Anomalous Hall Conductivity from the Dipole Mode of Spin-Orbit-Coupled Cold-Atom Systems

E. van der Bijl and R. A. Duine

Institute for Theoretical Physics, Utrecht University, Leuvenlaan 4, 3584 CE Utrecht, The Netherlands

(Received 11 July 2011; published 1 November 2011)

Motivated by recent experiments by Lin *et al.*, [Nature (London) **471**, 83 (2011)] that engineered spin-orbit coupling in ultracold mixtures of bosonic atoms, we study the dipole oscillation of trapped spin-orbit-coupled noncondensed Bose and Fermi gases. We find that different directions of oscillation are coupled by the spin-orbit interactions. The phase difference between oscillatory motion in orthogonal directions and the trapping frequencies of the modes are shown to be related to the anomalous Hall conductivity. Our results can be used to experimentally determine the anomalous Hall conductivity for cold-atom systems.

DOI: 10.1103/PhysRevLett.107.195302 PACS numbers: 67.85.−d, 03.75.−b, 05.30.Fk, 71.70.Ej

Introduction.—Transport phenomena play a crucial role in understanding and characterizing condensed-matter systems. Two of these phenomena, the Hall effect and the anomalous Hall effect (AHE) were both discovered in the late 19th century. That the Hall effect is due to the Lorentz force has been understood since those days. The AHE, a transverse voltage or current present in ferromagnets in the absence of a magnetic field, is related to spin-orbit (SO) coupling and has proven much more challenging to understand (for a review see Ref. [1]). Since SO coupling is responsible for the AHE, anomalous-Hall–like effects should also be present for particles that do not carry charge, and, indeed, such effects are observed for magnons [2], phonons [3,4], and photons [5]. Although these effects were observed using heat currents, and they should thus be called anomalous Righi-Leduc effects, their physical mechanism is similar to that of the AHE.

In this Letter, we consider the AHE in homogeneous and harmonically-trapped cold-atom systems. (The AHE was considered in cold-atom systems in the presence of an optical lattice in two dimensions by Dudarev *et al.* [6].) As the atoms are neutral, the AHE here refers to a mass current perpendicular to an applied force in the absence of a Coriolis force. (For cold-atom systems under rotation, the resulting Coriolis force mimics the Lorentz force.) Our investigation is motivated by the recent experiment by Lin *et al.* [7], where spin-orbit coupling in a Bose-Einstein condensate [8,9] was engineered via laser fields. This experiment is one of the latest achievements in studying phenomena known from solid-state physics in a cold-atom setting. Other examples are the Mott-insulator-to-superfluid phase transition [10], Bardeen-Cooper-Schrieffer superfluidity [11], the Berezinskii-Kosterlitz-Thouless phase transition [12,13], and Anderson localization [14]. An important feature of cold-atom systems is that new physical regimes (as compared to solid-state systems) can be explored. Furthermore, cold-atom systems are, in principle, disorder free and have a well-understood microscopic description making it worthwhile to undertake a detailed comparison between theory and experiment, whereas in solid-state materials typically a multitude of effects play a role which makes modeling harder.

In the case of the AHE, for example, the difficulty in understanding the effect lies in part in the interplay between so-called *intrinsic* and *extrinsic* contributions. Intrinsic contributions come from spin-orbit coupling effects in the band structure, whereas extrinsic contributions arise from disorder. A recent theoretical advancement in the understanding of the AHE is the semiclassical theory that yields equations of motion for Bloch wave packets [15,16]. In this description, the intrinsic contribution to the AHE stems from anomalous-velocity contributions to these semiclassical equations of motion [17]. In modern language, this anomalous velocity results from the Berry-phase curvature of the Bloch bands that in turn is determined by the topology of the band structure. The relation between band structure topology and the Hall conductivity was first emphasized by Thouless *et al.* [18], and has regained interest with the very recent discovery of topological insulators [19].

In a typical cold-atom experiment, steady-state currents are not readily created and transport coefficients can be measured only indirectly. In this Letter, we show that the anomalous Hall conductivity can be obtained from the properties of the dipole oscillation of a cloud of spin-orbit coupled cold atoms that is trapped in an external harmonic trapping potential. The dipole mode is a collective oscillation of the center-of-mass of the cloud. According to Kohn's theorem [20], the frequencies of the dipole oscillation are equal to the trap frequencies. The SO coupling, however, breaks the harmonic nature of the system and as a result Kohn's theorem for the dipole modes does not hold. We find that spin-orbit coupling modifies the oscillation frequencies and that different directions of oscillation are coupled by the spin-orbit interactions. The phase difference between oscillatory motion in different directions and the mode frequencies turn out to be related to the anomalous Hall conductivity. This result can be used to

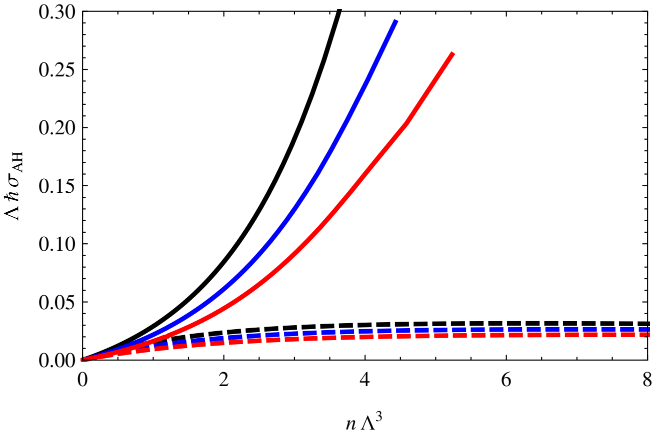

FIG. 1 (color online). The anomalous Hall conductivity for bosons (solid) and fermions (dashed) as a function of $n\Lambda^3$. The lines correspond from top to bottom to $\alpha/\beta = \{\infty, 2, 1.5\}$ where $\alpha^2 + \beta^2 = m\Lambda/\hbar^2$. The spin-splitting energy $\Delta = 0.2k_{\rm B}T$.

experimentally determine the anomalous Hall conductivity for cold-atom systems. Below we detail the semiclassical Boltzmann approach on which our findings are based, determine the anomalous Hall conductivity for homogeneous noncondensed Bose and Fermi gases, and show how this conductivity can be obtained from the dipole oscillation of trapped atomic gases.

Semiclassical equations of motion.—We consider atoms with effective spin-1/2 and mass m trapped in an external potential $V^{\rm ex}(\boldsymbol{x})$ in the presence of a generic spin-orbit coupling. The Hamiltonian is

$$\hat{H} = \frac{\hat{\boldsymbol{p}}^2}{2m} + V^{\rm ex}(\hat{\boldsymbol{x}}) - \boldsymbol{M}(\hat{\boldsymbol{p}})\cdot\boldsymbol{\tau}, \tag{1}$$

with $\hat{\boldsymbol{p}}$ and $\hat{\boldsymbol{x}}$, the momentum and position operators of the particles, and $\boldsymbol{\tau}$ the vector of Pauli matrices. The last term describes the SO coupling, which for spin one-half particles is without loss of generality given in terms of a momentum-dependent effective magnetic field $\boldsymbol{M}$.

At the semiclassical level, we consider in the first instance the dynamics of the expectation values of the position $\boldsymbol{x} = \langle\hat{\boldsymbol{x}}\rangle$, momentum $\boldsymbol{p} = \langle\hat{\boldsymbol{p}}\rangle$, and spin $\boldsymbol{s} = \hbar\langle\boldsymbol{\tau}\rangle/2$ degrees of freedom. We obtain the Heisenberg equations of motion

$$\dot{\boldsymbol{x}} = \frac{\boldsymbol{p}}{m} - \frac{2}{\hbar}\frac{\partial \boldsymbol{M}}{\partial \boldsymbol{p}}\cdot\boldsymbol{s}, \tag{2}$$

$$\dot{\boldsymbol{p}} = -\frac{\partial V^{\rm ex}}{\partial \boldsymbol{x}}, \tag{3}$$

$$\dot{\boldsymbol{s}} = \boldsymbol{s}\times\frac{\boldsymbol{M}}{\hbar}. \tag{4}$$

We proceed by assuming that the spin degree of freedom is much faster than the motion of the particles. Thus, we let the spin follow the effective magnetic field $\boldsymbol{M}$ adiabatically, and only allow for a small misalignment, between the spin and the effective magnetization, that is first order in time derivatives of the orbital dynamics. This approach is essentially exact in the linear-response regime. Hence, we solve the equation for the spin degree of freedom, Eq. (4), up to first order in time derivatives by $\boldsymbol{s} \propto \sum_i \boldsymbol{m} + \frac{\hbar}{|\boldsymbol{M}|}(\boldsymbol{m}\times\frac{\partial\boldsymbol{m}}{\partial p_i})\cdot\dot{p}_i$, with $\boldsymbol{m}(\boldsymbol{p}(t))$ the unit vector in the direction of $\boldsymbol{M}$. For spins opposite to the field the result is $-\boldsymbol{s}$. Insertion of the result for $\boldsymbol{s}$ into Eq. (2) gives [15,16]

$$\dot{\boldsymbol{x}}_k = \frac{\partial \epsilon_{\boldsymbol{p},k}}{\partial \boldsymbol{p}} + k\dot{\boldsymbol{p}}\times\boldsymbol{B}(\boldsymbol{p}), \tag{5}$$

where the band index k distinguishes between atoms with spin parallel (+) or antiparallel (−) to the field $\boldsymbol{M}$. Furthermore, the dispersion is given by $\epsilon_{\boldsymbol{p},k} = \boldsymbol{p}^2/2m - k|\boldsymbol{M}(\boldsymbol{p})|$, and the vector field $B_c(\boldsymbol{p}) = \hbar\sum_{a,b\in\{x,y,z\}}\epsilon^{abc}(\partial\boldsymbol{m}/\partial p_a\times\partial\boldsymbol{m}/\partial p_b)\cdot\boldsymbol{m}/2$ determines the anomalous-velocity contribution. We refer to $\boldsymbol{B}$ as the Berry magnetic field.

Boltzmann equation and anomalous Hall conductivity.—We proceed by calculating the anomalous Hall conductivity for a homogeneous gas from the Boltzmann equation for the distribution function $f_k(\boldsymbol{x}, \boldsymbol{p}, t)$ for atoms in band k, which is given by

$$\frac{\partial f_k}{\partial t} + \dot{\boldsymbol{x}}_k\cdot\frac{\partial f_k}{\partial \boldsymbol{x}} + \dot{\boldsymbol{p}}\cdot\frac{\partial f_k}{\partial \boldsymbol{p}} = 0, \tag{6}$$

where we ignored collisions as the intrinsic anomalous Hall conductivity does not depend on relaxation [1]. In the above, $\dot{\boldsymbol{x}}_k$ and $\dot{\boldsymbol{p}}$ are given by Eqs. (3) and (5), respectively. We consider a steady-state situation with a constant applied force $\boldsymbol{F} = -\partial V^{\rm ex}/\partial\boldsymbol{x}$ acting equally on atoms in both bands, and define the conductivity tensor $\boldsymbol{\sigma}$ by $\boldsymbol{j} = \boldsymbol{\sigma}\cdot\boldsymbol{F}$, where $\boldsymbol{j}$ is the particle current density given by $\boldsymbol{j} \equiv \sum_k\int\frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3}f_k(\boldsymbol{p})\dot{\boldsymbol{x}}_k$. The anomalous Hall conductivity $\sigma_{\rm AH}$ is the off-diagonal component of this conductivity tensor. The solution of the Boltzmann equation leads to the conductivity

$$\sigma_{\rm AH} = \sum_{k\in\{+,-\}}\int\frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3}kN(\epsilon_{\boldsymbol{p},k})B_z(\boldsymbol{p}), \tag{7}$$

where $N(\epsilon) = [e^{(\epsilon-\mu)/k_{\rm B}T}\pm 1]^{-1}$ (with $k_{\rm B}T$ the thermal energy and μ the chemical potential), is the Fermi-Dirac (+) or Bose-Einstein (−) distribution function that applies for fermions or bosons, respectively. The above expression for the anomalous Hall conductivity is the intrinsic contribution due to SO coupling effects in the band structure. In cold-atom atom systems there is, unless engineered [14], no disorder and thus extrinsic contributions are absent.

So far we have considered a generic SO coupling. In order to make a connection with experiments we will now consider a Rashba-Dresselhaus [21] form of the SO coupling so that the effective magnetic field reads

$$M(\boldsymbol{p}) = \left(\frac{\alpha}{\hbar}p_y - \frac{\beta}{\hbar}p_x, -\frac{\alpha}{\hbar}p_x + \frac{\beta}{\hbar}p_y, \frac{\Delta}{2}\right)^T, \quad (8)$$

where α and β are the coupling constants for Rashha and Dresselhaus SO coupling, respectively, and Δ is a spin-splitting energy. We then find for the Berry magnetic field

$$\boldsymbol{B}(\boldsymbol{p}) = \frac{4(\alpha^2 - \beta^2)\Delta\hbar^2}{[\Delta^2\hbar^2 + 4(\alpha^2 + \beta^2)\boldsymbol{p}^2 - 16\alpha\beta p_x p_y]^{3/2}}\hat{\boldsymbol{z}}. \quad (9)$$

Note that in the experiments by Lin *et al.* [7] an equal amount of Rashba and Dresselhaus coupling was realized, i.e., $\alpha = \pm\beta$. It follows that in this specific case $\boldsymbol{B}(\boldsymbol{p}) = 0$ and that $\sigma_{\mathrm{AH}} = 0$ [22]. It is, however, experimentally straightforward to create a more general SO coupling using additional lasers [7,23]. The anomalous Hall conductivity vanishes in the absence of a spin-splitting Δ, in agreement with the fact that the AHE occurs in ferromagnets.

In Fig. 1 we show results for the anomalous Hall conductivity of bosons and fermions. We only show the results for $\alpha > \beta$ since $\sigma_{\mathrm{AH}}(\pm\alpha, \pm\beta) = \sigma_{\mathrm{AH}}(\alpha, \beta)$ and $\sigma_{\mathrm{AH}}(\alpha, \beta) = -\sigma_{\mathrm{AH}}(\beta, \alpha)$. The results shown for bosons are above the critical temperature for Bose-Einstein condensation (this temperature depends on α, β, and Δ). Here, n is the density and $\Lambda \equiv (2\pi\hbar^2/mk_BT)^1/2$ the de Broglie wavelength.

Collective modes.—We now study the dipole oscillation of an atomic cloud of N_a atoms in an anisotropic harmonic trapping potential of the form $V^{\mathrm{ex}}(\boldsymbol{x}) = \frac{m}{2}[\omega_r^2(x^2 + y^2) + \omega_z^2 z^2]$, with ω_r and ω_z, the trapping frequencies. The dipole ($l = 1$) oscillations are pure translations of the cloud with no changes in its internal structure that are described by the equations of motion for the center-of-mass position $\boldsymbol{x}_0 \equiv \frac{1}{N_a}\sum_k \int d\boldsymbol{x} \int [d\boldsymbol{p}/(2\pi\hbar)^3] f_k \boldsymbol{x}_k$ and velocity $\boldsymbol{v}_0 \equiv \frac{1}{N_a}\sum_k \int d\boldsymbol{x} \int [d\boldsymbol{p}/(2\pi\hbar)^3] f_k \dot{\boldsymbol{x}}_k$. Hence, we make the following ansatz for the distibution function: $f_k(\boldsymbol{x}, \boldsymbol{p}, t) = n_k(\boldsymbol{x} - \boldsymbol{x}_0(t), \boldsymbol{p} - m\boldsymbol{v}_0(t))$, where $n_k(\boldsymbol{x}, \boldsymbol{p}) = N(\epsilon_{\boldsymbol{p},k} - \mu(\boldsymbol{x}))$ is the Bose-Einstein or Fermi-Dirac distribution function in the local-density approximation, with $\mu(\boldsymbol{x}) = \mu - V^{\mathrm{ex}}(\boldsymbol{x})$. From the Boltzmann equation we obtain the equations of motion for the center-of-mass coordinates. For displacements that are small compared to the size of the clouds, $\boldsymbol{x}_0/\mathrm{width} \ll 1$, it is sufficient to linearize these equations of motion [24] resulting in

$$\dot{\boldsymbol{x}}_0 = \bar{H} \cdot \boldsymbol{v}_0 - \boldsymbol{\nabla} V^{\mathrm{ex}}(\boldsymbol{x}_0) \times \bar{\boldsymbol{B}}, \quad (10)$$

$$m\dot{\boldsymbol{v}}_0 = -\boldsymbol{\nabla} V^{\mathrm{ex}}(\boldsymbol{x}_0), \quad (11)$$

where $\bar{H}$ is proportional to the Hessian matrix of the dispersion and $\bar{\boldsymbol{B}}$ is the Berry magnetic field averaged over the trap. These are given by

$$\bar{\boldsymbol{B}} = \frac{1}{N_a}\sum_{k\in\{+,-\}} \int d^3\boldsymbol{x} \int \frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3} k n_k(\boldsymbol{x}, \boldsymbol{p})\boldsymbol{B}(\boldsymbol{p}),$$

$$\bar{H}_{ij} = \frac{1}{N_a}\sum_{k\in\{+,-\}} \int d^3\boldsymbol{x} \int \frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3} n_k(\boldsymbol{x}, \boldsymbol{p}) H_{ij,k}(\boldsymbol{p}),$$

with $H_{ij,k} \equiv m\partial^2\epsilon_{\boldsymbol{p},k}/\partial p_i \partial p_j$. The linearized equations (10) and (11) have solutions with harmonic time dependence corresponding to collective modes of the atomic cloud. For the mode in the z direction we obtain the result $\omega = \omega_z$ as predicted by Kohn's theorem and expected since the SO coupling only affects the dynamics in the x-y plane. The modes in this plane have frequencies given by

$$\omega_\pm = \omega_r\sqrt{A \pm \sqrt{(\bar{H}_{xy}^2 - \bar{H}_{xx}\bar{H}_{yy}) + A^2}}, \quad (12)$$

with $A = (\bar{H}_{xx} + \bar{H}_{yy} + \bar{B}_z^2 m^2 \omega_r^2)/2$. When there is no SO coupling, $\alpha, \beta = 0$, we find $\omega = \omega_r$ as predicted by Kohn's theorem. This mode is doubly degenerate due to the two equivalent orthogonal directions of oscillation in the x-y plane. For nonzero SO coupling the degeneracy of these modes is lifted since the SO coupling breaks the rotational invariance in the plane, resulting in two different frequencies. The eigenmodes of oscillation are given by $\boldsymbol{x}_0^\pm(t) = (x_1 \sin(\omega_\pm t + \varphi_\pm), x_2 \sin(\omega_\pm t))^T$ with $\sin\varphi_\pm = m\bar{B}_z\omega_\pm/(\bar{H}_{xy}^2 + m^2\bar{B}_z^2\omega_\pm^2)^{1/2}$. In Fig. 2 we show the mode frequency ω_+ and angle φ_+ as a function of α; in the special case $\alpha = \pm\beta$ we have $\bar{B}_z = 0$ and find $\varphi_\pm = 0$. Another special case occurs when the SO coupling is of the pure Rashba or Dresselhaus form. Then $\bar{H}_{xy}$ vanishes, which results in $\varphi = \pi/2$. The phase difference between the two different directions of oscillations is determined by the Berry magnetic field. We can relate the average of the Berry magnetic field over the trap to the anomalous Hall conductivity for a homogeneous gas with a density equal to

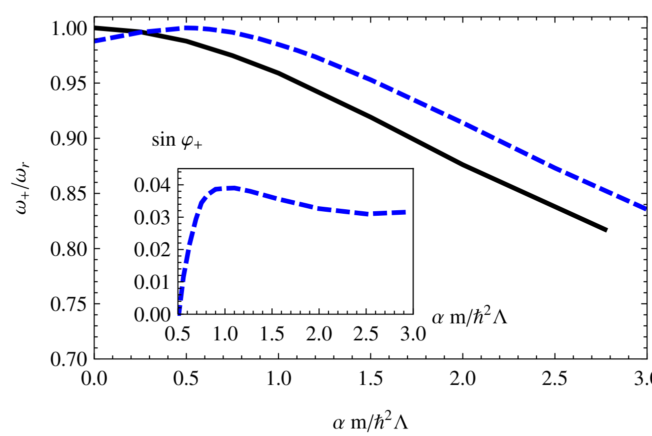

FIG. 2 (color online). Dipole mode frequency ω_+ as given by Eq. (12). The solid line is calculated for $\beta = 0$, the dashed lines correspond to $\beta = 0.5m/\hbar^2\Lambda$. The spin splitting is $\Delta = 0.2k_{\mathrm{B}}T$. The number of particles $N_a = 1.8 \times 10^5$ and temperature $T = 200$ nK. The inset shows the $\sin\varphi$ as a function of α.

the central density n_0 of the trapped cloud $\bar{B}_z \simeq \sigma_{\rm AH}(n_0)/n_0$, where $\sigma_{\rm AH}(n_0)$ is given by Eq. (7). This approximation is valid whenever the local-density approximation holds, i.e., when the de Broglie wavelengths of the particles are small compared to the size of the atomic cloud. This is typically the case in these experiments. This conductivity can therefore be experimentally determined by measuring the frequencies $\omega_\pm$ or the phase differences $\varphi_\pm$ of the modes.

Discussion and conclusions.—We have studied the dipole oscillation of a trapped gas of spin-orbit coupled cold atoms, and found that these oscillations can be used as an experimental probe for the anomalous Hall effect. In the experiments by Lin *et al.*[7] the SO coupling strength is $\alpha, \beta \simeq m\Lambda/\hbar^2$ and the Zeeman spin splitting is $\Delta \simeq 0.2 k_{\rm B} T$, using a temperature of $T \simeq 200$ nK. Taking values for α and β of this order, we find that the angle $\varphi_+ \simeq 0.03$ and that $(\omega_\pm - \omega_r)/\omega_r \simeq 10\%$ which appear to be observable. We also note that exciting the dipole mode by a sudden displacement of the trap will generally excite both modes, leading to a beating pattern with frequency $(\omega_+ - \omega_-)/2$, which may also be observed.

Up to this point we have not considered collisions between the atoms leading to a damping of collective oscillations. The harmonic nature of our system is explicitly broken by the SO coupling leading to a relaxation of the center-of-mass motion of the cloud (such relaxation is absent when the SO coupling is zero and Kohn's theorem prevails). This can be described phenomenologically by adding a term $-\boldsymbol{v}_0/\tau$ on the right-hand side of Eq. (11), which would lead to a damping of the dipole modes but does not affect the anomalous Hall conductivity. The frequencies of the damped system are $\Omega_\pm = \omega_\pm - i\gamma_\pm$ with the damping rate $\gamma_\pm$, up to first order in $1/\tau$, given by

$$\gamma_\pm = \frac{1}{\tau}\left(\frac{\omega_\pm^2 + \bar{H}_{xx}\omega_r^2}{2\omega_\pm^2 + 2\bar{H}_{xx}\omega_r^2 + \bar{B}_z^2 m^2 \omega_r^4} - 1\right).$$

We note that the relaxation time τ can, in principle, be calculated from the Boltzmann equation but considering this is, given the above remarks regarding its importance, beyond the scope of the present Letter.

In the adiabatic approximation that leads to the semiclassical equations of motion, spin directions transverse to the magnetic field $\boldsymbol{M}(\boldsymbol{p})$ are taken into account approximately and give rise to the anomalous-velocity terms. One could go beyond this adiabatic approximation and consider the (2×2)-distribution function $f_{\sigma\sigma'}(\boldsymbol{p})$ that allows for all possible spin directions. We have checked, by solving the Boltzmann equation for this distribution function in the collisionless limit, that our results for the anomalous Hall conductivity and the phases $\varphi_\pm$ are not altered.

Possible extensions of this work might consider the Bose-Einstein partially condensed phase for bosons, and the situation without Zeeman spin splitting Δ. In the latter case the AHE is absent, but there will be a spin Hall effect [25,26] that can be probed via the spin-dipole mode. (The spin Hall effect for cold atoms was proposed by Zhu *et al.* [27] for a cloud falling due to gravitation.) We also intend to investigate the effects of spin-orbit coupling on other collective modes, in particular, the quadrupole oscillation.

We would like to thank Henk Stoof and Sandy Fetter for useful remarks. This work was supported by the Stichting voor Fundamenteel Onderzoek der Materie (FOM), the Netherlands Organization for Scientific Research (NWO), and by the European Research Council (ERC).

[1] N. Nagaosa, J. Sinova, S. Onoda, A. H. MacDonald, and N. P. Ong, Rev. Mod. Phys. **82**, 1539 (2010).
[2] Y. Onose, T. Ideue, H. Katsura, Y. Shiomi, N. Nagaosa, and Y. Tokura, Science **329**, 297 (2010).
[3] C. Strohm, G. L. J. A. Rikken, and P. Wyder, Phys. Rev. Lett. **95**, 155901 (2005).
[4] L. Sheng, D. N. Sheng, and C. S. Ting, Phys. Rev. Lett. **96**, 155901 (2006).
[5] M. Onoda, S. Murakami, and N. Nagaosa, Phys. Rev. Lett. **93**, 083901 (2004).
[6] A. M. Dudarev, R. B. Diener, I. Carusotto, and Q. Niu, Phys. Rev. Lett. **92**, 153005 (2004).
[7] Y. J. Lin, K. Jiménez-García, and I. B. Spielman, Nature (London) **471**, 83 (2011).
[8] T. D. Stanescu, B. Anderson, and V. Galitski, Phys. Rev. A **78**, 023616 (2008).
[9] C. Wang, C. Gao, C.-M. Jian, and H. Zhai, Phys. Rev. Lett. **105**, 160403 (2010).
[10] M. Greiner, O. Mandel, T. Esslinger, T. Hänsch, and I. Bloch, Nature (London) **415**, 39 (2002).
[11] M. Greiner, C. Regal, and D. Jin, Nature (London) **426**, 537 (2003).
[12] Z. Hadzibabic, P. Krüger, M. Cheneau, B. Battelier, and J. Dalibard, Nature (London) **441**, 1118 (2006).
[13] V. Schweikhard, S. Tung, and E. A. Cornell, Phys. Rev. Lett. **99**, 030401 (2007).
[14] J. Billy, V. Josse, Z. Zuo, A. Bernard, B. Hambrecht, P. Lugan, D. Clément, L. Sanches-Palencia, P. Bouyer, and A. Aspect, Nature (London) **453**, 891 (2008).
[15] M.-C. Chang and Q. Niu, Phys. Rev. Lett. **75**, 1348 (1995); T. Jungwirth, Q. Niu, and A. H. MacDonald, *ibid.* **88**, 207208 (2002).
[16] N. A. Sinitsyn, J. Phys. Condens. Matter **20**, 023201 (2008).
[17] R. Karplus and J. M. Luttinger, Phys. Rev. **95**, 1154 (1954).
[18] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).
[19] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 146802 (2005); A. Roth, C. Brüne, H. Buhmann, L. W. Molenkamp, J. Maciejko, X.-L. Qi, and S.-C. Zhang, Science **325**, 294 (2009); M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).
[20] W. Kohn, Phys. Rev. **123**, 1242 (1961).

[21] E. I. Rashba, Fiz. Tverd. Tela (Leningrad) **2**, 1224 (1960) [Sov. Phys. Solid State **2**, 1109 (1960)].G. Dresselhaus, Phys. Rev. **100**, 580 (1955).
[22] J. Schliemann, J. C. Egues, and D. Loss, Phys. Rev. Lett. **90**, 146801 (2003).
[23] X.-J. Liu, M. F. Borunda, X. Liu, and J. Sinova, Phys. Rev. Lett. **102**, 046402 (2009).
[24] This linear-response regime can be experimentally accessed to determine the collective mode spectrum; see, e.g., D. S. Jin, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **77**, 420 (1996).
[25] S. Murakami, N. Nagaosa, and S.-C. Zhang, Science **301**, 1348 (2003).
[26] J. Sinova, D. Culcer, Q. Niu, N. A. Sinitsyn, T. Jungwirth, and A. H. MacDonald, Phys. Rev. Lett. **92**, 126603 (2004).
[27] S.-L. Zhu, H. Fu, C.-J. Wu, S.-C. Zhang, and L.-M. Duan, Phys. Rev. Lett. **97**, 240401 (2006).