Time transients in the quantum corrected Newtonian potential induced by a massless nonminimally coupled scalar field

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We calculate the one-loop graviton vacuum polarization induced by a massless, nonminimally coupled scalar field on Minkowski background. We make use of the Schwinger-Keldysh formalism, which allows us to study time dependent phenomena. As an application we compute the leading quantum correction to the Newtonian potential of a point particle. The novel aspect of the calculation is the use of the Schwinger-Keldysh formalism, within which we calculate the time transients induced by switching on the graviton-scalar coupling.

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I. INTRODUCTION

It has been known for a while that general relativity is, when viewed as a quantum theory, nonrenormalizable. Indeed, when gravity is coupled to scalar matter, new local counterterms are required to cancel the divergences occurring at one loop [1], while in the case of pure gravity, new counterterms occur at two loops [2,3]. That does not necessarily mean that quantizing gravity and studying perturbative corrections is meaningless, as long as one is interested in low energy effects, which are much below the Planck scale. This effective field theory view on quantum gravity has been fruitful, and has lead to useful results [4–6]. Observable effects occur primarily in cosmology, where as a result of tree-level quantization of gravity during inflation, one is led to the Newtonian potentials [7,8] that leave observable imprints on the relic cosmic microwave radiation, as well as the seeds for the large scale structure of the Universe. The question whether perturbative quantum gravitational effects can change the basic tree-level predictions of inflationary cosmology is still hotly debated [9,10].

On the other hand, there are well-known tree-level quantum effects occurring in curved space-times, the most famous one being the Hawking thermal radiation generated by black holes. There is even a flat space effect—the well known example being the Unruh effect—which constitutes thermal radiation seen by accelerating observers in Minkowski space. Neither Hawking nor the Unruh effect have so far been confirmed experimentally. The quantum corrections to these tree-level effects are as yet not well understood. In particular, it would be of interest to investigate whether such quantum corrections can induce a significant backreaction on the background space-time that would correct some undesirable features that occur when fields are quantized on black hole spaces, such as quasinormal modes (which essentially tell us that it is inconsistent to quantize massless fields on black hole backgrounds). The main message from these studies is that we shall be able to incorporate self-consistently the backreaction from fluctuating quantum fields on black hole backgrounds only if we allow space-times to become dynamical. In other words, a self-consistent semiclassical gravitational theory will require a (quantum) modification of the Birkhoff theorem (whose classical version states that a spherically symmetric distribution of matter can induce static, radially dependent, metrics only).

In this paper we do not address this very interesting question, but we point out that the Schwinger-Keldysh formalism [11–15] is suitable for such studies. As an example of how to apply the formalism, we provide a perturbative calculation of the quantum correction to the Newtonian potential generated by a massless scalar field. This represents a baby version of the more interesting problem of quantum corrections to black hole space-times. Our treatment generalizes the work of Park and Woodard [16], in that we consider not just the minimally coupled massless scalar field, but also include a nonminimal coupling of the scalar field to the Ricci curvature scalar. Another important difference is in that Park and Woodard assumed that gravity was turned on at $t_0 \to -\infty$, thus preventing any time transients from occurring. We, on the other hand, assume that the scalar field decouples from gravity at early times ($t < t_0$), and that the coupling turns on at $t = t_0$, where t_0 is some finite time. In practice, this can be realized, for example, by a Higgs mechanism, which gives a large mass m_ϕ to the scalar at $t < t_0$, and the mass vanishes at $t \geq t_0$. Such a scalar will effectively decouple from gravity at $t < t_0$ and on the length scales $L > 1/m_\phi$, and will induce a nontrivial gravitational effect for $t > t_0$. In this paper we demonstrate that the Schwinger-Keldysh formalism allows us to follow in a manifestly causal manner the gravitational field induced by such quantum scalar field fluctuations.

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Of course, our work is not the first to address quantum corrections to Newtonian potentials. Such studies were pioneered by Donoghue [17,18], and followed by many others [19,20], with differing results. The problem was again revived in the 2000s by Bjerrum-Bohr *et al.* [21,22], Butt [23], Faller [24] *etc.* where also some of the graviton vertex corrections were considered. Also the works of Dalvit, Mazzitelli, Satz and Alvarez [25,26] are of interest to us. Indeed, in Appendix C we show that there is a rather intricate connection between the methods used in Refs. [25,26] and our methods.

The paper is organized as follows: in Sec. II we briefly show how to obtain the classical Newtonian potential. Section III is devoted to calculation of the one-loop graviton vacuum polarization induced by a nonminimally coupled scalar field, while Sec. IV presents an application of the main result of Sec. III: the quantum one-loop correction to the Newtonian potential. In Sec. V we summarize our main results. In the Appendixes we review the Schwinger-Keldysh formalism (Appendix A); some subtleties in deriving the Newtonian potential of a point particle (Appendix B); we show how to expand the 2PI scalar bubble diagram to get the stress energy tensor and the graviton vacuum polarization tensor (Appendix C); and we present details of the one-loop vacuum polarization calculation (Appendix D).

Unless stated explicitly, we work in natural units where $\hbar = 1 = c$.

II. CLASSICAL NEWTONIAN POTENTIAL

We are interested in the gravitational response of a static, pointlike particle of a mass M in the particle's rest frame, with the classical stress energy tensor in Minkowski background:

$$T^{(c)}_{\mu\nu} = M\delta^0_\mu\delta^0_\nu\delta^{D-1}(\vec{x}), \tag{1}$$

where D denotes the dimension of space-time. For gravity we take the classical Einstein-Hilbert action:

$$S_{\rm EH} = -\frac{1}{\kappa^2}\int d^Dx\sqrt{-g}R \qquad (\kappa^2 = 16\pi G_N), \tag{2}$$

where R is the Ricci scalar, g is the determinant of the metric tensor $g_{\mu\nu}$, and G_N is the Newton constant. In order to get the classical Newtonian potential, we first expand the metric tensor around the flat Minkowski metric $\eta_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad \eta_{\mu\nu} = {\rm diag}(-1, \underbrace{1,\ldots,1}_{D-1}), \tag{3}$$

where, in general, $h_{\mu\nu} = h_{\mu\nu}(x)$, $x = (t, \vec{x})$. The full classical gravitational response to a point mass is obtained by varying the action (2) and setting it equal to the classical stress energy tensor, $\delta S_m/\delta h^{\mu\nu} = -\sqrt{-g}/2T_{\mu\nu}$. Here we are primarily interested in the leading gravitational response (Newtonian potential) to a point particle at rest, whose stress energy tensor is given by (1). To accomplish this we need the Einstein-Hilbert action to quadratic order in $h_{\mu\nu}$ (for detailed calculations see Appendix B):

$$S_{\rm EH} = \frac{1}{2\kappa^2}\int d^Dx\Big[h^{\mu\nu}L_{\mu\nu\rho\sigma}h^{\rho\sigma} + \mathcal{O}(h^3_{\mu\nu})\Big], \tag{4}$$

where $L_{\mu\nu\rho\sigma}$ stands for the Lichnerowicz operator in Minkowski background:

$$L_{\mu\nu\rho\sigma} = \partial_{(\rho}\eta_{\sigma)(\mu}\partial_{\nu)} - \frac{1}{2}\eta_{\mu(\rho}\eta_{\sigma)\nu}\partial^2 - \frac{1}{2}(\eta_{\mu\nu}\partial_\rho\partial_\sigma + \eta_{\rho\sigma}\partial_\mu\partial_\nu) + \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma}\partial^2. \tag{5}$$

For a later use we rewrite the Lichnerowicz operator as a combination of two simpler operators:

$$L_{\mu\nu\rho\sigma} = D_{\mu\nu\rho\sigma} - \frac{1}{2}\eta_{\mu\nu}D_{\rho\sigma}, \tag{6}$$

where

$$\begin{aligned} D_{\rho\sigma} &= \partial_\rho\partial_\sigma - \eta_{\rho\sigma}\partial^2, \\ D_{\mu\nu\rho\sigma} &= \partial_{(\rho}\eta_{\sigma)(\mu}\partial_{\nu)} - \frac{1}{2}\eta_{\mu(\rho}\eta_{\sigma)\nu}\partial^2 - \frac{1}{2}\eta_{\rho\sigma}\partial_\mu\partial_\nu. \end{aligned} \tag{7}$$

The Lichnerowicz operator (5) possesses the following symmetries: it is symmetric under the exchange of the first two indices ($\mu \leftrightarrow \nu$); of the last two indices ($\rho \leftrightarrow \sigma$), as well as under the exchange of the first two and the last two indices ($\mu\nu \leftrightarrow \rho\sigma$).

Now varying the action (4) with the stress energy tensor (1) yields

$$L_{\mu\nu\rho\sigma}h^{\rho\sigma}(x) = \frac{\kappa^2}{2}\delta^0_\mu\delta^0_\nu M\delta^{D-1}(\vec{x}). \tag{8}$$

The same equation can be obtained by linearizing the Einstein equation around the Minkowski background, which is a simpler procedure. As we show in Appendix B, the Newtonian potential in the longitudinal (Newton) gauge Eq. (8) gives the solution

$$h^{(0)}_{00}(x) = h^{(0)}_{ii}(x) = \frac{2G_NM}{r}, \qquad (i = 1, 2, 3), \tag{9}$$

where the superscript (0) emphasizes that the metric components $h^{(0)}_{\mu\nu}$ refer to the classical solution. In the following two sections we show how to use this solution to construct the quantum-corrected one-loop Newtonian potential.

III. GRAVITON VACUUM POLARIZATION

Here we derive the leading quantum contribution to the graviton vacuum polarization tensor due to a nonminimally coupled massless scalar field φ, with the action

$$S_\varphi = \int d^Dx\sqrt{-g}\Big\{-\frac{1}{2}(\partial_\mu\varphi)(\partial_\nu\varphi)g^{\mu\nu} - \frac{1}{2}\xi R\varphi^2\Big\}, \tag{10}$$

where ξ measures the coupling strength of φ to gravity through the Ricci scalar R. In this paper we focus on the quantum corrections around Minkowski space, hence we expand the metric tensor around the flat Minkowski metric (*cf.* Eq. (3)):

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}. \tag{11}$$

Since we are here primarily interested in the one-loop graviton vacuum polarization, in (11) we have extracted the gravitational coupling constant κ defined in Eq. (2), as κ^2 can be used as the loop counting parameter of perturbative gravity. Note that $h_{\mu\nu}(x)$ in Eq. (11) can be understood to contain both classical and quantum contributions to the Newtonian potential, which is in the spirit of the rest of the paper. One can write Eq. (11) in the more general form, $g_{\mu\nu} = g^{(b)}_{\mu\nu} + \kappa h_{\mu\nu}$, where $g^{(b)}_{\mu\nu}$ denotes a classical background metric around which one expands. In the context of this work, the natural choices for $g^{(b)}_{\mu\nu}$ are the metric of the classical Newtonian potential, $g^{(b)}_{\mu\nu} = \eta_{\mu\nu} + h^{(0)}_{\mu\nu}$ with the elements of $h^{(0)}_{\mu\nu}$ given in Eq. (9), or by the Schwarzschild black hole metric. Since choosing these background metrics would entail significant technical complications, we leave their treatment for a future work. In our model (2) and (10) the graviton acquires quantum contributions both from the graviton and scalar quantum fluctuations. To calculate the graviton one-loop vacuum polarization one needs the cubic and quartic graviton vertices, which can be quite straightforwardly extracted from Eq. (B1). For simplicity in this paper we focus on the one-loop scalar contribution to the graviton self-energy (vacuum polarization). To this end we need the cubic and quartic scalar-graviton vertices, which can be easily extracted from the scalar action (10):

$$S_{h\varphi\varphi} = -\frac{\kappa}{2}\int d^D x\bigg\{\bigg(\frac{1}{2}\eta^{\mu\nu}\eta_{\rho\sigma} - \delta^{(\mu}_{(\rho}\delta^{\nu)}_{\sigma)}\bigg)(\partial_\mu\varphi(x))(\partial_\nu\varphi(x)) + \xi(D_{\rho\sigma}\varphi^2(x))\bigg\}h^{\rho\sigma}(x) \tag{12}$$

$$\begin{aligned} S_{h^2\varphi^2} = -\frac{\kappa^2}{2}\int d^D x\bigg\{ & h^{\mu\nu}(x)\bigg[\bigg(\frac{1}{8}\eta^{\alpha\beta}\eta_{\mu\nu}\eta_{\rho\sigma} - \frac{1}{4}\eta_{\mu\nu}\delta^{\alpha}_{(\rho}\delta^{\beta}_{\sigma)} - \frac{1}{4}\eta_{\rho\sigma}\delta^{\alpha}_{(\mu}\delta^{\beta}_{\nu)} - \frac{1}{4}\eta^{\alpha\beta}\eta_{\mu(\rho}\eta_{\sigma)\nu} + \delta^{\alpha}_{(\mu}\eta_{\nu)(\rho}\delta^{\beta}_{\sigma)}\bigg) \\ & \times(\partial_\alpha\varphi(x))(\partial_\beta\varphi(x))\bigg]h^{\rho\sigma}(x) + \xi\varphi^2(x)\bigg[\bigg(\frac{h}{2}\bigg)(\partial_\mu\partial_\nu h^{\mu\nu} - \partial^2 h) - h_{\rho\sigma}(2\partial^\rho\partial^\mu h^{\sigma}{}_{\mu} - \partial^2 h^{\rho\sigma}) + h^{\mu\nu}\partial_\mu\partial_\nu h \\ & - \frac{3}{2}(\partial_\rho h^{\rho\sigma})(\partial^\mu h_{\mu\sigma}) - \frac{1}{4}(\partial_\mu h)(\partial^\mu h) + (\partial^\mu h)(\partial^\nu h_{\mu\nu}) + \frac{3}{4}(\partial^\mu h^{\nu\sigma})(\partial_\mu h_{\nu\sigma})\bigg]\bigg\}, \end{aligned} \tag{13}$$

where we made use of Eqs. (B3)–(7).

In the spirit of the Schwinger-Keldysh formalism, cubic and quartic parts contribute to the interaction action as

$$\begin{aligned} & S_{\rm int}[\varphi^+, h^+_{\alpha\beta}, \varphi^-, h^-_{\alpha\beta}] \\ & = S_{h\varphi\varphi}[\varphi^+, h^+_{\alpha\beta}] - S_{h\varphi\varphi}[\varphi^-, h^-_{\alpha\beta}] \\ & \quad + S_{h^2\varphi^2}[\varphi^+, h^+_{\alpha\beta}] - S_{h^2\varphi^2}[\varphi^-, h^-_{\alpha\beta}]. \end{aligned} \tag{14}$$

These interactions can be used to generate the graviton tadpole and the graviton vacuum polarization induced by the one-loop scalar fluctuations. The scalar diagram that contributes to the graviton tadpole is shown in Fig. 1.

The three scalar field diagrams that contribute to the one-particle irreducible (1PI) graviton vacuum polarization at the one-loop order are shown in Fig. 2. The first diagram is the local contribution generated by the quartic action (13), the second is the nonlocal contribution generated by the cubic action (12) squared, and the third is the counterterm.

In this paper we work with the 1PI effective action, which consists of the tree-level (Einstein-Hilbert) part, the matter free action, and the one-loop contributions with zero, one, two, three, etc. graviton insertions. The one-loop bubble diagrams with no graviton insertions are shown in Fig. 3. Formally, we can write the one-loop 1PI contributions to the effective action as

$$\begin{aligned} & \Gamma_1[h_{\mu\nu}, \Delta] \\ & = -\frac{\imath}{2}{\rm Tr}\ln[\Delta^{aa}(x;x)] - \frac{\imath}{2}{\rm Tr}\ln[{}_{\mu\nu}\Delta^{aa}_{\rho\sigma}(x;x)] \\ & \quad -\frac{1}{2}\sum_{a,b=\pm}\int d^D x h^{\mu\nu}_a(x) a\delta^{ab}T^{(1)ab}_{\mu\nu}(x) \\ & \quad -\frac{1}{2}\sum_{a,b=\pm}\int d^D x\int d^D x' h^{\mu\nu}_a(x)[{}^a_{\mu\nu}\Pi^b_{\rho\sigma}](x;x')h^{\rho\sigma}_b(x') \\ & \quad -\sum_{n=3}^{\infty}\frac{1}{n!}\sum_{a_1,a_2,\ldots,a_n=\pm}\int d^D x_1 h^{\mu_1\nu_1}_{a_1}(x_1)\int d^D x_2 h^{\mu_2\nu_2}_{a_2}(x_2)\cdots \\ & \quad \times\int d^D x_n h^{\mu_n\nu_n}_{a_n}(x_n)V^{a_1a_2\cdots a_n}_{\mu_1\nu_1\mu_2\nu_2\ldots\mu_n\nu_n}(x_1;x_2;\ldots;x_n), \end{aligned} \tag{15}$$

where $\Delta^{ab}(x;x')$ denotes the scalar Schwinger-Keldysh propagator, which for a massless scalar field on Minkowski space can be found in Eqs. (A15) and (A16) in Appendix A, Tr denotes both a trace over the space-time indices and an integration over space-time, $T^{(1)aa}_{\mu\nu}(x)$ denotes the tadpole in Fig. 1, $[{}^{\pm}_{\mu\nu}\Pi^{\pm}_{\rho\sigma}](x;x')$ is the graviton vacuum polarization tensor, for which the contributing one-loop diagrams are shown in Fig. 2, and $V^{a_1a_2\cdots a_n}_{\mu_1\nu_1\mu_2\nu_2\ldots\mu_n\nu_n}(x_1;x_2;\ldots;x_n)$ stand for the one-loop vertex corrections with $n\geq 3$ graviton insertions. An example of a vertex correction diagram is shown in Fig. 4. In order to

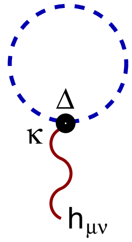

FIG. 1 (color online). The graviton one-loop tadpole. Only the scalar loop contribution is shown. The scalar propagator Δ is the blue dashed line, while the graviton insertion $h_{\mu\nu}$ is the red solid wavy line, and $\kappa = \sqrt{16\pi G_N}$ is the gravitational coupling constant.

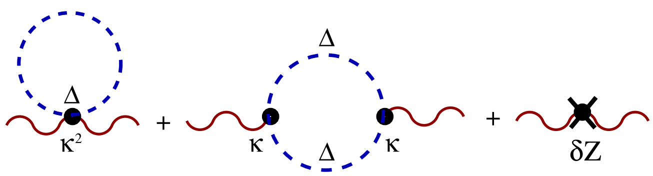

FIG. 2 (color online). The scalar diagrams that contribute to the graviton one-loop vacuum polarization. δZ stands for the one-loop counterterms.

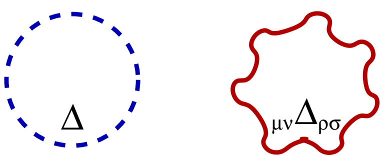

FIG. 3 (color online). The scalar and graviton one-loop bubble diagrams. $\rho\sigma\Delta_{\mu\nu}$ denotes the graviton propagator.

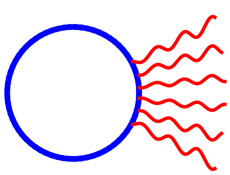

FIG. 4 (color online). A one-loop vertex diagram with one scalar loop and six graviton insertions.

get the complete one-loop correction to the graviton, one needs to evaluate all of the diagrams contributing to the one-loop effective action (15). This is hard. However, the corrections with more and more graviton insertions are expected to be progressively smaller and smaller, and hence it is often enough to calculate the leading nonvanishing correction. In this paper we evaluate this leading order correction, which corresponds to two-graviton insertions. In fact, all leading order quantum effects on the graviton are embodied in the graviton vacuum polarization tensor, and the contributing diagrams are shown in Fig. 2.

The effective action (15) can be derived by a Legendre transform of the free action. Here we sketch a heuristic derivation of (15), which misses the scalar and graviton bubble diagrams. The loop contributions to the effective action can loosely be defined as an expectation value of $e^{\imath S_{\rm int}}$, where $S_{\rm int}$ is the interaction action given in (14):

$$e^{\imath\Gamma_1} = \langle e^{\imath S_{\rm int}}\rangle = 1 + \imath\langle S^+_{h\varphi\varphi} - S^-_{h\varphi\varphi}\rangle + \imath\langle S^+_{h^2\varphi^2} - S^-_{h^2\varphi^2}\rangle - \frac{1}{2}\langle (S^+_{h\varphi\varphi} - S^-_{h\varphi\varphi})^2\rangle + \mathcal{O}((h^\pm_{\mu\nu})^3), \quad (16)$$

where $\langle\cdot\rangle$ denotes an expectation value with respect to a chosen state $|\Omega\rangle$, which in our case is the Minkowski vacuum, corrected for the fact that the interaction with the graviton switches on at a finite time t_0. This difference allows us to investigate the thereby induced time transients and, in particular, how causality affects and limits the growth in time of the terms induced by quantum effects. In Eq. (16) we introduced the notation, $S^\pm_{h\varphi\varphi} = S_{h\varphi\varphi}[\varphi^\pm, h^\pm_{\mu\nu}]$ and $S^\pm_{h^2\varphi^2} = S_{h^2\varphi^2}[\varphi^\pm, h^\pm_{\mu\nu}]$. In this work we neglect the vertex corrections to Γ_1, which occur at the order $\mathcal{O}(h^3_{\mu\nu})$ and higher. It would be of interest to investigate whether such corrections change any of the results presented in this work. For a discussion of the role of vertex corrections to the Newtonian potential see Refs. [21,27]. The first nontrivial term on the right-hand-side of (16) yields the one-loop graviton tadpole of Fig. 1, and thus the scalar one-loop contribution to the stress energy. The second contribution yields the local one-loop contribution to the graviton self-energy, shown in the first diagram of Fig. 2. The disconnected part of the third contribution is the tadpole squared, which is a part of the geometric series that is needed to reconstruct $\exp(\imath\Gamma_1)$; while the connected part of the third contribution represents the nonlocal contribution to the one-loop self-energy, diagrammatically shown by the middle diagram of Fig. 2, etc.

More economical is the two-particle irreducible (2PI) formalism, in which one formally writes the effective action in terms of scalar propagators $\Delta(x; x'; [g^{(b)}_{\mu\nu}])$ in a general background $g^{(b)}_{\mu\nu}$. The one-loop contributions to the 2PI effective action are then the scalar and graviton bubble diagrams from Fig. 3,

$$\Gamma^{(1)}_{2PI}[\Delta, {}_{\mu\nu}\Delta_{\rho\sigma}] = -\frac{\imath}{2}\,{\rm Tr}\ln[\Delta^{aa}(y; y; [g^{(b)}_{\mu\nu}])] - \frac{\imath}{2}\,{\rm Tr}\ln[{}_{\mu\nu}\Delta^{aa}_{\rho\sigma}(y; y; g^{(b)}_{\mu\nu})], \quad (17)$$

while the two-loop contributions come from the diagrams shown in Fig. 5, etc. For example, Refs. [25,26] vary the first term in Eq. (17) with respect to $g^{(b)}_{\mu\nu}$, and insert this into the classical linearized equation for the graviton with a static point mass source to study the leading quantum correction to the Newtonian potential, which in that approach originates from the quadratic term in the expansion

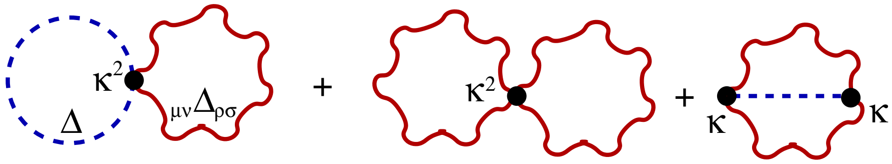

FIG. 5 (color online). The two-loop diagrams contributing to the 2PI effective action.

of the scalar propagator Δ^{ab} in Eq. (17) in powers of $h_{\mu\nu} = g^{(b)}_{\mu\nu} - \eta_{\mu\nu}$. In Appendix C we show that that approach must lead to the same answer for the quantum-corrected Newtonian potential as obtained from the one-loop graviton vacuum polarization, which is the approach advocated in this paper. A detailed comparison shows that the results indeed agree, representing a nontrivial check of our work.

Let us now go back and discuss various contributions to Eq. (15). When the scalar and graviton propagators are calculated on Minkowski vacuum, they do not depend on $h_{\mu\nu}$, and hence the one-loop bubble terms in (15) contribute to the effective action as a constant, and can be neglected. Therefore, the one-loop graviton effective action in our theory has the following form (*cf.* also Eq. (C6) in Appendix C):

$$\Gamma_1[h^{\pm}_{\mu\nu}] = S_{\rm EH}[h^{\pm}_{\mu\nu}] - \frac{1}{2}\int d^Dx h^{\mu\nu}_{\pm}(x) T^{(1)\pm}_{\mu\nu}(x) - \frac{1}{2}\int d^Dx d^Dx' h^{\mu\nu}_{\pm}(x)[^{\pm}_{\mu\nu}\Pi^{\pm}_{\rho\sigma}](x;x') h^{\rho\sigma}_{\pm}(x') + \mathcal{O}((h^{\pm}_{\mu\nu})^3). \quad (18)$$

We shall first argue that the tadpole and the local contribution to the graviton vacuum polarization vanish and then evaluate the nonlocal contribution to the one-loop graviton vacuum polarization.

A. Local contributions

Since we work in dimensional regularization, the local diagram—which yields a quadratic divergence—is automatically subtracted, and hence will not contribute. To see this in some detail, from Eq. (16) we infer that the tadpole contribution to the effective action yields

$$\Gamma_{\rm tadpole} = \Gamma_{h\varphi\varphi}[h^{\mu\nu}_{\pm}] = \langle S^{+}_{h\varphi\varphi} - S^{-}_{h\varphi\varphi}\rangle, \quad (19)$$

where $S^{\pm}_{h\varphi\varphi}$ are defined in Eqs. (12) and (14). If we take a look at the structure of $S^{\pm}_{h\varphi\varphi}$ in Eq. (12) we see that the contributing terms can be written as

$$\langle\varphi(x)^2\rangle = \imath\Delta_{++}(x;x) \quad \text{and}$$
$$\langle\partial_\alpha\varphi(x)\partial_\beta\varphi(x)\rangle = \partial_\alpha\partial'_\beta \imath\Delta_{++}(x;x')|_{x\to x'}. \quad (20)$$

The first term in Eq. (20) equals the coincident ($x = x'$) Feynman propagator $\imath\Delta_{++} \propto 1/\Delta x^{D-2}_{++}$ (see Eq. (A15)), which is in dimensional regularization defined as the analytic extension of the propagator where it is defined, i.e. in $\Re[D] < 2$, where $\imath\Delta_{++}(x;x) = 0$. Hence, this term vanishes in the dimensional regularization. In order to show that the contribution of the second term vanishes, first note that $\partial_\alpha\partial'_\beta\imath\Delta_{++}(x;x') = \delta^0_\alpha\delta^0_\beta\imath\delta^D(x-x')$. This implies that when either $\alpha \neq 0$ or $\beta \neq 0$, this term vanishes, since in dimensional regularization it equals the analytic extension of $\partial_\alpha\partial'_\beta\imath\Delta_{++}(x;x')|_{x\to x'}$ from $\Re[D] < 0$ (where it vanishes) to the whole complex plane. When both $\alpha = 0 = \beta$, then this term yields $\imath\delta^D(x-x')|_{x'\to x} = \imath\delta^D(0)$, which can be shown to vanish in dimensional regularization. We have thus shown that the tadpole vanishes in dimensional regularization:

$$\Gamma_{\rm tadpole} = 0. \quad (21)$$

Let us next consider the local contribution to the one-loop graviton vacuum polarization:

$$\Gamma_{\rm local} = \Gamma_{h^2\varphi^2}[h^{\mu\nu}_{\pm}] = \langle S^{+}_{h^2\varphi^2} - S^{-}_{h^2\varphi^2}\rangle. \quad (22)$$

From the structure of the quartic action (13) we see that the same terms as in Eq. (20) contribute to $\Gamma_{\rm local}$, and therefore, in dimensional regularization,

$$\Gamma_{\rm local} = 0. \quad (23)$$

B. Nonlocal contributions

As previously elaborated, the only term in Eq. (16) that contributes to the one-loop effective action (18) is contained in the Wick-contracted part of the expression $\imath\delta\Gamma_{h\varphi\varphi} \equiv -\frac{1}{2}\langle(S^{+}_{h\varphi\varphi} - S^{-}_{h\varphi\varphi})^2\rangle$:

$$\begin{aligned}
&\delta\Gamma_{h\varphi\varphi}[\varphi^{\pm}, h^{\pm}_{\alpha\beta}] \\
&= -\frac{1}{2}\int d^Dx\int d^Dx' h^{\mu\nu}_{\pm}(x)[^{\pm}_{\mu\nu}\Pi^{\pm}_{\rho\sigma}](x;x')h^{\rho\sigma}_{\pm}(x') \\
&= \frac{\imath}{2}\Big(\frac{\pm\kappa}{2}\Big)\Big(\frac{\pm\kappa}{2}\Big)\int d^Dx d^Dx' h^{\mu\nu}_{\pm}(x)\Big\langle\Big[\Big(\frac{1}{2}\eta^{\alpha\beta}\eta_{\mu\nu} - \delta^{(\alpha}_{(\mu}\delta^{\beta)}_{\nu)}\Big) \\
&\quad\times(\partial_\alpha\varphi_{\pm}(x))(\partial_\beta\varphi_{\pm}(x)) + \xi(D_{\mu\nu}\varphi^2_{\pm}(x))\Big] \\
&\quad\times\Big[\Big(\frac{1}{2}\eta^{\alpha'\beta'}\eta_{\rho\sigma} - \delta^{(\alpha'}_{(\rho}\delta^{\beta')}_{\sigma)}\Big)(\partial'_{\alpha'}\varphi_{\pm}(x'))(\partial'_{\beta'}\varphi_{\pm}(x')) \\
&\quad+ \xi(D'_{\alpha\beta}\varphi^2_{\pm}(x'))\Big]\Big\rangle h^{\rho\sigma}_{\pm}(x'),
\end{aligned} \quad (24)$$

where the primed partial derivatives stand for derivatives with respect to x'. Next, we perform a detailed calculation of the vacuum polarization tensor in Appendix D and note that the derivative operators appearing in the vacuum polarization tensor (D9) can be written in the useful form

$$D_{\alpha\beta\mu\nu}D^{\alpha\beta}{}_{\rho\sigma} = L_{\mu\nu\alpha\beta}\Big(D^{\alpha\beta}{}_{\rho\sigma} - \frac{1}{2}\eta^{\alpha\beta}D_{\rho\sigma}\Big) \qquad (25)$$

$$D_{\mu\nu}D_{\rho\sigma} = L_{\mu\nu\alpha\beta}(-\eta^{\alpha\beta}D_{\rho\sigma}), \qquad (26)$$

where $L_{\mu\nu\alpha\beta}$ is the Lichnerowicz operator (5)–(7). Making use of the results from Appendix D, we can split the graviton polarization tensor (D9) into a nonlocal finite part and a local divergent part as follows:

$$\iota[{}^{\pm}_{\mu\nu}\Pi^{\pm}_{\rho\sigma}](x;x') = \iota[{}^{\pm}_{\mu\nu}\Pi^{\pm}_{\rho\sigma}]^{(\mathrm{ren})}(x;x') + \iota[{}^{\pm}_{\mu\nu}\Pi^{\pm}_{\rho\sigma}]^{(\mathrm{div})}(x;x') \qquad (27)$$

where the renormalized graviton one-loop vacuum polarization tensor reads

$$\iota[{}^{\pm}_{\mu\nu}\Pi^{\pm}_{\rho\sigma}]^{(\mathrm{ren})}(x;x') = -(\pm)(\pm)\frac{\kappa^2}{30720\pi^4}L_{\mu\nu\alpha\beta}\Big\{D^{\alpha\beta}{}_{\rho\sigma} - \eta^{\alpha\beta}D_{\rho\sigma} \times\Big[\frac{1}{6}+30\Big(\xi-\frac{1}{6}\Big)^2\Big]\Big\}\partial^4[\ln^2(\mu^2\Delta x^2_{\pm\pm}) - 2\ln(\mu^2\Delta x^2_{\pm\pm})]. \qquad (28)$$

This is the generalization of the graviton vacuum polarization induced by a minimally coupled scalar ($\xi = 0$) calculated by Park and Woodard in [16] to the case of a massless, nonminimally coupled, scalar ($\xi \neq 0$), and it is one of the main results of this work. When $\xi = 0$, our result (28) is in perfect agreement with [16], representing a nontrivial check of our work.

The divergent part of the vacuum polarization tensor (D1) can be extracted by inserting the divergent part of Eq. (D13) into (D9). The result is

$$[{}^{\pm}_{\mu\nu}\Pi^{\pm}_{\rho\sigma}]^{(\mathrm{div})}(x;x') = (\sigma^3)^{\pm\pm}\frac{\kappa^2\Gamma(\frac{D}{2})\mu^{D-4}}{64\pi^{D/2}(D^2-1)(D-2)(D-3)(D-4)} \times\Big\{4D_{\alpha\beta\mu\nu}D^{\alpha\beta}{}_{\rho\sigma} + \Big[-\frac{D}{D-1}+8(D^2-1) \times\Big(\xi-\frac{D-2}{4(D-1)}\Big)^2\Big]D_{\mu\nu}D_{\rho\sigma}\Big\}\delta^D(x-x'). \qquad (29)$$

As we show in the next section, this term has the right structure such that it can be subtracted by local counterterms, which ought to be added to the effective action to complete the one-loop renormalization program.

C. Renormalization

In order to cancel the divergent part of the graviton vacuum polarization (29), from the form of the effective action (18) we see that it should comprise the counterterm (s) of the form

$$\delta\Gamma_{\mathrm{c.t.}} \propto \int d^Dxd^Dx'h^{\mu\nu}_{\pm}(x)[{}^{\pm}_{\mu\nu}\Pi^{\pm}_{\rho\sigma}]^{(\mathrm{div})}(x;x')h^{\rho\sigma}_{\pm}(x'). \qquad (30)$$

In spite of the appearance, this counterterm (30) is local (because of the delta function in Eq. (29)), as it should be.

As we shall now show, two counterterms that are quadratic in the curvature tensors are needed to subtract the divergences in (29). To see this, first note that the linearized form of Ricci scalar and Ricci tensor are (cf. Eq. (B1))

$$R^{(\mathrm{lin})}[h^{\pm}_{\mu\nu}] = \partial^\mu\partial^\nu h^{\pm}_{\mu\nu} - \partial^2 h_{\pm} = D_{\mu\nu}h^{\mu\nu}_{\pm} \qquad (31)$$

$$R^{(\mathrm{lin})}_{\mu\nu}[h^{\pm}_{\mu\nu}] = \Big(\partial_{(\rho}\eta_{\sigma)(\mu}\partial_{\nu)} - \frac{1}{2}\eta_{\mu(\rho}\eta_{\sigma)\nu}\partial^2 - \frac{1}{2}\eta_{\rho\sigma}\partial_\mu\partial_\nu\Big)h^{\rho\sigma}_{\pm} = D_{\mu\nu\rho\sigma}h^{\rho\sigma}_{\pm}, \qquad (32)$$

where we expanded the metric tensor around Minkowski space, $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$. To quadratic order in $h_{\mu\nu}$, the following two counterterms:

$$\delta\Gamma_{\mathrm{c.t.}} = \alpha_1\sum_{\pm}(\pm)\int d^D\sqrt{-g}xR[g^{\pm}_{\alpha\beta}]^2 + \alpha_2\sum_{\pm}(\pm)\int d^Dx\sqrt{-g}R_{\mu\nu}[g^{\pm}_{\alpha\beta}]R^{\mu\nu}[g^{\pm}_{\alpha\beta}], \qquad (33)$$

and can be recast as

$$\delta\Gamma_{\mathrm{c.t.}} = \sum_{\pm\pm}(\sigma^3)^{\pm\pm}\kappa^2\int d^Dxd^Dx'h^{\mu\nu}_{\pm}(x)[(\alpha_1 D_{\mu\nu}D'_{\rho\sigma} + \alpha_2 D_{\alpha\beta\mu\nu}D'^{\alpha\beta}{}_{\rho\sigma})\delta^D(x-x')]h^{\rho\sigma}_{\pm}(x'). \qquad (34)$$

Comparing this with Eqs. (29) and (29), we can read off the coefficients α_1 and α_2 in (34):

$$\alpha_1 = \frac{\Gamma(\frac{D}{2})\mu^{D-4}}{128\pi^{D/2}(D^2-1)(D-2)(D-3)(D-4)} \times\Big[-\frac{D}{D-1}+8(D^2-1)\Big(\xi-\frac{D-2}{4(D-1)}\Big)^2\Big] + \alpha_1^{(\mathrm{fin})} \qquad (35)$$

$$\alpha_2 = \frac{\Gamma(\frac{D}{2})\mu^{D-4}}{32\pi^{D/2}(D^2-1)(D-2)(D-3)(D-4)} + \alpha_2^{(\mathrm{fin})}, \qquad (36)$$

where $\alpha_1^{(\mathrm{fin})}$ and $\alpha_2^{(\mathrm{fin})}$ represent the finite parts of the counterterms, which are to be fixed by measurements. One often uses the minimal subtraction scheme, in which one expands the coefficients α_1 and α_2 around $1/(D-4)$. In this case their divergent parts are

$$\alpha_1^{(\mathrm{div})} = \frac{1}{960\pi^2}\Big[-\frac{1}{3} + 30\Big(\xi - \frac{1}{6}\Big)^2\Big]\frac{\mu^{D-4}}{D-4}, \quad \text{and}$$
$$\alpha_2^{(\mathrm{div})} = \frac{1}{960\pi^2}\frac{\mu^{D-4}}{(D-4)} \tag{37}$$

and—when $\xi = 0$—agrees with Eqs. (3.3) and (3.34) of Ref. [1].[1] This completes our discussion of renormalization.

D. Retarded self-energy

By varying the action (18) with respect to the fields $h_+^{\mu\nu}$ or $h_-^{\mu\nu}$ and setting $h_+^{\mu\nu} = h_-^{\mu\nu} = h^{\mu\nu}$:

$$\frac{\delta\Gamma[h_\pm^{\mu\nu}]}{\delta h_\pm^{\mu\nu}}\Bigg|_{h_+^{\mu\nu}=h_-^{\mu\nu}=h^{\mu\nu}} = 0, \tag{38}$$

the quantum-corrected equation of motion for the metric perturbation $h_{\mu\nu}$ is

$$L_{\mu\nu\rho\sigma}h^{\rho\sigma}(x) + \int d^4x'[{}_{\mu\nu}\Pi^{\mathrm{ret}}_{\rho\sigma}](x;x')h^{\rho\sigma}(x') + \mathcal{O}((h_{\mu\nu})^2) = \frac{\kappa^2}{2}\delta^0_\mu\delta^0_\nu M\delta^3(\vec{x}), \tag{39}$$

where

$$[{}_{\mu\nu}\Pi_{\rho\sigma}]^{\mathrm{ret}}(x;x') = [{}^+_{\mu\nu}\Pi^+_{\rho\sigma}]^{(\mathrm{ren})}(x;x') + [{}^+_{\mu\nu}\Pi^-_{\rho\sigma}]^{(\mathrm{ren})}(x;x') \tag{40}$$

is the retarded graviton vacuum polarization, which we use in the next section to compute the quantum-corrected Newtonian potential.

IV. THE QUANTUM-CORRECTED NEWTONIAN POTENTIAL

Rather than attempting to consistently solve Eq. (39), we shall solve it perturbatively, i.e. we make the following perturbative *Ansatz* for the graviton field:

$$h_{\mu\nu}(x) = h^{(0)}_{\mu\nu}(x) + h^{(1)}_{\mu\nu}(x), \tag{41}$$

where $h^{(0)}_{\mu\nu}$ stands for the classical solution (9) obtained by solving the classical part (8) of the full equation of motion (39).

To find the quantum correction $h^{(1)}_{\mu\nu}$ we need to solve the perturbative equation

$$L_{\mu\nu\rho\sigma}h^{\rho\sigma(1)}(x) + \int d^4x'[{}_{\mu\nu}\Pi_{\rho\sigma}]^{\mathrm{ret}}(x;x')h^{\rho\sigma(0)}(x') = 0, \tag{42}$$

where $[{}_{\mu\nu}\Pi_{\rho\sigma}]^{\mathrm{ret}}$ is given in Eqs. (28) and (40).[2]

In order to evaluate $[{}_{\mu\nu}\Pi_{\rho\sigma}]^{\mathrm{ret}}$, it is useful to split the logarithm function in (28) into its real and imaginary parts:

$$\begin{aligned}\ln(\mu^2\Delta x^2_{++}) &= \ln[\mu^2(-\Delta t^2 + \Delta r^2 + \iota\epsilon)]\\ &= \ln|\mu^2(-\Delta t^2 + \Delta r^2)| + \iota\pi\Theta(\Delta t^2 - \Delta r^2)\\ \ln(\mu^2\Delta x^2_{+-}) &= \ln[\mu^2(-\Delta t^2 + \Delta r^2 - \iota\epsilon\,\mathrm{sign}(\Delta t))]\\ &= \ln|\mu^2(-\Delta t^2 + \Delta r^2)|\\ &\quad - \iota\pi\Theta(\Delta t^2 - \Delta r^2)\mathrm{sign}(\Delta t),\end{aligned} \tag{43}$$

with the abbreviations $\Delta r = \| \vec{x} - \vec{x}' \|$ and $\Delta t = t - t'$. The retarded graviton vacuum polarization is then

$$\begin{aligned}[{}_{\mu\nu}\Pi^{\mathrm{ret}}_{\rho\sigma}](x;x') = -\frac{\kappa^2}{7680\pi^3} &\times L_{\mu\nu\alpha\beta}\Big\{D^{\alpha\beta}_{\rho\sigma} - \eta^{\alpha\beta}D_{\rho\sigma}\\ &\times\Big[\frac{1}{6} + 30\Big(\xi - \frac{1}{6}\Big)^2\Big]\Big\}\partial^4\Theta(\Delta t^2 - \Delta r^2)\\ &\times\Theta(\Delta t)[\ln|\mu^2(\Delta r^2 - \Delta t^2)| - 1].\end{aligned} \tag{44}$$

Recalling that $h^{(0)\rho\sigma}(x') = 2G_N M\delta^{\rho\sigma}/r'$ $(r' = \| \vec{x}' \|)$ for $\rho = \sigma = 0, \ldots, 3$ and inserting it in Eq. (42) together with (44) we get

$$\begin{aligned}L_{\mu\nu\rho\sigma}h^{\rho\sigma(1)}(x) = \frac{\kappa^2 G_N M}{3840\pi^3}L_{\mu\nu\rho\sigma}&\Big\{D^{\rho\sigma}{}_{\alpha\beta} - \eta^{\rho\sigma}D_{\alpha\beta}\\ &\times\Big(\frac{1}{6} + 30\Big(\xi - \frac{1}{6}\Big)^2\Big)\Big\}\partial^4F^{\alpha\beta},\end{aligned} \tag{45}$$

where

$$F^{\alpha\beta} \equiv F(r,t) = \int_{t_0}^t dt'\int d^3x'\Theta(\Delta t - \Delta r) \times[\ln\mu^2(\Delta t^2 - \Delta r^2) - 1]\frac{1}{r'}. \tag{46}$$

Here we are not interested in homogeneous solutions of the operator $L_{\mu\nu\rho\sigma}$, and thus $L_{\mu\nu\rho\sigma}$ can be dropped out from both sides of Eq. (45), by which (45) simplifies considerably.

It is much easier to carry out the integration in (45) if we change the variable of integration $\vec{x}'$ to $\Delta\vec{r} = \vec{x} - \vec{x}'$, i.e.,

[1] In fact, Veltman and 't Hooft get a factor two larger result for α_1 and α_2, but they work with a complex scalar field, which can be decomposed into two real scalar fields, thus explaining the difference.

[2] Had we attempted to find an exact solution to (39), we would need to solve an integral equation for $h^{00} = (2G_N M/r)H_0(\kappa r)$ $h^{00} = \delta^{ij}(2G_N M/r)H_1(\kappa r)$, where $H_0(z)$ and $H_1(z)$ are the sought-for functions, whose asymptotic series (around $r = \infty$) begins as $H_{1,2}(z) = 1 + \mathcal{O}(1/z^2)$. It would be of interest to find out what the coefficients are of this asymptotic series and whether it can be resummed, i.e. whether it can be analytically extended to the whole complex z plane. Understanding this function is important, because it would tell us whether quantum corrections can resolve the $r = 0$ singularity of Newtonian gravity. Pursuing this analysis is, however, beyond the scope of this work.

$d^3x' = d^3\Delta r = 2\pi(\Delta r)^2 d(\Delta r)\sin\theta d\theta$. The integration over θ yields

$$\int_0^\pi \sin\theta d\theta \frac{1}{r'} = \int_{-1}^1 \frac{d(\cos\theta)}{\sqrt{r^2+\Delta r^2 - 2r\Delta r\cos(\theta)}} = \frac{r+\Delta r - |r-\Delta r|}{r\Delta r}, \tag{47}$$

where $r = \| \vec{x} \|$ and we choose $\vec{r}' = \hat{z}r'$. We thus have

$$F(r,t) = 4\pi\int_{t_0}^t dt' \int_0^\infty d(\Delta r)(\Delta r)^2\{\ln[\mu^2(\Delta t^2 - \Delta r^2)] - 1\} \times \left\{\frac{1}{\Delta r}\Theta(\Delta r - r) + \frac{1}{r}\Theta(r-\Delta r)\right\}\Theta(\Delta t - \Delta r). \tag{48}$$

For simplicity, we decompose this integral into two parts, namely I_A and I_B:

$$F = I_A + I_B,$$
$$I_A = 4\pi\int_{t_0}^t dt' \int_0^\infty d(\Delta r)(\Delta r)\{\ln[\mu^2(\Delta t^2 - \Delta r^2)] - 1\} \times \Theta(\Delta r - r)\Theta(\Delta t - \Delta r) \tag{49}$$

$$I_B = \frac{4\pi}{r}\int_{t_0}^t dt' \int_0^\infty d(\Delta r)(\Delta r)^2\{\ln[\mu^2(\Delta t^2 - \Delta r^2)] - 1\} \times \Theta(r-\Delta r)\Theta(\Delta t - \Delta r). \tag{50}$$

If we now take $\Delta t_0 \equiv t - t_0$ and carefully treat the Θ-functions, for I_A we get the following result:

$$\begin{aligned} I_A &= 4\pi\int_0^{\Delta t_0} d(\Delta t)\int_r^{\Delta t} d(\Delta r)\Delta r\{\ln[\mu^2(\Delta t^2 - \Delta r^2)] - 1\} \times \Theta(\Delta t - r) \\ &= \frac{2\pi}{9}\left\{-8\Delta t_0^3 + 30\Delta t_0 r^2 - 22r^3 - 6r^3\ln\left(\frac{\Delta t_0 + r}{\Delta t_0 - r}\right) + 12r^3\ln(2\mu r) + 3\Delta t_0(\Delta t_0^2 - 3r^2) \times \ln[\mu^2(\Delta t_0^2 - r^2)]\right\}\Theta(\Delta t_0 - r). \end{aligned} \tag{51}$$

In order to calculate I_B, we first write it as a sum of two Θ-functions:

$$I_B = \frac{4\pi}{r}\int_0^{\Delta t_0} d\Delta t\left\{\int_0^r d(\Delta r)\Delta r^2\{\ln[\mu^2(\Delta t^2 - \Delta r^2)] - 1\} \times \Theta(\Delta t - r) + \int_0^{\Delta t} d(\Delta r)\Delta r^2\{\ln[\mu^2(\Delta t^2 - \Delta r^2)] - 1\} \times \Theta(r - \Delta t)\right\}. \tag{52}$$

Note that, after integrating over Δr, the second integral in I_B gets divided in two parts, thus explaining the origin of the time transients. Schematically this means

$$\int_0^{\Delta t_0} d\Delta t\{\cdots\}\Theta(r-\Delta t) \to \int_0^{\Delta t_0} d\Delta t\{\cdots\}\Theta(r - \Delta t_0) + \int_0^r d\Delta t\{\cdots\}\Theta(\Delta t_0 - r). \tag{53}$$

It is now convenient to break the overall result for F into a sum over two parts, each proportional to one Θ-function:

$$F(r,t) \equiv F_1(r,t)\Theta(\Delta t_0 - r) + F_2(r,t)\Theta(r - \Delta t_0), \tag{54}$$

where F_1 and F_2 evaluate to

$$F_1(r,t) \equiv 4\pi\frac{r^3}{6}\left[\ln(2\mu r) - \frac{25}{12}\right] \tag{55}$$

$$+ 4\pi\frac{\Delta t_0^2 - r^2}{6}\left[\frac{(\Delta t_0 + r)^2}{2r}\ln[\mu(\Delta t_0 + r)] - \frac{(\Delta t_0 - r)^2}{2r}\ln[\mu(\Delta t_0 - r)] - \frac{11}{3}\Delta t_0\right] \tag{56}$$

$$F_2(r,t) \equiv 4\pi\frac{\Delta t_0^4}{6r}\left[\ln(2\mu\Delta t_0) - \frac{25}{12}\right]. \tag{57}$$

One can easily verify that $\partial^2 F$ satisfies

$$\partial^2 F(r,t) = [\partial^2 F_1(r,t)]\Theta(\Delta t_0 - r) + [\partial^2 F_2(r,t)]\Theta(r - \Delta t_0), \tag{58}$$

and analogously for $\partial^4 F$. This simple result holds true because, not only F_1 and F_2 are continuous at the causal boundary $\Delta t_0 = r$, but also the pairs $\{(\partial_0 + \partial_r)F_1, (\partial_0 + \partial_r)F_2\}$, $\{\partial^2 F_1, \partial^2 F_2\}$ and $\{(\partial_0 + \partial_r)\partial^2 F_1, (\partial_0 + \partial_r)\partial^2 F_2\}$ are all continuous at $\Delta t_0 = r$. Finally, from (55)–(58) we obtain

$$\partial^4 F = 16\pi\left\{\frac{\ln(2\mu r)}{r}\Theta(\Delta t_0 - r) + \frac{\ln(2\mu\Delta t_0)}{r}\Theta(r - \Delta t_0)\right\}. \tag{59}$$

By inserting this into (45), we get the desired result for the quantum one-loop correction to the Newtonian potential:

$$h^{00(1)} = \frac{\kappa^2 G_N M}{160\pi^2}\left\{\frac{1}{3r^3}(1 + 2\tilde{\xi})\Theta(\Delta t_0 - r) - \frac{1}{r\Delta t_0^2}(-1 + 2\tilde{\xi})\Theta(r - \Delta t_0) + \frac{4}{3}\frac{1}{r\Delta t_0}(1 - \tilde{\xi})\delta(r - \Delta t_0)\right\} \tag{60}$$

$$h^{ij(1)} = \frac{\kappa^2 G_N M}{160\pi^2}\delta^{ij}\left\{-\frac{1}{3r^3}(-1+2\tilde{\xi})\Theta(\Delta t_0 - r) + \frac{1}{3r\Delta t_0^2}(-1+6\tilde{\xi})\Theta(r-\Delta t_0) + \frac{4\tilde{\xi}}{3r\Delta t_0}\delta(r-\Delta t_0)\right\} \quad (61)$$

$$h^{i0(1)} = \frac{\kappa^2 G_N M}{240\pi^2}\partial^i\left\{\frac{1}{r\Delta t_0}\Theta(r-\Delta t_0)\right\}, \quad (62)$$

where $\tilde{\xi}$ denotes

$$\tilde{\xi} \equiv \frac{1}{6} + 30\left(\xi - \frac{1}{6}\right)^2. \quad (63)$$

Equations (60)–(63) constitute our second main result. Apart from the static results, which are reproduced in the limit when $\Delta t_0 \to \infty$, Eqs. (60)–(62) contain time transients both on the light cone (where $\Delta t_0 = r$) and outside the light cone (where $\Delta t_0 < r$). The coincident time divergences $\propto 1/(\Delta t_0)^2$ and $\propto 1/(\Delta t_0)$ in (60)–(62) outside the light cone signal initial time divergences, which occur as a consequence of the sudden switching of the graviton-scalar coupling, and can be removed by either resorting to an adiabatic switching [28], or by a suitably modifying the initial state at $t = t_0$ [29,30]. However, performing this initial surface renormalization is beyond the scope of this work. The appearance of the time transients in (60)–(62) is the main novelty of our approach to quantum gravitational corrections on classical space-times, and can be of help to understanding the dynamical quantum gravitational backreaction.

In the limit when $\Delta t_0 \to \infty$ ($t_0 \to -\infty$) the second and the third part of $h^{00(1)}$ and $h^{ij(1)}$ in (60) and (61) vanish and all of $h^{i0(1)}$ in (62) vanishes, and one recovers the static result. This is to be expected, since in the limit when $t_0 \to -\infty$ time transients should disappear. In this limit our results reduce to the ones obtained by the *in-out* formalism, in which calculations are usually performed in momentum space, and no time transients are allowed. Indeed, Refs. [25,26] have evaluated the one-loop correction for a nonminimally coupled scalar in the same (longitudinal or Newtonian) gauge, albeit by different techniques (see Appendix C). Their results are shown in Eqs. (31)–(33). When compared with ours (60) and (61) (with $\Delta t_0 \to \infty$), the coefficients of $h^{00(1)}$ and $h^{ij(1)}$ agree in magnitude, but they have the opposite sign. Comparing, however, with Park and Woodard [16], which calculate the effect of a minimally coupled scalar only, our results agree perfectly. To show that, observe first that in the longitudinal gauge we use, and in the limit $\Delta t_0 \to \infty$ (*cf.* Eqs. (60), (61), and (B18)) we get the quantum-corrected Bardeen potentials

$$\Phi = -\frac{G_N M}{r} - \frac{G_N^2 M}{60\pi}\frac{1+2\tilde{\xi}}{r^3} + \mathcal{O}\left(\frac{1}{r^5}\right),$$
$$\Psi = -\frac{G_N M}{r} - \frac{G_N^2 M}{60\pi}\frac{1-2\tilde{\xi}}{r^3} + \mathcal{O}\left(\frac{1}{r^5}\right), \quad (64)$$

whose leading quantum parts in the $\xi = 0$ ($\tilde{\xi} = 1$) case are

$$\Phi^{(1)} \overset{\xi\to 0}{\to} -\frac{G_N^2 M}{20\pi r^3}, \qquad \Psi^{(1)} \overset{\xi\to 0}{\to} \frac{G_N^2 M}{60\pi r^3}. \quad (65)$$

On the other hand, relation (B25) and Eqs. (55–56) of Ref. [16] tell us that, in the gauge used in Ref. [16], $h^{(1)}_{00\mathrm{PW}} = G_N^2 M/(10\pi r^3)$, $h^{(1)}_{\mathrm{PW}} = -G_N^2 M/(10\pi r^3)$ and $\tilde{h}^{(1)}_{\mathrm{PW}} = -G_N^2 M/(30\pi r)$. Inserting these into Eqs. (B18) and (B26) we get $\Phi^{(1)}_{\mathrm{PW}} = -G_N^2 M/(20\pi r^3)$ and $\Psi^{(1)}_{\mathrm{PW}} = G_N^2 M/(60\pi r^3)$, which perfectly agree with (65).

An interesting question is: in which physical situations one expects to see time transients of the kind exhibited in Eqs. (60)–(62). One such situation would be a very massive scalar field which at time $t = t_0$ suddenly becomes massless (*e.g.* via a Higgs-like mechanism). Such a scalar would effectively decouple from gravity at $t < t_0$, and would begin acting gravitationally at $t = t_0$. Equations (60)–(62) describe a good approximation to the gravitational field generated by such a scalar. If the Higgs mechanism would induce a significant change in the vacuum energy, one would need to consider the change in the Newtonian potential from a de Sitter background to a Minkowski background. While certainly interesting, this calculation would be technically demanding. For a recent calculation of the one-loop vacuum polarization in de Sitter background see Ref. [31]. One may object that the work presented here is mainly of academic interest. While this is to an extent true, we emphasize that there are many similar situations that are realized in nature, where time transients are of crucial importance. One such example is a shell of collapsing matter, that eventually forms a black hole. While understanding the quantum backreaction in such a dynamical setting is beyond the scope of the (momentum space) *in-out* formalism, as the present work indicates, it is well within the scope of an *in-in* treatment.

V. CONCLUSIONS

We have calculated the one-loop graviton vacuum polarization induced by a massless nonminimally coupled scalar field (28) by making use of the Schwinger-Keldysh formalism. We have then applied our result to calculate the perturbative quantum correction to the Newtonian potential of a point particle. The novelty of our approach are the time transients (60)–(63), which naturally appear within the Schwinger-Keldysh formalism. Such transients would occur, for example, in the case when the scalar mass would change from a large value to zero by a Higgs-like mechanism.

There are many directions in which the calculation presented in this paper might be developed. One such direction is to consider quantum effects due to fermionic, vector and tensor fields. Other possible avenues comprise a study of time transients induced by quantum effects in the process of formation of compact stars, such as neutron stars, black holes, boson stars or gravastars [32]. In particular, these types of studies would improve our understanding of the question of quantum backreaction. For example, growing time transients would indicate that the background space-time ought to be modified.

APPENDIX A: SCHWINGER-KELDYSH FORMALISM

In this section we give a short overview of the Schwinger-Keldysh or *in-in* formalism [11–15] by making a comparison with the more conventional *in-out* formulation of quantum field theory (QFT). Here we shall follow mainly Refs. [10,13]. Within the *in-out* method the vacuum-to-vacuum amplitudes $\langle\Omega, t_{\rm out}|\Omega, t_{\rm in}\rangle_J$ are calculated in the presence of an external source J in virtue of the path integral representation of the in-out generating functional $Z[J]$:

$$\langle\Omega, t_{\rm out}|\Omega, t_{\rm in}\rangle_J = Z[J] = \int \mathcal{D}\varphi \exp\Big[\iota\Big(S[\varphi] + \int d^D x J\varphi\Big)\Big], \tag{A1}$$

where $|\Omega, t_{\rm in/out}\rangle$ represent the *in* and *out* states in the remote past and future, respectively, $S[\varphi]$ is the free (classical) action for a (scalar) field $\varphi(x)$, $J(x)$ is a source current, and $\mathcal{D}\varphi = \prod_x d\varphi(x)$ is a path integral integration measure. In flat space-time the *in* and *out* states are mostly expressed as a superposition of free plane waves defined globally; in curved spaces however the *in* and *out* states can differ (in that they are defined only on part of the space) and the *out* states may even not be known. For these type of situations the *in-in* formalism is more suitable. The partition function $Z[J]$ of the *in-out* formalism generates time-ordered correlation functions between different states:

$$\langle\Omega, t_{\rm out}|T\{\varphi(x_1)\cdots\varphi(x_n)\}|\Omega, t_{\rm in}\rangle = (-\iota)^n \frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)} Z[J]|_{J=0}. \tag{A2}$$

In nontrivial gravitational backgrounds the final state of a system usually cannot be predefined, and hence one resorts to calculating expectation values with respect to an *in* state. In doing so one first defines the *in-in* generating functional by means of the *two* external sources J_+ and J_-:

$$Z[J_-, J_+] \underset{J_-}{=} \langle\Omega, t_{\rm in}|\Omega, t_{\rm in}\rangle_{J_+} = \sum_\alpha \langle\Omega, t_{\rm in}|\alpha, t_{\rm out}\rangle_{J_-}\langle\alpha, t_{\rm out}|\Omega, t_{\rm in}\rangle_{J_+}, \tag{A3}$$

where the sum goes over a complete set of *out* states. While in the standard *in-out* formalism one evolves the *in* state in the presence of a source J and compares it with the *out* state, in the *in-in* framework one evolves the *in* state in the presence of the two different sources J_+ and J_- and compares the results in the remote future. In other words, expression (A3) can be understood as the *in*-vacuum going forward in time under influence of the source J_+ and then returning back in time under the influence of the source J_- (hence the term the *closed-time-path formalism* broadly used in literature [14]). From the path integral representation of the *in-in* generating functional (for details see *e.g.* [13]),

$$Z[J_-, J_+] = \int \mathcal{D}\varphi_+\mathcal{D}\varphi_- \exp\Big(\iota S[\varphi_+] - \iota S[\varphi_-] + \iota\int d^D x J_+\varphi_+ - \iota\int d^D x J_-\varphi_-\Big), \tag{A4}$$

one obtains the expectation values upon differentiating with respect to J_+ and J_- and then setting $J_+ = J_-$:

$$\frac{\delta}{-\iota\delta J_-(y_1)}\cdots\frac{\delta}{-\iota\delta J_-(y_m)} \times \frac{\delta}{\iota\delta J_+(x_1)}\cdots\frac{\delta}{\iota\delta J_+(x_n)} \times Z[J_-, J_+]_{J_-=J_+=0} = \langle\Omega, t_{\rm in}|\{\bar{T}\varphi(y_1)\cdots\varphi(y_m)\}\times\{T\varphi(x_1)\cdots\varphi(x_n)\}|\Omega, t_{\rm in}\rangle, \tag{A5}$$

with the boundary (initial) conditions for the *in*-vacuum state, and the constraint for the fields on the future boundary $\varphi_+(t_{\rm out}) = \varphi_-(t_{\rm out})$. In (A5) T and $\bar{T}$ denote time and antitime ordering operators, i.e., $T\varphi(x)\varphi(x') = \Theta(x^0 - x'^0)\varphi(x)\varphi(x') + \Theta(x'^0 - x^0)\times \varphi(x')\varphi(x)$ and $\bar{T}\varphi(x)\varphi(x') = \Theta(x'^0 - x^0)\varphi(x)\varphi(x') + \Theta(x^0 - x'^0)\varphi(x')\varphi(x)$.

The generating functional is now an integral over two fields which we shall conveniently combine into a column matrix:

$$\varphi = \begin{pmatrix}\varphi_+\\ \varphi_-\end{pmatrix}, \qquad \text{with the source fields:} \qquad \mathcal{J} = \begin{pmatrix}J_+\\ J_-\end{pmatrix}. \tag{A6}$$

To calculate the generating functional $Z[J_-, J_+]$ is a rather formidable task, which principally can be performed only in the perturbative setting. Hence we decompose the action $S[\varphi]$ into its free ($S_F[\varphi]$) and interacting part ($S_I[\varphi]$):

$$S[\varphi] = S_F[\varphi] + S_I[\varphi], \qquad S_F[\varphi] = \int d^D x \frac{1}{2}\varphi\mathcal{D}_0\varphi, \tag{A7}$$

where for a nonminimally coupled scalar field the kinetic operator $\mathcal{D}_0$ in (A7) takes the form $\Box\,\mathcal{D}_0 = -m^2 - \xi R$, where $\Box$ is the d'Alembertian operator, m is the scalar field mass, R is the Ricci scalar and D is the dimension of space-time. The *in-in* generating functional (A4) now becomes

$$Z[J_-, J_+] = e^{\iota S_I^+[-\iota\delta/\delta J_+] - \iota S_I^-[\iota\delta/\delta J_-]} \times \int \mathcal{D}\varphi_+ \mathcal{D}\varphi_- \exp \times \left\{ \int d^4x \left[\frac{1}{2}(\varphi_+, \varphi_-) \begin{pmatrix} \iota\mathcal{D}_0 & 0 \\ 0 & -\iota\mathcal{D}_0 \end{pmatrix} \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix} + (\varphi_+, \varphi_-) \begin{pmatrix} \iota J_+ \\ -\iota J_- \end{pmatrix} \right] \right\}. \tag{A8}$$

Next we shift the fields

$$\begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix} \to \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix} - \begin{pmatrix} \iota\Delta_{++} & \iota\Delta_{+-} \\ \iota\Delta_{-+} & \iota\Delta_{--} \end{pmatrix} \begin{pmatrix} \iota J_+ \\ -\iota J_- \end{pmatrix}, \tag{A9}$$

which results in

$$Z[J_-, J_+] = e^{\iota S_I^+[-\iota\delta/\delta J_+] - \iota S_I^-[\iota\delta/\delta J_-]} \times \exp\left\{ \int d^4x d^4x' \frac{1}{2}(\iota J_+, -\iota J_-)(x) \times \begin{pmatrix} \iota\Delta_{++} & \iota\Delta_{+-} \\ \iota\Delta_{-+} & \iota\Delta_{--} \end{pmatrix}(x; x') \begin{pmatrix} \iota J_+ \\ -\iota J_- \end{pmatrix}(x') \right\}, \tag{A10}$$

where the matrix of Δ's is a 2×2 Keldysh matrix of (free) propagators, defined as the inverse of $\mathcal{D}_0$'s:

$$\begin{pmatrix} \mathcal{D}_0 & 0 \\ 0 & -\mathcal{D}_0 \end{pmatrix}(x) \begin{pmatrix} \iota\Delta_{++} & \iota\Delta_{+-} \\ \iota\Delta_{-+} & \iota\Delta_{--} \end{pmatrix}(x; x') = \iota\delta^D(x - x'). \tag{A11}$$

Here $\iota\Delta_{++}$ is the free Feynman (time-ordered), $\iota\Delta_{--}$ is the free anti-Feynman (or Dyson) propagator and $\iota\Delta_{+-}$ and $\iota\Delta_{-+}$ are the (positive and negative frequency) free Wightman functions. If we now insert Eq. (A10) into Eq. (A5) we obtain relations between expectation values and propagators:

$$\begin{aligned} \iota\Delta_{++}(x, x') &= \langle\Omega|T(\varphi(x)\varphi(x'))|\Omega\rangle \\ &= \Theta(x^0 - x'^0)\iota\Delta_{-+}(x; x') \\ &\quad + \Theta(x'^0 - x^0)i\Delta_{+-}(x; x'), \end{aligned} \tag{A12}$$

$$\begin{aligned} \iota\Delta_{--}(x, x') &= \langle\Omega|\bar{T}(\varphi(x)\varphi(x'))|\Omega\rangle \\ &= \Theta(x^0 - x'^0)\iota\Delta_{+-}(x; x') \\ &\quad + \Theta(x'^0 - x^0)\iota\Delta_{-+}(x; x'). \end{aligned} \tag{A13}$$

It is worth noting that these relations hold both for free and dressed propagators. When φ is a massless minimally coupled scalar field, i.e. $\mathcal{D}_0 = \partial^2$, the free propagator equations reduce to

$$\begin{aligned} &\partial^2\iota\Delta_{++}(x; x') = \iota\delta^D(x - x'), \qquad \partial^2\iota\Delta_{+-}(x; x') = 0, \\ &\partial^2\iota\Delta_{--}(x; x') = -\iota\delta^D(x - x'), \qquad \partial^2\iota\Delta_{-+}(x; x') = 0. \end{aligned} \tag{A14}$$

The vacuum solution to all four propagators can be written compactly as

$$\iota\Delta_{\pm\pm}(x; x') = \frac{\Gamma(\frac{D}{2} - 1)}{4\pi^{D/2}} \left(\frac{1}{\Delta x_{\pm\pm}^2} \right)^{(D/2)-1}, \tag{A15}$$

where $\Delta x_{\pm\pm}$ are the four Schwinger-Keldysh length functions:

$$\begin{aligned} \Delta x_{++}^2 &= \| \vec{x} - \vec{x}' \|^2 - (|t - t'| - \iota\delta)^2, \\ \Delta x_{+-}^2 &= \| \vec{x} - \vec{x}' \|^2 - (t - t' + \iota\delta)^2, \\ \Delta x_{--}^2 &= \| \vec{x} - \vec{x}' \|^2 - (|t - t'| + \iota\delta)^2, \\ \Delta x_{-+}^2 &= \| \vec{x} - \vec{x}' \|^2 - (t - t' - \iota\delta)^2, \end{aligned} \tag{A16}$$

where $\delta > 0$ is an infinitesimal parameter, which defines the time integration contours needed when propagators act on functions or distributions. Notice that Δx_{++}^2 and Δx_{--}^2 are Lorentz invariant, such that the Feynman and anti-Feynman propagators in (A15) are manifestly Lorentz invariant.

APPENDIX B: CLASSICAL NEWTONIAN POTENTIAL

In order to obtain the Einstein-Hilbert action to quadratic order in $h_{\mu\nu}$ (see Sec. II) we first expand the Ricci scalar in powers of $h_{\mu\nu}$. The general result is (see, for example, Appendix A of Ref. [10]),

$$\begin{aligned} \sqrt{-g}R = \sqrt{-g}g^{\mu\nu}\Big\{ &g^{\rho\sigma}[\partial_\rho\partial_\mu h_{\sigma\nu} - \partial_\mu\partial_\nu h_{\rho\sigma}] \\ &+ g^{\rho\alpha}g^{\sigma\beta}\Big[-(\partial_\rho h_{\alpha\beta})(\partial_\mu h_{\sigma\nu}) - \frac{1}{4}(\partial_\sigma h_{\nu\alpha})(\partial_\mu h_{\beta\rho}) \\ &- \frac{1}{4}(\partial_\sigma h_{\nu\alpha})(\partial_\rho h_{\mu\beta}) - \frac{1}{4}(\partial_\beta h_{\mu\nu})(\partial_\sigma h_{\rho\alpha}) \\ &+ (\partial_\sigma h_{\mu\nu})(\partial_\rho h_{\alpha\beta}) + \frac{3}{4}(\partial_\mu h_{\rho\sigma})(\partial_\nu h_{\alpha\beta}) \Big]\Big\}, \end{aligned} \tag{B1}$$

where

$$\begin{aligned} \sqrt{-g} &= 1 + \frac{1}{2}h + \left(\frac{1}{8}h^2 - \frac{1}{4}h_{\mu\nu}h^{\mu\nu} \right) + \mathcal{O}(h_{\mu\nu}^3), \\ g^{\mu\nu} &= \eta^{\mu\nu} - h^{\mu\nu} + h_\rho^\mu h^{\rho\nu} + \mathcal{O}(h_{\mu\nu}^3). \end{aligned} \tag{B2}$$

Notice that (B1) contains all orders of $h_{\mu\nu}$, and hence can be used to obtain the cubic, quartic and higher order gravitational vertices. When Eqs. (B1) and (B2) are combined, we get for the linear and quadratic contribution to the Einstein-Hilbert action

$$\sqrt{-g}R = \left(1+\frac{h}{2}\right)(\partial_\mu\partial_\nu h^{\mu\nu} - \partial^2 h) - h_{\rho\sigma}(2\partial^\rho\partial^\mu h^\sigma_\mu - \partial^2 h^{\rho\sigma}) + h^{\mu\nu}\partial_\mu\partial_\nu h - \frac{3}{2}(\partial_\rho h^{\rho\sigma})(\partial^\mu h_{\mu\sigma}) - \frac{1}{4}(\partial_\mu h)(\partial^\mu h) + (\partial^\mu h)(\partial^\nu h_{\mu\nu}) + \frac{3}{4}(\partial^\mu h^{\nu\sigma})(\partial_\mu h_{\nu\sigma}) + \mathcal{O}(h^3_{\mu\nu})$$
$$= -\frac{1}{4}h\partial^2 h + \frac{1}{2}h\partial_\mu\partial_\nu h^{\mu\nu} - \frac{1}{2}h^{\mu\nu}\partial_\mu\partial_\sigma h^\sigma_\nu + \frac{1}{4}h^{\mu\nu}\partial^2 h_{\mu\nu} + \mathcal{O}(h^3_{\mu\nu}) + \mathtt{tot.der.terms}, \tag{B3}$$

where $\partial^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu$, $h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$, $\partial^\mu = \eta^{\mu\nu}\partial_\nu$, $h = \eta^{\mu\nu}h_{\mu\nu}$. In the last line of (B3) we dropped several total derivative terms (which contribute as boundary terms, and hence do not contribute to the equations of motion). The last line in (B3) can be used to recover the quadratic action (4).

Next, for pedagogical reasons here we present a derivation of the classical potentials (9), which solve Eq. (8):

$$L_{\mu\nu\rho\sigma}h^{\rho\sigma(0)}(x) = \frac{\kappa^2}{2}\delta^0_\mu\delta^0_\nu M\delta^{D-1}(\vec{x}), \tag{B4}$$

where $\kappa^2 = 16\pi G_N$ and the Lichnerowicz operator is given by

$$L_{\mu\nu\rho\sigma} = \partial_{(\rho}\eta_{\sigma)(\mu}\partial_{\nu)} - \frac{1}{2}\eta_{\mu(\rho}\eta_{\sigma)\nu}\partial^2 - \frac{1}{2}(\eta_{\rho\sigma}\partial_\mu\partial_\nu + \eta_{\mu\nu}\partial_\rho\partial_\sigma) + \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma}\partial^2. \tag{B5}$$

Let us first write the (00), $(0i)$ and (ij) components of Eq. (B4):

$$\frac{1}{2}(\partial_i\partial_j - \delta_{ij}\nabla^2)h_{ij} = \frac{\kappa^2}{2}M\delta^3(\vec{x}) \tag{B6}$$

$$-\frac{1}{2}\nabla^2 h_{0i} + \frac{1}{2}\partial_i\partial_j h_{0j} + \frac{1}{2}\partial_0(\partial_j h_{ij} - \partial_i h_{jj}) = 0 \tag{B7}$$

$$\frac{1}{2}(\partial_i\partial_j - \delta_{ij}\nabla^2)h_{00} - \partial_0\partial_{(i}h_{j)0} + \delta_{ij}\partial_0\partial_l h_{0l} + \partial_l\partial_{(i}h_{j)l} - \frac{1}{2}(-\partial_0^2 + \nabla^2)h_{ij} + \frac{1}{2}\delta_{ij}(-\partial_0^2 + \nabla^2)h_{ll} - \frac{1}{2}(\delta_{ij}\partial_k\partial_l h_{kl} + \partial_i\partial_j h_{ll}) = 0, \tag{B8}$$

where $\nabla^2 = \delta_{ij}\partial_i\partial_j \equiv \partial_i\partial_i$ and h_{ij} are the spatial components of $h_{\mu\nu}$.

Next we perform the standard scalar-vector-tensor decomposition of $h_{\mu\nu}$:

$$h_{0i} = n_i^T + \partial_i\sigma,$$
$$h_{ij} = \frac{\delta_{ij}}{3}h + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\tilde{h} + \partial_i h_j^T + \partial_j h_i^T + h_{ij}^{TT}, \tag{B9}$$

such that $\partial_i n_i^T = 0 = \partial_i h_i^T$, $h_{ii}^{TT} = 0$, $\partial_i h_{ij}^{TT} = 0$ and $\mathrm{Tr}[h_{ij}] = h$. Consider now an infinitesimal coordinate transformation, $x^\mu \to x^\mu + \xi^\mu(x)$. Then, as a consequence of the tensorial transformation law, the metric perturbation on Minkowski background transforms as

$$h_{\mu\nu}(x) \to h_{\mu\nu}(x) - \partial_\mu\xi_\nu(x) - \partial_\mu\xi_\nu(x) + \mathcal{O}(\xi^2, h^2, \xi h), \tag{B10}$$

(with $\xi_\mu = \eta_{\mu\nu}\xi^\nu$) where it is also known as the gauge transformation of general relativity. In the light of the metric decomposition (B9), it is also convenient to decompose ξ^μ into two scalars and a transverse vector as $\xi_\mu = (\xi_0, \xi_i^T, \partial_i\xi)$, where $\partial_i\xi_i^T = 0$. With this (B10) can be rewritten in components as

$$h_{00} \to h_{00} - 2\partial_0\xi_0, \quad n_i^T \to n_i^T - \partial_0\xi_i^T,$$
$$\sigma \to \sigma - \partial_0\xi - \xi_0, \quad h_{ij}^{TT} \to h_{ij}^{TT}, \quad h_i^T \to h_i^T - \xi_i^T, \tag{B11}$$
$$\tilde{h} \to \tilde{h} - 2\xi, \quad h \to h - 2\nabla^2\xi.$$

From these relations it then immediately follows that the following two scalars, one vector and the tensor, are gauge invariant:

$$h_{00} - 2\partial_0\sigma + \partial_0^2\tilde{h}, \quad h - \nabla^2\tilde{h}, \quad n_i^T - \partial_0 h_i^T, \quad h_{ij}^{TT}. \tag{B12}$$

This means that out of the ten components of $h_{\mu\nu}$, six are gauge invariant (and correspond to observables), while the remaining four components are gauge dependent, and have no independent physical meaning. It is now instructive to rewrite Eqs. (B7)–(B9) in terms of the scalar, vector and tensor components of $h_{\mu\nu}$ defined in (B9). The resulting equations are

$$\nabla^2(h - \nabla^2\tilde{h}) = -\frac{3\kappa^2}{2}M\delta^3(\vec{x}), \tag{B13}$$

$$\nabla^2(n_i^T - \partial_0 h_i^T) = 0, \qquad \partial_0\partial_i(h - \nabla^2\tilde{h}) = 0, \tag{B14}$$

$$(\partial_i\partial_j - \delta_{ij}\nabla^2)\left[\frac{1}{2}(h_{00} - 2\partial_0\sigma + \partial_0^2\tilde{h}) - \frac{1}{6}(h - \nabla^2\tilde{h})\right] = 0,$$
$$\partial_0^2(h - \nabla^2\tilde{h}) = 0, \qquad \partial_0\partial_i(n_j^T - \partial_0 h_j^T) = 0, \tag{B15}$$

$$(-\partial_0^2 + \nabla^2)h_{ij}^{TT} = 0. \tag{B16}$$

These equations possess a number of remarkable properties. First, Eq. (B17) is the only dynamical equation and it is the wave equation for gravitational waves, obeying the standard Lorentz covariant wave operator. Second, all equations are gauge invariant (see Eq. (B12)), and hence can be written in the gauge invariant form as

$$\nabla^2\Psi = 4\pi G_N M\delta^3(\vec{x}), \qquad \partial_0^2\Psi = \partial_0\partial_i\Psi = 0,$$
$$(\partial_i\partial_j - \delta_{ij}\nabla^2)(-\Phi + \Psi) = 0, \qquad \partial_0\partial_i\tilde{n}_i^T = \nabla^2\tilde{n}_i^T = 0, \tag{B17}$$

where we introduced the usual Bardeen potentials,

$$\Psi = -\frac{1}{6}(h - \nabla^2\tilde{h}), \qquad \Phi = -\frac{1}{2}(h_{00} - 2\partial_0\sigma + \partial_0^2\tilde{h}), \tag{B18}$$

and a gauge invariant shift vector,

$$\tilde{n}_i^T = n_i^T - \partial_0 h_i^T. \tag{B19}$$

Equations (B17) can be easily solved assuming that $\Phi - \Psi$ and $\tilde{n}_i^T$ vanish at spatial infinity:

$$\Phi = \Psi = -\frac{G_N M}{r}, \qquad \tilde{n}_i^T = 0, \tag{B20}$$

where $r = \| \vec{x} \|$ and we made use of the Green function $G(\vec{x}) = -1/(4\pi r)$ for $\nabla^2 G(\vec{x}) = \delta^3(\vec{x})$. Notice that there are 3 equations for Ψ in (B17). These equations are not redundant. In fact, they assure that the linearized version of the Birkhoff theorem holds: the potentials of linearized gravity generated by a static point mass must be time independent. Notice finally that, the linearized Eqs. (B17) and (B18) do not tell us anything about the gauge variant quantities (among which are two scalars and one vector), which is also what one expects.

Most of the literature does not make use of the (gauge invariant) Bardeen potentials to specify the gravitational response to a static point mass, but instead one fixes a gauge. The main purpose of this Appendix, in which we show how to construct classical gauge invariant potentials, is to facilitate comparison with the literature, which uses a variety of gauges. For example, in the longitudinal (Newton) gauge, the scalar metric sector is fixed to be

$$ds^2 = h_{00}dt^2 + \sum_{i=1}^{3} h_{ii}(dx^i)^2. \tag{B21}$$

In this gauge we have simply

$$\Phi = -\frac{1}{2}h_{00},$$
$$\Psi = -\frac{1}{6}\,\mathrm{Tr}[h_{ij}] = -\frac{1}{2}h_{ii} \quad \text{(no summation)}. \tag{B22}$$

Together with (B20), these equations imply (9), which is the classical Newtonian potential that we use in this work.

On the other hand, Park and Woodard [16] make use of the nondiagonal gauge

$$h_{00} = \frac{2G_N M}{r}, \qquad h_{ij} = \frac{2G_N M}{r}\hat{r}_i\hat{r}_j. \tag{B23}$$

This gauge can be obtained from the two scalars of h_{ij} in (B9) by observing that for a radial scalar function $\tilde{h} = \tilde{h}(r)$, $\partial_i = \hat{r}_i d/dr$ ($\hat{r}_i = x^i/r$) and $\partial_i\partial_j = (\delta_{ij}/r)\times(d/dr) + \hat{r}_i\hat{r}_j[(d^2/dr^2) - (1/r)(d/dr)]$, from which it follows,

$$(h_{ij})_{\text{scalar}} = \frac{\delta_{ij}}{3}h + \Big(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\Big)\tilde{h}$$
$$= \frac{\delta_{ij}}{3}\Big[h - \Big(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\Big)\tilde{h}\Big] + \hat{r}_i\hat{r}_j\Big(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\Big)\tilde{h}. \tag{B24}$$

Comparing this with (B23) reveals that in that gauge

$$h = \Big(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\Big)\tilde{h} = \frac{2G_N M}{r}. \tag{B25}$$

To check these relations, note first that, up to a constant, $\tilde{h} = -2G_N M r$. Equation (B18) then implies

$$\Psi = -\frac{1}{6}\Big[h - \Big(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\Big)\tilde{h}\Big] = -\frac{G_N M}{r}, \tag{B26}$$

which agrees with (B20), implying that Eqs. (B23) represent a correct classical solution.

APPENDIX C: EXPANDING THE BUBBLE DIAGRAM

In this Appendix we expand the scalar field bubble diagram in Eq. (17) shown in Fig. 3 in powers of small (quantum) metric perturbations $\delta g^{\mu\nu}(x) = -h^{\mu\nu}(x) + h^{\mu\mu'}(x)h^{\nu}_{\mu'}(x) + \mathcal{O}((h^{\mu\nu})^3)$.

Let us begin by noting that the scalar propagator in (17) obeys the equation

$$(-g^{(b)}(x))^{1/2}\Box_x^{(b)}\Delta^{ab}(x;x';[g^{(b)}_{\alpha'\beta'}])$$
$$= \partial_\mu g^{(b)\mu\nu}(x)[-g^{(b)}(x)]^{1/2}\partial_\nu\Delta^{ab}(x;x';[g^{(b)}_{\alpha'\beta'}])$$
$$= (\sigma^3)^{ab}\delta^D(x-x'), \tag{C1}$$

such that $\sqrt{-g^{(b)}(x)}\Box_x^{(b)}\delta^D(x-y)(\sigma^3)^{ab}$ is the operator inverse of $\Delta^{ab}(y;x';[g^{(b)}_{\alpha'\beta'}])$. $\sigma^3 = \mathrm{diag}(1,-1)$ in (C1) is the Pauli matrix.

Writing $g^{(b)}_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}$, we can expand the bubble diagram in (17) in powers of $\delta g_{\mu\nu}$. To proceed, let us vary the first bubble diagram in (17) with respect to $g^{\mu\nu}(x)$ (for notational simplicity we drop the Keldysh indices $a, b, \ldots = \pm$ and the background index (b)):

$$\frac{\delta}{\delta g^{\mu\nu}(x)}\left[-\frac{\iota}{2}\mathrm{Tr}\ln[\Delta_F(y;y;[g^{(b)}_{\alpha'\beta'}])]\right]$$
$$=\frac{\delta}{\delta g^{\mu\nu}(x)}\left[\frac{\iota}{2}\int d^Dy\{\ln[\sqrt{-g(y)}\Box_y\delta^D(y-y')|_{y'\to y}]\}\right]$$
$$=\frac{\iota}{2}\int d^Dy\int d^Dz\partial^y_\alpha\left[\left(-\frac{1}{2}\sqrt{-g(y)}g^{\alpha\beta}(y)g_{\mu\nu}(y)+\sqrt{-g(y)}\delta^\alpha_{(\mu}\delta^\beta_{\nu)}\right)\delta^D(y-x)\partial^y_\beta\delta^D(y-z)\right]\Delta_F(z;y;[g_{\alpha'\beta'}]), \tag{C2}$$

where $\Delta_F(z;y;[g_{\alpha'\beta'}])=\Delta^{++}(z;y;[g_{\alpha'\beta'}])$ denotes the Feynman propagator. Now, because of the $\delta^D(y-x)$ function, all of the y's in the square brackets can be replaced by x's, and a part can be pulled out of the integral to obtain

$$\left(-\frac{1}{2}\sqrt{-g(x)}g^{\alpha\beta}(x)g^{\mu\nu}(x)+\sqrt{-g(x)}\delta^\alpha_{(\mu}\delta^\beta_{\nu)}\right)\frac{\iota}{2}\times\int d^Dy\int d^Dz\delta^D(y-x)[\partial^x_\beta\delta^D(x-z)]\times\Delta_F(z;y;[g_{\alpha'\beta'}])[\partial^y_\alpha\delta^D(y-x)].$$

Next, the derivatives $\partial^x_\beta\delta^D(x-z)=-\partial^z_\beta\delta^D(x-z)$ and ∂^y_α can be moved by partial integration to act on the propagator $\Delta_F(z;y[g_{\alpha\beta}])$, and finally the two integrals can be performed on the expense of the δ-functions. The result is

$$\frac{\delta}{\delta g^{\mu\nu}(x)}\left[-\frac{\iota}{2}\mathrm{Tr}\ln[\Delta_F(y;y;[g_{\alpha'\beta'}])]\right]=\frac{1}{2}\left(\frac{1}{2}\sqrt{-g(x)}g^{\alpha\beta}(x)g_{\mu\nu}(x)-\sqrt{-g(x)}\delta^\alpha_{(\mu}\delta^\beta_{\nu)}\right)\times[\partial^x_\alpha\partial^y_\beta\iota\Delta_F(x;y;[g_{\alpha'\beta'}])]_{y\to x}. \tag{C3}$$

This is the general expression for the tadpole diagram, i.e. the linear coupling of the stress energy tensor to metric perturbations.

In order to get the vacuum polarization we need to vary (C3) one more time with respect to $g^{\rho\sigma}(x')$. To facilitate the variation procedure, recall the operator inverse identity:

$$\int d^Dz\partial^{z'}_\gamma\sqrt{-g(z')}g^{\gamma\delta}(z')\partial^{z'}_\delta\delta^D(z'-z)\Delta_F(z;y;[g_{\alpha'\beta'}])=\delta^D(z'-y).$$

Varying this with respect to $g^{\rho\sigma}(x')$ yields

$$\int d^Dz\int d^Dz'\Delta_F(x;z';[g_{\alpha'\beta'}])\partial^{z'}_\gamma\times\left[\left(-\frac{1}{2}\sqrt{-g(z')}g^{\gamma\delta}(z')g_{\rho\sigma}(z')\delta^D(z'-x')+\sqrt{-g(z')}\delta^\gamma_{(\rho}\delta^\delta_{\sigma)}\delta^D(z'-x')\right)\partial^{z'}_\delta\delta^D(z'-z)\right]\times\Delta_F(z;y;[g_{\alpha'\beta'}])+\int d^Dz\delta^D(x-z)\frac{\delta}{\delta g^{\rho\sigma}(x')}\times\Delta_F(z;y;[g_{\alpha'\beta'}])=0.$$

Setting $y\to x$ and performing similar steps as above, this expression can be recast as

$$\left[\frac{\delta}{\delta g^{\rho\sigma}(x')}\Delta_F(x;y;[g_{\alpha'\beta'}])\right]_{y\to x}=-\left(\frac{1}{2}\sqrt{-g(x')}g^{\gamma\delta}(x')g_{\rho\sigma}(x')-\sqrt{-g(x')}\delta^\gamma_{(\rho}\delta^\delta_{\sigma)}\right)\times[\partial^{x'}_\gamma\Delta_F(x;x';[g_{\alpha'\beta'}])][\partial^{x'}_\delta\Delta_F(x';x;[g_{\alpha'\beta'}])\Big]. \tag{C4}$$

Inserting this into (C3) we get for the second variation of the bubble diagram:

$$\frac{\delta}{\delta g^{\rho\sigma}(x')}\frac{\delta}{\delta g^{\mu\nu}(x)}\left[-\frac{\iota}{2}\mathrm{Tr}\ln[\Delta_F(y;y;[g_{\alpha'\beta'}])]\right]=\frac{\iota}{2}\sqrt{-g(x)}\left(\frac{1}{2}g^{\alpha\beta}(x)g_{\mu\nu}(x)-\delta^\alpha_{(\mu}\delta^\beta_{\nu)}\right)[\partial^x_\alpha\partial^{x'}_\gamma\iota\Delta_F(x;x';[g_{\alpha'\beta'}])]\times[\partial^x_\beta\partial^{x'}_\delta\iota\Delta_F(x';x;[g_{\alpha'\beta'}])]\sqrt{-g(x')}\left(\frac{1}{2}g^{\gamma\delta}(x')g_{\rho\sigma}(x')-\delta^\gamma_{(\rho}\delta^\delta_{\sigma)}\right)-\sqrt{-g(x)}\left[\frac{1}{8}g_{\rho\sigma}(x)g^{\alpha\beta}(x)g_{\mu\nu}(x)-\frac{1}{4}\delta^\alpha_{(\rho}\delta^\beta_{\sigma)}g_{\mu\nu}(x)-\frac{1}{4}\delta^\alpha_{(\mu}\delta^\beta_{\nu)}g_{\rho\sigma}(x)+\frac{1}{4}g^{\alpha\beta}(x)g_{\mu(\rho}(x)g_{\sigma)\nu}(x)\right]\delta^D(x-x')[\partial^x_\alpha\partial^{x'}_\beta\iota\Delta_F(x;x';[g_{\alpha'\beta'}])]. \tag{C5}$$

The last two lines come from varying the metric tensors in (C3). Putting everything together, we get the following expansion of the scalar bubble diagram:

$$-\frac{\iota}{2}\mathrm{Tr}\ln\left[\Delta_F(y;y;[g^{(b)}_{\alpha\beta}])\right]=-\frac{\iota}{2}\mathrm{Tr}\ln[\Delta_F(y;y;[g_{\alpha'\beta'}])\Big]+\frac{1}{2}\int d^Dx\sqrt{-g(x)}\delta g^{\mu\nu}(x)T_{\mu\nu}(x;[g_{\alpha'\beta'}])-\frac{1}{2}\int d^Dx\sqrt{-g(x)}\int d^Dx'\sqrt{-g(x')}\delta g^{\mu\nu}(x)[_{\mu\nu}\tilde{\Pi}_{\rho\sigma}](x;x';[g_{\alpha'\beta'}])\delta g^{\rho\sigma}(x')+\mathcal{O}((\delta g^{\alpha\beta})^3), \tag{C6}$$

where

$$T_{\mu\nu}(x;[g_{\alpha'\beta'}]) = \left(\frac{1}{2}g^{\alpha\beta}(x)g_{\mu\nu}(x) - \delta^{\alpha}_{(\mu}\delta^{\beta}_{\nu)}\right)[\partial^x_\alpha\partial^{x'}_\beta \iota\Delta_F(x;x';[g_{\alpha'\beta'}])]_{x'\to x} \tag{C7}$$

$$\begin{aligned}[{}_{\mu\nu}\tilde{\Pi}_{\rho\sigma}](x;x';[g_{\alpha'\beta'}]) = &-\frac{\iota}{2}\left(\frac{1}{2}g^{\alpha\beta}(x)g_{\mu\nu}(x) - \delta^{\alpha}_{(\mu}\delta^{\beta}_{\nu)}\right)\left(\frac{1}{2}g^{\gamma\delta}(x')g_{\rho\sigma}(x') - \delta^{\gamma}_{(\rho}\delta^{\delta}_{\sigma)}\right)[\partial^x_\alpha\partial^{x'}_\gamma \iota\Delta_F(x;x';[g_{\alpha'\beta'}])]\\ &\times[\partial^x_\beta\partial^{x'}_\delta \iota\Delta_F(x';x;[g_{\alpha'\beta'}])] + \left[\frac{1}{8}g_{\rho\sigma}(x)g^{\alpha\beta}(x)g_{\mu\nu}(x) - \frac{1}{4}\delta^{\alpha}_{(\rho}\delta^{\beta}_{\sigma)}g_{\mu\nu}(x) - \frac{1}{4}\delta^{\alpha}_{(\mu}\delta^{\beta}_{\nu)}g_{\rho\sigma}(x)\right.\\ &\left.+\frac{1}{4}g^{\alpha\beta}(x)g_{\mu(\rho}(x)g_{\sigma)\nu}(x)\right]\frac{\delta^D(x-x')}{\sqrt{-g(x)}}[\partial^x_\alpha\partial^{x'}_\beta \iota\Delta_F(x;x';[g_{\alpha'\beta'}])].\end{aligned} \tag{C8}$$

The interpretation of this result is simple. The first term in (C6) is the scalar 1PI bubble in a classical background $g_{\alpha\beta}$, the second term is the 1PI tadpole, with the metric perturbation $\delta g^{\mu\nu}(x) = -h^{\mu\nu}(x) = -g^{\mu\alpha}(x)g^{\nu\beta}(x)h_{\alpha\beta}(x)$ and the matter stress energy tensor $T_{\mu\nu}$ given in (C7). The last term in (C6) is the contribution that is quadratic in the metric perturbation $\delta g^{\mu\nu} \simeq -h^{\mu\nu}$, with ${}_{\mu\nu}\tilde{\Pi}_{\rho\sigma}(x;x';[g_{\alpha'\beta'}])$ being (a part of) the graviton vacuum polarization (calculated on a $g_{\alpha\beta}(x)$ background), whose explicit form is given in (C8). The first two lines in (C8) represent the nonlocal part of the graviton vacuum polarization shown by the middle diagram in Fig. 2, and they perfectly agree with the graviton vacuum polarization shown in the first line of Eq. (D1) in Sec. III ($\xi \to 0$ and $g_{\alpha\beta} \to \eta_{\alpha\beta}$). The last two lines in (C8) represent a part of the local contribution to the vacuum polarization. The other part is obtained by observing that the quadratic term in the expansion $\delta g^{\mu\nu}(x) = -h^{\mu\nu}(x) + h^{\mu}_{\rho}(x)h^{\rho\nu}(x) + \mathcal{O}((h^{\mu\nu}(x))^3)$ in the tadpole in (C6) yields the local contribution to the vacuum polarization of the form

$$\begin{aligned}&\delta[{}_{\mu\nu}\Pi_{\rho\sigma}](x;x';[g_{\alpha'\beta'}])\\ &= \left[-\frac{1}{2}g^{\alpha\beta}(x)g_{\nu(\rho}g_{\sigma)\mu}(x) + \delta^{\alpha}_{(\mu}g_{\nu)(\rho}\delta^{\beta}_{\sigma)}\right]\delta^D(x-x')\\ &\quad\times[\partial^x_\alpha\partial^{x'}_\beta \iota\Delta_F(x;x';[g_{\alpha'\beta'}])],\end{aligned} \tag{C9}$$

such that

$$\begin{aligned}&[{}_{\mu\nu}\Pi_{\rho\sigma}](x;x';[g_{\alpha'\beta'}])\\ &\quad= [{}_{\mu\nu}\tilde{\Pi}_{\rho\sigma}](x;x';[g_{\alpha'\beta'}]) + \delta[{}_{\mu\nu}\Pi_{\rho\sigma}](x;x';[g_{\alpha'\beta'}])\end{aligned} \tag{C10}$$

represents the complete graviton vacuum polarization, which correctly includes also the local contribution from the quartic interaction (13) (in the limit when $g_{\mu\nu} \to \eta_{\mu\nu}$ and $\xi \to 0$). A technique that is equivalent to expanding the 2PI bubble diagram around the Minkowski background was used to calculate the leading quantum correction to the Newtonian potential of a static point particle in Refs. [25,26]. In the limit when time transients are gone, our results presented in Sec. IV indeed agree with those of Refs. [25,26], just as one would expect from the analysis presented in this Appendix.

And finally one more remark. From Eqs. (C6) and (C7) and the preceding analysis it follows that the nonlocal part of the graviton vacuum polarization is obtained by

$$\begin{aligned}&\{[{}_{\mu\nu}\Pi_{\rho\sigma}](x;x')\}_{\text{nonlocal}}\\ &\quad= \frac{1}{\sqrt{g(x')}}\frac{\tilde{\delta}}{\tilde{\delta}g_{\rho\sigma}(x')}[T_{\mu\nu}(x;[g_{\alpha'\beta'}])],\end{aligned} \tag{C11}$$

where the tilde on the functional derivative ($\tilde{\delta}$) signifies that it acts on the propagator(s) in $T_{\mu\nu}$ only. A special case of this relation can be found in Eqs. (12–15) and (47) of Ref. [33], where the nonlocal part of the graviton vacuum polarization tensor around the Minkowski vacuum is expressed in terms of the nonlocal, Wick-contracted, stress energy—stress energy correlator.

APPENDIX D: NONLOCAL CONTRIBUTIONS

In this Appendix we carry out thorough calculation of the graviton vacuum polarization. By Wick-contracting all noncoincident scalars and making use of the propagator definitions (A12) and (A13), the graviton vacuum polarization inferred from (24) becomes

$$\begin{aligned}
\iota[{}^{\pm}_{\mu\nu}\Pi^{\pm}_{\rho\sigma}](x;x') = \frac{\pm\kappa}{2}\frac{\pm\kappa}{2}\bigg\{2\Big(\frac{1}{2}\eta^{\alpha\beta}\eta_{\mu\nu} - \delta^{(\alpha}_{(\mu}\delta^{\beta)}_{\nu)}\Big)\Big[\partial_\alpha\partial'_{(\alpha'}\iota\Delta_{\pm\pm}(x;x')\partial'_{\beta')}\partial_\beta\iota\Delta_{\pm\pm}(x;x')\Big]\Big(\frac{1}{2}\eta^{\alpha'\beta'}\eta_{\rho\sigma} - \delta^{(\alpha'}_{(\rho}\delta^{\beta')}_{\sigma)}\Big) \\
+ 4\xi\Big(\frac{1}{2}\eta^{\alpha\beta}\eta_{\mu\nu} - \delta^{(\alpha}_{(\mu}\delta^{\beta)}_{\nu)}\Big)\Big[\partial_\alpha\partial'_{(\rho}\iota\Delta_{\pm\pm}(x;x')\partial'_{\sigma)}\partial_\beta\iota\Delta_{\pm\pm}(x;x') \\
- \eta^{\gamma'\delta'}\eta_{\rho\sigma}\partial_\alpha\partial'_{(\gamma'}\iota\Delta_{\pm\pm}(x;x')\partial'_{\delta')}\partial_\beta\iota\Delta_{\pm\pm}(x;x') + \partial_{(\alpha}\iota\Delta_{\pm\pm}(x;x')\partial_{\beta)}D'_{\rho\sigma}\iota\Delta_{\pm\pm}(x;x')\Big] \\
+ 4\xi\Big[\partial'_{\alpha'}\partial_{(\mu}\iota\Delta_{\pm\pm}(x;x')\partial_{\nu)}\partial'_{\beta'}\iota\Delta_{\pm\pm}(x;x') - \eta^{\gamma\delta}\eta_{\mu\nu}\partial'_{\alpha'}\partial_{(\gamma}\iota\Delta_{\pm\pm}(x;x')\partial_{\delta)}\partial'_{\beta'}\iota\Delta_{\pm\pm}(x;x') \\
+ \partial'_{(\alpha'}\iota\Delta_{\pm\pm}(x;x')\partial'_{\beta')}D_{\mu\nu}\iota\Delta_{\pm\pm}(x;x')\Big]\Big(\frac{1}{2}\eta^{\alpha'\beta'}\eta_{\rho\sigma} - \delta^{(\alpha'}_{(\rho}\delta^{\beta')}_{\sigma)}\Big) \\
+ 8\xi^2\Big[\partial_\mu\partial'_{(\sigma}\iota\Delta_{\pm\pm}(x;x')\partial'_{\rho)}\partial_\nu\iota\Delta_{\pm\pm}(x;x') - \eta^{\gamma'\delta'}\eta_{\rho\sigma}\partial_\mu\partial'_{(\gamma'}\iota\Delta_{\pm\pm}(x;x')\partial'_{\delta')}\partial_\nu\iota\Delta_{\pm\pm}(x;x') \\
- \eta^{\gamma\delta}\eta_{\mu\nu}\partial_\gamma\partial'_{(\rho}\iota\Delta_{\pm\pm}(x;x')\partial'_{\sigma)}\partial_\delta\iota\Delta_{\pm\pm}(x;x') + \eta^{\gamma\delta}\eta_{\mu\nu}\eta^{\gamma'\delta'}\eta_{\rho\sigma}\partial_\gamma\partial'_{(\delta'}\iota\Delta_{\pm\pm}(x;x')\partial'_{\gamma')}\partial_\delta\iota\Delta_{\pm\pm}(x;x') \\
+ \partial_{(\mu}\iota\Delta_{\pm\pm}(x;x')\partial_{\nu)}D'_{\rho\sigma}\iota\Delta_{\pm\pm}(x;x') - \eta^{\gamma\delta}\eta_{\mu\nu}\partial_{(\gamma}\iota\Delta_{\pm\pm}(x;x')\partial_{\delta)}D'_{\rho\sigma}\iota\Delta_{\pm\pm}(x;x') \\
+ \partial'_{(\rho}\iota\Delta_{\pm\pm}(x;x')\partial'_{\sigma)}D_{\mu\nu}\iota\Delta_{\pm\pm}(x;x') - \eta^{\gamma'\delta'}\eta_{\rho\sigma}\partial'_{(\gamma'}\iota\Delta_{\pm\pm}(x;x')\partial'_{\delta')}D_{\mu\nu}\iota\Delta_{\pm\pm}(x;x') \\
+ \frac{1}{2}\iota\Delta_{\pm\pm}(x;x')D_{\mu\nu}D'_{\rho\sigma}\iota\Delta_{\pm\pm}(x;x') + \frac{1}{2}D_{\mu\nu}\iota\Delta_{\pm\pm}(x;x')D'_{\rho\sigma}\iota\Delta_{\pm\pm}(x;x')\Big]\bigg\}.
\end{aligned} \tag{D1}$$

The first line in (D1) corresponds to the graviton vacuum polarization tensor of a minimally coupled massless scalar field already calculated in [16]. Our analysis in Appendix C shows that this expression can be also obtained by expanding the 2PI bubble diagram around the Minkowski space, i.e. take Eq. (C8) in the limit when $g_{\mu\nu} \to \eta_{\mu\nu}$. The vacuum polarization (D1) contains a local divergence. In order to extract it, the following identities can be employed [33], which are easily derived from (A15), (A16), and (B1):

$$\begin{aligned}
\partial'_\beta\partial_\alpha\iota\Delta_{++}(x;x') &= \delta^0_\alpha\delta^0_\beta\iota\delta^D(x-x') + \frac{\Gamma(\frac{D}{2})}{2\pi^{D/2}}\Big[\frac{\eta_{\alpha\beta}}{(\Delta x^2_{++})^{D/2}} - D\frac{\Delta x_\alpha\Delta x_\beta}{(\Delta x^2_{++})^{(D/2)+1}}\Big], \\
\partial'_\beta\partial_\alpha\iota\Delta_{--}(x;x') &= -\delta^0_\alpha\delta^0_\beta i\delta^D(x-x') + \frac{\Gamma(\frac{D}{2})}{2\pi^{D/2}}\Big[\frac{\eta_{\alpha\beta}}{(\Delta x^2_{--})^{D/2}} - D\frac{\Delta x_\alpha\Delta x_\beta}{(\Delta x^2_{--})^{(D/2)+1}}\Big], \\
\partial'_\beta\partial_\alpha\iota\Delta_{-+}(x;x') &= \frac{\Gamma(\frac{D}{2})}{2\pi^{D/2}}\Big[\frac{\eta_{\alpha\beta}}{(\Delta x^2_{-+})^{D/2}} - D\frac{\Delta x_\alpha\Delta x_\beta}{(\Delta x^2_{-+})^{(D/2)+1}}\Big], \\
\partial'_\beta\partial_\alpha\iota\Delta_{+-}(x;x') &= \frac{\Gamma(\frac{D}{2})}{2\pi^{D/2}}\Big[\frac{\eta_{\alpha\beta}}{(\Delta x^2_{+-})^{D/2}} - D\frac{\Delta x_\alpha\Delta x_\beta}{(\Delta x^2_{+-})^{(D/2)+1}}\Big].
\end{aligned} \tag{D2}$$

In order to reduce the ξ-dependent terms we also need the following derivatives:

$$\begin{aligned}
&\partial_\sigma\partial'_{\rho'}\partial'_{\sigma'}\iota\Delta_{\pm\pm}(x;x') \\
&= (\delta^0_\sigma\delta^0_{\rho'}\partial'_{\sigma'} + \delta^0_\sigma\delta^0_{\sigma'}\partial'_{\rho'} - \delta^0_{\sigma'}\delta^0_{\rho'}\partial_\sigma)\iota\delta^D(x-x') \\
&\quad + D\frac{\Gamma(\frac{D}{2})}{2\pi^{D/2}}\Big[\frac{\eta_{\rho'\sigma'}\Delta x_\sigma + \eta_{\rho'\sigma}\Delta x_{\sigma'} + \eta_{\sigma'\sigma}\Delta x_{\rho'}}{\Delta x^{D+2}_{\pm\pm}} - (D+2)\frac{\Delta x_{\rho'}\Delta x_{\sigma'}\Delta x_\sigma}{\Delta x^{D+4}_{\pm\pm}}\Big] \\
&\partial_\rho\partial_\sigma\partial'_{\rho'}\partial'_{\sigma'}\iota\Delta_{\pm\pm}(x;x') \\
&= (\delta^0_\sigma\delta^0_{\rho'}\partial'_{\sigma'}\partial_\rho + \delta^0_{\sigma'}\delta^0_\rho\partial_\sigma\partial'_{\rho'} + \delta^0_\sigma\delta^0_{\sigma'}\partial'_{\rho'}\partial_\rho + \delta^0_{\rho'}\delta^0_\rho\partial_\sigma\partial'_{\sigma'} - \delta^0_\sigma\delta^0_\rho\partial'_{\sigma'}\partial'_{\rho'} \\
&\quad - \delta^0_{\sigma'}\delta^0_{\rho'}\partial_\sigma\partial_\rho)\iota\delta^D(x-x') + D\frac{\Gamma(\frac{D}{2})}{2\pi^{D/2}}\Big[\frac{\eta_{\rho'\sigma'}\eta_{\rho\sigma} + \eta_{\rho'\sigma}\eta_{\sigma'\rho} + \eta_{\sigma'\sigma}\eta_{\rho'\rho}}{\Delta x^{D+2}_{\pm\pm}} + (D+2)(D+4)\frac{\Delta x_\rho\Delta x_\sigma\Delta x_{\rho'}\Delta x_{\sigma'}}{\Delta x^{D+6}_{\pm\pm}} \\
&\quad - (D+2)\frac{\eta_{\rho'\sigma'}\Delta x_\rho\Delta x_\sigma + \eta_{\rho'\sigma}\Delta x_\rho\Delta x_{\sigma'} + \eta_{\rho'\rho}\Delta x_{\sigma'}\Delta x_\sigma + \eta_{\rho\sigma'}\Delta x_{\rho'}\Delta x_\sigma + \eta_{\rho\sigma}\Delta x_{\rho'}\Delta x_{\sigma'} + \eta_{\sigma\sigma'}\Delta x_\rho\Delta x_{\rho'}}{\Delta x^{D+4}_{\pm\pm}}\Big].
\end{aligned} \tag{D3}$$

When these expressions are inserted into the graviton vacuum polarization (D1), the terms in (D2) and (D3) that contain delta functions do not contribute in dimensional regularization, since they hit a propagator which contains a function of the type $(\Delta x)^{-\alpha}$, where $\alpha = \alpha(D)$ is a D-dependent power. One can show that rigorously by the analogous reasoning as was used in Sec. III A.

The next step in the reduction of the graviton vacuum polarization tensor is the extraction of derivatives. In the following we express various terms appearing in the vacuum polarization as an operator acting on the single function, $1/\Delta x^{2D-4}$ (which still contains the local divergence that we are after):

$$\frac{1}{\Delta x_{\pm\pm}^{2D}} = \left\{\frac{\partial^4}{4D(D-1)(D-2)^2}\right\}\frac{1}{\Delta x_{\pm\pm}^{2D-4}}, \tag{D4}$$

$$\frac{\Delta x_\mu \Delta x_\nu}{\Delta x_{\pm\pm}^{2D+2}} = \left\{\frac{\eta_{\mu\nu}\partial^4 + D\partial_\mu\partial_\nu\partial^2}{8D^2(D-1)(D-2)^2}\right\}\frac{1}{\Delta x_{\pm\pm}^{2D-4}}, \tag{D5}$$

$$\frac{\Delta x_\mu \Delta x_\nu \Delta x_\rho \Delta x_\sigma}{\Delta x_{\pm\pm}^{2D+4}} = \left\{\frac{(\eta_{\mu\nu}\eta_{\rho\sigma} + 2\eta_{\mu(\rho}\eta_{\sigma)\nu})\partial^4}{16D^2(D+1)(D-1)(D-2)^2}, + \frac{(\eta_{\mu\nu}\partial_\rho\partial_\sigma + 4\partial_{(\mu}\eta_{\nu)(\rho}\partial_{\sigma)} + \eta_{\rho\sigma}\partial_\mu\partial_\nu)\partial^2 + (D-2)\partial_\mu\partial_\nu\partial_\rho\partial_\sigma}{16D(D+1)(D-1)(D-2)^2}\right\}\frac{1}{\Delta x_{\pm\pm}^{2D-4}}. \tag{D6}$$

Before reducing all the terms of the tensor (D1) it is instructive to note the structure of the parts linear and quadratic in ξ which are basically the same up to a multiplicative constant. To be more precise, in the second and third terms that are linear in ξ the following operator can be extracted:

$$-(D^2 - D - 2)D_{\mu\nu}D_{\rho\sigma}, \tag{D7}$$

while in the term that is quadratic in ξ we can extract

$$4(D^2 - 1)D_{\mu\nu}D_{\rho\sigma}. \tag{D8}$$

After some algebra, and after combining the linear and quadratic contributions in ξ, the graviton vacuum polarization (D1) can be written as

$$\begin{aligned} \imath[{}^{\pm}_{\mu\nu}\Pi^{\pm}_{\rho\sigma}](x;x') &= \left(\frac{\pm\kappa}{2}\right)\left(\frac{\pm\kappa}{2}\right)\frac{\Gamma^2(\frac{D}{2})}{16\pi^D(D^2-1)(D-2)^2}\Big\{4D_{\alpha\beta\mu\nu}D^{\alpha\beta}{}_{\rho\sigma} \\ &\quad + \left[-\frac{D}{D-1} + 8(D^2-1)\left(\xi - \frac{D-2}{4(D-1)}\right)^2\right]D_{\mu\nu}D_{\rho\sigma}\Big\} \\ &\quad \times \frac{1}{\Delta x_{\pm\pm}^{2D-4}}. \end{aligned} \tag{D9}$$

Upon performing one more partial integration:

$$\frac{1}{\Delta x_{\pm\pm}^{2D-4}} = \frac{\partial^2}{2(D-3)(D-4)}\frac{1}{\Delta x_{\pm\pm}^{2D-6}}, \tag{D10}$$

we see that $(D-4)$ appears in the denominator, signifying a divergence. This divergence can be removed by recalling the identities (that were employed when constructing the propagators (A15)),

$$\partial^2\frac{1}{\Delta x_{++}^{D-2}} = \frac{4\pi^{D/2}}{\Gamma(\frac{D}{2}-1)}\imath\delta^D(x-x'), \qquad \partial^2\frac{1}{\Delta x_{+-}^{D-2}} = 0,$$
$$\partial^2\frac{1}{\Delta x_{--}^{D-2}} = -\frac{4\pi^{D/2}}{\Gamma(\frac{D}{2}-1)}\imath\delta^D(x-x'), \qquad \partial^2\frac{1}{\Delta x_{-+}^{D-2}} = 0. \tag{D11}$$

Based on these identities and Eq. (D10) we easily obtain

$$\begin{aligned} \frac{1}{\Delta x_{\pm\pm}^{2D-4}} &= \frac{\partial^2}{2(D-3)(D-4)}\left(\frac{1}{\Delta x_{\pm\pm}^{2D-6}} - \frac{\mu^{D-4}}{\Delta x_{\pm\pm}^{D-2}}\right) \\ &\quad + (\sigma^3)^{\pm\pm}\frac{2\pi^{D/2}\mu^{D-4}}{\Gamma(\frac{D}{2}-1)(D-3)(D-4)}\imath\delta^D(x-x'), \end{aligned} \tag{D12}$$

where $\sigma^3 = \mathrm{diag}(1,-1)$ is the Pauli matrix and μ is the mass scale appearing for dimensional reasons which signifies the renormalization scale. The identity (D12) is judiciously constructed, such that $(\Delta x_{\pm\pm}^2)^{2-D}$ has been split into a piece that is finite in $D=4$ and a local divergent piece. Indeed, when expanded around $D=4$, Eq. (D12) can be recast as

$$\begin{aligned} \frac{1}{\Delta x_{\pm\pm}^{2D-4}} &= -\frac{\mu^{2D-8}\partial^2}{32}[\ln^2(\mu^2\Delta x_{\pm\pm}^2) - 2\ln(\mu^2\Delta x_{\pm\pm}^2) \\ &\quad + \mathcal{O}(D-4)] + (\sigma^3)^{\pm\pm} \\ &\quad \times \frac{2\pi^{D/2}\mu^{D-4}}{\Gamma(\frac{D}{2}-1)(D-3)(D-4)}\imath\delta^D(x-x'), \end{aligned} \tag{D13}$$

where we made use of

$$\frac{1}{\Delta x_{\pm\pm}^{2D-6}} - \frac{\mu^{D-4}}{\Delta x_{\pm\pm}^{D-2}} = \frac{\mu^{2D-8}}{\Delta x_{\pm\pm}^2}\left[-\frac{D-4}{2}\ln(\mu^2\Delta x_{\pm\pm}^2) + \mathcal{O}((D-4)^2)\right] \tag{D14}$$

and

$$\frac{\ln(\mu^2\Delta x^2_{\pm\pm})}{\Delta x^2_{\pm\pm}} = \frac{1}{8}\partial^2\Big[\ln^2(\mu^2\Delta x^2_{\pm\pm}) - 2\ln(\mu^2\Delta x^2_{\pm\pm})\Big]. \tag{D15}$$

With the results outlined in this Appendix it is possible now to express a graviton polarization tensor in terms of the sum of the nonlocal finite and local divergent parts (27).

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