FROM GREEDY TO LAZY EXPANSIONS AND THEIR DRIVING DYNAMICS

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Abstract. In this paper we study the ergodic properties of non-greedy series expansions to non-integer bases \( \beta > 1 \). It is shown that the so-called 'lazy' expansion is isomorphic to the 'greedy' expansion. Furthermore, a class of expansions to base \( \beta > 1, \beta \notin \mathbb{Z} \), 'in between' the lazy and the greedy expansions are introduced and studied. It is shown that these expansions are isomorphic to expansions of the form \( T\alpha(x) = \beta x + \alpha \) (mod 1). Finally, for \( \beta \) equal to the 'Golden Mean', a random expansion to base \( \beta \) is given.

1. Introduction

As is well-known, it is quite straightforward to develop any \( x \in [0, 1) \) in a series expansion to any integer base \( r > 1 \). Almost every\(^1\) \( x \in [0, 1) \) has a unique series expansion

\[
(1.1) \quad x = \sum_{k=1}^{\infty} \frac{a_k}{r^k}, \quad a_k \in \{0, 1, \ldots, r-1\},
\]

denoted by \( x = b_1 b_2 \cdots b_n \cdots \). Only rationals \( p/q \) with \( q = p_1^{\ell_1} \cdots p_m^{\ell_m} \) (where the \( \ell_i \)'s are non-negative integers and the \( p_i \)'s are the prime divisors of \( r \)), have two different expansions of the form (1.1), one of them being finite while the other expansion ends in an infinite string of \( r-1 \)'s. Underlying these so-called \( r \)-ary expansions of the form (1.1) are maps \( T_r : [0, 1) \to [0, 1) \), given by

\[
T_r(x) = rx \pmod{1},
\]

and the digits \( a_k = a_k(x), k \geq 1 \), are given by

\[
a_k = \lfloor rT_r^{k-1}(x) \rfloor, \quad k \geq 1,
\]

where \( \lfloor \xi \rfloor \) denotes the largest integer not exceeding \( \xi \). Clearly \( T_r \) is related to the Bernoulli-shift on \( r \) symbols, and the Lebesgue measure \( \lambda \) is \( T_r \)-invariant.

In case of a non-integer \( \beta > 1 \) the situation is quite different. Again any number \( x \in [0, 1) \) can be expanded to base \( \beta \):

\[
(1.2) \quad x = \sum_{k=1}^{\infty} \frac{b_k}{\beta^k}, \quad b_k \in \{0, 1, \ldots, \lfloor \beta \rfloor \}.
\]

However, one easily sees that for a given non-integer \( \beta > 1 \) almost every \( x \in [0, 1) \) has infinitely many different series expansions of the form (1.2). As in the case of the \( r \)-ary expansion, an expansion of \( x \in [0, 1) \) of the form (1.2) can be obtained by using the map \( T_\beta : [0, 1) \to [0, 1) \), given by

\[
T_\beta(x) = \beta x \pmod{1},
\]

see also Figure 1. In this case we speak of the \( \beta \)-expansion of \( x \). In 1957, A. Rényi

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\(^1\)All almost all statements are with respect to Lebesgue measure \( \lambda \).
[R] introduced these maps $T_\beta$, and studied their ergodic properties. Rényi showed that 

$$([0,1), \mu_\beta, T_\beta)$$

forms an ergodic system, where $\mu_\beta$ is a $T_\beta$-invariant probability measure equivalent to $\lambda$ with density $h_\beta$, with

$$1 - \frac{1}{\beta} \leq h_\beta \leq \frac{1}{1 - \frac{1}{\beta}}.$$

Independently, A.O. Gel’fond (in 1959) [G] and W. Parry [P1] (in 1960) showed that

$$h_\beta(x) = \frac{1}{F(\beta)} \sum_{x < T^n(1)} \frac{1}{\beta^n} 1_{[0,1)}(x),$$

where $F(\beta) = \int_0^1 \left( \sum_{x < T^n(1)} \frac{1}{\beta^n} \right) dx$ is a normalizing constant. After Parry the ergodic properties of $T_\beta$ were studied by several authors. E.g., F. Hofbauer [H] showed that $\mu_\beta$ is the measure of maximal entropy, and M. Smorodinsky [Sm] “closed the gap” between the ergodic properties of $T_\beta$ for $\beta \in \mathbb{Z}$ and $\beta \notin \mathbb{Z}$, by showing that for each non-integer $\beta > 1$ the system $([0,1), \mu_\beta, T_\beta)$ is weakly Bernoulli, see also [DKS]. A deep result by N. Friedman and D. S. Ornstein [FO] then yields that the natural extension of $([0,1), \mu_\beta, T_\beta)$ is a Bernoulli automorphism.

The $\beta$-expansion of $x$ is also known as the greedy expansion of $x$. The digits $b_n, n \geq 1$, of the greedy expansion of $x$ are recursively given by

$$b_n = k \text{ (with } 0 \leq k \leq \lfloor \beta \rfloor \text{) } \iff \sum_{k=1}^{n-1} \frac{b_k}{\beta^k} + \frac{b}{\beta^n} \leq x < \sum_{k=1}^{n-1} \frac{b_k}{\beta^k} + \frac{b + 1}{\beta^n}.$$

Clearly this yields a series expansion of $x$ of the form (1.2), and setting

$$t_n = t_n(x) := \beta^n \sum_{k=n+1}^{\infty} \frac{b_k}{\beta^k}, \quad n \geq 0,$$

it is an exercise to show that $t_n = T^n_\beta(x)$ for $n \geq 0$.

For reasons which will become apparent in Sections 2 and 3 we expand the domain of $T_\beta$ from $[0,1)$ to $\Lambda_\beta := [0,[\beta]/(\beta - 1))$. Now let $T_\beta : \Lambda_\beta \to \Lambda_\beta$ be
defined by
\[ T_\beta(x) = \begin{cases} 
\beta x \mod 1, & 0 \leq x < 1, \\
\beta x - \lfloor \beta \rfloor, & 1 \leq x < \lfloor \beta \rfloor / (\beta - 1).
\end{cases} \]

Notice that for each \( x \in \Lambda_\beta \) there exists a unique integer \( n_0 = n_0(x) \) such that for all \( n \geq n_0 \) one has that \( T_\beta^n(x) \in [0,1) \). In view of this we let \( h_\beta \) be as before on \([0,1)\), and define \( \mu_\beta \left( \{1, \lfloor \beta \rfloor / (\beta - 1)\} \right) = 0 \). Due to this, the system
\[
\left( \Lambda_\beta := \left\{ \frac{\lfloor \beta \rfloor}{\beta - 1}, \mu_\beta, T_\beta \right\},
\]
is weak-Bernoulli, since the “original” system on \([0,1)\) is.

In the last decade an interest in expansions to non-integer bases \( \beta > 1 \) other than the greedy expansion has developed. In particular in papers by P. Erdős, M. and I. Joo, V. Komornik, P. Loreti, F. Schnitzer and others, the so-called lazy expansion to base \( \beta \in (1,2) \) has been studied, see e.g. [EJK], [KL1], [KL2], and (the references in) [JS]. In particular in these (and other) papers the lazy-expansion of 1, and its relation to the greedy-expansion of 1 has been thoroughly investigated.

In general, for a non-integer \( \beta > 1 \), the digits \( (\hat{b}_k)_{k \geq 1} \) of the lazy-expansion of \( x \in \Lambda_\beta \) are recursively given by
\[
\hat{b}_n = 0 \iff \sum_{k=1}^{n-1} \frac{\hat{b}_k}{\beta^k} + \frac{\lfloor \beta \rfloor}{\beta^{n+1}} + \frac{\lfloor \beta \rfloor}{\beta^{n+2}} + \cdots \geq x
\]
and \( \hat{b}_n = b \) (with \( 1 \leq b \leq \lfloor \beta \rfloor \)) if and only if both
\[
\sum_{k=1}^{n-1} \frac{\hat{b}_k}{\beta^k} + \frac{b-1}{\beta^n} + \frac{\lfloor \beta \rfloor}{\beta^{n+1}} + \frac{\lfloor \beta \rfloor}{\beta^{n+2}} + \cdots < x
\]
and
\[
\sum_{k=1}^{n} \frac{\hat{b}_k}{\beta^k} + \frac{b+1}{\beta^n} + \frac{\lfloor \beta \rfloor}{\beta^{n+1}} + \frac{\lfloor \beta \rfloor}{\beta^{n+2}} + \cdots \geq x
\]
are satisfied. By induction we always have that for \( n \in \mathbb{N} \)
\[
\sum_{k=1}^{n} \frac{\hat{b}_k}{\beta^k} \leq x \leq \sum_{k=1}^{n} \frac{\hat{b}_k}{\beta^k} + \frac{\lfloor \beta \rfloor}{\beta^{n+1}} \sum_{k=0}^{\infty} \frac{1}{\beta^k}
\]
Since
\[
\lim_{n \to \infty} \frac{\lfloor \beta \rfloor}{\beta^{n+1}} \sum_{k=0}^{\infty} \frac{1}{\beta^k} = \lim_{n \to \infty} \frac{\lfloor \beta \rfloor}{\beta^n} \frac{1}{\beta - 1} = 0,
\]
it follows that the series expansion \( \sum_{k=1}^{\infty} \hat{b}_k / \beta^k \) of \( x \) converges to \( x \).

In Section 2 we show that there is an ergodic map \( S_\beta \) underlying the lazy-expansion, which is isomorphic to (our extended version of) \( T_\beta \). From this, and the fact that the isomorphism can be given explicitly, several conclusions will be drawn.

In Section 3 we will introduce a new class of transformations \( S_{\beta,\alpha} \), each of them yielding a series-expansion (1.2) of any \( x \in \Lambda_\beta \) “in-between” the lazy-expansion and the greedy-expansion of \( x \). We will see that each \( S_{\beta,\alpha} \) is essentially isomorphic to
\[
T_{\beta,\alpha}(x) = \beta x + \alpha^* \mod 1,
\]
where \( \alpha^* = [\beta] 1 - (\alpha + 1)(\beta - 1) \). The maps \( T_{\beta,\alpha} \) were previously studied by Parry in 1964 [P2] and by R. Palmer in 1979 [Pa], see also [FL]. In Section 4 an
example of a “random expansion” to base $\beta$ (where $\beta$ equals the golden ratio $G$, i.e., $\beta = G = (1 + \sqrt{5})/2$) will be discussed. Loosely speaking, this “random expansion” will be a random mix of the greedy and lazy expansions.

2. Lazy expansions

Let $\beta > 1$, $\beta \notin \mathbb{Z}$, fixed. Setting for $x \in \Lambda_\beta = [0, [\beta]/(\beta - 1))$ and $n \in \mathbb{N}$:

$$\tilde{t}_{n-1} = \hat{t}_{n-1}(x) = \beta^{n-1} \sum_{k=n}^{\infty} \frac{\tilde{b}_k}{\beta^k},$$

where $\tilde{b}_k$ is for $k \geq 1$ defined as in Section 1. Since

$$x = \sum_{k=1}^{n-1} \frac{\tilde{b}_k}{\beta^k} + \sum_{k=n}^{\infty} \frac{\tilde{b}_k}{\beta^k} = \sum_{k=1}^{n-1} \frac{\tilde{b}_k}{\beta^k} + \frac{1}{\beta^{n-1}} \tilde{t}_{n-1},$$

it follows from (1.3), (1.4) and (1.5) that

$$\tilde{b}_n = 0 \iff \tilde{t}_{n-1} \leq \frac{[\beta]}{\beta(\beta - 1)},$$

and, if $d \in \{1, 2, \ldots, [\beta]\}$

$$\tilde{b}_n = d \iff \frac{(d-1)\beta + [\beta] - (d-1)}{\beta(\beta - 1)} < \tilde{t}_{n-1} \leq \frac{d\beta + [\beta] - d}{\beta(\beta - 1)}.$$

In view of this we define the lazy map $S_\beta : \Lambda_\beta \to \Lambda_\beta$ by

$$S_\beta(x) = \beta x - d, \quad \text{for } x \in \Delta(d),$$

where

$$\Delta(0) = \left[0, \frac{[\beta]}{\beta(\beta - 1)}\right]$$

and

$$\Delta(d) = \left(\frac{(d-1)\beta + [\beta] - (d-1)}{\beta(\beta - 1)}, \frac{d\beta + [\beta] - d}{\beta(\beta - 1)}\right), \quad d \in \{1, 2, \ldots, [\beta]\},$$

i.e., to get the time 0 partition one starts from $[\beta]/(\beta - 1)$ by taking $[\beta]$ intervals of length $1/\beta$ from right to left. The last interval with endpoints 0 and $([\beta] + 1 - \beta)/\beta(\beta - 1)$, corresponding to the lazy digit 0, is longer than the rest, see also Figure 2. As in the greedy case it is an easy exercise to show that
\[ \hat{t}_n(x) = S^n_\beta(x), \quad \text{for } n \geq 0. \]

Notice that from the dynamics of \( S_\beta \) it follows that for every \( x \in (0, \lfloor \beta \rfloor / \beta(\beta - 1)) \) there exists a unique \( n_0 = n_0(x) \in \mathbb{N} \) such that
\[ S^n_\beta(x) \notin \left[ 0, \frac{\lfloor \beta \rfloor + 1 - \beta}{\beta - 1} \right), \quad \text{for all } n \geq n_0, \]
i.e., the interval \( A_\beta = \left( \frac{\lfloor \beta \rfloor + 1 - \beta}{\beta - 1}, \frac{\lfloor \beta \rfloor}{\beta - 1} \right) \) is an 'attractor' of the map \( S_\beta \) (of length 1). Now let \( \psi : \Lambda_\beta \to \Lambda_\beta \) be given by
\[ \psi(x) = \frac{\lfloor \beta \rfloor}{\beta - 1} - x, \]
then \( \psi([0,1)) = A_\beta \), and for \( x \in [0,1) \) one has
\[ T_\beta(x) = \beta x - d, \quad \text{for } x \in \psi^{-1}(\Delta(\lfloor \beta \rfloor - d)). \]
We have the following result.

**Theorem 1.** The map \( \psi : \Lambda_\beta \to \Lambda_\beta \) is measurable and \( \psi T_\beta = S_\beta \psi \). Furthermore, the system
\[ \left( 0, \frac{\lfloor \beta \rfloor}{\beta - 1}, \rho_\beta, S_\beta \right) \]
is weak Bernoulli,
where \( \rho_\beta \) is a probability measure on \( \Lambda_\beta \), given by
\[ \rho_\beta(A) = \mu_\beta(\psi^{-1}(A)), \quad \text{for any Lebesgue set } A \subset \Lambda_\beta. \]

**Proof.** Clearly \( \psi \) is measurable since it takes cylinders to cylinders and if \( x \in C(d) = \lfloor (d - 1)/\beta, d/\beta \rfloor \), where \( d \in \{1, 2, \ldots, \lfloor \beta \rfloor \} \), then \( T_\beta(x) = \beta x - (d - 1) \), and we find that
\[ \psi(T_\beta(x)) = \frac{\lfloor \beta \rfloor}{\beta - 1} - \beta x + d - 1. \]
We also have that
\[ \psi(x) \in \left( \psi\left( \frac{d}{\beta} \right), \psi\left( \frac{d - 1}{\beta} \right) \right) = \left( \frac{(\lfloor \beta \rfloor - d)\beta + d}{\beta(\beta - 1)}, \frac{\lfloor \beta \rfloor - (d - 1)\beta + d - 1}{\beta(\beta - 1)} \right), \]
for \( d \in \{1, 2, \ldots, \lfloor \beta \rfloor \} \). Thus
\[ S_\beta \left( \psi(x) \right) = \frac{\lfloor \beta \rfloor}{\beta - 1} - \beta x + d - 1 = \psi(T_\beta(x)). \]
A similar reasoning works for the case that \( x \) is in the interval \( \lfloor (\lfloor \beta \rfloor / \beta), \lfloor \beta \rfloor / (\beta - 1) \rfloor \).
Since \( \psi : \Lambda_\beta \to \Lambda_\beta \) is a bijection, it follows by construction of \( \rho_\beta \) that \( \psi \) is a measure theoretical isomorphism. Hence \( (\Lambda_\beta, \rho_\beta, S_\beta) \) inherits the mixing properties of \( (\Lambda_\beta, \mu_\beta, T_\beta) \) and is therefore weak Bernoulli.

**Remarks 1.** 1. It was already noticed in [EJK] in the case \( 1 < \beta < 2 \) that if \( x \in [0,1) \) has a greedy expansion \( x = \ldots b_1 b_2 \ldots b_n \ldots \), then \( \psi(x) \) has a greedy expansion \( (1 - b_1)(1 - b_2) \ldots (1 - b_n) \ldots \), i.e., \( \tilde{b}_n = 1 - b_n \), for \( n \in \mathbb{N} \). Clearly a similar relation holds in general. I.e., if \( \beta > 1 \) is non-integer, and if \( x \in \Lambda_\beta \) has as greedy expansion \( x = \ldots b_1 b_2 \ldots b_n \ldots \), then \( \psi(x) \) has as lazy expansion \( (1 - b_1)(1 - b_2) \ldots (1 - b_n) \ldots \), i.e., \( \tilde{b}_n = \lfloor \beta \rfloor - b_n \), for \( n \in \mathbb{N} \).

2. By definition of the 'lazy measure' \( \rho_\beta \) one has that the density \( d_\beta \) of \( \rho_\beta \) equals
\[ d_\beta(x) = h_\beta (\psi^{-1}(x)), \quad \text{for } x \in A_\beta, \]
and \( d_\beta = 0 \) for \( x \notin A_\beta \).
3. Let $\beta > 1$, $\beta \notin \mathbb{Z}$ and let $x \in \Lambda_\beta$. For $n \in \mathbb{N}$, let $b_i \in \{0, 1, \ldots, |\beta|\}$, $1 \leq i \leq n$. Then we define the asymptotic density $D(b_1, b_2, \ldots, b_n; x)$ of the block $b_1, b_2, \ldots, b_n$ by

$$D(b_1, \ldots, b_n; x) := \lim_{N \to \infty} \frac{1}{N} \{ 0 \leq i \leq N - 1; b_{i+1}(x) = b_1, \ldots, b_{i+n}(x) = b_n \}.$$

Similarly the asymptotic density $\tilde{D}(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n; x)$ is defined for the lazy expansion. For instance, Rényi [R1] showed that in case $\beta = G$ for almost all $x$ one has that $D(1; x) = \frac{\beta^2}{\beta^2 + 1} = 0.7236\ldots$. In this case one also has that $D(11; x) = \tilde{D}(00; x) = 0$.

**Corollary 1.** Let $\beta > 1$, $\beta \notin \mathbb{Z}$ and let $n \in \mathbb{N}$. Furthermore, let $b_i \in \{0, 1, \ldots, |\beta|\}$ for $1 \leq i \leq n$. Then for almost all $x \in \Lambda_\beta$ one has that

$$D(b_1, b_2, \ldots, b_n; x) = D(|\beta| - b_1, |\beta| - b_2, \ldots, |\beta| - b_n; x)$$

For $x \in \Lambda_\beta$, we define the greedy resp. lazy convergents $C_n = C_n(x)$ resp. $\tilde{C}_n = \tilde{C}_n(x), n \geq 1$, of $x$ by

$$C_n := \sum_{k=1}^{n} \frac{b_k}{\beta^k}, \quad \text{resp.} \quad \tilde{C}_n := \sum_{k=1}^{n} \frac{\tilde{b}_k}{\beta^k}, \quad n \geq 1.$$

From the definitions of the greedy and lazy maps one might be tempted to think that one always has that

$$x - C_n \leq x - \tilde{C}_n, \quad \text{for } n \geq 1.$$

However, this is incorrect, as the following example shows. Let $\beta = 1.618, x = 0.619$, then using MAPLE one finds that the greedy expansion of $x$ equals

$$0.1000101001010111010110\ldots,$$

and the lazy expansion of $x$ is

$$0.01010101010110101110\ldots,$$

and that $C_n = C_1 = .6180469716\ldots$ for $n = 2, \ldots, 14, C_n = .6187803401\ldots$ for $n = 15, \ldots, 18$ and $C_{19} = .6188873461\ldots$ Furthermore, $\tilde{C}_n = .6188093591\ldots$ for $n = 17, 18$, and $C_{19} = .6189163651\ldots$. Thus we see that $\tilde{C}_n > C_n$ for $n = 17, 18, 19$. Notice that

$$C_n^* := \frac{1}{\beta^2} + \frac{1}{\beta^3} = 6180649139\ldots,$$

so there exist expansions of $x$ to base $\beta$ which are neither lazy nor greedy for which the convergents (sometimes) perform better than the greedy convergents.

In order to compare the quality of approximation of the two algorithms we define approximation coefficients $\theta_n = \theta_n(x)$ resp. $\tilde{\theta}_n = \tilde{\theta}_n(x)$ by

$$\theta_n = \theta_n(x) := \beta^n (x - C_n), \quad \tilde{\theta}_n = \tilde{\theta}_n(x) := \beta^n (x - \tilde{C}_n), \quad \text{for } n \geq 1.$$

Clearly $T^n_{\beta}(x) = \theta_n$ and $S^n_{\beta}(x) = \tilde{\theta}_n$ for $n \geq 0$. But then it follows from the ergodic theorem that the limits

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \theta_k(x) \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \tilde{\theta}_k(x)$$

exist and equal $M_{\text{greedy}} := \int_0^{\frac{1}{\beta^2}} x d\mu_{\beta}$ resp. $M_{\text{lazy}} := \int_0^{\frac{1}{\beta^3}} x d\rho_{\beta}$, for almost all $x$.

We have the following result, which states that on average for almost all $x$ the greedy convergents approximate $x$ (better) than the lazy convergents of $x$. 
Proposition 1. Let $\beta > 1$, $\beta \notin \mathbb{Z}$, then

\[ M_{\text{greedy}} + M_{\text{lazy}} = \frac{|\beta|}{\beta - 1} \text{ and } M_{\text{greedy}} < M_{\text{lazy}} \]

Proof. The first statement follows directly from the relation between $h_\beta$ and $d_\beta$, viz.,

\[
M_{\text{lazy}} = \int_{\Lambda_\beta} x \, d\rho_\beta(x) = \int_{0}^{\frac{1}{\beta^2}} x h_\beta \left( \frac{|\beta|}{\beta - 1} - x \right) \, dx \]
\[
= \int_{0}^{\frac{1}{\beta^2}} (\frac{|\beta|}{\beta - 1} - y) \, h_\beta(y) \, dy = \frac{|\beta|}{\beta - 1} - M_{\text{greedy}}
\]

For the second statement, notice that by definition of $M_{\text{greedy}}$ one has that

\[
M_{\text{greedy}} = \frac{1}{F(\beta)} \sum_{n=0}^{\infty} \int_{0}^{T_n^\beta(1)} x \, \frac{1}{\beta^n} \, dx = \frac{1}{F(\beta)} \sum_{n=0}^{\infty} \frac{(T_n^\beta(1))^2}{2\beta^n}.
\]

Furthermore, by definition of $d_\beta$ one has

\[
M_{\text{lazy}} = \int_{\Lambda_\beta} x d\beta(x) \, dx = \frac{1}{F(\beta)} \sum_{n=0}^{\infty} \int_{\psi(T_n^\beta(1))}^{T_n^\beta(1)} x \, \frac{1}{\beta^n} \, dx
\]
\[
= \frac{1}{F(\beta)} \sum_{n=0}^{\infty} \left( \frac{|\beta|}{\beta - 1} \right)^2 - \left( \psi(T_n^\beta(1)) \right)^2.
\]

The first result follows from the observation that for every $n \geq 0$ one has that

\[
(T_n^\beta(1))^2 < \left( \frac{|\beta|}{\beta - 1} \right)^2 - \left( \psi(T_n^\beta(1)) \right)^2,
\]

a statement equivalent to $T_n^\beta(1) < \lceil \beta \rceil / (\beta - 1)$, which is obviously correct for every $n \geq 0$. \hfill \Box

As an example, we consider here $\beta = G = (1 + \sqrt{5})/2$. In this case Rényi [R1] already showed that

\[ h_G(x) = \frac{G^2}{G^2 + 1} 1_{[0, G)}(x) + \frac{G^2}{G^2 + 1} 1_{[G, 1)}(x), \]

where $g = 1/G$. But then one finds that

\[ M_{\text{greedy}} = \frac{1}{\sqrt{5}} = .4472\ldots \quad \text{and} \quad M_{\text{greedy}} = G - \frac{1}{\sqrt{5}} = 1.17082\ldots. \]

3. $(\beta, \alpha)$ Expansions

In this section we will discuss a new class of series expansions to any non-integer base $\beta > 1$. Notice that both the greedy map $T_\beta$ and the lazy map $S_\beta$ have ‘attractors’ of length 1. For each

\[ \alpha \in \left[ 0, \frac{|\beta|}{\beta - 1} - 1 \right] \]

we will define a map $N_{\beta, \alpha}$ on $\Lambda_\beta$, which has as attractor the interval $[\alpha, \alpha + 1)$. Just as the greedy map $T_\beta$ and the lazy map $S_\beta$ the map $N_{\beta, \alpha}$ generates a series expansion (1.2) to base $\beta$. Let the partition points $d_1, \ldots, d_{[\beta]}$ be given by:

\[ d_i := \frac{\alpha + i}{\beta}, \quad i = 1, \ldots, [\beta], \]
Figure 3. $N_{\beta,a}$ for $\beta = \sqrt{3}$ and $\alpha = g^2$.

see also Figure 3, then $N_{\beta,a} : \Lambda_\beta \to \Lambda_\beta$ is defined by

$$N_{\beta,a}(x) := \begin{cases} 
\beta x, & x \in [0,d_1), \\
\beta x - i, & x \in [d_i,d_{i+1}), 1 \leq i < [\beta], \\
\beta x - [\beta], & x \in [d_{[\beta]}, \frac{[\beta]}{\beta - 1}).
\end{cases}$$

In order to understand the dynamical properties of $N_{\beta,a}$ we consider the map $\psi^* : [\alpha, \alpha + 1) \to [0,1]$, given by $\psi^*(x) := \alpha + 1 - x$. Setting

$$T^*(x) = \psi^*(N_{\beta,a}(\psi^{*-1}(x))).$$

We have the following lemma.

**Lemma 1.** Let $\beta > 1, \beta \notin \mathbb{Z}$, and let $\alpha \in [0, \frac{[\beta]}{\beta - 1} - 1)$. Then

$$T^*(x) = \beta x + \alpha^* \pmod{1},$$

where $\alpha^* = [\beta] - (\alpha + 1)(\beta - 1)$.

**Proof.** The proof is essentially the same as the first part of Theorem 1, and is therefore omitted. \hfill \Box

**Remarks 2** Maps $T(x) = \beta x + \alpha \pmod{1}$ were first introduced and studied by Parry [P2]. Parry showed that $T$ is ergodic with respect to the Lebesgue measure $\lambda$, and that there exists a unique $T$-invariant probability measure $\tau = \tau_{\beta,a} \ll \lambda$, with density

$$h_\tau(x) = K \left( \sum_{x < T^{n}(1)} \frac{1}{\beta^n} - \sum_{x < T^{n}(0)} \frac{1}{\beta^n} \right) 1_{(0,1)}(x),$$

where $K = K_{\beta,a}$ is a normalizing constant. In [Pa], R. Palmer extended results by R. Bowen [B], Parry [P2] and Smorodinsky [Sm] by giving the exact regions in the $(\beta, \alpha)$-plane in which $T$ is weakly Bernoulli (WB). Palmer also determined the eigenvalues of all those transformations $T$ which are not WB. Since Palmer’s thesis [Pa] was never published, we will recall here some of her results, see also [FL].

**Theorem 2.** (Palmer, 1979) Let $\beta > 1, 0 \leq \alpha < 1$. Then ....

From Lemma 1 and Palmer’s theorem we at once have the following corollary.
Corollary 2. Let $\beta > 1$, $0 \leq \alpha < 1$. Then 

4. Random expansions to base $\beta = (1 + \sqrt{5})/2$

Let $\beta = G = (1 + \sqrt{5})/2$ be the golden mean. In this section we consider the greedy map $T_G$ on $[0,1)$, and the lazy map $S_G$ on $[g,G)$, where $g = \beta - 1 = 1/G = (\sqrt{5} - 1)/2$. Let $L = [0,g)$, $M = [g,1)$ and $R = [1,G)$. For any $x \in [0,G)$ we will use the following “random” algorithm to generate expansions of $x$ to base $G$ which are neither greedy nor lazy nor an $(\beta, \alpha)$ expansion as described in the previous section. Note that the maps $T_G$ and $S_G$ overlap on the interval $M$. We will use $M$ as a “switch region”, where one is allowed to replace a digit 1 generated by the greedy algorithm to a 0 by switching the map to the corresponding lazy algorithm, and conversely. In the previous section this was done in a deterministic way, we now will do it in a random way. The digits are obtained as follows.

Start with a point $x \in [0,G)$,

* if $x \in L$, then set $d_1 = d_1(x) = 0$ and let $K(x) = T_G(x) = Gx$;

* if $x \in R$, then set $d_1 = d_1(x) = 1$ and let $K(x) = S_G(x) = Gx - 1$;

* if $x \in M$, then flip a coin with $P(\text{Heads}) = p$, where $0 \leq p \leq 1$. If the coin flip is Heads, then set $d_1 = d_1(x) = 1$ and let $K(x) = T_G(x) = Gx - 1$. If the coin toss is Tails, then set $d_1 = d_1(x) = 0$ and let $K(x) = S_G(x) = Gx$.

Summarizing,

$$d_1 = d_1(x) = \begin{cases} 0 & \text{if } x \in L \text{ or } x \in M \text{ and Tails}, \\ 1 & \text{if } x \in R \text{ or } x \in M \text{ and Heads}. \end{cases}$$

For $n \in \mathbb{N}$, let $d_n = d_n(x) = d_1(K^{n-1}x)$.

Proposition 2. Given $x \in [0,G)$, the digits $d_n$ given by the above procedure satisfy

$$x = \sum_{k=1}^{\infty} \frac{d_k}{\beta^k}.$$ 

Proof. Notice that $K(x) = Gx - d_1(x) = Gx - d_1$. Hence,

$$x = \frac{d_1(x)}{G} + \frac{K(x)}{G},$$

and iterating this $n$-times yields

$$x = \frac{d_1(x)}{G} + \frac{1}{G} \left( \frac{d_1(K(x))}{G} + \frac{K^2(x)}{G} \right) = \frac{d_1(x)}{G} + \frac{d_2(x)}{G^2} + \frac{K^2(x)}{G^2} = \ldots = \frac{d_1(x)}{G} + \frac{d_2(x)}{G^2} + \ldots + \frac{d_n(x)}{G^n} + \frac{K^n(x)}{G^n},$$

from which it follows that

$$\left| x - \sum_{k=1}^{n} \frac{d_k(x)}{G^k} \right| = \frac{1}{G^n} |K^n(x)| \leq \frac{G}{G^n} = \frac{G - 1}{G^n} \to 0 \quad \text{as } n \to \infty.$$ 

It is well-known that the dynamical system underlying the $\beta$-transformation or greedy algorithm on $[0,1)$ with $\beta = G$, can be symbolically described by the ergodic Markov chain on 2 symbols 0 and 1 and transition matrix given by

$$\left( \begin{array}{cc} g & g^2 \\ 1 & 0 \end{array} \right),$$
see e.g. [R1], or [R2]. The stationary distribution corresponds to the Parry measure. Analogously, the lazy transformation \( S_G \) on \([g, G]\) can be described by the ergodic Markov chain on 2 symbols 0 and 1, and transition matrix

\[
\begin{pmatrix}
0 & 1 \\
g & g^2
\end{pmatrix}.
\]

We will now describe how our random digits \( d_n \) given by (4.1) are generated by an ergodic Markov chain on 3 symbols \( \ell, m \) and \( r \), and transition matrix

\[
P = \begin{pmatrix}
g & g^2 & 0 \\
p & 0 & 1 - p \\
0 & g & g^2
\end{pmatrix}.
\]

This Markov chain has stationary distribution \( \pi = (\pi_\ell, \pi_m, \pi_r) \), given by

\[
\pi_\ell = \frac{pG^2}{G^2 + 1}, \quad \pi_m = \frac{1}{G^2 + 1} \quad \text{and} \quad \pi_r = \frac{(1 - p)G^2}{G^2 + 1}.
\]

Notice that if \( p = 1 \) one gets the Parry measure and if \( p = 0 \) one gets the lazy measure. Any sequence of \( \ell \)'s, \( m \)'s and \( r \)'s that is generated by this Markov chain corresponds to a random expansion to base \( \beta = G \) as described in the beginning of this section. To see this, let

\[X_1, X_2, X_3, \ldots\]

be a sequence generated by this Markov chain. Define a sequence \((d_n)_{n \in \mathbb{N}}\) of 0’s and 1’s as follows:

\[
d_n = \begin{cases}
0 & \text{if } X_n = \ell \quad \text{or} \quad (X_n = m \text{ and } X_{n+1} = r), \\
1 & \text{if } X_n = r \quad \text{or} \quad (X_n = m \text{ and } X_{n+1} = \ell).
\end{cases}
\]

Setting \( x = \sum_{k=1}^{\infty} d_k / G^k \), we now show that \((d_1, d_2, \ldots)\) can also be generated by the initial procedure (4.1) (here we know apriori the flip times and the results of the coin flips!). In other words, given the flip times and outcomes of the coin tosses, we want to show for \( n \in \mathbb{N} \) that

\[
K^{n-1}(x) \in \begin{cases}
L & \iff X_n = \ell, \\
M & \iff X_n = m, \\
R & \iff X_n = r.
\end{cases}
\]

For this it is enough to show that if \( n_0 \in \mathbb{N} \) is the first index \( n \) for which \( X_n = m \), then \( K^{n_0-1}(x) \in M \) and either \( x, K(x), \ldots, K^{n_0-2}(x) \in L \) or \( x, K(x), \ldots, K^{n_0-2}(x) \in R \). If this is shown, we begin again with the point \( K^{n_0}(x) \) and \( X_{n_0+1}, X_{n_0+2}, \ldots \).

We consider two cases:

\begin{itemize}
    \item if \( n_0 = 1 \), then either \((X_1 = m \text{ and } X_2 = r)\) (hence \( d_1 = 0, d_2 = 1 \)) or \((X_1 = m \text{ and } X_2 = \ell)\) (hence \( d_1 = 1, d_2 = 0 \)). In the first case (using that \( G^2 = G + 1 \))
      \[
      \frac{1}{G} = \frac{1}{G^2} + \frac{1}{G^3} + \sum_{k=4}^{\infty} \frac{0}{G^k} \leq x = \frac{1}{G^2} + \sum_{k=3}^{\infty} \frac{d_k}{G^k} \leq \sum_{k=2}^{\infty} \frac{1}{G^k} = 1.
      \]
      In the second case (i.e., the case \((X_1 = m, X_2 = \ell))\),
      \[
      \frac{1}{G} \leq x = \frac{1}{G} + \sum_{k=3}^{\infty} \frac{d_k}{G^k} \leq \frac{1}{G} + \sum_{k=3}^{\infty} \frac{1}{G} = \frac{1}{G} + \frac{1}{G(G-1)} = 1.
      \]
    \item if \( n_0 > 1 \), then either
      \begin{itemize}
          \item[(a)] \( X_1 = \ell = X_2 = \ldots = X_{n_0-1} \text{ and } (X_{n_0} = m \text{ and } X_{n_0+1} = r)\), which implies that \( d_1 = \ldots = d_{n_0-1} = 0, d_{n_0} = 0, d_{n_0+1} = 1 \),
          \item[(b)] \( X_1 = \ell = X_2 = \ldots = X_{n_0-1} \text{ and } (X_{n_0} = m \text{ and } X_{n_0+1} = \ell)\), which implies that \( d_1 = \ldots = d_{n_0-1} = 0, d_{n_0} = 1, d_{n_0+1} = 0 \),
      \end{itemize}
\end{itemize}
(c) \(X_1 = r = X_2 = \ldots = X_{m_0-1}\) and \((X_0 = m\) and \(X_{m_0+1} = r\), which implies that \(d_1 = \ldots = d_{m_0-1} = 1, d_{m_0} = 0, d_{m_0+1} = 1\).

(d) \(X_1 = r = X_2 = \ldots = X_{m_0-1}\) and \((X_0 = m\) and \(X_{m_0+1} = \ell\), which implies that \(d_1 = \ldots = d_{m_0-1} = 1, d_{m_0} = 1, d_{m_0+1} = 0\).

In case (a), we get \(x, K(x) = Gx, \ldots, K^{m_0-2}(x) = G^{m_0-2}x \in L\) and \(\frac{1}{G} = \frac{1}{G^2} + \frac{1}{G^3} \leq K^{m_0-1}(x) = G^{m_0-1}x = \frac{1}{G} + \sum_{k=3}^{\infty} \frac{d_{k+m_0-1}}{G^{k}} \leq 1\),

which yields that \(K^{m_0-1}(x) \in M\).

In case (b), we get \(x, K(x) = Gx, \ldots, K^{m_0-2}(x) = G^{m_0-2}x \in L\) and \(\frac{1}{G} \leq K^{m_0-1}(x) = G^{m_0-1}x = \frac{1}{G} + \sum_{k=3}^{\infty} \frac{d_{k+m_0-1}}{G^{k}} \leq 1\),

which yields that \(K^{m_0-1}(x) \in M\).

In case (c), \(x, K(x) = Gx - 1, \ldots, K^{m_0-2}(x) = G(K^{m_0-3}(x)) - 1 \in R\).

Since 
\[
x = \frac{1}{G} + \ldots + \frac{1}{G^{m_0-1}} + \frac{0}{G^{m_0}} + \frac{1}{G^{m_0+1}} + \frac{d_{m_0+2}}{G^{m_0+2}} + \ldots
\]

it follows that
\[
K(x) = \frac{1}{G} + \ldots + \frac{1}{G^{m_0-2}} + \frac{0}{G^{m_0-1}} + \frac{1}{G^{m_0}} + \frac{d_{m_0+2}}{G^{m_0+1}} + \ldots
\]

\[
\vdots
\]

\[
K^{m_0-2}(x) = \frac{1}{G} + \frac{0}{G^2} + \frac{1}{G^3} + \frac{d_{m_0+2}}{G^4} + \ldots \in R.
\]

Therefore,
\[
\frac{1}{G} \leq K^{m_0-1}(x) = \frac{0}{G} + \frac{1}{G^2} + \sum_{k=3}^{\infty} \frac{d_{k+m_0-1}}{G^{k}} \leq 1,
\]

and we see that \(K^{m_0-1}(x) \in M\). Finally, case (d) follows in a similar way.

Notice that if we are given the sequence of digits \((d_n)_{n \in \mathbb{N}}\) one is able to recover the original sequence of \(\ell\)'s, \(m\)'s and \(r\)'s in a unique way. Let \(n \in \mathbb{N}\) be the first index for which
\[
d_1 = \ldots = d_n \quad \text{and} \quad d_n \neq d_{n+1}.
\]
Mark the block \(d_n d_{n+1}\) and start again with \(d_{n+2}\); find the first \(m \geq 0\) such that
\[
d_{n+2} = \ldots = d_{n+m} \quad \text{and} \quad d_{n+m} \neq d_{n+m+1},
\]
mark the block \(d_{n+m} d_{n+m+1}\) and repeat this procedure beginning with \(d_{n+m+2}\). Once this blocking at ‘switch times’ is done, one is able to retrieve the original sequence.

For indices \(n\) that are not blocked, use the following correspondence:
\[
d_n = 0 \quad \leftrightarrow \quad X_n = \ell,
\]
\[
d_n = 1 \quad \leftrightarrow \quad X_n = r.
\]

For blocked indices \(d_n d_{n+1}\) use the following correspondence:
\[
01 \quad \leftrightarrow \quad mr,
\]
\[
10 \quad \leftrightarrow \quad ml.
\]

Here is an example:
\[
1 \ 10 \ 01 \ 10 \ 10 \ 01 \ 1 \ 01 \ 1 \ 10 \ 01 \ 1 \ldots
\]
Let \((X, \sigma_X)\) be the shift space consisting of all possible realizations of the above Markov chain on the symbols \(\{\ell, m, r\}\), and let \(\mu\) be the shift-invariant measure corresponding to the stationary distribution \(\pi\) and the transition matrix \(P\).

Similarly, let \((Y, \sigma_Y)\) be the shift space of all possible sequences of digits \((d_1, d_2, \cdots)\) obtained by using the above ‘random map’ \(K(x)\). Due to the above discussion we see that there exists a 2-block factor map \(\phi : X \to Y\) given by

\[
(\phi(x))_i = \begin{cases} 
0 & \text{if } X_i = \ell \text{ or } X_iX_{i+1} = mr, \\
1 & \text{if } X_i = r \text{ or } X_iX_{i+1} = m\ell,
\end{cases}
\]

see also (4.1) and (4.2). Then clearly \(\psi \circ \sigma_X = \sigma_Y \circ \psi\), and the measure \(\rho\) defined on \(Y\) by \(\rho(A) = \mu(\psi^{-1}(A))\) is \(\sigma_Y\) invariant. In other words, \(\rho\) is \(K\) invariant and ergodic.

Given this correspondence between sequences generated by the Markov chain and the random expansions to base \(\beta = G\), we are now able to describe the asymptotic as well as generic behaviour of these sequences.

As an example we give a number of stationary probabilities.

\[
P(d_n = 0) = P(X_n = \ell) + P(X_n = m, X_{n+1} = r) = \pi_\ell + P(X_{n+1} = r \mid X_n = m)\pi_m = \frac{pg^2}{G^2 + 1} + (1 - p) \frac{1}{G^2 + 1} = pG + 1 \frac{1}{G^2 + 1}.
\]

Note that if \(p = 1/2\) one finds that \(P(d_n = 0) = 1/2\), as one might expect beforehand due to symmetry.

\[
P(d_n = 0, d_{n+1} = 1) = P(X_n = \ell, X_{n+1} = m) + P(X_n = m, X_{n+1} = r) = P(X_{n+1} = m \mid X_n = \ell)P(X_n = \ell) + P(X_{n+1} = r \mid X_n = m)P(X_n = m) = g^2 \frac{pg^2}{G^2 + 1} + (1 - p) \frac{1}{G^2 + 1} = \frac{1}{G^2 + 1}.
\]

We can also calculate the expected return time to the region \(M\), i.e., the expected time between two flips of symbols \(1/\pi_m = G^2 + 1\), which is the same for all choices of \(p\).

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FROM GREEDY TO LAZY EXPANSIONS


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