon (1 + \varepsilon)\}
\]

(82)

\[
(q^\prime \cdot \lambda^\prime)\{T^\prime (1 - \varepsilon) \varepsilon (1 + \varepsilon)\}
\]

(82)

\[
(q^\prime \cdot \lambda^\prime)\{T^\prime (1 - \varepsilon) \varepsilon (1 + \varepsilon)\}
\]

(82)

\[
(q^\prime \cdot \lambda^\prime)\{T^\prime (1 - \varepsilon) \varepsilon (1 + \varepsilon)\}
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(82)

\[
(q^\prime \cdot \lambda^\prime)\{T^\prime (1 - \varepsilon) \varepsilon (1 + \varepsilon)\}
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(82)

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(q^\prime \cdot \lambda^\prime)\{T^\prime (1 - \varepsilon) \varepsilon (1 + \varepsilon)\}
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(82)

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(q^\prime \cdot \lambda^\prime)\{T^\prime (1 - \varepsilon) \varepsilon (1 + \varepsilon)\}
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(q^\prime \cdot \lambda^\prime)\{T^\prime (1 - \varepsilon) \varepsilon (1 + \varepsilon)\}
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(82)

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(q^\prime \cdot \lambda^\prime)\{T^\prime (1 - \varepsilon) \varepsilon (1 + \varepsilon)\}
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(82)

\[
(q^\prime \cdot \lambda^\prime)\{T^\prime (1 - \varepsilon) \varepsilon (1 + \varepsilon)\}
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(82)

\[
(q^\prime \cdot \lambda^\prime)\{T^\prime (1 - \varepsilon) \varepsilon (1 + \varepsilon)\}
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(82)

\[
(q^\prime \cdot \lambda^\prime)\{T^\prime (1 - \varepsilon) \varepsilon (1 + \varepsilon)\}
\]

(82)

\[
(q^\prime \cdot \lambda^\prime)\{T^\prime (1 - \varepsilon) \varepsilon (1 + \varepsilon)\}
\]

(82)

\[
(q^\prime \cdot \lambda^\prime)\{T^\prime (1 - \varepsilon) \varepsilon (1 + \varepsilon)\}
\]

(82)

\[
(q^\prime \cdot \lambda^\prime)\{T^\prime (1 - \varepsilon) \varepsilon (1 + \varepsilon)\}
\]

(82)
\[
\left( \frac{g}{\lambda} \right) \cdot \left( \frac{\partial Y}{\partial \theta} \right) \cdot \left( \frac{\partial V}{\partial \theta} \right) \cdot \left( \frac{\partial \Sigma}{\partial \theta} \right) = \frac{g(\lambda, Y)}{\lambda} \cdot \frac{g(V, \Sigma)}{V} = \frac{g(X, \lambda)}{X}
\]
Describing the phonon modes finally becomes interesting to include the interaction between the displacements and the physical fields 

\[ \sum_{\alpha} \chi_{\alpha} \sum_{\beta} \left( \psi_{\alpha} \psi_{\beta} \right) \psi_{\alpha} \psi_{\beta} = \sum_{\alpha} \chi_{\alpha} \sum_{\beta} \left( \psi_{\alpha} \psi_{\beta} \right) \psi_{\alpha} \psi_{\beta} \]

We see that there is indeed only an interaction with the \((\zeta - \zeta)\) and \((\zeta \zeta)\) components of

\[ \sum_{\alpha} \chi_{\alpha} \sum_{\beta} \left( \psi_{\alpha} \psi_{\beta} \right) \psi_{\alpha} \psi_{\beta} = \sum_{\alpha} \chi_{\alpha} \sum_{\beta} \left( \psi_{\alpha} \psi_{\beta} \right) \psi_{\alpha} \psi_{\beta} \]

where we have used

\[ \sum_{\alpha} \chi_{\alpha} \sum_{\beta} \left( \psi_{\alpha} \psi_{\beta} \right) \psi_{\alpha} \psi_{\beta} = \sum_{\alpha} \chi_{\alpha} \sum_{\beta} \left( \psi_{\alpha} \psi_{\beta} \right) \psi_{\alpha} \psi_{\beta} \]

and substitute this in the first term on the right-hand side of \( \Phi \).

We obtain

\[ \sum_{\alpha} \chi_{\alpha} \sum_{\beta} \left( \psi_{\alpha} \psi_{\beta} \right) \psi_{\alpha} \psi_{\beta} = \sum_{\alpha} \chi_{\alpha} \sum_{\beta} \left( \psi_{\alpha} \psi_{\beta} \right) \psi_{\alpha} \psi_{\beta} \]

in Eq. (34), i.e., to see this we use

\[ \sum_{\alpha} \chi_{\alpha} \sum_{\beta} \left( \psi_{\alpha} \psi_{\beta} \right) \psi_{\alpha} \psi_{\beta} = \sum_{\alpha} \chi_{\alpha} \sum_{\beta} \left( \psi_{\alpha} \psi_{\beta} \right) \psi_{\alpha} \psi_{\beta} \]

The second and third term on the right-hand side of the above equation will give an interaction between the components introduced in the interaction term involving \( \chi \). The first term together with the contraction \( \langle \psi_1 \rangle \Phi \) and annihilate the components \( \langle \psi_1 \psi \rangle \) and \( \langle \psi_1 \rangle \Phi \) as is obvious from the fact that the contractions \( \langle \psi_1 \rangle \Phi \) and \( \langle \psi_1 \rangle \Phi \) are the second and third term on the right-hand side of the above equation.
\[
\mathcal{S} = \int \mathcal{F} \cdot \mathcal{W} \cdot \mathcal{A}_{\mathcal{F}} + \sum_{\mathcal{N}} \mathcal{F} \cdot \mathcal{A}_{\mathcal{F}} \int_{y_1} \frac{c(y_1)}{\mathcal{L} P} \int_0^1 \mathcal{X} \mathcal{A}_{\mathcal{X}} = \mathcal{X} \mathcal{A}_{\mathcal{X}} \int
\]
\[ \nabla^* \theta \cdot \partial \psi = \left( f \cdot W \frac{\partial f}{\partial \xi} + j \cdot \nu \right) \cdot \frac{\partial \psi}{\partial \xi} = f \cdot \nabla^* \theta \]

Action \( S \) and are given by

In what follows we need the field equations for \( j \) and \( \nu \), which follow from the complete analogies of the Lamé constants \( \lambda \) and \( \mu \) in the isotropic case.

The elasticity tensor \( C_{ij} \) for the symmetric combinations \( \xi_{ij} \) constants the

The associated symmetric group is \( \text{Cm}_{2} \), which contains rotations about the \( x \)-axis and replaces the

from this point on the special case of a hexagonal close packed (HCP) crystal structure,

Because we are especially interested in the behavior of solid \( \nu \), we consider

descriptions. Hence we are especially interested in the behavior of solid \( \nu \), we consider

defines the vector \( g \) and matrix \( \Lambda \) will therefore be determined from symmetry con-

cerned in \( g \) and \( f \). But the HPC vector can only be a lattice vector. There are however subtleties involved in

given by Eq. (34). The symmetry of the crystal is then contained in \( g \) and \( f \). Because

possible to explicitly take the limit of a description loop shrinking to zero in the interaction

This symmetry contains the coefficients \( \xi_{ij} \) and the matrix \( \Lambda \). In principle it should be

form of the interaction is determined by the symmetry of the crystal under consideration.

Up to this point, no specific crystal symmetry has been assumed. However, the explicit

\[ f^* \theta = f^* \nabla^* \theta \]

defect density and defect current density therefore satisfy a continuity equation

which is conserved because defects and interstitials can locally be created in pairs. The

in fact the difference between the interstitial density and the vacancy density is

Thus, the net defect density is

In general there can be both vacancies and interstitials present, and the change in site density

\[ \left( (u_{i}) x - x \right) \gamma_{(u_i)} b_{(u_i)} x^* \theta | \sum_{u_i} = (x)^* \theta \]

\[ \sum_{u_i} = (x)^* \nabla \theta \]

where \( \gamma_{(u_i)} \) are given by \( \theta \); the defect density \( \nabla \theta \) and defect current density \( \gamma_{(u_i)} b_{(u_i)} \) are

are the matrix components of the defect current density and \( \Lambda^* \) is a matrix. If we have

\[ \sum_{u_i} \theta | (1-i) f^* \theta | (1+i) f^* \theta = \theta | (1-i) f^* \theta | (1+i) f^* \theta \]

containing an extra factor of \( \gamma \). The defect density is denoted by \( \gamma \).
\[ 0 = \mathbf{g} \cdot \left\{ [\mathbf{H}]_{1-\mathcal{G}} \cdot [\mathbf{H}]_{\{\eta, \xi\}} - [\mathbf{H}]_{1-\mathcal{G}} \cdot [\mathbf{H}]_{1-\mathcal{G}} \right\} \]

This means that Eq. (8) translates into

\[ \nabla N \mathbf{p} \cdot \mathbf{\tau} \mathcal{G} = \mathcal{F} \mathbf{\eta} = \mathcal{F} \mathbf{\xi} \]

we get

Eq. (8) into the equation for \( \mathbf{\tau} \mathcal{G} \) with \( \mathbf{g} = 0 \). i.e., considering a static defect.

Putting this into the equation introduced before and \( \mathbf{\eta} \) and \( \mathbf{\xi} \) are vectors.

\[ \nabla N \mathbf{p} \cdot \mathbf{\tau} = \mathcal{F} \mathbf{\eta} \]

be derived from Eq. (8) and read

In the case of a static defect we can solve for \( \mathbf{\tau} \) using the field equations for \( \mathcal{F} \). These can

\[ \nabla N \mathbf{p} \cdot \mathbf{\tau} = \mathcal{F} \mathbf{\eta} \]

which in Fourier language reads

\[ \mathbf{\phi} = (\mathbf{X} - \mathbf{S}) \mathbf{\eta} \]

get

This means that \( \mathbf{g} \) has to be invariant under the symmetry operations of \( \mathbf{\eta} \) and \( \mathbf{\xi} \). The fact that a single point defect has no orientation in follows from the stress tensor \( \mathcal{F} \) residing from a single point defect.

To determine the form of the interaction between the phonons and the defects we calculate

\[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathcal{H} & \mathcal{B} \\ \mathcal{B} & V \end{pmatrix} \cdot \begin{pmatrix} \mathcal{U} & \mathcal{D} \\ \mathcal{D} & d \end{pmatrix} \]

where

\[ \nabla N \mathbf{p} \cdot \mathbf{\tau} = \left( \mathcal{F} \cdot \mathcal{W} + \mathcal{G} \cdot \mathcal{N} \right) \cdot \mathcal{G} = \mathcal{F} \mathbf{\eta} \]
angle about the c-axes we are only considering the point \((0,0,1)\) as that \(C_\pi \neq I\). We note that these matrices transform under rotations over an

Thus, according to Sturm's lemma, they do not mix under the operator \(C_{\pi \theta}\), which means each of these matrices transforms according to an irreducible representation of \(G_{\pi \theta}\).

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & -1 \\
0 & -1 & 1 \\
\end{pmatrix}
\]

This means that here our helicity-basis becomes

\[
\begin{align*}
\Phi &= (1,0,0) = (\pi^\theta) \\
(0,1,0) &= (\pi^\theta) \\
(0,0,1) &= (1^\theta)
\end{align*}
\]

other point does not lead to any additional restrictions. In this point we choose the c-axes are indeed these two points. We only treat \((\pi^\theta)^\pi, \pi^\theta\) is not by construction, automatically satisfied for rotations about points where \( \Sigma \) is not, by construction, automatically satisfied for rotations about the sphere and the points \((\pi^\theta)^\pi, \pi^\theta\) are excluded. Therefore the only this way are invariant under rotations about the c-axes. Thus cannot be done for the entire sphere. In the radial direction, i.e. the other two in such a way that the vector holds we get in the radial direction, i.e. the other two in such a way that the vector holds we get in the helicity-basis. On each point of the unit sphere we choose these orthogonal vectors. One interaction. In order to do so, we must choose a particular form for the thus far unspecified interaction. Next we are going to translate this equation into a restriction on the coefficients \(a_i\) in the
when there are only point defects present. However, when there are dislocation present $f_q$, the side of this equation are two equivalent expressions for $-d^2 f_q /d^2 q$. Therefore it should be valid if the displacement field is single valued and continuous everywhere, the left and right hand

\[ (58) \quad \langle \phi f_d (t^2 \Phi q) \rangle d^2 q^2 = \langle \phi f_d (t^2 \Phi q) \rangle d^2 q^2 \]

following equality be valid.

Next we must determine the form of the matrix $W$. This is done by demanding the

\[ (59) \quad \left\{ f^T \cdot (\gamma + \sum_{i=1}^{3} X_i T \Phi \Delta \gamma) \right\} \frac{\mu \gamma}{\kappa p} \int_p \begin{vmatrix} \frac{\mu \gamma}{\kappa p} \end{vmatrix} = \left[ \lambda \gamma + \eta \right] \mu \gamma S \]

which means that our integration in first instance reduces to

\[ (60) \quad 0 = (\varepsilon - \varepsilon)^p = (\varepsilon \varepsilon)^p \]

The only solution to these equations is

\[ (61) \quad f \neq \| 0 \| \Rightarrow (\varepsilon - \varepsilon)^p + \frac{11}{12} (\varepsilon \varepsilon)^p \]

\[ (62) \quad \varepsilon \neq \| 0 \| \Rightarrow (\varepsilon - \varepsilon)^p - \frac{11}{12} (\varepsilon \varepsilon)^p \]

This equation has to be valid for all values of $f$ and $\varepsilon$. Therefore we get

\[ (63) \quad 0 = (\varepsilon)^p = (\varepsilon - \varepsilon)^p \]

Thus for infinitesimal rotations $\Phi_q$ reduces to

\[ (64) \quad (\gamma)^{\alpha \beta} \gamma \left( 1 - \gamma \right)^p = (\gamma_{\alpha \beta} \gamma - \gamma_{\alpha \beta} \gamma ) \left( 1 - \gamma \right)^p \]

It now follows that

\[ (65) \quad (\gamma)_{(\varepsilon - \varepsilon)^p} \gamma = (\gamma)_{(\varepsilon \varepsilon)^p} \gamma \]

\[ (66) \quad (\gamma)_{(\varepsilon \varepsilon)^p} \gamma = (\gamma)_{(\varepsilon \varepsilon)^p} \gamma \]

\[ (67) \quad (\gamma)_{(\varepsilon \varepsilon)^p} \gamma = (\gamma)_{(\varepsilon \varepsilon)^p} \gamma \]
Interception is now uniquely defined in terms of one parameter $\gamma$ and is given by

\begin{align*}
\gamma^N \theta \gamma^N \theta^N = e^\frac{1}{\hbar} W
\end{align*}

we see that $(\gamma_{\text{shift}}(n^+))^{-1} \theta = (\gamma_{\text{shift}}(n^-))^{-1} \theta$

(19) \begin{align*}
\langle \gamma_{\text{shift}}(n^+ \theta) \rangle \theta = \langle \gamma_{\text{shift}}(n^+ \theta) \rangle \theta d^4 \theta
\end{align*}

the expression for which reads

and the fact that only $\gamma$ and $\gamma$ are traceless. If we combine the result in Eq. (9)

(9) \begin{align*}
\gamma_{\text{shift}}(n^+ \theta) = \\
\left[ \begin{array}{cc}
\gamma_{\text{shift}}(1) \cdot \gamma_{\text{shift}}(1) \\
\gamma_{\text{shift}}(1) \cdot \gamma_{\text{shift}}(1)
\end{array} \right] = \mathcal{O} \cdot \gamma_{\text{shift}}(1) + d \cdot \gamma_{\text{shift}}(1)
\end{align*}

basis in writing

where we used the equations of motion in Eq. (9) and the completeness relation for the helicity

(9) \begin{align*}
\left\{ \left( \gamma_{\text{shift}}(1) \cdot \gamma_{\text{shift}}(1) \right) + \langle \gamma_{\text{shift}}(1) \gamma_{\text{shift}}(1) \rangle \right\} \theta = \langle \gamma_{\text{shift}}(1) \gamma_{\text{shift}}(1) \rangle \theta d^4 \theta
\end{align*}

and find

$\langle \gamma_{\text{shift}}(1) \gamma_{\text{shift}}(1) \rangle \theta d^4 \theta = (\gamma_{\text{shift}}(1 \theta))^{-1} \theta d^4 \theta$
This way of writing it will prove useful when determining the hydrodynamic equations of motion.

\[ (99) \quad (\varepsilon_{\alpha \beta} \theta^\alpha \theta^\beta)_u \frac{\partial^\gamma}{\partial t^\gamma} \frac{\partial}{\partial t^\gamma} = \frac{\partial}{\partial t^\gamma} \frac{\partial}{\partial t^\gamma} \]

where

\[ (99) \quad \cdot \quad (\varepsilon_{\alpha \beta} \theta^\alpha \theta^\beta)_u \frac{\partial^\gamma}{\partial t^\gamma} \frac{\partial}{\partial t^\gamma} (\varepsilon_{\alpha \beta} \theta^\alpha \theta^\beta)_u = \frac{\partial}{\partial t^\gamma} \frac{\partial}{\partial t^\gamma} \]

The equations of motion for \( x \) can also be written as

\[ (29) \quad \nabla \cdot \left( \nabla \gamma \theta \frac{\partial^\gamma}{\partial t^\gamma} - \nabla \gamma \theta \frac{\partial^\gamma}{\partial t^\gamma} \right) = \frac{\partial}{\partial t^\gamma} \frac{\partial}{\partial t^\gamma} \]

\[ (29) \quad \nabla \cdot \left( \nabla \gamma \theta \frac{\partial^\gamma}{\partial t^\gamma} - \nabla \gamma \theta \frac{\partial^\gamma}{\partial t^\gamma} \right) \cdot \frac{\partial^\gamma}{\partial t^\gamma} = \frac{\partial}{\partial t^\gamma} \frac{\partial}{\partial t^\gamma} \]

The field equations are now found by varying with respect to \( \omega \) and the positions \( x \) of the field. The form of the anisotropic mass \( m \) is constrained by the symmetry of the crystal. The defect is given by

\[ (49) \quad \cdot \quad \left( \theta^\alpha \theta^\beta \right)_u \frac{\partial^\gamma}{\partial t^\gamma} \int_{\varepsilon^2}^u \frac{\partial^\gamma}{\partial t^\gamma} = \left[ \left( x \right) \right] \phi \]

We have now completely specified the interaction of the point defects with the phonon field.
we cannot use this lone wavefunction result to find the interaction between the phonons

\[ \text{Equation (22)} \]

\[ \cdot [\varepsilon (\{ n \}) \mathcal{O} + (\{ u \})] \Lambda = (\{ n + u \}) \Lambda \]

potentials \((|\{ x \} - (\{ x \})| \Lambda) = (\{ x \}) \Lambda \)

Since the positions \((\{ u \})\) correspond to the equilibrium positions of the crystal, the total

\[ \text{Equation (12)} \]

\[ \cdot \left\{ |(\{ x \}) - (\{ x \})| \Lambda \sum_{\varepsilon} + \varepsilon (f(x)(n^2 \theta + y) \Lambda \sum_{\varepsilon} \frac{1}{\varepsilon} \right\} \int_{\varepsilon}^{\infty} = [n]S \]

where the defects are located at the positions \((\{ \{ x \} \})\). Inserting this decomposition of the

\[ \text{Equation (62)} \]

\[ f(x)(n + y) = (\{ n \}) \]

To explicitly include the point defects, we then decompose the displacement field into a part

\[ \text{Equation (69)} \]

\[ \cdot \left\{ |(\{ x \}) - (\{ x \})| \Lambda \sum_{\varepsilon} + \varepsilon (f(x)(n^2 \theta + y) \Lambda \sum_{\varepsilon} \frac{1}{\varepsilon} \right\} \int_{\varepsilon}^{\infty} = [n]S \]

By first instance, the microscopic action reads

approximations we implicitly made when we wrote down the free action of a point defect in

that the hydrodynamic momentum density is \( \mathcal{Q} \) and, as we will see below, what

range interaction between the individual atoms. In this approach, if is clear

relative to the ideal reference lattice and assumes an isotropic, short

\[ \text{Equation (86)} \]

\[ (\{ n \} + (\{ u \}) = (\{ x \}) \]

the atoms constituting the crystal by their positions

naming of the point defects and the phonons is to start from a microscopic action. It describes

An alternative approach to the derivation of an action which describes the coupled dp-

B, Microscopic Picture


By means of two Hubbard-Stredling transformations and following the same route as
action in terms of the stress tensor, as found in the previous section, which can be achieved
be treated as independent. It is important when writing to establish a connection with the
and we believe that it does not preserve the relationships between them. Therefore, they have to
between the coefficients of this action. However, renormalization changes these coefficients
associated with the creation of a defect. The microscopic action gives certain relations
for part to interact only through the phonon field. Furthermore, it denotes the energy
where we have neglected contributions with \( n \neq 0 \), assuming the defects to be sufficiently

\[
\begin{aligned}
\langle \Omega \rangle &= \left\{ F + \left[ \varepsilon \left( x(u) \right) \left[ \frac{\partial n^+}{\partial \theta} \right] + \left[ \frac{\partial n^+}{\partial \theta} \right] \right] \right\} \mathcal{F} \int \mathcal{F} \\
&+ \left\{ \frac{\varepsilon}{\frac{\partial n^+}{\partial \theta}} + \left( \frac{\partial n^+}{\partial \theta} \right) \right\} \mathcal{F} \int \mathcal{F} = \langle n \rangle S
\end{aligned}
\]

Explain the action in Fig. (13) up to second order in the displacements and making use

\[
\begin{aligned}
\langle \Omega \rangle &= \frac{1}{2} \langle n \rangle S
\end{aligned}
\]

For a moving point defect

\[
\langle \Omega \rangle \propto \mathcal{F} \int \mathcal{F} \mathcal{F}
\]

where

\[
\begin{aligned}
\langle \Omega \rangle &\propto \mathcal{F} \int \mathcal{F} \mathcal{F}
\end{aligned}
\]

we can always write for the displacement of a static defect

\[
\begin{aligned}
\langle \Omega \rangle \propto \mathcal{F} \int \mathcal{F} \mathcal{F}
\end{aligned}
\]
This equation is nothing but the constraints found in Sec. 2. In a perfect

\[ \dot{\theta} = \theta \]

\[ T \]

We can immediately write down the following equality

\[ \theta = \theta \]

unnecessarily complicates the notation.

In the remainder of this article we do not explicitly include the averaging brackets, since it

the point defects, because we have constructed the physical quantity \( \rho \) that was

was situated on the lattice sites. It is important to note that it includes the momentum of

microscopic point of view because locally it is just the momentum of the particles of mass

the hydrodynamic momentum density \( \rho \), which is obvious from a
dynamic modes. To find the equations of motion describing these modes we first identify

results in hydrodynamic Goldstone modes. Hence we expect to find a total of 8 hydro-

addition to the conservation laws, translationally symmetry is spontaneously broken, which

from the action and therefore in principle also include this mode into our considerations. In

that and we will not consider it here. Note however that we can obtain the Hamiltonian

an additional thermal diffusion mode. However, for our purposes it is relatively unimportant

modes. In principle we also need to take into account energy conservation, which would yield

of interactions and vacuums. The associated conservation laws result in hydrodynamic

the total momentum and the net number of defects \( N \), the difference between the number

charges and the number of broken symmetries. These conserved quantities are the total mass

number of hydrodynamic modes is fundamentally related to the number of conserved gauge

We can now derive the hydrodynamic equations for a crystal with point defects. The

C. Hydrodynamics

part of the defect current density.

particularly that there is indeed only an interaction between the phonons and the longitudinal

before introducing the gauge fields. The result turns out to be identical and shows in
\[
\left\{ \nabla N^2 \theta - \left( \frac{\hat{\theta}}{u^2 u^2} + \frac{\hat{\theta}}{v^2 v^2} \right) \frac{d}{d \theta} \right\} \frac{d}{d \theta} \frac{d}{d \theta} = \nabla^2 \theta
\]

(18)

\[
\nabla^2 \theta = \frac{d^2}{d \theta^2}
\]

We can split the above equation into three continuity equations as follows

(18)

\[
\left( \nabla N^2 \theta - \frac{d}{d \theta} \right) \frac{d}{d \theta} \frac{d}{d \theta} = \nabla^2 \theta
\]

Thus we find for the following equation of motion, describing the phonon modes and their interaction with the point defects

(18)

\[
\nabla^2 \theta = \frac{d^2}{d \theta^2}
\]

Therefore we need to know \( \frac{d^2}{d \theta^2} \theta \) which is easily calculated as

(18)

\[
\frac{d^2}{d \theta^2} \theta = \nabla^2 \theta
\]

of motion is found by taking the time derivative of the above equation crystal without defects, the hydrodynamic modes are the phonon modes, and their equation
where the following integrated equation of motion for $\nabla N^\varphi$ is used:}

\[
\frac{\partial}{\partial t} N^\varphi + \nabla \cdot \left( \nabla N^\varphi \right) = 0
\]

Note that if we had not introduced the dynamics of the defects

\[
\frac{1}{\hbar} \frac{\partial}{\partial t} H = \nabla \cdot \left( \nabla N^\varphi \right) + \nabla \cdot \left( \nabla \cdot \left( \nabla N^\varphi \right) \right) = 0
\]
\[ \mathcal{L} = \rho \frac{\partial}{\partial \phi} \rho^2 \phi \frac{\partial}{\partial \phi} \]

\[ \mathcal{L} = \rho \frac{\partial}{\partial \phi} \rho \frac{\partial}{\partial \phi} \]

Expected amount to a total of 5 continuity equations.

For completeness we write down the total set of hydrodynamic equations which are eX-

\[ \rho \frac{\partial}{\partial \phi} \rho \frac{\partial}{\partial \phi} \]

with the substitution \( \frac{\partial}{\partial \phi} \rho \frac{\partial}{\partial \phi} \). Therefore, Eq. (8) is the equation of the hydrodynamic

\[ \rho \frac{\partial}{\partial \phi} \rho \frac{\partial}{\partial \phi} \]

term quadratic in \( \phi \), which is the free part of the equation in Eq. (9).

Note that in Eq. (8) the

\[ \rho \frac{\partial}{\partial \phi} \rho \frac{\partial}{\partial \phi} \]

equations for the phonon modes and defining \( \rho \alpha \) the constant of the field

\[ \rho \frac{\partial}{\partial \phi} \rho \frac{\partial}{\partial \phi} \]

are found by writing down the field

\[ \rho \frac{\partial}{\partial \phi} \rho \frac{\partial}{\partial \phi} \]

where the associated action, the hydrodynamic equations are

\[ \rho \frac{\partial}{\partial \phi} \rho \frac{\partial}{\partial \phi} \]

when

\[ \rho \frac{\partial}{\partial \phi} \rho \frac{\partial}{\partial \phi} \]

action of which the Lagrangian density is given by

\[ \rho \frac{\partial}{\partial \phi} \rho \frac{\partial}{\partial \phi} \]

It is interesting to note that these equations can also be derived from a hydrodynamic

\[ \rho \frac{\partial}{\partial \phi} \rho \frac{\partial}{\partial \phi} \]

and

\[ \rho \frac{\partial}{\partial \phi} \rho \frac{\partial}{\partial \phi} \]

phonons, point defects, and their interactions for a HCP crystal. They are given by Eq. (8).

\[ \rho \frac{\partial}{\partial \phi} \rho \frac{\partial}{\partial \phi} \]

hydrodynamic equations. We have obtained a set of hydrodynamic equations describing

\[ \rho \frac{\partial}{\partial \phi} \rho \frac{\partial}{\partial \phi} \]

renewal in zero a microscopic time scale. This completes our discussion of the dissociation

\[ \rho \frac{\partial}{\partial \phi} \rho \frac{\partial}{\partial \phi} \]

transverse part is not a hydrodynamic variable and is neglected to

\[ \rho \frac{\partial}{\partial \phi} \rho \frac{\partial}{\partial \phi} \]

We stress that this is actually only an equation for the longitudinal part of the defect current.

\[ \rho \frac{\partial}{\partial \phi} \rho \frac{\partial}{\partial \phi} \]

\[ \rho \frac{\partial}{\partial \phi} \rho \frac{\partial}{\partial \phi} \]

order time derivatives. Therefore we rewrite Eq. (7) as a part of continuity equations in the

\[ \rho \frac{\partial}{\partial \phi} \rho \frac{\partial}{\partial \phi} \]

hydrodynamic equations are usually given as a set of continuity equations. It is with this

\[ \rho \frac{\partial}{\partial \phi} \rho \frac{\partial}{\partial \phi} \]

fundamental and illuminates deeply the underlying physics of this Lagrangian density. The

\[ \rho \frac{\partial}{\partial \phi} \rho \frac{\partial}{\partial \phi} \]

same equation by varying the action with respect to \( \mathcal{L} \). However, our approach is more
The two-dimensional equations of the equations found by Shool". These are indeed the three-dimensional equations, with the hydrodynamic equations for an isotropic three-dimensional fluid. Furthermore, we write the hydrodynamic equations in real time, which amounts to the substitution

\[
\nabla \cdot \nabla \varphi = \frac{\partial \varphi}{\partial t}
\]

This implies the following equations

\[

\frac{\partial \varphi}{\partial t} = \frac{\partial \varphi}{\partial t}
\]

\[
\theta \varphi = \theta \varphi
\]

\[
\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial \varphi} + \frac{\partial \varphi}{\partial t} = \frac{\partial \varphi}{\partial \varphi}
\]

By

\[
\varphi \varphi \varphi = \varphi \varphi \varphi
\]

\[
\theta \varphi = \theta \varphi
\]

\[
\theta \varphi = \theta \varphi
\]

To check heuristically if we end up with the right equations we write down the hydrodynamic equations, with the right equations for the two-dimensional equations of the equations found by Shool". These are indeed the three-dimensional equations, with the hydrodynamic equations for an isotropic three-dimensional fluid. Furthermore, we write the hydrodynamic equations in real time, which amounts to the substitution

\[
\nabla \cdot \nabla \varphi = \frac{\partial \varphi}{\partial t}
\]

This implies the following equations

\[

\frac{\partial \varphi}{\partial t} = \frac{\partial \varphi}{\partial t}
\]

\[
\theta \varphi = \theta \varphi
\]

\[
\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial \varphi} + \frac{\partial \varphi}{\partial t} = \frac{\partial \varphi}{\partial \varphi}
\]

By

\[
\varphi \varphi \varphi = \varphi \varphi \varphi
\]

\[
\theta \varphi = \theta \varphi
\]

\[
\theta \varphi = \theta \varphi
\]
the parameters is determined by the discrete symmetries of the system, which in the case

where we need that $\ell$ contains only a longitudinal degree of freedom. The specific form of

\begin{equation}
\left( \begin{array}{c}
\frac{d}{d\ell} \theta_{\ell} \\ell^2 \theta_{\ell} \\
\frac{d}{d\ell} \theta_{\ell} \\ell \theta_{\ell}
\end{array} \right) = \frac{1}{\ell} \left( \begin{array}{c}
\ell^2 \theta_{\ell} \\
\ell \theta_{\ell}
\end{array} \right)
\end{equation}

\begin{equation}
\left( \begin{array}{c}
\frac{d}{d\ell} \theta_{\ell} \\ell \theta_{\ell} \\
\frac{d}{d\ell} \theta_{\ell} \\ell^2 \theta_{\ell}
\end{array} \right) = \frac{1}{\ell} \left( \begin{array}{c}
\ell \theta_{\ell} \\
\ell^2 \theta_{\ell}
\end{array} \right)
\end{equation}

associated hydrodynamic variables are given by

requirement that the time reversal symmetry of the dissipative currents is opposite to the

then as a result, only the dissipative currents allowed by the

are associated with the 3 broken symmetries, whereas $\theta$, $\theta^\prime$ account for the conserved

and for dissipative part has already been determined. Roughly speaking the variables $\theta$ and $\theta^\prime$ where the superscript $D$ denotes the dissipative part of the stress tensors, and the non-

\begin{equation}
\theta_{\ell} \theta_{\ell} = \chi_{N}\theta
\end{equation}

in our hydrodynamic equations have the following form

\begin{equation}
\left( \begin{array}{c}
\frac{d}{d\ell} \theta_{\ell} \\ell \theta_{\ell} \\
\frac{d}{d\ell} \theta_{\ell} \\ell^2 \theta_{\ell}
\end{array} \right) = \frac{1}{\ell} \left( \begin{array}{c}
\ell \theta_{\ell} \\
\ell^2 \theta_{\ell}
\end{array} \right)
\end{equation}

\begin{equation}
\left( \begin{array}{c}
\frac{d}{d\ell} \theta_{\ell} \\ell^2 \theta_{\ell} \\
\frac{d}{d\ell} \theta_{\ell} \\ell \theta_{\ell}
\end{array} \right) = \frac{1}{\ell} \left( \begin{array}{c}
\ell^2 \theta_{\ell} \\
\ell \theta_{\ell}
\end{array} \right)
\end{equation}

where the treatment of this standard method is given by. A, B, and H. Similarly, the generally

in turn expanding the conjugate forces in terms of the coefficients of the symmetries of the system under consideration and then

coefficients to be compatible with the symmetries of the system under consideration and then

dissipative part of the stress tensor to linear order in the conjugate forces and requiring the
dissipation into our hydrodynamic equations in the standard way by first expanding the
do not mix the mode, but no dissipation. Therefore we include

the point here is no real dissipation because the bilinear coupling between the phonon and

the defect modes cause mixing of these modes, but no dissipation. Although there is a coupling between the phonon

and defects into our hydrodynamic equations. Although there is a coupling between the phonon

and defects into our hydrodynamic equations. Although there is a coupling between the phonon

in order to give a realistic description of the system, we need to include dissipational

isotropic crystal without dissipation.
(66)  \[
\cdot (\frac{1}{d} - \frac{1}{a}) \frac{d^2 \phi}{d \gamma^2} \frac{\phi^d}{d} + \frac{\partial^2 \phi}{d \gamma^2} + \frac{1}{d} \partial \phi = \frac{\partial \phi}{d}
\]

Furthermore, the dissipation part of the stress tensor becomes

\[
\frac{\partial \phi}{d}
\]

and the dynamics of the superfluid velocity has to be determined. Following the standard

logon to the gradient of the superfluid phase

where the superfluid velocity is purely longitudinal, i.e., because it is proper-

(66)  \[
\left( \frac{\epsilon}{d} a - \frac{\epsilon}{d} a \right) \frac{d}{d} + \frac{\epsilon}{d} a \frac{d}{d} = \frac{\partial \phi}{d}
\]

a normal part and a superfluid part according to

\[
\frac{\partial \phi}{d}
\]

Note that the tensorial nature of the densities is of importance in the case of an anisotropic

HYDRODYNAMICS

The equations for the normal phase, i.e., the density is split into a normal

is well known how we should proceed to include these additional degrees of freedom into the
degree of freedom, from microscopic theories developed for superfluid liquids and gases it
come to the point where we have to introduce into our hydrodynamic equations the superfluid
paper was also to formulate the hydrodynamic equations of superfluid Helium. Hence, we have

in view of the existing experiments by Lena and Goodkind, our aim in writing this

III. SUPERSOLID HYDRODYNAMICS

current is a relaxation to zero on a microscopic time scale.

differs from zero. As we have seen, the correct behavior of the transverse part of the defect
incorrectly assumed that the transverse part of the defect current behaves as a gas and
of the HCP crystal form the group H, it should be noted that in classical it was

H

CF
The large number of dissipative terms makes these equations look rather intricate, but in the presence of external forces, we can write:

\[
\begin{align*}
&\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)^2\phi + \phi \frac{\partial^2 \phi}{\partial x^2} - \phi \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{\partial^2 \phi}{\partial t^2}
\end{align*}
\]

\(\phi\) is a scalar field, typically representing the density or pressure of the system. The terms on the left-hand side of the equation describe the evolution of the field, while the right-hand side represents external forces or sources.

The structure of \(\phi\) suggests it is a solution to a wave equation, indicating that \(\phi\) could represent a wave-like phenomenon in the system. The external forces could be due to gravity, electromagnetism, or other interactions that influence the field. The equation is solved by applying appropriate boundary conditions and initial conditions, which depend on the specific system under consideration.
\[ (\mathbf{q} - \mathbf{q})_{\text{eff}} = 0 \text{ at } \tau = 0 \text{ and } \mathbf{q}'_{\text{eff}} = \mathbf{q}_{\text{eff}} \text{ for } \tau > 0. \]

In order to find the equations used by Lennard and Goodkind we first expand in the 6 phonon modes of the ideal crystal, and then divide by \( \mathbf{q} \).

\[
\text{We now turn to Eq. (66), which describes the phonon modes. First we define the matrix \( \left[ \begin{IEEEeqnarraybox} \IEEEeqnarrayleft\{ \IEEEeqnarraybox \{ \text{expression}\} \right] \right. \text{as follows.} \]
\]

The phonon mode structure present in our dissipationless hydrodynamic equations is described by a normal crystal with defects.

We show below that these equations essentially follow from our hydrodynamic equations.

oscillators that now precisely agree with the equations used by Lefevre and Coolkind.

Before, we can again eliminate two modes. We then find a coupled set of damped harmonic
oscillators. These, however, do not satisfy the equations of Fig. (3) completely. However,
we now add dissipation, the modes $\nu$ and $\omega$ no longer degenerate. Hence, the equations

\begin{align}
\frac{d}{d\tau} \gamma_{\nu} - \nu \frac{d}{d\tau} \gamma_{\omega} &= \tau F_{\nu,\omega} \\
\frac{d}{d\tau} \gamma_{\omega} + \nu \frac{d}{d\tau} \gamma_{\nu} &= \tau F_{\omega,\nu}
\end{align}

describe a set of coupled harmonic oscillators and agree with the dissipations

\begin{align}
\left( w \frac{d}{d\tau} \gamma_{\nu} + \nu \frac{d}{d\tau} \gamma_{\omega} \right)_{\nu,\omega} = \tau F_{\nu,\omega}
\end{align}

These equations still contain a separate mode. Therefore, solutions to these equations have

\begin{align}
\frac{d}{d\tau} \gamma_{\nu} - \nu \frac{d}{d\tau} \gamma_{\omega} &= \tau F_{\nu,\omega} \\
\frac{d}{d\tau} \gamma_{\omega} + \nu \frac{d}{d\tau} \gamma_{\nu} &= \tau F_{\omega,\nu}
\end{align}

Inserting this into Eq. (101) we find

\begin{align}
\frac{d}{d\tau} \gamma_{\nu} &= \nu \frac{d}{d\tau} \gamma_{\omega} + \nu \frac{d}{d\tau} \gamma_{\nu} = \nu \frac{d}{d\tau} \gamma_{\nu}
\end{align}

After Fourier transforming both the time variable the equations for $\nu \neq \omega$ are

\begin{align}
\frac{d}{d\tau} \gamma_{\nu} &= \nu \frac{d}{d\tau} \gamma_{\omega} + \nu \frac{d}{d\tau} \gamma_{\nu} = \nu \frac{d}{d\tau} \gamma_{\nu}
\end{align}

and insert these expressions into Eq. (101) and Eq. (102). After, with the eigenvalues $\nu$ and $\omega$ the second with $\nu < \omega$ this leads to the following equations

\begin{align}
\left( \frac{d}{d\tau} \gamma_{\nu} - \nu \frac{d}{d\tau} \gamma_{\omega} \right)_{\nu,\omega} &= \tau F_{\nu,\omega} \\
\frac{d}{d\tau} \gamma_{\omega} + \nu \frac{d}{d\tau} \gamma_{\nu} &= \tau F_{\omega,\nu}
\end{align}
This brings the total number of hydrodynamic modes to \( g + \frac{z}{2} + 8 = 8 \). Therefore we introduced this behavior in their experimental observation instead of the direct behavior of the usual countable argument, excluding a thermal diffusion mode. However, allowing and hydrodynamic modes, this is then in agreement with the \( 8 - 1 \) modes one expects from modes, the defects are usually assumed to have definite symmetries. Because we know that there are \( 6 \) photon defects into the hydrodynamic equations to find the right number of modes predicted by defects. It is well known that to describe the normal solid phase it is essential to include We have derived the hydrodynamic equations for the solid and superfluid phases of

\section{Conclusion}

If the modes associated with the superfluid degrees of freedom:

\begin{equation}
\begin{pmatrix}
\nabla N \\
\partial \phi
\end{pmatrix}
\begin{pmatrix}
\frac{\partial}{\partial t} \\
\frac{\partial}{\partial \phi}
\end{pmatrix}
\frac{\partial}{\partial \phi}
\begin{pmatrix}
\nabla N \\
\partial \phi
\end{pmatrix}
= \begin{pmatrix}
\nabla N \\
\partial \phi
\end{pmatrix}
\begin{pmatrix}
\frac{\partial}{\partial t} \\
\frac{\partial}{\partial \phi}
\end{pmatrix}
\frac{\partial}{\partial \phi}
\end{equation}

The hydrodynamic modes can in principle be found by diagonalizing the two matrices. If we consider the dissipationless hydrodynamic equations longitudinally part we find schematically the following equations describing an isotropic superfluid. The transverse phonon modes then decouple, and for the

To conclude this section, let us consider the dissipationless hydrodynamic equations
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existence of a superfluid phase than the analysis made by Leugers and Cookhind.

the phonons. In our opinion this would be a more convincing experimental proof for the

resonance in the attenuation and velocity of sound due to the coupling of these modes to

the results it should then be possible in principle to identify experimentally an additional

to one of these modes becoming predominant whereas the other will remain diffuse. Given

in stead of one, we expect on general grounds that including temperature fluctuations leads

of the usual two fluid hydrodynamics. As a result we end up with two second sound modes

in the standard way, we end up with what one might call a four fluid hydrodynamics instead

models. If we include these suppleld degrees of freedom into our hydrodynamic equations

down both fluctuations in the defects density and lattice vibrations to lead to superfluid

Furthermore, we have considered the hydrodynamic equations of superfluid $H_{4}$ by an-

of the observed collective mode as a propagating decaying mode.

used by Leugers and Cookhind to interpret their data, and lead them to the identification

symmetry. Indeed, our equations reproduce the set of coupled wave equations which were

and a recently specked associated with respectively a conservation law and a broken

we should also include the conservation of defects. Hence the continuity equations for $V_{A}$

that this is justified by noting that, when combining the number of correlated quantities,

another hydrodynamic variable, the longitudinal part of the defect momentum. We believe
REFERENCES
(1983).

27. TR. Kipplatik and IR. Dornman. Low. Temp. Phys. 38, 301 (1982); ibid. 38, 399


FIG. 1. Tentative sketch of the phase diagram of He.