Degenerations of cubic fourfolds
and
holomorphic symplectic geometry

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Cover illustration:
The cover shows the configuration of 27 lines on the Clebsch surface. This is a cubic surface, which may be most elegantly described by the homogeneous equations $x_0 + x_1 + x_2 + x_3 + x_4 = x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$ in $\mathbf{P}^4$. Since the second half of the 19th century cubic hypersurfaces have been studied by means of the configuration of lines on them. This thesis fits into this line of research; we study varieties of lines on cubic fourfolds. Unfortunately, these are hard to depict in any sensible way.

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en
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(met een samenvatting in het Nederlands)

Proefschrift

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Contents

Introduction

In this thesis we study the relation between the geometry of cubic hypersurfaces in 5-dimensional projective space and the geometry of so-called holomorphic symplectic manifolds. An essential ingredient will be the study of the variety of lines on the cubic hypersurfaces. In this introduction, we will try to give an idea of what these notions mean and why they are worth studying. We end with a short outline of the thesis.

Lines on cubic hypersurfaces

By a *cubic hypersurface* we mean a hypersurface of degree 3 in some complex projective space. In particular, a generic line in this ambient projective space will intersect the hypersurface in three distinct points. However, there may be lines that are completely contained in the hypersurface. It is an interesting problem to characterize the configuration of these lines and investigate what it tells you about the geometry of the hypersurface itself.

This problem is very classical: the characterization of lines on a cubic hypersurface of dimension 2, or cubic surface for short, goes back to the work of Cayley and Salmon in the 19^{th} century. In 1849 they published their now famous result that any smooth cubic surface contains precisely 27 lines. This result is much more than just an exercise in enumerative geometry. Indeed, as Dolgachev remarks in [11], it can be considered as the first non-trivial result on algebraic surfaces of degree higher than 2, and in fact, as the beginning of modern algebraic geometry.

An important aspect of the result is that the cubic surface is entirely determined by the configuration of the 27 lines. We thus obtain a dictionary between the geometry of cubic surfaces and properties of the configuration of the lines on it. This provides a very powerful tool in the study of cubic surfaces, which is still going on today.

For a smooth cubic hypersurface of dimension 3 or higher the set of lines contained in it is no longer finite, but it does have the structure of an algebraic variety. We call this the *variety of lines* on the cubic hypersurface. The study of such varieties of lines can again be a powerful tool to understand the geometry of cubic hypersurfaces. An example of this is provided by the celebrated paper "The intermediate Jacobian of the cubic threefold" ([8]) published by Clemens and Griffiths in 1972. In this paper they show how one can associate, in a natural way, to every smooth cubic hypersurface in $\mathbf{P}^4$ (cubic threefolds for short) an Abelian variety, which they call the intermediate Jacobian of the cubic threefold. They relate this intermediate Jacobian to the variety of lines on the cubic threefold, and then use this relation to show that a cubic threefold is completely determined by its intermediate Jacobian. This observation enabled them to prove strong statements on the geometry of cubic threefolds.

Going one dimension higher we arrive at smooth cubic hypersurfaces in $\mathbf{P}^5$, *cubic fourfolds* for short. In the 80's, Beauville and Donagi studied the varieties of lines on cubic fourfolds and showed that any such variety has the structure of an *irreducible holomorphic symplectic fourfold*, that is, it is smooth, simply connected, of dimension 4 and admits a non-degenerate holomorphic 2-form which is unique up to overall scaling by a constant (see [6]). Furthermore, they prove strong relations between the geometry of the cubic fourfolds and the holomorphic symplectic geometry of the varieties of lines on them. These results will play a central in this thesis. To state them and explain their relevance we need to introduce some extra notions; we ask the reader to bear with us for a moment.

Hodge structures and Torelli problems

Let X be a complex manifold of dimension n and $z_1, \ldots, z_n$ local complex coordinates on X. A differential form α with complex coefficients on X is said to be a (p,q)-form if it can locally be written as linear combination of forms $dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \ldots \wedge d\overline{z}_{j_q}$ with $1 \le i_1 < \ldots < i_p \le n$ and $1 \le j_1 < \ldots < j_q \le n$. Any differential k-form on X with complex coefficients uniquely decomposes as sum of (p,q)-forms, with $p+q=k$. It is a standard result in the theory of complex manifolds that for any complex projective manifold (or more generally, compact Kähler manifold) X this decomposition induces a natural decomposition of the complexification of the de Rham cohomology groups:

$$H^k(X,\mathbf{C}) = \bigoplus_{p,q\ge 0, p+q=k} H^{p,q}(X).$$

with the property that all $H^{p,q}(X)$ are finite dimensional complex vector spaces and $H^{p,q}(X)$ is naturally identified with the complex conjugate of $H^{q,p}(X)$. By the de Rham theorem we have a natural identification $H^k(X,\mathbf{C}) \cong H^k_{\mathrm{sing}}(X,\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C}$, where $H^k_{\mathrm{sing}}(X,\mathbf{Z})$ is the k^{th} singular cohomology group of X with coefficients in $\mathbf{Z}$. In what follows we will drop the subscript 'sing'.

We obtain a decomposition

$$H^k(X,\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C} = \bigoplus_{p,q\ge 0, p+q=k} H^{p,q}(X),$$

this is called a *Hodge decomposition.* Let M be the real manifold underlying X, then clearly $H^k(X,\mathbf{Z}) = H^k(M,\mathbf{Z})$; it does not depend on the complex structure. However, the Hodge decomposition *does* depend on the complex structure. More generally, if X' is a complex (Kähler) manifold that is *diffeomorphic* to X but not isomorphic as complex manifold, then there exists an isomorphism $\varphi : H^k(X',\mathbf{Z}) \to H^k(M,\mathbf{Z})$, but the induced Hodge decomposition

$$H^k(M,\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C} = \bigoplus_{p,q\geq 0, p+q=k} \varphi_{\mathbf{C}}(H^{p,q}(X))$$

may be different from the original Hodge decomposition

$$H^k(M,\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C} = \bigoplus_{p,q\geq 0, p+q=k} H^{p,q}(X).$$

That is, the Hodge decomposition may change *with respect to* the singular cohomology of the manifold. The data of the Hodge decomposition of a complex manifold X *and* singular cohomology of X (or equivalently, M) is called the *Hodge structure* of X.

In this way we can in principle reduce something as elusive as deformations of complex structures on the real manifold M to deformations of Hodge structures, which are essentially defined in the context of linear algebra. We say 'in principle': the question that remains is of course to what extent the complex structure is actually determined by the Hodge structure.

Torelli was amongst the first to study this question, in the specific case of smooth compact complex algebraic curves. Such curves have the property that the first singular cohomology group comes with a natural bilinear form: the intersection pairing. In 1913 Torelli proved the following now well-known and famous fact:

> *If C and C' are compact complex algebraic curves such that there exists an isomorphism $H^1(C,\mathbf{Z}) \to H^1(C',\mathbf{Z})$ that preserves the intersection pairing and whose extension $H^1(C,\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C} \to H^1(C',\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C}$ preserves the Hodge structure, then C is isomorphic to C'.*

The data of the intersection pairing on $H^1(C,\mathbf{Z})$ in addition to the Hodge structure on $H^1(C,\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C}$ is called a *polarized* Hodge structure. So Torelli's theorem tells us that a compact complex algebraic curve is determined, up to isomorphism, by its polarized Hodge structure. Nowadays, the question to what extent a complex manifold is determined by the (polarized[1]) Hodge structure on its cohomology is referred to as a *Torelli problem.*

[1] The notion of a *polarized* Hodge structure is more involved in the context of complex manifolds of dimension higher than one. We will not go into this here. The reader may consult [48] for an extensive and thorough exposition of Hodge theory.

The Torelli problem has been (and is still being) extensively studied for irreducible holomorphic symplectic manifolds. We will describe this in a bit more detail. It started with the study of irreducible holomorphic symplectic surfaces, better known as K3 surfaces. It can be shown that all such surfaces are Kähler and mutually diffeomorphic (see for example [4]). Furthermore, for any K3 surface S the second cohomology $H^2(S,\mathbf{Z})$ is torsion-free. The natural pairing on cohomology defines a non-degenerate integral bilinear form q on $H^2(S,\mathbf{Z})$. It makes $H^2(S,\mathbf{Z})$ into a *lattice*. Under the natural embedding $H^2(S,\mathbf{Z}) \hookrightarrow H^2_{dR}(S)$ this bilinear form can be characterized by the property $q(\alpha,\beta) = \int_S \widetilde{\alpha} \wedge \widetilde{\beta}$, where $\widetilde{\alpha}, \widetilde{\beta}$ are 2-forms representing the cohomology classes $\alpha, \beta \in H^2(S,\mathbf{Z})$.

The lattice structure on the second cohomology of S only depends on the real manifold S_r underlying S. The complex structure on S then defines a Hodge structure by the decomposition:

$$H^2(S_\mathrm{r},\mathbf{Z}) \otimes_\mathbf{Z} \mathbf{C} = H^{2,0}(S) \oplus H^{1,1}(S) \oplus H^{0,2}(S).$$

It can be rather easily shown that this decomposition has a number of particular properties (see for example [4]):

- $H^{2,0}(S) = \overline{H^{0,2}(S)}$ (this holds for all Hodge decompositions);
- $H^{2,0}(S) \oplus H^{0,2}(S)$ is orthogonal to $H^{1,1}(S)$ with respect to $q_\mathbf{C}$ (the complex bilinear extension of q);
- $H^{2,0}(S)$ is of dimension 1 (over $\mathbf{C}$);
- $H^{2,0}(S)$ is isotropic with respect to $q_\mathbf{C}$;
- For any $\alpha \in H^{2,0}(S)$ we have $q_\mathbf{C}(\alpha,\overline{\alpha}) > 0$.

In fact, the last three properties follow directly from the fact that by definition S admits a non-degenerate symplectic form that is unique up to scaling. The Hodge structures on $H^2(S_\mathrm{r},\mathbf{Z}) \otimes_\mathbf{Z} \mathbf{C}$ with these properties are parameterized by a complex space called the *period domain*. Let us denote it by Ω_{S_r}. Its elements are called *periods*

We mentioned before that all K3 surfaces are diffeomorphic, hence for any K3 surface S' we can find a lattice isomorphism $\varphi : H^2(S',\mathbf{Z}) \to H^2(S_\mathrm{r},\mathbf{Z})$. This is known as a *marking*. It is not unique since the lattice-automorphism group of $H^2(S_\mathrm{r},\mathbf{Z})$ is non-trivial. Through φ, the Hodge decomposition of S' defines a Hodge decomposition of $H^2(S_\mathrm{r},\mathbf{Z}) \otimes_\mathbf{Z} \mathbf{C}$ with the properties listed above, and hence a point in Ω_{S_r}. In particular, if $\mathfrak{M}$ is the moduli space of K3 surfaces with a marking (such a moduli space exists, we will not go into further details), then we obtain a map

$$\mathcal{P} : \mathfrak{M} \to \Omega_{S_\mathrm{r}}$$

called the *period map*.

This map has interesting properties that have been extensively studied. It can be shown to be a local isomorphism: that is, every small deformation of the complex structure on a K3 surface is captured by a deformation of the Hodge structure. Furthermore, it is surjective, that is, every decomposition of $H^2(S_r,\mathbf{Z})\otimes_{\mathbf{Z}}\mathbf{C}$ that satisfies the properties listed above is actually realized as the Hodge structure of a K3 surface. But most importantly, it has been proven that if two marked K3 surfaces (S,φ) and (S',φ') have the same period (that is, the same image under the period map), that then $S\cong S'$. In particular, a K3 surface is completely determined, up to isomorphism, by its polarized[2] Hodge structure. In fact, there exist even stronger statements that involve the relation between the markings of S and S' and the isomorphism between S and S', but we will not address those in this introduction. For a complete account of the Torelli problem for K3 surfaces, we refer the reader to [4] and [2].

This success has been a source of inspiration for the study of higher dimensional irreducible holomorphic symplectic manifolds. The theory of Hodge structures for such manifolds is strikingly similar to the case of K3 surfaces. The reason is that for any irreducible holomorphic symplectic manifold Y the second integral cohomology group $H^2(Y,\mathbf{Z})$ is torsion-free and can be naturally equipped with a non-degenerate integral bilinear form q such that the Hodge decomposition

$$H^2(Y,\mathbf{Z})\otimes_{\mathbf{Z}}\mathbf{C}=H^{2,0}(Y)\oplus H^{1,1}(Y)\oplus H^{0,2}(Y)$$

satisfies exactly the properties listed above for the case of K3 surfaces. This important observation was made by Beauville in the 80's, see [3]. The form q is called the *Beauville–Bogomolov form.* So completely analogous to the case of K3 surfaces, given a diffeomorphism class C of irreducible holomorphic symplectic manifolds, we have a period domain Ω_C that parameterizes all Hodge structures that satisfy the given properties. Furthermore, it can be shown that there exists a moduli space $\mathfrak{M}_C$ of *marked* irreducible holomorphic symplectic manifolds of the given diffeomorphism class, and a period map

$$\mathcal{P}_C:\mathfrak{M}_C\to\Omega_C$$

Beauville showed in [3] that this period map is a local isomorphism and Huybrechts showed in [25] that it is surjective. These results show that the Hodge structures of irreducible holomorphic symplectic manifolds encode a lot of geometric information. However, it is generally *not* true that an irreducible holomorphic symplectic manifold is completely determined by its Hodge structure; counterexamples exist. The question to what extent the Hodge structures *do* determine the manifolds is still very actively studied these days; we will address this in more detail in chapter 1.

[2]As in the case of algebraic curves, the intersection pairing on the cohomology group is an essential part of the data

The Beauville–Donagi construction

Let us return to what we started with: the study of cubic hypersurfaces. As we mentioned before, Beauville and Donagi proved that for any cubic fourfold X the variety of lines $F(X)$ naturally has the structure of an irreducible holomorphic symplectic fourfold. The fourfold $F(X)$ has an extra property over general irreducible holomorphic symplectic fourfolds; it is naturally contained in a Grassmannian space (namely the space that parameterizes lines in $\mathbf{P}^5$, or equivalently, 2-planes in $\mathbf{C}^6$) and therefore *projective*. Hence it comes with a natural line bundle, the polarization, which it inherits from the ambient Grassmannian space. We denote it by $\mathcal{O}_{F(X)}(1)$. The important point is that any line bundle has a first Chern class, which is both an element of the second integral cohomology *and* the cohomology class of a $(1,1)$-form. The existence of such a class puts restrictions on the possible Hodge structures.
More explicitly, let N be the real manifold underlying $F(X)$ for any cubic fourfold X (all $F(X)$ are diffeomorphic) and let $\alpha \in H^2(N,\mathbf{Z})$ be the first Chern class of $\mathcal{O}_{F(X)}(1)$. Let $\Omega_{\mathrm{BD}} \subset \Omega_N$ be the subspace of Hodge structures for which the Hodge decomposition

$$H^2(N,\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$$

is such that $\alpha \in H^{1,1}$. It can be shown that this is a complex subspace of Ω_N. Let $\mathfrak{M}_{\mathrm{BD}}$ be the moduli space of marked irreducible holomorphic symplectic manifolds, diffeomorphic to N, with period in Ω_{BD}. Of course we then have the restricted period map

$$\mathcal{P}_{\mathrm{BD}} : \mathfrak{M}_{\mathrm{BD}} \to \Omega_{\mathrm{BD}}.$$

It is surjective by construction and a local isomorphism.

In addition to what we mentioned earlier, and maybe more importantly, Beauville and Donagi prove the following: the construction of associating $F(X)$ to X induces a natural isomorphism $\Phi : H^4(X,\mathbf{Z}) \cong H^2(F(X),\mathbf{Z})$. This isomorphism has the following property: let $\mathcal{O}_X(1)$ be the polarizing line bundle on X inherited from the ambient projective space and let $h \in H^2(X,\mathbf{Z})$ be its first Chern class, then Φ sends $h^2 \in H^4(X,\mathbf{Z})$ to $\alpha \in H^2(F(X),\mathbf{Z})$. Furthermore, there is a natural lattice structure on $H^4(X,\mathbf{Z})$, and Φ induces a lattice isomorphism (up to a sign) from the orthocomplement of h^2 to the orthocomplement of α. Even stronger, Φ induces an isomorphism[3] of Hodge structures in the sense that $\Phi_{\mathbf{C}}(H^{p,q}(X)) = H^{p-1,q-1}(F(X))$. In particular $H^{4,0}(X) = H^{0,4}(X) = \{0\}$.

We can define a period domain for cubic fourfolds in the following way. Let X be a cubic fourfold, $F(X)$ its variety of lines and M and N the respective underlying real manifolds. Then by the isomorphism $\Phi : H^4(M,\mathbf{Z}) \cong H^2(N,\mathbf{Z})$ any Hodge decomposition of $H^2(N,\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C}$ in Ω_{BD} defines a Hodge decomposition of $H^4(M,\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C}$ compatible with the polarization class. We will denote the

[3] To be precise, the Hodge structures are isomorphic up to a Tate twist

space of Hodge structures on $H^4(M,\mathbf{Z})\otimes\mathbf{C}$ thus obtained by Ω_{cubic}; it is of course isomorphic to Ω_{BD}. We define a marking on a cubic fourfold X' to be a lattice isomorphism $H^4(X',\mathbf{Z})\to H^4(M,\mathbf{Z})$ which preserves the polarization classes.

Let $\mathfrak{M}_{\text{cubic}}$ be the moduli space of marked cubic fourfolds and

$$\mathcal{P}_{\text{cubic}}:\mathfrak{M}_{\text{cubic}}\to\Omega_{\text{cubic}}$$

the period map. Then the Beauville–Donagi construction yields the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{M}_{\text{cubic}} & \xrightarrow{F} & \mathfrak{M}_{\text{BD}} \\ \downarrow{\scriptstyle \mathcal{P}_{\text{cubic}}} & & \downarrow{\scriptstyle \mathcal{P}_{\text{BD}}} \\ \Omega_{\text{cubic}} & \xrightarrow[\cong]{} & \Omega_{\text{BD}} \end{array} \tag{1}$$

The strength of this results lies in the fact that it makes it possible to study the Hodge structure on the fourth cohomology of cubic fourfolds in terms of holomorphic symplectic fourfolds and their periods. This observation was the starting point for Voisin to solve the Torelli problem for cubic fourfolds: every cubic fourfold is uniquely determined by its polarized Hodge structure, up to projective transformations. See [47].

The story does not end there, however. Although the image of $\mathcal{P}_{\text{cubic}}$ is dense, as already follows from the results of Beauville and Donagi, Hassett showed in his thesis that not all Hodge structures in Ω_{cubic} are realized by cubic fourfolds. That is, the period map for cubic fourfolds is not surjective onto the period domain. Hassett then conjectured that the Hodge structures in the complement of the period map should correspond to degenerations of cubic fourfolds into cubics with an ordinary double point, or into the so-called determinantal cubic. This conjecture was proven independently by Looijenga [30] and Laza [29], by different methods. Together with the results by Voisin it gives a complete description of the period map for cubic fourfolds.

Relevance for holomorphic symplectic geometry

The Beauville–Donagi construction is not only relevant for the geometry of cubic fourfolds, but also for the theory of irreducible symplectic manifolds. Let us explain this in more detail.

Given an irreducible holomorphic symplectic manifold Y, we can obtain new examples of such manifolds by deforming the complex structure of Y. We say that all holomorphic symplectic manifolds thus obtained are in the same *deformation class* as Y. Manifolds in the same deformation class share many properties, in particular they are all diffeomorphic. One of the main problems in the field of holomorphic symplectic geometry is that, although the definition of irreducible holomorphic symplectic manifolds may seem rather innocent, only very few examples of such deformation classes are known. Indeed, for a long time the only known examples of holomorphic symplectic manifolds were K3 surfaces, which make up one deformation class, and in fact many believed that this was all.

It came as a surprise when, in the 80's, Fujiki constructed an example of an irreducible holomorphic symplectic of dimension 4, see [14]. Soon after, Beauville showed how to generalize this construction to higher dimensions (see [3]) and he used it to construct two deformation classes of irreducible holomorphic symplectic manifolds in every even[4] dimension. In the 90's, O'Grady constructed previously unknown deformation classes (even new *diffeomorphism* classes) in dimensions 6 and 10, see [39] and [38]. Until present no new deformation classes have been found, so that in every dimension there are at most 3 deformation classes of irreducible holomorphic symplectic manifolds known. Furthermore, only for the case of 2-dimensional such manifolds it is known that there is no more than one deformation class. In all other dimensions it is an open problem whether there are more deformation classes than those that are presently known.

When it comes to the explicit geometric construction of (algebraic) examples of irreducible holomorphic symplectic manifolds the situation is even worse. For most of the above mentioned deformation classes, the known constructions of algebraic representatives only yield varieties that are special, in the sense that their Picard rank is 2 or higher. However, generic algebraic representatives have Picard rank 1. What makes the Beauville–Donagi construction so interesting is that it not only yields algebraic examples of holomorphic symplectic manifolds, but also *all* of their local algebraic deformations. Constructions with this property are known as *locally complete projective constructions.* In particular, such constructions *do* yield algebraic holomorphic symplectic varieties with Picard rank 1.

For K3 surfaces there are several known examples of locally complete projective constructions, but for higher dimensional holomorphic symplectic varieties only four of such constructions are known at present (see [10] for an overview). In fact, all four yield holomorphic symplectic manifolds diffeomorphic to the blow-up of the diagonal of the symmetric self-product of a K3-surface. Such manifolds are said to be of $K3^{[2]}$*-type*.

What is in this thesis?

We have tried to explain the Beauville–Donagi construction in some detail, show how it connects the geometry of cubic fourfolds to the holomorphic symplectic geometry and give an idea of how both sides benefit from this connection. The main theme of this thesis is to exploit this connection even further and use results on the period map of cubic fourfolds to find geometric properties of deformations

[4]Note that the complex dimension of a holomorphic symplectic manifold is necessarily even.

of the varieties of lines on them.

Let us be more precise. Consider diagram (1); as Hassett argued, the period map $\mathcal{P}_{\text{cubic}}$ is not surjective, but $\mathcal{P}_{\text{BD}}$ *is* surjective. Since the period domains are isomorphic, the periods that are missed by $\mathcal{P}_{\text{cubic}}$ can be 'realized' by holomorphic symplectic manifolds. In chapter 2 we address the question of identifying those holomorphic symplectic manifolds and investigate why their periods cannot be realized by a cubic fourfold. In particular, we find why these holomorphic symplectic manifolds cannot be the variety of lines on a cubic fourfold.

Hassett already showed that although there are periods in Ω_{cubic} that are not in the image of the period map, these periods can be reobtained as limiting Hodge structures of deformations of certain singular cubics. In chapter 2 we identify the holomorphic symplectic manifolds corresponding to these periods. We already mentioned that they cannot be realized as the variety of lines on a cubic fourfold, but in the light of Hassett's observation we can try to reconstruct them from the deformations of singular cubics. In chapter 3 we will investigate to what extent this is possible in a *projective* way. We describe the cases for which there are obstructions, and perform an explicit construction in the other cases.

Before we describe the content of last chapter we observe the following: by construction, all holomorphic symplectic manifolds in $\mathfrak{M}_{\text{BD}}$ come equipped with an integral $(1,1)$-class α in the cohomology. It turns out that this class is fixed up to isometry by the following numerical data: it has even inner product (with respect to the Beauville–Bogomolov bilinear form) with any element in the second cohomology lattice and its self-product is 6. Furthermore, this class uniquely defines a line bundle on the manifold. It turns out that the results of chapter 2 gives information on obstructions and criteria for this line bundle to be very-ample, and the results in chapter 3 give information on obstructions and criteria for this line bundle to be ample. In chapter 4 we gather and extend these results to obtain a rather complete description of very-ampleness, ampleness and nefness of line bundles of the numerical type described above on holomorphic symplectic manifolds of $\text{K3}^{[2]}$-type.

Let us elaborate a bit on this last result. Criteria and obstructions for very-ampleness, ampleness and nefness of line bundles of a given numerical type have been extensively and exhaustively analyzed for K3 surfaces, in particular by Saint–Donat in [42]. The methods used by Saint–Donat are very particular to surfaces and cannot easily be generalized to higher dimensional holomorphic symplectic manifolds. The last chapter of this thesis provides an example of how locally complete projective constructions can be employed to find characterizations of (very-)ampleness and nefness of line bundles of given topological type on holomorphic symplectic manifolds of dimension bigger than 2. To the best of the author's knowledge it is the first such characterization.

Of course we are limited to line bundles of only one specific topological type, namely the type provided by the Beauville–Donagi construction. Furthermore, we do not only need the locally complete projective construction, but also other strong results, in this case on the period map of cubic fourfolds. Nevertheless, our hope is that the results in this thesis may serve as inspiration, or at least motivation, to exploit the other known locally complete projective construction to obtain similar results.

Chapter 1

Preliminary results

In this thesis we deal with the relation between cubic fourfolds and certain irreducible holomorphic symplectic fourfolds. In this chapter we introduce both cubic fourfolds and irreducible holomorphic symplectic manifolds, their properties and relations.

1.1 Holomorphic symplectic manifolds

The geometry of holomorphic symplectic manifolds is a very rich and extensively described in the literature. In this section we limit ourselves to a discussion of the results that will be of use and importance to us.

1.1.1 Definition and basic properties

Definition 1.1.1. For us an **irreducible holomorphic symplectic manifold** will be a simply connected compact complex manifold of Kähler type which admits a non-degenerate holomorphic two-form that is unique up to scaling.

By 'of Kähler type' we mean that the manifold admits a Kähler metric, but that a choice of such a metric is not part of the data. Neither do we consider a choice of holomorphic 2-form as part of the data. In particular, when we speak of deformations of a holomorphic symplectic manifold, this will refer to deformations of the complex structure.

Convention 1.1.2. The adjective 'irreducible' refers to the uniqueness of the holomorphic symplectic form (up to scaling). Since in this thesis only irreducible holomorphic symplectic manifolds occur we will simply refer to them as 'holomorphic symplectic manifolds'.

Remark 1.1.3. In the literature one also encounters the designation 'irreducible hyperkähler manifold' instead of 'irreducible holomorphic symplectic manifold'. To us, an **irreducible hyperkähler manifold** is a Riemannian manifold with holonomy group isomorphic to $\mathrm{Sp}(n)$, the group of automorphisms of $\mathbf{H}^n$, where **H** denotes the space of quaternions, that preserve the standard Hermitian metric. From the viewpoint of real differentiable manifolds the two notions coincide: every differentiable manifold with hyperkähler structure can be endowed the structure of a holomorphic symplectic manifold and vice versa, see for example Huybrechts' contribution to [18]. However, there is no natural 1-1 correspondence between these structures. Therefore we think it is best to separate the notions.

Holomorphic symplectic manifolds are extensively studied in the literature. In this section we will discuss some of their basic properties. We refer the reader to [3] and [25] for proofs and details. Every holomorphic symplectic manifold known at present is a deformation of one of the following examples:

- K3 surfaces. These may be defined as holomorphic symplectic manifolds of dimension 2. It can be shown (see for example [4]) that all K3 surfaces make up one deformation class, that is, given two K3 surfaces we can deform one into the other. Projective examples are given by 2:1 coverings of $\mathbf{P}^2$ branched along a smooth sextic, quartic hypersurfaces in $\mathbf{P}^3$, the generic intersection of a quadric and a cubic hypersurface in $\mathbf{P}^4$ and a generic intersection of 3 quadric hypersurfaces in $\mathbf{P}^5$. Many more such constructions are known, mainly due to Mukai, see for example [34] and [35].

- For any K3 surface S let $S^{[n]}$ denote the Hilbert scheme[1] of length n subschemes of S. Then $S^{[n]}$ is a holomorphic symplectic manifold of dimension $2n$. We will say that a holomorphic symplectic manifold is of **K3$^{[n]}$-type** if it is isomorphic to a deformation of $S^{[n]}$ for a K3 surface S. We remark here that for every $n \geq 2$ there exist holomorphic symplectic manifolds of K3$^{[n]}$-type that are not isomorphic to $S^{[n]}$ for any K3 surface S, see also [3].

- Let T be a complex 2-torus and let $T^{[n]}$ be the Hilbert scheme of length n subschemes of T. There is a natural summation map $\Sigma : T^{[n]} \to T$ which sends a length n subscheme of T to the sum of the elements of its support (with multiplicity). The fibers of Σ are isomorphic holomorphic symplectic manifolds.

- In [39] and [38] O'Grady constructed examples of holomorphic symplectic manifolds in dimension 6 and 10 respectively that are not diffeomorphic to any of the examples above.

[1] To be precise, in an analytic context 'Hilbert scheme' will have to be replaced by 'Douady space'.

It is unknown whether there are more diffeomorphism classes or even more deformation classes. Nevertheless, there is a rather extensive general theory for holomorphic symplectic manifolds. Some of the main results will be presented here.

Theorem 1.1.4. *Let X be a holomorphic symplectic manifold of (complex) dimension $2n$. Then $H^2(X,\mathbf{Z})$ is torsion-free. Moreover, it can be endowed with a non-degenerate primitive*[2] *quadratic form q with the property that there exists a positive number $c_X \in \mathbf{Q}$, depending only on the topology of X, such that*

$$\int_X \alpha^n = c_X q(\alpha)^n,$$

for all $\alpha \in H^2(X,\mathbf{Z})$. The lattice thus obtained is of signature $(3, b_2 - 3)$.

Proof. See [15] theorem 4.7. □

The quadratic form on the integral second cohomology is called the **Beauville–Bogomolov form**. On a K3 surface it coincides with the intersection form. The constant c_X is called the **Fujiki constant**. The Fujiki constant and second cohomology lattices have been identified for all known deformation classes of holomorphic symplectic manifolds, see [41] for an overview.

Note that on every holomorphic symplectic manifold X we have a weight 2 integral Hodge structure

$$H^2(X,\mathbf{Z}) \otimes \mathbf{C} = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X).$$

It is well known that K3 surfaces are completely determined up to isomorphism by their Hodge structure. Even stronger, the following holds:

Theorem 1.1.5. *Let S and S' be K3 surfaces and assume that there is a Hodge isometry $\varphi : H^2(S,\mathbf{C}) \to H^2(S',\mathbf{C})$, that is, a map that preserves both the Hodge structures and the underlying lattice. Then S and S' are isomorphic.*
If in addition there exists a Kähler class in $H^2(S,\mathbf{C})$ whose image under φ is a Kähler class on S', then there exists an isomorphism $f : S' \to S$ that induces φ. That is, $\varphi = f^*$.

The second part of this result is called the Strong Torelli Theorem for K3 surfaces, see [2] for a proof. This result does not generalize in full strength to higher dimensional holomorphic symplectic manifolds. One of the obstructions is a result by Huybrechts in [25] which states that two holomorphic symplectic manifolds that are bimeromorphic have Hodge-isometric second cohomology. Debarre (see [9]) has constructed examples of non-isomorphic holomorphic

[2] That is, the associated bilinear form is integer valued and not divisible by an integer bigger than 1

symplectic manifolds which are bimeromorphic, hence cannot be distinguished by their Hodge structure. There even exist examples of non-bimeromorphic holomorphic symplectic manifold that have Hodge-isometric second cohomology, see [36].

Nevertheless, in recent years progress has been made towards a Torelli-type theorem for holomorphic symplectic manifolds of any dimension, mainly by the work of Verbitsky ([46]) and Markman ([31]). To formulate it, Markman introduces the concept of **parallel transport operators** between the cohomology groups of two holomorphic symplectic manifolds Y and Y'. By definition these are isomorphisms of $H^*(Y,\mathbf{Z}) \to H^*(Y',\mathbf{Z})$ which can be obtained by parallel transport in the local system associated to some smooth, proper and flat family that contains Y and Y' as fibers. The Torelli theorem for holomorphic symplectic manifolds may be stated as follows (see [31, Theorem 1.3]):

Theorem 1.1.6. *Let Y and Y' be holomorphic symplectic manifolds which are deformation equivalent.*

1. *Y and Y' are bimeromorphic if and only if there exists a parallel transport operator $f : H^2(Y,\mathbf{Z}) \to H^2(Y',\mathbf{Z})$ which is also a a Hodge isometry.*
2. *Let $f : H^2(Y,\mathbf{Z}) \to H^2(Y',\mathbf{Z})$ be a parallel transport operator which is also a Hodge isometry. Then there exists an isomorphism $\widetilde{f} : Y' \to Y$ such that $f = \widetilde{f}^*$ if and only if f maps some Kähler class of Y to a Kähler class of Y'.*

Remark 1.1.7. In this thesis bimeromorphisms between holomorphic symplectic manifolds play an important role. From [18, Proposition III.21.6] it follows that any such bimeromorphism can be extended to an isomorphism in codimension 1. More explicitly, if $\psi : Y \dashrightarrow Y'$ is a bimeromorphism between holomorphic symplectic manifolds, then there exist analytic subsets $Z \subset Y$ and $Z' \subset Y'$, both of codimension at least 2, and a bimeromorphism $\widetilde{\psi} : Y \dashrightarrow Y'$ which extends ψ and defines an isomorphism of $Y \setminus Z$ onto $Y' \setminus Z'$. In what follows, when we introduce a bimeromorphism between holomorphic symplectic manifolds, we will use without explicit mention that it is an isomorphism in codimension 1.

Remark 1.1.8. Theorem 1.1.6 tells us that if $f : H^2(Y,\mathbf{Z}) \to H^2(Y',\mathbf{Z})$ is a parallel transport operator that is also a Hodge isometry, then there exists a bimeromorphic map $\psi : Y' \dashrightarrow Y$. By remark 1.1.7 this map is an isomorphism in codimension 1. Therefore it induces a Hodge isometry $\psi^* : H^2(Y,\mathbf{Z}) \to H^2(Y',\mathbf{Z})$. We stress here that it in general **not** the case that $f = \psi^*$ or even that $f = \pm\psi^*$. In fact, the set Hodge isometries from $H^2(Y,\mathbf{Z})$ and $H^2(Y',\mathbf{Z})$, the set of parallel transport operators and the set of isometries induced by bimeromorphisms will generically all be different and it may be difficult to gain control over these differences.

In [26], Huybrechts rephrases theorem 1.1.6 in a way that will also be of use to us. We will first introduce the concept of periods and period maps for holomorphic symplectic manifolds. By definition $H^{2,0}(X) = \mathbf{C}\cdot\sigma$, where σ is the cohomology class of a non-degenerate holomorphic 2-form on X. In particular $H^{2,0}(X)$ is of dimension one. Furthermore it follows from the properties of the Beauville–Bogomolov form that $H^{2,0}(X)\oplus H^{0,2}(X)$ is orthogonal to $H^{1,1}(X)$. Since $H^{2,0}(X) = \overline{H^{0,2}(X)}$ by definition of a Hodge structure, it follows that the Hodge structure is completely determined by the position of the line $H^{2,0}(X)$ in $H^2(X,\mathbf{Z})\otimes\mathbf{C}$. To keep track of this position we introduce the following concept of a marking.

Definition 1.1.9. Let X be a holomorphic symplectic manifold and Λ be a lattice isomorphic to $(H^2(X,\mathbf{Z}),q)$. An isomorphism $\varphi : H^2(X,\mathbf{Z}) \to \Lambda$ is called a **Λ-marking** of X (or simply a marking if the choice of Λ is clear from the context). We say that two Λ-marked holomorphic symplectic manifolds (X,φ) and (X',φ') are isomorphic if there exists an isomorphism $g : X \to X'$ such that $\varphi\circ g^* = \varphi'$.

Let (X,φ) be a Λ-marked holomorphic symplectic manifold and denote by

$$\varphi_{\mathbf{C}} : H^2(X,\mathbf{C}) \to \Lambda\otimes\mathbf{C}$$

the extension of φ. Let $\sigma\in H^{2,0}(X)$ non-zero. Then $\left[\varphi_{\mathbf{C}}(\sigma)\right]\in\mathbf{P}(\Lambda\otimes\mathbf{C})$ does not depend on the choice of σ and is also independent on the isomorphism class of (X,φ). Note that by the properties of the Beauville-Bogomolov form we have that $q(\sigma) = 0$ and $q(\sigma,\overline{\sigma}) > 0$. It follows that

$$\varphi_{\mathbf{C}}(\sigma)\in\left\{x\in\Lambda\otimes\mathbf{C}\,|\,x.x = 0,\ x.\overline{x} > 0\right\},$$

where the dot denotes (the complex bilinear extension of) the pairing on Λ.

Definition 1.1.10. The point $\left[\varphi_{\mathbf{C}}(\sigma)\right]\in\mathbf{P}(\Lambda\otimes\mathbf{C})$ is called the **period** of (X,φ). It is contained in the space

$$\Omega_\Lambda := \left\{[x]\in\mathbf{P}(\Lambda\otimes\mathbf{C})\,|\,x.x = 0,\ x.\overline{x} > 0\right\}$$

which is called the **period domain**. Let $\mathfrak{M}_\Lambda$ be the moduli space of Λ-marked holomorphic symplectic manifolds, then the map

$$\mathcal{P}_\Lambda : \mathfrak{M}_\Lambda \to \Omega_\Lambda$$

is called the **period map**

We recall some general properties of $\mathfrak{M}_\Lambda$.

Theorem 1.1.11. *Let Λ be a lattice and assume that $\mathfrak{M}_\Lambda$ is non-empty. Then the following holds:*

- *$\mathfrak{M}_\Lambda$ is locally a smooth complex manifold, but is possibly non-Hausdorff.*
- *$\mathfrak{M}_\Lambda$ is Hausdorff outside the preimage under $\mathcal{P}_\Lambda$ of the countable (but dense) union $\bigcup_{\alpha\in\Lambda}\mathbf{P}(\alpha^\perp)$ of hyperplanes in Ω_Λ.*
- *If two Λ-marked holomorphic symplectic manifolds (X,φ) and (X',φ') define non-separated points of $\mathfrak{M}_\Lambda$ then X and X' are bimeromorphic. Conversely, if X and X' are birational projective holomorphic symplectic manifolds with second cohomology lattice isomorphic to Λ, then there exist markings φ and φ' on X and X' respectively such that (X,φ) and (X',φ') define non-separated points of $\mathfrak{M}_\Lambda$.*

Proof. Proofs can be found in [26] and section 4 of [25]. □

Remark 1.1.12. As we mentioned before, Debarre has constructed holomorphic symplectic manifolds that are bimeromorphic (birational in fact) but not isomorphic (see [9]). This shows that the possibility for $\mathfrak{M}_\Lambda$ to be non-Hausdorff actually occurs.

We can define a relation $\sim$ on $\mathfrak{M}_\Lambda$ by declaring that $x\sim y$ if and only if x and y are non-separated points in $\mathfrak{M}_\Lambda$. Huybrechts shows in [26] that this is an equivalence relation. Let $\pi:\mathfrak{M}_\Lambda\to\overline{\mathfrak{M}}_\Lambda$ be the quotient map. Then $\overline{\mathfrak{M}}_\Lambda$ is a Hausdorff space and it is universal in the sense that any continuous map from $\mathfrak{M}_\Lambda$ to a Hausdorff space factors through π. Therefore we call $\overline{\mathfrak{M}}_\Lambda$ the **Hausdorff reduction** of $\mathfrak{M}_\Lambda$. Since in particular Ω_Λ is Hausdorff, it follows that $\mathcal{P}_\Lambda=\overline{\mathcal{P}}_\Lambda\circ\pi$, with $\overline{\mathcal{P}}_\Lambda:\overline{\mathfrak{M}}_\Lambda\to\Omega_\Lambda$. Huybrechts proves the following version of the Torelli theorem [26]:

Theorem 1.1.13. *$\overline{\mathcal{P}}_\Lambda$ maps the Hausdorff reduction of any connected component of $\mathfrak{M}_\Lambda$ isomorphically to Ω_Λ.*

1.1.2 Holomorphic symplectic manifolds with line bundle

The holomorphic symplectic fourfolds that we will consider come naturally equipped with a line bundle. So from now on we will deal with pairs of a holomorphic symplectic manifold and a line bundle, rather than with abstract holomorphic symplectic manifolds. We will adapt some of the theory from the previous sections to this situation.

Definition 1.1.14. Let Λ be a lattice and $h\in\Lambda$ a distinguished element. Let $(Y,\mathcal{L})$ be a holomorphic symplectic manifold with a line bundle $\mathcal{L}$. A **(Λ,h)-marking** of $(Y,\mathcal{L})$ is a lattice isomorphism $\varphi:H^2(Y,\mathbf{Z})\to\Lambda$ such that $\varphi(c_1(\mathcal{L}))=h$.
We say that $(Y,\mathcal{L},\varphi)$ and $(Y',\mathcal{L}',\varphi')$ are isomorphic as (Λ,h)-marked holomorphic symplectic manifolds if there exists an isomorphism $f:Y\to Y'$ such that $f^*\mathcal{L}'\cong\mathcal{L}$ and $\varphi\circ f^*=\varphi'$.

Every triple $(Y,\mathcal{L},\varphi)$ defines a point in $\mathfrak{M}_\Lambda$, namely the isomorphism class of (Y,φ). Remember that $H^{2,0}(Y)\oplus H^{0,2}(Y)$ is orthogonal to $H^{1,1}(Y)$ with respect to the Beauville-Bogomolov form. Since $c_1(\mathcal{L})\in H^{1,1}(Y)$ it follows that if φ is an (Λ,h)-marking of $(Y,\mathcal{L})$, then

$$\mathcal{P}_\Lambda(Y,\varphi)=\left[\varphi(H^{2,0}(Y))\right]\in\left[\varphi(c_1(\mathcal{L})^\perp)\right]\cap\Omega_\Lambda=:\Omega_{h^\perp},$$

where $h^\perp$ denotes the sublattice of Λ orthogonal to h and $\Omega_{h^\perp}$ is the associated period domain:

$$\Omega_{h^\perp}:=\left\{[x]\in\mathbf{P}(h^\perp\otimes\mathbf{C})\,|\,x.x=0,\ x.\overline{x}>0\right\}.$$

The following observation is elementary but important.

Lemma 1.1.15. *The map $(Y,\mathcal{L},\varphi)\mapsto(Y,\varphi)$ defines a bijection between the set of isomorphism classes of (Λ,h)-marked pairs $(Y,\mathcal{L})$ and $\mathcal{P}_\Lambda^{-1}(\Omega_{h^\perp})$.*

Proof. Let $S_{(\Lambda,h)}$ be the set of isomorphism classes of (Λ,h)-marked pairs and

$$\begin{array}{rcl} f:S_{(\Lambda,h)} & \to & \mathfrak{M}_\Lambda \\ (Y,\mathcal{L},\varphi) & \mapsto & (Y,\varphi) \end{array}$$

the 'forgetful map'. We have already seen that $f\big(S_{(\Lambda,h)}\big)\subseteq\mathcal{P}_\Lambda^{-1}(\Omega_{h^\perp})$.

For the rest of the proof the key ingredient is the fact that for every holomorphic symplectic manifold Y the map $c_1:\operatorname{Pic}Y\to H^2(Y,\mathbf{Z})\cap H^{1,1}(Y)$ that sends (the isomorphism class of) a line bundle to its first Chern class is an isomorphism of $\mathbf{Z}$-modules; injectivity follows from the exponential sheaf sequence and the fact that $H^1(\mathcal{O}_X)=\{0\}$, surjectivity follows from the Lefschetz theorem on $(1,1)$-classes (see also [3]).

Assume that $f(Y,\mathcal{L},\varphi)=f(Y',\mathcal{L}',\varphi')$. That is, there exists an isomorphism $g:Y\to Y'$ such that $\varphi'=\varphi\circ g^*$. Since φ and φ' are (Λ,h)-markings it follows that $c_1(\mathcal{L})=g^*c_1(\mathcal{L}')$. Hence $\mathcal{L}\cong g^*\mathcal{L}'$ and it follows that the triples $(Y,\mathcal{L},\varphi)$ and $(Y',\mathcal{L}',\varphi')$ are isomorphic. So f is injective.

Now let (Y,φ) be any Λ-marked holomorphic symplectic manifold and assume that $\mathcal{P}_\Lambda(Y,\varphi)\in\Omega_{h^\perp}$. Then $\varphi^{-1}(h)\in H^{1,1}(Y)\cap H^2(Y,\mathbf{Z})$, so there exists a line bundle $\mathcal{L}$ on Y such that $\varphi(c_1(\mathcal{L}))=h$. Then φ is a (Λ,h)-marking for the pair $(Y,\mathcal{L})$, and $f(Y,\mathcal{L},\varphi)=(Y,\varphi)$. Hence f maps $S_{(\Lambda,h)}$ surjectively onto $\mathcal{P}_\Lambda^{-1}(\Omega_{h^\perp})$. Since it is also injective, f is a bijection onto $\mathcal{P}_\Lambda^{-1}(\Omega_{h^\perp})$, and the lemma follows. □

We denote $\mathfrak{M}_{(\Lambda,h)}:=\mathcal{P}_\Lambda^{-1}(\Omega_{h^\perp})$. By the previous lemma it is the moduli space of (Λ,h)-marked pairs $(Y,\mathcal{L})$ consisting of a holomorphic symplectic manifold with a line bundle. We denote the restricted period map by

$$\mathcal{P}_{(\Lambda,h)}:\mathfrak{M}_{(\Lambda,h)}\to\Omega_{h^\perp},$$

the image of $\mathfrak{M}_{(\Lambda,h)}$ in $\overline{\mathfrak{M}}_\Lambda$ by $\overline{\mathfrak{M}}_{(\Lambda,h)}$ and the restriction of $\overline{\mathcal{P}}_\Lambda$ by

$$\overline{\mathcal{P}}_{(\Lambda,h)} : \overline{\mathfrak{M}}_{(\Lambda,h)} \to \Omega_{h^\perp}.$$

Proposition 1.1.16. *The following hold:*

- $\mathfrak{M}_{(\Lambda,h)}$ *is locally a complex submanifold of* $\mathfrak{M}_\Lambda$*, but possibly non-separated;*
- $\overline{\mathfrak{M}}_{(\Lambda,h)}$ *is a complex submanifold of* $\overline{\mathfrak{M}}_\Lambda$*;*
- $\overline{\mathfrak{M}}_{(\Lambda,h)}$ *is the Hausdorff reduction of* $\overline{\mathfrak{M}}_\Lambda$*;*
- $\overline{\mathcal{P}}_{(\Lambda,h)}$ *maps any connected component of* $\overline{\mathfrak{M}}_{(\Lambda,h)}$ *isomorphically onto a connected component of* $\Omega_{h^\perp}$*.*

Proof. We start with the last statement. Let $\overline{\mathfrak{M}}^0_{(\Lambda,h)}$ be a connected component of $\overline{\mathfrak{M}}_{(\Lambda,h)}$ and $\overline{\mathfrak{M}}^0_\Lambda$ the connected component of $\overline{\mathfrak{M}}_\Lambda$ in which $\overline{\mathfrak{M}}^0_{(\Lambda,h)}$ is contained. By theorem 1.1.13 $\overline{\mathcal{P}}_\Lambda$ maps $\overline{\mathfrak{M}}^0_\Lambda$ isomorphically onto Ω_Λ. From the definitions it follows that $\overline{\mathcal{P}}_{(\Lambda,h)}$ maps $\overline{\mathfrak{M}}^0_\Lambda \cap \overline{\mathfrak{M}}_{(\Lambda,h)}$ isomorphically onto $\Omega_{h^\perp}$. Since $\overline{\mathfrak{M}}^0_{(\Lambda,h)}$ is a connected component of $\overline{\mathfrak{M}}^0_\Lambda \cap \overline{\mathfrak{M}}_{(\Lambda,h)}$, the statement follows.

The third statement follows directly from the definition of Hausdorff reduction; it passes to subspaces. For the second statement observe that by construction $\Omega_{h^\perp}$ is a complex subspace of Ω_Λ. The statement now follows from the last statement. The first statement follows directly from the second and the third. □

Although the introduction of a marking makes it possible to compare the cohomology lattice of a holomorphic symplectic manifold to some abstract lattice, which is convenient in many cases, it is an auxiliary structure. We can get rid of it in the following way. Assume that $\mathfrak{M}_{(\Lambda,h)}$ is non-empty. Let $O(\Lambda,h) \subset O(\Lambda)$ be the group of isometries that fix h. Then $O(\Lambda,h)$ acts on $\mathfrak{M}_{(\Lambda,h)}$ by

$$\gamma \cdot (Y,\mathcal{L},\varphi) = (Y,\mathcal{L},\gamma\circ\varphi),\ \gamma \in O(\Lambda,h)$$

and this induces an action on $\overline{\mathfrak{M}}_{(\Lambda,h)}$. Let us denote

$$\begin{aligned} \mathfrak{N}_{(\Lambda,h)} &:= O(\Lambda,h)\backslash\mathfrak{M}_{(\Lambda,h)} \\ \overline{\mathfrak{N}}_{(\Lambda,h)} &:= O(\Lambda,h)\backslash\overline{\mathfrak{M}}_{(\Lambda,h)} \end{aligned}$$

Remark / definition 1.1.17. Note that two triples $(Y,\mathcal{L},\varphi)$ and $(Y',\mathcal{L}',\varphi')$ in $\mathfrak{M}_{(\Lambda,h)}$ lie in the same orbit of $O(\Lambda,h)$ if and only if $(Y,\mathcal{L}) \cong (Y',\mathcal{L}')$. We say that $\mathfrak{N}_{(\Lambda,h)}$ parameterizes isomorphism classes of pairs $(Y,\mathcal{L})$ of a holomorphic symplectic manifolds with a line bundle of **topological type** (Λ,h) (i.e. there exists an isomorphism $(H^2(Y,\mathbf{Z}),c_1(\mathcal{L})) \cong (\Lambda,h)$).

By definition the action of $O(\Lambda,h)$ on Λ preserves $h^{\perp}$ and hence defines an action on $\Omega_{h^{\perp}}$. We denote the quotient by $\mathcal{D}_{(\Lambda,h)}$. It follows from the definition of $\mathcal{P}_{(\Lambda,h)}$ and $\overline{\mathcal{P}}_{(\Lambda,h)}$ that both are $O(\Lambda,h)$-equivariant. Hence they descend to morphisms:

$$\begin{aligned} \mathcal{Q}_{(\Lambda,h)}:\mathfrak{N}_{(\Lambda,h)} &\rightarrow \mathcal{D}_{(\Lambda,h)} \\ \overline{\mathcal{Q}}_{(\Lambda,h)}:\overline{\mathfrak{N}}_{(\Lambda,h)} &\rightarrow \mathcal{D}_{(\Lambda,h)} \end{aligned}$$

Before we go further we assume in addition that $h.h>0$, where the dot denotes the product on Λ. Equivalently, we consider line bundles of which the first Chern class has positive square with respect to the Beauville–Bogomolov form.

Remark 1.1.18. The square of of the first Chern class of line bundles on holomorphic symplectic manifolds with respect to the Beauville–Bogomolov form will play an important role in this thesis. If for a given line bundle $\mathcal{L}$ this square is d we will simply say that $\mathcal{L}$ is of square d, without explicit mention of the first Chern class of $\mathcal{L}$ or the Beauville–Bogomolov form.

The assumption in h is a natural one to make in the context of this thesis. In fact, we will only consider pairs $(Y,\mathcal{L})$ which are deformations of polarized holomorphic symplectic manifolds, and on such pairs $\mathcal{L}$ is necessarily of positive square. More importantly, the moduli spaces described so far will be 'nicer' in this case, in a way that we will make precise.

For any holomorphic symplectic manifold Y, let $C_Y \subset H^{1,1}(Y)\cap H^2(Y,\mathbf{R})$ be the cone of elements of positive square. Since the signature of the Beauville–Bogomolov form on $H^{1,1}(Y)\cap H^2(Y,\mathbf{R})$ is $(1,b_2-3)$ (see theorem 1.1.4) it follows that C_Y has 2 connected components C_Y^+ and C_Y^-, where we define C_Y^+ to be the component that contains the Kähler classes and $C_Y^- := -1\cdot C_Y^+$. The component C_Y^+ is called the **positive cone**[3].

We can write

$$\mathfrak{M}_{(\Lambda,h)} = \mathfrak{M}^+_{(\Lambda,h)} \cup \mathfrak{M}^-_{(\Lambda,h)},$$

where $\mathfrak{M}^{\pm}_{(\Lambda,h)}$ is the subspace of triples $(Y,\mathcal{L},\varphi)$ for which $c_1(\mathcal{L})\in C_Y^{\pm}$. Since the property $c_1(\mathcal{L})\in C_Y^{\pm}$ is stable under deformation, $\mathfrak{M}^+_{(\Lambda,h)}$ and $\mathfrak{M}^-_{(\Lambda,h)}$ are disjoint. The subspaces are interchanged by the involution $(Y,\mathcal{L},\varphi)\mapsto(Y,\mathcal{L}^{\vee},-\varphi)$ and hence isomorphic.

Moreover, each of these subspaces is preserved by the action of $O(\Lambda,h)$. Indeed, $O(\Lambda,h)$ acts only on the marking in a triple $(Y,\mathcal{L},\varphi)$, hence cannot change whether $c_1(\mathcal{L})\in C_Y^+$ or not. In particular we can write

$$\mathfrak{N}_{(\Lambda,h)} = \mathfrak{N}^+_{(\Lambda,h)} \cup \mathfrak{N}^-_{(\Lambda,h)},$$

[3]Not to be confused with the cone of elements of positive square, which is the union of the positive cone and its reflection in the origin.

with $\mathfrak{N}^{\pm}_{(\Lambda,h)}$ disjoint and isomorphic.

Furthermore, since the Beauville-Bogomolov form has signature $(3, b_2-3)$ (see again theorem 1.1.4) the lattice $h^{\perp}$ is of signature $(2, b_2-3)$. It follows that $\Omega_{h^{\perp}}$ has two isomorphic connected components, each a symmetric homogeneous domain of type IV (see §6 of the appendix in [44]). It can be shown that the quotient $\mathcal{D}_{(\Lambda,h)} := O(\Lambda, h)\backslash\Omega_{h^{\perp}}$ is a complex analytic space with only finite quotient singularities. In particular it is Hausdorff in the analytic topology. Even stronger, by the famous results of Baily and Borel (see [1]) the quotient can be equipped with the structure of a quasi-projective variety. Note that $O(\Lambda, h)$ interchanges the connected components of $\Omega_{h^{\perp}}$, so the quotient $\mathcal{D}_{(\Lambda,h)}$ is in fact connected.

Proposition 1.1.19. *Let Λ be a lattice, $h \in \Lambda$ such that $h.h > 0$ and assume that $\mathfrak{N}_{(\Lambda,h)}$ is non-empty. Then the following hold:*

- *$\overline{\mathfrak{N}}_{(\Lambda,h)}$ is locally an analytic space;*
- *$\overline{\mathfrak{N}}_{(\Lambda,h)}$ is the Hausdorff reduction of $\mathfrak{N}_{(\Lambda,h)}$;*
- *$\overline{\mathcal{Q}}_{(\Lambda,h)}$ maps every connected component of $\overline{\mathfrak{N}}_{(\Lambda,h)}$ isomorphically to $\mathcal{D}_{(\Lambda,h)}$.*

Proof. We noted already that, under the assumptions given, $\mathcal{D}_{(\Lambda,h)}$ is connected and can be endowed with the structure of a quasi-projective variety. The last statement now follows from the last statement of proposition 1.1.16, the construction of $\overline{\mathcal{Q}}_{(\Lambda,h)}$ and $\overline{\mathfrak{N}}_{(\Lambda,h)}$ and the fact that $\mathcal{D}_{(\Lambda,h)}$ is connected. From the last statement and the fact that $\mathcal{D}_{(\Lambda,h)}$ can be endowed with the structure of a quasi-projective variety it follows that every connected component of $\overline{\mathfrak{N}}_{(\Lambda,h)}$ can be endowed with the structure of a quasi-projective variety. Hence in the analytic topology $\overline{\mathfrak{N}}_{(\Lambda,h)}$ has locally the structure of an analytic space.

To prove the second statement, let H be a Hausdorff space and $f : \mathfrak{N}_{(\Lambda,h)} \to H$ a continuous map. Let $\widehat{f} : \mathfrak{M}_{(\Lambda,h)} \to H$ be the composition of f with the quotient map $\mathfrak{M}_{(\Lambda,h)} \to \mathfrak{N}_{(\Lambda,h)}$. If we assume $O(\Lambda, h)$ to act trivially on H, $\widehat{f}$ is equivariant. By proposition 1.1.16 $\widehat{f}$ factors through a map $\widehat{f}' : \overline{\mathfrak{M}}_{(\Lambda,h)} \to H$. This map must also be equivariant, hence defines a map $f' : \overline{\mathfrak{N}}_{(\Lambda,h)} \to H$. Since $\overline{\mathfrak{N}}_{(\Lambda,h)}$ is locally an analytic space, it is Hausdorff, and it follows that it must be the Hausdorff reduction of $\mathfrak{N}_{(\Lambda,h)}$. □

The last statement of proposition 1.1.19 is a Torelli-type statement: two pairs $(Y, \mathcal{L})$ and $(Y', \mathcal{L}')$ that are deformation equivalent have the same period if and only their isomorphism classes are non-separated in $\mathfrak{N}_{(\Lambda,h)}$. Such a theorem is not very useful if we do not know what it means for two pairs to have non-separated isomorphism classes. The following theorem gives some information in this direction.

Proposition 1.1.20. *Let Λ be a lattice, $h \in \Lambda$ such that $h.h > 0$ and assume that $\mathfrak{N}_{(\Lambda,h)}$ is non-empty. Then the following are equivalent:*

- *$(Y,\mathcal{L})$ and $(Y',\mathcal{L}')$ are non-separated in $\mathfrak{N}_{(\Lambda,h)}$;*
- *$(Y,\mathcal{L})$ and $(Y',\mathcal{L}')$ have the same period and are in the same connected component of $\mathfrak{N}_{(\Lambda,h)}$;*
- *There exists a Hodge isometry $\chi : H^2(Y,\mathbf{Z}) \to H^2(Y,\mathbf{Z})$ which is also a parallel transport operator, such that $c_1(\mathcal{L}') = \chi(c_1(\mathcal{L}))$.*

If any of these hold, Y and Y' are bimeromorphic. Furthermore, if $(Y,\mathcal{L})$ and $Y',\mathcal{L}')$ are non-separated in $\mathfrak{N}_{(\Lambda,h)}$, $\mathcal{L}$ is ample on Y and $\mathcal{L}'$ is nef on Y', then $(Y,\mathcal{L})$ and $(Y',\mathcal{L}')$ are isomorphic as pairs. In particular $\mathcal{L}'$ is ample on Y'.

Proof. Let $(Y,\mathcal{L})$ and $(Y',\mathcal{L}')$ represent elements of $\mathfrak{N}_{(\Lambda,h)}$ that are non-separated. Then they are in the same connected component of $\mathfrak{N}_{(\Lambda,h)}$. Furthermore, they define the same point in $\overline{\mathfrak{N}}_{(\Lambda,h)}$ and hence have the same period.

Conversely, assume that $(Y,\mathcal{L})$ and $(Y',\mathcal{L}')$ lie in the same connected component of $\mathfrak{N}_{(\Lambda,h)}$ and have the same period. Let $y, y' \in \overline{\mathfrak{N}}_{(\Lambda,h)}$ be the points defined by $(Y,\mathcal{L})$ and $(Y',\mathcal{L}')$ respectively. Then y and y' lie in the same connected component and $\overline{\mathcal{P}}_{(\Lambda,h)}(y) = \overline{\mathcal{P}}_{(\Lambda,h)}(y')$. Since $\overline{\mathcal{P}}_{(\Lambda,h)}$ is an isomorphism on every connected component it follows that $y = y'$. Hence $(Y,\mathcal{L})$ and $(Y',\mathcal{L}')$ are non-separated in $\mathfrak{N}_{(\Lambda,h)}$, since $\overline{\mathfrak{N}}_{(\Lambda,h)}$ is its Hausdorff reduction. Equivalence of the first 2 items follows.

Assume again that $(Y,\mathcal{L})$ and $(Y',\mathcal{L}')$ have the same period and are in the same connected component of $\mathfrak{N}_{(\Lambda,h)}$. The assumption that they are in the same connected component implies that there exists a parallel transport operator $\chi : H^2(Y,\mathbf{Z}) \to H^2(Y',\mathbf{Z})$ such that $\chi(c_1(\mathcal{L})) = c_1(\mathcal{L}')$. The assumption that the pairs have the same period implies that χ is in fact a Hodge isometry.

Conversely, assume there exists a Hodge isometry $\chi : H^2(Y,\mathbf{Z}) \to H^2(Y,\mathbf{Z})$ with the properties mentioned. Since χ is a parallel transport operator, $(Y,\mathcal{L})$ and $(Y',\mathcal{L}')$ must belong to the same connected component. Let φ' be (Λ, h)-marking of $(Y',\mathcal{L}')$, then $\varphi := \varphi' \circ \chi$ is a (Λ, h)-marking of $(Y,\mathcal{L})$. Since χ is a Hodge isometry, it follows that $\left[\varphi(H^{2,0}(Y))\right] = \left[\varphi'(H^{2,0}(Y'))\right]$, hence the periods are the same. This proves equivalence of the second and third statement. By theorem 1.1.6 the last statement implies in particular that Y and Y' are bimeromorphic.

Now assume that $(Y,\mathcal{L})$ and $(Y',\mathcal{L}')$ are non-separated in $\mathfrak{N}_{(\Lambda,h)}$, $\mathcal{L}$ is ample on Y and $\mathcal{L}'$ is nef on Y'. By the first part of the proposition there exists a Hodge isometry $\chi : H^2(Y,\mathbf{Z}) \to H^2(Y',\mathbf{Z})$ which is also a parallel transport operator, such that $c_1(\mathcal{L}') = \chi(c_1(\mathcal{L}))$. In particular χ maps a class in the Kähler cone of Y (namely $c_1(\mathcal{L})$, which we assumed to be an ample class, hence Kähler) to a class in the closure of the Kähler cone on Y' (namely $c_1(\mathcal{L}')$, which we assumed to be a nef class). Then there must exist a Kähler class $\omega \in H^{1,1}(Y) \cap H^2(Y,\mathbf{R})$, close to $c_1(\mathcal{L})$, such that $\chi(\omega)$ is a Kähler class on Y'. By theorem 1.1.6 then there exists an isomorphism $f : Y' \to Y$ such that $\chi = f^*$. In particular $f^*\mathcal{L} \cong \mathcal{L}'$ and it follows that $(Y,\mathcal{L})$ and $(Y',\mathcal{L}')$ are isomorphic as pairs. □

Remark 1.1.21. Let us add a word of caution: the bimeromorphism $\psi : Y \dashrightarrow Y'$ that the previous proposition predicts to exist when $(Y,\mathcal{L})$ and $(Y',\mathcal{L}')$ are non-separated in $\mathfrak{N}_{(\Lambda,h)}$ in general will *not* induce χ on cohomology. It *does* induce a Hodge isometry $\psi^* : H^2(Y',\mathbf{Z}) \to H^2(Y,\mathbf{Z})$, as do all bimeromorphisms, but it may not be a parallel transport operator, or even map $c_1(\mathcal{L}')$ to $c_1(\mathcal{L})$.

Remark 1.1.22. Every point in $\overline{\mathfrak{N}}_{(\Lambda,h)}$ represents a set of isomorphism classes of pairs $(Y,\mathcal{L})$ that are non-separated in $\mathfrak{N}_{(\Lambda,h)}$. A priori there is no preferred isomorphism class in such a set. However, by proposition 1.1.20 there is at most one isomorphism class of pairs $(Y,\mathcal{L})$ in this set with the property that $\mathcal{L}$ is ample on Y. This may provide us with a preferred class.
In particular, if $\mathfrak{N}^{\mathrm{pol}}_{(\Lambda,h)}$ is the sublocus of $\mathfrak{N}_{(\Lambda,h)}$ of isomorphism classes of pairs $(Y,\mathcal{L})$ with $\mathcal{L}$ ample on Y and $\overline{\mathfrak{N}}^{\mathrm{pol}}_{(\Lambda,h)}$ its image in $\overline{\mathfrak{N}}_{(\Lambda,h)}$, then for every point in $\overline{\mathfrak{N}}^{\mathrm{pol}}_{(\Lambda,h)}$ there is a preferred isomorphism class of pairs $(Y,\mathcal{L})$ to represent it. In fact $\mathfrak{N}^{\mathrm{pol}}_{(\Lambda,h)} \cong \overline{\mathfrak{N}}^{\mathrm{pol}}_{(\Lambda,h)}$, and we find in particular that $\mathfrak{N}^{\mathrm{pol}}_{(\Lambda,h)}$ (locally) has the structure of a quasi-projective variety. This is well known from the theory of moduli spaces of projective varieties.

1.1.3 Holomorphic symplectic fourfolds of K3$^{[2]}$-type

Let S be a K3 surface. We mentioned before that the Hilbert scheme of length n subschemes of S is a holomorphic symplectic manifold, which we denote by $S^{[n]}$. In this thesis we will be concerned with holomorphic symplectic fourfolds that can be deformed into $S^{[2]}$ for some K3 surface S, that is, holomorphic symplectic fourfolds of K3$^{[2]}$-type. In this section we specify some of the general properties introduced in the previous sections to holomorphic symplectic manifolds of this type.

First we give an alternative construction of $S^{[2]}$. Let $\Delta \subset S \times S$ denote the diagonal and

$$\varepsilon_\Delta : \mathrm{Bl}_\Delta(S \times S) \to S \times S$$

the blow-up of Δ in $S \times S$. Let σ be the involution on $S \times S$ that interchanges the factors and $\widehat{\sigma}$ the lift of σ to $\mathrm{Bl}_\Delta(S \times S)$. Then $S^{[2]} \cong \mathrm{Bl}_\Delta(S \times S)/\widehat{\sigma}$. Moreover, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Bl}_\Delta(S \times S) & \xrightarrow{q_{\widehat{\sigma}}} & S^{[2]} \\ \downarrow{\scriptstyle \varepsilon_\Delta} & & \downarrow{\scriptstyle \eta} \\ S \times S & \xrightarrow[q_\sigma]{} & S^{(2)} \end{array} \tag{1.1.1}$$

where $S^{(2)} := (S \times S)/\sigma$, q_σ and $q_{\widehat{\sigma}}$ denote taking the quotient by σ and $\widehat{\sigma}$ respectively and η is the map that sends a length 2 subscheme to its support. In fact, the map η corresponds to the blow-up of the diagonal in $S^{(2)}$ and the exceptional divisor E of η corresponds precisely to the locus in $S^{[2]}$ that represents non-reduced length 2 subschemes of S.

Furthermore, there is an obvious incidence correspondence:

$$\begin{array}{ccc} & Z & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ S^{[2]} & & S \end{array} \tag{1.1.2}$$

where $S^{[2]} \times S \supset Z := \{(s,p) \mid p \in \mathrm{supp}(s)\}$ and p_1 and p_2 are the restricted projections. Since $p_1(p_2^{-1})(C) \subset S^{[2]}$ is a divisor for any curve $C \subset S$ it follows that $(p_1)_* p_2^*$ defines a map from $H^2(S,\mathbf{Z})$ to $H^2(S^{[2]},\mathbf{Z})$. In [3] Beauville proves the following:

Theorem 1.1.23. *The map*

$$(p_1)_* p_2^* : H^2(S,\mathbf{Z}) \to H^2(S^{[2]},\mathbf{Z})$$

is an embedding that preserves the Beauville-Bogomolov form and is compatible with the Hodge structures. Moreover, there is a lattice decomposition

$$H^2(S^{[2]},\mathbf{Z}) = (p_1)_* p_2^* \big(H^2(S,\mathbf{Z})\big) \oplus \mathbf{Z}\delta,$$

where 2δ is the cohomology class of the divisor of non-reduced length 2 subschemes of S. Furthermore, if q denotes the Beauville-Bogomolov form on $H^2(S^{[2]},\mathbf{Z})$, then $q(\delta) = -2$.

Notation 1.1.24. Let $(S,\mathcal{L})$ be a K3 surface with line bundle. In the rest of this document we will denote by $\mathcal{L}^{(a,b)}$ the line bundle on $S^{[2]}$ with first Chern class $a(p_1)_* p_2^* c_1(\mathcal{L}) + b\delta$.

It is well known (see for example [2]) that

$$H^2(S,\mathbf{Z}) \cong \widetilde{E8}^{\oplus 2} \oplus U^{\oplus 3},$$

where $\widetilde{E8}$ is the standard root lattice twisted by -1 and U is the rank 2 hyperbolic lattice. Usually the lattice $\widetilde{E8}$ is denoted as $E8(-1)$, but we chose this notation to avoid confusion with Tate twists in chapter 2, see also remark 2.1.2. Since the second cohomology lattice of a holomorphic symplectic manifold is a topological invariant, theorem 1.1.23 directly implies the following:

Proposition 1.1.25. *Let Y be a holomorphic symplectic manifold of $K3^{[2]}$-type. Then there is a lattice isomorphism*

$$H^2(Y,\mathbf{Z}) \cong \widetilde{E8}^{\oplus 2} \oplus U^{\oplus 3} \oplus \langle -2 \rangle,$$

where $\langle -2 \rangle$ denotes the rank one lattice such that the quadratic form takes value -2 on the generators.

For convenience we will denote the lattice $\widetilde{E8}^{\oplus 2} \oplus U^{\oplus 3} \oplus \langle -2\rangle$ by $\Lambda_{K3^{[2]}}$. Hence for any holomorphic symplectic manifold Y of K3$^{[2]}$-type a marking is a lattice isomorphism $\varphi : Y \to \Lambda_{K3^{[2]}}$. We will denote the moduli space of marked holomorphic symplectic manifolds of K3$^{[2]}$-type by $\mathfrak{M}_{K3^{[2]}}$. Note that this may be different from $\mathfrak{M}_{\Lambda_{K3^{[2]}}}$ (in the notation of section 1.1.1); there may exist holomorphic symplectic manifolds that have a second cohomology lattice isomorphic to $\Lambda_{K3^{[2]}}$, but nevertheless are *not* of K3$^{[2]}$-type. In fact, at present it is unknown if such holomorphic symplectic manifolds exist, although conjectures and partial results concerning this question can be found in [40]. For convenience, we will denote the period domain in $\mathbf{P}(\Lambda_{K3^{[2]}} \otimes \mathbf{C})$ by $\Omega_{K3^{[2]}}$. The period map $\mathfrak{M}_{K3^{[2]}} \to \Omega_{K3^{[2]}}$ will be denoted by $\mathcal{P}_{K3^{[2]}}$.

The holomorphic symplectic manifolds of K3$^{[2]}$-type that we will consider will always come with a line bundle $\mathcal{L}$ of positive square with respect to the Beauville–Bogomolov form. Let us specialize the notation and results introduced in section 1.1.2 to the case of holomorphic symplectic manifolds of K3$^{[2]}$-type.

Let $h \in \Lambda_{K3^{[2]}}$ be such that $h.h > 0$ and denote by $\mathfrak{M}_{K3^{[2]},h} \subset \mathfrak{M}_{K3^{[2]}}$ the moduli space of $(\Lambda_{K3^{[2]}}, h)$-marked holomorphic symplectic manifolds of K3$^{[2]}$-type. Let $\mathfrak{M}^{\mathrm{pol}}_{K3^{[2]},h} \subset \mathfrak{M}_{K3^{[2]},h}$ be the subspace of *polarized* such manifolds, that is, the space of triples $(Y, \mathcal{L}, \varphi)$ for which $\mathcal{L}$ is ample on Y. Let $\mathfrak{M}^{+}_{K3^{[2]},h}$ and $\mathfrak{M}^{-}_{K3^{[2]},h}$ be the subspaces with of triples $(Y, \mathcal{L}, \varphi)$ with $c_1(\mathcal{L}) \in C^{+}_{Y}$ and $c_1(\mathcal{L}) \in C^{-}_{Y}$ respectively. Let

$$\mathcal{P}_{K3^{[2]},h} : \mathfrak{M}_{K3^{[2]},h} \to \Omega_{h^{\perp}}$$

be the period map. As before we may take the quotient with respect to the group action of $O(\Lambda_{K3^{[2]}}, h)$ to obtain

$$\mathcal{Q}_{K3^{[2]},h} : \mathfrak{N}_{K3^{[2]},h} \to \mathcal{D}_{K3^{[2]},h}.$$

Also we write

$$\mathfrak{N}^{\pm}_{K3^{[2]},h} := O(\Lambda, h)\backslash \mathfrak{M}^{\pm}_{K3^{[2]},h}.$$

In the case of holomorphic symplectic fourfolds of K3$^{[2]}$-type the Torelli theorems as presented in the previous section can be improved. Indeed, in [31] Markman proves that if Y and Y' are two holomorphic symplectic manifolds of K3$^{[2]}$-type (or more generally, of K3$^{[n]}$-type with $n-1$ a prime power) then for any lattice isomorphism $f : H^2(Y, \mathbf{Z}) \to H^2(Y', \mathbf{Z})$ either f or $-f$ (but not both) is a parallel transport operator. Hence we have the following specific Torelli-type theorem:

Theorem 1.1.26. *Suppose that Y and Y' are holomorphic symplectic manifolds of K3$^{[2]}$-type.*

1. *If $f : H^2(Y, \mathbf{Z}) \to H^2(Y', \mathbf{Z})$ is a Hodge isometry, then Y and Y' are bimeromorphic.*

2. *If in addition f maps a Kähler class of Y to a Kähler class of Y', then there exists an isomorphism $\widetilde{f}: Y \to Y'$ such that $f = \widetilde{f}_*$.*

Proof. Since Y and Y' are assumed to be of K3$^{[2]}$-type, it follows that either f or $-f$ is a parallel transport operator. Since f is isomorphism of Hodge structures, so is $-f$, and it follows that there exists a map $f' : H^2(Y,\mathbf{Z}) \to H^2(Y',\mathbf{Z})$ that is both an isomorphism of Hodge structures and a parallel transport operator. In case that f maps a Kähler class to a Kähler class, we must have $f' = f$ (rather than $f' = -f$) since the positive cone is preserved by parallel transport. The theorem now follows from theorem 1.1.6. □

With this result we can improve proposition 1.1.20 for the case of holomorphic symplectic manifolds of K3$^{[2]}$-type. Before we state this improved result we make the following observation:

Proposition 1.1.27. *Let $h \in \Lambda_{K3^{[2]}}$ such that $h.h > 0$. Then the spaces $\mathfrak{N}^{\pm}_{K3^{[2]},h} \subset \mathfrak{N}_{K3^{[2]},h}$ are connected. In particular $\overline{\mathcal{Q}}_{K3^{[2]},h}$ maps $\overline{\mathfrak{N}}^{+}_{K3^{[2]},h}$ and $\overline{\mathfrak{N}}^{-}_{K3^{[2]},h}$ isomorphically onto $\mathcal{D}_{K3^{[2]},h}$ and restricts to an embedding on $\mathfrak{N}^{\mathrm{pol}}_{K3^{[2]},h}$.*

Proof. By proposition 1.1.19 the number of connected components of $\mathfrak{N}_{(\Lambda,h)}$ is the cardinality of $\overline{\mathcal{Q}}^{-1}_{K3^{[2]},h}(p)$ for any $p \in \mathcal{D}_{K3^{[2]},h}$. For the first statement it suffices to show that this number is 2.

Let $(Y,\mathcal{L})$ and $(Y',\mathcal{L}')$ be two pairs with isomorphism class in $\mathfrak{N}_{K3^{[2]},h}$ and assume that they have the same period. Then there exists a Hodge isometry $\chi : H^2(Y,\mathbf{Z}) \to H^2(Y',\mathbf{Z})$ such that $\chi(c_1(\mathcal{L})) = c_1(\mathcal{L}')$. By theorem 1.1.26 either χ or $-\chi$ is also a parallel transport operator. It now follows from proposition 1.1.20 that if χ is a parallel transport operator, $(Y,\mathcal{L})$ and $(Y',\mathcal{L}')$ must be in the same connected component, and if $-\chi$ is a parallel transport operator, $(Y,\mathcal{L}^\vee)$ and $(Y',\mathcal{L}')$ must be in the same connected component of $\mathfrak{N}_{K3^{[2]},h}$. It follows that $\mathfrak{N}_{K3^{[2]},h}$ has at most 2 connected components. Note that it must be at least 2, since $\mathfrak{N}^{+}_{K3^{[2]},h}$ and $\mathfrak{N}^{-}_{K3^{[2]},h}$ are disjoint. This proves the first part.

It now follows from proposition 1.1.19 that $\overline{\mathcal{Q}}_{K3^{[2]},h}$ maps $\overline{\mathfrak{N}}^{+}_{K3^{[2]},h}$ and $\overline{\mathfrak{N}}^{-}_{K3^{[2]},h}$ isomorphically onto $\mathcal{D}_{K3^{[2]},h}$. For the last statement, note that the first Chern class of an ample line bundle $\mathcal{L}$ must be contained in the positive cone. Hence $\mathfrak{N}^{\mathrm{pol}}_{K3^{[2]},h} \subset \mathfrak{N}^{+}_{K3^{[2]},h}$ and $\overline{\mathcal{Q}}_{K3^{[2]},h}$ restricts to an embedding on $\overline{\mathfrak{N}}^{\mathrm{pol}}_{K3^{[2]},h}$. But $\mathfrak{N}^{\mathrm{pol}}_{K3^{[2]},h}$ is isomorphic to $\overline{\mathfrak{N}}^{\mathrm{pol}}_{K3^{[2]},h}$ since it is Hausdorff, so $\mathcal{Q}_{K3^{[2]},h}$ embeds $\mathfrak{N}^{\mathrm{pol}}_{K3^{[2]},h}$ into $\mathcal{D}_{K3^{[2]},h}$. □

We will now prove an analogue to proposition 1.1.20.

Theorem 1.1.28. *Let $h \in \Lambda_{K3^{[2]}}$ with $h.h > 0$ and let $(Y,\mathcal{L})$ and $(Y',\mathcal{L}')$ be holomorphic symplectic manifolds of K3$^{[2]}$-type, with line bundles, such that the isomorphism classes of the pairs belong to $\mathfrak{N}_{K3^{[2]},h}$. Then $(Y,\mathcal{L})$ and $(Y',\mathcal{L}')$ have the* *same period in $\mathcal{D}_{K3^{[2]},h}$ if and only if there exists a bimeromorphism $\psi : Y \dashrightarrow Y'$ and Hodge isometry $\chi : H^2(Y,\mathbf{Z}) \to H^2(Y,\mathbf{Z})$ such that $\chi(c_1(\mathcal{L})) = \psi^*(c_1(\mathcal{L}'))$.*

Furthermore, $(Y,\mathcal{L})$ and $(Y',\mathcal{L}')$ are non-separated in $\mathfrak{N}_{K3^{[2]},h}$ if and only if there exists a bimeromorphism $\psi : Y \dashrightarrow Y'$ and Hodge isometry $\chi : H^2(Y,\mathbf{Z}) \to H^2(Y,\mathbf{Z})$ ***which preserves the positive cone*** *such that $\chi(c_1(\mathcal{L})) = \psi^*(c_1(\mathcal{L}'))$.*

Finally, if $\mathcal{L}$ is ample on Y, $\mathcal{L}'$ is nef on Y and $(Y,\mathcal{L})$ and $(Y',\mathcal{L}')$ have the same period in $\mathcal{D}_{K3^{[2]},h}$, then $(Y,\mathcal{L})$ and $(Y',\mathcal{L}')$ are isomorphic.

Proof. Assume that $(Y,\mathcal{L})$ and $(Y',\mathcal{L}')$ have the same period in $\mathcal{D}_{K3^{[2]},h}$. Then there exists a Hodge isometry $\chi' : H^2(Y,\mathbf{Z}) \to H^2(Y',\mathbf{Z})$ that sends $c_1(\mathcal{L})$ to $c_1(\mathcal{L}')$. It follows from theorem 1.1.26 that there exists a bimeromorphism $\psi : Y \dashrightarrow Y'$. Note that $\psi^* : H^2(Y',\mathbf{Z}) \to H^2(Y,\mathbf{Z})$ is a Hodge isometry. Now define $\chi := \psi^* \circ \chi'$. This is a composition of Hodge isometries, hence itself a Hodge isometry. Furthermore we have by construction that $\chi(c_1(\mathcal{L})) = \psi^*(c_1(\mathcal{L}'))$.

Conversely, assume that here exists a bimeromorphism $\psi : Y \dashrightarrow Y'$ and Hodge isometry $\chi : H^2(Y,\mathbf{Z}) \to H^2(Y,\mathbf{Z})$ such that $\chi(c_1(\mathcal{L})) = \psi^*(c_1(\mathcal{L}'))$. Let φ be any marking of Y and let $h := \varphi(c_1(\mathcal{L}))$. Then $\varphi' := \varphi \circ \chi^{-1} \circ \psi^*$ is a $(\Lambda_{K3^{[2]}}, h)$-marking of Y' and from the assumptions it follows that $\varphi^{-1} \circ \varphi'$ is a Hodge isometry. Therefore $(Y,\mathcal{L},\varphi)$ and $(Y',\mathcal{L}',\varphi')$ have the same period in $\Omega_{h^\perp}$ and thus $(Y,\mathcal{L})$ and $(Y,\mathcal{L})$ have the same period in $\mathcal{D}_{K3^{[2]},h}$. The first statement follows.

Now assume that $(Y,\mathcal{L})$ and $(Y',\mathcal{L}')$ are non-separated in $\mathfrak{N}_{K3^{[2]},h}$. Then in particular the periods in $\mathcal{D}_{K3^{[2]},h}$ are the same. It follows from the first part of the theorem that there exist a bimeromorphism $\psi : Y \dashrightarrow Y'$ and a Hodge isometry $\chi : H^2(Y,\mathbf{Z}) \to H^2(Y',\mathbf{Z})$ such that $\chi(c_1(\mathcal{L})) = \psi^*(c_1(\mathcal{L}'))$. The isomorphism classes of the pairs $(Y,\mathcal{L})$ and $(Y',\mathcal{L}')$ either both lie in $\mathfrak{N}^+_{K3^{[2]},h}$ or both in $\mathfrak{N}^-_{K3^{[2]},h}$, implying that either $c_1(\mathcal{L}) \in C^+_Y$ and $c_1(\mathcal{L}') \in C^+_{Y'}$ or $c_1(\mathcal{L}) \in C^-_Y$ and $c_1(\mathcal{L}') \in C^-_{Y'}$. In any case, the Hodge isometry $\chi^{-1} \circ \psi^*$ preserves the positive cones. Since ψ^* is induced by a bimeromorphism, it also preserves positive cones. Hence χ preserves the positive cone on Y.

Conversely, assume that there exists a bimeromorphism $\psi : Y \dashrightarrow Y'$ and Hodge isometry $\chi : H^2(Y,\mathbf{Z}) \to H^2(Y,\mathbf{Z})$ which preserves C^+_Y such that $\chi(c_1(\mathcal{L})) = \psi^*(c_1(\mathcal{L}'))$. Then in particular, by the first part of the theorem, $(Y,\mathcal{L})$ and $(Y',\mathcal{L},)$ have the same period. Furthermore, the additional assumption that χ preserves C^+_Y implies the isomorphism classes of that $(Y,\mathcal{L})$ and $(Y',\mathcal{L},)$ either both belong to $\mathfrak{N}^+_{K3^{[2]},h}$ or both belong to $\mathfrak{N}^-_{K3^{[2]},h}$. In particular, by proposition 1.1.27 both isomorphism classes lie in the same connected component of $\mathfrak{N}_{K3^{[2]},h}$. Since they also have the same period, it follows from proposition 1.1.20 that $(Y,\mathcal{L})$ and (Y',L') are non-separated in $\mathfrak{N}_{K3^{[2]},h}$. This proves the second statement.

The last statement follows directly from the last statement in proposition 1.1.20. □

Remark 1.1.29. We end this section with a notational remark. Let $h, h' \in \Lambda_{K3^{[2]}}$ be elements in the same $O(\Lambda_{K3^{[2]}})$-orbit. Then clearly $\mathfrak{M}_{K3^{[2]},h} \cong \mathfrak{M}_{K3^{[2]},h'}$ and the isomorphism is fixed by a choice of isometry of $\Lambda_{K3^{[2]}}$ that sends h to h'. However, such isometries are unique up to (pre-)composition with elements of $O(\Lambda_{K3^{[2]}}, h)$. It follows that $\mathfrak{N}_{K3^{[2]},h}$ and $\mathfrak{N}_{K3^{[2]},h'}$ are *canonically* isomorphic for any pair h, h' in the same $O(\Lambda_{K3^{[2]}})$-orbit.

By Eichler's criterion, see [12], the $O(\Lambda)$-orbit of an element $h \in \Lambda$ is numerically determined by $d := h.h$ and the positive generator n of the ideal $h.\Lambda \subseteq \mathbf{Z}$. Since by the observations above all h in the same orbit the spaces $\mathfrak{N}_{(\Lambda,h)}$ can be canonically identified, we may as well denote them by $\mathfrak{N}_{(\Lambda,d,n)}$.

For the special case of $\Lambda = \Lambda_{K3^{[2]}}$ we will denote $\mathfrak{N}_{K3^{[2]},d,n} := \mathfrak{N}_{(\Lambda_{K3^{[2]}},d,n)}$. It is shown in [17] that in this case n can be 1 or 2 whenever $d \equiv -2 \mod 8$ and n is always 1 for other values of d. The number d itself is always even. In case $n = 1$ for a certain line bundle we say that it is of **odd type**, if $n = 2$ we say that it is of **even type**.

1.2 Cubic fourfolds

By a cubic fourfold we will mean a smooth cubic hypersurface in $\mathbf{P}^5$. In this section we recall some of their properties that will be relevant to us.

Let X be a cubic fourfold. The intersection form endows $H^4(X,\mathbf{Z})$ with the structure of a lattice. Let $h \in H^2(X,\mathbf{Z})$ be the hyperplane class, then we will denote the primitive cohomology $(h^2)^\perp \subset H^4(X,\mathbf{Z})$ by $H^4(X,\mathbf{Z})^0$.

Theorem 1.2.1. *Let X be a cubic fourfold. Then there exist lattice isomorphisms:*

$$\begin{aligned} H^4(X,\mathbf{Z}) &\cong \langle +1\rangle^{\oplus 21} \oplus \langle -1\rangle^{\oplus 2} \\ H^4(X,\mathbf{Z})^0 &\cong \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \oplus E8^{\oplus 2} \oplus U^{\oplus 2}. \end{aligned}$$

Furthermore we have a Hodge decomposition

$$H^4(X,\mathbf{C}) \cong H^{3,1}(X) \oplus H^{2,2}(X) \oplus H^{1,3}(X)$$

and $H^{3,1}(X)$ is one-dimensional.

See [21] for a proof. For convenience we will denote $\Lambda_{\text{cubic}} := \langle +1\rangle^{\oplus 21} \oplus \langle -1\rangle^{\oplus 2}$ and we fix a class $g \in \Lambda_{\text{cubic}}$ with self-intersection 3 (all such classes are in one $O(\Lambda_{\text{cubic}})$-orbit since Λ_{cubic} is unimodular). We denote $\Lambda^0_{\text{cubic}} := g^\perp$.

We define a (Λ, g)-marking, or simply a marking, of a cubic fourfold to be a map

$$\varphi : H^4(X,\mathbf{Z}) \to \Lambda_{\text{cubic}}$$

such that $\varphi(h^2) = g$. Let $\mathfrak{M}_{\text{cubic}}$ be the moduli space of marked cubic fourfolds. It is a smooth complex manifold of dimension 20. By theorem 1.2.1 we can define a period map that is very similar to the period map for holomorphic symplectic manifolds:

$$\begin{array}{rcl}\mathcal{P}_{\text{cubic}}:\mathfrak{M}_{\text{cubic}} & \to & \mathbf{P}(\Lambda_{\text{cubic}}\otimes\mathbf{C})\\ (X,\varphi) & \mapsto & \varphi(H^{3,1}(X)).\end{array}$$

It is not hard to see that the following holds (see also [21]):

- $H^{3,1}(X)$ is isotropic with respect to the intersection form;
- $H^{3,1}(X)$ is orthogonal to $H^{2,2}(X)$ with respect to the intersection form;
- The pairing $u.\overline{u}$ is strictly negative for any non-zero $u \in H^{3,1}(X)$.

It follows that the image of $\mathcal{P}_{\text{cubic}}$ is contained in

$$\Omega_{\text{cubic}} := \left\{[x] \in \mathbf{P}(\Lambda_{\text{cubic}}\otimes\mathbf{C}) \,|\, (x,g) = (x,x) = 0, (x,\overline{x}) < 0\right\},$$

which we call the **period domain for cubic fourfolds**. It follows from theorem 1.2.1 that Ω_{cubic} has two (isomorphic) connected components, each of which is a symmetric domain of type IV.

In her thesis ([47]), Voisin proved that $\mathcal{P}_{\text{cubic}}$ is an embedding. For some time the question remained what the complement of the image of $\mathcal{P}_{\text{cubic}}$ in the period domain would be. This was proven independently by Looijenga in [30] and Laza in [29], based on a conjecture by Hassett. The result is the following:

Theorem 1.2.2. *The complement of the image of* $\mathcal{P}_{\text{cubic}}$ *in* Ω_{cubic} *is precisely* $\mathcal{H}_2 \cup \mathcal{H}_6$, *where*

$$\mathcal{H}_d := \bigcup_{K_d} \mathbf{P}(K_d^{\perp}\otimes\mathbf{C}) \cap \Omega_{\text{cubic}}$$

and the union runs over all primitive rank 2 sublattices $K_d \subset \Lambda_{\text{cubic}}$ *that contain* g *and have discriminant* d.

Remark 1.2.3. The period map can be extended (see [29] or [30]) to cubic hypersurfaces in $\mathbf{P}^5$ with mild singularities. Hassett shows in [21] that the generic element of $\mathcal{H}_6$ is the period of a cubic hypersurface in $\mathbf{P}^5$ with an isolated ordinary double point.
The elements of $\mathcal{H}_2$ are not periods of a cubic hypersurface. Hassett shows however that the generic element can be obtained as the limit of the periods of a one-parameter family of cubic fourfolds that degenerates to the so-called **determinantal cubic**. This is a specific singular cubic hypersurface that may be described as the space of singular conics on $\mathbf{P}^2$ in $|\mathcal{O}_{\mathbf{P}^2}(2)| \cong \mathbf{P}^5$. See also definition 2.1.1. It plays a central role in this thesis, especially in chapter 3.

As in the case of holomorphic symplectic manifolds there is an action of $O(\Lambda_{\text{cubic}}, g)$ on $\mathfrak{M}_{\text{cubic}}$ and Ω_{cubic}, and $\mathcal{P}_{\text{cubic}}$ intertwines this action. Moreover, it is obvious from the definition that for every $d \in \mathbf{N}$ the space $\mathcal{H}_d$ is preserved by the action of $O(\Lambda_{\text{cubic}}, g)$. We denote

$$\begin{aligned} \mathfrak{N}_{\text{cubic}} &:= O(\Lambda_{\text{cubic}}, g)\backslash\mathfrak{M}_{\text{cubic}} \\ \mathcal{D}_{\text{cubic}} &:= O(\Lambda_{\text{cubic}}, g)\backslash\Omega_{\text{cubic}} \\ \mathcal{D}_{\text{cubic},d} &:= O(\Lambda_{\text{cubic}}, g)\backslash\mathcal{H}_d \end{aligned}$$

and denote by $\mathcal{Q}_{\text{cubic}} : \mathfrak{N}_{\text{cubic}} \to \mathcal{D}_{\text{cubic}}$ the map induced by $\mathcal{P}_{\text{cubic}}$. Note that $\mathfrak{N}_{\text{cubic}}$ is precisely the moduli space of cubic fourfolds. It can be constructed as a GIT-quotient and has the structure of a quasi-projective variety. Also $\mathcal{D}_{\text{cubic}}$ has the structure of a quasi-projective variety, as follows from the theory by Baily and Borel. Furthermore, Hassett explains in [21] that $\mathcal{Q}_{\text{cubic}}$ is an algebraic map. We may rephrase theorem 1.2.2 without explicit reference to markings as follows:

Theorem 1.2.4. *The map $\mathcal{Q}_{\text{cubic}}$ defines an isomorphism of quasi-projective varieties*

$$\mathfrak{N}_{\text{cubic}} \cong \mathcal{D}_{\text{cubic}} \setminus (\mathcal{D}_{\text{cubic},2} \cup \mathcal{D}_{\text{cubic},6}).$$

1.3 From cubic fourfolds to holomorphic symplectic fourfolds

Let V be a complex vector space of dimension 6 and $X \subset \mathbf{P}(V)$ a cubic fourfold. The variety of lines on $\mathbf{P}(V)$ is isomorphic to the Grassmannian of 2-planes in V, which we will denote by $\mathrm{Gr}_2(V)$. Let $F(X) \subset \mathrm{Gr}_2(V)$ be the variety of lines that are contained in X.

The restriction of the Plücker line bundle on $\mathrm{Gr}_2(V)$ (that is, the pullback of $\mathcal{O}_{\mathbf{P}(\bigwedge^2 V)}(1)$ along the Pücker embedding $\mathrm{Gr}_2(V) \hookrightarrow \mathbf{P}(\bigwedge^2 V)$) defines a polarization on $F(X)$. In [6] Beauville and Donagi prove the following famous result:

Theorem 1.3.1. *For any cubic fourfold X the variety $F(X)$ is isomorphic to a holomorphic symplectic fourfold of $K3^{[2]}$-type. The natural polarization has square 6 (see remark 1.1.18) and is of even type with respect to the Beauville-Bogomolov form. Moreover, if $\mathcal{L}$ denotes the natural polarization of $F(X)$, then for any sufficiently small deformation $\mathcal{Y} \to B$ of the pair $(F(X), \mathcal{L})$ there exists a family of cubic fourfolds $\mathcal{X} \to B$ such that $\mathcal{Y}_t \cong F(\mathcal{X}_t)$ for all $t \in B$.*

In fact they prove more. Consider the incidence correspondence

$$\begin{array}{ccc} & Z & \\ {}^{p}\swarrow & & \searrow^{q} \\ X & & F(X) \end{array} \tag{1.3.1}$$

where

$$Z = \{(x,\ell) \in X \times F(X) \mid x \in \ell\}$$

and p and q are the obvious projections. Since q is a $\mathbf{P}^1$-bundle over $F(X)$ it follows that Z has dimension 5. We therefore have the following map on the level of cohomology, which Beauville and Donagi refer to as the Abel–Jacobi map:

$$\begin{aligned} \Phi_{\mathrm{AJ}} : H^4(X,\mathbf{Z}) &\rightarrow H^2(F(X),\mathbf{Z}) \\ \alpha &\mapsto q_* p^* \alpha. \end{aligned}$$

They then prove the following:

Theorem 1.3.2. *The Abel–Jacobi map Φ_{AJ} is an isomorphism of $\mathbf{Z}$-modules and maps the class in $H^4(X,\mathbf{Z})$ defined by the square of the hyperplane section to the polarization class in $H^2(F(X),\mathbf{Z})$.*
Moreover, Let $H^4(X,\mathbf{Z})^0$ and $H^2(X,\mathbf{Z})^0$ be the respective primitive sublattices. Then the restriction

$$\Phi^0_{\mathrm{AJ}} : H^4(X,\mathbf{Z})^0 \rightarrow H^2(X,\mathbf{Z})^0$$

*of Φ_{AJ} defines an isomorphism of **lattices** provided we change the sign of the quadratic form on $H^2(X,\mathbf{Z})^0$. In turn, this map extends to a Hodge isomorphism*

$$H^4(X,\mathbf{C})^0 \rightarrow H^2(F(X),\mathbf{C})^0(-1)$$

where (-1) denotes a Tate twist.

We rephrase this in terms of the notation we introduced in sections 1.1.2 and 1.1.3. Let X be a cubic fourfold, $F(X)$ its variety of lines and $\mathcal{L}$ the natural polarization on $F(X)$. By theorem 1.3.1 $\mathcal{L}$ has square 6 and is of even type. Hence the isomorphism class of $(F(X),\mathcal{L})$ defines a point of $\mathfrak{N}_{K3^{[2]},6,2}$ (in the notation introduced in remark 1.1.29). More precisely, since $\mathcal{L}$ is very ample on $F(X)$, the isomorphism class of $(F(X),\mathcal{L})$ belongs to $\mathfrak{N}^{\mathrm{pol}}_{K3^{[2]},6,2}$ which in turn is a subset of $\mathfrak{N}^{+}_{K3^{[2]},6,2}$. Because this space will occur a lot in this thesis, we will simply denote $\mathfrak{N}_{\mathrm{BD}} := \mathfrak{N}^{+}_{K3^{[2]},6,2}$, where the subscript 'BD' refers to 'the topological type of holomorphic symplectic manifolds obtained from the Beauville-Donagi construction'. Analogously, the Hausdorff reduction will be denoted by $\overline{\mathfrak{N}}_{\mathrm{BD}}$, the locus of polarized pairs by $\mathfrak{N}^{\mathrm{pol}}_{\mathrm{BD}}$, the quotient of the period domain by $\mathcal{D}_{\mathrm{BD}}$, the (unmarked) period map by $\mathcal{Q}_{\mathrm{BD}}$, and for any $h \in \Lambda_{K3^{[2]}}$ of square 6 and even type, the marked moduli space $\mathfrak{M}_{\Lambda_{K3^{[2]}},h}$ will be denoted by $\mathfrak{M}_{\mathrm{BD},h}$.
The construction of Beauville and Donagi now gives us a map

$$\begin{aligned} F : \mathfrak{N}_{\mathrm{cubic}} &\rightarrow \mathfrak{N}_{\mathrm{BD}} \\ X &\mapsto F(X). \end{aligned}$$

It can be shown that this is locally an isomorphism of quasi-projective varieties. Theorem 1.3.2 asserts that the Abel–Jacobi map induces an isomorphism Ψ_{AJ} : $\mathcal{D}_{\mathrm{cubic}} \to \mathcal{D}_{\mathrm{BD}}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{N}_{\mathrm{cubic}} & \xrightarrow{F} & \mathfrak{N}_{\mathrm{BD}} \\ \downarrow{\scriptstyle \mathcal{Q}_{\mathrm{cubic}}} & & \downarrow{\scriptstyle \mathcal{Q}_{\mathrm{BD}}} \\ \mathcal{D}_{\mathrm{cubic}} & \xrightarrow[\Psi_{\mathrm{AJ}}]{} & \mathcal{D}_{\mathrm{BD}} \end{array} \tag{1.3.2}$$

Here $\mathfrak{N}_{\mathrm{cubic}}$, $\mathcal{D}_{\mathrm{cubic}}$ and $\mathcal{D}_{\mathrm{BD}}$ are complex analytic spaces that can be naturally endowed with the structure of a quasi-projective variety. The space $\mathfrak{N}_{\mathrm{BD}}$ is locally complex analytic, but may be non-Hausdorff. However, diagram (1.3.2) can be reduced to the following:

$$\begin{array}{ccc} \mathfrak{N}_{\mathrm{cubic}} & \xrightarrow{\overline{F}} & \overline{\mathfrak{N}}_{\mathrm{BD}} \\ \downarrow{\scriptstyle \mathcal{Q}_{\mathrm{cubic}}} & & \downarrow{\scriptstyle \overline{\mathcal{Q}}_{\mathrm{BD}}} \\ \mathcal{D}_{\mathrm{cubic}} & \xrightarrow[\Psi_{\mathrm{AJ}}]{} & \mathcal{D}_{\mathrm{BD}} \end{array} \tag{1.3.3}$$

In this diagram all spaces are complex analytic with at most finite quotient singularities, and can all be endowed with a natural quasi-projective structure. In fact, Ψ_{AJ} is an isomorphism by theorem 1.3.2 and $\overline{\mathcal{Q}}_{\mathrm{BD}}$ is an isomorphism by proposition 1.1.27. By theorem 1.2.2 $\mathcal{Q}_{\mathrm{cubic}}$ defines an embedding of analytic spaces (and even of quasi-projective varieties). It follows that $\overline{F}$ is an embedding of analytic spaces as well. The map F is only an embedding of locally analytic spaces.

We may of course lift diagram (1.3.2) to one that involves moduli spaces of marked cubics and holomorphic symplectic fourfolds, but this is not canonical. More precisely, we have the following: fix $g \in \Lambda_{\mathrm{cubic}}$ of self-intersection 3 and $h \in \Lambda_{K3^{[2]}}$ such that $h.h = 6$ and $h.\Lambda_{K3^{[2]}} = 2\mathbf{Z}$. Then it follows from theorem 1.3.2 that there exists an isomorphism of $\mathbf{Z}$-modules $\Phi' : \Lambda_{\mathrm{cubic}} \to \Lambda_{K3^{[2]}}$ such that $\Phi'(g) = h$ and Φ' induces an isomorphism of *lattices*, up to a sign, from $h^{\perp}$ to $g^{\perp}$. If X is a cubic fourfold and φ a marking on X, then $\Phi' \circ \varphi \circ \Phi_{\mathrm{AJ}}^{-1}$ defines a marking on $F(X)$. We denote

$$\widetilde{F} : \mathfrak{M}_{\mathrm{cubic}} \to \mathfrak{M}_{\mathrm{BD},h},\ (X, \varphi) \mapsto (F(X), \Phi' \circ \varphi \circ \Phi_{\mathrm{AJ}}^{-1}).$$

Then diagram (1.3.2) lifts to the following:

$$\begin{array}{ccc} \mathfrak{M}_{\mathrm{cubic}} & \xrightarrow{\widetilde{F}} & \mathfrak{M}_{\mathrm{BD},h} \\ \downarrow{\scriptstyle \mathcal{P}_{\mathrm{cubic}}} & & \downarrow{\scriptstyle \mathcal{P}_{K3^{[2]},h}} \\ \Omega_{g^{\perp}} & \xrightarrow[\Phi'_{\Omega}]{} & \Omega_{h^{\perp}} \end{array} \tag{1.3.4}$$

where Φ'_{Ω} is the isomorphism of period domains induced by Φ'.

Chapter 2

Missing periods from holomorphic symplectic point of view

We use the notation of the previous chapter. Looijenga and Laza have proven that the complement of the image of $\mathcal{Q}_{\text{cubic}}$ is $\mathcal{D}_{\text{cubic},2} \cup \mathcal{D}_{\text{cubic},6}$; the Hodge structures for which the lattice of $(2,2)$-elements is of rank 2 and has discriminant 2 or 6 (see theorem 1.2.4). Denote $\mathcal{D}_{\text{BD},d} := \Psi_{\text{AJ}}(\mathcal{D}_{\text{cubic},d})$ and $\mathfrak{N}_{\text{BD},d} := \mathcal{Q}_{\text{BD}}^{-1}(\mathcal{D}_{\text{BD},d})$ for $d = 2,6$. By surjectivity of $\mathcal{Q}_{\text{BD}}$ every period in $\mathcal{D}_{\text{BD},2} \cup \mathcal{D}_{\text{BD},6}$ can be realized by at least one isomorphism class of a pair $(Y,\mathcal{L})$ in $\mathfrak{N}_{\text{BD},2} \cup \mathfrak{N}_{\text{BD},6}$.

In this chapter we explain how the results of Hassett in [21] lead to the identification of the isomorphism classes of pairs $(Y,\mathcal{L})$ in $\mathfrak{N}_{\text{BD},2}$ and $\mathfrak{N}_{\text{BD},6}$. Of course, it follows directly from theorem 1.2.2 that the pairs $(Y,\mathcal{L})$ in these loci cannot be obtained as variety of lines on a cubic fourfold. We will show how this fact also follows from the geometric properties of these specific holomorphic symplectic fourfolds and the line bundles of square 6 and of even type on them.

2.1 The case of $\mathfrak{N}_{\text{BD},2}$

2.1.1 Realization of the periods

Definition 2.1.1. Let W be a three-dimensional complex vector space. Note that there is a natural isomorphism

$$\begin{array}{ccc} \text{Sym}^2 W & \to & \text{Hom}_{\text{sym}}(W^\vee, W) \\ \xi & \mapsto & \varphi_\xi \end{array}$$

where φ_ξ is the symmetric map that sends $\alpha \in W^\vee$ to the contraction $i_\alpha \xi \in W$. Let $\mu \in \bigwedge^3 W$ be a volume element and let $D \in \text{Sym}^3\text{Sym}^2 W^\vee$ be the cubic form defined by

$$D_\mu(\xi) := \mu\Big(\bigwedge^3 \varphi_\xi(\mu)\Big).$$

Then D_μ is called the **determinantal cubic form**. We will often simply write D, since a different choice of μ only changes D by scaling, and that will not be relevant for us. The zero locus $Z(D) \subset \mathbf{P}(\mathrm{Sym}^2 W)$ is called the **determinantal cubic** and denoted by X_{det}.

In [21] Hassett proves that the periods in $\mathcal{D}_{\mathrm{cubic},2}$ are those that arise as limiting periods of deformations of the determinantal cubic. We explain his result here and show how it gives rise to Hilbert schemes of length 2 subschemes of K3 surfaces of degree 2.

Fix a volume form $\mu \in \bigwedge^3 W$ and let $D \in \mathrm{Sym}^3\mathrm{Sym}^2 W^\vee$ be the determinantal cubic form on $\mathrm{Sym}^2 W$ with respect to μ. Let $f \in \mathrm{Sym}^3\mathrm{Sym}^2 W^\vee$ be non-zero, then we can consider the one parameter linear deformation of D by f:

$$\begin{array}{ccc} \Delta & \to & \mathrm{Sym}^3\mathrm{Sym}^2 W^\vee \\ t & \mapsto & D + tf, \end{array}$$

where Δ is unit disc in the complex plane. This deformation defines a family of hypersurfaces, which we will denote by $\varphi_f : \mathcal{X}_f \to \Delta$. We will refer to it as a linear one-parameter deformation of the determinantal cubic by f

To a generic such deformation we can associate a K3 surface in the following way. Let

$$\begin{array}{rcl} v : \mathbf{P}(W) & \hookrightarrow & \mathbf{P}(\mathrm{Sym}^2 W) \\ {[w]} & \mapsto & [w^2] \end{array}$$

be the Veronese embedding. Then for generic choices of f, $v^* f$ is a non-zero sextic form on W and hence defines a sextic curve $\Sigma_f \subset W$. Let $\pi_f : S_f \to \mathbf{P}(W)$ be the double cover branched along Σ_f. In the generic case that Σ_f is smooth, S_f will be a K3 surface and $\pi^*\mathcal{O}_{\mathbf{P}(W)}(1)$ will define a polarization of S_f of degree 2.

Although the variation of Hodge structures over the punctured disc Δ^* of the family φ_f may have non-trivial monodromy, Hassett shows in [21] that a degree 2 base change suffices to obtain a monodromy-free variation of Hodge structures. He then calculates the limiting Hodge structure for a such variation of Hodge structures. To state his result more precisely we introduce some extra notation. Let K be the lattice generated by elements α, β with intersection matrix

	α	β
α	1	1
β	1	3.

There is a trivial Hodge structure of weight 0 on $K \otimes_{\mathbf{Z}} \mathbf{C}$, by declaring the $(0,0)$-part to be all of $K \otimes_{\mathbf{Z}} \mathbf{C}$. We denote the n^{th} Tate twist, as is customary, by $(K \otimes_{\mathbf{Z}} \mathbf{C})(n)$.

Remark 2.1.2. For a lattice Λ it is customary to denote by $\Lambda(-1)$ the lattice obtained from Λ by multiplying the quadratic form by -1. To avoid confusion with Tate twists, we will use the notation $\widetilde{\Lambda}$ instead of $\Lambda(-1)$. We reserve the bracket notation for Tate twists.

Proposition 2.1.3. *Let $f \in \mathrm{Sym}^3\mathrm{Sym}^2 W^\vee$ be such that it defines a smooth cubic hypersurface and such that S_f is smooth (hence K3). Let $\xi_f : \mathcal{X}_f \to \Delta$ be a linear one parameter deformation of the determinantal cubic by f. Let $\xi'_f : \mathcal{X}'_f \to \Delta$ be the resulting family after the base change $\Delta \to \Delta$, $t \mapsto t^2$. Then the limiting mixed Hodge structure $H^4_{\lim,\xi'_f}$ of ξ'_f is pure of weight 4. Moreover, there is an isometric embedding of $\mathbf{Z}$-Hodge structures of weight 4:*

$$(K \otimes_{\mathbf{Z}} \mathbf{C})(-2) \oplus H^2(S_f,\mathbf{C})^0(-1) \hookrightarrow H^4_{\lim,\xi'_f},$$

which is an isomorphism of $\mathbf{Q}$-Hodge structures and sends $\beta \in K$ to the limiting class of the polarization in $H^4_{\lim,\xi'_f}$.

We can rephrase it as follows. The local system $\mathbf{R}^4(\xi'_f)_*\mathbf{Z} \to \Delta^*$ is trivial over Δ^*, hence extends to a local system $\mathbf{L} \to \Delta$. Then $H^4_{\lim,\xi'_f}$ is the limiting Hodge structure on $\mathbf{L}_0 \otimes \mathbf{C}$. In particular there is an embedding of lattices

$$j : K \oplus \widetilde{H}^2(S_f,\mathbf{Z})^0 \hookrightarrow \mathbf{L}_0,$$

where again the tilde denotes twisting the intersection form by -1. Now let $\eta_f : \mathcal{F}(\mathcal{X}'_f) \to \Delta$ be the relative line variety[1] of ξ'_f. The local system $\mathbf{R}^2(\eta_f)_*\mathbf{Z}$ is trivial over Δ^* and extends to a local system $\mathbf{M} \to \Delta$. The Abel–Jacobi map (see section 1.3) defines an isomorphism of local systems:

$$\Phi : \mathbf{L} \to \mathbf{M}.$$

Note that although both $\mathbf{L}$ and $\mathbf{M}$ carry a relative lattice structure, Φ is not an isomorphism of relative lattices. However, by theorem 1.3.2, its restriction to primitive parts *does* induce an isomorphism

$$\Phi^0 : \mathbf{L}^0 \to \widetilde{\mathbf{M}^0}$$

of relative lattices (see remark 2.1.2 for the notation).

Again by theorem 1.3.2, for any $t \in \Delta^*$ the Abel–Jacobi induces an isomorphism of $\mathbf{Z}$-Hodge structures (but not an isometry!):

$$\widehat{\Phi}_t : H^4(X_t,\mathbf{C}) \to H^2(F(X_t),\mathbf{C})(-1)$$

where X_t is the fiber of ξ'_f over t. It follows that $H^2_{\lim,\eta_f}(-1) \cong \widehat{\Phi}_0 H^4_{\lim,\xi'_f}$, where $H^2_{\lim,\eta_f}$ is precisely the limiting Hodge structure on $\mathbf{M}_0$ of the variation of Hodge structures that comes from the family η_f. We have the following result:

[1] For now it suffices to remark that such an object exists and its fibers are the line varieties of the fibers of ξ'_f. In section 3.2 we will go into more details.

Lemma 2.1.4. *Denote the lattice $\langle 2\rangle \oplus \langle -2\rangle$ by K', then there is an imbedding of lattices*

$$K' \oplus H^2(S_f, \mathbf{Z})^0 \hookrightarrow \mathbf{M}_0$$

which extends to an isomorphism of Hodge structures

$$(K' \otimes_{\mathbf{Z}} \mathbf{C})(-1) \oplus H^2(S_f, \mathbf{C})^0 \hookrightarrow H^2_{\lim,\eta_f}$$

Proof. Let $i := \Phi_0 \circ j$, then we have an inclusion of $\mathbf{Z}$-modules

$$i(K) \oplus i(\widetilde{H}^2(S_f, \mathbf{Z})^0) \subseteq \mathbf{M}_0,$$

and both summands inherit a lattice structure. By proposition 2.1.3 and the properties of the Abel–Jacobi map it follows that this inclusion extends to an isometric inclusion of $\mathbf{Z}$-Hodge structures:

$$\big(i(K) \otimes_{\mathbf{Z}} \mathbf{C}\big)(-1) \oplus i(\widetilde{H}^2(S_f, \mathbf{Z})^0) \otimes_{\mathbf{Z}} \mathbf{C} \subseteq H^2_{\lim,\eta_f}.$$

Since, also by proposition 2.1.3, the limit of the polarization class in $\mathbf{L}_0$ is contained in $j(K)$, it follows that $j(\widetilde{H}^2(S_f, \mathbf{Z})^0)$ is contained in the primitive part $\mathbf{L}_0^0$ of $\mathbf{L}_0$. Since the Abel–Jacobi map induces a lattice isomorphism $\mathbf{L}_0^0 \cong \widetilde{\mathbf{M}_0^0}$ we conclude that $i(\widetilde{H}^2(S_f, \mathbf{Z})^0) \cong H^2(S_f, \mathbf{Z})$ as lattices.

It now suffices to prove that $i(K) \cong \langle 2\rangle \oplus \langle -2\rangle$ as lattices. From the definition of K it follows that $\beta^\perp \cap K = \mathbf{Z}(\beta - 3\alpha)$. Note that $\beta - 3\alpha$ has self-intersection 6. The properties of the Abel–Jacobi map, as given by theorem 1.3.2, now imply the following:

- $i(\beta) = (\Phi_0 \circ j)(\beta)$ has self-intersection 6;
- $i(\beta) \perp i(\beta^\perp)$;
- $i(\beta^\perp) \cong \widetilde{\beta^\perp}$, that is, $i(\beta) - 3i(\alpha)$ has self intersection -6.

From this we can calculate the intersection table of $i(K)$:

	$i(\alpha)$	$i(\beta)$
$i(\alpha)$	0	2
$i(\beta)$	2	6

Now define $\gamma := i(\beta) - i(\alpha)$ and $\delta := i(\beta) - 2i(\alpha)$. Then $i(K) = \langle \gamma, \delta\rangle$, and the intersection table is

	γ	δ
γ	2	0
δ	0	-2

Hence $i(K) \cong \langle 2\rangle \oplus \langle -2\rangle$. □

This lemma suggests that the limiting Hodge structure of η_f is isomorphic to the weight two Hodge structure on $H^2(S_f^{[2]},\mathbf{C})$, where $S_f^{[2]}$ is, as before, the Hilbert scheme of length two subschemes of S_f. Indeed, we know (see theorem 1.1.23) that there is an isomorphism

$$H^2(S_f^{[2]},\mathbf{Z}) \cong H^2(S_f,\mathbf{Z}) \oplus \mathbf{Z}\delta,$$

where δ is a class of self-intersection -2. It becomes an isomorphism of lattices if we endow $H^2(S_f^{[2]},\mathbf{Z})$ with the Beauville–Bogomolov quadratic form, and it induces an isometric isomorphism of weight 2 Hodge structures (if we declare δ to be a $(1,1)$ class).
Furthermore, S_f comes with a polarization of degree 2. Let $\gamma \in H^2(S_f,\mathbf{Z})$ be the polarization class, then we have an inclusion of lattices

$$\mathbf{Z}\gamma \oplus H^2(S_f,\mathbf{Z})^0 \subset H^2(S_f,\mathbf{Z}),$$

and hence an inclusion

$$K' \oplus H^2(S_f,\mathbf{Z})^0 \subset H^2(S_f^{[2]},\mathbf{Z})$$

where $K' := \langle\gamma,\delta\rangle \cong \langle 2\rangle \oplus \langle -2\rangle$. This inclusion induces an isomorphism of Hodge structures. Compare this to lemma 2.1.4 and note in addition that $\mathbf{M}_0$ and $H^2(S_f^{[2]},\mathbf{Z})$ are isomorphic as lattices. These observations motivate the following result.

Lemma 2.1.5. *Let f be such that S_f is smooth. Then there exists a lattice isomorphism*

$$\Phi : H^2(S_f^{[2]},\mathbf{Z}) \to \mathbf{M}_0$$

which extends to an isometric isomorphism of Hodge structures

$$\Phi_* : H^2(S_f^{[2]},\mathbf{C}) \to H^2_{\lim,\eta_f}.$$

Proof. We have two lattice embeddings

$$\begin{array}{ccc} & & \mathbf{M}_0 \\ & \nearrow & \\ K' \oplus H^2(S_f,\mathbf{Z})^0 & & \\ & \searrow & \\ & & H^2(S_f^{[2]},\mathbf{Z}), \end{array} \tag{2.1.1}$$

both of which induce isomorphisms of Hodge structures

$$\begin{array}{ccc} & & H^2_{\lim,\eta_f} \\ & \nearrow & \\ (K' \otimes \mathbf{C})(-1) \oplus H^2(S_f, \mathbf{C})^0 & & \\ & \searrow & \\ & & H^2(S_f^{[2]}, \mathbf{C}). \end{array}$$

Hence, to prove the theorem, it suffices to show that there exists a lattice isomorphism $\mathbf{M}_0 \to H^2(S_f^{[2]}, \mathbf{Z})$ which makes diagram 2.1.1 into a commutative diagram. To do this, we invoke the following result by Nikulin (for proof and terminology, see [37]):

Proposition 2.1.6. *Let Λ be a lattice, $\Lambda^* := \mathrm{Hom}(\Lambda, \mathbf{Z})$ its dual, and let $D_\Lambda := \Lambda^*/\Lambda$ be the discriminant group. Let q_Λ be the $\mathbf{Q}/2\mathbf{Z}$-valued quadratic form on D_Λ induced by the $\mathbf{Q}$-valued quadratic form on Λ^*.*

Let Λ' be a lattice of equal rank as Λ and $j' : \Lambda \hookrightarrow \Lambda'$ an inclusion of lattices. Then the subgroup $\Lambda'/\Lambda \subseteq D_\Lambda$ is isotropic with respect to q_Λ. Furthermore, if Λ'' is another lattice of equal rank as Λ and $j'' : \Lambda \hookrightarrow \Lambda''$ another inclusion of lattices, then there exists an isomorphism $\varphi : \Lambda' \to \Lambda''$ such that $j'' = \varphi \circ j'$ if and only if the subgroups $\Lambda'/\Lambda, \Lambda''/\Lambda \subseteq D_\Lambda$ are conjugate by a q_Λ-preserving group automorphism of D_Λ.

In our situation, we can take $\Lambda = K' \oplus H^2(S_f, \mathbf{Z})^0$ where $K' = \mathbf{Z}\gamma \oplus \mathbf{Z}\delta = \langle 2 \rangle \oplus \langle -2 \rangle$. Before we determine Λ^* we look more closely at $H^2(S_f, \mathbf{Z})^0$. Remember that

$$H^2(S_f, \mathbf{Z}) \cong \widetilde{E8}^{\oplus 2} \oplus U^{\oplus 3}, \tag{2.1.2}$$

where $\widetilde{E8}$ is the $E8$ lattice with sign reversed and U is the rank 2 hyperbolic lattice, which has isotropic generators e_1, e_2 such that $\langle e_1, e_2 \rangle = 1$. The lattice $H^2(S_f, \mathbf{Z})$ is unimodular, hence by Eichler's criterion (see [12]) two elements lie in the same orbit of the isometry group if and only if they have the same self-intersection. Therefore we can choose the identification (2.1.2) in such a way that the polarization class γ becomes identified with $e_1 + e_2$, where e_1, e_2 are isotropic generators of one of the hyperbolic lattices (note that $e_1 + e_2$ squares to 2, as does γ). It then easily follows that

$$H^2(S_f, \mathbf{Z})^0 \cong \widetilde{E8}^{\oplus 2} \oplus U^{\oplus 2} \oplus \mathbf{Z}\varepsilon,$$

where $\varepsilon = e_1 - e_2$ squares to -2.
We find that

$$\Lambda \cong \mathbf{Z}\gamma \oplus \mathbf{Z}\delta \oplus \mathbf{Z}\varepsilon \oplus M,$$

where $M = \widetilde{E8}^{\oplus 2} \oplus U^{\oplus 2}$ is unimodular. Hence

$$\Lambda^* \cong \mathbf{Z}\gamma^* \oplus \mathbf{Z}\delta^* \oplus \mathbf{Z}\varepsilon^* \oplus M,$$

where $\gamma^* \in \mathrm{Hom}(S, \mathbf{Z})$ is the characterized by $\gamma^*(\gamma) = 1, \gamma^*(\gamma^\perp) = 0$ and δ^* and ε^* are characterized similarly.

Under the natural inclusion $\Lambda \hookrightarrow \Lambda^*$ we have identifications $\gamma = 2\gamma^*$, $\delta = -2\delta^*$ and $\varepsilon = -2\varepsilon^*$. Hence

$$D_\Lambda \cong (\mathbf{Z}/2\mathbf{Z})^{\oplus 3}$$

with generators $\overline{\gamma}^*, \overline{\delta}^*$ and $\overline{\varepsilon}^*$, where $\overline{\gamma}^* \equiv \gamma^* \pmod{\gamma}$, etcetera. The generators are orthogonal with respect to q_Λ, and we have

$$\begin{aligned} q_\Lambda(\overline{\gamma}^*) &\equiv \frac{1}{2} \pmod 2 \\ q_\Lambda(\overline{\delta}^*) &\equiv -\frac{1}{2} \pmod 2 \\ q_\Lambda(\overline{\varepsilon}^*) &\equiv -\frac{1}{2} \pmod 2. \end{aligned}$$

It now easily follows that there are only two non-trivial isotropic subgroups of D_Λ, namely $\langle \overline{\gamma}^* + \overline{\delta}^* \rangle$ and $\langle \overline{\gamma}^* + \overline{\varepsilon}^* \rangle$. However, these subgroups are interchanged by the automorphism of D_Λ that fixes $\overline{\gamma}^*$ and interchanges $\overline{\delta}^*$ and $\overline{\varepsilon}^*$. Note that this automorphism preserves q_Λ. So Nikulin's result now implies that if Λ', Λ'' are two lattices of the same rank as Λ and $j' : \Lambda \hookrightarrow \Lambda'$, $j'' : \Lambda \hookrightarrow \Lambda''$ are two *strict* embeddings, then there must exist an isomorphism $\varphi : \Lambda' \to \Lambda''$ such that $j'' = \varphi \circ j'$.

So to prove the lemma we only need to show that the embeddings in diagram 2.1.1 are strict, that is, that they are not isomorphisms. However, from the calculations above it follows that the discriminant of $\Lambda = K' \oplus H^2(S_f, \mathbf{Z})^0$ is 8, whereas the discriminants of $\mathbf{M}_0$ and $H^2(S_f^{[2]}, \mathbf{Z})$ are both equal to -2. Therefore the embeddings cannot be isomorphisms and the lemma is proven. □

We can now state the main result of this subsection:

Theorem 2.1.7. *The locus $\overline{\mathfrak{N}}_{\mathrm{BD},2}$ in $\overline{\mathfrak{N}}_{\mathrm{BD}}$ is precisely the image of the map*

$$\begin{aligned} q_2 : \overline{\mathfrak{N}}_{K3,2} &\to \overline{\mathfrak{N}}_{\mathrm{BD}} \\ [S, \mathcal{L}] &\mapsto \left[S^{[2]}, \mathcal{L}^{(2,-1)}\right] \end{aligned}$$

where $\overline{\mathfrak{N}}_{K3,2}$ denotes the Hausdorff reduction of the moduli space of K3 surfaces with line bundle of square 2 of which the class is contained in the positive cone, $\mathcal{L}^{(2,-1)}$ denotes[2] *the line bundle on $S^{[2]}$ with first Chern class $2c_1(\mathcal{L}) - \delta$ and the brackets $[Y, \mathcal{L}]$ denote the image of the isomorphism class of the holomorphic symplectic manifold Y with line bundle $\mathcal{L}$ in the Hausdorff reduction of the moduli space.*

[2]see also 1.1.24

Proof. The map q_2 fits into the following commutative diagram:

$$\begin{array}{ccc} \overline{\mathfrak{N}}_{K3,2} & \xrightarrow{q_2} & \overline{\mathfrak{N}}_{\mathrm{BD}} \\ \Big\downarrow{\scriptstyle \overline{\mathcal{Q}}_{K3,2}} & & \Big\downarrow{\scriptstyle \overline{\mathcal{Q}}_{\mathrm{BD}}} \\ \mathcal{D}_{K3,2} & \xrightarrow[d]{} & \mathcal{D}_{\mathrm{BD}} \end{array}$$

where $\mathcal{D}_{K3,2}$ is the period domain of K3 surfaces of degree 2 and $\mathcal{Q}_{K3,2}$ is the period map. The map d is defined as follows. First of all there we can identify

$$\Lambda_{K3^{[2]}} = \Lambda_{K3} \oplus \delta\mathbf{Z},$$

with Λ_{K3} isomorphic to the second cohomology lattice of a K3 surface and with $\delta.\delta = -2$. Let $h \in \Lambda_{K3}$ be a class of square 2 (all such classes lie in one orbit of $O(\Lambda_{K3})$). The identification then gives an embedding of lattices

$$h^\perp \hookrightarrow (2h-\delta)^\perp,$$

and hence an embedding of period domains

$$\Omega_{h^\perp} \hookrightarrow \Omega_{(2h-\delta)^\perp} \cong \Omega_{\mathrm{BD}}$$

where the last identification comes from the fact that $2h-\delta$ is a lattice element of square 6 and of even type. Note that the image of this embedding is closed in Ω_{BD}. It is not hard to see that this map induces a map on the quotients:

$$d : \mathcal{D}_{K3,2} := O(\Lambda_{K3}, h)\backslash\Omega_{h^\perp} \to O(\Lambda_{K3^{[2]}}, 2h-\delta)\backslash\Omega_{\mathrm{BD}} =: \mathcal{D}_{\mathrm{BD}}.$$

This is precisely the map d that makes the diagram above commute. It is a morphism of quasi-projective varieties and the image is closed in $\mathcal{D}_{\mathrm{BD}}$ and of dimension 19.

From the theory of $K3$ surfaces it follows that $\overline{\mathcal{Q}}_{K3,2}$ is an isomorphism. Moreover, $\overline{\mathcal{Q}}_{\mathrm{BD}}$ is an isomorphism by proposition 1.1.27 and the definition of $\mathfrak{N}_{\mathrm{BD}}$. It follows in particular that q_2 has closed image in $\overline{\mathfrak{N}}_{\mathrm{BD}}$.

Furthermore, by definition of $\overline{\mathfrak{N}}_{\mathrm{BD},2}$ it follows that $\overline{\mathfrak{N}}_{\mathrm{BD},2} \cong \mathcal{D}_{\mathrm{cubic},2}$. The latter is shown to be irreducible and of dimension 19 by Hassett in [21]. Hence, to show that $\operatorname{im} q_2 = \overline{\mathfrak{N}}_{\mathrm{BD},2}$ it suffices to show that q_2 maps generic elements of $\mathfrak{N}_{K3,2}$ into $\overline{\mathfrak{N}}_{\mathrm{BD},2}$. This is what we will show.

Let $(S,\mathcal{L})$ be a generic K3 surface of degree 2 such that $|\mathcal{L}|$ is base point free and the morphism $\pi : S \to |\mathcal{L}|^\vee$ a 2:1 covering branched along a smooth sextic curve. Choose an element $\sigma \in \mathrm{Sym}^6 W^\vee$ and an identification $|\mathcal{L}|^\vee \cong \mathbf{P}(W)$ such that the zero locus of σ in $\mathbf{P}(W)$ corresponds to the branch curve.
The natural equivariant embedding $\mathrm{Sym}^6 W^\vee \hookrightarrow \mathrm{Sym}^3\mathrm{Sym}^2 W^\vee$ maps σ to a cubic form f_σ on $\mathrm{Sym}^2 W$ with the property that $v^* f_\sigma = \sigma$, where as before $v : W \to$

$\mathrm{Sym}^2 W$ is the affine Veronese map. Then proposition 2.1.3 implies that $t \mapsto D + t^2 f_\sigma$ defines a deformation of the determinantal cubic of which the limiting Hodge structure exists and defines a period in $\mathcal{D}_{\mathrm{cubic},2}$. By diagram (1.3.3) this period corresponds to a unique point in $\overline{\mathfrak{N}}_{\mathrm{BD},2} \subset \overline{\mathfrak{N}}_{\mathrm{BD}}$, and by lemma 2.1.5 this point must be $\left[S^{[2]}, \mathcal{L}^{(2,-1)}\right]$. □

Corollary 2.1.8. *A pair $(Y, \mathcal{L})$ defines a point in $\mathfrak{N}_{\mathrm{BD},2}$ if and only if there exist a K3 surface S with line bundle $\mathcal{L}_2$ of square 2, a bimeromorphic map $\psi : S^{[2]} \dashrightarrow Y$ and a Hodge isometry $\chi : H^2(S^{[2]}, \mathbf{Z}) \to H^2(S^{[2]}, \mathbf{Z})$ which preserves the positive cone and has the property that it maps $c_1\left(\mathcal{L}_2^{(2,-1)}\right)$ to $\psi^*(c_1(\mathcal{L}))$.*

Proof. This follows directly from the definition of $\mathfrak{N}_{\mathrm{BD},2}$, theorem 1.1.28 and theorem 2.1.7. □

2.1.2 The ample cone for generic varieties in $\mathfrak{N}_{\mathrm{BD},2}$

Before we show why the varieties in $\mathfrak{N}_{\mathrm{BD},2}$ cannot be realized as variety of lines on a cubic fourfold we will look more closely at the Hilbert scheme of length 2 subschemes of generic K3 surfaces S of degree 2. In this section S will be a K3 surface of degree 2 and with Picard rank 1. We will describe the ample cone of $S^{[2]}$. This has been done by Hassett and Tschinkel in [22], but their exposition is rather compact. We fill in some details here.
Remember that we have

$$\mathrm{Pic}(S^{[2]}) \cong \mathrm{Pic}(S) \oplus \mathbf{Z}\delta,$$

where δ is such that 2δ is the class of the divisor $E \subset S^{[2]}$ that parameterizes non-reduced length 2 subschemes of S (see theorem 1.1.23). By assumption we have $\mathrm{Pic}(S) = \mathbf{Z}\gamma'$, where γ' is the class of a genus 2 curve G on S. Let γ be the image of γ' in $\mathrm{Pic}(S^{[2]})$, then γ is the class of the divisor of length 2 subschemes on S whose support intersect G non-trivially.

A necessary condition for a line bundle on a smooth algebraic variety to be ample is that the pairing of its first Chern class with the homology class of any algebraic curve is positive. We can construct the following (homology classes of) curves on $S^{[2]}$:

- Let $p \in S$ be an arbitrary point. We denote by C_p the locus of length 2 subschemes of S with support p. It is a smooth rational curve in $S^{[2]}$, contained in E. Its cohomology class is clearly independent of the choice of p and we will denote it by c_δ.

- Let $p \in S$ be an arbitrary point and $G \subset S$ a smooth curve in the linear system of the polarization on S and assume that G does not pass through p. We denote by $C_{p,G} \subset S^{[2]}$ the locus of length 2 subschemes of S whose support contains p and intersects G. Then $C_{p,G}$ is isomorphic to G, hence generically a smooth genus 2 curve. Its cohomology class is independent of the choice of p and G; we will denote it by c_γ.

- Since S is assumed to have Picard rank 1, the linear system of its polarization defines a map $\pi : S \to \mathbf{P}^2$. Let $\ell \subset \mathbf{P}^2$ be a line. We denote by C_ℓ the locus length 2 subschemes over ℓ that are contained in a fiber of π (we call such subschemes **vertical** with respect to π). It is another smooth rational curve in $S^{[2]}$. Its cohomology class is independent of the choice of ℓ. We will denote it by c_v.

Lemma 2.1.9. *We have the following intersection table:*

$\int$	γ	δ
c_δ	*0*	*-1*
c_γ	*2*	*0*
c_v	*2*	*3*

where, by slight abuse of notation, we use $\int$ to denote the duality pairing of homology and cohomology classes.

Proof. For any curve $C \subset S$ denote by $D(C) \subset S^{[2]}$ the divisor of length 2 subschemes of S that have some support on C. Let $\mathcal{L}_{\gamma'}$ denote the line bundle on S with first Chern class γ' and let $|\mathcal{L}_{\gamma'}|$ denote its linear system. Note that for any curve $G \in |\mathcal{L}_{\gamma'}|$ the divisor $D(G)$ on $S^{[2]}$ has cohomology class γ. Equivalently, for any line $\ell \subset \mathbf{P}^2$ the divisor $D(\pi^{-1}(\ell))$ has cohomology class γ.

The easiest case is that of c_γ. Choose $G \subset |\mathcal{L}_{\gamma'}|$ and choose $p \in S$ outside G. Then any length 2 subscheme in $C_{p,G}$ is reduced, hence $C_{p,G}$ does not intersect E. Since the cohomology class of E is 2δ it follows that $\int_{c_\gamma} \delta = 0$.
Furthermore, let $G' \subset |\mathcal{L}_{\gamma'}|$ be another curve such that it meets G transversally and does not pass through p. Then $D(G') \cap C_{p,G}$ is the set of length 2 subschemes of S the support of which contains p and intersects $G \cap G'$. Since $G \cap G'$ consists of 2 points, it follows that

$$\int_{c_\gamma} \gamma = |D(G') \cap C_{p,G}| = 2.$$

Let $\ell \subset \mathbf{P}^2$ be arbitrary and choose $p \in S \setminus \pi^{-1}(\ell)$. In that case $D(\pi^{-1}(\ell)) \cap C_p = \emptyset$ and it follows that $\int_{C_\delta} \gamma = 0$.

Furthermore, the branch locus B of π is smooth and of degree 6, hence we can choose $\ell \subset \mathbf{P}^2$ such that it intersects B transversally in 6 points. Since a vertical length 2 subschemes in S is non-reduced if and only if it is supported over B, it follows that

$$|C_\ell \cap E| = |\ell \cap B| = 6.$$

Since C_ℓ has class c_v and E has cohomology class 2δ it follows that $\int_{c_v}\delta = 3$.
For the remaining two cases, remember that $S^{[2]}$ naturally fits into the diagram:

$$\begin{array}{ccc} & \widetilde{S\times S} & \\ {}^{q_\sigma}\swarrow & & \searrow^{\varepsilon} \\ S^{[2]} & & S\times S \end{array}$$

where ε is the blow-up of the diagonal in $S\times S$, σ denotes the involution on $\widetilde{S\times S}$ induced by the map on $S\times S$ that interchanges the factors and where q_σ is taking the quotient by σ. Let $\widehat{E}$ be the exceptional divisor of ε. Then $\varepsilon|_{\widehat{E}}$ has the structure of a $\mathbf{P}^1$-bundle over the diagonal of $S\times S$. Let $p\in S$ be arbitrary and let $F_p\subset\widehat{E}$ be the fiber over $(p,p)\in S\times S$. Then F_p is a fiber of $\varepsilon|_{\widehat{E}}$ and we have that $[F_p]\cap[\widehat{E}] = -1[x]$, where $\cdot\cap\cdot$ denotes the intersection product on $H_*(S^{[2]},\mathbf{Z})$ and $[x]$ denotes the class of a point on $\widetilde{S\times S}$. This follows from standard theory on blowing up subvarieties of smooth varieties, see for example [16].
Now observe that $C_p = q_\sigma(F_p)$. It follows that

$$q_\sigma^* c_\delta = q_\sigma^*[C_p] = 2[F_p],$$

where the factor 2 comes form the fact that F_p is in the ramification locus of q_σ. Analogously, since $\widehat{E} = q_\sigma^{-1}(E)$ and $\widehat{E}$ precisely is the fixed locus of σ, we have

$$q_\sigma^*[E] = 2[\widehat{E}].$$

Furthermore, pull-back preserves the intersection product on homology (see [16]), so

$$q_\sigma^*([E]\cap c_p) = 4[\widehat{E}]\cap[F_p] = -4[x] = -2q_\sigma^*[y],$$

where $[y]$ is the class of a point on $S^{[2]}$. Hence $[E]\cap c_p = -2[y]$ and since $[E]$ is Poincaré dual to -2δ it follows that $\int_{c_\delta}\delta = -1$.

Finally, denote for any subvariety $Y\subseteq S\times S$ by $\widehat{Y}$ the strict transform of Y in $\widetilde{S\times S}$. Let $\ell\subset\mathbf{P}^2$ be an arbitrary line. Then note that $D(\pi^{-1}(\ell)) = q_\sigma(\pi^{-1}(\ell)\times S)^\wedge$ (where the hat denotes strict transform). Furthermore, let $\ell'\subset\mathbf{P}^2$ be a second line that intersects ℓ transversally outside B (which is the generic situation). Define

$$V_{\ell'} := \{(s_1,s_2)\in S\times S\,|\,\pi(s_1) = \pi(s_2)\in\ell'\}.$$

Then $C_{\ell'} = q_\sigma\widehat{V}_{\ell'}$. We have:

$$\begin{array}{rcl} q_\sigma^* c_v & = & q_\sigma^*[C_{\ell'}] = [\widehat{V}_{\ell'}] \\ q_\sigma^*[D(\pi^{-1}(\ell))] & = & [(\widehat{\pi^{-1}(\ell)\times S})] + [(\widehat{S\times\pi^{-1}(\ell)}] \end{array}$$

Note that by the properties of a blow-up map we have for all subvarieties $X,Y\subset S\times S$ that $[\widehat{X}]\cap[\widehat{Y}] = \varepsilon^*([X]\cap[Y])$. So we find:

$$q_\sigma^*\big(c_v\cap[D(\pi^{-1}(\ell))]\big) = \varepsilon^*\big([V_{\ell'}]\cap[\widehat{\pi^{-1}(\ell)}\times S] + [V_{\ell'}]\cap[S\times\widehat{\pi^{-1}(\ell)}]\big).$$

Let $x \in \mathbf{P}^2$ be the point of intersection of ℓ and ℓ'. Then

$$V_{\ell'} \cap (\widehat{\pi^{-1}(\ell)} \times S) = \{(s_1, s_2) \in S \times S \,|\, \pi(s_1) = \pi(s_2) = x\}.$$

This intersection has cardinality 2, and so has the intersection $V_{\ell'} \cap (S \times \widehat{\pi^{-1}(\ell)})$ by symmetry. Hence

$$q_\sigma^*\big(c_v \cap [D(\pi^{-1}(\ell))]\big) = 4\varepsilon^*[z] = 4[y] = 2q_\sigma^*[x],$$

where $[z]$ is the class of a point on $S \times S$ and $[y]$ and $[x]$ are as before. We used here that ε generically of degree 1. Since the cohomology class of $D(\pi^{-1}(\ell))$ is γ, it follows that $\int_{c_v} \gamma = 2$. □

Proposition 2.1.10. *The ample cone of $S^{[2]}$ is contained in the convex cone in $H^{1,1}(S^{[2]})$ spanned by γ and $3\gamma - 2\delta$.*

Proof. Let $\mathcal{L}$ be an ample line bundle. Since by assumption $\mathrm{Pic}(S^{[2]}) = \langle \gamma, \delta \rangle$, it follows that there exist $a, b \in \mathbf{Z}$ such that $c_1(\mathcal{L}) = a\gamma + b\delta$. We can rewrite this as

$$c_1(\mathcal{L}) = \frac{1}{2}(2a + 3b)\gamma + \frac{1}{2}(-b)(3\gamma - 2\delta). \tag{2.1.3}$$

Since $\mathcal{L}$ is assumed to be ample we must have $\int_C c_1(\mathcal{L}) > 0$ for any curve $C \subset S^{[2]}$. In particular, using lemma 2.1.9 we find:

$$\begin{aligned} -b &= \int_{c_\delta} c_1(\mathcal{L}) > 0 \\ 2a + 3b &= \int_{c_v} c_1(\mathcal{L}) > 0 \end{aligned}$$

Hence the coefficients in the left hand side of equation 2.1.3 are positive, so $c_1(\mathcal{L})$ is contained in the convex cone spanned by γ and $3\gamma - 2\delta$. This proves the proposition. □

We claim that the ample cone is in fact *the interior* of the convex cone spanned by γ and $3\gamma - 2\delta$. To see this, we construct morphisms associated to γ and $3\gamma - 2\delta$.

More precisely, let S be as before, denote by $\mathcal{L}_{\gamma'}$ the line bundle corresponding to the polarization class γ'. Set $W := H^0(S, \mathcal{L}_{\gamma'})^\vee$, then the genericity assumptions on S imply that the linear system $|\mathcal{L}_{\gamma'}|$ is base-point free, $\pi : S \to |\mathcal{L}_{\gamma'}| \cong \mathbf{P}(W)$ is well defined and 2:1 (see [42]). Let $S^{(2)} : (S \times S)/\sigma$ where σ is the involution that interchanges the factors of $S \times S$. Then π naturally induces a map

$$\begin{aligned} \pi^{(2)} : S^{(2)} &\to \mathbf{P}(\mathrm{Sym}^2 W) \\ \{p, q\} &\mapsto [\pi(p) \cdot \pi(q)] \end{aligned}$$

Remember that there is a natural resolution map $\eta : S^{[2]} \to S^{(2)}$ corresponding to the blow-up of the singular locus in $S^{(2)}$. Let $\varphi_1 := \pi^{(2)} \circ \eta$.

Lemma 2.1.11. $c_1(\varphi_1^*\mathcal{O}_{\mathbf{P}(\mathrm{Sym}^2 W)}(1)) = \gamma$.

Proof. First note that for any $p \in S$ the curve C_p (as defined before) is contracted by η and hence by φ_1. Therefore

$$\int_{C_\delta} c_1(\varphi_1^*\mathcal{O}_{\mathbf{P}(\mathrm{Sym}^2 W)}(1)) = \int_{C_p} c_1(\varphi_1^*\mathcal{O}_{\mathbf{P}(\mathrm{Sym}^2 W)}(1)) = 0.$$

Furthermore, let $\ell \subset \mathbf{P}(W)$ be a line and $C_\ell \subset S^{[2]}$ the locus of vertical length 2 subschemes in S over ℓ. Then

$$\varphi_1(C_\ell) = \left\{[v^2] \in \mathbf{P}(\mathrm{Sym}^2 W) \mid [v] \in \ell\right\}.$$

Since $\varphi_1|_{C_\ell}$ is 1:1 onto its image it follows that

$$\int_{\varphi_1(C_\ell)} c_1\left(\mathcal{O}_{\mathbf{P}^2(\mathrm{Sym}^2 W)}(1)\right) = 2.$$

Now from our genericity assumptions on S it follows that there are $a, b \in \mathbf{Z}$ such that

$$c_1(\varphi_1^*\mathcal{O}_{\mathbf{P}(\mathrm{Sym}^2 W)}(1)) = a\gamma + b\delta.$$

The calculations above and lemma 2.1.9 then imply that $a = 1$ and $b = 0$. This proves the lemma. □

We construct a second map. First note that the assumption that S is of Picard rank 1 implies that $\mathcal{L}_{\gamma'}$ is ample. By [42, theorem 8.3] then $\mathcal{L}_{\gamma'}^{\otimes 3}$ is very ample. Hence if $V := H^0(S, \mathcal{L}_{\gamma'}^{\otimes 3})^\vee$ then the natural map $\psi : S \to \mathbf{P}(V)$ is an embedding.

Let I_2 be the space of quadratic forms on V that vanish on $\psi(S)$. Let $s \subset S$ be a length 2 subscheme, then $\psi(s)$ spans a line $\ell_s \subset \mathbf{P}(V)$. Let $Q_s \subset I_2$ be the space of quadratic forms that vanish on ℓ_s, then Q_s is a hyperplane unless ℓ_s is contained in S. The latter cannot happen by our genericity assumption: a K3 surface of degree 2 with Picard rank 1 cannot contain smooth rational curves, so $\psi(S)$ will not contain any lines. Hence we obtain a well defined map

$$\begin{aligned} \varphi_2 : S^{[2]} &\to \mathbf{P}(I_2^\vee) \\ s &\mapsto Q_s. \end{aligned}$$

Lemma 2.1.12. $c_1(\varphi_2^*\mathcal{O}_{\mathbf{P}(I_2^\vee)}(1)) = 3\gamma - 2\delta$

Proof. Let $q \in I_2$ be arbitrary. It defines a hyperplane $H_q \subset \mathbf{P}(I_2^\vee)$. If $C \subset S^{[2]}$ is a curve and H_q and $\psi(C)$ intersect transversally, it follows that

$$\int_C c_1(\varphi_2^*\mathcal{O}_{\mathbf{P}(I_2^\vee)}(1)) = |H_q \cap \psi(C)| = |\{s \in C \mid q|_{\ell_s} = 0\}|$$

Let $p \in S$ be arbitrary. Then for every length 2 subscheme $s \subset S$ we have that $s \in C_p$ if and only if the line ℓ_s is tangent to $\psi(S)$ at $\psi(p)$. Let $T_p \subset \mathbf{P}(V)$ be the

2-plane tangent to $\psi(S)$ at $\psi(p)$. Then for generic $q \in I_2$ we have that $q|_{T_p}$ is non-vanishing, and in that case the zero locus is a degenerate conic with vertex at $\psi(p)$. Hence the number (with multiplicity) of lines tangent to $\psi(S)$ at $\psi(p)$ on which q vanishes is 2. Therefore

$$\int_{C_p} c_1(\varphi_2^* \mathcal{O}_{\mathbf{P}(I_2^\vee)}(1)) = 2.$$

Furthermore, note that there is a natural map $\mathrm{Sym}^3 H^0(S, \mathcal{L}_{\gamma'}) \to H^0(S, \mathcal{L}_{\gamma'}^{\otimes 3})$ which induces a rational projection $p : \mathbf{P}(V) \to \mathbf{P}(\mathrm{Sym}^3 W)$. This fits into the following commutative diagram:

$$\begin{array}{ccc} S & \overset{\psi}{\hookrightarrow} & \mathbf{P}(V) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbf{P}(W) & \underset{\nu_3}{\hookrightarrow} & \mathbf{P}(\mathrm{Sym}^3 W) \end{array}$$

where ν_3 denotes the cubic Veronese map. Since p is linear, its image is the linear span of the image of $\nu_3 \circ \pi$. Since π is surjective onto $\mathbf{P}(W)$ this image of $\nu_3 \circ \pi$ is precisely the cubic Veronese surface in $\mathbf{P}(\mathrm{Sym}^3 W)$ and it is well known that its linear span is all of $\mathbf{P}(\mathrm{Sym}^3 W)$. Hence p is surjective. Furthermore, a Riemann-Roch calculation shows that $\mathbf{P}(V)$ is of dimension 10, whereas $\mathbf{P}(\mathrm{Sym}^3 W)$ is of dimension 9. Hence p is defined everywhere outside a certain point $e \in \mathbf{P}(V)$. Note that $e \notin \psi(S)$.

Let $s \subset S$ be a length 2 subscheme vertical with respect to π and let $w \in W$ be such that $\pi(\sigma) = [w]$. Then p maps $\psi(s)$ to the point $[w^3]$, and it follows that the line ℓ_s spanned by $\psi(s)$ is equal to the closure of $p^{-1}([w^3])$. In particular $e \in \ell_s$.

We noted before that our genericity assumption on S implies that $\psi(S)$ does not contain any lines. In particular $\ell_s \not\subset \psi(S)$. It follows that there must exist a $q \in I_2$ which is not identically zero on ℓ_s. Since $e \in \ell_s$ but $e \notin S$, q is non-zero at e. But our argument shows that e is contained in the linear span of $\psi(s')$ for every length 2 subscheme $s' \subset S$ which is vertical with respect to π. It follows that q does not vanish identically on any line spanned by the image under ψ of a vertical length 2 subscheme in S. In particular, if $\ell' \subset \mathbf{P}(W)$ is a line and $C_{\ell'} \subset S^{[2]}$ as before the locus of vertical length 2 subschemes of S over ℓ' then

$$\{s \in C_{\ell'} \mid q|_{\ell_s} = 0\} = \emptyset,$$

and it follows that

$$\int_{C_{\ell'}} c_1(\varphi_2^* \mathcal{O}_{\mathbf{P}(I_2^\vee)}(1)) = 0.$$

Now again we may write $c_1(\varphi_2^* \mathcal{O}_{\mathbf{P}(I_2^\vee)}(1)) = a\gamma + b\delta$ for $a, b \in \mathbf{Z}$ and it follows from lemma 2.1.9 that $a = 3, b = -2$. This completes the proof. □

Having performed these calculations, we may now prove the main result of this section.

Theorem 2.1.13. *Let S be a K3 surface of degree 2 with Picard rank 1. Let $S^{[2]}$ be the Hilbert scheme of length 2 subschemes of S. Let $\gamma \in \operatorname{Pic} S^{[2]}$ be the class induced from the polarization of S and $\delta \in \operatorname{Pic} S^{[2]}$ be such that 2δ is the class corresponding to the locus of non-reduced length 2 subschemes in S. Then $\operatorname{Pic} S^{[2]} = \langle \gamma, \delta \rangle$ and the ample cone of $S^{[2]}$ is the interior of the convex cone spanned by γ and $3\gamma - 2\delta$.*

Proof. We have already seen that $\operatorname{Pic} S^{[2]}$ is generated by γ and δ under the assumptions on S. Furthermore, proposition 2.1.10 tells us that the ample cone is contained in the positive span of γ and $3\gamma - 2\delta$. Therefore it suffices to prove that the line bundles $\mathcal{L}_\gamma$ and $\mathcal{L}_{3\gamma-2\delta}$ that correspond to γ and $3\gamma - 2\delta$ respectively are nef. But it follows from lemmas 2.1.11 and 2.1.12 that both are globally generated, hence certainly nef. This concludes the proof. □

2.1.3 Obstructions to very-ampleness

The aim of this section is to explain why a line bundle of square 6 on a holomorphic symplectic manifold that is bimeromorphic to the Hilbert scheme of length 2 subschemes of a K3 surface of degree 2 can never be very ample. This will then explain, in a sense, why the pairs $(Y, \mathcal{L})$ that are parameterized by $\mathfrak{N}_{\mathrm{BD},2}$ cannot be realized as variety of lines on a cubic fourfold.

First consider again the generic situation: let S be a K3-surface of degree 2 and Picard rank 1 and let $\gamma, \delta \in \operatorname{Pic} S^{[2]}$ be the generators that we introduced earlier. Let $\mathcal{L}$ be a line bundle on $S^{[2]}$ of square 6, then there are $a, b \in \mathbf{Z}$ such that $c_1(\mathcal{L}) = a\gamma + b\delta$ and

$$q(a\gamma + b\delta) = a^2 q(\gamma) + b^2 q(\delta) = 2a^2 - 2b^2 = 6,$$

where q is the Beauville-Bogomolov form on $H^2(S^{[2]}, \mathbf{Z})$. It follows that there are four possibilities for (a, b), namely $a = \pm 2$ and $b = \pm 1$. However we have the following result:

Proposition 2.1.14. *Let S be a K3 surface of degree 2 with Picard rank 1. Let Y be a holomorphic symplectic manifold bimeromorphic to $S^{[2]}$ and $\mathcal{L}$ a line bundle of square 6 on Y. Then $\mathcal{L}$ is ample on Y if and only if there exists an isomorphism $\varphi : S^{[2]} \to Y$ such that $\varphi^* c_1(\mathcal{L}) = 2\gamma - \delta$.*

Proof. Let Y and $\mathcal{L}$ be as asserted and assume that there exists an isomorphism $\varphi : S^{[2]} \to Y$ such that $\varphi^* c_1(\mathcal{L}) = 2\gamma - \delta$. By theorem 2.1.13 the class $2\gamma - \delta$ is contained in the ample cone. It follows that $\varphi^* \mathcal{L}$ is ample on $S^{[2]}$, hence $\mathcal{L}$ is ample on Y since φ is an isomorphism.

For the converse, we first introduce some extra notation. Let $\psi : S^{[2]} \dashrightarrow Y$ be a bimeromorphism. By remark 1.1.7 the indeterminacy locus of ψ is of codimension at least 2, hence $\psi^* : H^2(Y, \mathbf{Z}) \to H^2(S^{[2]}, \mathbf{Z})$ is a Hodge isometry. It follows that

$$\psi^* c_1(\mathcal{L}) = \pm 2\gamma \pm \delta.$$

Now assume that $\mathcal{L}$ is ample on Y. Then in particular $c_1(\mathcal{L})$ is contained in the positive cone C_Y^+ of $\mathbf{R}\otimes \operatorname{Pic} Y$. Since ψ^* preserves positive cones (see for example [25]), it sends C_Y^+ to

$$C^+_{S^{[2]}} = \{a\gamma + b\delta \mid a^2 - b^2 > 0,\ a, b \in \mathbf{R}\},$$

hence we must have that

$$\psi^* c_1(\mathcal{L}) = 2\gamma \pm \delta.$$

We deal with the two cases separately. First assume that $\psi^* c_1(\mathcal{L}) = 2\gamma - \delta$. Then

$$\psi^* : H^2(Y, \mathbf{Z}) \to H^2(S^{[2]}, \mathbf{Z})$$

is a Hodge isometry that maps an ample class on Y (namely $c_1(\mathcal{L})$) to an ample class on $S^{[2]}$ (namely $2\gamma - \delta$). In particular ψ^* maps a Kähler class to a Kähler class. It follows from theorem 1.1.26 that Y and $S^{[2]}$ are isomorphic. Let $\varphi : S^{[2]} \to Y$ be any isomorphism, then $\varphi^* c_1(\mathcal{L})$ is an ample class of square 6, hence equal to $2\gamma - \delta$.

Now assume that $\psi^* c_1(\mathcal{L}) = 2\gamma + \delta$. Remember that there is a natural lattice decomposition

$$H^2(S^{[2]}, \mathbf{Z}) = H^2(S, \mathbf{Z}) \oplus \mathbf{Z}\delta.$$

Let $R_\delta : H^2(S^{[2]}, \mathbf{Z}) \to H^2(S^{[2]}, \mathbf{Z})$ be the map that sends δ to $-\delta$ and is the identity on $H^2(S, \mathbf{Z})$. This is a lattice involution and, since δ is a (1,1)-class, a Hodge isometry. Moreover it preserves the positive cone. Hence the composition $R_\delta \circ \psi^*$ is a Hodge isometry which preserves the positive cone. It maps $c_1(\mathcal{L})$, which is a Kähler class by assumption, to the ample, hence Kähler, class $2\gamma - \delta$. So again by theorem 1.1.26 Y and $S^{[2]}$ are isomorphic, and for any isomorphism $\varphi : S^{[2]} \to Y$ we must have $\varphi^* c_1(\mathcal{L}) = 2\gamma - \delta$. □

We have already seen, see corollary 2.1.8, that the limiting Hodge structures of deformations of the determinantal cubic can generically be realized by a pair $(Y, \mathcal{L})$ of holomorphic symplectic fourfold with line bundle if and only if this pair is non-separated from $\left(S^{[2]}, \mathcal{L}_2^{(2,-1)}\right)$in $\mathfrak{N}_{\mathrm{BD}}$. The proposition above makes $\left(S^{[2]}, \mathcal{L}_2^{(2,-1)}\right)$ into a generically preferred pair in the set of non-separated pairs; it is generically the only pair on which the limiting class of the polarization can be ample.

In particular, to understand why this limiting class cannot be *very* ample on any holomorphic symplectic fourfold which realizes the limiting Hodge structure, it suffices to understand why the class $2\gamma - \delta$ fails to be very ample on $S^{[2]}$.
Let $\mathcal{L}$ be the line bundle on $S^{[2]}$ with first Chern class $2\gamma - \delta$, then we may construct the morphism $\varphi_{\mathcal{L}} : S^{[2]} \dashrightarrow |\mathcal{L}|^\vee$ explicitly. Indeed, as before let γ' be the ample generator of $\operatorname{Pic} S$, let $\mathcal{L}_{\gamma'}$ be the corresponding line bundle and set

$W := H^0(S, \mathcal{L}_{g'})^\vee$. Let $\pi : S \to \mathbf{P}(W)$ be the 2:1 surjective morphism, and let

$$\begin{array}{rcl} \pi^{[2]} : S^{[2]} & \dashrightarrow & \mathbf{P}(W)^{[2]} \\ s \subset S & \mapsto & \pi(s). \end{array}$$

Note that it is well-defined outside the locus of length 2 subschemes in S that are vertical with respect to π. Furthermore, let $v_2 : \mathbf{P}(W) \to \mathbf{P}(\mathrm{Sym}^2 W)$ be the Veronese map. It is an embedding, hence defines an embedding

$$v_2^{[2]} : \mathbf{P}(W)^{[2]} \to \mathbf{P}(\mathrm{Sym}^2 W)^{[2]}.$$

Finally, any length two subscheme in $\mathbf{P}(\mathrm{Sym}^2 W)$ spans a line, which in turn defines a point of $\mathbf{P}(\bigwedge^2 \mathrm{Sym}^2 W)$. This gives rise to a map

$$\lambda : \mathbf{P}(\mathrm{Sym}^2 W)^{[2]} \to \mathbf{P}(\bigwedge^2 \mathrm{Sym}^2 W).$$

Let $\psi := \lambda \circ v_2^{[2]} \circ \pi^{[2]}$. Then we have the following result.

Lemma 2.1.15. *Assume that S has Picard rank 1. Then there exists an isomorphism*

$$\chi : \mathbf{P}(\bigwedge^2 \mathrm{Sym}^2 W) \to |\mathcal{L}|^\vee$$

such that $\varphi_{\mathcal{L}} = \chi \circ \psi$.

Proof. From calculations in [13] (in the proof of lemma 5.1), it follows that

$$H^0(S^{[2]}, \mathcal{L}) \cong \bigwedge^2 H^0(S, \mathcal{L}_{\gamma'}^{\otimes 2}),$$

where $\mathcal{L}_{\gamma'}$ is the polarizing line bundle on S. By [42, proposition 5.6] we have

$$H^0(S, \mathcal{L}_{\gamma'}^{\otimes 2}) \cong \mathrm{Sym}^2 H^0(S, \mathcal{L}_{\gamma'}).$$

It follows that

$$|\mathcal{L}|^\vee = \mathbf{P}(H^0(S^{[2]}, \mathcal{L})^\vee) \cong \mathbf{P}(\bigwedge^2 \mathrm{Sym}^2 H^0(S, \mathcal{L}_{\gamma'})^\vee) = \mathbf{P}(\bigwedge^2 \mathrm{Sym}^2 W).$$

Hence to prove the lemma it suffices to show that

$$\psi^* H = 2\gamma - \delta,$$

where H is the hyperplane class on $\mathbf{P}(\bigwedge^2 \mathrm{Sym}^2 W)$.

Since we have assumed S to be of Picard rank 1 we can write $\psi^* H = a\gamma + b\delta$ for some $a, b \in \mathbf{Z}$. We use contraction with curves and lemma 2.1.9 to find the values of a and b.

Let $p \in S$ be outside the ramification locus of π. Let $P \subset W$ be a 2-plane such that $\mathbf{P}(P)$ does not contain $\pi(p)$. Then tracing the construction of ψ gives that it embeds C_p into $\mathbf{P}(\bigwedge^2 \mathrm{Sym}^2 W)$ and the image is

$$\{[\pi(p)^2 \wedge \pi(p) \cdot w] \mid w \in P\},$$

which is of degree 1. It follows that

$$\int_{C_p} \psi^* H = -b = 1.$$

Furthermore, let $\ell \subset \mathbf{P}(W)$ be a line and let $p \in S$ be outside $\pi^{-1}(\ell)$. Then ψ maps $C_{p,\pi^{-1}\ell}$ 2:1 onto the degree 2 curve

$$\left\{[\pi(p)^2 \wedge w^2] \mid [w] \in \ell\right\} \subset \mathbf{P}(\bigwedge^2 \mathrm{Sym}^2 W).$$

It follows that

$$\int_{C_{p,\pi^{-1}(\ell)}} \psi^* H = 2a = 4.$$

Hence $\psi^* H = 2\gamma - \delta$. This proves the lemma. □

Proposition 2.1.16. *The rational morphism $\varphi_{\mathcal{L}} : S^{[2]} \dashrightarrow |\mathcal{L}|^\vee$ has the following properties*

- *Its locus of indeterminacy is the locus $V \subset S^{[2]}$ of length 2 subschemes of $S^{[2]}$ that are vertical with respect to $\pi : S \to \mathbf{P}(W)$;*
- *It contracts curves $C_p \subset S^{[2]}$ of length 2 subschemes supported on $p \in S$ if p is in the ramification locus of π;*
- *It is generically 4:1;*
- *There exists an isomorphism $\chi : \mathbf{P}(\bigwedge^2 \mathrm{Sym}^2 W) \to |\mathcal{L}|^\vee$ such that closure of the image $\chi^{-1} \circ \varphi_{\mathcal{L}}$ is the Plücker embedding of the locus of lines that are secant to the Veronese surface in $\mathbf{P}(\mathrm{Sym}^2 W)$.*

In particular, the line bundle $\mathcal{L}$ is not very ample.

Proof. It suffices to prove the lemma for generic choice of S, and we will choose S to be of Picard rank 1. By lemma 2.1.15 it suffices to prove that $\psi = \lambda \circ v_2^{[2]} \circ \pi^{[2]}$ has the properties listed. Note that both $v_2^{[2]}$ and λ are everywhere defined, and that the composition is injective. Indeed, assume that $t_1, t_2 \subset \mathbf{P}(W)$ are two different length 2 subschemes such that their images under $\lambda \circ v_2^{[2]}$ coincide. That would mean that $v_2(t_1)$ and $v_2(t_2)$ span the same line in $\mathbf{P}(\mathrm{Sym}^2 W)$. In particular there would be a line $\ell \subset \mathbf{P}(\mathrm{Sym}^2 W)$ that intersects the image of v_2 with multiplicity at least 3. However, the image of v_2 is the Veronese surface, whose vanishing ideal is generated in degree 2. It follows that ℓ is contained in the Veronese. This is a contradiction, since the Veronese surface does not contain any lines. Hence $\lambda \circ \nu_2^{[2]}$ is injective.

So in fact it suffices to prove all the properties for the map $\pi_2^{[2]}$. It is easy to see from its definition (and we already noted it) that its locus of indeterminacy is V. For the second statement, note that for any smooth variety X and any $x \in X$, any length 2 subscheme of X with support x canonically defines a point in $\mathbf{P}(T_x X)$ and vice-versa. Let $p \in S$ be arbitrary, then C_p becomes naturally identified with $\mathbf{P}(T_p S)$ and $\pi^{[2]}(C_p)$ becomes naturally identified with $\mathbf{P}(D_p \pi T_p S) \subset \mathbf{P}(T_{\pi(p)} \mathbf{P}(W))$. But if p lies in the ramification locus, the rank of $D_p \pi$ drops from 2 to 1, hence $\mathbf{P}(D_p \pi T_p S)$ is a single point. It follows that $\pi^{[2]}$ contracts C_p, hence ψ contracts C_p.

Furthermore, note that the set of length 2 subschemes of $\mathbf{P}(W)$ that are reduced and whose support is disjoint from the branch locus of π is dense in $\mathbf{P}(W)^{[2]}$. Let $t \subset \mathbf{P}(W)$ be such a subscheme, let $\{[w_1],[w_2]\}$ be its support. Denote $\pi^{-1}([w_i]) =: \{s_{i1}, s_{i2}\}$, $i = 1,2$, then by assumption $s_{ij} = s_{kl}$ if and only if $i = k$ and $j = l$. It follows that $\left(\pi^{[2]}\right)^{-1}(t)$ consists of 4 points, namely the subschemes of S with supports $\{s_{1i}, s_{2j}\}$ where $i = 1,2$ and $j = 1,2$. It follows that $\pi^{[2]}$, and hence ψ, is generically of degree 4.

Finally, for the last assertion we can pick χ as in lemma 2.1.15 and it suffices to describe the closure of the image of ψ. By construction the closure of the image of ψ is the Plücker embedding of the locus of lines that are spanned by a length 2 subscheme of the Veronese surface in $\mathbf{P}(\mathrm{Sym}^2 W)$. But the definition of a line being secant to subvariety of a projective space is precisely that it is spanned by a length 2 subscheme of that subvariety.

These properties imply that ψ, and hence $\varphi_{\mathcal{L}}$, is far from being an embedding. It follows that $\mathcal{L}$ is not very ample. □

Remark 2.1.17. The last item of the previous proposition shows us that the obstruction to $\mathcal{L}$ being very ample is in a sense quite severe; the closure of the image of $\varphi_{\mathcal{L}}$ has no moduli and there is no way to reconstruct $S^{[2]}$ from it. We will show in section 3.2.1 that in fact the determinantal cubic *is* the secant variety of the Veronese surface. In some sense this explains why the period map for cubic fourfolds cannot be surjective onto the period domain, even if we allow singular cubics; moduli and hence Hodge structures are 'swallowed' by the determinantal cubic.

Corollary 2.1.18. *Let $(Y, \mathcal{L})$ be any pair with isomorphism class in $\mathfrak{N}_{\mathrm{BD},2}$. Then $\mathcal{L}$ is not very ample on Y.*

Proof. Assume that there exists a pair $(Y, \mathcal{L})$ in $\mathfrak{N}_{\mathrm{BD},2}$ such that $\mathcal{L}$ is very ample on Y. By corollary 2.1.8 there exists K3 surface S of with line bundle $\mathcal{L}_2$ of square 2, a bimeromorphism $\psi : S^{[2]} \dashrightarrow Y$ and a Hodge isometry $\chi : H^2(S^{[2]}, \mathbf{Z}) \to H^2(S^{[2]}, \mathbf{Z})$

such that χ preserves the positive cone and $\chi(\mathcal{L}_2^{(2,-1)}) = \psi^*\mathcal{L}$. Since for any small deformation of $(Y,\mathcal{L})$ the line bundle will remain very ample, we may assume that S is generic. In particular we may assume that it has Picard rank 1. Since $\mathcal{L}$ is in particular ample on Y, it follows from proposition 2.1.14 that $(Y,\mathcal{L}) \cong \left(S^{[2]}, \mathcal{L}_2^{(2,-1)}\right)$, where $\mathcal{L}_2$ is a line bundle of square 2 on S. But by proposition 2.1.16 $\mathcal{L}_2^{(2,-1)}$ is not very ample on $S^{[2]}$. It follows that $\mathcal{L}$ cannot be very ample on Y. □

2.1.4 Alternative obstruction to very-ampleness

Proposition 2.1.16 shows in a very direct way why $\mathcal{L}_2^{(2,-1)}$ cannot be very ample on $S^{[2]}$ for a K3 surface S with line bundle $\mathcal{L}_2$ of square 2. There is however an alternative way to prove this, which we will sketch here because it compares nicely to the paper [42] by Saint–Donat, in which he analyzes obstructions to ampleness and very-ampleness of line bundles on K3 surfaces.

Let us make the following observations, which are based on example 0.6 in [33] (proofs can be found there). Any length 2 subscheme in $\mathbf{P}(W)$ spans a line. We can naturally interpret $\mathbf{P}(W^\vee)$ as the space of lines on $\mathbf{P}(W)$, hence we get a map $\mathbf{P}(W)^{[2]} \to \mathbf{P}(W^\vee)$. Composing this map with the $\pi^{[2]}$ gives a rational map

$$\eta : S^{[2]} \dashrightarrow \mathbf{P}(W^\vee)$$

which is well defined outside V. One can show that on the blow-up morphism $\varepsilon_V : \mathrm{Bl}_V(S^{[2]}) \to S^{[2]}$ of V in $S^{[2]}$ the map η extends to a well-defined morphism. As it turns out, the exceptional locus in $\mathrm{Bl}_V(S^{[2]})$ admits 2 rulings, one of which is induced by ε_V. We can blow down along the other ruling to a smooth fourfold Y, which is a holomorphic symplectic fourfold birational to $S^{[2]}$. This process is known as an **elementary transformation** or **flop**.

It can be shown that η extends to a well-defined map $\overline{\eta} : Y \to \mathbf{P}(W^\vee)$. Let $f : Y \dashrightarrow S^{[2]}$ be the birational map corresponding to the flop, then we have moreover that

$$c_1(\overline{\eta}^*\mathcal{O}_{\mathbf{P}(W^\vee)}(1)) = f^*(\gamma - \delta).$$

By [32, theorem 2] the generic fiber A is a smooth Abelian surface. We can calculate:

$$\begin{aligned}
\int_A c_1(f^*\mathcal{L})^2 &= \int_Y \left(f^*(\gamma-\delta)\right) c_1(f^*\mathcal{L})^2 \\
&= \int_{S^{[2]}} (\gamma-\delta)^2(2\gamma-\delta)^2 \\
&= \frac{1}{12}\left(\int_{S^{[2]}}(3\gamma-2\delta)^4 + \int_{S^{[2]}}\gamma^4 - 2\int_{S^{[2]}}(2\gamma-\delta)^4 - 2\int_{S^{[2]}}(\gamma-\delta)^4\right) \\
&= \frac{1}{4}\left(q(3\gamma-2\delta)^2 + q(\gamma)^2 - 2q(\gamma-\delta)^2 - 2q(2\gamma-\delta)^2\right) \\
&= 8
\end{aligned}$$

From Riemann-Roch and vanishing results for cohomology of Abelian varieties, see for example [7], it follows that

$$h^0(A, f^*\mathcal{L}|_{\mathcal{L}}) = \frac{1}{2}\int_A c_1(f^*\mathcal{L})^2 = 4.$$

As is explained in [7], to every line bundle $\mathcal{L}'$ on an Abelian variety A' of dimension n one can associate a **type** $(d_1, \ldots, d_n)$, which is a set of integers with the property that $d_1|d_2|\ldots|d_n$ and

$$d_1 \cdot \ldots \cdot d_n = h^0(A', \mathcal{L}').$$

The type is a numerical invariant which may be used to characterize properties of the linear system of $\mathcal{L}'$ on A'.

Let (d_1, d_2) be the type of the line bundle $f^*\mathcal{L}|_A$ on the Abelian variety A. Since $h^0(A, f^*\mathcal{L}|_{\mathcal{L}}) = 4$ it follows that the type is either $(1,4)$ or $(2,2)$. In both cases the morphism associated to the linear system has degree bigger than 1, again see [7]. Let A' be the closure of $f(A)$ in $S^{[2]}$. Then A and A' are birational and isomorphic outside $V \cap A$, which is a finite set of points. It follows that the morphism $A' \dashrightarrow |\mathcal{L}|_{A'}|$ is of degree bigger than 1, hence $\mathcal{L}$ cannot be very ample.

Note that this is very similar to the situation described by Saint-Donat for K3 surfaces of degree 2; he shows that the morphism associated to a line bundle of square 2 is 2:1 to the projective plane unless its linear system contains a divisor $2E + C$, where C is a smooth rational curve, E a smooth elliptic curve and $[E] \cap [C] = 1$ (that is, C forms a section of the elliptic fibration associated to E). In that case the restriction of the line bundle to an elliptic curve is of degree $[2E + C].E = 1$, hence the restricted linear system consists of 1 point, so the map associated to it cannot be 2:1. In this situation the map that associates to a K3 surface of degree 2 the sextic curve that forms the branch locus of the associated 2:1 covering fails (since there is no 2:1 covering). Moreover, for any generic family of degree 2 K3 surfaces that degenerates to this situation, the associated branch curve degenerates to the triple conic, which has no moduli. In a sense the triple conic 'swallows' all moduli of degree 2 K3 surfaces that have a divisor of type $2E + C$ in their linear system.

Compare this to our situation; the line bundle $\mathcal{L} = \mathcal{L}_2^{(2,-1)}$ on $S^{[2]}$ has first Chern class $2\gamma - \delta$. This we may write as $2(\gamma - \delta) + \delta$, where $\gamma - \delta$ induces an Abelian fibration (on a holomorphic symplectic fourfold birational to $S^{[2]}$). As we calculated above, the intersection $q(\gamma - \delta, 2\gamma - \delta)$ is too low for the restriction of $\mathcal{L}$ to A' to induce an injective map. Hence the map that associates a cubic fourfold to a holomorphic symplectic fourfold of $\mathrm{K3}^{[2]}$-type endowed with a line bundle of square 6, must fail for $S^{[2]}$. Moreover, for any family of such holomorphic symplectic fourfolds degenerating to $S^{[2]}$, the limiting cubic fourfold is the determinantal cubic fourfold, which has no moduli. In a sense the cubic fourfold 'swallows' the moduli of all holomorphic symplectic fourfolds that are isomorphic to the Hilbert scheme of length 2 subschemes of a K3 surface of degree 2.

2.2 The case of $\mathfrak{N}_{\mathrm{BD},6}$

2.2.1 Realization of the periods

In this section we show that for any period in $\mathcal{D}_{\mathrm{BD},6}$ there exists a K3 surface S of degree 6 and a line bundle $\mathcal{L}$ on $S^{[2]}$ such that the pair $(S^{[2]},\mathcal{L})$ belongs to $\mathfrak{N}_{\mathrm{BD},6}$ and realizes the given period. This case is very analogous to the case of $\mathfrak{N}_{\mathrm{BD},2}$ treated in section 2.1.1. We use results from section 4.2 in [21].

Let $X \subset \mathbf{P}^5$ be a cubic with an isolated ordinary double point p (that is, a singular point at which the projectivized tangent cone is a smooth quadric). Hassett argues that the lines contained in X passing through p are parameterized by a K3 surface S_p of degree 6. Indeed, if we choose local coordinates $x_1,\dots,x_5$ for $\mathbf{P}^5$ such that p corresponds to $(0,0,0,0,0)$ then the equation of X' is given by

$$f_2(x_1,\dots,x_5)+f_3(x_1,\dots,x_5)$$

for homogeneous polynomials f_2 and f_2 of degree 2 and 3 respectively. The line spanned by an element $x \in \mathbf{C}^5 \setminus \{0\}$ is contained in X if and only $f_2(x) = f_3(x) = 0$. Let us identify the set of lines in $\mathbf{P}^5$ through p with $\mathbf{P}(T_p\mathbf{P}^5)$. Then the set of lines through p contained in X is the intersection S_p of a quadric and a cubic in $\mathbf{P}(T_p\mathbf{P}^5) = \mathbf{P}^4$, defined by f_2 and f_3 respectively. If we assume that p is the only singular point of X, then S_p must be smooth. As is well known, any smooth intersection of a quadric and a cubic in $\mathbf{P}^4$, hence in particular S_p, is a K3 surface of degree 6.

Let K_6 be the lattice generated by α and β with intersection matrix

	α	β
α	3	0
β	0	2

In his thesis ([21]) Hassett shows the following, based on an argument given by Voisin in her thesis ([47]).

Proposition 2.2.1. *Let X be a cubic with a single ordinary double point $p \in X$ and let S_p be the associated K3 surface of degree 6. Let $\xi : \mathcal{X} \to \Delta$ be a deformation of X with generically smooth fibers. Then (possibly after a base change of order 2) the limiting Hodge structure $H^4_{\lim,\xi}$ is pure of weight 4. Then there is an embedding of $\mathbf{Z}$-Hodge structures*

$$i : (K_6 \otimes \mathbf{C})(-2) \oplus H^2(S_p,\mathbf{C})^0(-1) \hookrightarrow H^4_{\lim,\xi}$$

such that $i(\alpha)$ is the limit of the polarization class on the family ξ.

Let X and ξ be as in the proposition. Let $\eta : \mathcal{Y} \to \Delta$ be the relative variety of lines on ξ, $H^2_{\lim,\eta}$ its limiting Hodge structure and γ the limit of the polarization class. In a way similar to that of section 2.1.1 we wish to identify a holomorphic symplectic manifold with Hodge structure $H^2_{\lim,\eta}$, based on the information given by proposition 2.2.1. Let $\mathbf{L}_0$ be the lattice underlying $H^4_{\lim,\xi}$ and $\mathbf{M}_0$ the lattice underlying $H^2_{\lim,\eta}$. The Abel–Jacobi map (or rather the limit of the relative Abel–Jacobi map between the local systems associated to the families ξ and η, see section 2.1.1) gives a isomorphism of $\mathbf{Z}$-modules

$$\Phi : \mathbf{L}_0 \to \mathbf{M}_0$$

with the properties that it sends $i(\alpha)$ to γ, it defines a lattice isometry from $\alpha^\perp$ to $\widetilde{\gamma^\perp}$ (the tilde indicates that the sign of the pairing is reversed, see remark 2.1.2) and extends to a Hodge isometry $H^4_{\lim,\xi} \cong H^2_{\lim,\eta}(-1)$. We have the following result:

Lemma 2.2.2. *Denote* $K_6' := \langle 6\rangle \oplus \langle -2\rangle$. *Then there is an extension of lattices*

$$K_6' \oplus H^2(S_p,\mathbf{Z})^0 \hookrightarrow \mathbf{M}_0$$

which extends to an isomorphism of Hodge structures

$$(K_6' \otimes \mathbf{C})(-1) \oplus H^2(S_p,\mathbf{Z}) \hookrightarrow H^2_{\lim,\eta}.$$

Proof. The embedding i in proposition 2.2.1 restricts to an embedding of lattices

$$i' : \alpha\mathbf{Z} \oplus \beta\mathbf{Z} \oplus \widetilde{H}^2(S_p,\mathbf{Z})^0 \hookrightarrow \mathbf{L}_0,$$

where $\widetilde{H}^2(S_p,\mathbf{Z})^0$ is the primitive cohomology lattice with the sign of the intersection form reversed. This embedding is an extension of lattices since the ranks are equal. Let $j' := \Phi \circ i'$ the we obtain an extension

$$j'(\alpha)\mathbf{Z} \oplus j'(\beta)\mathbf{Z} \oplus j'(\widetilde{H}^2(S_p,\mathbf{Z})^0) \hookrightarrow \mathbf{M}_0.$$

By the properties of Φ this extends to an isometry of Hodge structures. Hence it suffices to show that

$$j'(\alpha)\mathbf{Z} \oplus j'(\beta)\mathbf{Z} \oplus j'(\widetilde{H}^2(S_p,\mathbf{Z})^0) \cong K_6' \oplus H^2(S_p,\mathbf{Z})^0 \tag{2.2.1}$$

as lattices. The map Φ is a Hodge isometry form $i'(\alpha)^\perp$ to $\widetilde{j'(\alpha)^\perp}$, so we have

$$j'(\beta)\mathbf{Z} \oplus j'(\widetilde{H}^2(S_p,\mathbf{Z})^0) \cong \langle -2\rangle \oplus H^2(S_p,\mathbf{Z})^0.$$

Furthermore, $j'(\alpha) = \gamma \in \mathbf{M}_0$, so it has self-intersection 6. In particular

$$j'(\alpha)\mathbf{Z} \oplus j'(\beta) \cong \langle 6\rangle \oplus \langle -2\rangle$$

and equation 2.2.1 follows. □

Let K_6' be as in lemma 2.2.2 and α', β' orthogonal generators of self-intersection 6 and -2 respectively. The extension given in the lemma sends α' to $\gamma \in \mathbf{M}_0$, the limit of the polarization on the family η of holomorphic symplectic fourfolds. In particular, by theorem 1.3.1, the image of α' in $\mathbf{M}_0$ of *even type*. This property characterizes the extension up to isometry. More precisely we have the following result:

Lemma 2.2.3. *Let*

$$\varphi_1, \varphi_2 : K_6' \oplus H^2(S_p, \mathbf{Z})^0 \hookrightarrow \mathbf{M}_0$$

be lattice extensions such that both φ_1 and φ_2 map $\alpha' \in K_6'$ to an element of even type. Then there exists a lattice automorphism λ of $\mathbf{M}_0$ such that $\varphi_2 = \lambda \circ \varphi_1$.

Proof. First remember that

$$\mathbf{M}_0 \cong \Lambda_{K3^{[2]}} \cong \widetilde{E8}^{\oplus 2} \oplus U^{\oplus 3} \oplus \langle -2 \rangle. \tag{2.2.2}$$

Furthermore observe that since $\varphi_1(\alpha')$ and $\varphi_2(\alpha')$ both have self intersection 6 and both are of even type by assumption, they must be contained in the same orbit. In particular, if e, f denote the isotropic generators of the hyperbolic lattice U and δ a generator of $\langle -2 \rangle$, we may assume under the decomposition (2.2.2) that

$$\varphi_1(\alpha') = \varphi_2(\alpha') = 2e + 2f - \delta.$$

Indeed, it can be easily checked that the right hand side has self-intersection 6 and is of even type in $U \oplus \langle -2 \rangle$, and hence of even type in $\mathbf{M}_0$. The extensions φ_1 and φ_2 now induce extensions

$$\varphi_1^{\perp}, \varphi_2^{\perp} : (\alpha)' = \beta' \mathbf{Z} \oplus H^0(S_p, \mathbf{Z})^0 \hookrightarrow (2e + 2f - \delta)^{\perp} = \langle e - \delta, f - \delta \rangle \oplus \widetilde{E8}^{\oplus 2} \oplus U^{\oplus 2}.$$

It suffices to show that these extensions are unique up to isometry. We invoke again the results of Nikulin, see proposition 2.1.6. Note that

$$\beta' \mathbf{Z} \oplus H^0(S_p, \mathbf{Z}) \cong \beta' \mathbf{Z} \oplus \zeta \mathbf{Z} \oplus M,$$

where $M = \widetilde{E8}^{\oplus 2} \oplus U^{\oplus 2}$ is unimodular. Denote the right hand side by Λ. Then

$$\Lambda^* = (\beta')^* \mathbf{Z} \oplus \zeta^* \mathbf{Z} \oplus M$$

where $(\beta')^* \in \mathrm{Hom}(\Lambda, \mathbf{Z})$ maps β' to 1, its orthocomplement to 0 and similarly for ζ^*. It follows that

$$\Lambda^* / \Lambda = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/6\mathbf{Z}$$

with generators $\overline{(\beta')}^*$ and $\overline{\zeta}^*$. Let q be the natural $\mathbf{Q}/2\mathbf{Z}$-valued quadratic form on Λ/Λ^*, then

$$q(a\overline{(\beta')}^* + b\overline{\zeta}^*) \equiv -\frac{a^2}{2} - \frac{b^2}{6}.$$

It follows that there is only one non-trivial isotropic subgroup of Λ/Λ^*, namely the one generated by $\overline{(\beta')}^* + 3\overline{\zeta}^*$. Proposition 2.1.6 now implies that up to isometry there is only one non-trivial extension of Λ.

To complete the proof we must show that $\varphi_1^\perp$ and $\varphi_2^\perp$ are non-trivial extensions, that is, are not isomorphisms. But it is easy to calculate that

$$\mathrm{discr}(\beta'\mathbf{Z} \oplus H^0(S_p,\mathbf{Z})^0) = 12$$
$$\mathrm{discr}(\langle e-\delta, f-\delta\rangle \oplus \widetilde{E8}^{\oplus 2} \oplus U^{\oplus 2}) = 3$$

so $\varphi_1^\perp$ and $\varphi_2^\perp$ cannot be isomorphisms. □

This result enables us to identify a holomorphic symplectic fourfold with weight 2 Hodge structure isomorphic to $H^2_{\mathrm{lim},\eta}$. From the results sofar an obvious candidate arises. Indeed, Let $\gamma \in H^2(S_p,\mathbf{Z})$ be the polarization class, then there is a lattice extension

$$\gamma\mathbf{Z} \oplus H^2(S_p,\mathbf{Z})^0 = \langle 6\rangle \oplus H^2(S_p,\mathbf{Z})^0 \hookrightarrow H^2(S_p,\mathbf{Z})$$

which induces a Hodge isomorphism. Since $H^2(S_p^{[2]},\mathbf{Z}) \cong H^2(S_p,\mathbf{Z}) \oplus \delta$ with $\delta^2 = -2$ it follows that there is a lattice extension

$$j' : K_6' \oplus H^2(S_p,\mathbf{Z})^0 \hookrightarrow H^2(S_p^{[2]},\mathbf{Z})$$

that sends α' to γ and β' to δ (or rather their images in $H^2(S_p^{[2]},\mathbf{Z})$). Let us stress here that this is **not** the extension we want (and it took the author some time to realize this). The reason is that γ is not of even type: since it lies in $H^2(S_p,\mathbf{Z})$, which is unimodular, there must exist an element $\gamma' \in H^2(S_p,\mathbf{Z})$ and hence in $H^2(S_p^{[2]},\mathbf{Z})$ such that $\gamma.\gamma' = 1$.
This issue is quickly resolved by the following observation. The lattice $\langle\gamma,\delta\rangle$ has intersection matrix

	γ	δ
γ	6	0
δ	0	2

It is then easy to check, by calculating intersection numbers, that the map defined by

$$\gamma \mapsto 2\gamma - 3\delta$$
$$\delta \mapsto \gamma + 2\delta$$

is a lattice automorphism of $\langle\gamma,\delta\rangle$. Now define the lattice extension

$$j : K_6' \oplus H^2(S_p,\mathbf{Z})^0 \hookrightarrow H^2(S_p^{[2]},\mathbf{Z})$$

in the same way as j', with the difference that it maps α' to $2\gamma-3\delta$ and β' to $\gamma+2\delta$. It also extends to a Hodge isomorphism. The point of this exercise is that $2\gamma-3\delta$ *is* of even type in $H^2(S_p^{[2]},\mathbf{Z})$. This observation leads to the following identification of Hodge structures.

Proposition 2.2.4. *There exists an isomorphism of lattices*

$$\mathbf{M}_0 \cong H^2(S_p^{[2]},\mathbf{Z})$$

which maps the limit of the polarization class on the family η *in* $\mathbf{M}_0$ *to the class* $2\gamma-3\delta \in H^2(S_p^{[2]},\mathbf{Z})$. *Furthermore, it extends to an isomorphism of Hodge structures*

$$H^2_{\lim,\eta} \cong H^2(S_p^{[2]},\mathbf{C}).$$

Proof. We have constructed non-trivial lattice extensions

$$\begin{array}{ccc} & & \mathbf{M}_0 \\ & \overset{j_1}{\nearrow} & \\ K_6' \oplus H^2(S_p,\mathbf{Z})^0 & & \\ & \underset{j_2}{\searrow} & \\ & & H^2(S_p^{[2]},\mathbf{Z}) \end{array}$$

where j_1 is as in lemma 2.2.2 and $j_2 := j$ as constructed above. Both map $\beta' \in K_6'$ to an element of even type and both extend to Hodge isomorphisms. It follows from lemma 2.2.3 that there exists a lattice isomorphism $\psi : \mathbf{M}_0 \cong H^2(S_p^{[2]},\mathbf{Z})$ that makes this diagram commute. It then also extends to a Hodge isomorphism. Finally note that by construction j_1 maps $\alpha' \in K_6'$ to the limiting class of the polarization on η and j_2 maps α' to the class $2\gamma-3\delta \in H^2(S_p^{[2]},\mathbf{Z})$. Hence ψ maps the limit of the polarization class to $2\gamma-3\delta$. □

We can now state the main result of this section, which is analogous to theorem 2.1.7.

Theorem 2.2.5. *The locus* $\overline{\mathfrak{N}}_{\mathrm{BD},6}$ *in* $\overline{\mathfrak{N}}_{\mathrm{BD}}$ *is precisely the image of the map*

$$\begin{array}{rcl} q_6 : \overline{\mathfrak{N}}_{K3,6} & \to & \overline{\mathfrak{N}}_{\mathrm{BD}} \\ [S,\mathcal{L}] & \mapsto & \left[S^{[2]},\mathcal{L}^{(2,-3)}\right] \end{array}$$

where $\overline{\mathfrak{N}}_{K3,6}$ *denotes the moduli space of K3 surfaces with line bundle of square 6,* $\mathcal{L}^{(2,-3)}$ *denotes the line bundle on* $S^{[2]}$ *with first Chern class* $2c_1(\mathcal{L})-3\delta$ *and* $[Y,\mathcal{L}]$ *denotes the image the isomorphism class of a pair* $(Y,\mathcal{L})$ *in the Hausdorff reduction of the moduli space.*

Proof. By the same reasoning as in the proof of theorem 2.1.7 it suffices to show that q_6 maps generic elements of $\overline{\mathfrak{N}}_{K3,6}$ into $\overline{\mathfrak{N}}_{\mathrm{BD},6}$.

Let $(S,\mathcal{L})$ be a generic K3 surface of degree 6. We may assume that $\mathcal{L}$ is very ample and that $S \subset |\mathcal{L}^{\vee}| \cong \mathbf{P}^4$ is the intersection of a smooth quadric hypersurface and a smooth cubic hypersurface. Let f_2 and f_3 be homogeneous form that define the quadric and the cubic respectively. Now let $(x_0,\dots,x_5)$ be a homogeneous coordinate system on $\mathbf{P}^5$ and define the cubic form f by

$$f(x_0,\dots,x_5) = x_0 f_2(x_1,\dots,x_5) + f_3(x_1,\dots x_5).$$

Let X be the cubic defined by f, then X has a single ordinary double point in

$$p = [1:0:0:0:0:0].$$

Furthermore, the set of lines S_p on X through p is canonically identified with S, see the argument in the introduction of this section.

Let $\xi : \mathcal{X} \to \Delta$ be a deformation of X with smooth generic fiber. Its relative variety of lines $\eta : \mathcal{Y} \to \Delta$ defines a limiting Hodge structure (possible after base change) $H^2_{\lim,\eta}$ which is isomorphic to $H^2(S^{[2]},\mathbf{C})$, as follows from lemma 2.2.4. Moreover, it follows from that lemma that the primitive part $(H^2_{\lim,\eta})^0$ is isomorphic to the primitive part $H^2(S^{[2]},\mathbf{C})^0$ with respect to (the first Chern class of) the line bundle $\mathcal{L}^{(2,-3)}$ on $S^{[2]}$.

Let $P \in \mathcal{D}_{\mathrm{BD}}$ be the period defined by the Hodge structure $H^2_{\lim,\eta}$, then we find that

$$\overline{\mathcal{Q}}_{\mathrm{BD}}^{-1}(P) = \left[S^{[2]},\mathcal{L}^{(2,-3)}\right] \in \overline{\mathfrak{N}}_{\mathrm{BD}}$$

(remember that $\overline{\mathcal{Q}}_{\mathrm{BD}}$ is an isomorphism). On the other hand, by definition of $\mathfrak{N}_{\mathrm{BD},6}$ we must have that $\overline{\mathcal{Q}}_{\mathrm{BD}}^{-1}(P) \in \overline{\mathfrak{N}}_{\mathrm{BD},6}$, since P is the image under the Abel–Jacobi map of the period defined by $H^4_{\lim,\xi}$, which lies in $\mathcal{D}_{\mathrm{cubic},6}$ by construction. Hence $q_6[S,\mathcal{L}] \in \overline{\mathfrak{N}}_{\mathrm{BD},6}$. Since this works for generic pairs $(S,\mathcal{L}) \in \mathfrak{N}_{K3,6}$, we find that q_6 maps a dense subset of $\overline{\mathfrak{N}}_{K3,6}$, and hence all of $\overline{\mathfrak{N}}_{K3,6}$, onto $\overline{\mathfrak{N}}_{\mathrm{BD},6}$. □

Remark 2.2.6. Let X be a generic nodal cubic, S the associated K3 surface and $\mathcal{L}$ the polarization on S. One can explicitly construct a birational map $\varphi : S^{[2]} \to F(X)$ such that $\varphi^* H = \mathcal{L}^{(2,-3)}$, where H is the natural polarization on $F(X)$ (we will treat this construction in some detail in section 4.4). Claire Voisin pointed out to us that one can use this observation, in combination with proposition 4.2 in [10], to prove theorem 2.2.5 without an explicit consideration of variation of Hodge structures. The argument would then run roughly as follows. Given a smoothening of X, its relative variety of lines, let us call it η, is a flat projective family of holomorphic symplectic fourfolds of K3$^{[2]}$-type which degenerates to $F(X)$. The polarization on the smooth fibers of η is of square 6 with respect to the Beauville–Bogomolov form, and the same holds for $\varphi^* H = \mathcal{L}^{(2,-3)}$ on $S^{[2]}$. The Hilbert polynomial of a line bundles on a holomorphic symplectic manifold only depends on the square of the line bundle and the deformation type of the manifold, see for example [25]. In particular the Hilbert polynomial of $\mathcal{L}^{(2,-3)}$ must coincide with that of the polarization on smooth fibers of η. By flatness of η it then follows that $\chi(H^{\otimes k}) = \chi((\mathcal{L}^{(2,-3)})^{\otimes k})$ for all $k \in \mathbf{Z}$. Proposition 4.2 in [10] now tells us that a general small deformation of $\left(S^{[2]}, \mathcal{L}^{(2,-3)}\right)$ is isomorphic to a (smooth) small deformation of $(F(X), H)$. This can be used in the proof of theorem 2.2.5 to show that q_6 maps generic elements of $\overline{\mathfrak{N}}_{K3,6}$ to $\overline{\mathfrak{N}}_{\mathrm{BD},6}$.

We end this section with a characterization of the elements of $\mathfrak{N}_{\mathrm{BD},6}$.

Corollary 2.2.7. *A pair $(Y, \mathcal{L})$ defines a point in $\mathfrak{N}_{\mathrm{BD},6}$ if and only if there exist a K3 surface S with line bundle $\mathcal{L}_6$ of square 6, a bimeromorphic map $\psi : S^{[2]} \dashrightarrow Y$ and a Hodge isometry $\chi : H^2(S^{[2]}, \mathbf{Z}) \to H^2(S^{[2]}, \mathbf{Z})$ which preserves $C^+_{S^{[2]}}$ and has the property that it maps $c_1\left(\mathcal{L}_6^{(2,-3)}\right)$ to $\psi^*(c_1(\mathcal{L}))$.*

Proof. This follows directly from the definition of $\mathfrak{N}_{\mathrm{BD},6}$, theorem 1.1.28 and the previous theorem. □

2.2.2 The birational Kähler cone for generic varieties in $\mathfrak{N}_{\mathrm{BD},6}$

Before we show why the elements of $\mathfrak{N}_{\mathrm{BD},6}$ cannot be realized as variety of lines on a cubic fourfold we will look more closely at the Hilbert scheme of length 2 subschemes of generic K3 surfaces S of degree 6.

In this section S will be a K3 surface of degree 6 and with Picard rank 1. We will find some properties of the so-called **birational Kähler cone** $\mathcal{BK}_{S^{[2]}}$ of $S^{[2]}$. This is by definition the set of classes $\alpha \in H^{1,1}(S^{[2]}, \mathbf{R}) := H^2(S^{[2]}, \mathbf{R}) \cap H^{1,1}(S^{[2]})$ for which there exist a holomorphic symplectic manifold Y and a bimeromorphism $\psi : Y \dashrightarrow S^{[2]}$ such that $\psi^*\alpha$ is a Kähler class on Y. See also [27].

In fact it may happen that the birational Kähler cone is not a cone but rather a union of cones. Nevertheless its closure *is* a cone. In [27] Huybrechts gives the following characterization of the closure of the birational Kähler cone:

Proposition 2.2.8 ([27], Prop 4.2)**.** *Let Y be a compact holomorphic symplectic manifold and q the Beauville–Bogomolov quadratic form on $H^2(Y, \mathbf{Z})$. Then $\alpha \in H^{1,1}(S^{[2]}, \mathbf{R})$ is contained in the closure of the birational Kähler cone if and only if $q(\alpha) > 0$ and $q(\alpha, [D]) \geq 0$ for all uniruled divisors $D \subset X$, where $[D]$ denotes the class in $H^2(Y, \mathbf{Z})$ Poincaré dual to the homology class of D.*

So to study the birational Kähler cone of $S^{[2]}$ we must identify uniruled divisors in $S^{[2]}$. There is one obvious such divisor present: the divisor $N \subset S^{[2]}$ of non-reduced length 2 subschemes is uniruled. In fact $N \cong \mathbf{P}(TS)$, and hence has in particular the structure of a $\mathbf{P}^1$-bundle over S. Remember that $[N] = 2\delta$, where $\delta \in H^2(S^{[2]}, \mathbf{Z})$ is as before.

We will construct a second collection of rational curves on $S^{[2]}$, parameterized by S (and hence giving rise to a uniruled divisor on S). First note that by the genericity assumption we may assume that $S \subset \mathbf{P}^4$ and is equal to the transversal intersection of a conic hypersurface $Q \subset \mathbf{P}^4$ and a cubic hypersurface $K \subset \mathbf{P}^4$. Let $s \in S$ be arbitrary. Since $S = Q \cap K$ it follows in particular that $s \in Q$. Let ℓ be any line through s that is contained in Q. By the assumption that S is of Picard rank 1 it follows that ℓ is not contained in S, hence not in K. We have that $\ell \cap S = \ell \cap K = s \cup \sigma_{\ell,s}$, where $\sigma_{\ell,s}$ is a length 2 subscheme in S with the property that it spans ℓ.

Let $T_sQ \subset P^4$ be the tangent space of Q at s. The set of lines through s contained in Q sweep out the set $T_sQ \cap Q$, which is a 2-dimensional cone in T_sQ with vertex at s and will be denoted by T_s. In particular the set of lines in Q through s are parameterized by a $\mathbf{P}^1$.
Define

$$S^{[2]} \supset C_s := \bigcup_{\ell \subset T_s} \sigma_{\ell,s}.$$

Then C_s is a rational curve in $S^{[2]}$. It has the following numerical property:

Lemma 2.2.9. *Let $a, b \in \mathbf{Z}$ be arbitrary, then*

$$\int_{C_s} c_1(\mathcal{L}_6^{(a,b)}) = 6a + 4b$$

Proof. It suffices to do the calculation for $c_1(\mathcal{L}_6^{(1,0)})$ and $c_1(\mathcal{L}_6^{(0,1)})$. Let $H \subset \mathbf{P}^4$ be a hyperplane, then $c_1(\mathcal{L}_6^{(1,0)})$ represents a divisor $D_H \subset S^{[2]}$ of length 2 subschemes of S that have some support on $S \cap H$. Define

$$\widetilde{C}_s := \bigcup_{\ell \subset T_s} \operatorname{supp}(\sigma_{\ell,s}) \subset S.$$

For generic choices of H and s we have

$$\int_{C_s} c_1(\mathcal{L}_6^{(1,0)}) = |H \cap \widetilde{C}_s|.$$

The right hand side is precisely the degree of $\widetilde{C}_s$ in $\mathbf{P}^4$, which can be easily computed: $\widetilde{C}_s$ is contained in the linear subspace T_sQ, and in this subspace it is the intersection of the quadric hypersurface T_s and the cubic hypersurface $K \cap T_sQ$. Hence $\widetilde{C}_s$ is of degree 6 and thus

$$\int_{C_s} c_1(\mathcal{L}_6^{(1,0)}) = 6.$$

What remains is to calculate the pairing of $[C_s]$ with $c_1(\mathcal{L}_6^{(0,1)}) = \delta$. Let $N \subset S^{[2]}$ be the divisor of non-reduced length 2 subschemes of S, then for generic choice of s

$$2\int_{C_s} \delta = |N \cap C_s|.$$

We will calculate the right hand side. It is equal to the number of non-reduced length 2 subschemes in C_s. We can rephrase this as follows. Let $\varepsilon_s : \widetilde{T}_s \to T_s$ be the blow-up of the cone T_s at its vertex and denote by E the exceptional divisor. Let $\widehat{C}_s$ be the strict transform of $\widetilde{C}_s$ in $\widetilde{T}_s$. Generically this will be a smooth curve. There is natural projection $\widetilde{T}_s \to E$ which restricts to a 2:1 covering map $\rho : \widetilde{C}_s \to E$. The fibers of ρ are precisely the length 2 subschemes parameterized by C_s. They are non-reduced precisely over the branch points of ρ. Hence $|N \cap C_s|$ is in fact the number of branch points of ρ.

Since E has genus 0 it follows from the topological Hirzebruch-Riemann-Roch theorem that the number of branch points of ρ is equal to $2g(\widetilde{C}_s) + 2$, where $g(\widetilde{C}_s)$ is the genus of $\widetilde{C}_s$. Determining this is basically an exercise in the theory of complex algebraic surfaces, see for example [5]. From the exact sequence

$$0 \to \mathcal{O}_{\widetilde{T}_s}(-\widetilde{C}_s) \to \mathcal{O}_{\widetilde{T}_s} \to \mathcal{O}_{\widetilde{C}_s} \to 0$$

and the identity

$$\chi(\mathcal{O}_{\widetilde{T}_s}(-\widetilde{C}_s)) = \chi(\mathcal{O}_{\widetilde{T}_s}) + \frac{1}{2}([\widetilde{C}_s].[\widetilde{C}_s] + [\widetilde{C}_s].K_{\widetilde{T}_s}),$$

where K_{T_s} is the canonical divisor on $\widetilde{T}_s$, it follows that

$$g(\widetilde{C}_s) = 1 - \chi(\mathcal{O}_{\widetilde{C}_s}) = 1 + \frac{1}{2}([\widetilde{C}_s].[\widetilde{C}_s] + [\widetilde{C}_s].K_{\widetilde{T}_s}).$$

Now observe that T_s is isomorphic to a cone in $\mathbf{P}^3$ over an smooth conic in $\mathbf{P}^2$. It follows that $\widetilde{T}_s \cong \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(2)) =: \mathbf{F}_2$ is a Hirzebruch surface. From the general theory of ruled surfaces (see for example [5] chapter III) it follows that $H^2(\mathbf{F}_2, \mathbf{Z}) = \langle Z, F \rangle$, where Z is the pull-back to $\widetilde{T}_s$ of a hyperplane section of T_s and F is the strict transform of a line in T_s through the vertex. The intersection matrix is given by

	Z	F
Z	2	1
F	1	0

Furthermore $[E] = Z - 2F$ and $K_{\widetilde{T}_s} = -2Z$. We need to determine the class of $\widehat{C}_s$ in terms of Z and F. Remember that by construction $\widetilde{C}_s \setminus s = T_s \cap K$, where K is as before the cubic such that $S = K \cap Q$. It follows that $\varepsilon_s^{-1}(K) = E \cup \widetilde{C}_s$, since $\widehat{C}_s$ is the strict transform of $\widetilde{C}_s$. By definition of Z and F we have that

$$[\varepsilon_s^{-1}(K)].Z = 6$$
$$[\varepsilon_s^{-1}(K)].F = 3$$

from which immediately follows that

$$[\widetilde{C}_s].Z = 6$$
$$[\widetilde{C}_s].F = 2.$$

Hence $[\widetilde{C}_s] = 2Z + 2F$ and we can compute:

$$\begin{aligned} g(\widetilde{C}_s) &= 1 + \frac{1}{2}([\widetilde{C}_s].[\widetilde{C}_s] + [\widetilde{C}_s].K_{\widetilde{T}_s}) \\ &= 1 + \frac{1}{2}(16 - 12) \\ &= 3. \end{aligned}$$

Hence the number of branch points, and hence $|N \cap C_s|$ equals $2 \cdot 3 + 2 = 8$. we conclude that

$$\int_{C_s} c_1(\mathcal{L}_6^{(0,1)}) = \int_{C_s} \delta = \frac{1}{2}|N \cap C_s| = 4.$$

This completes the proof. □

The previous result enables us to prove the main theorem of this subsection.

Proposition 2.2.10. *Let S be a K3 surface of Picard rank 1 with line bundle $\mathcal{L}_6$ of square 6. Then the birational Kähler cone of $S^{[2]}$ is contained in the convex cone in $H^{1,1}(S^{[2]}, \mathbf{R})$ spanned by $c_1(\mathcal{L}_6^{(1,0)})$ and $c_1(\mathcal{L}_6^{(2,-3)})$.*

Proof. Write $\gamma := c_1(\mathcal{L}_6^{(1,0)})$ and $\delta := c_1(\mathcal{L}_6^{(2,-3)})$. Then $H^{1,1}(S^{[2]}) = \langle \gamma, \delta \rangle$ with intersection matrix

q	γ	δ
γ	6	0
δ	0	-2

Let $\alpha \in \mathcal{BK}_{S^{[2]}}$ be arbitrary. Then $\alpha = a\gamma + b\delta$ for some $a, b \in \mathbf{R}$. We can rewrite this as

$$\alpha = (a + \frac{2}{3}b)\gamma - \frac{1}{3}b(2\gamma - 3\delta) \tag{2.2.3}$$

$$= (a + \frac{2}{3}b)\, c_1(\mathcal{L}_6^{(1,0)}) - \frac{1}{3}b\, c_1(\mathcal{L}_6^{(2,-3)}). \tag{2.2.4}$$

Hence to complete the proof it suffices to show that $a + \frac{2}{3}b \geq 0$ and $b \leq 0$.

Let $N \subset S^{[2]}$ as before the divisor of non-reduced length 2 subschemes of S. Then by proposition 2.2.8 $q(\alpha, [N]) \geq 0$. Since $[N] = 2\delta$ it follows that $b \leq 0$.

Furthermore, since α lies in the birational Kähler cone there exist a holomorphic symplectic manifold Y and a bimeromorphism $\psi : Y \dashrightarrow S^{[2]}$ such that $\psi^*\alpha$ is Kähler on Y. Let $s \in S$ be such that C_s is not contained in the complement of the image of ψ. This is possible since the union $\bigcup_{s \in S} C_s$ is of codimension 1 in $S^{[2]}$ whereas the complement of the image of ψ is of codimension at least 2, by remark 1.1.7. Let C_s' the curve in Y that is the closure of $\psi^{-1}(C_s)$. Then $\int_{C_s} \psi^*\alpha > 0$ since $\psi^*\alpha$ is Kähler. Since ψ is an isomorphism outside loci of codimension at least 2 we have that

$$\int_{C_s} \alpha = \int_{C_s'} \psi^*\alpha => 0.$$

By lemma 2.2.9 we then have that $6a + 4b > 0$, in particular $a + \frac{2}{3}b \geq 0$. This completes the proof. □

Remark 2.2.11. This result is strong enough to analyze the obstructions to very ampleness of line bundles of square 6 on manifolds bimeromorphic to $S^{[2]}$. In chapter 4 we will reconsider the birational Kähler cone and prove some further properties.

Proposition 2.2.10 has an important consequence:

Proposition 2.2.12. *Let S be a K3 surface of degree 6 and Picard rank 1. Then the birational Kähler cone of $S^{[2]}$ does not contain any integral classes of square 6.*

Proof. Let $\mathcal{L}_6$ be the line bundle of square 6 on S. Write $\gamma := c_1(\mathcal{L}_6^{(1,0)})$ and $\delta := c_1(\mathcal{L}_6^{(2,-3)})$. Then $H^{1,1}(S^{[2]}) = \langle \gamma, \delta \rangle$ with intersection matrix

q	γ	δ
γ	6	0
δ	0	-2

Let α be an integer class in the birational Kähler cone of $S^{[2]}$. The $\alpha = a\gamma + b\delta$ with $a, b \in \mathbf{Z}$. From proposition 2.2.10 it follows that $b < 0$ and $3a + 2b > 0$. In particular it follows that

$$\frac{a^2}{b^2} > \frac{4}{9}. \tag{2.2.5}$$

Now assume that in addition $q(\alpha) = 6$, or equivalently, that $3a^2 - b^2 = 3$. It follows that $b = -3n$ for some $n \in \mathbf{N}$ (with a minus sign since $b \leq 0$). Furthermore we find that

$$\frac{a^2}{b^2} = \frac{1}{b^2} + \frac{1}{3} = \frac{1}{9n^2} + \frac{1}{3} \leq \frac{4}{9}.$$

This contradicts inequality (2.2.5). Hence the birational Kähler cone does not contain any classes of square 6. □

2.2.3 Obstructions to very-ampleness

We are now in position to show that for any pair $(Y, \mathcal{L})$ in $\mathfrak{N}_{\mathrm{BD},6}$ the line bundle $\mathcal{L}$ cannot be very ample. In fact we will show that it cannot even be ample. These results give an explanation for the fact that pairs in $\mathfrak{N}_{\mathrm{BD},6}$ of holomorphic symplectic fourfolds and line bundle cannot be isomorphic to the variety of lines on a cubic fourfold.

Theorem 2.2.13. *Let $(Y, \mathcal{L})$ be a pair belonging to $\mathfrak{N}_{\mathrm{BD},6}$. Then $\mathcal{L}$ cannot be ample on Y*

Proof. By corollary 2.2.7 there exist a K3 surface S with line bundle $\mathcal{L}_6$ of square 6 and a bimeromorphism $\psi^* : Y \dashrightarrow S^{[2]}$. Now assume that $\mathcal{L}$ is ample on Y. Since ampleness is preserved by any small deformation of the pair $(Y, \mathcal{L})$, we may replace $(Y, \mathcal{L})$ by a pair $(Y', \mathcal{L}')$ such that $\mathcal{L}'$ is ample on Y' and such that there exist a K3 surface *S of Picard rank 1* with line bundle $\mathcal{L}_6$ of square 6 and a bimeromorphism $\psi : Y' \dashrightarrow S^{[2]}$ as above. Since $\mathcal{L}'$ is assumed to be ample on Y, $\psi^* c_1(\mathcal{L}')$ must be contained in the birational cone of $S^{[2]}$. But this is in contradiction with proposition 2.2.12, since $\psi^* c_1(\mathcal{L}')$ is of square 6. □

Chapter 3

Projective reconstruction of holomorphic symplectic fourfolds from families of cubics

In this section we investigate to what extent we can reconstruct projective holomorphic symplectic manifolds from degenerations of cubic fourfolds. More precisely, let $\xi : \mathcal{X} \to \Delta$ be a one-parameter family of cubic hypersurfaces in $\mathbf{P}^5$ with smooth fibers over the punctured disc $\Delta^* \subset \Delta$, can we then explicitly construct a projective family $\zeta : \mathcal{Z} \to \Delta$ of holomorphic symplectic fourfolds such that for $t \in \Delta^*$ we have an isomorphism $F(\mathcal{X}_t) \cong \mathcal{Y}_t$ compatible with the polarizations[1]? Note that this is only possible if the family of periods defined by ξ has a limit that is contained in $\mathcal{D}_{\text{cubic}}$. If this is the case, we have the following possibilities:

- ξ extends to a family of cubic fourfolds over the full disc Δ. In this case, we may simply take ζ to be the relative variety of lines on the extended family over Δ;

- The limiting period of ξ is contained in $\mathcal{D}_{\text{cubic},6}$. In that case, if a family ζ would exist, the isomorphism class of the central fiber, with polarization, would be contained in $\mathfrak{N}_{\text{BD},6}$. We have shown however that for such pairs the line bundle cannot be a polarization (see theorem 2.2.13). We conclude that, in this situation, a family ζ as desired does not exist;

- The limiting period of ξ is contained in $\mathcal{D}_{\text{cubic},2}$. In that case, if a family ζ would exist, the isomorphism class of the central fiber, with polarization, would be contained in $\mathfrak{N}_{\text{BD},2}$. In particular, by theorem 2.1.7, the isomorphism class would be non-separated from a pair $\left(S^{[2]}, \mathcal{L}_2^{(2,-1)}\right)$ for some K3

[1] By 'compatible' we mean that the polarizations have a common power.

surface S with line bundle $\mathcal{L}_2$ of square 2. We have seen (theorem 2.1.13) that for such pairs the line bundle is generically ample, so it may in fact be possible construct a family ζ a desired.

Remark 3.0.14. Let us stress here that we want ζ to be a *projective* family of holomorphic symplectic fourfolds. If we drop projectivity, a family of holomorphic symplectic manifolds with the properties mentioned always exists (abstractly). Indeed, ξ defines a family of Hodge structures in the period domain for cubic fourfolds. Via the Beauville–Donagi construction this then defines a family of Hodge structures in the period domain for holomorphic symplectic manifolds of $K3^{[2]}$-type. The period map for holomorphic symplectic manifolds is known to be a proper local isomorphism (see for example [26]). Therefore this family of Hodge structures can be realized, at least locally, by a family of holomorphic symplectic manifolds. The generic fiber of this family will be isomorphic to the variety of lines of a fiber of ξ. However, this family may be non-projective.

We will show that in case the limiting Hodge structure of ξ lies in $\mathcal{D}_{\text{cubic},2}$, we can indeed generically construct a family ζ as desired. We will first prove this for linear one-parameter deformations of the determinantal cubic (see section 2.1.1 for a definition); a precise statement of the result as well as an outline of the proof can be found in section 3.1. Sections 3.2 to 3.5 are devoted to the actual proof. The choice to consider one-parameter linear deformations may seem rather restrictive, but we will show in section 3.6 that it can be globalized to a quasi-universal result (we will of course explain what we mean by 'quasi-universal'). In section 3.7 we will perform a short analysis of the natural polarization on the families of projective holomorphic symplectic fourfolds that we have constructed. We have also adopted a list of notations that appear throughout this chapter.

3.1 Reconstruction from linear deformations

As before, let W be a complex vector space of dimension 3, $\mu \in \bigwedge^3 W$ a volume form and $D \in \mathrm{Sym}^3\mathrm{Sym}^2 W^\vee$ the determinantal cubic form with respect to μ. It defines the determinantal cubic in $\mathbf{P}(\mathrm{Sym}^2 W)$; we will denote it by $X_{\det}$. We consider deformations of $X_{\det}$ over a one-parameter basis Δ. Since the setting of this chapter is algebraic we will take $\Delta = \operatorname{Spec} \mathbf{C}[t]_{\langle t\rangle}$; the spectrum of the localization of $\mathbf{C}[t]$ at $\langle t\rangle$. The more analytically inclined reader may interpret Δ as a small open disc around the origin of the complex plane.

For any $f \in \mathrm{Sym}^3\mathrm{Sym}^2 W^\vee$ we defined the linear one-parameter deformation of $X_{\det}$ by f to be the family

$$\xi : \mathcal{X} \to \Delta$$

where $\mathcal{X} \subset \mathbf{P}(\mathrm{Sym}^2 W) \times \Delta$ is defined by the form $D + tf$, for $t \in \Delta$. See also section 2.1.1.

Furthermore, let $v : W \to \mathrm{Sym}^2 W$ be the affine Veronese map (that sends $w \mapsto w^2$). It induces a projection

$$v^* : \mathrm{Sym}^3\mathrm{Sym}^2 W^\vee \to \mathrm{Sym}^6 W^\vee$$

by pull-back. The largest part of this chapter is devoted to the proof of the following result:

Proposition 3.1.1. *Let $f \in \mathrm{Sym}^3\mathrm{Sym}^2 W$ be such that $v^* f \neq 0$, let $\Sigma_f \subset \mathbf{P}(W)$ be the sextic curve defined by $v^* f$ and $S_f \to \mathbf{P}(W)$ the double cover branched along Σ_f. Furthermore, let $\xi_f : \mathcal{X}_f \to \Delta$ be a one-parameter deformation of X_{det} by f. Let $\xi'_f : \mathcal{X}_f \to \Delta$ be the family obtained by a pull-back along the base change $\Delta \to \Delta, t \mapsto t^2$.*

Then there exists a natural projective family $\zeta_f : \mathcal{Z}_f \to \Delta$ such that the generic fiber of ζ_f is the variety of lines on the generic fiber of ξ'_f and the central fiber $\mathcal{Y}_{f,0}$ is isomorphic to $S_f^{[2]}$.
In particular, if Σ_f is smooth, ζ_f is a projective family of holomorphic symplectic fourfolds.

Remark 3.1.2. Given a family of cubic fourfolds $\xi_f : \mathcal{X}_f \to \Delta$, to be able to extend the relative variety of lines on ξ_f over Δ^* to a family of holomorphic symplectic fourfolds over Δ it is necessary that the limiting Hodge structure of ξ_f is pure. As Hassett already showed, for a generic (linear) deformation of the determinantal cubic we will need a base change of order 2 to achieve this. So the base change in theorem 3.1.1 cannot be omitted.

We will prove proposition 3.1.1 by explicitly constructing the family ζ_f from ξ'_f. We do this in three steps:
Step 1: relative variety of lines. We take the relative variety of lines $\varphi : \mathcal{F} \to \Delta$ associated to ξ'_f. In particular, $\mathcal{F}_0 = F(X_{\mathrm{det}})$ parameterizes the lines on the determinantal cubic. We will show (proposition 3.2.3) that $F(X_{\mathrm{det}})$ consists of two components, whose reductions we denote by F_1 and F_2, with the properties that $F_1 \cong \mathbf{P}(W)^{[2]}$ and $F_2 \cong \mathbf{P}(W) \times \mathbf{P}(W^\vee)$. More details and proofs can be found in section 3.2.
Step 2: blow-up of F_1. The next step is to blow up F_1 in $\mathcal{F}$, which will be dealt with in section 3.3. Write $\widehat{\mathcal{F}} := \mathrm{Bl}(\mathcal{F}_1, \mathcal{F})$, then the composition of φ with the blow-up morphism defines a family $\widehat{\varphi} : \widehat{\mathcal{F}} \to \Delta$. We will show that $\widehat{\mathcal{F}}_0$ is a normal crossing of the exceptional divisor E of the blow-up and the strict transform $\widehat{F}_2$ of F_2 in $\widehat{\mathcal{F}}$ (theorem 3.3.7). Furthermore, let $V \subset S_f^{[2]}$ be the locus of length 2 subschemes of S_f that are vertical with respect to the natural projection $S_f \to \mathbf{P}(W)$. We will show in section 3.4 that canonically $E \cong \mathrm{Bl}(V, S_f^{[2]})$.
Step 3: blow-down of $\widehat{F}_2$. We will show that it is possible to contract $\widehat{F}_2$ projectively and smoothly within $\widehat{\mathcal{F}}$ to a lower dimensional subvariety. More precisely,

we will construct a scheme $\overline{\mathcal{F}}$ and a projective morphism $\gamma:\widehat{\mathcal{F}}\to\overline{\mathcal{F}}$ such that $\overline{\mathcal{F}}$ is smooth along $\gamma(\widehat{F}_2)$. Moreover, we show that $\widehat{\varphi}$ factors through γ, so that we obtain a new projective family

$$\overline{\varphi}:\overline{\mathcal{F}}\to\Delta.$$

These results will be proven in section 3.5.

In section 3.5.3 we prove proposition 3.6.4 by showing that we can take ζ to be the family $\overline{\varphi}$, the construction of which we just described.

3.1.1 Notation and conventions

The construction of the family ζ_f in proposition 3.1.1 along the procedure outlined above involves quite a bit of notation. At the end of this chapter we have included a list of the symbols that occur the most. Let us stress here that the whole construction depends on the choice of a cubic form $f\in\mathrm{Sym}^3\mathrm{Sym}^2W^\vee$ (for which $v^*f\neq 0$). We will fix such an f, but we will not make it explicit in our notation, only in the symbols that appear in the statement of proposition 3.1.1.

We will also need to perform calculations in symmetric and exterior algebras of vector spaces. We fix the notation here. Let V be a vector space over $\mathbf{C}$ (the restriction to $\mathbf{C}$ is not essential, but natural since we work in a complex setting). For $n\in\mathbf{N}$ we will view the n^{th} symmetric power Sym^nV and the n^{th}s exterior power $\bigwedge^n V$ of V as subspaces of $V^{\otimes n}$:

$$\mathrm{Sym}^nV=\left\langle\{\sum_{\sigma\in\mathcal{S}_n}v_{\sigma(1)}\otimes\ldots\otimes v_{\sigma(n)}\mid v_1,\ldots,v_n\in V\}\right\rangle$$
$$\bigwedge^n V=\left\langle\{\sum_{\sigma\in\mathcal{S}_n}\mathrm{sgn}(\sigma)\,v_{\sigma(1)}\otimes\ldots\otimes v_{\sigma(n)}\mid v_1,\ldots,v_n\in V\}\right\rangle,$$

where $\mathcal{S}_n$ is the permutation group of the set $\{1,\ldots,n\}$, $\mathrm{sgn}(\sigma)$ is the sign of the permutation $\sigma\in\mathcal{S}_n$ and the brackets denote the linear span over $\mathbf{C}$. For $v_1,\ldots v_n\in V$ we denote

$$v_1\cdot\ldots\cdot v_n:=\frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n}v_{\sigma(1)}\otimes\ldots\otimes v_{\sigma(n)}$$
$$v_1\wedge\ldots\wedge v_n:=\frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n}\mathrm{sgn}(\sigma)\,v_{\sigma(1)}\otimes\ldots\otimes v_{\sigma(n)},$$

note the combinatorial factors in front of the summation. Sometimes we will denote v_1v_2 instead of $v_1\cdot v_2$, v_1^2 instead of $v_1\cdot v_1$, and similarly for higher powers. In case of repeated symmetric or exterior powers we will separate the products by brackets. For example, elements of $\mathrm{Sym}^3\mathrm{Sym}^2V$ will be denoted by $(v_1\cdot v_2)\cdot(v_3\cdot v_4)\cdot(v_5\cdot v_6)$ (and sums of such expressions). The natural map $\mathrm{Sym}^3\mathrm{Sym}^2V\to\mathrm{Sym}^6V$ is then given by

$$(v_1\cdot v_2)\cdot(v_3\cdot v_4)\cdot(v_5\cdot v_6)\mapsto v_1\cdot\ldots\cdot v_6.$$

Let $V^\vee$ be the dual vector space of V. For contractions of elements in the tensor algebras of V and $V^\vee$ we use the following convention: let $n, m \in \mathbf{N}$, $v_1, \ldots v_{n+m} \in V$ and $\alpha_1, \ldots, \alpha_{n+m} \in V^\vee$, then

$$(\alpha_n \otimes \ldots \alpha_1)(v_1 \otimes \ldots v_{n+m}) = \alpha_1(v_1)\alpha_2(v_2)\cdots\alpha_n(v_n)\, v_{n+1} \otimes \ldots \otimes v_{n+m}$$
$$(\alpha_{n+m} \otimes \ldots \otimes \alpha_1)(v_1 \otimes \ldots \otimes v_n) = \alpha_1(v_1)\alpha_2(v_2)\cdots\alpha_n(v_n)\, \alpha_{n+m} \otimes \ldots \otimes \alpha_{n+1}.$$

Note the change of ordering. This induces contraction conventions for elements in symmetric powers or antisymmetric powers of V and $V^\vee$. The issues with ordering play no role for symmetric powers, but for antisymmetric powers it may give rise to signs.

Inevitably, combinatorial factors will occur in contractions. For example, it is important to note that if $\{e_1, \ldots, e_k\}$ is a basis for V with dual basis $\{e_1^\vee, \ldots, e_2^\vee\}$, then the basis

$$\{e_i^\vee \cdot e_j^\vee \mid 1 \le i \le j \le k\}$$

of $\mathrm{Sym}^2 V^\vee$ will *not* be dual to the basis $\{e_i \cdot e_j \mid 1 \le i \le j \le k\}$ of $\mathrm{Sym}^2 V$ when $k \ge 2$. Indeed, we have

$$(e_1^\vee \cdot e_2^\vee)(e_1 \cdot e_2) = \frac{1}{2} \ne 1.$$

Instead, $\{e_i \cdot e_j \mid 1 \le i \le j \le k\}$ has dual basis

$$\{e_i^2 \mid 1 \le i \le k\} \cup \{2e_i \cdot e_j \mid 1 \le i < j \le k\}.$$

We end this section by a remark on the specific case of $\mathrm{Sym}^2 V$. There is a natural identification

$$\begin{aligned} \mathrm{Sym}^2 V &\cong \mathrm{Hom}_{\mathrm{Sym}}(V^\vee, V) \\ b &\mapsto \big(\varphi_b : \alpha \mapsto \alpha(b)\big), \end{aligned}$$

where $\mathrm{Hom}_{\mathrm{Sym}}(V^\vee, V)$ is the space of symmetric linear maps from $V^\vee$ to V. Let $\{e_1, \ldots, e_k\}$ be a basis for V and $\{e_1^\vee, \ldots, e_k^\vee\}$ the dual basis, then the elements of $\mathrm{Hom}_{\mathrm{Sym}}(V^\vee, V)$ can be represented by symmetric matrices with respect to these bases. At times we will use the identification to express elements of $\mathrm{Sym}^2 V$ as symmetric matrix, if a basis of V is given. For example, if V is two-dimensional and $\{e_1, e_2\}$ is a basis, then $ae_1^2 + be_2^2 + ce_1e_2 \in \mathrm{Sym}^2 V$ corresponds to the matrix

$$\begin{pmatrix} a & \frac{1}{2}c \\ \frac{1}{2}c & b \end{pmatrix}$$

3.2 The relative variety of lines

Let $f \in \mathrm{Sym}^3 \mathrm{Sym}^2 W^\vee$ be such that $v^* f \ne 0$ and let

$$\xi'_f : \mathcal{X}'_f \to \Delta$$

be the family of cubics defined by the homogeneous equation $D + t^2 f = 0$. Define

$$F : \Delta \to \mathrm{Sym}^3\mathrm{Sym}^2 W^\vee, t \mapsto D + t^2 f$$

Furthermore, let $\mathcal{T}$ be the tautological bundle on the Grassmannian $\mathrm{Gr}_2(\mathrm{Sym}^2 W)$ of 2-dimensional planes in $\mathrm{Sym}^2 W$ and $\widehat{\mathcal{T}}$ its pullback to $\mathrm{Gr}_2(\mathrm{Sym}^2 W) \times \Delta$, then F defines a global section σ_F of $\mathrm{Sym}^3 \widehat{\mathcal{T}}^\vee$ by the property that $\sigma_F(P, t) = F(t)|_P$ for any $t \in \Delta$ and 2-plane $P \subset \mathrm{Sym}^2 W$. In particular, $\sigma_F(P, t) = 0$ if and only if $F(t)|_P = 0$, that is, if and only if the line $\mathbf{P}(P) \subset \mathbf{P}(\mathrm{Sym}^2 W)$ is contained in the cubic $\mathcal{X}'_t$.

Let $\mathcal{F} \subset \mathrm{Gr}_2(\mathrm{Sym}^2 W) \times \Delta$ be the zero locus of σ_F and let $\varphi : \mathcal{F} \to \Delta$ be the restriction to $\mathcal{F}$ of the projection to Δ. We say that φ is the **relative variety of lines** on ξ'_f. The aim of this section is to find local equations for $\mathcal{F}$ along the central fiber.

3.2.1 The variety of lines on X_{det}

By construction, the central fiber of $\varphi : \mathcal{F} \to \Delta$ is $F(X_{\mathrm{det}})$, the variety of lines on the determinantal cubic X_{det}. The aim of this section is to describe the structure of $F(X_{\mathrm{det}})$. For convenience we give some alternative characterizations of X_{det}.

Proposition 3.2.1. *Let $V_2 \subset \mathbf{P}(\mathrm{Sym}^2 W)$ be the Veronese, that is, the image of the Veronese map $\mathbf{P}(W) \hookrightarrow \mathbf{P}(\mathrm{Sym}^2 W)$. Then we have the following characterizations:*

1. *X_{det} is equal to $S_1 V_2$, the first secant variety of the image of V_2. That is, X_{det} is equal to the Zariski closure of the union of all lines in $\mathbf{P}(\mathrm{Sym}^2 W)$ that intersect V_2 in two distinct points.*

2. *X_{det} is equal to TV_2, the tangent variety of V_2 in $\mathbf{P}(\mathrm{Sym}^2 W)$. That is, X_{det} is equal to the union of all 2-planes in $\mathbf{P}(\mathrm{Sym}^2 W)$ that are tangent to V_2.*

3. *Let $X'_{\mathrm{det}} \subset \mathrm{Sym}^2 W$ be the affine cone over X_{det}. Then X'_{det} precisely parameterizes those $\alpha \in \mathrm{Sym}^2 W$ for which the associated symmetric form on $W^\vee$ is degenerate.*

Proof. We start with the last characterization, which is almost immediate: by definition $\alpha \in X'_{\mathrm{det}}$ if and only if $\bigwedge^3 \varphi_\alpha$ vanishes, where $\varphi_\alpha : W^\vee \to W$ is the symmetric map associated to α. This vanishing occurs if and only if φ_α has non-trivial kernel, which is equivalent to the statement that the symmetric form on $W^\vee$ associated to α is degenerate. The last characterization follows.

Now suppose $\alpha \in X'_{\mathrm{det}}$. Since symmetric forms can be diagonalized, we know that there are $u_1, u_2, u_3 \in W$ such that $\alpha = u_1^2 + u_2^2 + u_3^2$. But we also know that this symmetric form is degenerate, so it follows that there are elements $w_1, w_2 \in W$ (which could be equal) such that $\alpha = w_1^2 + w_2^2$. Hence $[\alpha] \in X_{\mathrm{det}} \subset \mathbf{P}(\mathrm{Sym}^2 W)$ lies on a line $\langle [w_1^2], [w_2^2] \rangle$ through $[w_1^2]$ and $[w_2^2]$, which are points on the Veronese variety V_2. So $[\alpha] \in S_1 V_2$. On the other hand, let $p_1, p_2 \in V_2$ be distinct and $z \in \langle p_1, p_2 \rangle$. Then there are $w_1, w_2 \in W$ and $s, t \in \mathbf{C}$ such that $p_1 = [w_1^2], p_2 = [w_2^2]$ and $z = [sw_1^2 + tw_2^2]$. In particular z is represented by a degenerate symmetric form on $W^\vee$, so $z \in X_{\mathrm{det}}$. From this the first statement follows.

For the second characterization, first note the following. Let $w \in W$ be arbitrary, then the derivative at w of the Veronese map $v : W \to \mathrm{Sym}^2 W$ is given by

$$D_w v : W \to \mathrm{Sym}_2 W, u \mapsto 2w \cdot u.$$

So the tangent variety of V_2 in $\mathbf{P}(\mathrm{Sym}^2 W)$ is given by $\{[w \cdot u] \mid u, w \in W\}$. We know that $p \in X_{\mathrm{det}}$ if and only if there exist $w_1, w_2 \in W$ such that $p = [w_1^2 + w_2^2]$. But $w_1^2 + w_2^2 = (w_1 + \mathrm{i} w_2)(w_1 - \mathrm{i} w_2)$, so $p \in X_{\mathrm{det}}$ if and only if p lies in the tangent variety of V_2. The second statement follows. □

We introduce two subsets of $\mathrm{Gr}_2(\mathrm{Sym}^2 W)$:

- Let $F_1 \subset \mathrm{Gr}_2(\mathrm{Sym}^2 W)$ be the set of lines secant to V_2, that is, the set of lines that intersect V_2 with multiplicity bigger than or equal to 2;
- Let $F_2 \subset \mathrm{Gr}_2(\mathrm{Sym}^2 W)$ be the set of lines of the form $[w] \cdot \ell \in \mathbf{P}(\mathrm{Sym}^2 W)$ for $[w] \in \mathbf{P}(W)$ a point and $\ell \subset \mathbf{P}(W)$ a line.

Remark 3.2.2. The set F_1 is in fact equal to the set of lines that intersect V_2 with multiplicity precisely 2. Indeed, if $\ell \subset \mathbf{P}(\mathrm{Sym}^2 W)$ intersects V_2 with multiplicity strictly bigger than 2, it follows that every quadratic form that vanishes on V_2 also vanishes on ℓ. Since the homogeneous vanishing ideal of V_2 is generated in degree 2, this would imply that $\ell \subset V_2$. This is absurd, since the Veronese surface does not contain any lines.

Proposition 3.2.3. *Set-theoretically* $F(X_{\mathrm{det}}) = F_1 \cup F_2$.

Proof. First we prove that $F_1, F_2 \subseteq F(X_{\mathrm{det}})$. By the first statement in proposition 3.2.1 we know that X_{det} is isomorphic to the first secant variety of the Veronese surface V_2. The secant variety contains by definition the set of secant lines, so it directly follows that F_1 must be a subvariety of $F(X_{\mathrm{det}})$. Similarly, by the second characterization in proposition 3.2.1 it follows directly that F_2 is a subvariety of $F(X_{\mathrm{det}})$.

To complete the proof it suffices to show that $F(X_{\mathrm{det}}) \subset F_1 \cup F_2$. Let ℓ be a line in X_{det}. By characterization 3 in proposition 3.2.1 the line ℓ represents a pencil of singular quadrics in $\mathbf{P}(W^\vee)$. Bertini's theorem[2] states that the generic element of a pencil is smooth outside the base locus, so for this pencil the base locus must be non-empty.

First assume that the base locus B is of dimension 1. Then it must be a line in $\mathbf{P}(W^\vee)$ and in this case the pencil of quadrics is simply the union of this line

[2]We thank Claire Voisin for her suggestion to use Bertini's theorem to prove this proposition.

with a pencil of lines in $\mathbf{P}(W^\vee)$. Let $[w_B] \in \mathbf{P}(W)$ be dual to B and let $\ell' \subset \mathbf{P}(W)$ represent the pencil of lines, then $\ell = [w_B] \cdot \ell'$. In particular $\ell \in F_2$.

The remaining case is that B is of dimension 0. The generic quadric in the pencil defined by ℓ is of rank 2 (otherwise ℓ would be contained in the Veronese in $\mathbf{P}(\mathrm{Sym}^2 W)$, which is not possible) and hence has 1 singular point. By Bertini's theorem this point must be contained in B. It follows that if B is zero-dimensional, all quadrics in the pencil have a singular point in common. Denote this point by $[\alpha_B] \in \mathbf{P}(W^\vee)$. We must have that $\ell \subset \mathbf{P}(\mathrm{Sym}^2 \ker \alpha_B)$. The intersection of the Veronese in $\mathbf{P}(\mathrm{Sym}^2 W)$ with $\mathbf{P}(\mathrm{Sym}^2 \ker \alpha_B)$ is of codimension 1 and degree 2 in $\mathbf{P}(\mathrm{Sym}^2 \ker \alpha_B)$, hence ℓ must be secant to the Veronese. It follows that $\ell \in F_1$. This completes the proof.

□

Let us study the structure of F_1 and F_2 more closely.

Proposition 3.2.4. *There are natural identifications*

- $F_1 \cong \mathbf{P}(W)^{[2]}$, *where the righthand side denotes the smooth Hilbert scheme of length 2 subschemes of* $\mathbf{P}(W)$*;*
- $F_1 \cong \mathbf{P}(\mathrm{Sym}^2 \mathcal{K}^\vee)$, *where* $\mathcal{K} \subset W \otimes \mathcal{O}_{\mathbf{P}(W^\vee)}$ *is the rank 2 subbundle with the property that* $\mathcal{K}_\alpha = \ker \alpha \subset W$ *for all* $\alpha \in W^\vee$*;*
- $F_2 \cong \mathbf{P}(W) \times \mathbf{P}(W^\vee)$.

Moreover

- *As a subspace of* F_1 *and under the identification* $F_1 \cong \mathbf{P}(W)^{[2]}$, *the intersection* $F_1 \cap F_2$ *corresponds to the locus* $\mathbf{P}(W)^{[2]}_{\text{non-red}}$ *of non-reduced length 2 subschemes of* $\mathbf{P}(W)$*;*
- *As a subspace of* F_2 *and under the identification* $F_2 \cong \mathbf{P}(W) \times \mathbf{P}(W^\vee)$, *the intersection* $F_1 \cap F_2$ *corresponds to the incidence variety* $\{([w],[\alpha]) \mid w \in \ker \alpha\} \subset \mathbf{P}(W) \times \mathbf{P}(W^\vee)$*;*
- *At any point* $P \in F_1 \cap F_2$ *the dimension of the intersection of* $T_P F_1$ *and* $T_P F_2$ *in* $T_P \mathrm{Gr}_2(\mathrm{Sym}_2 W)$ *equals the dimension of the intersection of* F_1 *and* F_2.

Proof. By definition, F_2 is the image of the map

$$\begin{aligned} \varphi_2 : \mathbf{P}(W) \times \mathbf{P}(W^\vee) &\to \mathrm{Gr}_2(\mathrm{Sym}^2 W), \\ ([w],[\alpha]) &\mapsto [w \cdot \ker \alpha]. \end{aligned}$$

Let $p : \mathrm{Gr}_2(\mathrm{Sym}^2 W) \hookrightarrow \mathbf{P}(\bigwedge^2 \mathrm{Sym}^2 W)$ be the Plücker embedding. Using the natural identification $\mathbf{P}(W) \cong \mathbf{P}(\bigwedge^2 W)$, $p \circ \varphi_2$ can be rewritten as

$$\begin{aligned} \mathbf{P}(W) \times \mathbf{P}(\textstyle\bigwedge^2 W) &\to P(\textstyle\bigwedge^2 \mathrm{Sym}^2 W) \\ ([w_1],[w_2 \wedge w_3]) &\mapsto [(w_1 \cdot w_2) \wedge (w_1 \cdot w_3)]. \end{aligned}$$

It is easily checked that this composition is an embedding, hence φ_2 itself is an embedding and it provides an isomorphism $\mathbf{P}(W) \times \mathbf{P}(W^\vee) \cong F_2$.

For F_1 note that there is a natural map $\varphi_1 : \mathbf{P}(W)^{[2]} \to F_1$ which associates to a length 2 subscheme $s \subset \mathbf{P}(W)$ the line $\langle v(s)\rangle \subset \mathbf{P}(\mathrm{Sym}^2 W)$, where $v : \mathbf{P}(W) \hookrightarrow \mathbf{P}(\mathrm{Sym}^2 W)$ is the Veronese embedding. Remark 3.2.2 implies that this map is injective. Even stronger, it asserts that the map that sends a line in F_1 to its intersection with V_2 is a well defined map from F_1 to $\mathbf{P}(W)^{[2]}$. This map is clearly an inverse to φ_1, hence φ_1 is an isomorphism.

For the rest of the first part of the theorem it now suffices to show that $\mathbf{P}(W)^{[2]} \cong \mathbf{P}(\mathrm{Sym}^2 \mathcal{K}^\vee)$. Note that any length 2 subscheme in $\mathbf{P}(W)$ spans a line. This gives rise to a natural map $\lambda : \mathbf{P}(W)^{[2]} \to \mathbf{P}(W^\vee)$. For every $\alpha \in W^\vee$ we have that $\lambda^{-1}([\alpha]) = \mathbf{P}(\ker\alpha)^{[2]}$. Since there is a natural identification $\mathbf{P}(\ker\alpha)^{[2]} \cong \mathbf{P}(\mathrm{Sym}^2(\ker\alpha)^\vee)$ we obtain an isomorphism $\mathbf{P}(W)^{[2]} \cong \mathbf{P}(\mathrm{Sym}^2\mathcal{K}^\vee)$ respecting the projections to $\mathbf{P}(W^\vee)$.

The next step is to give a description of the intersection locus $I := F_1 \cap F_2$. Let $J_1 \subset F_1$ be the set parameterizing lines tangent to the Veronese V_2 and let $J_2 \subset F_2$ be given by

$$J_2 = \{[w \cdot \ker\alpha] \mid w \in W, \alpha \in W^\vee, w \in \ker\alpha\}.$$

We claim that $J_1 = J_2 = I$.

Indeed, let $P \in J_1$ be represented by the line $\ell_P \subset \mathbf{P}(\mathrm{Sym}^2 W)$. By definition, ℓ_P is tangent to V_2, hence there are $w_1, w_2 \in W$ such that $\ell_P = \langle [w_1^2], [w_1 \cdot w_2]\rangle$. There also exists a nonzero $\alpha \in W^\vee$ such that $\ker\alpha = \langle w_1, w_2\rangle$, and we see that $\ell_P = \mathbf{P}(w_1 \cdot \ker\alpha)$ and in particular $w_1 \in \ker\alpha$. So $P \in J_2$. This argument is completely reversible, hence $J_1 = J_2$, and in particular $J_1, J_2 \subseteq I$.

Now let $P \in F_1 \setminus J_1$ be arbitrary and let $\ell_P \subset \mathbf{P}(\mathrm{Sym}^2 W)$ be the corresponding line. Then there are $w_1, w_2 \in W$, linearly independent, such that $\ell_P = \langle [w_1^2], [w_2^2]\rangle$. Suppose $P \in I$, then in particular $P \in F_2$ and hence there exist $x \in W$ and $\alpha \in W^\vee$ such that $\ell_P = \mathbf{P}(x \cdot \ker\alpha)$. Hence there must exist $y_1, y_2 \in \ker\alpha$ such that $[w_1^2] = [x \cdot y_1]$ and $[w_2^2] = [x \cdot y_2]$. This implies that $[x] = [w_1] = [w_2]$, which is absurd since w_1 and w_2 are linearly independent. It follows that $P \in F \setminus J_1$ implies that $P \notin I$, or equivalently, $I \subseteq J_1$. Combining this with earlier results it follows that $J_1 = J_2 = I$.

Finally, let $P \in I$ be arbitrary. To complete the proof it suffices to show that $T_P F_1 \cap T_P F_2$ is 3-dimensional. Choose a basis $\{w_1, w_2, w_3\} \subset W$ such that the plane P is spanned by w_1^2 and $w_1 w_2$. A local parametrization of $\mathbf{P}(W) \times \mathbf{P}(\bigwedge^2 W)$ is given by

$$(a, b, c, d) \mapsto [w_1 + a w_2 + b w_3, (w_1 + c w_3) \wedge (w_2 + d w_3)].$$

By composing this map with $p \circ \varphi_2$ we obtain a parametrization of $p(F_2)$ with the property that the origin maps to $p(P)$. The tangent space $T_{p(P)}p(F_2)$ is described by the linear part of this map, which is given by:

$$(a,b,c,d) \mapsto \big[w_1^2 \wedge w_1 w_2 + a w_1^2 \wedge w_2^2 + b(w_1 w_3 \wedge w_1 w_2 + w_1^2 \wedge w_2 w_3) \\ + c w_1 w_3 \wedge w_1 w_2 + d w_1^2 \wedge w_1 w_3\big].$$

Similarly, for $\sigma \in \mathbf{P}(W)^{[2]}$ reduced we have that

$$(p \circ \varphi_1)(\sigma) = s_1(\sigma)^2 \wedge s_2(\sigma)^2,$$

where $\{[s_1(\sigma)], [s_2(\sigma)]\}$ is the support of σ. Now we can locally parameterize reduced length 2 subschemes by their support:

$$(a+b, ab, c, d) \mapsto \{[w_1 + a w_2 + (c+ad) w_3], [w_1 + b w_2 + (c+bd) w_3]\},$$

for $a \neq b$. It is well-defined since the image is an unordered pair. Composition with $p \circ \varphi_1$ then gives a local parametrization of $p(F_1)$. It has the following linearization:

$$(a+b, ab, c, d) \mapsto \big[w_1^2 \wedge w_1 w_2 + (a+b) w_1^2 \wedge w_2^2 + 2ab w_1 w_2 \wedge w_2^2 \\ + 2c(w_1^2 \wedge w_2 w_3 - 2 w_1 w_2 \wedge w_1 w_3) \\ + 2d(w_1^2 \wedge w_1 w_3)\big].$$

Note that it extends for $a = b$ and that the origin maps to $p(P)$. Comparing the descriptions, it follows directly that the tangent spaces intersect in a 3-dimensional subspace. □

3.2.2 Local equations for F_1, F_2 and $F(X_{\text{det}})$

It will be convenient to have explicit local equations for F_1, F_2 and $F(X_{\text{det}})$ in $\text{Gr}_2(\text{Sym}_2 W)$ at our disposal. Let $P \subset \text{Sym}^2 W$ be any 2-plane and let $P^\perp$ be a complementary 4-plane. Then we have an embedding

$$\begin{array}{rcl} \kappa_{P,P^\perp} : \text{Hom}(P, P^\perp) & \hookrightarrow & \text{Gr}_2(\text{Sym}^2 W) \\ A & \mapsto & (\text{Id}_P + A)(P) \end{array}$$

where Id_P is the identity on P. Let us denote the image of $\kappa_{P,P^\perp}$ by $U_{P,P^\perp}$. For any choice of basis $\beta := \{\beta_1, \dots, \beta_6\}$ of $\text{Sym}^2 W$ such that $P = \langle \beta_1, \beta_2 \rangle$ and $P^\perp = \langle \beta_3, \dots, \beta_6 \rangle$ we can define functions

$$\begin{array}{rcl} a_{i,j} : U_{P,P^\perp} & \to & \mathbf{C} \\ u & \mapsto & \beta_j^\vee\big(\kappa_{P,P^\perp}^{-1}(u)\beta_i\big) \end{array}$$

where $\beta^\vee$ is the dual basis for β. Then the set $\{a_{i,j} \mid 1 \le i \le 2, 3 \le j \le 6\}$ is system of coordinates on $U_{P,P^\perp}$.

We use such coordinate systems to express equations for F_1 and F_2. We will consider the coordinates systems for three choices of P: the case $P \in F_1 \cap F_2$, the case $P \in F_1 \setminus F_2$ and the case $P \in F_2 \setminus F_1$. We will find local equations for F_1 in the first and second case and local equations for F_2 in the second and third case.

Case 1: $P \in F_1 \cap F_2$. Let $P \subset \mathrm{Sym}^2 W$ be a subspace of dimension 2 such that $P \in F_1 \setminus F_2$. Then by proposition 3.2.4 we can find a basis $\{e_1, e_2, e_3\}$ of W such that $P = \langle e_1^2, e_1 e_2\rangle$ and the volume form $\mu = e_1 \wedge e_2 \wedge e_3$. Let $P^\perp := \langle e_2^2, e_1 e_3, e_2 e_3, e_3^2\rangle$. Let $\{e_1^\vee, e_2^\vee, e_3^\vee\} \subset W^\vee$ be a dual basis, then define

$$a_{i,jk} : \mathrm{im}\,\kappa \to \mathbf{C}, x \mapsto (e_j^\vee e_k^\vee)(\kappa^{-1}x)(e_1 e_i).$$

The set

$$\{a_{i,jk} \mid i = 1,2, jk = 22,13,23,33\}$$

is a coordinate system on $U_{P,P^\perp}$. We will denote the germs at P of the functions $a_{i,jk}$ by the same symbols. We then have the following result:

Lemma 3.2.5. *Let F_1^P be the germ of F_1 at P. It is defined by equations*

$$\begin{aligned} a_{1,23} &= a_{1,22}a_{2,13} \\ a_{2,23} &= a_{1,13} + a_{2,22}a_{2,13} \\ a_{1,33} &= a_{1,13}^2 + a_{1,22}a_{2,13}^2 \\ a_{2,33} &= 2a_{1,13}a_{2,13} + a_{2,22}a_{2,13}^2 \end{aligned}$$

in $\mathcal{O}_{\mathrm{Gr}_2(\mathrm{Sym}^2 W),P}$.

Proof. For $(a,b,c,d) \in \mathbf{C}^4$, set $w_1 := e_1 + ae_2 + (c+ad)e_3$ and $w_2 := e_1 + be_2 + (c+bd)e_3$. That is, a, b parameterize deformations of $[w_1]$ and $[w_2]$ respectively within $\langle [w_1],[w_2]\rangle \subset \mathbf{P}(W)$ and c, d parameterize deformations of the line $\langle [w_1],[w_2]\rangle$ with respect to $\langle [e_1],[e_2]\rangle$. This may seem to be a somewhat cumbersome way to parameterize two points in $\mathbf{P}(W)$, but it will be convenient in what follows. If $a \neq b$ we have that w_1^2 and w_2^2 span a two-plane in $\mathrm{Sym}^2 W$. This plane is parameterized by the map

$$(s,t) \mapsto \begin{pmatrix} s+t & as+bt & (c+ad)s+(c+bd)t \\ as+bt & a^2s+b^2t & a(c+ad)s+b(c+bd)t \\ (c+ad)s+(c+bd)t & a(c+ad)s+b(c+bd)t & (c+ad)^2s+(c+bd)^2t \end{pmatrix},$$

where we used the basis of W to represent elements of $\mathrm{Sym}^2 W$ by a symmetric matrix. By construction this plane represents an element of F_1. On the other hand, in the chosen local coordinate system of $\mathrm{Gr}_2(\mathrm{Sym}^2 W)$ the planes are parameterized by:

$$(x,y) \mapsto \begin{pmatrix} x & y & a_{1,13}x + a_{2,13}y \\ y & a_{1,22}x + a_{2,22}y & a_{1,23}x + a_{2,23}y \\ a_{1,13}x + a_{2,13}y & a_{1,23}x + a_{2,23}y & a_{1,33}x + a_{2,33}y \end{pmatrix}.$$

To find implicit local equations for F_1 we first equate the images of the parameterizations and solve for s,t. This gives

$$s = \frac{bx-y}{b-a}$$
$$t = \frac{-ax+y}{b-a}$$

and consequently we find that there exists an open subset $U \subset F_1$ such that a point in $\mathrm{Gr}_2(\mathrm{Sym}^2 W)$ represented by the local coordinates $\{a_{i,jk}\}$ lies in U if and only if there are $a,b,c,d \in \mathbf{C}$ such that:

$$\begin{aligned}
a_{1,13} &= \frac{b(c+ad)-a(c+bd)}{b-a} = c \\
a_{1,22} &= \frac{a^2b-b^2a}{b-a} = -ab \\
a_{1,23} &= \frac{ab(c+ad)-ab(c+bd)}{b-a} = -abd \\
a_{1,33} &= \frac{(c+ad)^2b-(c+bd)^2a}{b-a} = c^2-abd^2 \\
a_{2,13} &= \frac{(c+bd)-(c+ad)}{b-a} = d \\
a_{2,22} &= \frac{b^2-a^2}{b-a} = a+b \\
a_{2,23} &= \frac{b(c+bd)-a(c+ad)}{b-a} = c+d(a+b) \\
a_{2,33} &= \frac{(c+bd)^2-(c+ad)^2}{b-a} = 2cd+d^2(a+b).
\end{aligned}$$

Note that all these equations extend to the case $a=b$. Since $F_1 \cong \mathbf{P}(W)^{[2]}$ is irreducible they will indeed provide us with equations for F_1. We can solve:

$$\begin{aligned}
ab &= = -a_{1,22} \\
a+b &= = a_{2,22} \\
c &= a_{1,13} \\
d &= a_{2,13}
\end{aligned}$$

And hence we can find the following local equations for F_1:

$$\begin{aligned}
a_{1,23} &= a_{1,22}a_{2,13} \\
a_{2,23} &= a_{1,13}+a_{2,13}a_{2,22} \\
a_{1,33} &= a_{1,13}^2+a_{1,22}a_{2,13}^2 \\
a_{2,33} &= 2a_{1,13}a_{2,13}+a_{2,22}a_{2,13}^2.
\end{aligned}$$

These are local equations for F_1 in $\mathrm{Gr}_2(\mathrm{Sym}^2 w)$. The lemma follows. □

We can also find local equations for F_2 in this coordinate system.

Lemma 3.2.6. *Let $P \in F_1 \cap F_2$ and let F_2^P be the germ of F_2 at P. It is defined by the equations*

$$\begin{aligned}
a_{1,22} &= -\frac{1}{4}a_{2,22}^2 \\
a_{1,23} &= \frac{1}{4}a_{2,22}(2a_{1,13} - 2a_{2,23} + a_{2,22}a_{2,13}) \\
a_{1,33} &= \frac{1}{4}(2a_{2,23} - a_{2,22}a_{2,13})(4a_{1,13} - 2a_{2,23} + a_{2,22}a_{2,13}) \\
a_{2,33} &= 2a_{2,23}a_{2,13} - a_{2,22}a_{2,13}^2
\end{aligned}$$

in the stalk $\mathcal{O}_{\mathrm{Gr}_2(\mathrm{Sym}^2 W),P}$.

Proof. We can locally parameterize F_2 (as subspace of $\mathrm{Gr}_2(\mathrm{Sym}^2 W)$ by the map

$$\begin{aligned}
\rho : \mathbf{C}^4 &\to \mathrm{Gr}_2(\mathrm{Sym}^2 W) \\
(a,b,c,d) &\mapsto \langle (e_1 + ae_2 + be_3)(e_1 + ce_2), (e_1 + ae_2 + be_3)(e_1 + ce_2) \rangle.
\end{aligned}$$

Hence the 2-plane in $\mathrm{Sym}^2 W$ that corresponds to $\rho(a,b,c,d)$ is parameterized by

$$(s,t) \mapsto \begin{pmatrix} s & \frac{1}{2}(as+t) & \frac{1}{2}((b+c)s+dt) \\ \frac{1}{2}(as+t)t & at & \frac{1}{2}(acs+(b+ad)t) \\ \frac{1}{2}((b+c)s+dt) & \frac{1}{2}(acs+(b+ad)t) & bcs+bdt \end{pmatrix},$$

where we used the basis of W to represent elements of $\mathrm{Sym}^2 W$ by a symmetric matrix. On the other hand, 2-planes near P are parameterized by

$$(x,y) \mapsto \begin{pmatrix} x & a_{1,12}x + a_{2,12}y & a_{1,13}x + a_{2,13}y \\ a_{1,12}x + a_{2,12}y & y & a_{1,23}x + a_{2,23}y \\ a_{1,13}x + a_{2,13}y & a_{1,23}x + a_{2,23}y & a_{1,33}x + a_{2,33}y \end{pmatrix}.$$

To find local equations for F_2 we equate these parameterizations. We find $s = x$, $t = 2y - ax$ and

$$\begin{aligned}
a_{1,13} &= \frac{1}{2}(b + c - ad) \\
a_{1,22} &= -a^2 \\
a_{1,23} &= \frac{a}{2}(c - b - ad) \\
a_{1,33} &= b(c - ad) \\
a_{2,13} &= d \\
a_{2,22} &= 2a \\
a_{2,23} &= b + ad \\
a_{2,33} &= 2bd.
\end{aligned}$$

Using the first, fifth, sixth and seventh equation we can solve for (a,b,c,d):

$$\begin{aligned}
a &= \frac{1}{2}a_{2,22} \\
b &= a_{2,23}-\frac{1}{2}a_{2,22}a_{2,13} \\
c &= 2a_{1,13}-a_{2,23}+a_{2,22}a_{2,13} \\
d &= a_{2,13}.
\end{aligned}$$

Substituting this back into the remaining equations eliminates a, b, c and d, and gives us local equations for F_2 in $\mathrm{Gr}_2(\mathrm{Sym}^2 W)$ □

Finally, we can identify local equations for $F(X_{\det})$ at P. For $\kappa^{-1}_{P,P^\perp}(F(X_{\det}))$ in $\mathrm{Hom}(P,P^\perp)$ the defining equation is

$$D\big(x(e_1^2+A(e_1^2))+y(e_1e_2+A(e_1e_2))\big)=0 \text{ for all } x,y\in\mathbf{C}$$

This is equivalent to the vanishing of the determinant of the symmetric matrix that represents $xA(e_1^2)+yA(e_1e_2)\in\mathrm{Sym}^2W$, that is:

$$\begin{vmatrix} x & y & a_{1,13}x+a_{2,13}y \\ y & a_{1,22}x+a_{2,22}y & a_{1,23}x+a_{2,23}y \\ a_{1,13}x+a_{2,13}y & a_{1,23}x+a_{2,23}y & a_{1,33}x+a_{2,33}y \end{vmatrix} = 0 \text{ for all } x,y\in C \qquad (3.2.1)$$

The left hand side is a homogeneous cubic polynomial in x and y. Vanishing for all x and y is equivalent to vanishing of all coefficients, so by expanding the determinant we easily obtain the following:

Lemma 3.2.7. *The germ of $F(X_{\det})$ at P is determined by the following equations in $\mathcal{O}_{\mathrm{Gr}_2(\mathrm{Sym}^2W),P}$:*

$$\begin{aligned}
a_{1,13}a_{1,22}-a_{1,23}^2-a_{1,13}^2a_{1,22}&=0\\
2a_{1,13}a_{1,23}-a_{1,13}a_{2,22}-2a_{1,23}a_{2,23}+a_{1,22}a_{2,33}&\\
-2a_{1,13}a_{1,22}a_{2,13}-a_{1,13}^2a_{2,22}&=0\\
-a_{1,13}+2a_{1,23}a_{2,13}+2a_{1,13}a_{2,23}-a_{2,23}^2+a_{2,22}a_{2,33}&\\
-a_{1,22}a_{2,13}^2-2a_{1,13}a_{2,13}a_{2,22}&=0\\
-a_{2,33}+2a_{2,13}a_{2,23}-a_{2,13}^2a_{2,22}&=0
\end{aligned}$$

Case 2: $P\in F_1\setminus F_2$**.** Let $P\subset\mathrm{Sym}^2W$ be a subspace of dimension two such that $P\in F_1\cap F_2$. Then we can find a basis $\{e_1,e_2,e_3\}$ of W such that $P=\langle e_1^2,e_2^2\rangle$ and the volume form $\mu=e_1\wedge e_2\wedge e_3$. Let $P^\perp:=\langle e_1e_2,e_1e_3,e_2e_3,e_3^2\rangle$. Using the basis of W we can introduce explicit coordinate functions on $U_{P,P^\perp}$: let $\{e_1^\vee,e_2^\vee,e_3^\vee\}\subset W^\vee$ be a dual basis, then define

$$a_{i,jk}:\mathrm{im}\,\kappa\to\mathbf{C}, x\mapsto(e_j^\vee e_k^\vee)(\kappa^{-1}x)(e_1e_i).$$

The set

$$\{a_{i,jk} \mid i = 1,2, jk = 12,13,23,33\}$$

is a coordinate system on $U_{P,P^\perp}$. Again we will denote the germs at P of the functions $a_{i,jk}$ by the same symbols. We then have the following result:

Lemma 3.2.8. *Let $P \in F_1 \setminus F_2$ and F_1^P the germ of F_1 at P. It is defined by the following equations:*

$$\begin{aligned}
a_{1,23} &= \frac{a_{1,12}a_{1,13} - a_{1,12}^2 a_{2,23}}{1 - a_{1,12}a_{2,12}}\\
a_{2,13} &= \frac{a_{2,12}a_{2,23} - a_{2,12}^2 a_{1,13}}{1 - a_{1,12}a_{2,12}}\\
a_{1,33} &= \frac{a_{1,13}^2 - a_{1,12}^2 a_{2,23}^2 - 2a_{1,12}a_{2,12}a_{1,13}^2 + 2a_{1,12}^2 a_{2,12}a_{1,13}a_{2,23}}{(1 - a_{1,12}a_{2,12})^2}\\
a_{2,33} &= \frac{a_{2,23}^2 - a_{2,12}^2 a_{1,13}^2 - 2a_{1,12}a_{2,12}a_{2,23}^2 + 2a_{1,12}a_{2,12}^2 a_{1,13}a_{2,23}}{(1 - a_{1,12}a_{2,12})^2}
\end{aligned}$$

in $\mathcal{O}_{\mathrm{Gr}_2(\mathrm{Sym}^2 W),P}$.

Proof. For $(a,b,c,d) \in \mathbf{C}^4$, set $w_1 := e_1 + ae_2 + (c+ad)e_3$ and $w_2 := e_2 + be_1 + (bc+d)e_3$. Just as in case 1, a,b parameterize deformations of $[w_1]$ and $[w_2]$ respectively within $\langle [w_1],[w_2]\rangle \subset \mathbf{P}(W)$ and c,d parameterize deformations of the line $\langle [w_1],[w_2]\rangle$ with respect to $\langle [e_1],[e_2]\rangle$. For (a,b,c,d) close to 0 we have that w_1^2 and w_2^2 span a two-plane in $\mathrm{Sym}^2 W$. This plane is parameterized by the following map

$$(s,t) \mapsto \begin{pmatrix} s+b^2t & as+bt & (c+ad)s+b(bc+d)t \\ as+bt & a^2s+t & a(c+ad)s+(bc+d)t \\ (c+ad)s+b(bc+d)t & a(c+ad)s+(bc+d)t & (c+ad)^2s+(bc+d)^2t \end{pmatrix},$$

where we used the basis of W to represent elements of $\mathrm{Sym}^2 W$ by a symmetric matrix. By construction this plane represents an element of F_1. On the other hand, in the local coordinate system which we have chosen for $\mathrm{Gr}_2(\mathrm{Sym}^2 W)$ the planes are parameterized by:

$$(x,y) \mapsto \begin{pmatrix} x & a_{1,12}x + a_{2,12}y & a_{1,13}x + a_{2,13}y \\ a_{1,12}x + a_{2,12}y & y & a_{1,23}x + a_{2,23}y \\ a_{1,13}x + a_{2,13}y & a_{1,23}x + a_{2,23}y & a_{1,33}x + a_{2,33}y \end{pmatrix}.$$

To find implicit local equations for F_1 we first equate the images of the parameterizations and solve for s,t. This gives

$$s = \frac{x - b^2 y}{1 - a^2b^2}$$
$$t = \frac{-a^2x + y}{1 - a^2b^2}$$

and consequently we find that there exists an open subset $U \subset F_1$ such that a plane represented by the local coordinates $\{a_{i,jk}\}$ lies in U if and only if there are $a, b, c, d \in \mathbf{C}$ such that:

$$\begin{aligned}
a_{1,12} &= \frac{a - a^2 b}{1 - a^2 b^2} = \frac{a}{1+ab} \\
a_{1,13} &= \frac{(c+ad) - a^2 b(bc+d)}{1 - a^2 b^2} = c + \frac{a}{1+ab} d \\
a_{1,23} &= \frac{a(c+ad) - a^2 (bc+d)}{1 - a^2 b^2} = \frac{a}{1+ab} c \\
a_{1,33} &= \frac{(c+ad)^2 - a^2 (bc+d)^2}{1 - a^2 b^2} = c^2 + 2\frac{a}{1+ab} cd \\
a_{2,12} &= \frac{-b^2 a + b}{1 - a^2 b^2} = \frac{b}{1+ab} \\
a_{2,13} &= \frac{-b^2 (c+ad) + b(bc+d)}{1 - a^2 b^2} = \frac{b}{1+ab} d \\
a_{2,23} &= \frac{-b^2 a(c+ad) + (bc+d)}{1 - a^2 b^2} = d + \frac{b}{1+ab} c \\
a_{2,33} &= \frac{-b^2 (c+ad)^2 + (bc+d)^2}{1 - a^2 b^2} = d^2 + 2\frac{b}{1+ab} cd.
\end{aligned}$$

We can solve:

$$\begin{aligned}
\frac{a}{1+ab} &= a_{1,12} \\
\frac{b}{1+ab} &= a_{2,12} \\
c &= \frac{a_{1,13} - a_{1,12} a_{2,23}}{1 - a_{1,12} a_{2,12}} \\
d &= \frac{a_{2,23} - a_{2,12} a_{1,13}}{1 - a_{1,12} a_{2,12}}
\end{aligned}$$

And hence we can find the following local equations for F_1:

$$\begin{aligned}
a_{1,23} &= \frac{a_{1,12} a_{1,13} - a_{1,12}^2 a_{2,23}}{1 - a_{1,12} a_{2,12}} \\
a_{2,13} &= \frac{a_{2,12} a_{2,23} - a_{2,12}^2 a_{1,13}}{1 - a_{1,12} a_{2,12}} \\
a_{1,33} &= \frac{a_{1,13}^2 - a_{1,12}^2 a_{2,23}^2 - 2a_{1,12} a_{2,12} a_{1,13}^2 + 2a_{1,12}^2 a_{2,12} a_{1,13} a_{2,23}}{(1 - a_{1,12} a_{2,12})^2} \\
a_{2,33} &= \frac{a_{2,23}^2 - a_{2,12}^2 a_{1,13}^2 - 2a_{1,12} a_{2,12} a_{2,23}^2 + 2a_{1,12} a_{2,12}^2 a_{1,13} a_{2,23}}{(1 - a_{1,12} a_{2,12})^2}
\end{aligned}$$

These are local equations for F_1 in $\mathrm{Gr}_2(\mathrm{Sym}^2 w)$. The lemma follows by taking the germs in $\mathcal{O}_{\mathrm{Gr}_2(\mathrm{Sym}^2 W),P}$ of the equations. □

We identify local equations for $F(X_{\mathrm{det}})$ at P. In this case the defining equation is

$$\begin{vmatrix} x & a_{1,12}x+a_{2,12}y & a_{1,13}x+a_{2,13}y \\ a_{1,12}x+a_{2,12}y & y & a_{1,23}x+a_{2,23}y \\ a_{1,13}x+a_{2,13}y & a_{1,23}x+a_{2,23}y & a_{1,33}x+a_{2,33}y \end{vmatrix} = 0 \text{ for all } x,y\in\mathbf{C} \tag{3.2.2}$$

Again the left hand side is a homogeneous cubic polynomial in x and y. Vanishing for all x and y is equivalent to vanishing of all coefficients, so by expanding the determinant we easily obtain the following:

Lemma 3.2.9. *The germ $F(X_{\mathrm{det}})^P$ is determined by the following equations in $\mathcal{O}_{\mathrm{Gr}_2(\mathrm{Sym}^2 W),P}$:*

$$\begin{aligned} -a_{1,23}^2+2a_{1,12}a_{1,13}a_{1,23}-a_{1,12}^2a_{1,33}&=0\\ a_{1,33}-2a_{1,23}a_{2,23}-a_{1,13}^2+2a_{1,13}a_{1,23}a_{2,12}-2a_{1,12}a_{1,33}a_{2,12}&\\ +2a_{1,12}a_{1,23}a_{2,13}+2a_{1,12}a_{1,13}a_{2,23}-a_{1,12}^2a_{2,33}&=0\\ a_{2,33}-a_{2,23}^2-2a_{1,13}a_{2,13}-a_{1,33}a_{2,12}^2+2a_{1,23}a_{2,12}a_{2,13}&\\ +2a_{1,13}a_{2,12}a_{2,23}+2a_{1,12}a_{2,13}a_{2,23}-2a_{1,12}a_{2,12}a_{2,33}&=0\\ -a_{2,13}^2+2a_{2,12}a_{2,13}a_{2,23}-a_{2,12}^2a_{2,33}&=0 \end{aligned}$$

Remark 3.2.10. We find here in particular that $F(X_{\mathrm{det}})$ is not reduced along F_1; the tangent cone of $F(X_{\mathrm{det}})$ at any point in $F_1\setminus F_2$ is the complete intersection of two linear and two quadratic polynomials. Hence the multiplicity of $F(X_{\mathrm{det}})$ along F_1 is 4.

Case 3: $P\in F_2\setminus F_1$**.** We will only need local equations for $F(X_{\mathrm{det}})$ along $F_2\setminus F_1$. Let $P\subset \mathrm{Sym}^2 W$ be a subspace of dimension two such that $P\in F_2\cap F_1$. Then we can find a basis $\{e_1,e_2,e_3\}$ of W such that $P=\langle e_1e_3,e_2e_3\rangle$ and the volume form $\mu=e_1\wedge e_2\wedge e_3$. Let $P^\perp:=\langle e_1^2,e_2^2,e_1e_2,e_3^2\rangle$. Using the basis of W we can introduce explicit coordinate functions on $U_{P,P^\perp}$: let $\{e_1^\vee,e_2^\vee,e_3^\vee\}\subset W^\vee$ be a dual basis, then define

$$a_{i,jk}:\mathrm{im}\,\kappa\to\mathbf{C},\, x\mapsto (e_j^\vee e_k^\vee)(\kappa^{-1}x)(e_ie_3).$$

The set

$$\{a_{i,jk}\mid i=1,2,\, jk=11,22,12,33\}$$

is a coordinate system on $U_{P,P^\perp}$. The local equations for $F(X_{\mathrm{det}})$ at P are summarized by:

$$\begin{vmatrix} a_{1,11}x+a_{2,11}y & a_{1,12}x+a_{2,12}y & x \\ a_{1,12}x+a_{2,12}y & a_{1,22}x+a_{2,22}y & y \\ x & y & a_{1,33}x+a_{2,33}y \end{vmatrix} = 0 \text{ for all } x,y\in\mathbf{C} \tag{3.2.3}$$

Expanding this equation gives us:

Lemma 3.2.11. *The germ $F(X_{\det})^P$ is given by the following equations:*

$$a_{1,22}(-1+a_{1,11}a_{1,33})-a_{1,12}a_{1,33}^2=0$$
$$a_{1,12}(2-2a_{1,33}a_{2,12})+a_{2,22}(-1+a_{1,11}a_{1,33})$$
$$-a_{1,12}^2a_{2,33}+a_{1,22}(a_{1,33}a_{2,11}+a_{1,11}a_{2,33})=0$$
$$a_{1,11}(-1+a_{2,22}a_{2,33})+a_{2,12}(2-a_{1,12}a_{2,33})$$
$$-a_{2,12}^2a_{1,33}+a_{2,11}(a_{1,33}a_{2,22}+a_{1,22}a_{2,33})=0$$
$$a_{2,11}(-1+a_{2,22}a_{2,33})-a_{2,12}^2a_{2,33}=0$$

In particular $F(X_{\det})$ is a smooth subscheme of $\mathrm{Gr}_2(\mathrm{Sym}_2 W)$ *at $P \in F_2 \setminus F_1$.*

Proof. The equations follow from expansion. Then differentials of the polynomials are clearly linearly independent. Hence $F(X_{\det})$ is smooth at every $P \in F_2 \setminus F_1$. □

Remark 3.2.12. Since $F(X_{\det})$ is smooth at $P \in F_2 \setminus F_1$ and $F(X_{\det}) = F_1 \cup F_2$ set-theoretically, it follows that the equations of $F(X_{\det})$ at such P are automatically local equations for F_2.

3.2.3 Local equations for $\mathcal{F}$

From now on we will view F_1, F_2 as subspaces of the central fiber of $\mathcal{F}$. In particular $F_1, F_2 \subset \mathrm{Gr}_2(\mathrm{Sym}^2 W) \times \{0\} \subset \mathrm{Gr}_2(\mathrm{Sym}^2 W) \times \Delta$. To describe the effect on $\mathcal{F}$ of the blow-up and blow-down map that we mentioned in section 3.1 we have to find local equations of $\mathcal{F}$ near points in $F_1 \setminus F_2$ and $F_1 \cap F_2$.

Let $P \subset \mathrm{Sym}^2 W$ be a 2-plane and $P^\perp$ a complementary 4-plane. Let $\kappa_{P,P^\perp}$ as introduced in section 3.2.2. Then the defining equation of

$$(\kappa_{P,P^\perp} \times \mathrm{id}_\Delta)^{-1}\mathcal{F} \subset \mathrm{Hom}(P, P^\perp) \times \Delta$$

is given by

$$(D + t^2 f)|_{(I+A)P} = 0 \tag{3.2.4}$$

where $(A, t) \in \mathrm{Hom}(P, P^\perp) \times \Delta$ and $I : P \to \mathrm{Sym}^2(W)$ denotes the embedding. We will make this equation more explicit by expressing it in local coordinates. We will split the calculation in 3 cases.

Case 1: points in $F_1 \cap F_2$. Let $(P, 0) \in F_1 \cap F_2$. We will re-express equation (3.2.4) in local coordinates at $(P, 0)$. Pick a basis $\{e_1, e_2, e_3\}$ of W such that $P = \langle e_1^2, e_1 e_2\rangle$ and the volume form $\mu = e_1 \wedge e_2 \wedge e_3$. Let $P^\perp := \langle e_2^2, e_1e_3, e_2e_3, e_3^2\rangle$. Then $U_{P,P^\perp} \times \Delta$ defines an open neighborhood of $(P, 0)$. The functions:

$$a_{i,jk} : U_{P,P^\perp} \to \mathbf{C}, x \mapsto (e_j^\vee e_k^\vee)(\kappa_{P,P^\perp}^{-1} x)(e_1 e_i)$$

that we introduced in section 3.2.2 (Case 1) trivially lift to functions on $U_{P,P^\perp} \times \Delta$, which, by abuse of notation, we will denote by the same symbols. Let t be the coordinate on Δ; it lifts to a coordinate function on $U_{P,P^\perp} \times \Delta$, which again we denote by the same symbol. Finally, we will use the shorthand notation $f_{ij,kl,mn} := f(e_i e_j, e_k e_l, e_m e_n)$.

Equation (3.2.4) may be rewritten as

$$D\big(x_1(e_1^2 + A(e_1^2)) + x_2(e_1e_2 + A(e_1e_2))\big) = -t^2 f\big(x_1(e_1^2 + A(e_1^2)) + x_2(e_1e_2 + A(e_1e_2))\big)$$

for all $x_1, x_2 \in \mathbf{C}$. The left hand side was already expressed in terms of $a_{i,jk}$ in lemma 3.2.7. We can straightforwardly expand the right hand side in terms of $a_{i,jk}$, t and $f_{ij,kl,mn}$; this gives rise to a large expression. We may express it somewhat cryptically as follows:

$$-t^2 \sum_{i_1,i_2,i_3=1}^{2} \prod_{n=1}^{3} x_{i_n}(f_{1i_n} + \sum_{jk} a_{i_n,jk} f_{jk}) \tag{3.2.5}$$

with the convention that $f_{ij}f_{kl}f_{mn} := f_{ij,kl,mn}$ and where jk runs over 22, 13, 23 and 33. In any case, we have the following result:

Lemma 3.2.13. *Denote $\mathcal{G} = \mathrm{Gr}_2(\mathrm{Sym}^2 W) \times U'_H$ and let $(P,0) \in F_1 \cap F_2 \subset \mathcal{G}$. Then local equations for $\mathcal{F}$ in the stalk $\mathcal{O}_{\mathcal{G},(P,0)}$ are given by:*

$$\begin{aligned}
a_{1,13}a_{1,22} - a_{1,23}^2 - a_{1,13}^2 a_{1,22} + t^2(f_{11,11,11} + r_0) &= 0\\
2a_{1,13}a_{1,23} - a_{1,13}a_{2,22} - 2a_{1,23}a_{2,23} + a_{1,22}a_{2,33}&\\
-2a_{1,13}a_{1,22}a_{2,13} - a_{1,13}^2 a_{2,22} + t^2(3f_{11,11,12} + r_1) &= 0\\
-a_{1,13} + 2a_{1,23}a_{2,13} + 2a_{1,13}a_{2,23} - a_{2,23}^2&\\
+a_{2,22}a_{2,33} - a_{1,22}a_{2,13}^2 - 2a_{1,13}a_{2,13}a_{2,22} + t^2(3f_{11,12,12} + r_2) &= 0\\
-a_{2,33} + 2a_{2,13}a_{2,23} - a_{2,13}^2 a_{2,22} + t^2(f_{12,12,12} + r_3) &= 0
\end{aligned}$$

where the r_i are contained in the maximal ideal of $\mathcal{O}_{\mathcal{G},(P,0)}$

Proof. Equation (3.2.5) may be rewritten as

$$-t^2\big((f_{11,11,11} + r_0)x_1^3 + 3(f_{11,11,12} + r_1)x_1^2 x_2 + 3(f_{11,12,12} + r_2)x_1 x_2^2 + (f_{12,12,12} + r_3)x_2^3\big)$$

where the r_i are contained in the ideal generated by the $a_{i,jk}$ (for $i = 1,2$ and $jk = 22, 13, 23, 33$). In particular, their germs (denoted by the same symbols) are contained in the maximal ideal of $\mathcal{O}_{\mathcal{G},(P,0)}$. The lemma now follows from lemma 3.2.7. □

Case 2: points in $F_1 \setminus F_2$**.** Now let $(P,0) \in F_1 \setminus F_2$. Again we will re-express equation (3.2.4) in local coordinates at $(P,0)$. Pick a basis $\{e_1, e_2, e_3\}$ of W such that $P = \langle e_1^2, e_2^2 \rangle$ and the volume form $\mu = e_1 \wedge e_2 \wedge e_3$. Let $P^\perp := \langle e_1e_2, e_1e_3, e_2e_3, e_3^2 \rangle$. Then $U_{P,P^\perp} \times \Delta$ defines an open neighborhood of $(P,0)$. Let t and $f_{ij,kl,mn}$ be as in case 1. Furthermore, note that the functions

$$a_{i,jk} : U_{P,P^\perp} \to \mathbf{C}, x \mapsto (e_j^\vee, e_k^\vee)(\kappa_{P,P^\perp}^{-1} x)(e_i^2)$$

that we introduced in section 3.2.2 trivially lift to functions on $U_{P,P^\perp} \times \Delta$. Again we will denote those by the same symbols.

Equation (3.2.4) may be rewritten as

$$D\big(x_1(e_1^2 + A(e_1^2)) + x_2(e_2^2 + A(e_2^2))\big) = -t^2 f\big(x_1(e_1^2 + A(e_1^2)) + x_2(e_2^2 + A(e_2^2))\big)$$

for all $x_1, x_2 \in \mathbf{C}$.
The left hand side was already expressed in terms of $a_{i,jk}$ in lemma 3.2.9. We may express the righthand side somewhat cryptically as follows:

$$-t^2 \sum_{i_1,i_2,i_3=1}^{2} \prod_{n=1}^{3} x_{i_n}(f_{i_n i_n} + \sum_{jk} a_{i_n,jk} f_{jk}) \tag{3.2.6}$$

with the convention that $f_{ij} f_{kl} f_{mn} := f_{ij,kl,mn}$ and where jk runs over 12, 13, 23 and 33. In any case, we have the following result:

Lemma 3.2.14. *Denote $\mathcal{G} = \mathrm{Gr}_2(\mathrm{Sym}^2 W) \times U'_H$ and let $(P,0) \in F_1 \setminus F_2 \subset \mathcal{G}$. Then local equations for $\mathcal{F}$ in the stalk $\mathcal{O}_{\mathcal{G},(P,0)}$ are given by:*

$$
\begin{aligned}
-a_{1,23}^2 + 2a_{1,12}a_{1,13}a_{1,23} - a_{1,12}^2 a_{1,33} + t^2(f_{11,11,11} + r_0) &= 0\\
a_{1,33} - 2a_{1,23}a_{2,23} - a_{1,13}^2 + 2a_{1,13}a_{1,23}a_{2,12} - 2a_{1,12}a_{1,33}a_{2,12} + 2a_{1,12}a_{1,23}a_{2,13}&\\
+2a_{1,12}a_{1,13}a_{2,23} - a_{1,12}^2 a_{2,33} + t^2(3f_{11,11,22} + r_1) &= 0\\
a_{2,33} - a_{2,23}^2 - 2a_{1,13}a_{2,13} - a_{1,33}a_{2,12}^2 + 2a_{1,23}a_{2,12}a_{2,13} + 2a_{1,13}a_{2,12}a_{2,23}&\\
+2a_{1,12}a_{2,13}a_{2,23} - 2a_{1,12}a_{2,12}a_{2,33} + t^2(3f_{11,22,22} + r_2) &= 0\\
-a_{2,13}^2 + 2a_{2,12}a_{2,13}a_{2,23} - a_{2,12}^2 a_{2,33} + t^2(f_{22,22,22} + r_3) &= 0
\end{aligned}
$$

where the r_i are contained in the maximal ideal of $\mathcal{O}_{\mathcal{G},(P,0)}$

Proof. Exactly analogous to Case 1. □

Case 3: points in $F_2 \setminus F_1$. Finally, let $(P,0) \in F_2 \setminus F_1$. In principle we can find local equations of $\mathcal{F}$ at $(P,0)$ in a way completely analogous to the previous two cases, using the explicit equations of $F(X_{\mathrm{det}})$ at $P \in F_2 \setminus F_1$ as given by lemma 3.2.11. However, we will only use the following result.

Lemma 3.2.15. *The space $\mathcal{F}$ is smooth along $F_2 \setminus F_1$.*

Proof. Let $(P,0) \in F_2 \setminus F_1$. Then modulo t^2 the local equations of $\mathcal{F}$ at $(P,0)$ are the same as the local equations of $F(X_{\mathrm{det}})$ at P as given by lemma 3.2.11, if we interpret the coordinate functions $a_{i,jk}$ as functions on $\mathcal{G}$. It follows that the differentials of the defining polynomials are linearly independent. Hence $\mathcal{F}$ is smooth at $(P,0)$. □

3.3 Blow-up

In the previous section we defined the relative variety of lines on $\xi'_f : \mathcal{X}_f \to \Delta$, which we denoted by $\varphi : \mathcal{F} \to \Delta$, and found that set-theoretically $\mathcal{X}_0 = F_1 \cup F_2$. Let

$$\varepsilon_{F_1} : \widehat{\mathcal{F}} \to \mathcal{F}$$

be the blow-up of $\mathcal{F}$ along F_1 and denote $\widehat{\varphi} := \varphi \circ \varepsilon_{F_1}$. In this section we present local equations for the exceptional locus and use these to derive some general properties of $\widehat{\mathcal{F}}$.

3.3.1 Local equations of the exceptional locus

We refer to appendix A for definitions, conventions and notation regarding the blow-up of a subscheme in a scheme. As before we denote $\mathcal{G} := \mathrm{Gr}_2(\mathrm{Sym}^2 W) \times \Delta$. First observe that since $F_1 \subset \mathcal{F} \subset \mathcal{G}$ the variety $\mathrm{Bl}(F_1, \mathcal{F})$ is naturally identified with the strict transform of $\mathcal{F}$ in $\mathrm{Bl}(F_1, \mathcal{G})$; therefore the notation $\widehat{\mathcal{F}} := \mathrm{Bl}(F_1, \mathcal{F})$ causes no confusion. Furthermore the exceptional divisor $E := E(F_1, \mathcal{F}) \subset \widehat{\mathcal{F}}$ is precisely the intersection of $\widehat{\mathcal{F}}$ and $E(F_1, \mathcal{G})$.

It follows from propositions 3.2.5 and 3.2.8 that F_1 is smooth in $\mathrm{Gr}_2(\mathrm{Sym}^2 W)$, hence F_1 is smooth in $\mathcal{G}$. In particular the normal cone (see remark A.2.5) of $\nu_{F_1/\mathcal{G}}$ of F_1 in $\mathcal{G}$ is a vector bundle over F_1 (of rank 5) and $E(F_1, \mathcal{G}) = \mathbf{P}(\nu_{F_1/\mathcal{G}})$. The aim of this section is to locally describe E as subvariety of $\mathbf{P}(\nu_{F_1/\mathcal{G}})$.

Denote by $\mathcal{I}$ and $\mathcal{J}$ the vanishing ideals on $\mathcal{G}$ of F_1 and $\mathcal{F}$ respectively. Let $p := (P, 0) \in F_1$ be arbitrary. It suffices to describe the blow-up at stalk level. That is, we wish to

1. find a set $\{\mathbf{x}, \mathbf{y}\}$ (where $\mathbf{x} = \{x_0, \ldots\}$ and $\mathbf{y} = \{y_0, \ldots\}$) of generators of the maximal ideal $\mathfrak{m}_p \subset \mathcal{O}_{\mathcal{G},p}$ such that $\mathcal{I}_p = \langle \mathbf{x} \rangle$. Equivalently, find local coordinates on $\mathcal{G}$ at p in which F_1 is linear; this is possible since F_1 is smooth in $\mathcal{G}$.

2. in these coordinates, find an $\mathcal{I}_p$-standard base $\{g_1, \ldots, g_k\}$ for $\mathcal{J}_p$ (see the appendix for definition)

For any subvariety $Y \subset \mathcal{G}$ that contains p we will denote by Y^p its germ at p. Then given the data above we have by definition (see section A.1 in the appendix for details)

$$\mathrm{Bl}(F_1^p, \mathcal{G}^p) = \mathrm{Proj}\left(\bigoplus_{n \geq 0} \mathcal{I}_p^n s^n\right).$$

The exceptional locus is given by

$$E(F_1^p, \mathcal{G}^p) = \mathrm{Proj}\left(\bigoplus_{n \geq 0} \mathcal{I}_p^n / \mathcal{I}_p^{n+1}\right) = \mathrm{Proj}\, \mathcal{O}_{F_1,p}[\mathbf{x}].$$

Observe that we may naturally view the germs y_i as elements of $\mathcal{O}_{F_1,p}$, and as such they generate the maximal ideal of $\mathcal{O}_{F_1,p}$. We also see here explicitly that

$E(F_1^p, \mathcal{G}^p)$ is a bundle of projective spaces over F_1^p.
The intersection of $E(F_1^p, \mathcal{G}^p)$ with $\widehat{\mathcal{F}}$ is described by the ideal

$$\bigoplus_{n\geq 0}(\mathcal{I}_p^n \cap \mathcal{J}_p + \mathcal{I}_p^{n+1})/\mathcal{I}_p^{n+1} =: \mathrm{In}_{\mathcal{I}_p}(\mathcal{J}_p) = \mathrm{In}_{\mathcal{I}_p}\langle g_1, \dots, g_k\rangle.$$

see remark A.1.4. If $\{g_1, ..., g_k\}$ is a standard base, then by definition we have

$$\mathrm{In}_{\mathcal{I}_p}\langle g_1, \dots, g_k\rangle = \langle \mathrm{In}_{\mathcal{I}_p} g_1, \dots, \mathrm{In}_{\mathcal{I}_p} g_k\rangle,$$

which then gives us an explicit description the ideal of $\widehat{\mathcal{F}}$ in $E(F_1^p, \mathcal{G}^p)$.

The main objective is now to find local coordinates in $\mathcal{G}$ in which F_1 linearizes and to find an $\mathcal{I}_p$-standard basis for $\mathcal{J}_p$, for every $p \in F_1$. Once again we will split the calculations into two cases: $p \in F_1 \cap F_2$ and $p \in F_1 \setminus F_2$.

Case 1: $p \in F_1 \cap F_2$**.** Assume that $p = (P, 0) \in F_1 \cap F_2$. Let $\{e_1, e_2, e_3\}$ be a basis such that P is spanned by e_1^2 and $e_1 e_2$, and let $P^\perp$ be the plane spanned by the remaining degree 2 monomials in e_1, e_2 and e_3.Let $a_{i,jk}$, $f_{ij,kl,mn}$ and t be as in section 3.2.3 Case 1. We have the following result:

Lemma 3.3.1. *Let F_1^p be the germ of F_1 at p. It is defined by equations*

$$\begin{aligned} a_{1,23} &= a_{1,22}a_{2,13} \\ a_{2,23} &= a_{1,13} + a_{2,22}a_{2,13} \\ a_{1,33} &= a_{1,13}^2 + a_{1,22}a_{2,13}^2 \\ a_{2,33} &= 2a_{1,13}a_{2,13} + a_{2,22}a_{2,13}^2 \\ t &= 0 \end{aligned}$$

in $\mathcal{O}_{\mathcal{G},p}$.

Proof. This follows directly from the definition of F_1 and lemma 3.2.5. □

Define:

$$\begin{aligned} b_{1,23} &= a_{1,23} - a_{1,22}a_{2,13} \\ b_{2,23} &= a_{2,23} - a_{1,13} - a_{2,13}a_{2,22} \\ b_{1,33} &= a_{1,33} - a_{1,13}^2 - a_{1,22}a_{2,13}^2 \\ b_{2,33} &= a_{2,33} - 2a_{1,13}a_{2,13} - a_{2,22}a_{2,13}^2. \end{aligned} \tag{3.3.1}$$

Then

$$\mathcal{I}_p = \langle b_{1,23}, b_{2,23}, b_{1,33}, b_{2,33}, t\rangle \subset \mathcal{O}_{\mathcal{G},p}.$$

Hence we have in this case:

$$G(\mathcal{I}_p, \mathcal{O}_{\mathcal{G},p}) := \bigoplus_{n\geq 0}\mathcal{I}_p^n/\mathcal{I}_p^{n+1} = \mathcal{O}_{F_1,p}[b_{1,23}, b_{2,23}, b_{1,33}, b_{2,33}, t]. \tag{3.3.2}$$

We already found equations of $\mathcal{F}$ at p in section 3.2.3 Case 1. If we substitute the coordinate transformation 3.3.1 we can find four generators of $\mathcal{J}_p$. We give them in compressed form:

$$\begin{aligned}
g_0 &= -2a_{1,22}a_{2,13}b_{1,23} + a_{1,22}b_{1,33} - b_{1,23}^2 + t^2\big(f_{11,11,11} + r_0'\big) + \text{h.d.t} \\
g_1 &= -2a_{2,13}a_{2,22}b_{1,23} + a_{2,22}b_{1,33} - 2a_{1,22}a_{2,13}b_{2,23} + a_{1,22}b_{2,33} - 2b_{1,23}b_{2,23} \\
&\quad +3t^2\big(f_{11,11,12} + r_1'\big) + \text{h.d.t} \\
g_2 &= -b_{1,33} + 2a_{2,13}b_{1,23} - 2a_{2,13}a_{2,22}b_{2,23} + a_{2,22}b_{2,33} - b_{2,23}^2 \\
&\quad +3t^2\big(f_{11,12,12} + r_2'\big) + \text{h.d.t} \\
g_3 &= -b_{2,33} + 2a_{2,13}b_{2,23} + t^2\big(f_{12,12,12} + r_3'\big) + \text{h.d.t},
\end{aligned}$$

where "h.d.t." denotes terms of higher $\mathcal{I}_p$-degree and the r_i' do not depend on $b_{1,23}$, $b_{2,23}$, $b_{1,33}$, $b_{2,33}$ or t.

This set $\{g_i\}_{k=0}^3$ forms a basis of $\mathcal{J}_p$, but this basis does not yet have the property that the parts of lowest $\mathcal{I}_p$-degree form a regular sequence, hence this may not yet be a standard base (see theorem A.1.7). However, by a simple elimination process we can find the following set of generators:

$$\begin{aligned}
h_0 &:= g_0 + a_{1,22}g_2 + a_{1,22}a_{2,22}g_3 = -b_{1,23}^2 - a_{1,22}b_{2,23}^2 + t^2\big(f_{11,11,11} + r_0\big) + \text{h.d.t} \\
h_1 &:= g_1 + a_{2,22}g_2 + (a_{1,22} + a_{2,22}^2)g_3 \\
&\quad = -2b_{1,23}b_{2,23} - a_{2,22}b_{2,23}^2 + 3t^2\big(f_{11,11,12} + r_1\big) + \text{h.d.t} \\
h_2 &:= g_2 + a_{2,22}g_3 = -b_{1,33} + 2a_{2,13}b_{1,23} + \text{h.d.t} \\
h_3 &:= g_3 = -b_{2,33} + 2a_{2,13}b_{2,23} + \text{h.d.t.}
\end{aligned}$$

We then have the following result:

Proposition 3.3.2. *The set $\{h_0, h_1, h_2, h_3\}$ forms a standard base of J_p. That is,*

$$\mathrm{In}_{\mathcal{I}_p} J_p = \langle \mathrm{In}_{\mathcal{I}_p}(h_0), \mathrm{In}_{\mathcal{I}_p}(h_1), \mathrm{In}_{\mathcal{I}_p}(h_2), \mathrm{In}_{\mathcal{I}_p}(h_3)\rangle.$$

Proof. It suffices to show, by theorem A.1.7, that the sequence

$$(\mathrm{In}_{\mathcal{I}_p}(h_0), \mathrm{In}_{\mathcal{I}_p}(h_1), \mathrm{In}_{\mathcal{I}_p}(h_2), \mathrm{In}_{\mathcal{I}_p}(h_3)),$$

or a permutation of it, is regular in the ring

$$G(\mathcal{I}_p, \mathcal{O}_{\mathcal{G},p}) = \bigoplus_{n\geq 0} \mathcal{I}_p^n/\mathcal{I}_p^{n+1} = \mathcal{O}_{F_1,p}[b_{1,23}, b_{2,23}, b_{1,33}, b_{2,33}, t].$$

We have

$$\mathrm{In}_{\mathcal{I}_p}(h_0) = -b_{1,23}^2 - a_{1,22}b_{2,23}^2 + t^2(f_{11,11,11} + r_0') \tag{3.3.3}$$

$$\mathrm{In}_{\mathcal{I}_p}(h_1) = -2b_{1,23}b_{2,23} - a_{2,22}b_{2,23}^2 + 3t^2(f_{11,11,12} + r_1') \tag{3.3.4}$$

$$\mathrm{In}_{\mathcal{I}_p}(h_2) = -b_{1,33} + 2a_{2,13}b_{1,23} \tag{3.3.5}$$

$$\mathrm{In}_{\mathcal{I}_p}(h_3) = -b_{2,33} + 2a_{2,13}b_{2,23} \tag{3.3.6}$$

where we now interpret the $a_{i,jk}$ and r'_i as elements of $\mathcal{O}_{F_1,p}$. Note that in that case the $a_{i,jk}$ and r'_i are contained in the maximal ideal of $\mathcal{O}_{F_1,p}$. For convenience denote:

$$S_{(i_1,\dots,i_n)} := G(\mathcal{I}_p, \mathcal{O}_{\mathcal{G},p})/\langle \mathrm{In}_{\mathcal{I}_p}(h_{i_1}),\dots,\mathrm{In}_{\mathcal{I}_p}(h_{i_n})\rangle.$$

Then clearly:

$$\begin{aligned} S_{(2)} &\cong G(\mathcal{I}_p, \mathcal{O}_{\mathcal{G},p})/\langle b_{1,33}\rangle \\ S_{(2,3)} &\cong G(\mathcal{I}_p, \mathcal{O}_{\mathcal{G},p})/\langle b_{1,33}, b_{2,33}\rangle. \end{aligned}$$

These rings are integral domains, so in particular the image of $\mathrm{In}_{\mathcal{I}_p}(h_0)$ in $S_{(2,3)}$ is not a zero-divisor. Furthermore, the image of $\mathrm{In}_{\mathcal{I}_p}(h_0)$ is prime in $S_{(2,3)}$; if it would factor, the image of $a_{1,22}$ in $S_{(2,3)}$ would have to be a square, which it is not. Hence the ring $S_{(2,3,0)}$ is an integral domain, so in particular the image of $\mathrm{In}_{\mathcal{I}_p}(h_0)$ in that ring is not a zero-divisor. It is neither a unit, and from this it follows that the sequence

$$(\mathrm{In}_{\mathcal{I}_p}(h_2), \mathrm{In}_{\mathcal{I}_p}(h_3), \mathrm{In}_{\mathcal{I}_p}(h_0), \mathrm{In}_{\mathcal{I}_p}(h_1))$$

is regular. So by theorem A.1.7 $\{h_0, h_1, h_2, h_3\}$ is a standard base for J_p. That is,

$$\mathrm{In}_{\mathcal{I}_p} J_p = \langle \mathrm{In}_{\mathcal{I}_p}(h_0), \mathrm{In}_{\mathcal{I}_p}(h_1), \mathrm{In}_{\mathcal{I}_p}(h_2), \mathrm{In}_{\mathcal{I}_p}(h_3)\rangle.$$

□

Theorem 3.3.3. *Let $p \in F_1 \cap F_2$ and let $E(F_1,\mathcal{F})_p$ and $E(F_1,\mathcal{G})_p$ be the fibers over p of the natural projections $E(F_1,\mathcal{F}) \to F_1$ and $E(F_1,\mathcal{G}) \to F_1$ respectively. Then*

$$E(F_1,\mathcal{G})_p \cong \mathrm{Proj}\,\mathbf{C}[b_{1,23}, b_{2,23}, b_{1,33}, b_{2,33}, t],$$

and under this identification $E(F_1,\mathcal{F})_p$ is given by the ideal

$$\langle b_{1,23}^2 - t^2 f_{11,11,11}, 2b_{1,23}b_{2,23} - 3t^3 f_{11,11,12}, b_{1,33}, b_{2,33}\rangle.$$

Proof. The embedding $\mathcal{G}^p \hookrightarrow \mathcal{G}$ induces an embedding $\mathrm{Bl}(F_1^p, \mathcal{G}^p) \hookrightarrow \mathrm{Bl}(F_1, \mathcal{G})$ which identifies $E(F_1,\mathcal{G})_p$ and $E(F_1,\mathcal{F})_p$ with $E(F_1^p,\mathcal{G}^p)_p$ and $E(F_1^p,\mathcal{F}^p)_p$ respectively. Hence we may proceed with the proof by considering the blow-up of germs. The ring

$$G(\mathcal{I}_p, \mathcal{O}_{\mathcal{G},p}) := \bigoplus_{n\geq 0} \mathcal{I}_p^n/\mathcal{I}_p^{n+1}$$

is a local homogenous coordinate ring of the exceptional divisor $E(F_1^p, \mathcal{G}^p)$. We have seen before that it is naturally isomorphic to $\mathcal{O}_{F_1,p}[b_{1,23}, b_{2,23}, b_{1,33}, b_{2,33}, t]$. It follows directly that

$$E(F_1^p,\mathcal{G}^p)_p \cong \mathrm{Proj}\,\mathbf{C}[b_{1,23}, b_{2,23}, b_{1,33}, b_{2,33}, t]. \tag{3.3.7}$$

Under this identification the inclusion

$$E(F_1^p, \mathcal{G}^p)_p \hookrightarrow E(F_1^p, \mathcal{G}^p)$$

corresponds to the projection

$$\pi : \mathcal{O}_{F_1,p}[b_{1,23}, b_{2,23}, b_{1,33}, b_{2,33}, t] \to \mathbf{C}[b_{1,23}, b_{2,23}, b_{1,33}, b_{2,33}, t]$$

that sends elements of the maximal ideal of $\mathcal{O}_{F_1,p}$ to 0.

The homogeneous ideal

$$\mathrm{In}_{\mathcal{I}_p} \mathcal{J}_p \subset G(\mathcal{I}_p, \mathcal{O}_{\mathcal{G},p})$$

is the vanishing ideal of $E(F_1^p, \mathcal{F}^p) \subset E(F_1^p, \mathcal{G}^p)$. Lemma 3.3.2 gives generators of this ideal, see equations (3.3.3) to (3.3.6) for explicit expressions. Let $J_{E,p} \subset \mathbf{C}[b_{1,23}, b_{2,23}, b_{1,33}, b_{2,33}, t]$ be the homogeneous vanishing ideal of $E(F_1^p, \mathcal{G}^p)_p$ in $E(F_1^p, \mathcal{G}^p)_p$. Then $J_{E,p} = \pi(\mathrm{In}_{\mathcal{I}_p} \mathcal{J}_p)$ and in particular

$$J_{E,p} = \langle \pi(\mathrm{In}_{\mathcal{I}_p} h_0), \ldots \pi(\mathrm{In}_{\mathcal{I}_p} h_3) \rangle.$$

This amounts to setting the $a_{i,jk}$ and r'_i to 0 in equations (3.3.3) to (3.3.6) and evaluating the functions $f_{ij,kl,mn}$ at p. It follows that

$$J_{E,p} = \langle b_{1,23}^2 - t^2 f_{11,11,11}, 2b_{1,23}b_{2,23} - 3t^3 f_{11,11,12}, b_{1,33}, b_{2,33} \rangle.$$

□

Case 2: $p \in F_1 \setminus F_2$**.** Assume that $p = (P, 0) \in F_1 \cap F_2$. Let $\{e_1, e_2, e_3\}$ be a basis for W such that $P = \langle e_1^2, e_2^2 \rangle$ and set $P^\perp = \langle e_1 e_2, e_1 e_3, e_2 e_3, e_3^2 \rangle$. Let $a_{i,jk}$, $f_{ij,kl,mn}$ and t be as in section 3.2.3 Case 2. We have the following result:

Lemma 3.3.4. *Let F_1^p be the germ of F_1 at p. It is defined by equations*

$$\begin{aligned}
a_{1,23} &= \frac{a_{1,12}a_{1,13} - a_{1,12}^2 a_{2,23}}{1 - a_{1,12}a_{2,12}} \\
a_{2,13} &= \frac{a_{2,12}a_{2,23} - a_{2,12}^2 a_{1,13}}{1 - a_{1,12}a_{2,12}} \\
a_{1,33} &= \frac{a_{1,13}^2 - a_{1,12}^2 a_{2,23}^2 - 2a_{1,12}a_{2,12}a_{1,13}^2 + 2a_{1,12}^2 a_{2,12}a_{1,13}a_{2,23}}{(1 - a_{1,12}a_{2,12})^2} \\
a_{2,33} &= \frac{a_{2,23}^2 - a_{2,12}^2 a_{1,13}^2 - 2a_{1,12}a_{2,12}a_{2,23}^2 + 2a_{1,12}a_{2,12}^2 a_{1,13}a_{2,23}}{(1 - a_{1,12}a_{2,12})^2} \\
t &= 0
\end{aligned}$$

in $\mathcal{O}_{\mathcal{G},p}$.

Proof. This follows directly from the definition of F_2 and lemma 3.2.5. □

Define:

$$
\begin{aligned}
b_{1,23} &= a_{1,23} - \frac{a_{1,12}a_{1,13} - a_{1,12}^2 a_{2,23}}{1 - a_{1,12}a_{2,12}} \\
b_{2,13} &= a_{2,13} - \frac{a_{2,12}a_{2,23} - a_{2,12}^2 a_{1,13}}{1 - a_{1,12}a_{2,12}} \\
b_{1,33} &= a_{1,33} - \frac{a_{1,13}^2 - a_{1,12}^2 a_{2,23}^2 - 2a_{1,12}a_{2,12}a_{1,13}^2 + 2a_{1,12}^2 a_{2,12}a_{1,13}a_{2,23}}{(1 - a_{1,12}a_{2,12})^2} \\
b_{2,33} &= a_{2,33} - \frac{a_{2,23}^2 - a_{2,12}^2 a_{1,13}^2 - 2a_{1,12}a_{2,12}a_{2,23}^2 + 2a_{1,12}a_{2,12}^2 a_{1,13}a_{2,23}}{(1 - a_{1,12}a_{2,12})^2}.
\end{aligned}
\tag{3.3.8}
$$

Then

$$\mathcal{I}_p = \langle b_{1,23}, b_{2,13}, b_{1,33}, b_{2,33}, t\rangle \subset \mathcal{O}_{\mathcal{G},p}.$$

Hence we have in this case

$$G(\mathcal{I}_p, \mathcal{O}_{\mathcal{G},p}) = \bigoplus_{n\geq 0} \mathcal{I}_p^n / \mathcal{I}_p^{n+1} = \mathcal{O}_{F_1,p}[b_{1,23}, b_{2,13}, b_{1,33}, b_{2,33}, t].$$

We found equations of $\mathcal{F}$ at p in section 3.2.3 case 2. If we substitute the coordinate transformation 3.3.8 we can find four generators of $\mathcal{J}_p$. It is straightforward to write out the full generators explicitly in the coordinates, but this will give rise to large and intransparent expressions[3]. To compress things, we introduce the following notation:

$$
\begin{aligned}
A_1 &:= -1 + a_{1,12}a_{2,12} \\
A_2 &:= -1 + 2a_{1,12}a_{2,12} \\
A_3 &:= -1 + 3a_{1,12}a_{2,12} \\
A_4 &:= a_{1,13} - a_{1,12}a_{2,23} \\
A_5 &:= a_{2,23} - a_{2,12}a_{1,13}.
\end{aligned}
$$

Then it can be verified directly that:

$$
\begin{aligned}
g_0 &= A_1 h_0 + a_{1,12}^2 h_1 \\
g_1 &= A_2 h_1 + a_{1,12}^2 h_2 \\
g_2 &= a_{2,12}^2 h_1 + A_2 h_2 \\
g_3 &= a_{2,12}^2 h_2 + A_1 h_3,
\end{aligned}
\tag{3.3.9}
$$

[3] Yes, even larger and more intransparent than the expression already present in this thesis!

where

$$\begin{aligned}
h_0 &:= b_{1,23}^2 + \frac{2a_{1,12}^3}{A_3} b_{1,23} b_{2,13} - t^2\big(f_{11,11,11} + r_0'\big) + \text{h.d.t.}\\
h_1 &:= 2A_5 b_{1,23} + A_1 b_{1,33} - \frac{2a_{1,12}A_1}{A_3} b_{1,23} b_{2,13} - 3t^2\big(f(w_1^2, w_1^2, w_2^2) + r_1'\big) + \text{h.d.t.}\\
h_2 &:= A_4 b_{2,13} + A_1 b_{2,33} - \frac{2a_{2,12}A_1}{A_3} b_{1,23} b_{2,13} - 3t^2\big(f(w_1^2, w_2^2, w_2^2) + r_2'\big) + \text{h.d.t.}\\
h_3 &:= b_{2,13}^2 + \frac{2a_{2,12}^3}{A_3} b_{1,23} b_{2,13} - t^2\big(f_{22,22,22} + r_3'\big) + \text{h.d.t.}
\end{aligned}$$

where again the “h.d.t.” denotes terms of higher $\mathcal{I}_p$-degree and the r_i' do not depend on the $b_{i,jk}$ nor on t. Note that A_3 is a unit in the local ring $\mathcal{O}_{\mathcal{G},p}$, so division by A_3 is defined.

It is not hard to see that the transformation (3.3.9) is invertible in $\mathcal{O}_{\mathcal{G},p}$, so it follows that $J_p = \langle h_0, h_1, h_2, h_3\rangle$. We now have the following result:

Theorem 3.3.5. *The set $\{h_0, h_1, h_2, h_3\}$ forms a standard base of J_p. In particular, if $E(F_1,\mathcal{F})_p$ and $E(F_1,\mathcal{G})_p$ are the fibers over p of the natural projections $E(F_1,\mathcal{F}) \to F_1$ resp. $E(F_1,\mathcal{G}) \to F_1$, then*

$$E(F_1,\mathcal{G})_p \cong \operatorname{Proj} \mathbf{C}[b_{1,23}, b_{2,13}, b_{1,33}, b_{2,33}, t],$$

and under this identification $E(F_1,\mathcal{F})_p$ is given by the ideal

$$\langle b_{1,23}^2 - t^2 f_{11,11,11}, b_{1,33}, b_{2,33}, b_{2,13}^2 - t^2 f_{11,11,22}\rangle.$$

Proof. Just as in case 1 (theorem 3.3.3), it follows from theorem A.1.7 that it suffices for the first statement to show that the sequence

$$(\mathrm{In}_{\mathcal{I}_p}(h_0), \mathrm{In}_{\mathcal{I}_p}(h_1), \mathrm{In}_{\mathcal{I}_p}(h_2), \mathrm{In}_{\mathcal{I}_p}(h_3)),$$

or a permutation of it, is regular in $G(\mathcal{I}_p, \mathcal{O}_{\mathcal{G},p})$. Remember that we can identify

$$G(\mathcal{I}_p, \mathcal{O}_{\mathcal{G},p}) = \mathcal{O}_{F_1,p}[b_{1,23}, b_{2,13}, b_{1,33}, b_{2,33}, t],$$

We have

$$\begin{aligned}
\mathrm{In}_{\mathcal{I}_p}(h_0) &:= b_{1,23}^2 + \frac{2a_{1,12}^3}{A_3} b_{1,23} b_{2,13} - t^2\big(f_{11,11,11} + r_0'\big)\\
\mathrm{In}_{\mathcal{I}_p}(h_1) &:= 2A_5 b_{1,23} + A_1 b_{1,33}\\
\mathrm{In}_{\mathcal{I}_p}(h_2) &:= A_4 b_{2,13} + A_1 b_{2,33}\\
\mathrm{In}_{\mathcal{I}_p}(h_3) &:= b_{2,13}^2 + \frac{2a_{2,12}^3}{A_3} b_{1,23} b_{2,13} - t^2\big(f_{22,22,22} - r_3'\big).
\end{aligned}$$

where the $a_{i,jk}$, A_i and r_i' are as before, but now viewed as elements of $\mathcal{O}_{F_1,p}$.

For convenience denote (as in case 1):

$$S_{(i_1,\dots,i_n)} := G(\mathcal{I}_p, \mathcal{O}_{\mathcal{G},p})/\langle \mathrm{In}_{\mathcal{I}_p}(h_{i_1}),\dots,\mathrm{In}_{\mathcal{I}_p}(h_{i_n})\rangle,$$

and let

$$\pi_{(i_1,\dots,i_n)} : G(\mathcal{I}_p, \mathcal{O}_{\mathcal{G},p}) \to S_{i_1,\dots,i_n}$$

be the corresponding projection. Then clearly:

$$\begin{aligned} S_{(2)} &\cong G(\mathcal{I}_p, \mathcal{O}_{\mathcal{G},p})/\langle b_{1,33}\rangle \cong \mathcal{O}_{F_1,p}[b_{1,23}, b_{2,13}, b_{2,33}] \qquad (3.3.10)\\ S_{(2,3)} &\cong G(\mathcal{I}_p, \mathcal{O}_{\mathcal{G},p})/\langle b_{1,33}, b_{2,33}\rangle \cong \mathcal{O}_{F_1,p}[b_{1,23}, b_{2,13}]. \end{aligned}$$

These rings are integral domains, so in particular the image of $\mathrm{In}_{\mathcal{I}_p}(h_0)$ in $S_{(2,3)}$ is not a zero-divisor. Furthermore, the image of $\mathrm{In}_{\mathcal{I}_p}(h_0)$ in $S_{(2,3)}$ is prime. Hence $S_{(2,3,0)}$ is an integral domain. In particular the image of $\mathrm{In}_{\mathcal{I}_p}(h_3)$ in $S_{(2,3,0)}$ is not a zero-divisor. It is neither a unit, so we can conclude that

$$\{\mathrm{In}_{\mathcal{I}_p}(h_1), \mathrm{In}_{\mathcal{I}_p}(h_2), \mathrm{In}_{\mathcal{I}_p}(h_0), \mathrm{In}_{\mathcal{I}_p}(h_3)\}$$

is a regular sequence. Hence in this case the first statement of the theorem follows.

The proof of the second part of the theorem runs completely analogously to that of theorem 3.3.3. Again the generators of the homogeneous ideal

$$J_{E,p} \subset \mathbf{C}[b_{1,23}, b_{2,13}, b_{1,33}, b_{2,33}, t]$$

of $E(F_1^p, \mathcal{F}^p)_p$ in $E(F_1^p, \mathcal{G}^p)_p$ are obtained from the generators $\mathrm{In}_{\mathcal{I}_p} h_0, \dots \mathrm{In}_{\mathcal{I}_p} h_3$ of $\mathrm{In}_{\mathcal{I}_p}\mathcal{J}_p$ by setting the r_i' and $a_{i,jk}$ to 0, which means that A_1, A_2 and A_3 become 1 and A_4 and A_5 become 0. Hence

$$J_{E,p} = \langle b_{1,23}^2 - t^2 f_{11,11,11}, b_{1,33}, b_{2,33}, b_{2,13}^2 - t^2 f_{11,11,22}\rangle.$$

□

Remark 3.3.6. As noted before the space $E(F_1, \mathcal{G})$ can be naturally identified with $\mathbf{P}(\nu_{F_1/\mathcal{G}})$. It will be convenient to view E as a subvariety of $\mathbf{P}(\nu_{F_1/\mathcal{G}})$. For any $p \in F_1$ we have

$$\mathbf{P}(\nu_{F_1/\mathcal{G}})_p = \mathrm{Proj}\,\mathrm{Sym}^*(\nu_{F_1/\mathcal{G}}^\vee)_p.$$

Furthermore, if

$$N : T\mathcal{G}|_{F_1} \to \nu_{F_1/\mathcal{G}}$$

denotes the usual projection, we have an embedding

$$\mathrm{Sym}^* N_p^\vee : \mathrm{Sym}^*(\nu_{F_1/\mathcal{G}}^\vee)_p \hookrightarrow \mathrm{Sym}^* T_p^\vee \mathcal{G}.$$

The target algebra is generated by powers of local differential 1-forms on $\mathcal{G}$, evaluated at p. It follows that we can express the homogeneous vanishing ideal

of E_p in terms of differentials at p.
More explicitly we have the following description. First assume that $p \in F_1 \cap F_2$. Then it follows from lemma 3.3.1 and the coordinate transformation 3.3.1 that

$$\operatorname{im} N_p^\vee = \langle d_p b_{1,23}, d_p b_{2,23}, d_p b_{1,33}, d_p b_{2,33}, d_p t \rangle.$$

These linear forms factorize as $d_p b_{i,jk} = \beta_{i,jk} \circ N_p$ and $d_p t = \tau \circ N_p$. It follows from theorem 3.3.3 that $E_p \subset \operatorname{Sym}^\bullet(\nu^\vee_{F_1/\mathcal{G}})_p$ is given by the following equations:

$$\beta_{1,23}^2 = \tau^2 f_{11,11,11} \tag{3.3.11}$$
$$2\beta_{1,23}\beta_{2,23} = 3\tau^2 f_{11,11,12} \tag{3.3.12}$$
$$\beta_{1,33} = 0 \tag{3.3.13}$$
$$\beta_{2,33} = 0 \tag{3.3.14}$$

Analogously, if $p \in F_1 \setminus F_2$ it follows from lemma 3.3.4 and the coordinate transformation 3.3.8 that

$$\operatorname{im} N_p^\vee = \langle d_p b_{1,23}, d_p b_{2,13}, d_p b_{1,33}, d_p b_{2,33}, d_p t \rangle.$$

If we define the linear forms $\beta_{i,jk}$ and τ on $(\nu_{F_1/\mathcal{G}})_p$ analogously, it follows from theorem 3.3.5 that E_p is defined by the homogeneous equations

$$\beta_{1,23}^2 = \tau^2 f_{11,11,11} \tag{3.3.15}$$
$$\beta_{1,33} = 0 \tag{3.3.16}$$
$$\beta_{2,33} = 0 \tag{3.3.17}$$
$$\beta_{2,13}^2 = \tau^2 f_{11,11,22} \tag{3.3.18}$$

3.3.2 Properties of $\widehat{\mathcal{F}}$

Composition of $\varphi : \mathcal{F} \to \Delta$ and the blow-up $\varepsilon_{F_1} : \widehat{\mathcal{F}} \to \mathcal{F}$ gives rise to a new family $\widehat{\varphi} : \widehat{\mathcal{F}} \to \Delta$. The central fiber $\widehat{\mathcal{F}}_0$ is the union of the exceptional divisor E and the strict transform $\widehat{F}_2$ of F_2. The following theorem summarizes the results that we will prove in this section.

Theorem 3.3.7.

1. *$\widehat{\mathcal{F}}_0$ is a normal crossing of E and $\widehat{F}_2$;*

2. *The restriction $\varepsilon_{F_1}|_{\widehat{F}_2} : \widehat{F}_2 \to F_2$ is an isomorphism. In particular $\widehat{F}_2$ is smooth and naturally isomorphic to $\mathbf{P}(W) \times \mathbf{P}(W)^\vee$;*

3. *Let $\Gamma \subset \mathbf{P}(W) \times \mathbf{P}(W)^\vee$ be the incidence variety. Then under the natural identification $\widehat{F}_2 \cong \mathbf{P}(W) \times \mathbf{P}(W)^\vee$ we have that $E \cap \widehat{F}_2$ corresponds to Γ;*

4. *$\widehat{\mathcal{F}}$ is smooth along $\widehat{F}_2$.*

We need some preliminary results. Let $p = (P, 0) \in F_1 \cap F_2$; we may choose a basis $\{e_1, e_2, e_3\} \subset W$ such that P is spanned by e_1^2 and $e_1 e_2$. For convenience we will denote the vanishing ideal sheaf of F_1 on $\mathcal{G}$ by $\mathcal{I}_1$ instead of $\mathcal{I}$. Let $\mathcal{I}_2$ be the vanishing ideal sheaf on $\mathcal{G}$ of F_2. Denote by $\mathcal{I}_{1,p}, \mathcal{I}_{2,p} \subset \mathcal{O}_{\mathcal{G},p}$ the stalks at p of these sheaves.

Let $\{a_{i,jk}\}$ be the local functions at p in $\mathcal{G}$ as we defined them in section 3.2.3. In the section 3.3.1 we found local coordinate functions $b_{1,23}, b_{2,23}, b_{1,33}, b_{2,33}, t$ on $\mathcal{G}$ at p such that $\mathcal{I}_{1,p} = \langle b_{1,23}, b_{2,23}, b_{1,33}, b_{2,33}, t\rangle$, see the coordinate transformation (3.3.1). In these coordinates we can find generators of $\mathcal{I}_{2,p}$:

Lemma 3.3.8. $\mathcal{I}_{2,p} = \langle r_1, r_2, r_3, r_4, r_5\rangle$ *where:*

$$\begin{aligned} r_1 &= 4a_{1,22} + a_{2,22}^2 \\ r_2 &= 4b_{1,23} + 2a_{2,22}b_{2,23} \\ r_3 &= 4b_{1,33} + 4a_{2,13}a2,22b_{2,23} + 4b_{2,23}^2 \\ r_4 &= 2a_{2,13}b_{2,23} - b_{2,33} \\ r_5 &= t \end{aligned}$$

In particular, this is a standard basis with respect to $\mathcal{I}_{1,p}$.

Proof. It follows from lemma 3.2.6 and the definition of F_2 that $\mathcal{I}_{2,p}$ is generated by

$$\begin{aligned} & a_{1,22} + \frac{1}{4}a_{2,22}^2, \\ & a_{1,23} - \frac{1}{4}a_{2,22}(2a_{1,13} - 2a_{2,23} - a_{2,22}a_{2,13}), \\ & a_{1,33} - \frac{1}{4}(2a_{2,23} - a_{2,22}a_{2,13})(4a_{1,13} - 2a_{2,23} + a_{2,22}a_{2,13}), \\ & a_{2,33} - 2a_{2,23}a_{2,13} + a_{2,22}a_{2,13}^2, \\ & \text{and } t \end{aligned}$$

Now apply substitutions 3.3.1 to obtain the generators in the statement of the lemma. We have:

$$\begin{aligned} \mathrm{In}_{\mathcal{I}_{1,p}} r_1 &= 4a_{1,22} + a_{2,22}^2 \\ \mathrm{In}_{\mathcal{I}_{1,p}} r_2 &= 4b_{1,23} + 2a_{2,22}b_{2,23} \\ \mathrm{In}_{\mathcal{I}_{1,p}} r_3 &= 4b_{1,33} + 4a_{2,13}a2,22b_{2,23} \\ \mathrm{In}_{\mathcal{I}_{1,p}} r_4 &= 2a_{2,13}b_{2,23} - b_{2,33} \\ \mathrm{In}_{\mathcal{I}_{1,p}} r_5 &= t \end{aligned}$$

where the $a_{i,jk}$ should be viewed as elements of $\mathcal{O}_{F_1,p}$. It is easy to see that

$$\{\mathrm{In}_{\mathcal{I}_{1,p}} r_1, \ldots, \mathrm{In}_{\mathcal{I}_{1,p}} r_5\}$$

is a regular sequence. Hence by theorem A.1.7 $\{r_1,\dots,r_5\}$ is a $\mathcal{I}_{1,p}$-standard base for $\mathcal{I}_{2,p}$. □

The following lemma is a direct consequence.

Lemma 3.3.9. *Let $E(F_1,\mathcal{G})_p$ be the fiber of $\varepsilon : \mathrm{Bl}(F_1,\mathcal{G}) \to \mathcal{G}$ over p. Under the identification $E(F_1,\mathcal{G})_p \cong \mathrm{Proj}\,\mathbf{C}[b_{1,23}, b_{2,23}, b_{1,33}, b_{2,33}, t]$ the intersection of $E(F_1,\mathcal{G})_p$ and $\widehat{F}_2$ is the point given by*

$$b_{1,23} = b_{1,33} = b_{2,33} = t = 0.$$

Proof. For any subscheme $Y \subseteq \mathcal{G}$ denote, as before, the germ of Y at p by Y^p. We know that

$$E(F_1^p, \mathcal{G}^p) = \mathrm{Proj}\, G(F_1^p, \mathcal{G}^p)$$

where

$$G(F_1^p, \mathcal{G}^p) := \bigoplus_{n\geq 0} \mathcal{I}_{1,p}^n / \mathcal{I}_{1,p}^{n+1} \cong \mathcal{O}_{F_1,p}[b_{1,23}, b_{2,23}, b_{1,33}, b_{2,33}, t],$$

see equation (3.3.2).

Let Z denote the intersection of $\widehat{F}_2^p$ and $E(F_1^p, \mathcal{G}^p)$. By remark A.1.4 in the appendix it then follows that Z, as subspace of $E(F_1^p, \mathcal{G}^p)$, has vanishing ideal

$$I_Z := \mathrm{In}_{\mathcal{I}_{1,p}} \mathcal{I}_{2,p} \subset G(F_1^p, \mathcal{G}^p).$$

By lemma 3.3.8 this ideal is generated by

$$\begin{aligned}
\mathrm{In}_{\mathcal{I}_{1,p}} r_1 &= 4a_{1,22} + a_{2,22}^2 \\
\mathrm{In}_{\mathcal{I}_{1,p}} r_2 &= 4b_{1,23} + 2a_{2,22} b_{2,23} \\
\mathrm{In}_{\mathcal{I}_{1,p}} r_3 &= 4b_{1,33} + 4a_{2,13} a2,22 b_{2,23} \\
\mathrm{In}_{\mathcal{I}_{1,p}} r_4 &= 2a_{2,13} b_{2,23} - b_{2,33} \\
\mathrm{In}_{\mathcal{I}_{1,p}} r_5 &= t
\end{aligned}$$

if we interpret the $a_{i,jk}$ as elements of $\mathcal{O}_{F_1,p}$. Let $Z_p := Z \cap E(F_1,\mathcal{G})_p$ and remember that

$$E(F_1,\mathcal{G})_p \cong \mathrm{Proj}\,\mathbf{C}[b_{1,23}, b_{2,23}, b_{1,33}, b_{2,33}, t],$$

see for example the proof of theorem 3.3.3. Let $I_{Z,p}$ be the homogeneous vanishing ideal of Z_p in $\mathbf{C}[b_{1,23}, b_{2,23}, b_{1,33}, b_{2,33}, t]$. Then $I_{Z,p}$ may be obtained from I_Z by setting the elements of the maximal ideal in $\mathcal{O}_{F_1,p}$ to 0. It follows that

$$I_{Z,p} = \langle b_{1,23}, b_{1,33}, b_{2,33}, t \rangle.$$

In particular Z_p is a single point in the projective space $E(F_1,\mathcal{G})_p$, defined by the linear equations

$$b_{1,23} = b_{1,33} = b_{2,33} = t = 0.$$

□

Denote by z the intersection point of $\widehat{F}_2$ and $E(F_1,\mathcal{G})_p$. To prove the main result of this section, we wish to study the local ring $\mathcal{O}_{\widehat{\mathcal{F}},z}$. Since $\widehat{\mathcal{F}} \subset \mathrm{Bl}(F_1,\mathcal{G})$ we may equivalently study the vanishing ideal in $\mathcal{O}_{\mathrm{Bl}(F_1,\mathcal{G}),z}$ of the germ of $\widehat{\mathcal{F}}$ at z. Locally near $E(F_1,\mathcal{G})_p$ the space $\mathrm{Bl}(F_1,\mathcal{G})$ is described by

$$\mathrm{Bl}(F_1^p,\mathcal{G}^p) = \mathrm{Proj}\, B(\mathcal{I}_{1,p},\mathcal{O}_{\mathcal{G},p}) =: \mathrm{Proj} \bigoplus_{k\geq 0} \mathcal{I}_{1,p}^k s^k.$$

Let $I_z \subset B(\mathcal{I}_{1,p},\mathcal{O}_{\mathcal{G},p})$ be the homogeneous vanishing ideal of $z \in \mathrm{Bl}(F_1^p,\mathcal{G}^p)$. Then $\mathcal{O}_{\mathrm{Bl}(F_1,\mathcal{G}),z}$ is given by degree 0 part the homogeneous localization of $\bigoplus_{k\geq 0}\mathcal{I}_{1,p}^k s^k$ away from I_z. That is, as a $\mathcal{O}_{\mathcal{G},p}$-module it is generated by elements of the form g/h, with $g,h \in \bigoplus_{k\geq 0}\mathcal{I}_{1,p}^k s^k$ of equal degree and $h \notin I_z$.

Lemma 3.3.10. *Let $J \subseteq B(\mathcal{I}_{1,p},\mathcal{O}_{\mathcal{G},p})$ be any homogeneous ideal and $J_z \subset \mathcal{O}_{\mathrm{Bl}(F_1,\mathcal{G}),z}$ its localization at z. Then $J_z = \langle g/b_{2,23}^{\deg g} : g \in J\rangle$. If $J = \langle g_1,\dots,g_n\rangle$ then $J_z = \langle g_1/b_{2,23}^{d_1},\dots,g_n/b_{2,23}^{d_n}\rangle$, where d_i is the $\mathcal{I}_{1,p}$-degree of g_i.*

Proof. Clearly

$$J_z = \langle g/h : g \in J, h \in B(\mathcal{I}_{1,p},\mathcal{O}_{\mathcal{G},p}) \setminus I_z, \deg g = \deg h\rangle.$$

So it suffices to show that for any pair $g,h \in B(\mathcal{I}_{1,p},\mathcal{O}_{\mathcal{G},p})$ of equal degree k and such that $h \notin I_p$ there exists a $g' \in B(\mathcal{I}_{1,p},\mathcal{O}_{\mathcal{G},p})$ of degree k and $r \in \mathcal{O}_{\mathrm{Bl}(F_1,\mathcal{G}),z}$ such that $g/h = r\cdot g'/b_{2,23}^k$. The inclusion

$$E(F_1,\mathcal{G})_p \hookrightarrow \mathrm{Bl}(F_1^p,\mathcal{G}^p)$$

induces a projection of graded rings

$$\bigoplus_{k\geq 0}\mathcal{I}_{1,p}^k s^k \to \mathbf{C}[b_{1,23},b_{2,23},b_{1,33},b_{2,33},t],$$

where as before we used that

$$E(F_1,\mathcal{G})_p \cong \mathrm{Proj}\,\mathbf{C}[b_{1,23},b_{2,23},b_{1,33},b_{2,33},t].$$

It follows from lemma 3.3.9 that I_z is the preimage under this projection of the ideal $\langle b_{1,23},b_{1,33},b_{2,33},t\rangle$. Since $\mathcal{I}_{1,p} = \langle b_{1,23},b_{1,33},b_{2,33},b_{2,23},t\rangle$ it follows in particular that for every integer $k \geq 0$ we have a decomposition

$$\mathcal{I}_{1,p}^k s^k = (I_z \cap \mathcal{I}_{1,z}^k s^k) \oplus \mathbf{C} b_{2,23}^k s^k \tag{3.3.19}$$

of $\mathbf{C}$-vector spaces. It also follows that for any $g \in I_z$ of degree k the element $g/b_{2,23}^k$ is a non-invertible element of $\mathcal{O}_{\mathrm{Bl}(F_1,\mathcal{G}),z}$. Indeed, otherwise there would exists $q,r \in B(\mathcal{I}_{1,p},\mathcal{O}_{\mathcal{G},p})$ of equal degree and with $r \notin I_z$ such that $gq = b_{2,23}^k r$. Since the left hand side is in I_z but the left hand side is not, this is a contradiction.

Let $h \in \mathcal{I}_{1,p}^k s^k \setminus I_z$ for some $k \geq 0$. By equation 3.3.19 we may write $h = h_1 + c \cdot b_{2,23}^k$ with $h_1 \in I_z \cap \mathcal{I}_{1,p}^k$ and $c \in \mathbf{C} \setminus 0$. Then $h_1 / b_{2,23}^k$ is non-invertible in $\mathcal{O}_{\mathrm{Bl}(F_1,\mathcal{G}),z}$. Since $(h - h_1)/b_{2,23}^k = c$ is invertible, it follows that $h/b_{2,23}^k$ is invertible; otherwise c would be the sum of two non-invertible elements, which in a local ring is always non-invertible. This would be a contradiction. Now let $g \in \mathcal{I}_{1,p}^k s^k$ be arbitrary, h as before, and set $r = \left(h/b_{2,23}^k\right)^{-1}$. Then $g/h = r \cdot g/b_{2,23}^k$ and the lemma follows.

□

Proposition 3.3.11. *$\widehat{\mathcal{F}}$ is smooth along the intersection of E and $\widehat{F}_2$.*

Proof. As before let $\mathcal{J}$ be the vanishing ideal sheaf of $\mathcal{F}$ on $\mathcal{G}$ and $\mathcal{I}$ the vanishing ideal sheaf of F_1 on $\mathcal{G}$. Let $p \in F_1 \cap F_2$. The vanishing ideal sheaf of $\widehat{\mathcal{F}^p}$ on $\mathrm{Bl}(F_1^p, \mathcal{G}^p)$ is given by

$$\bigoplus_{k\geq 0} \mathcal{I}_p^k s^k \cap \mathcal{J}_p.$$

In section 3.3.1 (Case 1) we found generators $h_0, \ldots, h_3$ of $\mathcal{J}_p$ which form a $\mathcal{I}_p$-standard basis, see proposition 3.3.2. From theorem 13.7 in [24] it follows that

$$\mathcal{I}_p s \cap \mathcal{J}_p = \mathcal{O}_{\mathcal{G},p} h_3 s + \mathcal{O}_{\mathcal{G},p} h_4 s \text{ and} \tag{3.3.20}$$

$$\mathcal{I}_p^k s^k \cap \mathcal{J}_p = \sum_{i=1}^{4} \mathcal{I}_p^{k-d_i} h_i s^k \text{ for } k \geq 2 \tag{3.3.21}$$

where $d_i = \deg_{\mathcal{I}_p} h_i$.

Let $\widehat{\mathcal{J}}$ be the vanishing ideal sheaf on $\mathrm{Bl}(F_1, \mathcal{G})$ of $\widehat{\mathcal{F}}$, let z as before be the intersection point of $E = E(F_1, \mathcal{F})$ and $\widehat{F}_2$. Then it follows from lemma 3.3.10 that the stalk $\widehat{\mathcal{J}}_z$ as ideal in $\mathcal{O}_{\mathrm{Bl}(F_1,\mathcal{G}),z}$ is generated by elements of the form $g/b_{2,23}^k$, where $g \in \mathcal{I}_p^k \cap \mathcal{J}_p$. From the descriptions (3.3.20) and (3.3.21) and the explicit expressions for $h_0, \ldots, h_3$ we immediately find that $\widehat{\mathcal{J}}_p$ is generated by

$$\widehat{h}_0 := \frac{h_0}{b_{2,23}^2} = -a_{1,22} + \text{h.d.t} \tag{3.3.22}$$

$$\widehat{h}_1 := \frac{h_1}{b_{2,23}^2} = -a_{2,22} - 2\frac{b_{1,23}}{b_{2,23}} + \text{h.d.t} \tag{3.3.23}$$

$$\widehat{h}_2 := \frac{h_2}{b_{2,23}} = -\frac{b_{1,33}}{b_{2,23}} + \text{h.d.t} \tag{3.3.24}$$

$$\widehat{h}_3 := \frac{h_3}{b_{2,23}} = 2a_{2,13} - \frac{b_{2,33}}{b_{2,23}} + \text{h.d.t} \tag{3.3.25}$$

where "h.d.t" represents terms of higher degree with respect to the maximal ideal $\mathfrak{m}_z$ of $\mathcal{O}_{\mathrm{Bl}(F_1,\mathcal{G}),z}$. Observe that the set $\{\widehat{h}_0 \mod \mathfrak{m}_p, \ldots, \widehat{h}_3 \mod \mathfrak{m}_p\}$ is linear independent (over $\mathcal{O}_{\mathrm{Bl}(F_1,\mathcal{G}),z}$) in $\mathfrak{m}_p/\mathfrak{m}_p^2$. Since $\mathcal{O}_{\mathrm{Bl}(F_1,\mathcal{G}),z}$ is regular, it follows (see e.g. [49, theorem 26, page 303]) that $\mathcal{O}_{\mathrm{Bl}(F_1,\mathcal{G}),p}/\widehat{\mathcal{J}}_p = \mathcal{O}_{\widehat{Z},p}$ is regular. This proves the theorem.

□

proof of theorem 3.3.7. Remember that we have denoted $\mathcal{G} = \mathrm{Gr}_2(\mathrm{Sym}^2 W) \times \Delta$. Denote in addition $\widehat{\mathcal{G}} := \mathrm{Bl}(F_1, \mathcal{G})$ and let $\widehat{\mathcal{G}}_0$ be the central fiber of the projection $\widehat{\mathcal{G}} \to \Delta$. Let $p \in F_1 \cap F_2$ be arbitrary. Let z be the intersection point of $E := E(F_1, \mathcal{G})_p$ and $\widehat{F}_2$. Then in $\mathcal{O}_{\widehat{\mathcal{G}},z}$ the germ of $\widehat{\mathcal{G}}_0$ is given by the ideal $\langle t \rangle = \langle \frac{t}{b_{2,23}} \cdot b_{2,23} \rangle$. Since $\frac{t}{b_{2,23}}$ and $b_{2,23}$ are linearly independent modulo $\mathfrak{m}_z^2$, the space $\widehat{\mathcal{G}}_0$ is a normal crossing at z.

Note that $\widehat{\mathcal{F}}_0 = \widehat{\mathcal{G}}_0 \cap \widehat{\mathcal{F}}$. To prove that $\widehat{\mathcal{F}}_0$ is a normal crossing at z it suffices to show that $\widehat{\mathcal{F}}$ intersects the irreducible components of $\widehat{\mathcal{G}}_0$ transversally at z; this suffices since $\mathcal{F}$ is smooth at z by proposition 3.3.11.

The $\mathbf{C}$-vector space $\mathfrak{m}_z / \mathfrak{m}_z^2$ is naturally identified with the cotangent space of $\widehat{\mathcal{G}}$ at z. In it lies the annihilator of the Zariski tangent space of $\widehat{\mathcal{G}}_0$. By the previous calculations this annihilator is given by

$$\mathrm{span}_{\mathbf{C}}\{b_{2,23} \mod \mathfrak{m}_z^2, \frac{t}{b_{2,23}} \mod \mathfrak{m}_z^2\}.$$

Let $\widehat{J}_z \subset \mathcal{O}_{\widehat{\mathcal{G}},z}$ be the vanishing ideal of the germ of $\widehat{\mathcal{F}}$ at z. The annihilator in $\mathfrak{m}_z / \mathfrak{m}_z^2$ of the tangent space of $\widehat{\mathcal{F}}$ at z is given by $(\widehat{J}_z + \mathfrak{m}_z^2)/\mathfrak{m}_z^2$. We found explicit generators (3.3.22) to (3.3.25) of $\widehat{J}_z$, from which directly follows that

$$\mathrm{span}_{\mathbf{C}}\{b_{2,23} \mod \mathfrak{m}_z^2, \frac{t}{b_{2,23}} \mod \mathfrak{m}_z\} \cap (\widehat{J}_z + \mathfrak{m}_z^2)/\mathfrak{m}_z^2 = \{0\}.$$

It follows that $\widehat{\mathcal{F}}$ and the irreducible components of $\widehat{\mathcal{G}}_E$ intersect the transversally at z, hence $\widehat{\mathcal{F}}_0$ is a normal crossing at z. This holds for all $z \in E \cap \widehat{F}_2$, so the first statement of theorem 3.3.7 follows.

For the second statement remember that by remark A.1.4 the restriction $\varepsilon|_{\widehat{F}_2} : \widehat{F}_2 \to F_2$ corresponds to the blow-up of F_2 along $F_1 \cap F_2$. By proposition 3.2.4 F_2 is smooth and $F_1 \cap F_2$ is smooth and of codimension 1. Hence $\varepsilon|_{\widehat{F}_2}$ must be an isomorphism. Since F_2 is naturally isomorphic to $\mathbf{P}(W) \times \mathbf{P}(W^\vee)$, so is $\widehat{F}_2$. Furthermore, by proposition 3.2.4 $F_1 \cap F_2$ as a subspace of F_2 corresponds to the incidence variety Γ if we identify $F_2 \cong \mathbf{P}(W) \times \mathbf{P}(W^\vee)$. It follows from remark A.1.4 that $E \cap \widehat{F}_2 = (\varepsilon|_{\widehat{F}_2})^{-1}(F_1 \cap F_2)$. Since $\varepsilon|_{\widehat{F}_2}$ is an isomorphism, it follows that $E \cap \widehat{F}_2$ corresponds to Γ under the identification $\widehat{F}_2 \cong \mathbf{P}(W) \times \mathbf{P}(W^\vee)$. The third statement follows.

For the final statement remember that by lemma 3.2.15 $\mathcal{F}$ is smooth along $F_2 \setminus F_1$. This immediately implies that $\widehat{\mathcal{F}}$ is smooth along $\widehat{F}_2 \setminus E$. From theorem 3.3.11 it then follows that $\widehat{\mathcal{F}}$ is smooth along all of $\widehat{F}_2$, and the statement is proven. □

3.4 Geometrical description of E

By theorem 3.3.7 the locus $\widehat{\mathcal{F}}_0$ is a normal crossing of E and $\widehat{F}_2$. The geometrical structure of $\widehat{F}_2$ is clear: it is naturally isomorphic to $\mathbf{P}(W) \times \mathbf{P}(W^\vee)$. For E we only have obtained local homogeneous equations. The aim of this section is to give a geometric characterization of E. More precisely, we prove

Proposition 3.4.1. *Let $S_f \to \mathbf{P}(W)$ be the double cover branched along $v^* f$. Let $V \subset S_f^{[2]}$ be the locus of length 2 subschemes of S_f that are vertical with respect to the projection $S_f \to \mathbf{P}(W)$, that is, contained in a fiber. Then there is a natural isomorphism $E \cong \mathrm{Bl}(V, S_f^{[2]})$.*

To prove this result we will rephrase the local equations of E in terms of morphisms of vector bundles on F_1. We make this precise in sections 3.4.1 and 3.4.2. In section 3.4.3 we will then use this description to prove the theorem.

3.4.1 A subbundle of $\nu_{F_1/\mathcal{G}}$

As noted in remark 3.3.6 we may naturally view E as a subvariety of $\mathbf{P}(\nu_{F_1/\mathcal{G}})$, the projectivization of (the total space of) the normal bundle of F_1 in $\mathcal{G}$. We will first study the geometric properties of this normal bundle. By definition we have $\mathcal{G} = \mathrm{Gr}_2(\mathrm{Sym}^2 W) \times \Delta$, hence

$$\nu_{F_1/\mathcal{G}} = \nu_{F_1/\mathrm{Gr}_2(\mathrm{Sym}^2 W)} \oplus \mathcal{O}_{F_1} \tag{3.4.1}$$

Let us study the structure of $\nu_{F_1/\mathrm{Gr}_2(\mathrm{Sym}^2 W)}$. By proposition 3.2.4 there is a natural isomorphism $F_1 \cong \mathbf{P}(W)^{[2]}$, which maps every $P \in F_1 \subset \mathrm{Gr}_2(\mathrm{Sym}^2 W)$ to the length 2 subscheme of $\mathbf{P}(W)$ defined by the intersection of the line $\mathbf{P}(P) \subset \mathbf{P}(\mathrm{Sym}^2 W)$ with the Veronese surface. The bundle $\nu_{F_1/\mathrm{Gr}_2(\mathrm{Sym}^2 W)}$ can be expressed in terms of natural bundles on $\mathbf{P}(W)^{[2]}$, which we introduce first:

- There is a natural projection $F_1 \cong \mathbf{P}(W)^{[2]} \to \mathrm{Gr}_2(W)$ that maps a length 2 subscheme s of $\mathbf{P}(W)$ to the affine plane over the line in $\mathbf{P}(W)$ spanned by s. Let $\mathcal{V}$ be the pull-back to F_1 of the tautological plane bundle on $\mathrm{Gr}_2(W)$ along this projection. Explicitly, let $P \in \mathrm{Gr}_2(\mathrm{Sym}^2 W)$ be such that $P \in F_1$. Then either $P = L_1^2 + L_2^2$ with L_1, L_2 two distinct lines in W, or $P = L \cdot W$ for a complete flag $L \subset V \subset W$. In the first case $\mathcal{V}_P$ can be naturally identified with $L_1 + L_2 \subset W$, in the second case $\mathcal{V}_P$ can be naturally identified with V.
- Let $\mathcal{P}$ be the pullback of the tautological plane bundle on $\mathrm{Gr}_2(\mathrm{Sym}^2 W)$ along $F_1 \to \mathrm{Gr}_2(\mathrm{Sym}^2 W)$. So $\mathcal{P}_P = P$ for $P \in F_1$
- Let $\mathcal{W} := \mathcal{O}_{F_1} \otimes W$.
- Let $\gamma := \mathcal{W}/\mathcal{V}$.

Remark 3.4.2. Observe that $\mathcal{P} \subset \mathrm{Sym}^2 \mathcal{V}$. This gives rise to a bundle homomorphism $\mathcal{V}^\vee \to \mathcal{H}om(\mathcal{P}, \mathcal{V})$. Let us check that this is fiberwise injective and hence maps isomorphically onto a rank two subbundle of $\mathcal{H}om(\mathcal{P}, \mathcal{V})$. In the generic case when $P = L_1^2 + L_2^2$ and $V = L_1 + L_2$, then choose a basis (e_1, e_2) of V such that

$e_i \in L_i$. If $(e_1^\vee, e_2^\vee)$ denotes the basis of $V^\vee$ dual to (e_1, e_2), then to $e_1^\vee$ is assigned the map $P \to V$ defined by $e_1^2 \mapsto e_1$ and $e_2^2 \mapsto 0$ and likewise for $e_2^\vee$. In the special case when $P = V \circ L$, choose a basis (e_1, e_2) of V such that $e_1 \in L$. Then to $e_1^\vee$ is assigned the map $e_1^2 \mapsto e_1$, $e_1 e_2 \mapsto \frac{1}{2} e_2$ and to $e_2^\vee$ is assigned the map $e_1^2 \mapsto 0$, $e_1 e_2 \mapsto \frac{1}{2} e_1$. In either case, $V^\vee \to \mathrm{Hom}(P, V)$ is injective.

Let us denote by $\overline{\mathcal{H}om}(\mathcal{P}, \mathcal{V})$ and $\overline{\mathcal{H}om}(\mathcal{P}, \mathcal{W})$ the cokernel of $\mathcal{V}^\vee \to \mathcal{H}om(\mathcal{P}, \mathcal{V})$ and $\mathcal{V}^\vee \to \mathcal{H}om(\mathcal{P}, \mathcal{W})$ respectively. We then have a short exact sequence

$$0 \to \overline{\mathcal{H}om}(\mathcal{P}, \mathcal{V}) \to \overline{\mathcal{H}om}(\mathcal{P}, \mathcal{W}) \to \mathcal{H}om(\mathcal{P}, \gamma) \to 0. \tag{3.4.2}$$

Proposition 3.4.3. *The normal bundle $\nu_{F_1/\mathrm{Gr}_2(\mathrm{Sym}^2 W)}$ is naturally isomorphic to*

$$\overline{\mathcal{H}om}(\mathcal{P}, \mathcal{W}) \otimes \gamma,$$

hence fits in a natural short exact sequence

$$0 \to \overline{\mathcal{H}om}(\mathcal{P}, \mathcal{V}) \otimes \gamma \to \nu_{F_1/\mathrm{Gr}_2(\mathrm{Sym}^2 W)} \to \mathcal{H}om(\mathcal{P}, \gamma^2) \to 0.$$

Proof. By exactness of sequence 3.4.2 and the fact that tensoring with γ is exact it suffices to show that

$$\nu_{F_1/\mathrm{Gr}_2(\mathrm{Sym}^2 W)} \cong \overline{\mathcal{H}om}(\mathcal{P}, \mathcal{W}) \otimes \gamma.$$

Using the natural identification

$$T\mathrm{Gr}_2(\mathrm{Sym}^2 W) \cong \mathcal{H}om(\mathcal{P}, \mathrm{Sym}^2 \mathcal{W}/\mathcal{P})$$

it suffice to show that there exists a projection

$$\pi : \mathcal{H}om(\mathcal{P}, \mathrm{Sym}^2 \mathcal{W}/\mathcal{P}) \to \overline{\mathcal{H}om}(\mathcal{P}, \mathcal{W}) \otimes \gamma$$

such that the kernel of π is the tangent bundle of F_1. The map π is constructed as follows. There is an obvious quotient map

$$q : \mathrm{Sym}^2 \mathcal{W} \to \mathcal{W} \otimes \gamma$$

which maps $w^2 \in \mathrm{Sym}^2 \mathcal{W}_P$ to $w \otimes (w \mod \mathcal{V}_P)$. The kernel of this map is precisely $\mathrm{Sym}^2 \mathcal{V}$. Since $\mathcal{P} \subset \mathrm{Sym}^2 \mathcal{V}$ it follows that q factors through

$$\mathrm{Sym}^2 \mathcal{W}/\mathcal{P} \to \mathcal{W} \otimes \gamma,$$

which defines a map

$$q' : \mathcal{H}om(\mathcal{P}, \mathrm{Sym}^2 \mathcal{W}/\mathcal{P}) \to \mathcal{H}om(\mathcal{P}, \mathcal{W}) \otimes \gamma.$$

By definition of $\overline{\mathcal{H}om}(\mathcal{P}, \mathcal{W})$ there exist a projection

$$q'' : \mathcal{H}om(\mathcal{P}, \mathcal{W}) \otimes \gamma \to \overline{\mathcal{H}om}(\mathcal{P}, \mathcal{W}) \otimes \gamma.$$

We take $\pi = q'' \circ q'$.

It suffices to show that $T_P F_1 = \ker \pi_P$ for all $P \in F_1$. We distinguish as always the cases $P \in F_1 \cap F_2$ and $P \in F_1 \setminus F_2$.

Case 1. Let $P \in F_1 \cap F_2$ be arbitrary. Denote $V := \mathcal{V}_P$. Let $\{e_1, e_2, e_3\} \subset W$ be a basis such that $P = \langle e_1^2, e_1 e_2 \rangle$. We have:

$$\ker q' = \{e_1^2 \mapsto a e_2^2 \bmod P, e_1 e_2 \mapsto b e_2^2 \mid a, b \in \mathbf{C}\}$$

and

$$\ker q'' = \{e_1^2 \mapsto c e_1 \otimes (e_3 \bmod V), e_1 e_2 \mapsto (c e_2 + d e_1) \otimes (e_3 \bmod V) \mid c, d \in \mathbf{C}\},$$

precisely the image of $V^\vee$ in $\mathrm{Hom}(P, W)$, tensored with $\gamma_P = W/V$. It follows that

$$\ker \pi_P = \\ \{e_1^2 \mapsto (a e_2^2 + c e_1 e_3) \bmod P,\ e_1 e_2 \mapsto (b e_2^2 + c e_2 e_3 + d e_1 e_3) \bmod P \mid a, b, c, d \in \mathbf{C}\}.$$

On the other hand, in lemma 3.2.5 we found local equations for F_1 near P. From these we can immediately see that $T_P F_1$, as subspace of $\mathrm{Hom}(P, \mathrm{Sym}_2 W/P)$, coincides with $\ker \pi_P$.

Case 2. Let $P \in F_1 \setminus F_2$ be arbitrary. Denote again $V := \mathcal{V}_P$. Let $\{e_1, e_2, e_3\} \subset W$ be a basis such that $P = \langle e_1^2, e_2^2 \rangle$. We have:

$$\ker q' = \{e_1^2 \mapsto a e_1 e_2 \mod P, e_2^2 \mapsto b e_1 e_2 \mid a, b \in \mathbf{C}\}$$

and

$$\ker q'' = \{e_1^2 \mapsto c e_1 \otimes (e_3 \mod V), e_2^2 \mapsto d e_2) \otimes (e_3 \mod V) \mid c, d \in \mathbf{C}\},$$

precisely the image of $V^\vee$ in $\mathrm{Hom}(P, W)$, tensored with $\gamma_P = W/V$. It follows that

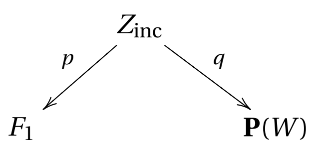
$$\ker \pi_P = \{e_1^2 \mapsto (a e_1 e_2 + c e_1 e_3) \bmod P, e_2^2 \mapsto (b e_1 e_2 + d e_2 e_3) \bmod P \mid a, b, c, d \in \mathbf{C}\}.$$

On the other hand, in lemma 3.2.8 we found local equations for F_1 near P. From these we can immediately see that $T_P F_1$, as subspace of $\mathrm{Hom}(P, \mathrm{Sym}_2 W/P)$, coincides with $\ker \pi_P$.

□

In particular, this result identifies a rank 2 subbundle $\overline{\mathcal{H}om}(\mathcal{P}, \mathcal{V}) \otimes \gamma$ of the normal bundle of $F_1 \subset \mathrm{Gr}_2(\mathrm{Sym}^2 W)$. We will investigate this subbundle more closely.

Since $F_1 \cong \mathbf{P}(W)^{[2]}$ we have the following incidence correspondence:



(3.4.3)

where $Z_{\text{inc}} \subset F_1 \times \mathbf{P}(W)$ corresponds to the tautological subscheme of $\mathbf{P}(W)^{[2]} \times \mathbf{P}(W)$ under the identification $F_1 \times \mathbf{P}(W) \cong \mathbf{P}(W)^{[2]} \times \mathbf{P}(W)$. Any line bundle $\mathcal{L}$ on $\mathbf{P}(W)$ defines a rank 2 vector bundle $p_* q^* \mathcal{L}$ on F_1. The fibre of this vector bundle over a point $x \in F_1$ is naturally identified with $H^0(s_x, \mathcal{L}|_{s_x})$, where $s_x \subset \mathbf{P}(W)$ denotes the length 2 subscheme corresponding to x. In words, the fiber of $p_* q^* \mathcal{L}$ over x is precisely the space of sections of $\mathcal{L}$ over s_x.

Proposition 3.4.4. *We can naturally identify* $\overline{\mathcal{H}om}(\mathcal{P}, \mathcal{V}) \otimes \gamma \cong p_* q^* (\mathcal{O}_{\mathbf{P}(W)}(3) \otimes \det W)$.

Proof. Let $P \in F_1$ be arbitrary, $s_P \subset \mathbf{P}(W)$ the corresponding subscheme. Then we have natural identifications

$$\begin{aligned} p_* q^* (\mathcal{O}_{\mathbf{P}(W)}(3) \otimes \det W)_P &\cong H^0(s_P, \mathcal{O}_{s_P}(3) \otimes \det W) \\ &\cong H^0(s_P, \mathcal{H}om(\mathcal{O}_{s_P}(-3), \det W)). \end{aligned}$$

Assume that s_P is reduced. Let $L_1, L_2 \subset W$ be 1-dimensional subspaces such that s_P is supported on $\mathbf{P}(L_1 \cup L_2)$ and set $V = \langle L_1, L_2 \rangle$. Then

$$H^0(s_P, \mathcal{H}om(\mathcal{O}_{s_P}(-3), \det W)) \cong \mathrm{Hom}(L_1^3, \det W) \oplus \mathrm{Hom}(L_2^3, \det W).$$

On the other hand, we have in this case

$$(\overline{\mathcal{H}om}(\mathcal{P}, \mathcal{V}) \otimes \gamma)_P \cong \mathrm{Hom}(L_1^2, V/L_1 \otimes W/V) \oplus \mathrm{Hom}(L_2^2, V/L_2 \otimes W/V).$$

This follows from the definition of $\overline{\mathcal{H}om}(\mathcal{P}, \mathcal{V})$, see also remark 3.4.2. Now note that there is a natural isomorphism $L_i \otimes V/L_i \otimes W/V \cong \det W$ $(i = 1, 2)$, which in turn gives a natural isomorphism

$$\begin{gathered} \mathrm{Hom}(L_i^2, V/L_i \otimes W/V) \cong \mathrm{Hom}(L_i^3, \det W) \\ \big(u^2 \mapsto (v \bmod L_i) \otimes (w \bmod V)\big) \mapsto \big(u^3 \mapsto u \wedge v \wedge w\big) \end{gathered}$$

for $u \in L_i$, $v \in V$ and $w \in W$; it is not hard to see that this is well-defined. We thus obtain a natural identification:

$$\begin{aligned} (\overline{\mathcal{H}om}(\mathcal{P}, \mathcal{V}) \otimes \gamma)_P &\cong \mathrm{Hom}(L_1^3, \det W) \oplus \mathrm{Hom}(L_2^3, \det W) \\ &= p_* q^* (\mathcal{O}_{\mathbf{P}(W)}(3) \otimes \det W)_P \end{aligned}$$

Now assume that s_P is non-reduced. Let $L \subset V \subset W$ be the associated complete flag such that s_P is supported on $\mathbf{P}(L)$ and has linear span $\mathbf{P}(V)$. Then

$$H^0(s_P, \mathcal{H}om(\mathcal{O}_{s_P}(-3), \det W)) \cong \mathrm{Hom}(L^2 \cdot V, \det W).$$

It follows from remark 3.4.2 that in this case we can identify

$$(\overline{\mathcal{H}om}(\mathcal{P}, \mathcal{V}) \otimes \gamma)_P \cong \mathrm{Hom}(L \cdot V, V/L \otimes W/V).$$

Again, by the fact that $L\otimes V/L\otimes W/V \cong \det W$ we obtain a natural identification:

$$\begin{gathered}\mathrm{Hom}(L\cdot V, V/L\otimes W/V) \cong \mathrm{Hom}(L^2\cdot V, \det W)\\ \big(u\cdot v\mapsto (v \bmod L)\otimes(w \bmod V)\big)\mapsto \big(u^2\cdot v\mapsto u\wedge v\wedge w\big)\end{gathered}$$

for $u\in L$, $v\in V$ and $w\in W$. By naturality these identifications extend to a natural identification

$$\begin{aligned}(\overline{\mathcal{H}om}(\mathcal{P},\mathcal{V})\otimes\gamma)_P &\cong \mathrm{Hom}(L^2\cdot V,\det W)\\ &= p_*q^*(\mathcal{O}_{\mathbf{P}(W)}(3)\otimes\det W)_P\end{aligned}$$

□

By identification (3.4.1) the subbundle $\overline{\mathcal{H}om}(\mathcal{P},\mathcal{V})\otimes\gamma$ defines a subbundle of $\nu_{\mathcal{F}_1/\mathcal{G}}$. We will denote it by $\mathcal{A}$. By proposition 3.4.4 there is a natural identification

$$\mathcal{A}\cong p_*q^*(O_{\mathbf{P}(W)}(3)\otimes\det W) \tag{3.4.4}$$

with p,q as in diagram (3.4.3). This identification will play a key role in the rewriting of the equations for E.

Remark 3.4.5. The definition of $\mathcal{A}$ involves the identification of some bundles. Although all of these identifications are natural, it can be hard to keep track of them. Let us try to make it more explicit. First assume that $p=(P,0)\in F_1\setminus F_2$. Choose a basis $\{e_1,e_2,e_2\}$ of W such that $P=\langle e_1^2,e_2^2\rangle$ and let $P^\perp:=\langle e_1e_2,e_1e_3,e_2e_3,e_3^2\rangle$. We can identify

$$T_p\mathcal{G}=\mathrm{Hom}(P,P^\perp)\oplus\mathbf{A}^1=P^\perp\otimes P^\vee\oplus\mathbf{A}^1.$$

Let $N_p: T_p\mathcal{G}\to(\nu_{F_1/\mathcal{G}})_p$ be the projection, then

$$\mathcal{A}_p=N_p\langle(e_2\cdot e_3\otimes(e_1^\vee)^2,0),(e_1\cdot e_3\otimes(e_2^\vee)^2,0)\rangle; \tag{3.4.5}$$

this follows from tracing the definition. Furthermore, let $L_1=\langle e_1\rangle$, $L_2=\langle e_2\rangle$ and $V=L_1+L_2$, then we have an identification

$$\big(p_*q^*O_{\mathbf{P}(W)}(3)\otimes\det W\big)_p=\mathrm{Hom}(L_1^3,\det W)\oplus\mathrm{Hom}(L_2^2,\det W),$$

and the isomorphism (3.4.4) is given by

$$\begin{aligned}N_p((e_2\cdot e_3\otimes(e_1^\vee)^2,0)&\mapsto(e_1\wedge e_2\wedge e_3)\otimes(e_1^\vee)^3,\\ N_p((e_1\cdot e_3\otimes(e_2^\vee)^2,0)&\mapsto(e_1\wedge e_2\wedge e_3)\otimes(e_2^\vee)^3.\end{aligned}$$

Now assume that $p=(P,0)\in F_1\cap F_2$. Choose a basis $\{e_1,e_2,e_2\}$ of W such that $P=\langle e_1^2,e_1e_2\rangle$ and let $P^\perp:=\langle e_2^2,e_1e_3,e_2e_3,e_3^2\rangle$. We can identify

$$T_p\mathcal{G}=\mathrm{Hom}(P,P^\perp)\oplus\mathbf{A}^1=P^\perp\otimes P^\vee\oplus\mathbf{A}^1.$$

Let $N_p : T_p\mathcal{G} \to (\nu_{F_1/\mathcal{G}})_p$ be the projection, then

$$\mathcal{A}_p = N_p\langle (e_2\cdot e_3 \otimes (e_1^\vee)^2, 0), (e_2\cdot e_3 \otimes e_1^\vee e_2^\vee, 0)\rangle; \tag{3.4.6}$$

again this follows from tracing the definition. Furthermore, let $L = \langle e_1\rangle$ and $V = \langle e_1, e_2\rangle$, then we have an identification

$$\left(\pi_1^* p_* q^* O_{\mathbf{P}(W)}(3) \otimes \det W\right)_p = \mathrm{Hom}(L^2\cdot V, \det W),$$

and the isomorphism (3.4.4) is given by

$$N_p((e_2\cdot e_3 \otimes (e_1^\vee)^2, 0) \mapsto (e_1\wedge e_2\wedge e_3)\otimes (e_1^\vee)^3,$$
$$N_p((e_2\cdot e_3 \otimes e_1^\vee e_2^\vee, 0) \mapsto (e_1\wedge e_2\wedge e_3)\otimes (e_1^\vee)^2\cdot e_2^\vee.$$

3.4.2 Rewriting local equations for E

Remember that we view E as subvariety of $\mathbf{P}(\nu_{\mathcal{F}_1/\mathcal{G}})$. In this section we will rewrite the equations for E in terms of maps between bundles on $\mathcal{F}_1$. First we make the following observation:

Lemma 3.4.6. $E \subset \mathbf{P}(\mathcal{A}\oplus\mathcal{O}_{F_1})$.

Proof. This follows by comparing remarks 3.4.5 and 3.3.6. Let us explain this in some detail. First assume $p = (P, 0) \in F_1 \cap F_2$ and let $\{e_1, e_2, e_3\}$ (as always) be a basis of W such that $P = \langle e_1^2, e_1e_2\rangle$ and $P^\perp := \langle e_2^2, e_1e_3, e_2e_3, e_3^2\rangle$. We denote by

$$N_p : T_p\mathcal{G} \to (\nu_{F_1/\mathcal{G}})_p$$

the projection and identify

$$T_p\mathcal{G} \cong \mathrm{Hom}(P, P^\perp)\oplus\mathbf{A}^1 \cong P^\perp\otimes P^\vee \oplus \mathbf{A}^1.$$

Let $\beta_{i,jk}$, for $(i, jk) = (1,23), (2,23), (1,33), (2,33)$, be the functions as defined in remark 3.3.6 for the case $p \in F_1\cap F_2$. By equations 3.3.13 and 3.3.14 E_p is contained in the linear subspace of $\mathbf{P}(\nu_{F_1/\mathcal{G}})_p$ on which $\beta_{1,33}$ and $\beta_{2,33}$ vanish. We claim that this linear subspace is precisely $\mathbf{P}(\mathcal{A}\oplus\mathcal{O}_{F_1})_p$, so that it follows that

$$E_p \subset \mathbf{P}(\mathcal{A}\oplus\mathcal{O}_{F_1})_p.$$

Indeed, $\beta_{1,33}$ and $\beta_{2,33}$ obviously annihilate $\mathrm{Tot}(\mathcal{O}_{F_1})_p$. Furthermore, note that we have:

$$(\beta_{1,33}\circ N_p)(e_2e_3\otimes(e_1^\vee)^2, 0) = d_p b_{1,33}(e_2e_3\otimes(e_1^\vee)^2, 0) = 0$$
$$(\beta_{1,33}\circ N_p)(e_2e_3\otimes e_1e_2, 0) = d_p b_{1,33}(e_2e_3\otimes e_1^\vee e_2^\vee, 0) = 0$$
$$(\beta_{2,33}\circ N_p)(e_2e_3\otimes(e_1^\vee)^2, 0) = d_p b_{2,33}(e_2e_3\otimes(e_1^\vee)^2, 0) = 0$$
$$(\beta_{2,33}\circ N_p)(e_2e_3\otimes e_1e_2, 0) = d_p b_{2,33}(e_2e_3\otimes e_1^\vee e_2^\vee, 0) = 0,$$

as easily follows from the definition of the local functions $b_{i,jk}$. It then follows from identity (3.4.5) that $\mathcal{A}_p$ is annihilated by $\beta_{1,33}$ and $\beta_{2,33}$. The claim follows from a dimension count and the fact that $\beta_{1,33}$ and $\beta_{2,33}$ are linearly independent differentials.

Now assume $p = (P,0) \in F_1 \setminus F_2$ and let $\{e_1, e_2, e_3\}$ (as always) be a basis of W such that $P = \langle e_1^2, e_2^2\rangle$ and $P^\perp := \langle e_1e_2, e_1e_3, e_2e_3, e_3^2\rangle$. We denote by

$$N_p : T_p\mathcal{G} \to (\nu_{F_1/\mathcal{G}})_p$$

the projection and identify

$$T_p\mathcal{G} \cong \mathrm{Hom}(P, P^\perp) \oplus \mathbf{A}^1 \cong P^\perp \otimes P^\vee \oplus \mathbf{A}^1.$$

Let $\beta_{i,jk}$, for $(i, jk) = (1,23), (2,13), (1,33), (2,33)$, be the functions as defined in remark 3.3.6 for the case $p \in F_1 \setminus F_2$. By equations 3.3.16 and 3.3.17 E_p is contained in the linear subspace of $\mathbf{P}(\nu_{F_1/\mathcal{G}})_p$ on which $\beta_{1,33}$ and $\beta_{2,33}$ vanish.
We claim that again this linear subspace is precisely $\mathbf{P}(\mathcal{A} \oplus \mathcal{O}_{F_1})_p$, so that

$$E_p \subset \mathbf{P}(\mathcal{A} \oplus \mathcal{O}_{F_1})_p.$$

Indeed, again $\beta_{1,33}$ and $\beta_{2,33}$ obviously annihilate $\mathrm{Tot}(\mathcal{O}_{F_1})_p$. Furthermore, note that we have:

$$\begin{aligned}
(\beta_{1,33} \circ N_p)(e_2e_3 \otimes (e_1^\vee)^2, 0) &= d_p b_{1,33}(e_2e_3 \otimes (e_1^\vee)^2, 0) = 0\\
(\beta_{1,33} \circ N_p)(e_1e_3 \otimes (e_2^\vee)^2, 0) &= d_p b_{1,33}(e_2e_3 \otimes (e_2^\vee)^2, 0) = 0\\
(\beta_{2,33} \circ N_p)(e_2e_3 \otimes (e_1^\vee)^2, 0) &= d_p b_{2,33}(e_2e_3 \otimes (e_1^\vee)^2, 0) = 0\\
(\beta_{2,33} \circ N_p)(e_1e_3 \otimes (e_2^\vee)^2, 0) &= d_p b_{2,33}(e_2e_3 \otimes (e_2^\vee)^2, 0) = 0,
\end{aligned}$$

as easily follows from the definition of the local functions $b_{i,jk}$. It then follows from identity (3.4.5) that $\mathcal{A}_p$ is annihilated by $\beta_{1,33}$ and $\beta_{2,33}$. The claim follows from a dimension count and the fact that $\beta_{1,33}$ and $\beta_{2,33}$ are linearly independent differentials. □

Now let

$$Q : \mathcal{O}_{\mathbf{P}(W)}(3) \otimes \det W \to \mathcal{O}_{\mathbf{P}(W)}(6) \otimes (\det W)^2$$

be the natural quadratic map. Let p, q be as in the incidence diagram (3.4.3). Then Q induces a quadratic map

$$\widetilde{Q} : p_*q^*(\mathcal{O}_{\mathbf{P}(W)}(3) \otimes \det W) \to p_*q^*(\mathcal{O}_{\mathbf{P}(W)}(6) \otimes (\det W)^2)$$

of bundles on $\mathcal{F}_1$. Let $p = (P,0) \in F_1 \setminus F_2$, $L_1, L_2 \subset W$ such that $P = L_1^2 + L_2^2$, then under the identification

$$(p_*q^*(\mathcal{O}_{\mathbf{P}(W)}(3) \otimes \det W)_p = \mathrm{Hom}(L_1^3 \oplus L_2^3, \det W)$$

we have

$$\begin{array}{rcl}\widetilde{Q}_P : \mathrm{Hom}(L_1^3 \oplus L_2^3, \det W) & \to & \mathrm{Hom}(L_1^6 \oplus L_2^6, (\det W)^2) \\ (\varphi_1, \varphi_2) & \mapsto & (\varphi_1^2, \varphi_2^2).\end{array}$$

For $p \in F_1 \cap F_2$ let $L \subset V \subset W$ be the flag such that $P = L \cdot V$, then we may naturally identify

$$p_* q^* (\mathcal{O}_{\mathbf{P}(W)}(3) \otimes \det W)_p = \mathrm{Hom}(L^2 \cdot V, \det W).$$

Notice that $L^5 \cdot V \subset \mathrm{Sym}^2(L^2 \cdot V)$. Therefore any element $\varphi \in \mathrm{Hom}(L^2 \cdot V, \det W)$ defines an element $\varphi^{(2)} \in \mathrm{Hom}(L^5 \cdot V, (\det W)^2)$ by restricting φ^2 to $L^5 \cdot V$. In this case $\widetilde{Q}_P$ is precisely the map $\varphi \mapsto \varphi^{(2)}$. Remember (see equation (3.4.4)) that we have a natural identification

$$\mathcal{A} \cong p_* q^* (\mathcal{O}_{\mathbf{P}(W)}(3) \otimes \det W).$$

By slight abuse of notation we will denote the quadratic map

$$\mathcal{A} \to p_* q^* (\mathcal{O}_{\mathbf{P}(W)}(6) \otimes (\det W)^2)$$

induced by $\widetilde{Q}$ by the same symbol.

The cubic form f that we have fixed (see section 3.1.1) was chosen such that $v^* f \neq 0$. It uniquely defines a non-zero section τ_f of $\mathcal{O}_{\mathbf{P}(W)}(6)$. In turn, τ_f defines a global section $\widetilde{\tau}_f$ of $p_* q^* \mathcal{O}_{\mathbf{P}(W)}(6)$. Indeed, let $P \in F_1$ be arbitrary and $s_P \subset \mathbf{P}(W)$ be the length 2 subscheme defined by the natural identification $F_1 \cong \mathbf{P}(W)^{[2]}$. Then

$$\widetilde{\tau}_f(P) := \tau_f|_{s_P} \in H^0(s_P, \mathcal{O}_{\mathbf{P}(W)}(6)|_{s_P}) = \big(p_* q^* \mathcal{O}_{\mathbf{P}(W)}(6)\big)_P.$$

Theorem 3.4.7. *Let $\mu \in \det W$ be a volume form. Let $\alpha : \mathrm{Tot}(\mathcal{A}) \to F_1$ be the projection map and let $T : \mathrm{Tot}(\mathcal{O}_{F_1}) \to \mathbf{A}^1$ be the trivializing morphism induced by the parameter t on Δ. The variety $E \subset \mathbf{P}(\mathcal{A} \oplus \mathcal{O}_{F_1})$ is defined by the homogeneous quadratic equation*

$$\widetilde{Q}(y) = 4T(z)^2(\mu^2 \otimes \widetilde{\tau}_f \circ \alpha(y)) \tag{3.4.7}$$

for $(y, z) \in \mathrm{Tot}(\mathcal{A} \oplus \mathcal{O}_{F_1})$.

Proof. We prove it fiberwise. First let $p = (P, 0) \in F_1 \setminus F_2$. Let $\{e_1, e_2, e_3\} \subset W$ be a basis such that $P = \langle e_1^2, e_2^2 \rangle$ and $\mu = e_1 \wedge e_2 \wedge e_3$. Let $\{e_1^\vee, e_2^\vee, e_3^\vee\} \subset W^\vee$ be the dual basis. Let $y \in \mathcal{A}_p$ be arbitrary. By remark 3.4.5 we may write

$$y = a_y N_p(e_2 \cdot e_3 \otimes (e_1^\vee)^2, 0) + b_y N_p(e_1 \cdot e_3 \otimes (e_2^\vee)^2, 0)$$

for $a_y, b_y \in \mathbf{C}$. We then have

$$\widetilde{Q}(y) = a_y^2(\mu^2 \otimes (e_1^\vee)^6) + b_y^2(\mu^2 \otimes (e_2^\vee)^6.$$

Let $\beta_{i,jk}$ be the linear forms on $(\nu_{F_1/\mathcal{G}})_p$ as introduced in remark 3.3.6 for the case $p \in F_1 \setminus F_2$. Then we have

$$\beta_{1,23}(y) = d_p b_{1,23}\big(a_y\, e_2 \cdot e_3 \otimes (e_1^\vee)^2 + b_y\, e_1 \cdot e_3 \otimes (e_2^\vee)^2, 0\big) = \frac{1}{2} a_y$$
$$\beta_{2,13}(y) = d_p b_{2,13}\big(a_y\, e_2 \cdot e_3 \otimes (e_1^\vee)^2 + b_y\, e_1 \cdot e_3 \otimes (e_2^\vee)^2, 0\big) = \frac{1}{2} b_y,$$

hence we can write

$$\widetilde{Q}(y) = 4\beta_{1,23}(y)^2(\mu^2 \otimes (e_1^\vee)^6) + 4\beta_{2,13}(y)^2(\mu^2 \otimes (e_2^\vee)^6.$$

On the other hand, let $L_1 = \langle e_1 \rangle$ and $L_2 = \langle e_2 \rangle$. Then $(\widetilde{\tau}_f \circ \alpha)(y) = \widetilde{\tau}_f(P)$ is the restriction of the sextic in $\mathrm{Sym}^6 W^\vee$, defined by f, to the subspace $L_1^6 \oplus L_1^6 \subset \mathrm{Sym}^6 W$. Explicitly, we can write

$$\tau_f(P) = f_{11,11,11}(e_1^\vee)^6 + f_{22,22,22}(e_2^\vee)^6,$$

where the functions $f_{ij,kl,mn}$ are as introduced in section 3.2.3 Case 2. We find that equation (3.4.7) is equivalent to the set of equations

$$\beta_{1,23}^2 = T^2 f_{11,11,11}$$
$$\beta_{2,13}^2 = T^2 f_{22,22,22}$$

in $\mathbf{P}(\mathcal{A} \oplus \mathcal{O}_{F_1})_p$. Compare this to equations 3.3.15 to 3.3.18 and remember that $\mathbf{P}(\mathcal{A} \oplus \mathcal{O}_{F_1})_p$ as a subspace of $\mathbf{P}(\nu_{F_1/\mathcal{G}})_p$ is cut out by the equations $\beta_{1,33}$ and $\beta_{2,33}$; it follows that equation (3.4.7) defines E_p.

Now let $p = (P, 0) \in F_1 \cap F_2$. Let $\{e_1, e_2, e_3\} \subset W$ be a basis such that $P = \langle e_1^2, e_1 e_2 \rangle$ and $\mu = e_1 \wedge e_2 \wedge e_3$. Let $\{e_1^\vee, e_2^\vee, e_3^\vee\} \subset W^\vee$ be the dual basis. Let $y \in \mathcal{A}_p$ be arbitrary. By remark 3.4.5 we may write

$$y = a_y N_p(e_2 \cdot e_3 \otimes (e_1^\vee)^2, 0) + b_y N_p(e_2 \cdot e_3 \otimes e_1^\vee e_2^\vee, 0)$$

for $a_y, b_y \in \mathbf{C}$. We then have

$$\widetilde{Q}(y) = a_y^2(\mu^2 \otimes (e_1^\vee)^6) + 2a_y b_y(\mu^2 \otimes (e_1^\vee)^5 e_2^\vee).$$

Let $\beta_{i,jk}$ be the linear forms on $(\nu_{F_1/\mathcal{G}})_p$ as introduced in remark 3.3.6 for the case $p \in F_1 \cap F_2$. Then we have

$$\beta_{1,23}(y) = d_p b_{1,23}\big(a_y\, e_2 \cdot e_3 \otimes (e_1^\vee)^2 + b_y\, e_2 \cdot e_3 \otimes e_1^\vee e_2^\vee, 0\big) = \frac{1}{2} a_y$$
$$\beta_{2,23}(y) = d_p b_{2,13}\big(a_y\, e_2 \cdot e_3 \otimes (e_1^\vee)^2 + b_y\, e_2 \cdot e_3 \otimes e_1^\vee e_2^\vee, 0\big) = \frac{1}{4} b_y.$$

Note the difference in prefactors here; they are a consequence of our conventions for contractions of elements of symmetric products of vector spaces, see section 3.1.1. We can write

$$\widetilde{Q}(y) = 4\beta_{1,23}(y)^2(\mu^2 \otimes (e_1^\vee)^6) + 16\beta_{1,23}\beta_{2,23}(y)(\mu^2 \otimes (e_2^\vee)^6.$$

On the other hand, let $L = \langle e_1\rangle$ and $V = \langle e_1, e_2\rangle$. Then $(\widetilde{\tau}_f \circ \alpha)(y) = \widetilde{\tau}_f(P)$ is the restriction of the sextic in $\mathrm{Sym}^6 W^\vee$, defined by f, to the subspace $L^5 \cdot V \subset \mathrm{Sym}^6 W$. Using the function $f_{ij,kl,mn}$ as introduced in section 3.2.3 (Case 1) we can write explicitly:

$$\tau_f(P) = f_{11,11,11}(p)(e_1^\vee)^6 + 6 f_{11,11,12}(p)(e_1^\vee)^5 \cdot e_2^\vee.$$

Again, note the prefactor in the second term. We find that equation (3.4.7) is equivalent to the set of equations

$$\begin{aligned} \beta_{1,23}^2 &= T^2 f_{11,11,11} \\ 2\beta_{1,23}\beta_{2,23} &= 3T^2 f_{22,22,22} \end{aligned}$$

in $\mathbf{P}(\mathcal{A} \oplus \mathcal{O}_{F_1})_p$. Compare this to equations 3.3.11 to 3.3.12 and remember that $\mathbf{P}(\mathcal{A} \oplus \mathcal{O}_{F_1})_p$ as a subspace of $\mathbf{P}(\nu_{F_1/\mathcal{G}})_p$ is cut out by the equations $\beta_{1,33}$ and $\beta_{2,33}$; it follows that equation (3.4.7) defines E_p. □

3.4.3 Identification of E

We will show in this section that there is a natural isomorphism

$$E \cong \mathrm{Bl}(V, S_f^{[2]}),$$

where S_f is the double plane branched along sextic curve defined by $v^* f$ and $V \subset S_f^{[2]}$ is the locus of length 2 subschemes in S_f that are the fibers of the covering map $S_f \to \mathbf{P}(W)$.

First we introduce some notation. Note that we have an embedding:

$$\begin{aligned} i_1 : \mathcal{A} &\to \mathbf{P}(\mathcal{A} \oplus \mathcal{O}_{F_1}) \\ y &\mapsto (y : 1). \end{aligned}$$

Let E^0 be the intersection of E with the image of i_1 and E^∞ the complement of E^0 in E. Using the results of the previous subsection we can identify the structure of E^0 and E^∞:

Theorem 3.4.8. *Let $\pi : S_f \to \mathbf{P}(W)$ be the double cover branched along the sextic defined by $v^* f$. Then $E^0 \cong S_f^{[2],hor}$, the space of length two subschemes of S_f that are horizontal with respect to π, that is, not contained in a fiber of π.*

Proof. As in section 3.4.2 let τ_f be the section of $\mathcal{O}_{\mathbf{P}(W)}(6)$ defined by $v^* f$ and let $\widetilde{\tau}_f$ be the induced section of $p_* q^* \mathcal{O}_{\mathbf{P}(W)}(6)$. It suffices to prove that $i_1^{-1} E^0 \cong S_f^{[2],\mathrm{hor}}$. From theorem 3.4.7 and the definition of i_1 it follows that

$$i_1^{-1} E^0 = \widetilde{Q}^{-1} \Gamma_{\widetilde{\tau}_f \otimes \mu^2},$$

where $\Gamma_{\widetilde{\tau}_f \otimes \mu^2}$ is the graph of the section $\widetilde{\tau}_f \otimes \mu^2$ of $\left(\pi_1^* p_* q^* \mathcal{O}_{\mathbf{P}(W)}(6)\right) \otimes (\det W)^2$. We claim that

$$\widetilde{Q}^{-1} \Gamma_{\widetilde{\tau}_f \otimes \mu^2} = \left(Q^{-1} \Gamma_{\tau_f \otimes \mu^2}\right)^{[2],\mathrm{hor}}, \tag{3.4.8}$$

where $\Gamma_{\tau_f\otimes\mu^2}$ is de graph of the section $\tau_f\otimes\mu^2$ of $\mathcal{O}_{\mathbf{P}(W)}(6)\otimes(\det W)^2$. This implies the statement of the theorem, since the natural map $Q^{-1}\Gamma_{\tau_f\otimes\mu^2}\to\mathbf{P}(W)$ is precisely a double cover of $\mathbf{P}(W)$ branched along the zero-scheme of τ_f, so that $S_f\cong Q^{-1}\Gamma_{\tau_f\otimes\mu^2}$.

We prove the claim. Let $\lambda:\mathrm{Tot}(\mathcal{O}_{\mathbf{P}(W)}(3)\otimes\det W)\to\mathbf{P}(W)$ be the natural projection and

$$\widetilde{\lambda}:\mathrm{Tot}\big(p_*q^*(\mathcal{O}_{\mathbf{P}(W)}(3)\otimes\det W)\big)\to F_1$$

the map induced by λ. The defining equation for the scheme $Q^{-1}\Gamma_{\tau_f\otimes\mu^2}$ is

$$Q(y)=(\tau_f\circ\lambda)(y)\otimes\mu^2$$

for $y\in\mathrm{Tot}(\mathcal{O}_{\mathbf{P}(W)}(3)\otimes\det W)$. Now note that for any locally free sheaf $\mathcal{R}$ on $\mathbf{P}(W)$ the horizontal length 2 subschemes of $\mathrm{Tot}\,\mathcal{R}$ are precisely those subschemes that are of the form $\sigma(s)$, for $s\subset\mathbf{P}(W)$ a length 2 subscheme and $\sigma\in H^0(s,\mathcal{R}|_s)$. Hence there is a natural identification $(\mathrm{Tot}\,\mathcal{R})^{[2],\mathrm{hor}}\cong\mathrm{Tot}(p_*q^*\mathcal{R})$.
In particular $(Q^{-1}\Gamma_{\tau_f\otimes\mu^2})^{[2],\mathrm{hor}}\subset\mathrm{Tot}(p_*q^*\mathcal{O}_{\mathbf{P}(W)}(3)\otimes\det W)$, and its defining equation is

$$Q\circ\varphi=\tau_f|_{s_\varphi}\otimes\mu^2, \tag{3.4.9}$$

for $\varphi\in\mathrm{Tot}(p_*q^*\mathcal{O}_{\mathbf{P}(W)}(3)\otimes\det W)$, where $s_\varphi=q(p^{-1}\widetilde{\lambda}(\varphi))$ is the subscheme of $\mathbf{P}(W)$ corresponding to the point $\widetilde{\lambda}(\varphi)$ and φ is interpreted as an element of $H^0(s_\varphi,\mathcal{O}_{\mathbf{P}(W)}(3)|_{s_\varphi})$. By definition of $\widetilde{Q}$ and $\widetilde{\tau}_f$ this equation can be rewritten as

$$\widetilde{Q}(\varphi)=(\widetilde{\tau}\circ\widetilde{\lambda})(\varphi)\otimes\mu^2,$$

for $\varphi\in\mathrm{Tot}(p_*q^*\mathcal{O}_{\mathbf{P}(W)}(3)\otimes\det W)$. This is the defining equation of $\widetilde{Q}^{-1}\Gamma_{\widetilde{\tau}\otimes\mu^2}$. □

Theorem 3.4.9. *The locus E^∞ has the following properties:*

1. *The restricted projection map $E^\infty\to F_1$ is an embedding, with as image precisely $F_1\cap F_2$. Equivalently, under the natural identification $F_1\cong\mathbf{P}(W)^{[2]}$ the image of E^∞ is the locus of non-reduced length 2 subschemes.;*

2. *E^∞ equals the intersection in $\widehat{\mathcal{G}}$ of E and $\widehat{F}_2$;*

3. *E^∞ is contained in the smooth locus of E;*

Proof. Let $p=(P,0)\in F_1\setminus F_2$ and let $L_1,L_2\subset W$ be such that $P=\langle L_1^2,L_2^2\rangle$. We may identify:

$$\mathcal{A}_p\cong\mathrm{Hom}(L_1^3,\det W)\oplus\mathrm{Hom}(L_2^3,\det W).$$

By this identification we write $\mathcal{A}_p\ni\varphi=(\varphi_1,\varphi_2)$ with $\varphi_i\in\mathrm{Hom}(L_i^3,\det W)$. Then

$$\widetilde{Q}(\varphi)=(\varphi_1^2,\varphi_2^2)\in\mathrm{Hom}(L_1^6,(\det W)^2)\oplus\mathrm{Hom}(L_2^6,(\det W)^2).$$

It follows that $\widetilde{Q}(\varphi) = 0$ if and only if $\varphi = 0$, hence the fiber of $i_0^{-1}E^\infty$ over p is empty.
It follows that

$$i_0^{-1}E^\infty \subseteq \mathbf{P}(\mathcal{A})|_{F_1\cap F_2}.$$

Let $p = (P, 0) \in F_1 \cap F_2$. Let $L \subset V \subset W$ be flag associated to P (such that $P = L \cdot V \subset \mathrm{Sym}^2 W$). We may identify:

$$\mathcal{A}_p \cong \mathrm{Hom}(L^2 \cdot V, \det W),$$

and under this identification we have for $\varphi \in \mathcal{A}_p$ that $\widetilde{Q}(\varphi) = \varphi^{(2)}$, the restriction of $\varphi^2 \in \mathrm{Hom}(\mathrm{Sym}^2(L^2 \cdot V), (\det W)^2)$ to the space $L^5 \cdot V$. So $\widetilde{Q}(\varphi) = 0$ if and only if $\varphi(e_1^3)\varphi(e_1^2 e_2) = 0$ for all $e_1 \in L$ and $e_2 \in V$. Since $L \subseteq V$ this is equivalent to $\varphi|_{L^3} = 0$. Hence $i_0^{-1}E^\infty$ and the fiber $\mathbf{P}(\mathcal{A})_p$ intersect in the point

$$[(e_1^\vee)^2 e_2] \in \mathbf{P}(\mathrm{Hom}(L^2 V, \det W)) \cong \mathbf{P}(\mathcal{A})_p.$$

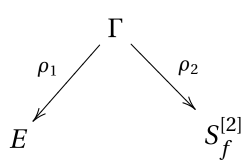
It follows that the natural map $i_0^{-1}E^\infty \to F_1 \cap F_2$ is 1:1, hence an isomorphism, since $F_1 \cap F_2$ is normal. This proves the second statement.

The third statement follows directly from the second and theorem 3.3.7.

For the fourth statement note that T only occurs quadratically in the equations for E, as given by theorem 3.4.7 (here T is a parameter on the fibres of the total space of $\mathcal{O}_{F_1}$). Since E^∞ is precisely the locus where $T = 0$, it follows that E is smooth along E^∞ precisely if E^∞ is smooth itself. Smoothness of E^∞ follows from statement 2 and the fact the $F_1 \cap F_2$ is smooth. □

Corollary 3.4.10. *Let $\pi : S_f \to \mathbf{P}(W)$ the double cover branched along the sextic curve defined by $\nu^* f$. Let $V \subset S_f^{[2]}$ be the set of length 2 subschemes that are the fibres of π. There is a birational morphism $\psi : E \to S_f^{[2]}$ which is an isomorphism over E^0 and contracts E^∞ to V. In particular, if the sextic curve is smooth, E is smooth and birational to a holomorphic symplectic fourfold.*

Proof. According to theorem 3.4.8 there is an isomorphism $E^0 \to S_f^{[2],\mathrm{hor}}$. Let $\Gamma^0 \subset E \times S^{[2]}$ be the graph of this isomorphism and Γ its closure. We obtain a correspondence



We claim that ρ_1 has finite fibers. Note that since ρ_1 and ρ_2 are induced by projections of $E \times S_f^{[2]}$ to the factors, it follows that for every $h \in E$ the map ρ_2 embeds the fiber $\rho_1^{-1}(h)$ into $S_f^{[2]}$. Hence to show that ρ_1 has finite fibers, it suffices to show that $\rho_2\rho_1^{-1}(h)$ is finite in $S_f^{[2]}$. This is clear over E^0, since by construction $\rho_2 \circ \rho_1^{-1}$ is well-defined and an isomorphism there (hence the fibers over E^0 have in fact length 1). So we investigate the fibers over $E^\infty = E \setminus E^0$. Let $\eta : E \to F_1 \cong \mathbf{P}(W)^{[2]}$ be the natural projection and

$$\pi^{[2]} : S_f^{[2],\mathrm{hor}} \to \mathbf{P}(W)^{[2]}$$

the projection induced by π. By construction the following diagram commutes:

$$\begin{array}{ccc} E^0 & \xrightarrow[\cong]{\rho_2 \circ \rho_1^{-1}} & S_f^{[2],\mathrm{hor}} \\ & {\scriptstyle \eta|_{E^0}} \searrow \quad \swarrow {\scriptstyle \pi^{[2]}} & \\ & \mathbf{P}(W)^{[2]} & \end{array}$$

So for any $h \in E^0$ the support of the length 2 subscheme $\rho_2\rho_1^{-1}(h) \in S_f^{[2]}$ of S_f is contained in $\pi^{-1}\operatorname{supp}\eta(h)$. Note that the closure of E^0 in E is precisely E (since E is smooth along the complement of E^0 by theorem 3.4.9). Using valuations not is not hard to show from these observations that for any $h \in E$ the support of any length 2 subscheme $s \in \rho_2\rho_1^{-1}(h) \in S_f^{[2]}$ must be contained in $\pi^{-1}\operatorname{supp}\eta(h)$.

Now let $h \in E \setminus E^0 = E^\infty$. Then the locus $\rho_2\rho_1^{-1}(h)$ consists of length 2 subschemes with support in $\pi^{-1}\operatorname{supp}\eta(h)$. From statement 2 in theorem 3.4.9 it follows that $\operatorname{supp}\eta(h)$ is actually a point, so that $\pi^{-1}\operatorname{supp}\eta(h)$ consists of length 2 subschemes with support in *one* fiber of π. Finally, since $\rho_2 \circ \rho_1^{-1} : E^0 \to S_f^{[2],\mathrm{hor}}$ is an isomorphism and $h \notin E^0$ it follows that $\rho_2 \circ \rho_1^{-1}(h)$ consists of length 2 subschemes that are not only supported in one fibre, but actually contained in the fiber (we use here that a length 2 subscheme that is not horizontal must be contained in a fiber). But since π is a degree 2 mapping, every fiber of π contains precisely one length 2 subscheme (namely, the fiber itself). We conclude that $\rho_2\rho_1^{-1}(h)$ consists of at most one point, and by properness of ρ_1 it must be exactly one point. In particular, ρ_1 has finite fibers. Because it is a proper morphism, it must be finite.

Let $E^* \subset E$ be the locus of regular points, Γ^* the intersection of Γ and $E^* \times S_f^{[2]}$ and ρ_1^* the restriction of ρ_1 to Γ^*. Then ρ_1^* is a finite map. Moreover it is birational since it is an isomorphism on dense open subsets of domain and image. Since by construction E^* is normal, it follows that ρ_1^* is an isomorphism.

We know:

- $\rho_1|_{\Gamma \cap E^0 \times S_f^{[2]}} \to E^0$ is an isomorphism;
- $\rho_1|_{\Gamma \cap E^* \times S_f^{[2]}} \to E^*$ is an isomorphism;
- $E^0 \cup E^* = E$ (since by theorem 3.4.9 $E^\infty \subseteq E^*$ and by definition $E = E^0 \cup E^\infty$).

It follows that ρ_1 itself is an isomorphism. Set $\psi := \rho_2 \circ \rho_1^{-1} : E \to S_f^{[2]}$, then ψ is a well defined morphism such that $\psi|_{E^0} \to S_f^{[2],\mathrm{hor}}$ is an isomorphism. Since E^0 and $S_f^{[2],\mathrm{hor}}$ are dense open subsets in E and $S_f^{[2]}$ respectively, it follows that ψ is a birational morphism. This morphism contracts E^∞ to the locus V of 'vertical' length 2 subschemes of S_f, i.e. schemes contained in the fibers of π. Note that we can naturally identify $V \cong \mathbf{P}(W)$.

For the final remark, we note that in case the sextic curve is smooth, so is S_f. In that case S_f is a K3 surface of degree 2, and $S_f^{[2]}$ is a holomorphic symplectic fourfold. It follows that $E^0 \cong S_f^{[2],\mathrm{hor}}$, and since E is smooth along E^∞, E itself must be smooth. □

Remark 3.4.11. For arbitrary $s \in V$ denote $C_s := \psi^{-1}(s) \subset E^\infty$. These are precisely the curves contracted by ψ. As before, let $\eta : E \to F_1 \cong \mathbf{P}(W)^{[2]}$ be the projection. Note that $\eta(C_s) \subset \mathbf{P}(W)^{[2]}$ is precisely the rational curve of length 2 subschemes of $\mathbf{P}(W)$ that are supported on the point $\pi(\operatorname{supp} s) \in \mathbf{P}(W)$. It follows from theorem 3.4.9 that the restriction of η to E^∞ is an embedding, so in particular $C_s \cong \mathbf{P}^1$ for all $s \in V$.

Furthermore, by theorem 3.4.9 $E^\infty = E \cap \widehat{F}_2$. In fact, by theorems 3.4.9 and 3.3.7 we have the following commutative diagram

$$\begin{array}{ccccccc}
 & E & \longleftarrow & E^\infty & \hookrightarrow & \widehat{F}_2 & \\
 \swarrow^{\eta} & & \swarrow^{\cong} & & \searrow^{\cong} & & \searrow^{\cong} \\
\mathbf{P}(W)^{[2]} & \hookleftarrow & \mathbf{P}(W)^{[2]}_{n.r} & \xrightarrow{\cong} & Z & \hookrightarrow & \mathbf{P}(W) \times \mathbf{P}(W^\vee)
\end{array}$$

where $\mathbf{P}(W)^{[2]}_{n.r}$ is the locus of non-reduced length 2 subschemes of $\mathbf{P}(W)^{[2]}$,

$$Z = \{(p,\ell) \mid p \in \ell\} \subset \mathbf{P}(W) \times \mathbf{P}(W^\vee)$$

is the incidence variety and the isomorphism $\mathbf{P}(W)^{[2]}_{n.r.} \to Z$ is given by $s' \mapsto (\operatorname{supp} s', \langle s' \rangle)$. In particular, the fibers of the projection $Z \to \mathbf{P}(W)$ correspond precisely to the curves in $\mathbf{P}(W)^{[2]}$ that parameterize length 2 subschemes supported on the same point. It follows that under the embedding $E^\infty \hookrightarrow \widehat{F}_2$ and the identification $\widehat{F}_2 \cong \mathbf{P}(W) \times \mathbf{P}(W^\vee)$ the curves C_s are precisely the fibers of the induced projection $E^\infty \to \mathbf{P}(W)$.

The morphism ψ can in fact be identified as a blow-up morphism. To prove this we need the following elementary result:

Proposition 3.4.12. *Let B be a smooth variety over $\mathbf{C}$. Let $\pi : Y \to B$ be a surjective double cover branched along some divisor in $D \subset B$. Let $V \subseteq Y^{[2]}$ be the locus of length 2 subschemes of Y that are vertical with respect to π. Then $Y^{[2]}$ is smooth along V.*

Proof. It suffices to prove the proposition for the following situation:

- $B := \operatorname{Spec} \mathbf{C}[\mathbf{x}]$ where $\mathbf{x}$ denotes $x_1, \dots, x_n$;
- $I_D := \langle f \rangle \subset \mathbf{C}[\mathbf{x}]$, $D := \operatorname{Spec} \mathbf{C}[\mathbf{x}]/I_D$;
- $Y := \operatorname{Spec} \mathbf{C}[\mathbf{x}, y]/\langle y^2 - f \rangle$;
- $\pi : Y \to B$ the natural projection.

Let σ_0 be the vertical length 2 subscheme over $0 \in B$. It suffices to prove that $\mathcal{O}_{Y^{[2]},\sigma_0}$ is regular. Denote $T := \operatorname{Spec} \mathbf{C}[\mathbf{x}, y]$ and let $H_c \subset T$ be subspaces defined by the equation $y - c = 0$ for $c \in \mathbf{C}$. Let $\widetilde{S}$ be the scheme of length 2 subschemes of B that are 'tilted' in the sense that they are not contained in H_c for any $c \in \mathbf{C}$. Let $\widetilde{S}_Y \subset \widetilde{S}$ be the subspace of tilted length 2 subschemes contained in Y. It is an open subspace of $Y^{[2]}$ and clearly contains the space V of length 2 subschemes that are vertical with respect to π. In particular it suffices to prove that $\mathcal{O}_{\widetilde{S}_Y,\sigma_0}$ is regular.

Let $\widetilde{L}$ be the scheme of lines (that is, one dimensional linear subspaces) in T that are tilted in the same sense as above. Note that there is an isomorphism

$$\begin{array}{rcl} \varphi : \operatorname{Spec} \mathbf{C}[\mathbf{a}, \mathbf{b}] & \to & \widetilde{L} \\ (\mathbf{a}, \mathbf{b}) & \mapsto & \text{line in } T \text{ defined by ideal } \langle \{a_i y + b_i - x_i\}_{1 \le i \le n} \rangle, \end{array}$$

where $\mathbf{a} := \{a_1, \dots, a_n\}$ and $\mathbf{b} := \{b_1, \dots, b_n\}$.

Since any tilted length 2 subspace uniquely spans a tilted line in T we have a natural map $\lambda : \widetilde{S} \to \widetilde{L}$. Note that $\lambda^{-1}(\ell) = \ell^{[2]} \cong \mathbf{A}^2_{\mathbf{C}}$, so λ defines a bundle of affine spaces over $\widetilde{L}$. We have the following trivializing diagram:

$$\begin{array}{ccc} \operatorname{Spec} \mathbf{C}[\mathbf{a}, \mathbf{b}, s, t] & \xrightarrow[\cong]{\psi} & \widetilde{S} \\ \downarrow & & \downarrow \lambda \\ \operatorname{Spec} \mathbf{C}[\mathbf{a}, \mathbf{b}] & \xrightarrow[\cong]{\varphi} & \widetilde{L} \end{array}$$

where ψ maps a point with coordinates $(\mathbf{a}, \mathbf{b}, s, t)$ to the length 2 subscheme of T with vanishing ideal $\langle \{a_i y + b_i - x_i\}_{1 \le i \le n}, y^2 + sy + t \rangle$. Clearly ψ is an isomorphism. It follows that as a set

$$\begin{array}{rcl} \psi^{-1}(\widetilde{S}_Y) & = & \{(\mathbf{a}, \mathbf{b}, s, t) \mid y^2 - f(\mathbf{x}) \in \langle \{a_i y + b_i - x_i\}_{1 \le i \le n}, y^2 + sy + t \rangle\} \\ & = & \{(\mathbf{a}, \mathbf{b}, s, t) \mid y^2 - f(\mathbf{a} \cdot y + \mathbf{b}) \in \langle y^2 + sy + t \rangle\} \\ & = & \{(\mathbf{a}, \mathbf{b}, s, t) \mid y^2 - f(\mathbf{a} \cdot y + \mathbf{b}) \equiv 0 \pmod{y^2 + sy + t}\} \end{array}$$

Let $g_0, g_1 \in \mathbf{C}[\mathbf{a}, \mathbf{b}, s, t]$ be the unique elements such that

$$y^2 - f(\mathbf{a} \cdot y + \mathbf{b}) \equiv g_0 + g_1 y \pmod{y^2 + sy + t},$$

then it follows that the vanishing ideal of $\varphi^{-1}(\widetilde{S}_Y)$ is generated by g_0 and g_1. We have:

$$\begin{aligned} y^2 &\equiv -sy - t \pmod{y^2 + sy + t} \\ f(\mathbf{a}\cdot y + \mathbf{b}) &\equiv r_0 + r_1 y \pmod{y^2 + sy + t}, \end{aligned}$$

where $r_0, r_1 \in \langle \mathbf{a}, \mathbf{b}\rangle \subset \mathbf{C}[\mathbf{a}, \mathbf{b}, s, t]$. Let $\mathfrak{m} \subset \mathcal{O}_{\operatorname{Spec}\mathbf{C}[\mathbf{a},\mathbf{b},s,t],0}$, then it follows that the images of $g_0 = -s - r_0$ and $g_1 = -t - r_1$ in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent (over $\mathbf{C}$). Hence the local ring $\mathcal{O}_{\psi^{-1}(\widetilde{S}_Y),0} = \mathcal{O}_{\operatorname{Spec}\mathbf{C}[\mathbf{a},\mathbf{b},s,t],0}/\langle g_0, g_1\rangle$ is regular, and hence $\widetilde{S}_Y$ is regular at $\psi(0) = \sigma_0$. This completes the proof. □

Theorem 3.4.13. *Let $\varepsilon : \mathrm{Bl}(V, S_f^{[2]}) \to S_f^{[2]}$ be the blow-up morphism. There is a unique isomorphism $\chi : E \to \mathrm{Bl}(V, S_f^{[2]})$ such that $\psi = \varepsilon \circ \chi$.*

Proof. First note that it follows from theorem 3.4.9 that $E^\infty \subset E$ is a Cartier divisor. Let $\mathcal{J}_V$ be the ideal sheaf of V on $S_f^{[2]}$. Since $\psi(E^\infty) = V$ and E^∞ is a Cartier divisor, it follows that $\psi^{-1}\mathcal{J}_V \cdot \mathcal{O}_E$ is an invertible sheaf on E. By the universal property of blow-up (see for example [20], proposition 7.14) it follows that there exists a unique morphism $\chi : E \to \mathrm{Bl}(V, S_f^{[2]})$ such that $\psi = \varepsilon \circ \chi$.

It remains to show that χ is in fact an isomorphism. First note that outside E^∞, χ is an isomorphism onto its image by construction. Next we show that χ is surjective. Let $p \in \varepsilon^{-1}(V)$ be arbitrary and $\gamma : \mathbf{C}[[t]] \to \mathrm{Bl}(V, S_f^{[2]})$ be such that it maps the special point to p and the generic point to $\varepsilon^{-1}(S_f^{[2],\mathrm{hor}})$. Since χ is an isomorphism outside E^∞ and E is proper, there exists a curve $\widehat{\gamma} : \mathbf{C}[[t]] \to E$ such that $\chi \circ \widehat{\gamma} = \gamma$ at the generic point. Since $\mathrm{Bl}(V, S_f^{[2]})$ is separated we must have $p = \chi(\widehat{\gamma}(0))$, so χ is indeed surjective.

Furthermore, χ is finite. It suffices to show that it has finite fibres (since it is projective) and it suffices to show this over $\varepsilon^{-1}(V)$. Let $v \in V$ be arbitrary. Then by its construction and its surjectivity χ maps $\psi^{-1}(v)$ surjectively onto $\varepsilon^{-1}(v)$. From the construction of ψ it follows that $\psi^{-1}(v) \cong \mathbf{P}^1$. We claim that $\varepsilon^{-1}(v) \cong \mathbf{P}^1$ as well. Indeed, from proposition 3.4.12 it follows that $S_f^{[2]}$ is smooth along V. Hence $\varepsilon^{-1}(v) \cong \mathbf{P}(\nu_{V/S_f^{[2]}})_v \cong \mathbf{P}^1$. Every surjective map from the projective line to itself is finite, which implies finiteness of χ.

Now since $S_f^{[2]}$ is smooth along V we can find a smooth open subset $U \subset S_f^{[2]}$ such that $V \subset U$ and $\psi^{-1}(U)$ is smooth. Note that $\varepsilon^{-1}(U)$ is smooth as well, so $\chi|_{\psi^{-1}(U)}$ is a finite birational morphism onto a normal variety, hence an isomorphism. This concludes the proof. □

3.5 Blow-down of $\widehat{F}_2$

The next step in the process outlined in section 3.1 is the contraction of $\widehat{F}_2$ within $\widehat{\mathcal{F}}$. We will use a contraction criterion by Ishii to ensure that a contraction can be performed in such a way that the resulting variety is projective and smooth. The contraction is induced by a linear system associated to an invertible sheaf on $\widehat{\mathcal{F}}$ with specific properties; in this section we will be mainly concerned with the construction of this invertible sheaf.

3.5.1 Algebraic contraction criterion

In [28] Ishii gives necessary and sufficient conditions for a subvariety of a non-singular projective variety to be contractible in such a way that the resulting space is again a non-singular projective variety. For convenience we restate part of his results here. To do so, we follow his notation and definitions:

Definition 3.5.1. Let X', Y be varieties (reduced and irreducible k-schemes) and let $L \subset X'$ be a proper closed subvariety. We say that L is **contractible to** Y **within** X' is there exists a variety X and a proper birational morphism $f : X' \to X$ such that

1. $f(L) \cong Y$ and
2. L is the closed subset of X' of points where f is not an isomorphism,

The triple (X', f, X) is called a **contraction of** L **to** Y. We say that the contraction is normal, projective or regular if X is. Finally, we say that two contractions (X', f_1, X_1) and (X', f_2, X_2) of L to Y are isomorphic if there exists an isomorphism $\alpha : X_1 \to X_2$ such that $f_2 = \alpha \circ f_1$.

Then Ishii proves the following results:

Theorem 3.5.2. *Let X' be an n-dimensional non-singular projective variety, L an effective divisor on X' and Y an r-dimensional non-singular projective variety with $r < n-1$.*
Assume that the following holds:

1. *L is isomorphic to a projective bundle $\mathbf{P}(E)$ for a vector bundle E on Y. Let p denote the induced projection of L to Y;*
2. *There exists an invertible sheaf $\mathcal{L}$ on Y such that the conormal sheaf of L in X' is isomorphic to $p^*\mathcal{L} \otimes \mathcal{O}_p(1)$;*
3. *There is an ample invertible sheaf $\mathcal{M}$ on Y and an invertible sheaf $\mathcal{M}'$ on X' generated by its global sections such that $\mathcal{M}'|_L \cong p^*\mathcal{M}$.*

Then there exists a regular projective contraction (X', f, X) of L to Y such that $f|_L : L \to Y$ corresponds to p. Furthermore, up to isomorphism this contraction is unique with that property. Finally, (X', f) is the blow up of X with center Y.

Proof. Theorem 2, theorem 3 and Corollary 2 in [28] □

This theorem can in fact be slightly generalized in the following way:

Theorem 3.5.3. *Let X' be an n-dimensional projective variety, L an effective divisor contained in the regular locus of X' and Y an r-dimensional non-singular projective variety with $r < n-1$. Assume that the three conditions in theorem 3.5.2 are satisfied.*

Then there exists a projective contraction (X, f, X') of L to Y such that $f|_L : L \to Y$ corresponds to p and such that $f(L)$ is contained in the regular locus of X. Furthermore, up to isomorphism this contraction is unique with these properties. Finally, (X', f) is the blow up of X with center Y.

Proof. The theorem follows from the local nature of the proofs of theorem 2, theorem 3 and corollary 2 in [28]. More precisely, to reach the conclusion in theorem 3.5.2 that X is regular, we indeed need to assume that X' itself is regular. But if we only ask for regularity of X locally along $f(L)$, then it suffices to require X to be regular along L. We will not repeat the proofs in [28] here, but the reader is invited to check that they indeed work in this local situation as well. □

The main problem is the construction of $\mathcal{M}'$ and showing that its linear system is base point free. We do this in the next section.

3.5.2 Construction of invertible sheaf

We first identify some invertible sheaves on $\widehat{\mathcal{F}}$. By theorem 3.3.7 we know that $\widehat{\mathcal{F}}$ is smooth along $\widehat{F}_2$. Hence $\widehat{F}_2$ is a Cartier divisor and there is an associated invertible sheaf $\mathcal{O}_{\widehat{\mathcal{F}}}(\widehat{F}_2)$. Similarly E is a Cartier divisor, since it is the exceptional divisor of a blow-up, and has an associated invertible sheaf $\mathcal{O}_{\widehat{\mathcal{F}}}(E)$. Furthermore, let $\mathcal{L}_p$ be the pullback of the relative tautological bundle on $\mathbf{P}(\bigwedge^2 \mathrm{Sym}^2 W) \times \Delta$ along the composition

$$\widehat{\mathcal{F}} \hookrightarrow \mathrm{Bl}(F_1, \mathcal{G}) \to \mathcal{G} \to \mathrm{Gr}_2(\mathrm{Sym}^2 W) \times \Delta \hookrightarrow \mathbf{P}(\bigwedge^2 \mathrm{Sym}^2 W) \times \Delta,$$

where the last inclusion is the Plücker morphism.

Let $i : \mathbf{P}(W) \times \mathbf{P}(W^\vee) \cong \widehat{F}_2$ be the natural identification. We wish to find an invertible sheaf $\mathcal{L}$ on $\widehat{\mathcal{F}}$ such that the associated rational morphism is in fact everywhere defined, and contracts $i(\{[w]\} \times \mathbf{P}(W^\vee)) \subset \widehat{F}_2$ to a point for every $[w] \in \mathbf{P}(W)$. Equivalently, $|\mathcal{L}|$ has to be base-point free and $i^*\mathcal{L}$ must be of bidegree $(d, 0)$, for some $d \in \mathbf{N}$. Let us calculate the bidegrees of the pullbacks of the sheaves introduced above.

Proposition 3.5.4.

- *$i^*\mathcal{L}_p$ is of bidegree (2,1);*
- *$i^*\mathcal{O}_{\widehat{\mathcal{F}}}(E)$ is of bidegree (1,1);*

- *$i^*\mathcal{L}_{\widehat{\mathcal{F}}}(\widehat{F}_2)$ is of bidegree (-1,-1).*

Proof. We first show that $i^*\mathcal{L}_p$ is of bidegree $(2,1)$. Let τ be the tautological bundle on $\mathbf{P}(\bigwedge^2 \mathrm{Sym}^2 W)$ and $j : \mathbf{P}(W) \times \mathbf{P}(W^\vee) \hookrightarrow \mathbf{P}(\bigwedge^2 \mathrm{Sym}^2 W)$ the embedding with image equal to the Plücker embedding of F_2. Then $j^*\tau = i^*\mathcal{L}_p$ (by definition of $\mathcal{L}_p$). If we interpret $\mathbf{P}(W^\vee)$ as the space parameterizing hyperplanes in $\mathbf{P}(W)$ then j may be described explicitly by

$$j : ([w_1],[w_2 \wedge w_3]) \mapsto [(w_1 \cdot w_2) \wedge (w_1 \cdot w_3)]$$

for $w_1, w_2, w_3 \in W \setminus \{0\}$ and $[w_2] \neq [w_3]$. Pick $u_1, \ldots, u_4 \in W \setminus \{0\}$ such that $[u_1] \neq [u_2], [u_3] \neq [u_4]$ and define the following maps:

$$\begin{array}{rcl} \lambda_1 : \mathbf{P}^1 & \to & \mathbf{P}(W) \times \mathbf{P}(W^\vee) \\ {[s:t]} & \mapsto & ([su_1 + tu_2],[u_3 \wedge u_4]) \\ \lambda_2 : \mathbf{P}^1 & \to & \mathbf{P}(W) \times \mathbf{P}(W^\vee) \\ {[s:t]} & \mapsto & ([u_1],[u_2 \wedge (su_3 + tu_4)]). \end{array}$$

Then $\operatorname{im}\lambda_1$ and $\operatorname{im}\lambda_2$ are lines contained in a fiber of projections $\mathbf{P}(W)\times\mathbf{P}(W^\vee) \to \mathbf{P}(W^\vee)$ and $\mathbf{P}(W) \times \mathbf{P}(W^\vee) \to \mathbf{P}(W)$ respectively. Therefore the bidegree of $j^*\tau$, and hence that of $i^*\mathcal{L}$, is given by $\big(\deg(j \circ \lambda_1)^*\tau, \deg(j \circ \lambda_2)^*\tau\big)$.
Now note that

$$\begin{array}{rcl} (j \circ \lambda_1)([s:t]) & = & \big[(su_1 \cdot u_3 + tu_2 \cdot u_3) \wedge (su_1 + tu_2) \cdot u_4\big] \\ (j \circ \lambda_2)([s:t]) & = & \big[(u_1 \cdot u_2) \wedge (su_1 \cdot u_3 + tu_1 \cdot u_4)\big]. \end{array}$$

Hence $j \circ \lambda_1$ is quadratic, $j \circ \lambda_2$ is linear, and it follows that the bidegree is (2,1).

For $\mathcal{O}_{\widehat{\mathcal{F}}}(E)$ note that by theorem 3.3.7

$$i^{-1}E = i^{-1}(E \cap \widehat{F}_2) = \{([w],[\alpha]) \mid w \in \ker\alpha\} \subset \mathbf{P}(W) \times \mathbf{P}(W^\vee).$$

Hence natural projections of $i^{-1}E$ to $\mathbf{P}(W)$ and $\mathbf{P}(W^\vee)$ are linear subspaces. It follows that $i^*\mathcal{O}_{\widehat{\mathcal{F}}}(E)$ is of bidegree (1,1).

To calculate the bidegree of $i^*\mathcal{O}_{\widehat{\mathcal{F}}}(\widehat{F}_2)$ remember that $\widehat{\mathcal{F}}_0$ is a normal crossing of E and $\widehat{F}_2$ (by theorem 3.3.7). Furthermore we have that

$$\mathcal{O}_{\widehat{\mathcal{F}}}(E) \otimes \mathcal{O}_{\widehat{\mathcal{F}}}(\widehat{F}_2) = \mathcal{O}_{\widehat{\mathcal{F}}}(\widehat{\mathcal{F}}_0) \cong \mathcal{O}_{\widehat{\mathcal{F}}}.$$

It follows that the bidegrees of $i^*\mathcal{O}_{\widehat{\mathcal{F}}}(E)$ and $i^*\mathcal{O}_{\widehat{\mathcal{F}}}(\widehat{F}_2)$ sum up to $(0,0)$, hence $i^*\mathcal{O}_{\widehat{\mathcal{F}}}(\widehat{F}_2)$ is of bidegree (-1,-1).

□

We start our construction with an invertible sheaf on $\widehat{\mathcal{F}}$ which is very ample with respect to $\widehat{\varphi} : \widehat{\mathcal{F}} \to \Delta$. The existence of such a sheaf is guaranteed by the following result.

Proposition 3.5.5. *There exists an $n \in \mathbf{N}$ such that the invertible sheaf $\mathcal{O}_{\widehat{\mathcal{F}}}(-E) \otimes \mathcal{L}_p^{\otimes n}$ is very ample with respect to $\widehat{\varphi}$.*

Proof. As we saw above there is a closed immersion $\mathcal{F} \hookrightarrow \mathbf{P}(\bigwedge^2 \mathrm{Sym}^2 W) \times \Delta$. Let $\mathcal{O}_{\mathcal{F}}(1)$ be the restriction of the relative tautological sheaf. It is very ample by construction, hence in particular ample. Remember that $\widehat{\mathcal{F}}$ is obtained from $\mathcal{F}$ by blowing up $F_1 \subset \mathcal{F}$. Let $\varepsilon : \widehat{\mathcal{F}} \to \mathcal{F}$ be the associated morphism. Let $\mathcal{O}_{\widehat{\mathcal{F}}}(1)$ be the relative twisting sheaf on $\widehat{\mathcal{F}}$ over $\mathcal{F}$. By [20, proposition II.7.10.b] there exists a $k \in \mathbf{N}$ such that the invertible sheaf $\mathcal{O}_{\widehat{\mathcal{F}}}(1) \otimes \varepsilon^* \mathcal{O}_{\mathcal{F}}(k)$ is very ample on $\widehat{\mathcal{F}}$ over $\mathcal{F}$. Since $\widehat{\mathcal{F}}$ is obtained by blowing up $\mathcal{F}$, the sheaf $\mathcal{O}_{\widehat{\mathcal{F}}}(1)$ and the vanishing ideal sheaf $\mathcal{I}_E$ of the exceptional divisor become naturally identified. Moreover, $\mathcal{I}_E \cong \mathcal{O}_{\widehat{\mathcal{F}}}(-E)$ and $\varepsilon^* \mathcal{O}_{\mathcal{F}}(k) \cong \mathcal{L}_p^{\otimes k}$. We thus obtain that for k as above the invertible sheaf $\mathcal{L}_p^{\otimes k}(-E)$ is very ample with respect to $\varepsilon : \widehat{\mathcal{F}} \to \mathcal{F}$. Note that $\mathcal{O}_{\mathcal{F}}(1)$ is very ample with respect to $\varphi : \mathcal{F} \to \Delta$. It now follows from [19, proposition 4.4.10.ii] that there exists an $m \in \mathbf{N}$ such that $\mathcal{O}_{\widehat{\mathcal{F}}}(-E) \otimes \mathcal{L}_p^{\otimes k} \otimes \varepsilon^* \mathcal{O}_{\mathcal{F}}(m) = \mathcal{L}_p^{\otimes k+m}(-E)$ is very ample with respect to $\widehat{\varphi}$. We take $n = k + m$ to complete the proof. □

Let $\mathcal{L}_n := \mathcal{L}_p^{\otimes n}(-E)$. By proposition 3.5.5 it is very ample for appropriate n. By proposition 3.5.4 the bidegree of $i^* \mathcal{L}_n$ is $(2n-1, n-1)$. Let $\mathcal{L}'_n := \mathcal{L}_n\big((n-1)\widehat{F}_2\big)$, then $i^* \mathcal{L}'_n$ is of bidegree $(n, 0)$, as we want. We now wish to show that we can choose n such that there is a power of $\mathcal{L}'_n$ whose linear system is base-point free. To do so we need some properties of $\widehat{F}_2$.

Proposition 3.5.6. *Let $m\widehat{F}_2 \subset \widehat{\mathcal{F}}$ be the scheme with multiplicity m which is set-theoretically equal to $\widehat{F}_2$. Then for every $m \in N$ and for every invertible sheaf $\mathcal{K}$ on $\widehat{\mathcal{F}}$ the restriction map*

$$H^0(\widehat{F}_2, \mathcal{K} \otimes \mathcal{O}_{m\widehat{F}_2}) \to H^0(\widehat{F}_2, \mathcal{K} \otimes \mathcal{O}_{\widehat{F}_2})$$

is surjective.

Proof. Let $\mathcal{I}_{\widehat{F}_2}$ be the vanishing ideal sheaf on $\widehat{\mathcal{F}}$ of $\widehat{F}_2$. We have the following exact sequence:

$$0 \to \mathcal{I}^m_{\widehat{F}_2} / \mathcal{I}^{m+1}_{\widehat{F}_2} \to \mathcal{O}_{(m+1)\widehat{F}_2} \to \mathcal{O}_{m\widehat{F}_2} \to 0.$$

We can twist it with any invertible sheaf $\mathcal{K}$ on $\widehat{F}_2$ to obtain the exact sequence

$$0 \to \mathcal{K} \otimes \mathcal{I}^m_{\widehat{F}_2} / \mathcal{I}^{m+1}_{\widehat{F}_2} \to \mathcal{K} \otimes \mathcal{O}_{(m+1)\widehat{F}_2} \to \mathcal{K} \otimes \mathcal{O}_{m\widehat{F}_2} \to 0. \qquad (3.5.1)$$

By theorem 3.3.7 $\widehat{F}_2$ is non-singular and $\widehat{\mathcal{F}}$ is non-singular along $\widehat{F}_2$. Therefore the conormal sheaf $\nu^\vee_{\widehat{F}_2/\widehat{\mathcal{F}}} := \mathcal{I}_{\widehat{F}_2} / \mathcal{I}^2_{\widehat{F}_2}$ is an invertible sheaf on $\widehat{F}_2$ and hence

$$\mathcal{I}^m_{\widehat{F}_2} / \mathcal{I}^{m+1}_{\widehat{F}_2} = (\nu^\vee_{\widehat{F}_2/\widehat{\mathcal{F}}})^{\otimes m},$$

is also invertible.

Now note that every invertible sheaf on $\mathbf{P}^2 \times \mathbf{P}^2$ has vanishing first cohomology; this follows from the fact that every invertible sheaf on $\mathbf{P}^2$ has vanishing first cohomology (see [20, theorem III.5.1]) and the Künneth formula for sheaf cohomology of coherent algebraic sheaves (see for example [43]). Since $\widehat{F}_2 \cong \mathbf{P}^2 \times \mathbf{P}^2$ it follows in particular that $H^1(\widehat{F}_2, \mathcal{K} \otimes \mathcal{I}^m_{\widehat{F}_2} / \mathcal{I}^{m+1}_{\widehat{F}_2}) = 0$. From the long exact sequence associated to (3.5.1) we find that the restriction map

$$H^0\big((m+1)\widehat{F}_2, \mathcal{K} \otimes \mathcal{O}_{(m+1)\widehat{F}_2}\big) \to H^0\big(m\widehat{F}_2, \mathcal{K} \otimes \mathcal{O}_{m\widehat{F}_2}\big)$$

is surjective. This holds for all $m \in \mathbf{N}$. Since the composition of surjective maps is surjective, the proposition follows from repeated application of the argument above. □

We are now in position to prove the following:

Proposition 3.5.7. *There exists an $n, k \in \mathbf{N}$ such that the linear system $|(\mathcal{L}'_n)^{\otimes k}|$ is base-point free.*

Proof. Remember that we defined $\mathcal{L}'_n = \mathcal{L}_n\big((n-1)\widehat{F}_2\big)$. Let $n \in N$ be such that $\mathcal{L}_n$ is very ample; this is possible by proposition 3.5.5. Let σ_n be a non-trivial global section of $\mathcal{O}_{\widehat{\mathcal{F}}}\big((n-1)\widehat{F}_2\big)$ vanishing only on $\widehat{F}_2$. Tensoring with σ_n defines a map $t_\sigma : \mathcal{L}_n \to \mathcal{L}'_n$ which is clearly injective. Since $\mathcal{L}_n$ is very ample, its linear system is base-point free. Since σ_n is nonzero outside $\widehat{F}_2$ it follows that the base locus of $|\mathcal{L}'_n|$, and hence that of $|(\mathcal{L}'_n)^{\otimes k}|$ for any $k \in \mathbf{N}$, must be contained in $\widehat{F}_2$.

By definition $\mathcal{L}_n \cong \mathcal{L}'_n\big((n-1)\widehat{F}_2\big)$, hence we have for any $k \in \mathbf{N}$ the short exact sequence

$$0 \to \mathcal{L}_n^{\otimes k} \to (\mathcal{L}'_n)^{\otimes k} \to (\mathcal{L}'_n)^{\otimes k} \otimes \mathcal{O}_{(n-1)k\widehat{F}_2} \to 0.$$

The sheaf $\mathcal{L}_n$ is very ample with respect to $\widehat{\varphi}$, therefore we can choose $k \in \mathbf{N}$ such that $H^1\big(\widehat{\mathcal{F}}, \mathcal{L}_n^{\otimes k}\big) = 0$ (see for example [20], theorem III.5.2). From the long exact sequence associated to the sequence above it then follows that the restriction map $H^0\big(\widehat{\mathcal{F}}, (\mathcal{L}'_n)^{\otimes k}\big) \to H^0\big(\widehat{F}_2, (\mathcal{L}'_n)^{\otimes k}|_{(n-1)k\widehat{F}_2}\big)$ is surjective. With proposition 3.5.6 it follows that the restriction map

$$H^0\big(\widehat{\mathcal{F}}, (\mathcal{L}'_n)^{\otimes k}\big) \to H^0\big(\widehat{F}_2, (\mathcal{L}'_n)^{\otimes k}|_{\widehat{F}_2}\big)$$

is surjective.

The surjectivity implies that every section of $(\mathcal{L}'_n)^{\otimes k}|_{\widehat{F}_2}$ can be extended to a section of $(\mathcal{L}'_n)^{\otimes k}$ on $\widehat{\mathcal{F}}$. From proposition 3.5.4 it follows that $i^*(\mathcal{L}'_n)^{\otimes k}$ is of bidegree $(nk, 0)$ on $\mathbf{P}(W) \times \mathbf{P}(W^\vee)$. Hence it has non-trivial sections, even stronger, the linear system is base-point free. Since i is an isomorphism, it follows that the linear system of $(\mathcal{L}'_n)^{\otimes k}|_{\widehat{F}_2}$ has no base points. Since every section of $(\mathcal{L}'_n)^{\otimes k}|_{\widehat{F}_2}$ extends to a section of $(\mathcal{L}'_n)^{\otimes k}$ the base-locus of $|(\mathcal{L}'_n)^{\otimes k}|$ is contained in $\widehat{\mathcal{F}} \setminus \widehat{F}_2$. We found before that the base-locus must be contained in $\widehat{F}_2$, hence the base locus is empty. □

We can now prove the following result:

Theorem 3.5.8. *There exist a family $\overline{\varphi}:\overline{\mathcal{F}}\to\Delta$, an embedding $j:\mathbf{P}(W)\hookrightarrow\overline{\mathcal{F}}$ and a birational map of families $\gamma:\widehat{\mathcal{F}}\to\overline{\mathcal{F}}$ such that $(\widehat{\mathcal{F}},\gamma,\overline{\mathcal{F}})$ is a contraction of $\widehat{F}_2$ to $\mathbf{P}(W)$ and $j\circ\pi_1=\gamma|_{\widehat{F}_2}\circ i$, where $i:\mathbf{P}(W)\times\mathbf{P}(W^\vee)\cong\widehat{F}_2$ is the natural identification and $\pi_1:\mathbf{P}(W)\times\mathbf{P}(W^\vee)\to\mathbf{P}(W)$ is the projection away to the first factor. Moreover, $(\widehat{\mathcal{F}},\gamma)$ corresponds to the blow-up of $\overline{\mathcal{F}}$ with center* $\operatorname{im} j$.

Proof. We check the conditions of theorem 3.5.3. First note that by theorem 3.3.7 $\widehat{F}_2$ is contained in the regular locus of $\widehat{\mathcal{F}}$. Furthermore:

1. Take $p:=\pi_1\circ i^{-1}$, then p endows $\widehat{F}_2$ with the structure of a trivial fiber bundle with fibers naturally isomorphic to $\mathbf{P}(W^\vee)$.
2. Since $\widehat{\mathcal{F}}$ is smooth along $\widehat{F}_2$ and $\widehat{F}_2$ is of codimension 1 the conormal sheaf $\nu_{\widehat{F}_2/\widehat{\mathcal{F}}}$ is the invertible sheaf given by the restriction of $\mathcal{O}_{\widehat{\mathcal{F}}}(-\widehat{F}_2)$ to $\widehat{F}_2$. By proposition 3.5.4 $i^*\nu_{\widehat{F}_2/\widehat{\mathcal{F}}}$ is of bidegree (1,1), so $\nu_{\widehat{F}_2/\widehat{\mathcal{F}}}\cong\mathcal{O}_p(1)\otimes p^*\mathcal{O}_{\mathbf{P}(W)}(1)$;
3. Let $\mathcal{L}'_n$ be as before, and choose k such that $(\mathcal{L}'_n)^{\otimes k}$ is generated by its global sections; this is possible by proposition 3.5.7. By construction of $\mathcal{L}'_n$ we know that $i^*(\mathcal{L}'_n)^{\otimes k}$ has bidegree $(nk,0)$, hence $i^*(\mathcal{L}'_n)^{\otimes k}\cong\pi_1^*\mathcal{O}_{\mathbf{P}(W)}(nk)$, from which directly follows that the restriction of $(\mathcal{L}'_n)^{\otimes k}$ to $\widehat{F}_2$ is isomorphic to the pull back along p of an ample sheaf, namely $\mathcal{O}_{\mathbf{P}(W)}(nk)$.

Hence by theorem 3.5.3 there exists a projective contraction $(\widehat{\mathcal{F}},\gamma,\overline{\mathcal{F}})$ such that there exists an embedding $j:\mathbf{P}(W)\to\overline{\mathcal{F}}$ with the property that $\gamma|_{\widehat{F}_2}=j\circ p$ and such that $\overline{\mathcal{F}}$ is regular along $\gamma(\widehat{F}_2)$.

From the construction it is clear that the map $\widehat{\varphi}:\widehat{\mathcal{F}}\to\Delta$ factors through γ, hence there exists a map $\overline{\varphi}:\overline{\mathcal{F}}\to\Delta$ that gives $\overline{\mathcal{F}}$ the structure of a family. This completes the proof. □

3.5.3 The central fiber of $\overline{\varphi}$

We have constructed a morphism $\gamma:\widehat{\mathcal{F}}\to\overline{\mathcal{F}}$ of families which contracts the component $\widehat{F}_2$ of $\widehat{\mathcal{F}}_0$. More precisely, γ can be identified with the blow-up of a subspace of $\overline{\mathcal{F}}$ isomorphic to $\mathbf{P}(W)$, such that the exceptional locus is precisely $\widehat{F}_2$. The final step of our construction, and the proof of proposition 3.1.1 is to show that the central fiber $\mathcal{F}_0$ of $\overline{\varphi}$ is indeed isomorphic to the scheme of length 2 subschemes of S_f, the double plane branched along the sextic curve defined by $\upsilon^* f$ (see section 3.6 for the notation). This is the aim of this section.

Remember (see theorem 3.3.7) that $\widehat{\mathcal{F}}_0$ is a normal crossing of E and $\widehat{F}_2$. It also follows from theorem 3.3.7 that E intersects every fiber of the fibration

$$\gamma|_{\widehat{F}_2}:\widehat{F}_2\to\mathbf{P}(W).$$

Hence $\overline{\mathcal{F}}_0=\gamma(E)$. We claim that $\gamma(E)$ is smooth along $\gamma(\widehat{F}_2)$. For this we need the following technical result.

Lemma 3.5.9. *Let X be a non-singular algebraic variety, $Y \subset X$ a non-singular subvariety and $\varepsilon : \mathrm{Bl}(Y, X) \to X$ the blow-up of X along Y. Let $T \subset \mathrm{Bl}(Y, X)$ be a closed irreducible subvariety, not necessarily smooth, such that:*

- *T is of codimension 1;*
- *$Y \subset \varepsilon(T)$;*
- *T is locally irreducible along $T \cap \varepsilon^{-1}(Y)$;*
- *The restriction of $\mathcal{O}_X(T)$ to $\varepsilon^{-1}(Y)$ is isomorphic to $\mathcal{O}_\varepsilon(1)$.*

Then $\varepsilon(T)$ is smooth along Y. Moreover $\varepsilon|_T : T \to \varepsilon(T)$ corresponds to the blow-up of $\varepsilon(T)$ with center Y.

Proof. It suffices to prove the lemma locally. Let $y \in Y$ be arbitrary, let $\mathcal{O}_{X,y}$ be the stalk at y of the structure sheaf, $\mathfrak{m}_y$ its maximal ideal. Let $\mathcal{I}$ be the vanishing ideal sheaf of Y on X and $\mathcal{I}_y \subset \mathcal{O}_{X,y}$ its stalk at y.

Let X_y, Y_y be the germs of X and Y at y and let $\varepsilon_y : \mathrm{Bl}(Y_y, X_y) \to X_y$ be the blow-up of X_y with center Y_y. Then

$$\mathrm{Bl}(Y_y, X_y) = \mathrm{Proj} \bigoplus_{k \geq 0} \mathcal{I}_y^k \to \mathrm{Spec}\, \mathcal{O}_{X,y}$$

and the (local) exceptional divisor E_y is given by

$$\mathrm{Proj} \bigoplus_{k \geq 0} \mathcal{I}_y^k / \mathcal{I}_y^{k+1}.$$

Let T_y be the image of T in $\mathrm{Bl}(Y_y, X_y)$. Note that since ε_y is proper (blow-up morphisms are proper by construction) it follows that $\varepsilon_y(T_y)$ is an irreducible algebraic codimension 1 subvariety of Y_y, hence, since X_y is regular, a Cartier divisor in X_y. It follows that the stalk of the ideal sheaf of $\varepsilon_y(T_y)$ is generated by a single element in $\mathcal{O}_{X,y}$, we will denote it by f.

Since by assumption $Y_y \subset \varepsilon_y(T_y)$ it follows that $f \in \mathcal{I}_y$. Furthermore, since T is assumed to be irreducible along $T \cap \varepsilon^{-1}(Y)$, it follows that $T \setminus \varepsilon^{-1}(Y)$ is dense in T. Hence in particular T_y equals the strict transform of $\varepsilon_y(T_y)$ in $\mathrm{Bl}(Y_y, X_y)$. By remark A.1.4 the vanishing ideal of $T_y \cap E_y$ in E_y is given by

$$\mathrm{In}_I \langle f \rangle := \bigoplus_{k \geq 0} (\mathcal{I}_y^k \cap \langle f \rangle + \mathcal{I}_y^{k+1}) / \mathcal{I}_y^{k+1} \subset \bigoplus_{k \geq 0} \mathcal{I}_y^k / \mathcal{I}_y^{k+1}.$$

Clearly, $\mathrm{In}_I \langle f \rangle = \langle \mathrm{In}_I f \rangle$ (remember that this is specific to principal ideals; in case of more than one generator it may not be true, and it can be a cumbersome exercise to determine if it is true for a specific ideal and set of generators. See also [24]). The linearity assumption on the intersection of T_y with $\varepsilon_y^{-1}(Y_y)$ now implies that $\mathrm{In}_I f \in \mathcal{I}_y / \mathcal{I}_y^2$, from which follows that $f \in \mathcal{I}_y$ but $f \notin \mathcal{I}_y^2$.

We claim that $f \notin \mathfrak{m}_y^2$. Indeed, as a consequence of the smoothness assumptions the natural map $\mathcal{I}_y/\mathcal{I}_y^2 \to \mathfrak{m}_y/\mathfrak{m}_y^2$ is injective (see e.g. [20]). Since f is an element of $\mathcal{I}_y$ but not of $\mathcal{I}_y^2$, it follows that $f \notin \mathfrak{m}_y^2$.

Let $\mathfrak{m}_{f,y}$ be the maximal ideal in $\mathcal{O}_{X,y}/\langle f\rangle$. Let $\overline{f}$ be the image of f in $\mathfrak{m}_y/\mathfrak{m}_y^2$. Then by the argument above $\overline{f} \neq 0$. Moreover, since

$$\mathfrak{m}_{f,y}/\mathfrak{m}_{f,y}^2 = \left(\mathfrak{m}_y/\mathfrak{m}_y^2\right)/\langle \overline{f}\rangle,$$

it follows that

$$\dim \mathfrak{m}_{f,y}/\mathfrak{m}_{f,y}^2 = \dim \mathfrak{m}_y/\mathfrak{m}_y^2 - 1 = \dim X_y - 1 = \dim T_y.$$

So T is smooth at y.

The last statement of the theorem follows directly from the fact that T is the strict transform of $\varepsilon(T)$. See section A.1 in the appendix for details. □

Lemma 3.5.10. *The subvariety $\overline{\mathcal{F}}_0$ of $\overline{\mathcal{F}}$ is smooth along $\gamma(\widehat{F}_2)$*

Proof. Denote $Y := \gamma(\widehat{F}_2)$. According to theorem 3.5.8 we can find an open subspace $U \subset \overline{\mathcal{F}}$ such that U is smooth and contains Y (which is also smooth). Moreover, there exists an identification $\gamma^{-1}(U) \to \mathrm{Bl}(Y,U)$ such that γ corresponds to the blow-up morphism. Set $S := \gamma^{-1}(U) \cap E$. Then S is of codimension 1 in U, $Y \subseteq \gamma(S)$, S is smooth, hence locally irreducible, along the intersection with $\widehat{F}_2$. Furthermore, we know that the projection $\gamma : \widehat{F}_2 \to Y$ is a fibration into projective spaces, and it follows from theorem 3.3.7 that S intersects these fibers in a linear subspace. We may now apply lemma 3.5.9 with $X = U$, $Y = Y$ and $S = S$ to conclude that $\gamma(E) = \overline{\mathcal{F}}_E$ is smooth along $\gamma(\widehat{F}_2)$. □

Remember that by theorem 3.4.13 there exists a commutative diagram

$$\begin{array}{ccccc} & & E & \xrightarrow[\chi]{\cong} & \mathrm{Bl}(V, S_f^{[2]}) \\ & \swarrow^{\gamma|_E} & & \searrow^{\psi} & \downarrow \varepsilon \\ \overline{\mathcal{F}}_0 & & & & S_f^{[2]} \end{array} \tag{3.5.2}$$

where $V \subset S_f^{[2]}$ is the variety of length 2 subschemes of S_f that are vertical with respect to $S_f \to \mathbf{P}(W)$. The main result of this section is the following:

Theorem 3.5.11. *There exists an isomorphism $S_f^{[2]} \to \overline{\mathcal{F}}_0$ which extends diagram (3.5.2) to a commutative diagram.*

Proof. Write $E = E^0 \cup E^\infty$ as in section 3.4.3. Note that both ψ and γ are birational morphisms, and for both the exceptional locus is E^∞. In particular $\overline{\mathcal{F}}_0$ and S_f are birational, let $\rho : S_f^{[2]} \dashrightarrow \overline{\mathcal{F}}_0$ be the rational map and $\Gamma_\rho \subset S_f^{[2]} \times \overline{\mathcal{F}}_0$ the closure of the graph of ρ. Let π_1, π_2 be the projections of Γ_ρ to $S_f^{[2]}$ and $\overline{\mathcal{F}}_0$ respectively. Clearly these are also birational morphisms.

From remark 3.4.11 and theorem 3.4.13 it follows that a curve in E is contracted by γ if and only if it is contracted by ψ. This implies that both π_1 and π_2 are quasi-finite, hence finite by properness. Since π_1 and π_2 are isomorphisms over $\psi(E^0)$ and $\gamma(E^0)$ respectively, to prove the theorem it suffices to show that $S_f^{[2]}$ is normal along $\psi(E^\infty)$ and $\overline{\mathcal{F}}_0$ is normal along $\gamma(E^\infty)$, since it then follows from previous observations that π_1 and π_2 must be isomorphisms.

We have already seen (proposition 3.4.12) that $S_f^{[2]}$ is smooth, hence in particular normal, along $V = \psi(E^\infty)$. Note that $\gamma(E^\infty) = \gamma(\widehat{F}_2 \cap E) = \gamma(\widehat{F}_2)$. Hence from lemma 3.5.10 it follows that $\overline{\mathcal{F}}_0$ is smooth and in particular normal along $\gamma(E^\infty)$. This completes the proof. □

The proof of proposition 3.1.1 is now straightforward:

Proof of proposition 3.1.1. The generic fiber of $\overline{\varphi}$ is by construction isomorphic to φ, which is the relative variety of lines on ξ'_f. The central fiber of $\overline{\varphi}$ is isomorphic to $S_f^{[2]}$ by theorem 3.5.11. So if we take $\mathcal{Z}_f = \overline{\mathcal{F}}$ and $\zeta_f = \overline{\varphi}$, then the family $\zeta_f : \mathcal{Z}_f \to \Delta$ satisfies the statement of the theorem 3.6.4.

In particular, if the the sextic defined by $v^* f$ is smooth, then $\mathcal{Z}_{f,0} = S_f^{[2]}$ is a projective holomorphic symplectic fourfold. Since Δ is the spectrum of a local ring, it follows that in that case ζ_f is a projective family of holomorphic symplectic fourfolds. This completes its proof. □

3.6 Quasi-universality

In this section we show how we can generalize proposition 3.1.1 to a quasi-universal reconstruction result that treats all one-parameter deformations (almost) at the same time. To state this result more precisely we need to introduce some notation.

As before, let W be a complex vector space of dimension 3, $\mu \in \Lambda^3 W$ a volume form, $D \in \mathrm{Sym}^3\mathrm{Sym}^2 W^\vee$ the determinantal cubic form and X_{det} the determinantal cubic. Let $V_{\mathrm{det}} \subset X_{\mathrm{det}}$ be the singular locus. It follows from proposition 3.2.1 that this is precisely the Veronese surface, and in particular it can be naturally identified with $\mathbf{P}(W)$.

Let $[D] \in \mathbf{P}(\mathrm{Sym}^3\mathrm{Sym}^2 W^\vee)$ be the point corresponding to D after projectivization. We can identify

$$T_{[D]}\mathbf{P}(\mathrm{Sym}^3\mathrm{Sym}^2 W^\vee) \cong \mathrm{Hom}(\mathbf{C}D, \mathrm{Sym}^3\mathrm{Sym}^2 W^\vee).$$

Let $\mathbf{D}$ denote $\mathrm{PGL}(\mathrm{Sym}^2 W)$-orbit of D. A line $t \mapsto [D_t f]$ is tangent to $\mathbf{D}$ at D if and only if f is contained in the ideal generated by the partial derivatives of D. This is the case if and only if f vanishes on the singular locus V_{det} of X_{det}. It follows that we have the following natural identifications

$$\begin{aligned} T_{[D]}\mathbf{D} &\cong \operatorname{Hom}(\mathbf{C}D, I_3(V_{\mathrm{det}})) \\ \big(\nu_{\mathbf{D}}\big)_{[D]} &\cong \operatorname{Hom}(\mathbf{C}D, \operatorname{Sym}^3\operatorname{Sym}^2 W^\vee / I_3(V_{\mathrm{det}})) \\ &\cong \operatorname{Hom}(\mathbf{C}D, H^0(V_{\mathrm{det}}, \mathcal{O}_{V_{\mathrm{det}}}(3))) \end{aligned}$$

where $\nu_{\mathbf{D}}$ is the normal bundle of $\mathbf{D}$ in $\mathbf{P}(\operatorname{Sym}^3\operatorname{Sym}^2 W^\vee)$ and $\mathcal{O}_{V_{\mathrm{det}}}(3)$ is the restriction of $\mathcal{O}_{P(\operatorname{Sym}^2 W)}(3)$ to V_{det}. The reader be warned: observe that since the Veronese embedding $\mathbf{P}(W) \hookrightarrow \mathbf{P}(\operatorname{Sym}_2 W)$ is quadratic, $\mathcal{O}_{V_{\mathrm{det}}}(3)$ corresponds to $\mathcal{O}_{\mathbf{P}(W)}(6)$ under the natural isomorphism $\mathbf{P}(W) \cong V_{\mathrm{det}}$. In particular we can naturally identify

$$\mathbf{P}\big(\nu_{\mathbf{D}}\big)_{[D]} \cong \mathbf{P}\big(H^0(V_{\mathrm{det}}, \mathcal{O}_{V_{\mathrm{det}}}(3))\big) \cong \mathbf{P}\big(H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(6))\big), \tag{3.6.1}$$

which corresponds to the linear system of sextic curves on $\mathbf{P}(W)$.

We can rather easily give a global description of $\mathbf{P}(\nu_{\mathbf{D}})$. Let

$$k : \mathcal{K} \to \mathbf{P}(\operatorname{Sym}^3\operatorname{Sym}^2 W^\vee)$$

be the universal cubic hypersurface and $k_{\mathbf{D}} : \mathcal{K}_{\mathbf{D}} \to \mathbf{D}$ the restriction of k over $\mathbf{D}$. In particular, for $g \in \mathrm{PGL}(\operatorname{Sym}^2 W)$ we have $(\mathcal{K}_{\mathbf{D}})_{g\cdot[D]} = g \cdot X_{\mathrm{det}}$. Let $\mathbf{V} \subset \mathcal{K}_{\mathbf{D}}$ be the singular locus. Then from the previous discussion it follows that we can naturally identify

$$\mathbf{P}(\nu_{\mathbf{D}}) \cong \mathbf{P}\big((k_{\mathbf{D}})_* \mathcal{O}_{\mathbf{V}}(3)\big),$$

where $\mathcal{O}_{\mathbf{V}}(3)$ is the third tensor power of the natural polarization on $\mathbf{V}$ induced by the polarization on $\mathcal{K}$ relative to k.

Remark 3.6.1. Note that in particular for any $g \in \mathrm{PGL}(\operatorname{Sym}^2 W)$ we have $\mathbf{V}_{g\cdot[D]} = g \cdot V_{\mathrm{det}}$, so that $k_{\mathbf{D}}|_{\mathbf{V}} : \mathbf{V} \to \mathbf{D}$ is a $\mathbf{P}^2$-bundle of $\mathbf{D}$. Let $\mathcal{L}_{\mathbf{V}}$ be the ample generator of the Picard group, then we obtain a natural isomorphism over $\mathbf{D}$:

$$\mathbf{V} \cong \mathbf{P}\big((k_{\mathbf{D}})_* \mathcal{L}_{\mathbf{V}}\big)^\vee.$$

Furthermore, by the remarks we made above it follows that $\mathcal{O}_{\mathbf{V}}(3) = \mathcal{L}_{\mathbf{V}}^{\otimes 6}$.

Let $\varepsilon_{\mathbf{D}} : B \to \mathbf{P}(\operatorname{Sym}^3\operatorname{Sym}^2 W^\vee)$ be the blow-up of $\mathbf{D}$ in $\mathbf{P}(\operatorname{Sym}^3\operatorname{Sym}^2 W^\vee)$ and let $N := \varepsilon_{\mathbf{D}}^{-1}(\mathbf{D})$ be the exceptional divisor. We can naturally identify

$$N \cong \mathbf{P}(\nu_{\mathbf{D}}) \cong \mathbf{P}\big((k_{\mathbf{D}})_* \mathcal{L}_{\mathbf{V}}^{\otimes 6}\big).$$

In particular, every $e \in N$ corresponds precisely to a curve Σ_e in $\mathbf{V}_{\varepsilon_{\mathbf{D}}(e)}$ of degree 6 with respect to $\mathcal{L}_{\mathbf{V}}$. Let

$$\sigma : \Sigma \to N$$

be the universal such curve. If $\widetilde{\mathbf{V}} \to N$ is the pull-back of $\mathbf{V} \to \mathbf{D}$ along $\varepsilon_{\mathbf{D}}|_N$, then $\Sigma \subset \widetilde{\mathbf{V}}$ and σ corresponds to the projection to N.

Note that in particular we have a natural identification

$$N_{[D]} \cong \mathbf{P}\big(H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(6))\big)$$

and under this identification the restriction of σ over $E_{[D]}$ corresponds to the universal sextic plane curve.

Let $U \subset N$ be an affine open subset and $\widetilde{\mathbf{V}}_U \to U$ the restriction of $\widetilde{\mathbf{V}} \to N$ over U. Then there exists a double cover $\mathcal{S}_U \to \widetilde{\mathbf{V}}_U$ branched along $\Sigma \cap \widetilde{\mathbf{V}_U}$. Composition with projection to U gives a family

$$s_U : \mathcal{S}_U \to U.$$

Let

$$s_U^{[2]} : \mathcal{S}_U^{[2]} \to U$$

be the relative Hilbert scheme of length 2 subschemes of s_U (that is, the scheme of length 2 subschemes of $\mathcal{S}_U$ that are contained in the fibers of s_U).

Remark 3.6.2. Idealy one would like to construct a global double cover of $\widetilde{\mathbf{V}}$ branched along Σ. There are topological obstructions to such a double cover. Indeed, its existence would imply the existence of a double cover of $\mathbf{P}(W) \times \mathbf{P}\big(H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(6))\big)$ branched along the universal sextic plane curve $\mathcal{C}_6$. If such a double cover were to exist, the bidegree of $\mathcal{C}_6$ would have to be divisible by 2. It is not hard to show that the bidegree of $\mathcal{C}_6$ is (6,1), hence cannot be the branch locus of a double cover. However, we *can* construct a double cover of $\mathbf{P}(W) \times U$ branched along $\mathcal{C}_6 \cap U$ if $U \subset \mathbf{P}\big(H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(6))\big)$ is an affine open subset.

Remark 3.6.3. By construction, for any $e \in U$ the fiber $\mathcal{S}_e$ of σ over e is a double cover of $\mathbf{V}_{\varepsilon_{\mathbf{D}}(e)}$ branched along Σ_e. Since $\mathbf{V}_{\varepsilon_{\mathbf{D}}(e)} \cong \mathbf{P}^2$ and Σ_e is a curve of degree 6 with respect to the ample generator of the Picard group of $\mathbf{V}_{\varepsilon_{\mathbf{D}}(e)}$ it follows that if Σ_e is smooth, which is generically the case, $\mathcal{S}_e$ is a K3 surface of degree 2 and $\mathcal{S}_e^{[2]}$ is a holomorphic symplectic fourfold.

Finally, let

$$\widehat{k} : \widehat{\mathcal{K}} \to B$$

be the pull back of the universal cubic k along the blow-up morphism

$$\varepsilon_{\mathbf{D}} : B \to \mathbf{P}(\mathrm{Sym}^3 \mathrm{Sym}^2 W^\vee).$$

For any affine open subset $U \subset B$ we denote by

$$\widehat{k}_U : \widehat{\mathcal{K}}_U \to U$$

the restriction of k over U.

We can state the quasi-universal version of proposition 3.1.1:

Theorem 3.6.4. *Let $U \subset B$ an affine open subset and let $\delta : U' \to U$ be the double cover of U branched along $U \cap N$. Let*

$$\widehat{k}'_U : \widehat{\mathcal{K}}'_U \to U'$$

be the pull-back of k'_U along δ. Then there exists a projective family

$$\zeta : \mathcal{Z} \to U'$$

such that $\mathcal{Z}_x$ is canonically isomorphic to the variety of lines on the cubic $(\widehat{\mathcal{K}}'_U)_x$ for $x \in U' \setminus N$ and canonically isomorphic to $(\mathcal{S}_U^{[2]})_x$ for $x \in U' \cap N$.

Remark 3.6.5. Theorem 3.6.4 is not completely universal since we have to pass to affine open subsets. The reason is that a base change of order 2 is necessary (see also remark 3.1.2) but cannot be performed globally. Indeed, this would amount to constructing a (smooth) double cover $B' \to B$ branched along N. If such a double cover were to exist, it would follow that $c_1(\nu_{N/B}) = 2c_1(\nu_{N/B'})$, where $\nu_{X/Y}$ denotes the normal bundle of X in Y. However, since N is the exceptional locus of the blow-up of a point, $\nu_{N/B} = \tau_N$, the relative tautological bundle on $N \to \mathbf{D}$. This is a generator of the relative Picard group; in particular its first Chern class is not divisible by 2. Hence a global double cover of B branched along N does not exist.

Proof. Let W, $D \in \mathrm{Sym}^3\mathrm{Sym}^2 W^\vee$ and $\mathbf{D}$ as before. The stabilizer of $[D]$ under the action of $\mathrm{PGL}(\mathrm{Sym}^2 W)$ on $\mathbf{P}(\mathrm{Sym}^3\mathrm{Sym}^2 W^\vee)$ is precisely $\mathrm{PGL}(W) \subset \mathrm{PGL}(\mathrm{Sym}^2 W)$. Let $\Gamma \subset \mathrm{PGL}(\mathrm{Sym}^2 W)$ be a small slice near the identity transversal to the $\mathrm{PGL}(W)$ cosets. We then get a map

$$\begin{aligned} \varphi_\Gamma : \Gamma \times \mathrm{Sym}^6 W^\vee &\to \mathbf{P}(\mathrm{Sym}^3\mathrm{Sym}^2 W^\vee) \\ (g, f) &\mapsto g \cdot [D + f] \end{aligned}$$

This map is an embedding near $(\mathrm{id}, 0)$ and has the property that it maps $\Gamma \times \{0\}$ into $\mathbf{D}$. As is easy to see, φ_Γ lifts to a map

$$\widehat{\varphi}_\Gamma : \Gamma \times \mathrm{Bl}_0(\mathrm{Sym}^6 W^\vee) \to B$$

where $\mathrm{Bl}_0(\mathrm{Sym}^6 W^\vee)$ is the blow up of $\mathrm{Sym}^6 W^\vee$ at the origin. We can naturally interpret $\mathrm{Bl}_0(\mathrm{Sym}^6 W^\vee)$ as the space of pointed lines through the origin in $\mathrm{Sym}^6 W^\vee$. Let $H \subset \mathrm{Sym}^6 W^\vee$ be an affine hyperplane not through the origin, then we have an embedding

$$\begin{aligned} i_H : \Gamma \times H \times \mathbf{A}^1 &\to \Gamma \times \mathrm{Bl}_0(\mathrm{Sym}^6 W^\vee) \\ (g, h, t) &\mapsto \big(g, (\mathbf{C} \cdot h, th)\big) \end{aligned}$$

where $(\mathbf{C} \cdot h, th)$ represents the line $\mathbf{C} \cdot h$ pointed at th. Denote $j_{\Gamma,H} := \widehat{\varphi_\Gamma} \circ i_H$, let $U \subset B$ be the image of $j_{\Gamma,h}$ and $\delta : U' \to U$ the double cover branched along

$N \cap U$. Then the map $j_{\Gamma,H}$ lifts to a map $j'_{\Gamma,H} : \Gamma \times H \times \mathbf{A}^1$ such that the following diagram commutes:

$$\begin{array}{ccccc} (g,h,t) \in & & \Gamma \times H \times \mathbf{A}^1 & \xrightarrow{j'_{\Gamma,H}} & U' \\ \downarrow & & \downarrow & & \downarrow \delta \\ (g,h,t^2) \in & & \Gamma \times H \times \mathbf{A}^1 & \xrightarrow[j_{\Gamma,H}]{} & U \end{array}$$

Let $k^{\Gamma,H} : \mathcal{K}^{\Gamma,H} \to \Gamma \times H \times \mathbf{A}^1$ be the pull-back of the universal cubic along $j'_{\Gamma,H} \circ \delta \circ \varepsilon_{\mathbf{D}}$. For $h \in H$, let $\widehat{h}$ be the image of h under the natural embedding

$$\mathrm{Sym}^6 W^\vee \hookrightarrow \mathrm{Sym}^3 \mathrm{Sym}^2 W^\vee.$$

Then for $(g,h,t) \in \Gamma \times H \times \mathbf{A}^1$ we have that

$$\mathbf{P}(\mathrm{Sym}^2 W) \supset \mathcal{K}^{\Gamma,H}_{(g,h,t)} = g \cdot Z_{D+t^2\widehat{h}},$$

where $Z_{D+t^2\widehat{h}}$ is the cubic hypersurface defined by $D + t^2\widehat{h}$. We can easily get rid of the twist by Γ; define the following map:

$$\tau : \mathbf{P}(\mathrm{Sym}^2 W) \times \Gamma \times H \times \mathbf{A}^1 \to \mathbf{P}(\mathrm{Sym}^2 W) \times \Gamma \times H \times \mathbf{A}^1 \tag{3.6.2}$$

$$(p,g,h,t) \mapsto (g^{-1} \cdot p, g, h, t) \tag{3.6.3}$$

and let $\widetilde{k}^{\Gamma,H} : \widetilde{\mathcal{K}}^{\Gamma,H} \to \Gamma \times H \times \mathbf{A}^1$ be the family obtained by pulling back $k^{\Gamma,H}$ along τ. Then, clearly, for $(g,h,t) \in \Gamma \times H \times \mathbf{A}^1$ we have that

$$\widetilde{\mathcal{K}}^{\Gamma,H}_{g,h,t} = Z_{D+t^2\widehat{h}}.$$

Furthermore, the preimage of $\widetilde{\mathbf{V}} \to N$ under $\tau \circ j'_{\Gamma,H}$ is precisely the projection

$$V_{\det} \times \Gamma \times H \times \{0\} \to \Gamma \times H \times \{0\}.$$

Let $\widetilde{\Sigma}^{\Gamma,H} \subset V_{\det} \times \Gamma \times H \times \{0\}$ the preimage of Σ. Then $\widetilde{\Sigma}^{\Gamma,H}_{g,h,0} \subset V_{\det} \cong \mathbf{P}(W)$ is precisely the sextic curve defined by h. To prove theorem 3.6.4 it suffices to prove the following statement:

> *There exists a projective family $\zeta : \mathcal{Z} \to \Gamma \times H \times \mathbf{A}^1$ such that for $g \in \Gamma$ and $h \in H$ the fiber $\mathcal{Z}_{g,h,t}$ is the variety of lines on the cubic defined by $D + t^2\widehat{h}$ if $t \neq 0$ and $\mathcal{Z}_{g,h,0}$ is isomorphic to the Hilbert scheme of length 2 subschemes of the double plane branched along the sextic curve defined by h.*

For every $h \in H$, such a family is provided over $\Gamma \times \{h\} \times \mathbf{A}^1$ by proposition 3.1.1; simply take it to be the family ζ_f for $f = \widehat{h}$. Since the construction of ζ_f is natural, projective and only depends on f, it extends to a family over $\Gamma \times H \times \mathbf{A}^1$ with the desired properties. □

3.7 The polarization

The family $\zeta : \mathcal{Z} \to U'$ as given by theorem 3.6.4 is in particular a projective family and comes equipped with a polarizing sheaf $\mathcal{O}_{\mathcal{Z}}(1)$. We end this chapter with a discussion of this polarization.

Proposition 3.7.1. *Let $u \in U'$ be such that $\mathcal{Z}_u$ is smooth. Then $\mathcal{Z}_u$ is a holomorphic symplectic manifold and the restriction $\mathcal{O}_{\mathcal{Z}}(1)$ to $\mathcal{Z}_u$ is a very ample line bundle that is a power of a line bundle of square 6 and of even type.*
If furthermore $u \in U' \cap N$ then there is a K3 surface $(S, \mathcal{L})$ with line bundle of square 2 such that $\mathcal{Z}_u \cong S^{[2]}$ and $\mathcal{O}_{\mathcal{Z}}(1)|_{\mathcal{Z}_u}$ is a non-trivial power of $\mathcal{L}^{(2,-1)}$ under this identification.

Proof. Let $U'_{sm} \subset U'$ be the dense subset over which the fibers of ζ are smooth. For $u \in U'_{sm} \setminus N$ the fiber $\mathcal{Z}_u$ is the variety of lines on a smooth cubic fourfold, hence a holomorphic symplectic fourfold by theorem 1.3.1. For $u \in U'_{sm} \cap N$ the fiber $\mathcal{Z}_u$ is isomorphic to $\mathcal{S}_u^{[2]}$ by theorem 3.6.4. The fiber $\mathcal{Z}_u$ is smooth if and only if $\mathcal{S}_u$ is smooth, in which case the latter must be a K3 surface. Then $\mathcal{Z}_u$ is a holomorphic symplectic fourfold.

Let $U'_1 \subset U'_{sm}$ be the subset over which the fibers have Picard rank 1. Since $U'_1 \cap N$ is empty, it follows that for any $u \in U'_1$ the holomorphic symplectic fourfold $\mathcal{Z}_u$ is isomorphic to the variety of lines on a cubic fourfold. It follows from theorem 1.3.1 and the assumption that the Picard rank is 1 that the Picard group of $\mathcal{Z}_u$is generated by a line bundle of square 6 and of even type. Hence $\mathcal{O}_{\mathcal{Z}}(1)|_{\mathcal{Z}_u}$ is a non-trivial power of a line bundle of square 6 and of even type (it must be non-trivial since $\mathcal{O}_{\mathcal{Z}}(1)|_{\mathcal{Z}_u}$ is very ample, hence non-trivial). Since U'_1 is dense in U'_{sm} it follows that for any $u \in U'_{sm}$ the restriction of $\mathcal{O}_{\mathcal{Z}}(1)$ to $\mathcal{Z}_u$ is a non-trivial power of an ample line bundle of square 6 and of even type. The first part of the statement follows.

Let $U'_2 \subset U'_{sm}$ be the locus over which the fibers are of Picard rank 2. Note that $U'_2 \cap N$ is dense in $U'_{sm} \cap N$. First assume that $u \in U'_2 \cap N$. Then $\mathcal{S}_u$ is a K3 surface of degree 2 and Picard rank 1. Let $\mathcal{L}_2$ be a line bundle of square 2 on $\mathcal{S}_u$. It follows from proposition 2.1.14 that up to isomorphism the only ample line of square 6 on $\mathcal{S}_u^{[2]}$ is $\mathcal{L}^{(2,-1)}$. Hence from the first part of the proposition it follows that the restriction $\mathcal{O}_{\mathcal{Z}}(1)|_{\mathcal{Z}_u}$ is a non-trivial power $\mathcal{L}^{(2,-1)}$.

Since $U'_2 \cap N$ is dense in $U'_{sm} \cap N$, it follows that for every $u \in U'_{sm} \cap N$ the restriction of $\mathcal{O}_{\mathcal{Z}}(1)$ to $\mathcal{Z}_u$ is a non-trivial power of $\mathcal{L}_2^{(2,-1)}$, where $\mathcal{L}_2$ is a line bundle of square 2 on $\mathcal{S}_u$. This completes the proof. □

3.8 List of notation

This is a short list of symbols and expression used throughout this chapter.

D	determinantal cubic form	34
Δ	spectrum of localization of $\mathbf{C}[t]$ at $\langle t\rangle$	68
$\varepsilon_{F_1}:\widehat{\mathcal{F}}\to\mathcal{F}$	blow-up of F_1 in $\mathcal{F}$	87
E	exceptional divisor of ε_{F_1} in $\widehat{\mathcal{F}}$	87
f	element of $\mathrm{Sym}^3\mathrm{Sym}^2 W^\vee$ such that $v^* f\neq 0$	69
F_1	component of X_{det} isomorphic to $\mathbf{P}(W)^{[2]}$	73
F_2	component of X_{det} isomorphic to $\mathbf{P}(W)\times\mathbf{P}(W^\vee)$	73
$\widehat{F}_2$	strict transform of F_2 in $\widehat{\mathcal{F}}$	95
$\varphi:\mathcal{F}\to\Delta$	relative line variety of ξ'_f	72
$\widehat{\varphi}:\widehat{\mathcal{F}}\to\Delta$	composition of ε_{F_1} and φ	87
$\overline{\varphi}:\overline{\mathcal{F}}\to\Delta$	factorization of $\widehat{\varphi}$ through contraction $\gamma:\widehat{\mathcal{F}}\to\overline{\mathcal{F}}$	122
$\gamma:\widehat{\mathcal{F}}\to\overline{\mathcal{F}}$	contraction of $\widehat{\mathcal{F}}_2$ in $\widehat{\mathcal{F}}$	122
$\mathcal{G}$	short for $\mathrm{Gr}_2(\mathrm{Sym}^2 W)\times\Delta$, natural ambient space for $\mathcal{F}$	85
$\widehat{\mathcal{G}}$	blow-up of $\mathcal{G}$ along F_1, natural ambient space for $\widehat{\mathcal{F}}$	100
S_f	double cover of $\mathbf{P}(W)$ branched along Σ_f	69
$S_f^{[2]}$	Hilbert scheme of length 2 subschemes of S_f	69
Σ_f	sextic curve in $\mathbf{P}(W)$ defined by $v^* f$	69
v^*	pull-back along Veronese map $W\to\mathrm{Sym}^2 W$	69
$V\subset S_f^{[2]}$	locus of length 2 subschemes contained in a fiber of $S_f\to\mathbf{P}(W)$	101
W	complex vector space of dimension 3	68
X_{det}	determinantal cubic in $\mathbf{P}(\mathrm{Sym}^2 W)$	68
$\xi_f:\mathcal{X}_f\to\Delta$	one-parameter linear deformation of X_{det} by f	69
$\xi'_f:\mathcal{X}'_f\to\Delta$	pull-back of ξ_f along base change $\Delta\to\Delta$, $t\mapsto t^2$	69

Chapter 4

Line bundles of square 6 and of even type on holomorphic symplectic fourfolds of $K3^{[2]}$-type

The results that we have obtained so far enable us to give a complete characterization of (very-)ampleness of line bundles that are of square 6 and of even type on holomorphic symplectic fourfolds of $K3^{[2]}$-type. More precisely, we will characterize the locus of pairs $(Y, \mathcal{L})$ in $\mathfrak{N}_{\mathrm{BD}}$ for which the line bundle is very ample, the locus for which it is ample and the locus for which it is nef. In sections 4.1 and 4.2 we will improve the ampleness results that we have obtained sofar for elements of $\mathfrak{N}_{\mathrm{BD},2}$ and $\mathfrak{N}_{\mathrm{BD},6}$. The main result will be presented in section 4.3. We end this chapter with some words on the cases that are not covered by the main result and some thoughts on possibilities of continuing research.

4.1 Reduction to ampleness on K3 surfaces

Let $(S, \mathcal{L})$ be a K3 surface with line bundle of square 2. From proposition 2.1.14 it follows that if S has Picard rank 1, then $\mathcal{L}^{(2,-1)}$ is ample on $S^{[2]}$. The results from the previous chapter enables us prove a much more precise statement:

Proposition 4.1.1. *Let $(S, \mathcal{L})$ be a K3 surface with line bundle of square 2. Then $\mathcal{L}^{(2,-1)}$ is ample on $S^{[2]}$ if and only if $\mathcal{L}$ is ample on S.*

Proof. First assume that $\mathcal{L}$ is not ample on S. By Kleiman's criterion for ampleness this means that there is a class $c \in \overline{NE}(S)$, the closure of the cone of curves, such that $\int_c c_1(\mathcal{L}) \leq 0$ (we use the integral sign to denote the duality pairing between homology and cohomology). Let $C \subset S$ be any curve and $p \in S$ a point outside C. Let $\widetilde{C}_p \subset S^{[2]}$ be the curve of length 2 subschemes of S whose support non-trivially intersects both C and p. The homology class of $\widetilde{C}_p$ does not depend on the choice of p, so the map of homology classes $[C] \mapsto [\widetilde{C}_p]$ defines a map $NE(S) \to NE(S^{[2]})$. In fact this map comes from an embedding

$$i : H_2(S) \to H_2(S^{[2]})$$

which is basically defined by replacing the curves by cycles. This map i is dual to the projection $i^\vee : H^2(S^{[2]}, \mathbf{Z}) \to H^2(S, \mathbf{Z})$ that comes form the natural decomposition

$$H^2(S^{[2]}, \mathbf{Z}) \cong H^2(S, \mathbf{Z}) \oplus \mathbf{Z}\delta$$

that we mentioned earlier. It follows that

$$\begin{aligned} \int_{i(c)} c_1(\mathcal{L}^{(2,-1)}) &= \int_c i^\vee\big(c_1(\mathcal{L}^{(2,-1)})\big) \\ &= \int_c i^\vee(2c_1(\mathcal{L}^{(1,0)}) - \delta) \\ &= 2\int_c c_1(\mathcal{L}) \\ &\le 0, \end{aligned}$$

where we used that by definition $i^\vee c_1(\mathcal{L}^{(1,0)}) = c_1(\mathcal{L})$, see notation 1.1.24. Since $c \in \overline{NE}(S)$ it follows that $i(c) \in \overline{NE}(S^{[2]})$. By Kleiman's criterion $\mathcal{L}^{(2,-1)}$ cannot be ample on $S^{[2]}$.

Now assume that $\mathcal{L}$ is ample on S. Then the linear system $|\mathcal{L}|$ is base point free (see [42]) and the associated morphism $\varphi_{\mathcal{L}} : S \to |\mathcal{L}|^\vee$ is a double covering branched along a smooth sextic Σ. Choose an isomorphism $|\mathcal{L}|^\vee \cong \mathbf{P}(W)$ and let $f \in \mathrm{Sym}^3\mathrm{Sym}^2 W^\vee$ be such that Σ is the zero locus of the pull-back of f along the Veronese.

Let $t \mapsto D + tf$, $t \in \Delta$ be a linear deformation of the determinantal cubic form by f and let $\mathcal{X}_f \to \Delta$ be corresponding family of cubics. Then according to proposition 3.1.1 there exists a projective family $\zeta : \mathcal{Z} \to \Delta$ such that $\mathcal{Z}_t$ is the variety of lines on the cubic defined by $D + t^2 f = 0$ (note the square) and $\mathcal{Z}_0 \cong S^{[2]}$. Furthermore, under this identification, the restriction of $\mathcal{O}_{\mathcal{Z}}(1)$ to $\mathcal{Z}_0$ corresponds to a power of $\mathcal{L}^{(2,-1)}$ by proposition 3.7.1. Since $\mathcal{O}_{\mathcal{Z}}(1)|_{\mathcal{Z}_0}$ is very ample, it follows that $\mathcal{L}^{(2,-1)}$ is ample on $S^{[2]}$. This completes the proof. □

This result relates ampleness of $\mathcal{L}^{(2,-1)}$ on $S^{[2]}$ to ampleness of $\mathcal{L}$ on S. The advantage is that ampleness of line bundles on K3 surfaces is well understood, and is governed by the presence of rational curves on S; a class $\alpha \in H^2(S, \mathbf{Z})$ is ample if and only if $\int_C \alpha > 0$ for all non-singular rational curves $C \subset S$, see for example [4]. This enables us to prove more precise results for the pair $\big(S^{[2]}, \mathcal{L}^{(2,-1)}\big)$ in case that $\mathcal{L}$ may not be ample on S, summarized in the two following propositions.

Proposition 4.1.2. *Let $(S, \mathcal{L})$ be a K3 surface with line bundle of square 2. Assume that $\int_C c_1(\mathcal{L}) \neq 0$ for all non-singular rational curves $C \subset S$. Then there exists a K3* *surface S' with line bundle $\mathcal{L}'$ of square 2 such that $(\mathcal{L}')^{(2,-1)}$ is ample on $(S')^{[2]}$ and the pairs $\left(S^{[2]}, \mathcal{L}^{(2,-1)}\right)$ and $\left((S')^{[2]}, (\mathcal{L}')^{(2,-1)}\right)$ are bimeromorphic.*

Proof. By the assumption that $\int_C c_1(\mathcal{L}) \neq 0$ for all non-singular rational curves $C \subset S$ there exists a sequence

$$S =: S_0 \overset{\varphi_1}{\dashrightarrow} S_1 \overset{\varphi_2}{\dashrightarrow} \cdots \overset{\varphi_k}{\dashrightarrow} S_k$$

of Atiyah flops such that $\mathcal{L}' := (\varphi_k \circ \ldots \circ \varphi_1)_* \mathcal{L}$ is ample on S_k (see [4] for a proof of this statement). In particular, by proposition 4.1.1 $(\mathcal{L}')^{(2,-1)}$ is ample on $S_k^{[2]}$. The bimeromorphic map $\varphi := \varphi_k \circ \ldots \circ \varphi_1$ induces a bimeromorphic map $\widetilde{\varphi} : S^{[2]} \dashrightarrow S_k^{[2]}$ such that $\widetilde{\varphi}_* \mathcal{L}^{(2,-1)} \cong (\mathcal{L}')^{(2,-1)}$ is ample. Take $S' = S_k$ and $\psi = \widetilde{\varphi}^{-1}$, then the proposition follows. □

Proposition 4.1.3. *Let S be a K3 surface of Picard rank 2 with a line bundle $\mathcal{L}$ of square 2 and a rational curve $C \subset S$ such that $\int_C c_1(\mathcal{L}) = 0$. Then $c_1(\mathcal{L}^{(2,-1)})$ is in the boundary of the closure of the birational Kähler cone*[1] *on $S^{[2]}$.*

Proof. As usual we have the decomposition

$$\operatorname{Pic} S^{[2]} \cong \operatorname{Pic} S \oplus \mathbf{Z}\delta$$

with δ as before half the class of the divisor of non-reduced length 2 subschemes. Let $D_C \subset S^{[2]}$ be the divisor of length 2 subschemes of S that have some support on C and $[D_C]$ its cohomology class. Under the decomposition above $[D_C] \in \operatorname{Pic} S^{[2]}$ corresponds to $c_1(\mathcal{O}_S(C)) \in \operatorname{Pic} S$. Hence $q_{S^{[2]}}([D_C], \delta) = 0$, where $q_{S^{[2]}}$ denotes the Beauville–Bogomolov form on $S^{[2]}$. Even stronger, since by assumption we have

$$q_S\big(c_1(\mathcal{O}_S(C)), c_1(\mathcal{L})\big) = \int_C c_1(\mathcal{L}) = 0,$$

where q_S is the Beauville–Bogomolov form on S, and since the decomposition above preserves the Beauville–Bogomolov forms, it follows that

$$q_{S^{[2]}}\big(c_1(\mathcal{L}^{(a,b)}, [D_C]\big) = 0$$

for all $a, b \in \mathbf{Z}$.

Note that by construction D_C is an uniruled divisor. It follows now from proposition 4.2 in [27] that $c_1(\mathcal{L}^{(a,b)})$ cannot be contained in the interior of the birational Kähler cone for any $a, b \in \mathbf{Z}$.

Hence to prove the proposition it suffices to show that $c_1(\mathcal{L}^{(2,-1)})$ is contained in the closure of the birational Kähler cone on $S^{[2]}$. To do this we invoke the following theorem, proven by Hassett and Tschinkel in [23] (we reformulated it to fit into the context):

[1] see section 2.2.2 for a definition

Theorem 4.1.4. *Let Y be a holomorphic symplectic manifold and $\mathcal{A}$ a line bundle on Y. If the base locus of the linear system of $\mathcal{A}$ is of codimension at least 2, then $c_1(\mathcal{A})$ is contained in the closure of the birational Kähler cone of Y.*

Let $\varphi : S \dashrightarrow \mathbf{P}\big(H^0(S, \mathcal{L}^{\otimes 2})^\vee\big)$ (note the tensor power!) be the map defined by the linear system of $\mathcal{L}$. Let

$$\varphi^{[2]} : S^{[2]} \dashrightarrow \mathbf{P}\left(\bigwedge^2 H^0(S, \mathcal{L}^{\otimes 2})^\vee\right)$$

be the map that sends a subscheme σ of S to the Plücker image of the line spanned by $\varphi(\sigma)$ (whenever defined). It follows from lemma 2.1.15 that if $\varphi^{[2]}$ is defined on the complement of a locus of codimension at least 2 in $S^{[2]}$, that then $\mathcal{L}^{(2,-1)} \cong (\varphi^{[2]})^* \mathcal{O}(1)$. Hence it suffices to show that $\varphi^{[2]}$ is well defined outside a locus of codimension 2. To show this we need to know what φ looks like under the assumptions that we made on the K3 surface. By assumption the rank of the Picard lattice of S is 2 and contains $\langle c_1(\mathcal{L}), c_1(\mathcal{O}_S(C))\rangle$ as sublattice, where $q_S\big(c_1(\mathcal{L}_2)\big) = 2$, $q_S\big(c_1(\mathcal{O}_S(C))\big) = -2$ and $q_S\big(c_1(\mathcal{L}), c_1(\mathcal{O}_S(C))\big) = 0$. We can distinguish 2 possibilities:

1. $\operatorname{Pic} S = \langle c_1(\mathcal{L}), c_1(\mathcal{O}_S(C))\rangle \cong \langle 2\rangle \oplus \langle -2\rangle$. Such K3 surfaces occur as the resolution of a double plane branched along a sextic with a double point. See also [45].

2. $\operatorname{Pic} S \cong U$, the hyperbolic plane. If e, f are orthogonal isotropic generators of U we can write $c_1(\mathcal{L}) = e + f$ and $c_1(\mathcal{O}_S(C)) = e - f$. Such K3 surfaces occur as limits of a family of double planes branched along a family of sextic curves that degenerates to the triple conic. See [45] and [42].

It follows from results in [42] that in case 1 the linear system $|\mathcal{L}|$ is base-point free, hence also $|\mathcal{L}^{\otimes 2}|$ is base-point free. The map $\varphi : S \to \mathbf{P}(H^0(S, \mathcal{L}^{\otimes 2})^\vee)$ has as image the Veronese surface (see [42, Prop. 5.6]) in $\mathbf{P}\big(H^0(S, \mathcal{L}^{\otimes 2})^\vee\big) \cong \mathbf{P}^5$, is of degree 2 outside C and contracts C. In the second case we are in case (i) of prop. 5.7 in [42], with $L = \mathcal{L}^{\otimes 2}$. Again $\varphi : S \to \mathbf{P}\big(H^0(S, \mathcal{L}^{\otimes 2})^\vee\big)$ is well-defined and has as image a cone in $\mathbf{P}^5$ over a rational normal twisted quartic in $\mathbf{P}^4$, is of degree 2 outside C and contracts C to the vertex of the cone (see also [45]).

For both cases, let $F \subset S^{[2]}$ be the locus of length 2 subschemes of S that are contained in a fiber of φ. Observe that $\varphi^{[2]}$ is well defined on $S^{[2]} \setminus F$, since for any length 2 subscheme σ not contained in a fiber of φ we have that $\varphi(\sigma)$ is a length 2 subscheme in $\operatorname{im}\varphi$. Furthermore, $F = V \cup C^{[2]}$, where $V \subset S^{[2]}$ is the set of length 2 subschemes of the form $\varphi^{-1}(p)$ for $p \in \operatorname{im}\varphi \setminus \varphi(C)$ and $C^{[2]}$ is the set of length 2 subschemes contained in C. Clearly both $C^{[2]}$ and V are of dimension 2, so F is of codimension 2 in $S^{[2]}$.

Since $\mathcal{L}^{(2,-1)} \cong (\varphi^{[2]})^* \mathcal{O}(1)$ it follows that the base locus of $|\mathcal{L}^{(2,-1)}|$ is of codimension at least 2. By theorem 4.1.4 we have that $c_1(\mathcal{L}^{(2,-1)})$ is contained in the closure of the birational Kähler cone of $S^{[2]}$. This finishes the proof. □

4.2 Ampleness for elements of $\mathfrak{N}_{\mathrm{BD},2}$

The results from the previous section enables us to analyze precisely for what pairs $(Y,\mathcal{L})$ with isomorphism class in $\mathfrak{N}_{\mathrm{BD},2}$ the line bundle $\mathcal{L}$ is ample on Y. Before stating the main result we show how proposition 4.1.3 provides an obstruction for ampleness of $\mathcal{L}$ on Y.

Lemma 4.2.1. *Let $(Y,\mathcal{L})$ be a holomorphic symplectic manifold of $K3^{[2]}$-type with line bundle. Assume that there exists a K3 surface S with line bundle $\mathcal{L}_2$ of square 2, a rational curve $C \subset S$ such that $\int_C c_1(\mathcal{L}_2) = 0$ and a Hodge isometry $\eta : H^2(Y,\mathbf{Z}) \to H^2(S^{[2]},\mathbf{Z})$ such that $\eta(c_1(\mathcal{L})) = c_1(\mathcal{L}_2^{(2,-1)})$. Then $\mathcal{L}$ cannot be ample on Y.*

Proof. Let $(Y,\mathcal{L})$, $S,\mathcal{L}_2,C$ and η as in the statement and assume in addition that $\mathcal{L}$ is ample on Y. Since ampleness is preserved by small deformations of the pair $(Y,\mathcal{L})$ we may assume that the triple $(S,\mathcal{L}_2,C)$ is generic in the sense that the Picard rank of S is 2 (see before). Then by proposition 4.1.3 $c_1(\mathcal{L}_2^{(2,-1)})$ is in the boundary of the closure of the birational Kähler cone of $S^{[2]}$. This implies that there exists holomorphic symplectic fourfold Y' and a bimeromorphic map $\psi : Y' \dashrightarrow S^{[2]}$ such that $\psi^* c_1(\mathcal{L}_2^{(2,-1)})$ is in the boundary of the closure of the Kähler cone of Y'. It follows that the Hodge isometry $\psi^* \circ \eta$ maps the $c_1(\mathcal{L})$, which is a Kähler class on Y' by assumption, to a class on the boundary of the Kähler cone of Y'. This is impossible by theorem 1.1.28. Hence $\mathcal{L}$ cannot be ample on Y. □

Theorem 4.2.2. *Let $(Y,\mathcal{L})$ be a holomorphic symplectic manifold with line bundle such that the isomorphism class of the pair belongs to $\mathfrak{N}_{\mathrm{BD},2}$. Then $\mathcal{L}$ is ample on Y if and only if there exists a K3 surface S with ample line bundle $\mathcal{L}_2$ of square 2 and an isomorphism $\varphi : S^{[2]} \to Y$ such that $\mathcal{L} \cong \varphi^* \mathcal{L}_2^{(2,-1)}$.*

Proof. First assume that there exist a K3 surface S with ample line bundle $\mathcal{L}_2$ of square 2 and an isomorphism $\varphi : Y \to S^{[2]}$ such that $\mathcal{L} \cong \varphi^* \mathcal{L}_2^{(2,-1)}$. Since $\mathcal{L}_2$ is assumed to be ample on S it follows from proposition 4.1.1 that $\mathcal{L}_2^{(2,-1)}$ is ample on $S^{[2]}$. Hence $\varphi^* \mathcal{L}_2^{(2,-1)} \cong \mathcal{L}$ is ample on Y.

We prove the converse. By theorem 2.1.8 the assumption that $(Y,\mathcal{L})$ lies in $\mathfrak{N}_{\mathrm{BD},2}$ implies that there exist a K3 surface S with line bundle $\mathcal{L}_2$ of square 2, a bimeromorphism $\psi : S^{[2]} \dashrightarrow Y$ and a Hodge isometry $\chi : H^2(S^{[2]},\mathbf{Z}) \to H^2(S^{[2]},\mathbf{Z})$ such that $\psi^* c_1(\mathcal{L}) = \chi\big(c_1(\mathcal{L}_2^{(2,-1)})\big)$.

Since we assumed $\mathcal{L}$ to be ample on Y, it follows from lemma 4.2.1 that S cannot contain a rational curve C such that $\int_C c_1(\mathcal{L}_2) = 0$. Then by proposition 4.1.2 there exists a K3 surface S' with line bundle $\mathcal{L}_2'$ of square 2 such that $(\mathcal{L}_2')^{(2,-1)}$ is ample on $(S')^{[2]}$ and there exists a bimeromorphic map $\psi_2 : (S')^{[2]} \dashrightarrow S^{[2]}$ such that $(\mathcal{L}_2')^{(2,-1)} = \psi_2^*(\mathcal{L}_2)^{(2,-1)}$. The composition $\psi_2^* \circ \chi^{-1} \circ \psi^*$ now is a Hodge isometry from $H^2(Y,\mathbf{Z})$ to $H^2((S')^{[2]},\mathbf{Z})$ that sends the class $c_1(\mathcal{L})$ to $c_1\big((\mathcal{L}_2')^{(2,-1)}\big)$. Both are classes of ample line bundles, hence Kähler. By theorem 1.1.28 there exists an isomorphism $f : (S')^{[2]} \to Y$ such that $f^* = \psi_2^* \circ \chi^{-1} \circ \psi^*$. This completes the proof.

□

4.3 Characterizations of ampleness and nefness

We are now in position to prove the main result that characterizes the ampleness of line bundles of square 6 and even type on holomorphic symplectic manifolds of $\mathrm{K3}^{[2]}$-type.

Theorem 4.3.1. *Let Y be a holomorphic symplectic fourfold of $\mathrm{K3}^{[2]}$-type and $\mathcal{L}$ a line bundle of sqaure 6 and even type. Then the following hold:*

- *$\mathcal{L}$ is very ample on Y if and only if there exists a cubic fourfold X and an isomorphism $\varphi : F(X) \to Y$ such that $\varphi^*(\mathcal{L})$ is the natural polarization on $F(X)$;*
- *$\mathcal{L}$ is ample but not very ample on Y if and only if there exists a K3 surface S with ample line bundle $\mathcal{L}_2$ of square 2 and an isomorphism $\varphi : S^{[2]} \to Y$ such that $\varphi^* \mathcal{L} \cong \mathcal{L}_2^{(2,-1)}$.*

Furthermore if $\mathcal{L}$ is nef but not ample on Y then there are two possibilities:

- *There is a K3 surface S with line bundle $\mathcal{L}_2$ of square 2, a rational curve $C \subset S$ such that $\int_C c_1(\mathcal{L}_2) = 0$, a bimeromorphic map $\psi : S^{[2]} \dashrightarrow Y$ and a Hodge isometry $\chi : H^2(S^{[2]}, \mathbf{Z}) \to H^2(S^{[2]}, \mathbf{Z})$ such that $(\chi \circ \psi^*)(c_1(\mathcal{L})) = c_1(\mathcal{L}_2^{(2,-1)})$;*
- *There is a K3 surface S with line bundle $\mathcal{L}_6$ of square 6, a bimeromorphic map $\psi : S^{[2]} \dashrightarrow Y$ and a Hodge isometry $\chi : H^2(S^{[2]}, \mathbf{Z}) \to H^2(S^{[2]}, \mathbf{Z})$ such that $(\chi \circ \psi^*)(c_1(\mathcal{L})) = c_1(\mathcal{L}_6^{(2,-3)})$;*

Proof. As before, let $\mathfrak{N}_{\mathrm{BD}}$ be the moduli space of holomorphic symplectic fourfolds of type $\mathrm{K3}^{[2]}$ with a line bundle of square 6 and of even type with first Chern class in the positive cone. Let $\overline{\mathfrak{N}}_{\mathrm{BD}}$ be the Hausdorff reduction.

Let Y be a holomorphic symplectic fourfold of $\mathrm{K3}^{[2]}$-type with a line bundle $\mathcal{L}$, such that the isomorphism class of the pair $(Y, \mathcal{L})$ lies in $\mathfrak{N}_{\mathrm{BD}}$. First assume that $(Y, \mathcal{L})$ is isomorphic to the variety of lines (endowed with its natural line bundle of square 6) on a cubic fourfold. Then $\mathcal{L}$ is very ample on Y by theorem 1.3.1.

Conversely assume that $\mathcal{L}$ is very ample on Y. Then by propositions 2.1.16 and 2.2.13 the isomorphism class of $(Y, \mathcal{L})$ cannot be contained in $\mathfrak{N}_{\mathrm{BD},2} \cup \mathfrak{N}_{\mathrm{BD},6}$ and hence there must exist a cubic fourfold X such that the isomorphism class of $\big(F(X), \mathcal{O}_{F(X)}(1)\big)$ is non-separated in $\mathfrak{N}_{\mathrm{BD}}$ from the isomorphism class of $(Y, \mathcal{L})$. Since $\mathcal{L}$ is ample on Y and $\mathcal{O}_{F(X)}(1)$ is ample on $F(X)$ the pairs must be isomorphic by theorem 1.1.28. This proves the first item of theorem 4.3.1

Now assume that $\mathcal{L}$ is ample on Y but not very ample. Again by propositions 2.1.16 and 2.2.13 it follows that the isomorphism class of $(Y,\mathcal{L})$ is contained in $\mathfrak{N}_{\mathrm{BD},2}$. By theorem 4.2.2 there exists a K3 surface S with ample line bundle $\mathcal{L}_2$ of square 2 such that $(Y,\mathcal{L}) \cong \left(S^{[2]}, \mathcal{L}_2^{(2,-1)}\right)$.

Conversely, if S is a K3 surface with line bundle $\mathcal{L}_2$ of square 2, then by proposition 4.1.1 $\mathcal{L}_2^{(2,-1)}$ is ample on $S^{[2]}$. However it cannot be very ample by proposition 2.1.16. Hence if $(Y,\mathcal{L})$ is isomorphic to $\left(S^{[2]}, \mathcal{L}_2^{(2,-1)}\right)$ for such $(S,\mathcal{L}_2)$, then $\mathcal{L}$ is ample but not very ample on Y. The second part of theorem 4.3.1 follows.

Finally assume that $\mathcal{L}$ is nef but not ample on Y. Then for any pair $(Y',\mathcal{L}')$ which is non-separated from $(Y,\mathcal{L})$, $\mathcal{L}'$ cannot be ample on Y' by theorem 1.1.28. By the first part of the theorem it follows that the isomorphism class of $(Y,\mathcal{L})$ must be contained in $\mathfrak{N}_{\mathrm{BD},2}$ or $\mathfrak{N}_{\mathrm{BD},6}$, with the additional property that in the first case there exists a K3 surface S with line bundle $\mathcal{L}_2$ of square 2 and rational curve $C \subset S$ such that $\int_C c_1(\mathcal{L}_2) = 0$ and Y is bimeromorphic to $S^{[2]}$. The theorem now follows from the characterizations of $\mathfrak{N}_{\mathrm{BD},2}$ and $\mathfrak{N}_{\mathrm{BD},6}$ given by theorems 2.1.8 and 2.2.7. □

4.4 Some words on nef non-ample line bundles

Theorem 4.3.1 gives necessary and sufficient conditions for ampleness and very-ampleness of line bundles $\mathcal{L}$ on holomorphic symplectic manifolds such that $(Y,\mathcal{L})$ belongs to $\mathfrak{N}_{\mathrm{BD}}$. For nefness it only gives necessary conditions. These conditions are too weak to be sufficient. In this section we look more closely at the case that $\mathcal{L}$ is nef but not ample on Y and show how we can obtain stronger conditions at the cost of imposing extra conditions on the pair $(Y,\mathcal{L})$.

Proposition 4.4.1. *Assume the isomorphism class of the pair $(Y,\mathcal{L})$ is contained in $\mathfrak{N}_{\mathrm{BD}}$ and assume that Y has Picard rank at most 2. Then $\mathcal{L}$ is nef but not ample on Y if and only if there exists a K3 surface S with line bundle of $\mathcal{L}_6$ square 6 and an isomorphism $\varphi : S_6^{[2]} \to Y$ such that $\varphi^*\mathcal{L} \cong \mathcal{L}_6^{(2,-3)}$.*

Proof. Let S be a K3 surface with line bundle $\mathcal{L}_6$ of square 6. Assume furthermore that the Picard lattice of S is of rank 1. This implies that $\mathcal{L}_6$ is ample on S, and hence we may view S as the intersection of a quadric hypersurface Q and a cubic hypersurface K in $\mathbf{P}^4$. The assumptions imply that S does not contain any lines.

Let $x_1,\dots x_5$ be homogeneous coordinates and let $f_2, f_3 \in \mathbf{C}[x_1,\dots x_5]$ be homogeneous forms that define Q and K respectively. Define

$$f := x_0 f_2 + f_3 \in \mathbf{C}[x_0,\dots,x_5],$$

then the zero locus X_f of f is a cubic in $\mathbf{P}^5$ with ordinary double point at $p := [1:0:0:0:0:0]$. Let $\pi : \mathbf{P}^5 \dashrightarrow \mathbf{P}^4$ be the projection away from p. The lines in

X_f through p sweep out a cone over $S \subset \mathbf{P}^4$ with vertex p. In particular the line $\overline{\pi^{-1}(s)}$ through p is contained in X_f if and only if $s \in S$.

Let $\sigma \subset S$ be a length 2 subscheme. It spans a line $\langle\sigma\rangle \subset \mathbf{P}^4$. Let $P_\sigma \subset \mathbf{P}^5$ be the closure of the pull-back of $\langle\sigma\rangle$ along π; it is a 2-plane through p. Since S does not contain any lines, in particular it does not contain $\langle\sigma\rangle$. It follows that the intersection $X_f \cap P_\sigma$ is a cubic curve in P_σ which contains the pair of lines $\overline{\pi^{-1}(\sigma)}$ (it is a double line if and only if σ is non-reduced). Hence $X_f \cap P_\sigma = \overline{\pi^{-1}(\sigma)} \cup \ell_\sigma$, a union of three lines.

We can define a morphism

$$\begin{aligned} \lambda : S^{[2]} &\to F(X_f) \\ \sigma &\mapsto \ell_\sigma. \end{aligned}$$

It is defined on all of $S^{[2]}$ and clearly non-constant. Let $\mathcal{L}$ be the natural polarization on $F(X_f)$, then $\lambda^*\mathcal{L}$ is non-trivial and nef, and its first Chern class is contained in the positive cone. In section 2.2.2 we defined for every $s \in S$ a rational curve $C_s \subset S^{[2]}$. From the definition of these curves it is not hard to see that $\lambda(C_s) = \overline{\pi^{-1}(s)}$, that is, λ contracts these curves to a point. By lemma 2.2.9 we then have that $\lambda^*\mathcal{L} \cong \mathcal{L}_6^{(2k,-3k)}$ for some $k \in \mathbf{Z}$. As we mentioned above $\lambda^*\mathcal{L}$ has first Chern class in the positive cone, and the same holds for $\mathcal{L}_6^{(2,-3)}$ by proposition 2.2.10. Therefore $k > 0$ (in fact it is not hard to show that $k = 1$, although we do not need that here). Since a line bundle is nef if and only if some positive multiple is nef, nefness of $\mathcal{L}_6^{(2,-3)}$ follows. It is however not ample by proposition 2.2.10.

So assume that $(Y,\mathcal{L})$ has isomorphism class in $\mathfrak{N}_{\mathrm{BD}}$, that Y has Picard at most 2 and that there exists a K3 surface S with line bundle $\mathcal{L}_6$ such that $(Y,\mathcal{L}) \cong \left(S^{[2]}, \mathcal{L}_6^{(2,-3)}\right)$. Then in particular $S^{[2]}$ is of Picard rank 2, hence S is of Picard rank 1. By the argument above $\mathcal{L}_6^{(2,-3)}$ is nef but not ample, hence the same holds for $\mathcal{L}$ on Y. Now assume that $(Y,\mathcal{L})$ has isomorphism class in $\mathfrak{N}_{\mathrm{BD}}$, that Y has Picard at most 2 and that $\mathcal{L}$ is nef but not ample on Y. By theorem 4.3.1 there are 2 possibilities:

- There is a K3 surface S with line bundle $\mathcal{L}_2$ of square 2, a rational curve $C \subset S$ such that $\int_C c_1(\mathcal{L}_2) = 0$, a bimeromorphic map $\psi : S^{[2]} \dashrightarrow Y$ and a Hodge isometry $\chi : H^2(S^{[2]},\mathbf{Z}) \to H^2(S^{[2]},\mathbf{Z})$ such that $(\chi \circ \psi^*)(c_1(\mathcal{L})) = c_1(\mathcal{L}_2^{(2,-1)})$;
- There is a K3 surface S with line bundle $\mathcal{L}_6$ of square 6, a bimeromorphic map $\psi : S^{[2]} \dashrightarrow Y$ and a Hodge isometry $\chi : H^2(S^{[2]},\mathbf{Z}) \to H^2(S^{[2]},\mathbf{Z})$ such that $(\chi \circ \psi^*)(c_1(\mathcal{L})) = c_1(\mathcal{L}_6^{(2,-3)})$;

The first one is ruled out by the assumption that Y has Picard rank at most 2, so the second case must hold. Since Y has Picard rank at most 2, it follows that S has Picard rank 1 in this case, and hence that $\mathcal{L}_6^{(2,-3)}$ is nef but not ample on $S^{[2]}$.

Furthermore, since ψ is a bimeromorphism from $S^{[2]}$ to Y and $\mathcal{L}$ is assumed to be nef on Y, $\psi^* c_1(\mathcal{L})$ must be contained in the closure of the birational Kähler cone of $S^{[2]}$. By proposition 2.2.10 there can be only one class of square 6 and even type in the closure of the birational Kähler cone of $S^{[2]}$, namely $c_1(\mathcal{L}_6^{(2,-3)})$. Hence $\psi^* c_1(\mathcal{L}) = c_1(\mathcal{L}_6^{(2,-3)})$. But $c_1(\mathcal{L}_6^{(2,-3)})$ is also in the *boundary* of the closure of the birational Kähler cone of $S^{[2]}$ and since the Picard rank is 2, the only possibility is that ψ^* maps the Kähler cone of Y to the Kähler cone of $S^{[2]}$. Hence ψ extends to an isomorphism $f : S^{[2]} \to Y$ such that $f^*\mathcal{L} \cong \mathcal{L}_6^{(2,-3)}$. This completes the proof. □

It would be interesting to find results of this nature for pairs $(Y, \mathcal{L})$ with Picard rank bigger than 2. Clearly, the analysis of the Kähler cone and birational Kähler cone is much harder in those cases.

4.5 Outlook

We end this chapter with some thoughts on possibilities for future research, based on the results in this chapter. Theorem 4.3.1 gives a precise characterization of of obstructions to (very-)ampleness of line bundles of square 6 and even type on holomorphic symplectic manifolds of K3$^{[2]}$-type. The result is somewhat in the same spirit as results obtained by Saint–Donat in [42].

Let us reflect on the proof of theorem 4.3.1. Our method of proving is based on two important ingredients:

- The Beauville–Donagi construction, since it provides a 'dictionary' between holomorphic symplectic manifolds of K3$^{[2]}$ with a line bundle of square 6 and even type on the one hand and cubic fourfolds on the other hand. In particular it relates the periods of holomorphic symplectic manifolds of K3$^{[2]}$ on this line bundle fails to be very ample, to the periods in the complement of the image of the period map for cubic fourfolds.

- The work of Voisin ([47]), Hassett ([21]), Laza ([29]) and Looijenga ([30]) on the period map for cubic fourfolds, in particular the identification of the complement of the image of the period map.

These two ingredients tell us precisely for what holomorphic symplectic fourfolds we could expect obstructions to (very-)ampleness. We carefully analyzed these holomorphic symplectic fourfolds in chapters 2 and 3 and obtained enough information to prove results on (very-)ampleness and nefness.

Important for the Beauville–Donagi construction to provide the 'dictionary' we mentioned is that it is a locally complete projective construction; we recall that this means that any small projective deformation of the variety of lines on a cubic fourfold (which is a holomorphic symplectic manifold of the type mentioned above) is again the variety of lines on a cubic fourfold (see also theorem 1.3.1). As we mentioned already in the introduction of this thesis, there are three other such constructions known which yield holomorphic symplectic manifolds that are not K3 surfaces (in fact all of these yield holomorphic symplectic manifolds of $K3^{[2]}$-type).

In each of these constructions the holomorphic symplectic manifolds come equipped with a line bundle of specific numerical type. Perhaps the constructions can be exploited to obtain information on (very-)ampleness and nefness of line bundles of the numerical types in question, in ways analogous to those in this thesis. Of course we would be restricted to line bundles of very specific numerical type, but research in this direction may provide clues for more general results.

Appendix A

Blowing up subschemes: a short review

In this section we will shortly summarize the setup of the theory of blow-up of subschemes, mainly to set the notation. A comprehensive review of this subject can be found in [24]. All details we leave out can be found there.

A.1 Affine schemes

Let R be a commutative ring and $I \subset R$ an ideal. We introduce the following graded rings:

$$\begin{aligned} B(I,R) &:= \bigoplus_{n\geq 0} I^n t^n \\ G(I,R) &:= \bigoplus_{n\geq 0} I^n / I^{n+1}, \end{aligned}$$

where t is an auxiliary parameter and we use the convention that $I^0 = R$.

Definition A.1.1. The natural morphism

$$\varepsilon_I : \mathrm{Proj}\,(B(I,R)) \to \mathrm{Spec}\,(R)$$

is called the **blow-up** (or monoidal transform) of R with center I. If $Z_I \subset \mathrm{Spec}\,(R)$ is the subscheme defined by I then we also say that ε_I is the blow-up (or monoidal transform) of $\mathrm{Spec}\,(R)$ with center Z_I. We will denote the scheme $\mathrm{Proj}\,(B(I,R))$ by $\mathrm{Bl}(I,R)$.

Proposition / definition A.1.2. *$G(I,R)$ is equal to the quotient of $B(I,R)$ by the homogeneous ideal $\bigoplus_{n\geq 0} I^{n+1} t^n$, hence $\mathrm{Proj}\,(G(I,R))$ is a closed subscheme of $\mathrm{Proj}\,(B(I,R)) =: \mathrm{Bl}(I,R)$. Moreover $\mathrm{Proj}\,(G(I,R)) = \varepsilon_I^{-1}(Z_I)$ and is a Cartier divisor in $\mathrm{Bl}(I,R)$. The morphism ε_I restricted to the complement of $\mathrm{Proj}\,(G(I,R))$ is an isomorphism onto the complement of Z_I. We call $\mathrm{Proj}\,(G(I,R))$ the **exceptional divisor** of the blow-up ε_I and will denote it by $E(I,R)$.*

Let $J \subset R$ be another ideal in R. We define:

$$\begin{aligned} B(I, J \subseteq R) &:= \bigoplus_{n \geq 0} (I^n \cap J) t^n \\ G(I, J \subseteq R) &:= \bigoplus_{n \geq 0} (I^n \cap J + I^{n+1}) / I^{n+1}. \end{aligned}$$

Note that $B(I, J \subseteq R)$ is a homogeneous ideal in the graded ring $B(I, R)$ and that

$$B(I, R) / B(I, J \subseteq R) \cong B\big((I + J)/J, R/J\big).$$

So $\mathrm{Bl}\big((I+J)/J, R/J\big) = \mathrm{Proj}\,\big(B((I+J)/J, R/J)\big)$ lies in $\mathrm{Bl}(I, R)$ as a closed subscheme. In fact, $\mathrm{Bl}\big((I+J)/J, R/J\big)$ is the closure in $\mathrm{Bl}(I, R)$ of $\varepsilon_I^{-1}(Z_J \setminus Z_I)$.

Definition A.1.3. The subscheme $\mathrm{Bl}\big((I+J)/J, R/J\big) \hookrightarrow \mathrm{Bl}(I, R)$ defined by the ideal $B(I, J \subseteq R)$ is called the **strict transform** of Z_J. We denote it by $\widehat{Z}_J$.

Remark A.1.4. The restriction of ε_I to $\widehat{Z}_J$ is precisely the blow-up morphism $\varepsilon_{(I+J)/J} : \widehat{Z}_J \to Z_J$, that is, it is the blow-up of Z_J with center $Z_J \cap Z_I$. Moreover, the exceptional divisor $E\big((I+J)/J, R/J\big) \subset \widehat{Z}_J$ is precisely the intersection of $E(I, R)$ and $\widehat{Z}_J$. We thus have the following commutative diagram:

$$\begin{array}{ccccc} E((I+J)/J, R/J) & \hookrightarrow & \widehat{Z}_J & \xrightarrow{\varepsilon_{(I+J)/J}} & Z_J \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ E(I, R) & \hookrightarrow & \mathrm{Bl}(I, R) & \xrightarrow[\varepsilon_I]{} & \mathrm{Spec}\,(R) \end{array}$$

Since $\widehat{Z}_J \subset \mathrm{Bl}(I, R)$ is defined by the ideal $B(I, J \subseteq R) \subset B(I, R)$ and since $E(I, R) \subset \mathrm{Bl}(I, R)$ is defined by the ideal $\bigoplus_{n \geq 0} I^{n+1} t^n$, we find that $\widehat{Z}_J \cap E(I, R)$, as subscheme of $E(I, R)$, is defined by the ideal

$$\begin{aligned} \Big(B(I, J \subseteq R) + \bigoplus_{n \geq 0} I^{n+1} t^n\Big) \Big/ \bigoplus_{n \geq 0} I^{n+1} t^n &= \bigoplus_{n \geq 0} (I^n \cap J + I^{n+1}) / I^{n+1} \\ &= G(I, J \subseteq R) \end{aligned}$$

in the ring $G(I, R)$. In our application of the results above, in the end it comes down to giving a description of this ideal. Because of this we take some time to recall some properties, also following [24]. First, we define the degree of an element $r \in R$ with respect to I as

$$\deg_I r := \max\{n \geq 0 \mid r \in I^n\}.$$

Then we can define a map:

$$\begin{aligned} \mathrm{In}_I : R &\to G(I, R) \\ r &\mapsto \big[r + I^{1+\deg_I r}\big] \in I^{\deg_I r} / I^{1+\deg_I r}. \end{aligned}$$

The reader be warned that this is not a morphism of rings. However, it does respect the product structure. In any case, it is clear that $\mathrm{In}_I(J) = G(I, J \subseteq R)$. In principle this provides a way to calculate $G(I, J \subseteq R)$ from J. However, the fact that In_I is not a morphism of rings complicates matters. Indeed, the ideal J will usually be described by generators: $J = \langle g_1, ..., g_k \rangle$. However, although clearly $\langle \mathrm{In}_I g_1, ..., \mathrm{In}_I g_k \rangle \subseteq \mathrm{In}_I(J)$, there is no guarantee that equality holds.

Example A.1.5. This problem is easily illustrated by the following example. Take $R = \mathbf{C}[x, y]$ and $I = \langle x, y \rangle$. Then $G(I, R)$ can be identified with R, and under this identification In_I maps elements of R to their lowest degree part. Now choose $g_1 = x^2 + y^3, g_2 = xy$ and $J = \langle g_1, g_2 \rangle$. Then $yg_1 - xg_2 = y^4 \in J$, hence $\mathrm{In}_I(y^4) = y^4 \in \mathrm{In}_I(J)$. But the ideal $\langle \mathrm{In}_I(g_1), \mathrm{In}_I(g_2) \rangle = \langle x^2, xy \rangle$ in $G(I, R)$ clearly does not contain y^4, since all its elements are divisible by x. Hence in this case $\langle \mathrm{In}_I(g_1), \mathrm{In}_I(g_2) \rangle \neq \mathrm{In}_I(J)$.

Definition A.1.6. Let be given a ring R, ideals $I, J \subset R$ and a set $g = \{g_1, ..., g_n\} \subset R$ of generators of J. Then g is called an **(I,R)-standard base** of J if

$$\mathrm{In}_I(J) = \langle \mathrm{In}_I(g_1), ..., \mathrm{In}_I(g_n) \rangle.$$

To apply the results that we recollected sofar in practice, we will need criteria on a set of generators g of J to be a standard base. The previous example suggests that an obstruction for a set of generators to be a standard base is related to the fact that the initial components of the generators might have common factors. This may give some idea why the following result is true:

Theorem A.1.7. *Let R be a Noetherian ring, I, J ideals in R and $g_1, ..., g_n$ generators of J. Then the following are equivalent:*

1. $(g_1, ..., g_2)$ *is a regular sequence in R and an (I, R)-standard base of J;*
2. $\big(\mathrm{In}_I(g_1), ..., \mathrm{In}_I(g_n)\big)$ *is a regular sequence in $G(I, R)$.*

Proof. This statement is precisely Theorem 13.10 in [24] (page 96). □

Referring to example A.1.5, note that indeed $(\mathrm{In}_I(g_1) = x^2, \mathrm{In}_I(g_1) = xy)$ does not form a regular sequence in $\mathbf{C}[x, y]$, since the image of x^2 in $\mathbf{C}[x, y]/\langle xy \rangle$ is annihilated by the (non-zero) image of y in $\mathbf{C}[x, y]/\langle xy \rangle$. However, $(g_1 = x^2 + y^3, g_2 = xy)$ *does* form a regular sequence. So the set $\{g_1, g_2\}$ cannot be a standard base.

A.2 Blow-up of schemes

The results mentioned sofar apply to affine schemes. However, the construction naturally extends to (Noetherian) schemes in general. More precisely, let X be a Noetherian scheme and $\mathcal{F}$ a quasi-coherent sheaf of $\mathcal{O}_X$-modules with the structure of a sheaf of graded $\mathcal{O}_X$-algebras. Furthermore assume that $\mathcal{F}_0 = \mathcal{O}_X$, that $\mathcal{F}_1$ is coherent and in addition locally generates $\mathcal{F}$ as $\mathcal{O}_X$-algebra.

Then there exists a natural X-scheme $Y_{\mathcal{F}} \to X$ such that for every affine open subset $U \hookrightarrow X$ there exist a (natural) X-isomorphism $\varphi_U : Y_{\mathcal{F}} \times_X U \to \operatorname{Proj}(\mathcal{F}(U))$. See Hartshorne's book [20] for the details. The scheme $Y_{\mathcal{F}}$ is called the **relative proj** of $\mathcal{F}$ and denoted by $\mathbf{Proj}(\mathcal{F})$.

Now let X be a scheme and $Y, Z \subseteq X$ closed subschemes with vanishing ideal sheaves $\mathcal{I}, \mathcal{J} \subseteq \mathcal{O}_X$ respectively. We can define the following sheaves of graded $\mathcal{O}_X$-algebras:

$$\begin{aligned} \mathcal{B}(\mathcal{I}, \mathcal{O}_X) &:= \bigoplus_{n\geq 0} \mathcal{I}^n t^n; \\ \mathcal{G}(\mathcal{I}, \mathcal{O}_X) &:= \bigoplus_{n\geq 0} \mathcal{I}^n / \mathcal{I}^{n+1} \end{aligned}$$

and graded ideal sheaves:

$$\begin{aligned} \mathcal{B}(\mathcal{I}, \mathcal{J}) &:= \bigoplus_{n\geq 0} (\mathcal{I}^n \cap \mathcal{J}) t^n; \\ \mathcal{G}(\mathcal{I}, \mathcal{J}) &:= \bigoplus_{n\geq 0} (\mathcal{I}^n \cap \mathcal{J} + \mathcal{I}^{n+1}) / \mathcal{I}^{n+1} \end{aligned}$$

We have the following definitions and results, completely analogous to the affine case:

Definition A.2.1. The natural projection

$$\varepsilon_Y : \mathbf{Proj}\big(\mathcal{B}(\mathcal{I}, \mathcal{O}_X)\big) \to X$$

is called the **blow-up** of X with center Y. We will denote $\mathbf{Proj}\big(\mathcal{B}(\mathcal{I}, \mathcal{O}_X)\big)$ by $\operatorname{Bl}(Y, X)$.

Note that this definition is the globalization of definition A.1.1.

Proposition / definition A.2.2. $\mathbf{Proj}(\mathcal{G}(\mathcal{I}, \mathcal{O}_X))$ *can be naturally identified with a closed subscheme of* $\operatorname{Bl}(Y, X)$. *Moreover* $\mathbf{Proj}(\mathcal{G}(\mathcal{I}, \mathcal{O}_X)) = \varepsilon_Y^{-1}(Y)$ *and is a Cartier divisor in* $\operatorname{Bl}(Y, X)$. *The morphism* ε_Y *maps the complement of* $\mathbf{Proj}(\mathcal{G}(\mathcal{I}, \mathcal{O}_X))$ *isomorphically onto the complement of* Y *in* X. *We call* $\mathbf{Proj}(\mathcal{G}(\mathcal{I}, \mathcal{O}_X))$ *the* ***exceptional divisor*** *of the blow-up* ε_Y *and will denote it by* $E(Y, X)$.

Proposition / definition A.2.3. $\mathcal{B}(\mathcal{I}, \mathcal{J})$ *defines an ideal sheaf on* $\operatorname{Bl}(Y, X)$ *and as such a closed subscheme* $\widehat{Z}$ *of* $\operatorname{Bl}(Y, X)$. *This subscheme is called the* ***strict transform*** *of* Z *in* $\operatorname{Bl}(Y, X)$. *It is equal to the closure of* $\varepsilon_Y^{-1}(Z \setminus Y)$ *in* $\operatorname{Bl}(Y, X)$.

Proposition A.2.4. $\mathcal{G}(\mathcal{I}, \mathcal{J})$ *defines an ideal sheaf on* $E(\mathcal{I}, \mathcal{J})$. *This is the ideal sheaf of the closed subscheme* $\widehat{Z} \cap E(\mathcal{I}, \mathcal{J})$ *in* $E(\mathcal{I}, \mathcal{J})$.

All results can be proven by localizing and applying the results from for blow-up of affine schemes.

Remark A.2.5. Similar to **Proj** there is a the notation of relative Spec for a sheaf $\mathcal{F}$ on X, which we denote by $\mathbf{Spec}\mathcal{F}$. In particular $\mathbf{Spec}\mathcal{F}$ is the relative affine cone over $\mathbf{Proj}\mathcal{F}$.
Hence, in the same notation as above, the Y-scheme $\mathbf{Spec}\mathcal{G}(\mathcal{I}, \mathcal{O}_X) \to Y$ is the relative affine cone over the exceptional divisor $E(Y, X)$. This affine cone is also called the **normal cone** of Y in X, denoted by $N_{Y/X}$. In the case that both X and Y are smooth, one can show that there is a natural identification $N_{Y/X} = TX|_Y / TY$, that is, the normal cone is the total space of the normal bundle of Y in X. In particular it follows that $E(Y, X) = \mathbf{P}(N_{X/Y})$ has the structure of a bundle of projective spaces over Y.

List of notation

Note that there is a separate list of notations for chapter 3, see section 3.8.

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Samenvatting

Gedurende de vier jaar dat ik als promovendus werkzaam ben geweest hebben vele mensen mij gevraagd waar mijn onderzoek nu eigenlijk over gaat. Een lastige vraag; de motivatie, gebruikte methoden en toepassingen van mijn onderzoek liggen allemaal binnen de theoretische wiskunde. In deze samenvatting wil ik een beeld schetsen van die theoretisch-wiskundige context van mijn onderzoek, en de plaats die het onderzoek zelf daarin inneemt.

Vooral de niet-wiskundigen onder u zullen tijdens het lezen waarschijnlijk geplaagd worden door vragen als 'waarom wil je je hier mee bezighouden?'. Ik kan lange en technische antwoorden proberen te geven, maar eigenlijk is het heel simpel: pure nieuwsgierigheid! Want alle mogelijke toepassingen ten spijt, voor veel wetenschappelijk onderzoekers is nieuwsgierigheid toch verreweg de belangrijkste motivatie.

Meetkunde met polynomen

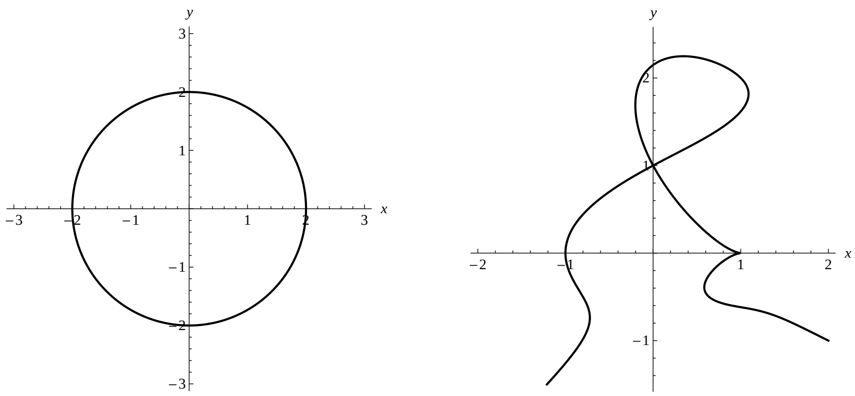
De meeste mensen die ik ken zijn op de middelbare school in contact gekomen met 'algebra' (formules en vergelijkingen) en 'meetkunde' (lijnen, cirkels, kubussen, etcetera). Het vakgebied waarin mijn onderzoek plaatsvindt heet 'algebraïsche meetkunde', wat een combinatie is van beide. De formules die daarin worden gebruikt zijn *polynomen*; dit zijn formules die opgebouwd kunnen worden met alleen optellen, aftrekken en vermenigvuldiging, bijvoorbeeld $x^3 - 4x^2 + x + 6$. Bedenk hierbij dat machtsverheffen herhaald vermenigvuldigen is (bijvoorbeeld $x^3 = x \cdot x \cdot x$), dus deze formule is inderdaad opgebouwd met alleen vermenigvuldiging, optellen en aftrekken. De letter x heet de *variabele* en markeert de plekken waar we waarden kunnen invullen. Een typisch algebra-vraagstuk is:

Voor welke waarden van x geldt $x^3 - 4x^2 + x + 6 = 0$?

De lezer kan nagaan dat dit geldt voor $x = -1, x = 2$ en $x = 3$. Deze waarden heten de *nulpunten* van de polynoom $x^3 - 4x^2 + x + 6$.

De delen van de formule die gescheiden worden door + en/of − tekens heten *termen*, deze bestaan alleen uit vermenigvuldigingen. Iedere term heeft een *graad*; dit is het aantal keer dat de variabele er in voorkomt, ermee rekening houdend dat machtsverheffing herhaald vermenigvuldigen is. Zo komt in de term $4x^2 = 4x \cdot x$ de variabele twee keer voor; de graad van deze term is dus 2. De graad van de laatste term is 0; de variabele komt er namelijk niet in voor. De *graad van een polynoom* is het maximum van de graden van de afzonderlijke termen. Ik kom hier later nog op terug.

Algebraïsche meetkunde is in feite de studie van de verzameling nulpunten van polynomen. In het bovenbeschreven voorbeeld bestaat de verzameling nulpunten uit drie waarden. Dit is weer te geven door drie punten op een lijn; niet zo interessant vanuit meetkundig oogpunt. Het wordt interessanter als we meerdere variabelen toelaten in de polynomen. Een voorbeeld van een polynoom met twee variabelen is $x^2 + y^2 - 4$. Een nulpunt van deze polynoom is een combinatie van waarden voor x én y waarvoor $x^2 + y^2 - 4$ gelijk aan 0 wordt.



(a) $x^2 + y^2 - 4$

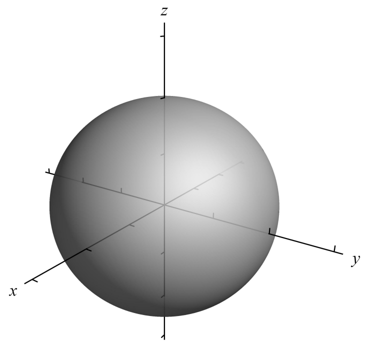
(b) $1 - 2x + 2x^3 - x^4 - 2y^2 + 2xy^2 - 3x^2y^2 + x^3y^2 - y^3 + 3y^4 - y^5$

Figuur 1: Nulpuntsverzamelingen

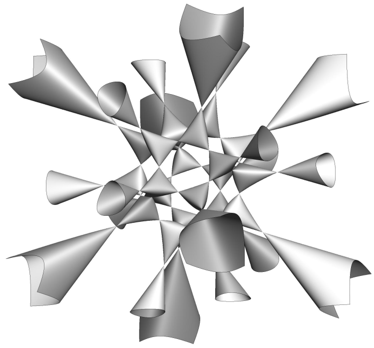
Bijvoorbeeld, als je voor x de waarde 2 invult en voor y de waarde 0, dan is $x^2 + y^2 - 4$ gelijk aan 0; de combinatie $(x, y) = (2, 0)$ is dus een nulpunt van $x^2 + y^2 - 4$. Andere voorbeelden van zulke waarden zijn $(x, y) = (0, 2)$, $(x, y) = (\frac{6}{5}, \frac{8}{5})$ en $(x, y) = (\frac{10}{13}, \frac{24}{13})$. Er blijken oneindig veel nulpunten te zijn voor deze polynoom te zijn. Als we de x- en y-waarden van de nulpunten opvatten als coördinaten kunnen we de verzameling nulpunten grafisch weergeven in een assenstelsel, zie figuur 1a. Het blijkt een bekende meetkundige figuur te zijn: een cirkel. Dit is een vrij eenvoudig voorbeeld, er zijn polynomen met veel gecompliceerdere nulpuntsverzamelingen. Zie bijvoorbeeld figuur 1b voor een relatief grillige nulpuntsverzameling van een nog vrij eenvoudige polynoom. Overigens is voor dit laatste voorbeeld de nulpuntsverzameling niet begrensd, we kunnen dus maar een deel ervan laten zien.

Het wordt meetkundig gezien nog interessanter als we meer variabelen toelaten. Bijvoorbeeld, de verzameling nulpunten van de polynoom $x^2 + y^2 + z^2 - 4$

bestaat uit die waarden voor x, y én z waarvoor $x^2 + y^2 + z^2 - 4 = 0$. Als we deze x, y en z-waarden opvatten als coördinaten, dan kunnen we een plaatje maken van de nulpuntsverzameling in een driedimensionaal assenstelsel, zie figuur 2a. In dit specifieke geval is de verzameling nulpunten het oppervlak van een bol met straal 2.



(a) $x^2 + y^2 + z^2 - 4$



(b) Barth oppervlak

Figuur 2: Nulpuntsverzamelingen

Wederom kan de vorm van de nulpuntsverzameling zeer ingewikkeld zijn, afhankelijk van de keuze van de polynoom. Zie bijvoorbeeld figuur 2b, het zogeheten Barth oppervlak. Dit is de nulpuntsverzameling van een specifieke polynoom van graad 6 (de polynoom is verder niet belangrijk voor mijn verhaal). Overigens is ook deze nulpuntsverzameling onbegrensd, en kan dus maar voor een deel worden weergegeven. In het algemeen kan de nulpuntsverzameling van een polynoom in drie variabelen weergegeven worden als een gekromd oppervlak in de ruimte.

Wiskundig gezien is er geen reden te stoppen bij drie variabelen; ook van een polynoom in bijvoorbeeld vijf variabelen x, y, z, u en w, zoals $3x^6 + 5x^2y^3 + 7z^2u^2 - w^3$, kunnen we ons afvragen voor welke waarden van de variabelen deze polynoom gelijk is aan nul. De verzameling van dergelijke waarden is dan de nulpuntsverzameling voor deze polynoom. Als we een vijfdimensionaal assenstelsel konden tekenen, zouden we er een plaatje van kunnen maken op dezelfde manier als in voorgaande voorbeelden. Helaas is er geen inzichtelijke manier om dat te doen. Echter, de theorie van algebraïsche meetkunde biedt gereedschappen om toch meetkundige eigenschappen van zulke nulpuntsverzamelingen te onderzoeken. Het is daarbij de kunst om algebraïsche eigenschappen van de polynoom te vertalen naar meetkundige eigenschappen van zijn nulpuntsverzameling en omgekeerd. Dit is ruwweg waar algebraïsche meetkunde zich mee bezighoudt.

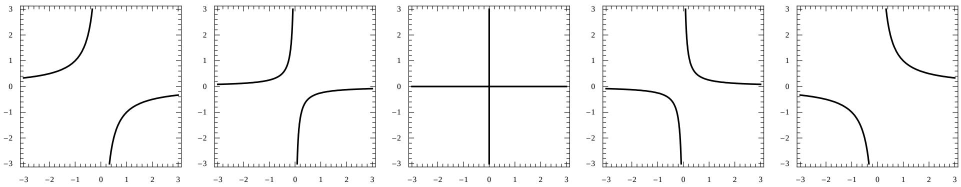
Meer terminologie

Voor we ons verhaal vervolgen hebben we wat meer begrippen nodig om eigenschappen van nulpuntsverzamelingen te beschrijven. Vergelijk de figuren 1a en 1b. Een belangrijk verschil tussen beide figuren is het volgende. Als we een punt op figuur 1a kiezen en daarop inzoomen, dan gaat de figuur in de buurt van dat punt steeds meer op een rechte lijn lijken. Voor figuur 1b is dit niet overal het geval. Deze figuur heeft een scherpe spits en een punt waar zij door zichzelf heensnijdt. Hoe ver we ook inzoomen, de figuur zal in de buurt van zulke punten nooit gaan lijken op een rechte lijn. Zulke punten worden *singuliere punten* van de nulpuntsverzameling genoemd. Een nulpuntsverzameling met singuliere punten wordt, heel toepasselijk, *singulier* genoemd. Figuur 1b is dus een voorbeeld van een singuliere nulpuntsverzameling. Een nulpuntsverzameling zonder singuliere punten heet *glad*. Ook een toepasselijke naam, als we kijken naar figuur 1a.

De begrippen 'glad' en 'singulier' zijn niet beperkt tot nulpuntsverzamelingen van polynomen in twee variabelen. Bijvoorbeeld, voor nulpuntsverzamelingen van polynomen in drie variabelen moeten we in bovenstaande beschrijving van de begrippen 'glad' en 'singulier' de woorden 'rechte lijn' vervangen door 'plat vlak'. Zo is de nulpuntsverzameling in figuur 2a glad; als we een punt kiezen en daarop gaan inzoomen, gaat het in de buurt van dat punt steeds meer op een plat vlak lijken. Dit is niet het geval voor de nulpuntsverzameling in figuur 2b; deze figuur bevat meerdere punten waar twee scherpe hoeken elkaar raken. Op zulke punten kunnen we blijven inzoomen zonder dat de figuur in de buurt van dat punt op een plat vlak gaat lijken. Figuur 2b is dus een voorbeeld van een singuliere nulpuntsverzameling.

Voor nulpuntsverzamelingen van polynomen met meer dan drie variabelen moet 'rechte lijn' en 'plat vlak' vervangen worden door hoger dimensionale versies van rechte lijnen en platte vlakken. Helaas kunnen we dit niet meer met plaatjes inzichtelijk maken. Wel merken we nog het volgende op: er kunnen criteria gegeven worden voor het al dan niet glad zijn van de nulpuntsverzameling van een polynoom, ongeacht het aantal variabelen, puur in termen van de polynoom zelf. Dit maakt het mogelijk om nulpuntsverzamelingen te onderzoeken op het al dan niet glad zijn, zonder dat het nodig is die verzameling grafisch weer te geven.

Een ander belangrijk begrip is dat van *vervormingen* van een nulpuntsverzameling. We beginnen met een voorbeeld: bekijk de polynoom $xy - t$. Dit is een polynoom in drie variabelen x, y en t. We kunnen echter een waarde voor t kiezen, bijvoorbeeld $t = 1$. We houden dan een polynoom met twee variabelen over, namelijk $xy - 1$. Voor iedere andere mogelijke waarde van t krijgen we een ander polynoom met twee variabelen en dus ook een andere nulpuntsverzameling, zie figuur 3. Dit wordt een *familie* van nulpuntsverzamelingen genoemd, en t heet de *parameter* van de familie; zijn waarde onderscheidt de leden van de familie. De familie beschrijft *vervormingen* (engels: *deformations*) van nulpuntsverzamelingen. Voor alle waarden van t ongelijk aan nul lijken de bijbehorende nulpuntsverzamelingen in vorm veel op elkaar. In ieder geval zijn ze allemaal glad. Voor $t = 0$ gebeurt er iets bijzonders: de bijbehorende nulpuntsverzameling heeft een singulier punt. We zeggen dat de familie *ontaardt* (engels: *degenerates*) voor t gelijk aan 0, de singuliere nulpuntsverzameling zelf heet een *ontaarding* van de familie.



Figuur 3: Nulpuntsverzameling van $xy - t$ voor $t = -1, -\frac{1}{4}, 0, \frac{1}{4}$ en 1.

Het gedrag van een familie van nulpuntsverzamelingen in de buurt van een ontaarding geeft veel informatie over de singuliere punten van de ontaarding zelf. Sterker nog, in de algebraïsche meetkunde vindt onderzoek naar singuliere punten van de nulpuntsverzameling van een polynoom vaak plaats door naar families te kijken waarin die nulpuntsverzameling als ontaarding optreedt. Dit principe speelt ook in dit proefschrift een belangrijke rol, we komen hier verderop nog op terug.

Van algebra naar meetkunde: een voorbeeld

Eerder heb ik beschreven dat algebraïsche meetkunde ruwweg gaat over het verband tussen algebraïsche eigenschappen van polynomen en meetkundige eigenschappen van hun nulpuntsverzamelingen. Ik zal een voorbeeld geven van een dergelijk verband. Ik heb het begrip *graad* van termen van een polynoom met één variabele al geïntroduceerd. De graad van een term van een polynoom met meer variabelen is als volgt gedefinieerd. Iedere variabele komt een aantal keer voor in de term (rekening houdend met machtsverheffingen), de graad van de term is de som van die aantallen. Neem bijvoorbeeld de polynoom $x^2y^7 + x^3y^3z - z^4$. Deze heeft drie termen, de eerste met graad $2+7 = 9$, de tweede met graad $3+3+1 = 7$ en de derde met graad 4. De graad van een polynoom is wederom het maximum van de graden van de termen van de polynoom, in dit geval 9.
Het interessante is nu dat de graad van een polynoom een meetkundige betekenis heeft:

Stelling A. *Kies een polynoom van graad d. Een willekeurige rechte lijn heeft ten hoogste d punten gemeenschappelijk met de nulpuntsverzameling van die polynoom, tenzij de lijn in zijn geheel in de nulpuntsverzameling ligt.*

De punten die een lijn en een nulpuntsverzameling gemeenschappelijk hebben worden *snijpunten* van de lijn met de nulpuntsverzameling genoemd. Het is goed om deze stelling met wat voorbeelden te illustreren. Eerder beschreef ik het feit dat de nulpuntsverzameling van de polynoom $x^2 + y^2 - 4$ een cirkel is. De polynoom heeft 3 termen, respectievelijk van graad 2, 2 en 0; de graad van de polynoom is dus 2. Inderdaad heeft iedere rechte lijn in het vlak hooguit 2 snijpunten met de cirkel. Een ander voorbeeld is de bol in figuur 2a, die de nulpuntsverzameling is van de polynoom $x^2 + y^2 + z^2 - 4$. Ook deze polynoom heeft graad 2, en inderdaad heeft iedere rechte lijn hooguit twee punten gemeenschappelijk met de bol. Verder is figuur 1b de nulpuntsverzameling van een polynoom van graad 5. Het is inderdaad mogelijk in het plaatje rechte lijnen te tekenen die precies 5 snijpunten met de figuur hebben. Ik nodig de lezer uit dit te proberen.

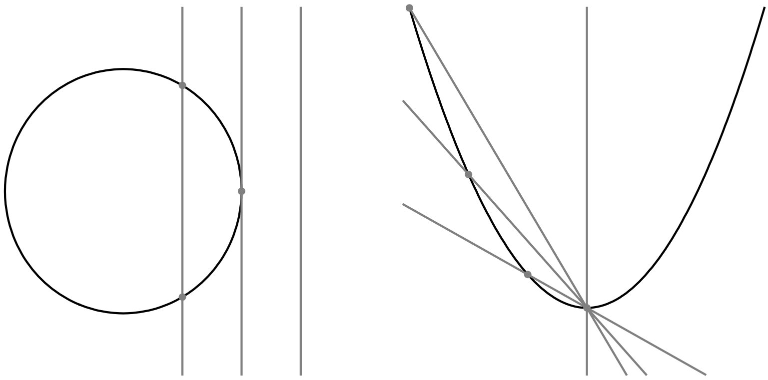
Dit is de plaats om wat uit te weiden over de woorden ‘ten hoogste’ in stelling A. Het aantal snijpunten van een rechte lijn met de nulpuntsverzameling van een polynoom kan in het algemeen alles zijn van nul tot en met de graad van de polynoom, dit is ook te zien in bovengenoemde voorbeelden. Wanneer het aantal snijpunten kleiner is dan de graad van de polynoom kan dit verschillende oorzaken hebben:

- Er kunnen snijpunten zijn samengevallen. Kijk bijvoorbeeld naar figuur 4a. De linker lijn heeft twee snijpunten met de figuur. Als we hem naar de middelste lijn opschuiven bewegen de snijpunten naar elkaar toe en vallen uiteindelijk samen. We houden dan weliswaar maar één snijpunt over, maar als we de lijn een klein beetje bewegen kunnen we er weer twee krijgen. Wiskundigen zeggen dan dat dit een snijpunt is met *multipliciteit 2*.

- Er kunnen snijpunten ‘naar oneindig’ zijn verdwenen. Met figuur 4b laat ik zien wat dit betekent. Deze figuur is de nulpuntsverzameling van een polynoom van graad 2. De onderste lijn heeft inderdaad 2 snijpunten met de figuur. Echter, als we de lijn ‘rechtop’ draaien loopt een van de twee snijpunten steeds verder weg. Op het moment dat de lijn rechtop staat is het snijpunt verdwenen, hoe ver we ook uitzoomen. We zeggen dat het snijpunt ‘op oneindig’ ligt.

- Snijpunten kunnen ook echt verdwijnen. Kijk weer naar figuur 4a. Als we de linker lijn naar de rechter lijn schuiven, verdwijnen de snijpunten.

De eerste twee situaties in bovenstaande opsomming zijn *randgevallen*; dat wil zeggen, als een van deze situaties optreedt voor een gegeven rechte lijn, dan kunnen we die lijn altijd een heel klein beetje verschuiven en/of draaien zodat het aantal snijpunten weer gelijk is aan de graad van de polynoom. In de laatstgenoemde situatie is dit niet altijd het geval.



(a) cirkel en snijlijnen

(b) parabool en snijlijnen

Figuur 4: Nulpuntsverzamelingen

Nu komt een belangrijk punt: in de algebraïsche meetkunde zijn algemene methoden om nulpuntsverzamelingen van polynomen zodanig uit te breiden dat de laatste twee genoemde situaties niet kunnen voorkomen. De tweede situatie wordt opgelost door 'punten op oneindig' aan de nulpuntsverzameling toe te voegen. Dit klinkt misschien flauw, maar er is een uiterst elegante manier om dit te doen door middel van zogenoemde *projectieve meetkunde*. Helaas is er hier geen ruimte om daar verder op in te gaan.

De derde situatie wordt opgelost door voor de variabelen van de polynomen meer waarden toe te staan dan we tot nu toe deden. Tot nu heb ik met het woord 'waarde' altijd getallen bedoeld die een hoeveelheid kunnen uitdrukken. Wiskundigen noemen deze getallen *reëel*. Een groot nadeel is dat we geen wortel kunnen trekken uit negatieve reële getallen; een feit dat u mogelijk wel bekend is. Dit heeft tot gevolg dat er polynomen bestaan *zonder nulpunten*, bijvoorbeeld $x^2 + 1$. Wiskundigen laten zich echter niet eenvoudig tegenhouden; het blijkt dat het systeem van reële getallen kan worden uitgebreid tot een getallensysteem met de eigenschap dat *elke polynoom* een nulpunt heeft in dit getallensysteem. We noemen de getallen in deze uitbreiding *complexe getallen*. Met deze getallen kan gerekend worden op dezelfde manier als met reële getallen.

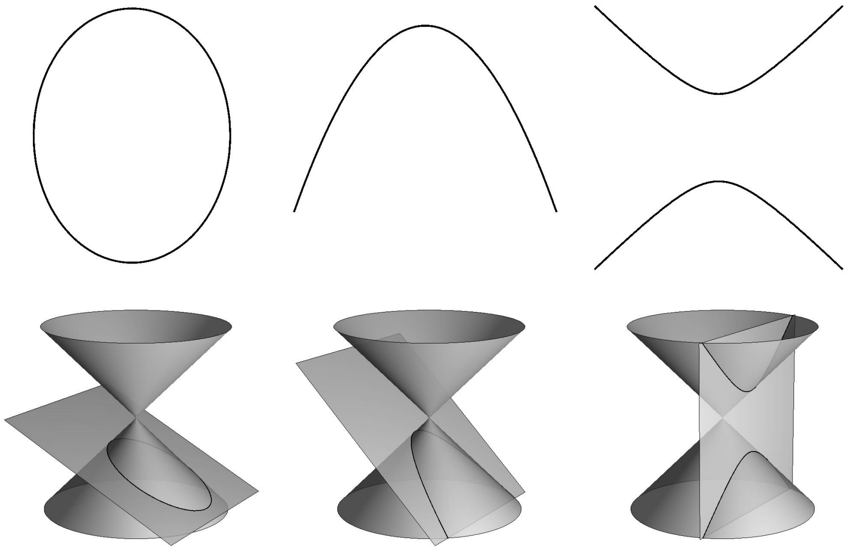
Hoewel de voorgaande korte uitleg waarschijnlijk anders doet vermoeden, is het bestuderen van deze projectief en complex uitgebreide nulpuntsverzamelingen eenvoudiger dan het bestuderen van de oorspronkelijke nulpuntsverzameling. De reden is dat er minder uitzonderingssituaties zijn. Bijvoorbeeld, stelling A wordt eenvoudiger:

Stelling A'. *Kies een polynoom van graad d. Iedere rechte lijn heeft d snijpunten met de projectief en complex uitgebreide nulpuntsverzameling van deze polynoom, samenvallende snijpunten dubbel geteld, tenzij de lijn in zijn geheel in die uitgebreide nulpuntsverzameling ligt.*

Nulpuntsverzamelingen van lage graad

De graad van een polynoom is een maat voor de complexiteit van zijn verzameling nulpunten. Bijvoorbeeld, als de graad van een polynoom laag is, kan een willekeurige rechte lijn maar weinig punten gemeenschappelijk hebben met de nulpuntsverzameling (tenzij die lijn in de nulpuntsverzameling ligt). Dat betekent dat de nulpuntsverzameling niet al te 'wild' kan zijn. Bij polynomen van hoge graad kan de nulpuntsverzameling echter bijzonder wild worden (zie bijvoorbeeld figuur 2b, hoewel dit de nulpuntsverzameling van een polynoom van slechts graad 6 is).

Het ligt dus voor de hand om bij een systematisch onderzoek van meetkundige eigenschappen van nulpuntsverzamelingen van polynomen te werken van lage graad naar hoge graad. Graad 1 is meetkundig gezien niet zo spannend: de nulpuntsverzameling van een polynoom van graad 1 in één variabele is altijd een enkel punt, bij twee variabelen is het een rechte lijn, bij drie variabelen is het een plat vlak. Bij meer variabelen zijn de nulpuntsverzamelingen hogerdimensionale versies van lijnen en vlakken (zogenoemde *affiene ruimtes*).



Figuur 5: Nulpuntsverzamelingen van polynomen van graad 2 in twee variabelen en hun kegelsneden.

Het geval van polynomen van graad 2 is al interessanter. De nulpuntsverzamelingen kunnen nu kwalitatief verschillende vormen hebben. In figuur 5 zijn drie nulpuntsverzamelingen weergegeven van polynomen van graad 2 met twee variabelen. Deze nulpuntsverzamelingen hebben een speciale eigenschap: ze kunnen allemaal verkregen worden als doorsnede van een (dubbele) kegel en een vlak. In figuur 5 zijn ook deze doorsnedes weergegeven. Alle nulpuntsverzamelingen van polynomen van graad 2 met twee variabelen kunnen op deze manier verkregen kunnen worden. Dit feit geeft een handvat om dergelijke figuren structureel te onderzoeken; de interessante meetkunde komt van één en hetzelfde object, namelijk de (dubbele) kegel. In zekere zin kunnen we al deze nulpuntsverzamelingen meetkundig gezien op een gelijke manier beschouwen. Dit zet zich voort voor polynomen van meer variabelen: ook de nulpuntsverzamelingen van polynomen van graad 2 met meer dan twee variabelen kunnen verkregen worden als doorsnede van het hoger dimensionaal analogon van een kegel met het hoger dimensionaal analogon van een vlak (een affiene ruimte). Dit maakt dat nulpuntsverzamelingen van polynomen van graad 2 vrij eenvoudig te bestuderen zijn.

De situatie verandert drastisch als we van graad 2 naar 3 gaan. Nulpuntsverzamelingen van polynomen van graad 2 zijn meetkundig gezien 'vergelijkbaar' in de zin dat ze van één en hetzelfde object afkomen, zoals ik eerder beschreef. Voor de nulpuntsverzameling van polynomen van graad drie en hoger (met twee of meer variabelen) is dit niet meer het geval; zulke nulpuntsverzamelingen meetkundig gezien echt kunnen variëren.

Nulpuntsverzamelingen van polynomen van graad 3 en hoger zijn rijk aan interessante meetkundige eigenschappen. Het nadeel is dat hoe hoger de graad van een polynoom is, hoe lastiger deze te analyseren is. Dit bemoeilijkt de studie van de bijbehorende nulpuntsverzameling.

Polynomen van graad 3 zijn echter nog relatief 'handelbaar' in vergelijking met polynomen van hogere graad. Toch zijn hun nulpuntsverzamelingen rijk aan meetkundige eigenschappen. Zulke polynomen zijn daarom uitgebreid bestudeerd binnen de wiskunde, met grote resultaten tot gevolg. Bijvoorbeeld, de studie van polynomen van graad 3 met twee variabelen en hun nulpuntsverzamelingen, begonnen in de 19^e eeuw, heeft de basis gevormd voor het bewijs van de beroemde stelling van Fermat[1].

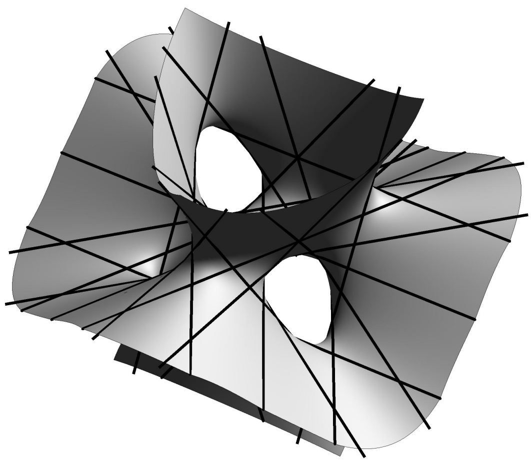
Ook polynomen van graad 3 met *drie* variabelen werden in de 19^e eeuw uitgebreid bestudeerd. De nulpuntsverzamelingen van dergelijke polynomen worden *kubische oppervlakken* genoemd. Het ging er vooral om de verschillende vormen van kubische oppervlakken te classificeren. Door de grote verscheidenheid van deze vormen was dit een moeilijk probleem.
Een belangrijke stap in dit onderzoek werd gezet in twee artikelen uit 1849 van Cayley en Salmon. Zoals gezegd worden kubische oppervlakken bepaald door een polynoom van graad 3. Stelling A vertelt ons dus dat een willekeurige rechte lijn hooguit 3 punten gemeenschappelijk heeft met het kubisch oppervlak, tenzij die lijn helemaal in dit oppervlak ligt. Caley en Salmon vroegen zich af of dit

[1] Dit is de uitspraak dat voor ieder geheel getal n groter dan 2 er geen oplossing van de vergelijking $x^n + y^n = z^n$ bestaat met x, y en z geheeltallig en ongelijk aan 0. In 1637 claimde de wiskundige Fermat een bewijs te hebben, maar hij heeft dit nooit gepubliceerd. De uitspraak werd pas in 1994 bewezen door de Britse wiskundige Andrew Wiles.

laatste kan voorkomen. Ze vonden dat er inderdaad rechte lijnen in hun geheel in een kubisch oppervlak kunnen liggen, maar dat dat alleen gebeurt voor rechte lijnen in zeer specifieke posities. Om precies te zijn bewezen ze het volgende:

> *Ieder glad kubisch oppervlak bevat een eindig aantal rechte lijnen, ten hoogste 27.*

Als we de nulpuntsverzameling van eerdergenoemde kubische polynomen projectief en complex uitbreiden is de uitspraak zelfs sterker: op zo'n uitgebreid kubisch oppervlak liggen *altijd* 27 lijnen! Maar ook op 'niet-uitgebreide' kubische oppervlakken kan het maximum van 27 rechte lijnen echt voorkomen, dat wil zeggen, er bestaan kubische oppervlakken met precies 27 rechte lijnen erop. Zie figuur 6 voor een voorbeeld.



Figuur 6: Kubisch oppervlak met 27 rechte lijnen erop

Het gaat niet om 'zomaar' 27 lijnen; ze vormen een zeer specifieke configuratie. Het plaatje op de omslag van dit proefschrift geeft een indruk van deze configuratie voor het kubisch oppervlak in figuur 6. Het belang van het resultaat van Cayley en Salmon is dat de configuratie van de rechte lijnen op een kubisch oppervlak veel, in veel gevallen volledige, informatie geeft over de vorm van het kubisch oppervlak. Op deze manier kunnen meetkundige eigenschappen van kubische oppervlakken bestudeerd worden aan de hand van configuraties van de lijnen erop. Dit lijkt misschien weinig winst, omdat de lijnenconfiguraties erg ingewikkeld kunnen zijn. Toch biedt het een zeer concrete manier om kubische oppervlakken te onderzoeken, en dit inzicht is zeer vruchtbaar gebleken. In het bijzonder heeft het geleid tot een classificatie van de mogelijke vormen die kubische oppervlakken kunnen hebben.

Waar gaat dit proefschrift over?

Het idee dat het interessant kan zijn om te kijken naar rechte lijnen die in hun geheel in de nulpuntsverzameling van een polynoom van graad 3 liggen was een belangrijke motivatie voor mijn onderzoek. Ik heb gekeken naar polynomen van graad 3 met *vijf* variabelen en hun nulpuntsverzamelingen. Om precies te zijn heb ik me beziggehouden met de projectieve en complexe uitbreiding van zulke nulpuntsverzamelingen, zoals ik die eerder noemde. Deze objecten worden *kubische viervouden* genoemd.

De verzameling van rechte lijnen op een kubisch viervoud is niet eindig, maar is zelf een meetkundig object van dimensie 4. Dit object wordt ook wel de *lijnenvariëteit* van het kubische viervoud genoemd. In de jaren 80 bewezen de wiskundigen Arnaud Beauville en Ron Donagi dat lijnenvariëteiten van kubische viervouden een zeer bijzondere meetkundige structuur hebben: een zogenaamde *holomorfe symplectische* structuur. Objecten met een dergelijke structuur worden *holomorfe symplectische variëteiten* genoemd.
Het zou me teveel tijd kosten om zelfs maar ruwweg uit te leggen wat deze termen betekenen. Het belangrijkste is dat objecten met een holomorfe symplectische structuur erg zeldzaam lijken te zijn: er zijn tot op heden maar relatief weinig voorbeelden van objecten met zo'n structuur gevonden en het is nog onduidelijk of er meer voorbeelden zouden moeten bestaan.

In dit opzicht is het resultaat van Beauville en Donagi dus erg bijzonder: het geeft een manier om specifieke holomorfe symplectische variëteiten te maken. Maar het belang van het resultaat gaat verder. Namelijk, ondanks het feit dat er maar weinig expliciete voorbeelden bekend zijn van irreducibele holomorfe symplectische variëteiten, is er juist veel bekend over hun meetkundige eigenschappen. Via de constructie van Beauville en Donagi vertellen deze eigenschappen veel over kubische viervouden zelf. Dit geeft een manier om kubische viervouden te onderzoeken. Een van de belangrijkste voorbeelden hiervan kan teruggevonden worden in het proefschrift van de Franse wiskundige Claire Voisin: zij heeft de constructie van Beauville en Donagi gebruikt om de zogeheten Torelli stelling voor kubische viervouden te bewijzen. Wederom leg ik hier niet uit wat dit precies betekent, de geïnteresseerde lezer verwijs ik naar de inleiding en hoofdstuk 1.

In dit proefschrift gebruik ik de resultaten van Beauville en Donagi in omgekeerde richting: ik gebruik meetkundige eigenschappen van kubische viervouden (onder andere de resultaten van Voisin) om informatie te krijgen over hun lijnenvariëteiten. Specifieker: ik kijk naar vervormingen (zie eerder) van deze objecten. In het eerdergenoemde artikel van Beauville en Donagi wordt bewezen dat als je de lijnenvariëteit van een kubisch viervoud een klein beetje vervormt, het resultaat ook weer een lijnenvariëteit is van een ander kubisch viervoud. Maar dit geldt alleen voor *kleine* vervormingen; het blijkt mogelijk te zijn een lijnenvariëteit zover te vervormen dat het resultaat niet langer een lijnenvariëteit van een kubisch viervoud is. Laten we deze vervormingen voor nu *speciale vervormingen* noemen.

In dit proefschrift onderzoek ik precies deze speciale vervormingen. Ik probeer te begrijpen welke eigenschappen van deze speciale vervormingen maken dat ze niet kunnen voorkomen als lijnenvariëteit van een kubisch viervoud. Het blijkt dat deze vervormingen gerelateerd zijn aan ontaardingen van kubische viervouden, dus kubische viervouden met singulariteiten. Ik onderzoek dit verband en laat zien hoe sommige speciale vervormingen te reconstrueren zijn uit die ontaardingen van kubische viervouden. Tot slot bekijk ik wat deze resultaten ons vertellen over de meetkunde van de lijnenvariëteiten zelf: het blijkt mogelijk te zijn om precies aan te geven welke vervormingen van lijnenvariëteiten gegeven kunnen worden als nulpuntsverzameling van polynomen en welke niet.

Dankwoord / Acknowledgement

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gedeeld, met name (in willekeurige volgorde) Alex Boer, Alex Quintero–Velez, Rogier, Sander, Tammo–Jan, Arthur, Vincent, Bas Janssens, Charlene, Jeroen, Aleksandra, Albert Jan, Jan Willem, Esther, Sebastian, Dana, Arjen, Bas Fagginger Auer en Sebastiaan. Ik heb het met en door jullie erg naar mijn zin gehad. Ik hoop jullie in de toekomst nog eens tegen te komen.

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Curriculum Vitae

Bart van den Dries werd op 19 juli 1983 geboren te Zoetermeer. In 2001 behaalde hij summa cum laude zijn diploma aan het Gymnasium Camphusianum te Gorinchem. Aanvankelijk was zijn plan sterrenkunde te gaan studeren. Na deelname aan de Natuurkunde Olympiade veranderde dit; hij besloot het TWIN-programma natuurkunde–wiskunde te gaan volgen aan de Universiteit Utrecht. In 2007 rondde hij beide studies cum laude af met zijn scriptie *Supergravity on Complex–Symplectic Manifolds*, geschreven onder begeleiding van prof. Bernard de Wit en prof. Eduard Looijenga.

Op basis van de ervaringen tijdens zijn afstuderen koos Bart verder te gaan in de wiskunde. Aansluitend aan zijn afstuderen begon hij met een promotieonderzoek bij prof. Eduard Looijenga op het gebied van holomorfe symplectische meetkunde. De resultaten van dit onderzoek zijn beschreven in dit proefschrift.

Sinds januari 2012 is Bart werkzaam aan de Hogeschool van Amsterdam als docent bij de lerarenopleiding wiskunde.