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CORRESPONDENCE ANALYSIS OF INCOMPLETE CONTINGENCY TABLES

JAN DE LEEUW

DEPARTMENT OF DATA THEORY
UNIVERSITY OF LEIDEN

PETER G. M. VAN DER HEIJDEN

DEPARTMENT OF PSYCHOMETRICS
UNIVERSITY OF LEIDEN

Correspondence analysis can be described as a technique which decomposes the departure from independence in a two-way contingency table. In this paper a form of correspondence analysis is proposed which decomposes the departure from the quasi-independence model. This form seems to be a good alternative to ordinary correspondence analysis in cases where the use of the latter is either impossible or not recommended, for example, in case of missing data or structural zeros. It is shown that Nora's reconstitution of order zero, a procedure well-known in the French literature, is formally identical to our correspondence analysis of incomplete tables. Therefore, reconstitution of order zero can also be interpreted as providing a decomposition of the residuals from the quasi-independence model. Furthermore, correspondence analysis of incomplete tables can be performed using existing programs for ordinary correspondence analysis.

Key words: Correspondence analysis, data analysis, quasi-independence, structural zeros.

1. Introduction

In this paper we introduce a modification of correspondence analysis (CA) which can be used in combination with the quasi-independence models familiar from loglinear analysis. The technique we propose decomposes the residuals that are left after fitting a quasi-independence model. The decomposed residuals are represented geometrically. Thus our paper interprets CA as a technique which can be used *complementary* to loglinear analysis. A similar approach has been adopted by Daudin and Trécourt (1980), Israëls and Sikkel (1982), Lauro and Decarli (1982), and Caussinus and de Falguerolles (1986). It was also suggested by Aitkin (discussion of Deville & Malinvaud, 1983). CA can also be introduced as a model in its own right, or as an approximation to existing models. This is the approach taken by Goodman (1985, 1986), for example.

The French approach to CA, originated by Benzécri (1973, 1980), and described in considerable detail by Greenacre (1984), interprets CA as a multidimensional scaling technique which makes pictures of data matrices. In this presentation no statistical model is involved. Although we think that this geometrical interpretation of CA is in many cases the most natural one, we also think that combination and comparison with current statistical modeling approaches for frequency tables is quite useful. This is illustrated in van der Heijden and de Leeuw (1985) and van der Heijden and Worsley (1988). In this complementary interpretation of CA we study it as a technique to represent residuals of a

Requests for reprints should be sent to Peter G. M. van der Heijden, Department of Psychometrics, Subfaculty of Psychology, Faculty of Social Sciences, University of Leiden, Hooigracht 15, 2312 KM Leiden, THE NETHERLANDS.

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loglinear analysis in a picture. Both the geometrical and the statistical aspects are present in this approach, but clearly the statistics are predominant. We only apply CA to the variation that is left after the model is fitted. A model with a good fit leaves very little variation, and thus CA will be quite useless in such cases. This is more or less true by definition: A model fits well if there is no systematic variation in the residuals. As a consequence CA is most useful in combination with models that do not fit well. Thus we must combine the use of CA with the use of fairly restrictive models. This agrees closely with recommendations made by Aitkin: "CA would be particularly useful when considerable structure remains, as indicated by a large deviance, but no useful explanatory variables are available. The component plots may help identify the nature of the structure and other variables which should have been measured" (discussion of Deville & Malinvaud, 1983, p. 357). Ordinary CA is, in our interpretation, complementary to the complete independence model, which is of course highly restrictive.

We shall make use of a generalization of CA to decompose residuals from the quasi-independence model. It is supposed in this paper that the reader is familiar with the theory and applications of quasi-independence models for two-way tables. We merely indicate our notation. The model states that the theoretical probabilities π_{ij} in a bivariate contingency table satisfy $\pi_{ij} = \alpha_i \beta_j$ for a subset S of all index pairs (i, j). There are various reasons why we may not want to require $\pi_{ij} = \alpha_i \beta_j$ for all pairs. The first one is that some elements of the table are missing. A second one is that some elements may be zero by definition, the so-called structural zeros. Thirdly we may know from a previous analysis that some cells fit the independence model badly. And finally we may have the idea that for some parts of the table the independence model may be true, while for other parts (for instance the diagonal) independence is not plausible at all. For a thorough discussion we refer to Caussinus (1965), Mosteller (1968), Goodman (1968), Bishop, Fienberg and Holland (1975, pp. 177–210), and Haberman (1979, pp. 444–486).

2. Correspondence Analysis

CA will be discussed briefly here. For a longer discussion from a comparable perspective we refer to van der Heijden and de Leeuw (1985). In order to discuss correspondence analysis (CA) of incomplete tables later, we first define ordinary CA in terms of the *Fisher-Lancaster* decomposition of an observed table. This is sometimes called the *canonical analysis of a contingency table* (for instance in Kendall & Stuart, 1967, chap. 33), while the French call it the *reconstitution formula*. Suppose P is the observed table, with positive entries that add up to one. The diagonal matrix D_r contains row marginals, D_c contains the column margins, t is a vector with all elements equal to one. Then we can find R and C such that $t'D_r R = 0$, $t'D_c C = 0$, $R'D_r R = I$, $C'D_c C = I$, and

$$P = D_r(tt' + R\Lambda C')D_c, \tag{1}$$

with Λ diagonal. The sum of squares of the elements of Λ is equal to Pearson's index of mean square contingency. If P is based on a sample of size n, then n times this index is equal to the chi-square statistic for testing independence. Thus we can say that CA, if interpreted as computing the Fisher-Lancaster decomposition (1), studies the deviations from the independence model.

In the introduction we said that CA gives a geometrical representation of the residuals, in this case of the residuals from independence. This can be explained most easily by introducing the chi-square distances between the rows of $D_r^{-1}P$. Rows of $D_r^{-1}P$ add up to 1, and are usually referred to as *profiles* (Benzécri, 1973, 1980; Greenacre, 1984). The distances between these profiles are defined in (1) and (6) of van der Heijden and de Leeuw (1985).

In the French literature R and C are called *factor matrices*, and $\tilde{R} = R\Lambda$ and $\tilde{C} = C\Lambda$ are called *principal components matrices.* The chi-square distances between the profiles of rows i and i' of P is equal to the ordinary Euclidean distance between rows i and i' of $\tilde{R}$. Thus we can represent the row profiles of P using the rows of $\tilde{R}$ as coordinates, and we can approximate the chi-square distance by dropping the last column(s) of $\tilde{R}$. It is clear that dually we can also define distances between column profiles of P, and approximate them by ordinary Euclidean distances between rows of $\tilde{C} = C\Lambda$.

In van der Heijden and de Leeuw (1985, p. 431) three ways are discussed for making simultaneous representations of row and column points, namely by using $(R, \tilde{C})$, $(\tilde{R}, C)$ or $(R\Lambda^{1/2}, C\Lambda^{1/2})$. In the French CA literature it is quite customary to make joint plots of the pair $(\tilde{R}, \tilde{C})$ (Baccini, 1984). This has some rather serious disadvantages, because distances between row- and column-points cannot be interpreted in terms of the transition formulas (Equation (5) in van der Heijden & de Leeuw, 1985). Moreover the inner products of row- and column-vectors do not reproduce residuals any more. However, both the distances between different row-points and the distances between different column-points approximate the chi-square distances while the distance of any point to the origin, weighted with its margin, approximates its contribution to the total chi square (also called *inertia*). A reason for not using the joint plots $(\tilde{R}, C)$ and $(R, \tilde{C})$, is that these plots are impractical for small eigenvalues, because, for example, the dispersion of the rows plotted with $\tilde{R} = R\Lambda$ is much smaller than the dispersion of the columns plotted with C.

3. Correspondence Analysis of Incomplete Tables

Now suppose P and Q are two contingency tables. We suppose P and Q have the same margins. The interpretation we have in mind is to take P as the observed data matrix and Q as the maximum likelihood estimates under quasi-independence. The technique we discuss is more general, however, because Q could also consist of maximum likelihood estimates under models such as the quasi-symmetry model (see van der Heijden, 1987). The idea of using a model to generalize correspondence analysis has been discussed in Escofier (1984) and van der Heijden and de Leeuw (1985).

If we start with the singular value decomposition

$$S_r^{-1/2}(P - Q)S_c^{-1/2} = U\Lambda V', \tag{2}$$

we find, analogous to (1), that

$$P = Q + S_r R\Lambda C' S_c, \tag{3}$$

with $\tilde{R} = S_r^{-1/2}U\Lambda$. In the French literature (2) and (3) are typically interpreted in terms of a *duality diagram.* (See Tenenhaus & Young, 1985, for a useful discussion of this approach.) We prefer the more algebraic presentation in terms of the singular value decomposition.

A proper choice must still be made for the diagonal elements of S_r and S_c. Such a choice is to take for S_r the values $\hat{\alpha}_i$ and for S_c the values $\hat{\beta}_j$, the maximum likelihood estimates of the quasi-independence parameters. In this way the sum of squares of the singular values becomes

$$\sum_i \sum_j \left\{ \frac{(p_{ij} - q_{ij})^2}{q_{ij}} \,\middle|\, (i, j) \in S \right\}, \tag{4}$$

which is, of course, the chi-square statistic for testing quasi-independence divided by n. In this way (3) provides us with a decomposition of the residuals which corresponds to the appropriate chi-square statistic, similar to ordinary CA.

In van der Heijden and de Leeuw (1985, Equation (20)) it is shown that the interpretation in terms of chi-square distances can still be maintained. The centroid principle occurs in a somewhat different way, but is still easy to understand geometrically (see their Equation (18)). This equation shows that $R\Lambda = \tilde{R}$ is equal to the difference of the two centroids $S_r^{-1}PC$ and $S_r^{-1}QC$. And reciprocally, $\tilde{C}$ is the difference of the two centroids $S_c^{-1}PR$ and $S_c^{-1}QX$. This means that it is interesting to plot these centroids, and their difference, as vectors in one joint plot. If $P - Q$ is small, then $\tilde{R}$ and $\tilde{C}$ will also be small. Thus, for the same reasons as above, we may decide to look at joint plots which are scaled differently.

It is clear from our results so far that if the quasi-independence model fits well, then $P - Q$ is small. Thus the singular values are small, and $\tilde{R}$ and $\tilde{C}$ will be small. This brings us back to the point mentioned in the introduction: if the fit of the model is too good, then there will be no interesting variation left for CA. Because structural zeros or nonrestricted cells do not contribute to $P - Q$, this means that we will need a fair number of restricted cells in the analysis.

4. Reconstitution of Order Zero

In the French literature a technique for CA of incomplete tables has been proposed by Nora (1975). It is also discussed in Benzécri et al. (1980, Vol. 2, chap. III, No. 8), and by Greenacre (1984, pp. 236–244). The technique is specifically intended for tables with missing data, because for such tables ordinary CA is not feasible. The idea is very similar to the way communalities are treated in least squares factor analysis: they are estimated, then a principal component analysis is performed on the reduced correlation matrix, then the results are used to improve the communality estimates, and so on. When we translate this to the context of CA, we find the following algorithm. First choose the dimensionality h and an initial estimate $X^{(0)}$ which satisfies $x_{ij}^{(0)} = p_{ij}$ for (i, j) in S, and is arbitrary for (i, j) not in S. Then *reconstitution of order h* is the iterative process

$$x_{ij}^{(m+1)} = x_{i*}^{(m)} x_{*j}^{(m)} \frac{1 + \sum_{\alpha=1}^{h} \lambda_\alpha^{(m)} r_{i\alpha}^{(m)} c_{j\alpha}^{(m)}}{x_{**}^{(m)}}, \tag{5}$$

which is applied for all (i, j) not in S. For (i, j) in S we simply set $x_{ij}^{(m)} = p_{ij}$ for all m. The solution will, in general, depend on the choice of the dimensionality h. Benzécri himself seems to favor iterative reconstitution of order zero, that is, for all (i, j) not in S we set

$$x_{ij}^{(m+1)} = \frac{x_{i*}^{(m)} x_{*j}^{(m)}}{x_{**}^{(m)}}. \tag{6}$$

It can be shown that (6) converges, say to X. Let Y be the matrix with expected values estimated under the hypothesis of independence corresponding with observed values in X. Now $(X - Y) = (P - Q)$, where $(X - Y)$ contains the residuals of R when the independence matrix Y is subtracted, and $(P - Q)$ contains the residuals from quasi-independence. This is because the cells in S, $q_{ij} = y_{ij} = \hat{\alpha}_i \hat{\beta}_j$, and therefore for the cells not in S we find $x_{ij} = y_{ij} = \hat{\alpha}_i \hat{\beta}_j$ for these cells S.

Two things can be concluded from this. First of all, the procedure which is known in the French literature as reconstitution of order zero can actually be interpreted as providing a decomposition of the residuals from quasi-independence. So, as for ordinary CA, it is possible to replace the model-free interpretation of Nora's CA of incomplete tables by a model-based interpretation as given above. Secondly, we can use this finding for practical purposes, since now it is clear that by using X we can actually do CA of incomplete tables with ordinary CA programs.

6. Examples

We discuss two examples here. The first example is a square matrix of order 15×15, and we are interested in the $15^2 - 15 = 210$ off-diagonal cells. The second example is a three-way matrix, of order $2 \times 4 \times 16$, which has 128 cells of which 12 are structurally zero. In the latter example we will show how CA of a two-way coding of this incomplete matrix corresponds to the analysis of residuals from a specific loglinear model for the three-way table.

Example 1: A Square Table

Ordinary CA is not appropriate for square tables where the diagonal cells are not defined or not of interest, for example transition matrices, import-export tables, confusion matrices, and migration tables. In the French literature CA of square tables has been given considerable attention. Burtchy (1984) and Foucart (1985) review the various approaches that have been used in combination with CA. They either replace the diagonal with values chosen on theoretical grounds, or they complete the diagonal by iterative reconstitution. Subsequently an ordinary CA is performed.

Here we show the analysis of a home-work traffic table published in Foucart (1985). The matrix is shown in Table 1. In a cell of this matrix a frequency gives the number of persons which live in one south-eastern suburb of Paris and work in another. Since home-work traffic can cause traffic problems especially in those cases that people do not live and work in the same suburb, we want to restrict attention to the off-diagonal cells. We can do this by studying the decomposition of the departure from a quasi-independence model, in which we take for the diagonal cells $\pi_{ij} = p_{ij}$, and for the off-diagonal cells $\pi_{ij} = \alpha_i \beta_j$. We then apply CA of incomplete tables to study the residuals from the quasi-independence model. Fitting this quasi-independence model yields a Pearson chi-square of 65535 (d.f. is 196). This departure is very large. CA of this incomplete table is useful to try to find the main structure in this departure.

The first four singular values with their percentage of the chi-square are .610 (34.5%), .439 (17.8%), .359 (12.0%), and .335 (10.4%). 52% of the chi-square is decomposed in the first two dimensions. A plot of these dimensions is shown in Figure 1. We used the 'French' normalization ($\tilde{R}$, $\tilde{C}$) (compare section 2). A small label indicates "living in", a large one denotes "working in". A horseshoe-like curve can be seen, with JOInville, BONneuil, CHArenton, ALFort, SUCy, and VALenton—suburbs lying most east—on the left, via THIais, CHOisy, ORLy and IVRy—suburbs lying in the middle—to KREmlin, GENtilly, RUNgis, and FREsnes—suburbs lying most western—on the right. Briefly, we can conclude from this figure that, those people who are not working and living in the same suburb are in general living more often than average in a suburb that is nearby the suburb in which they work. But a closer look at Figure 1 shows that corresponding row and column points are rather far apart sometimes. This is especially the case for RUNgis and SUCy. People living in RUNgis (small label) work relatively more often in KREmlin, GENtilly and FREsnes than the other way around (concerning people working in RUNgis). People living in SUCy work relatively more in CHArenton, ALFort, BONneuil and JOInville and relatively less in VALenton than the other way around. It is clear that such more precise interpretations are somewhat dangerous, since we are looking at only 52 % of the chi-square. Therefore it is advisable to check such findings in the data.

Table 1: Migration in the suburbs of Paris; rows are destinations, columns are origins.

	CHA	IVR	KRE	GEN	VIT	ALF	CHO	BON	VAL	ORL	RUN	FRE	THI	JOI	SUC	margins without diagonal
Charenton	6238	269	45	14	204	824	57	250	70	76	16	36	0	403	189	2453
Ivry	270	11268	1113	1113	257	2483	530	708	166	878	166	205	281	457	174	8801
Kremlin	34	585	11353	1001	1493	32	143	62	133	207	327	549	226	133	0	4925
Gentilly	0	106	1389	10695	425	100	99	220	27	111	215	1037	26	152	117	4024
Vitry	186	667	894	281	11263	1009	1577	148	123	1021	154	265	860	314	90	7589
Alfort	713	258	134	75	632	16420	595	1675	563	250	29	0	118	507	297	5846
Choisy	0	181	78	41	763	148	5590	24	396	964	104	38	745	25	87	3594
Bonneuil	51	81	68	0	133	1094	109	9235	107	92	0	28	39	1831	491	4124
Valenton	31	34	34	28	34	316	271	148	6161	628	0	0	59	83	228	1894
Orly	14	108	492	177	353	104	528	209	568	6461	315	408	551	191	130	4148
Rungis	0	21	160	83	81	33	23	20	64	248	1455	110	106	21	0	970
Fresnes	0	53	310	260	156	0	0	0	0	82	481	3889	131	0	0	1473
Thiais	0	66	21	0	151	40	421	24	43	248	26	0	1498	25	0	1067
Joinville	327	43	0	63	206	801	42	1362	0	40	54	90	35	17045	774	3837
Sucy	0	0	0	26	26	20	28	159	591	102	0	0	0	403	5624	1355
Without diagonal	1626	2472	4738	3164	4914	7004	4423	5009	2851	4947	1887	2766	3177	4545	2577	

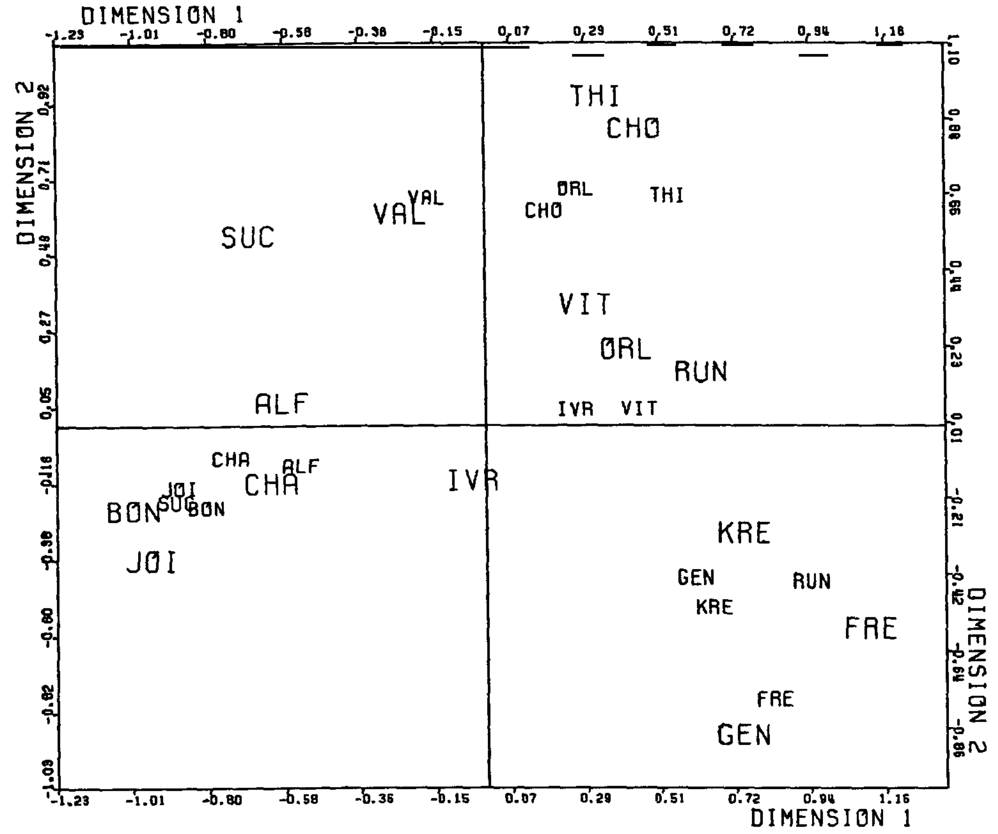

FIGURE 1
CA of migration table without diagonal cells

Example 2: Current Age × Age at First Marriage: Structural Zeros

The second example we will discuss is taken from Haberman (1979, pp. 455–471). There are three variables: age at first marriage (A), current age (C), and sex (S), and the three-way table is given in Table 2. There are structural zeros since the age at first marriage cannot exceed the current age. Following van der Heijden and de Leeuw (1985) we analyze three-way tables with frequencies f_{ijk} by coding two of the variables into a new variable, thus obtaining a two-way table. Van der Heijden and de Leeuw speak of "multiple tables". They show that ordinary CA of a multiple table with frequencies $f_{i(jk)}$—here j and k are coded interactively—decomposes residuals from the loglinear model $[I]$ $[JK]$. These residuals contain information on two first-order interactions and the second-order interaction. One consideration to choose for one of the three possible multiple tables is that one is less interested in the first-order interaction between the two variables coded interactively.

Since Haberman (1979) reports that in the three-way matrix of Table 2 the first-order interaction between current age (C) and sex (S) is not very large, the three-way table may be treated as a two-way table of order 4 × 16. It is easy to see that fitting a quasi-independence model to this two-way table is equivalent to fitting the loglinear model $[A][CS]$ (including structural zeros) to the three-way table. The chi-square equals 245 (d.f. is 33). We must conclude that age at first marriage is not independent of current age and

Table 2:

Age at first marriage x current age x sex

		Age at first marriage			
		≦20	21-25	26-30	≧31
Female	≦20	9	-	-	-
	21-25	43	20	-	-
	26-30	51	40	3	-
	31-40	103	53	4	1
	41-50	68	45	5	3
	51-60	65	43	7	9
	61-70	39	24	12	4
	≧71	22	26	7	4
Male	≦20	2	-	-	-
	21-25	24	23	-	-
	26-30	21	34	3	-
	31-40	30	61	10	4
	41-50	22	49	20	10
	51-60	19	50	27	15
	61-70	16	38	23	17
	≧71	11	19	19	11

sex jointly. We use CA of incomplete tables to decompose the departure from model [*A*][*CS*]. Note that residuals from this model do not contain first-order interaction between current age and sex, which was small according to Haberman. Thus we use CA of incomplete tables to make a plot of the main (larger) interactions in the data.

Figure 2 shows the original category numbers of current age set out against the

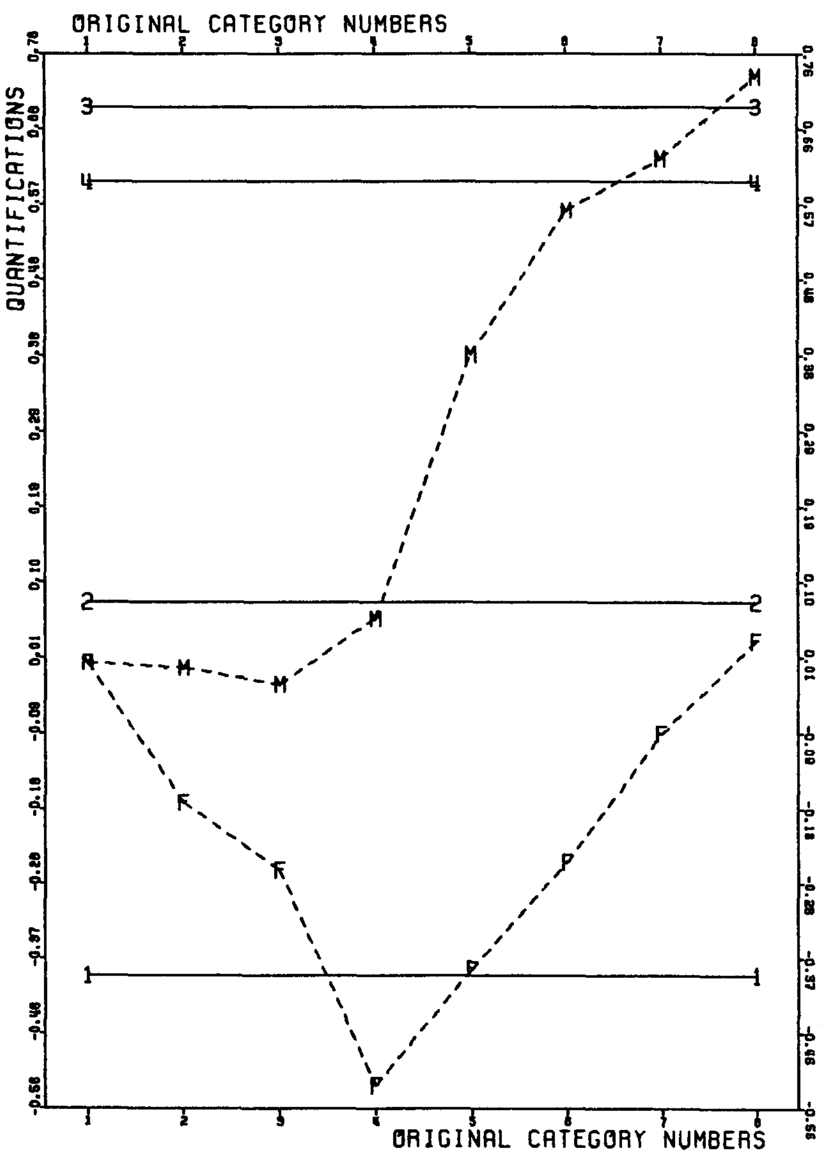

FIGURE 2
CA of table Age at first marriage × Current age × Sex original category numbers vs. first quantifications. Current age-line is horizontal, age at first marriage-categories are dotted lines.

category quantifications for the first dimension ($\lambda_1 = .385$). This dimension displays 81% of the chi-square, and therefore we restrict attention to one dimension only. Column points for age younger than 20 are given quantification 0, since for these columns the residuals are zero for all cells. The plot shows us that the age-at-first-marriage categories are quantified roughly linearly from $\leq$ 20 (1), via 21–25 (2) to 26 and older (3,4). For each current age men have their first marriage at an older age than women: the male-line lies above the female-line for all ages. This corresponds to first-order interaction between sex and age at first marriage. Furthermore, for both men and women we see a tendency that, from 31–40 and older, their age-at-first marriage tends to become higher as the respondents have an older current age. This corresponds to first-order interaction between current age and age at first marriage. Second-order interaction seems to be revealed by the fact that for a current age of 21 to 40, men and women seem to differ: for men the age at first marriage remains stable, while for women it goes down as their age increases. We can conclude that CA of incomplete tables facilitates the interpretation of patterns in the matrix of residuals.

7. Conclusion

We think that in general CA can be helpful for the analysis of residuals from independence models. It tries to find structure in the residual cells, which is especially useful for large tables. Whether some structure is found will be revealed by the singular values, and the proportion of chi-square they account for.

The procedure proposed, correspondence analysis of incomplete tables, seems to be a good alternative to correspondence analysis in all cases where the study of departure from quasi-independence seems more logical, or appropriate, than from independence. Thus the scope of CA is broadened to an important class of applications. Our CA of incomplete tables, using the quasi-independence model, is equivalent to the French procedure for missing data called "reconstitution of order zero", and this further justifies the French procedure. It also follows that CA of incomplete tables can be done using computer programs for ordinary CA. It is only necessary to construct the proper input matrix for the CA program (see section 4).

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