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Localization & Exact Holography

Atish Dabholkar[1], **João Gomes**[1], **and Sameer Murthy**[2]

[1] *Laboratoire de Physique Théorique et Hautes Energies (LPTHE)*
Université Pierre et Marie Curie-Paris 6; CNRS UMR 7589
Tour 13-14, 5ème étage, Boite 126, 4 Place Jussieu
75252 Paris Cedex 05, France

[2] *Institute for Theoretical Physics and Spinoza Institute*
University Utrecht, Leuvenlaan 4
3584 CC Utrecht, The Netherlands

Emails : atish at lpthe.jussieu.fr, gomes at lpthe.jussieu.fr, S.V.Murthy at uu.nl

Abstract: We consider the AdS_2/CFT_1 holographic correspondence near the horizon of big four-dimensional black holes preserving four supersymmetries in toroidally compactified Type-II string theory. The boundary partition function of CFT_1 is given by the known quantum degeneracies of these black holes. The bulk partition function is given by a functional integral over string fields in AdS_2. Using recent results on localization we reduce the infinite-dimensional functional integral to a finite number of ordinary integrals over a space of localizing instantons. Under reasonable assumptions about the relevant terms in the effective action, these integrals can be evaluated exactly to obtain a bulk partition function. It precisely reproduces all terms in the exact Rademacher expansion of the boundary partition function as nontrivial functions of charges except for the Kloosterman sum which can in principle follow from an analysis of phases in the background of orbifolded instantons. Our results can be regarded as a step towards proving 'exact holography' in that the bulk and boundary partition functions computed independently agree for finite charges. Since the bulk partition function defines the quantum entropy of the black hole, our results enable the evaluation of perturbative as well as nonperturbative quantum corrections to the Bekenstein-Hawking-Wald entropy of these black holes.

Keywords: black holes, superstrings, holography.

Contents

1. Introduction

In any consistent quantum theory of gravity such as string theory, it should be possible to view a black hole as a statistical ensemble of quantum states. This implies an extremely stringent theoretical constraint on the theory that the *exact* statistical entropy of this ensemble must equal an appropriately defined quantum entropy of the black hole. Such a constraint is also *universal* in that it must hold in any 'phase' or compactification of the theory that admits a black hole. It is therefore a particularly useful guide in our explorations of string theory in the absence of direct experimental guidance, especially given the fact that we do not know which phase of the theory might describe the real world.

The notion of exact statistical entropy is *a priori* well-defined as the logarithm of the dimension of the quantum Hilbert subspace corresponding to the ensemble. The notion of exact quantum entropy of a black hole is more subtle but should be definable as a quantum generalization of the Bekenstein-Hawking-Wald entropy [1, 2, 3, 4, 5]. A definition has recently been proposed by Sen [6, 7] for extremal black holes using holography in the two dimensional anti de Sitter (AdS_2) background near the horizon of the black hole. For a black hole of charge vector (q, p), its quantum entropy is defined as the logarithm of the expectation value $W(q, p)$ of a Wilson line inserted on the boundary of the Euclidean AdS_2 space.

This definition expresses the exact quantum entropy as a formal functional integral over all (spacetime) string fields in AdS_2. It is conceptually satisfying since it takes into account quantum effects from integrating over massless fields, keeps manifest all symmetries of the theory, and reduces to the Wald entropy in the appropriate limit. At the same time, it is rather difficult to work with, unless one can figure out an efficient way to compute the functional integral. For supersymmetric black holes, it is possible to use localization techniques [8, 9, 10, 11, 12] to simplify this infinite-dimensional functional integral enormously and reduce it to a finite number of ordinary integrals [13]. In the present work we apply these results in the concrete context of supersymmetric black holes preserving four supersymmetries in $\mathcal{N} = 8$ supersymmetric compactifications of string theory to four spacetime dimensions. Since the structure of the $\mathcal{N} = 8$ theory is particularly simple, it enables us to analytically perform the ordinary integrals that remain after localization and evaluate $W(q, p)$ even after including nonpertubative effects. The resulting $W(q, p)$ matches in remarkable details with the quantum degeneracies $d(q, p)$ of these black holes that are known independently. These results are interesting from two related perspectives.

- Our results could be viewed as a step towards 'proving' holography in the context of AdS_2/CFT_1 correspondence. Holography [14, 15] has emerged as one of the central concepts concerning the microscopic degrees of freedom of quantum gravity. The heuristic principle that the degrees of freedom of a quantum theory of gravity must scale with area rather than with volume has found its most precise realization in the AdS_{d+1}/CFT_d correspondence [16, 17, 18]. Given the fundamental significance of the concept of holography, it is desirable to seek simple examples as we consider in this paper where it might be possible to prove such an equivalence. We will compute the partition functions independently both in the bulk and the boundary for arbitrary finite charges.
- Our results could be viewed as the computation of finite size quantum corrections to the leading Bekenstein-Hawking entropy of a black hole. The Bekenstein-Hawking entropy formula is valid in the limit of large horizon area or large charges. Since it follows from the two-derivative Einstein-Hilbert action, it is independent of the 'phase' or the compactification under consideration. By contrast, the finite size corrections depend sensitively on the phase and contain a wealth of information about the details of compactification, the structure of higher-derivative effective action, as well as the spectrum of nonperturbative states in the theory. They are therefore very interesting as a sensitive probe of the microscopic structure of the theory.

With these motivations we first review the quantum entropy formalism and its relation with the AdS_2/CFT_1 correspondence in §2. In §3 we describe the supersymmetric microstates in the

$\mathcal{N} = 8$ theory and their degeneracies. In §4 we review the results of [13] on the localization of the resulting functional integral and describe the explicit evaluation of the localizing integrals for the system under consideration in §5 and conclude with remarks on open problems in §6.

2. Quantum Entropy and Holography

We now review the definition of the quantum entropy [6, 7] in the framework of AdS_2/CFT_1 holographic correspondence in the near-horizon region of the black hole. In general, AdS_{d+1}/CFT_d correspondence is obtained by focusing onto the near horizon degrees of freedom in the spacetime around a (d-1)-dimensional extremal black brane. The dual CFT_d is obtained by focusing onto low-energy excitations in the world-volume theory of the brane configuration. Quantum gravity in the near horizon AdS_{d+1} geometry is then expected to be equivalent to the quantum field theory of these low-energy excitations [16, 17, 18].

In our case, we have an extremal black 0-brane or a black hole with electric charges $\{q_I\}$ and magnetic charges $\{p^I\}$. The near horizon geometry is $AdS_2 \times S^2 \times T^6$. One can regard this as a compactification on $S^2 \times T^6$ to obtain an effective theory on AdS_2 with an infinite number of fields. Magnetic charges are given by fluxes on S^2 of this compactification. The massless bosonic sector contains the metric, gauge fields A^I with field strengths F^I, and scalar fields X^I. Classically, the metric on the Euclidean AdS_2 factor is

$$ds^2 = v_* \left[(r^2 - 1) d\theta^2 + \frac{dr^2}{r^2 - 1} \right] \qquad 1 \leq r < \infty; \qquad 0 \leq \theta < 2\pi \ . \tag{2.1}$$

The scale v_* of the horizon as well as the values of the scalar fields X^I_* and the electric fields e^I_* at the horizon are determined in terms of the charges (q, p) by the attractor mechanism.

Quantum mechanically, the AdS_2 functional integral is defined by summing over all field configurations which asymptote to the these attractor values with the fall-off conditions [6, 7, 19]

$$ds^2 = v_* \left[\left(r^2 + \mathcal{O}(1) \right) d\theta^2 + \frac{dr^2}{r^2 + \mathcal{O}(1)} \right] \ . \tag{2.2}$$

$$X^I = X^I_* + \mathcal{O}(1/r) \ , \qquad A^I = -i\, e^I_* (r - \mathcal{O}(1)) d\theta \ . \tag{2.3}$$

All massive fields asymptote to zero because of their mass term.

The functional integral for the partition function would be weighted by the exponential of the classical action given by a Wilsonian effective action at some scale such as the string scale. To make the classical variational problem well-defined, it is necessary to add a boundary term

$$-iq^I \int A_I \tag{2.4}$$

to the action to cancel the boundary terms arising from the variation of the bulk action for the gauge field. With this boundary term, the quantum bulk partition can be naturally interpreted as an expectation value of a Wilson line inserted at the boundary

$$W(q,p) = \left\langle \exp[-i\, q_I \oint_\theta A^I] \right\rangle^{finite}_{\mathrm{AdS}_2} \ , \tag{2.5}$$

where the superscript refers to a finite piece obtained by a procedure that we describe below.

The functional integral (2.5) has a well-known divergence as a consequence of the infinite volume of AdS_2. This can be removed by regularization and holographic renormalization. We introduce a cutoff at $r = r_0$ for a large r_0 to regularize the action. The proper length of the boundary scales as $2\pi\sqrt{v_*}r_0$. Since the classical action is an integral of a *local* Lagrangian, it scales as $C_1 r_0 + C_0 + \mathcal{O}(r_0^{-1})$. The linearly divergent part can now be renormalized away by a boundary counter-term which basically sets the origin of boundary energy. After this renormalization we can take the cut-off to infinity to obtain a finite functional integral weighted by the exponential of the finite piece C_0. We refer to C_0 as the renormalized action S_{ren} which is a functional of all fields and contains arbitrary higher-derivative terms[1].

It is worth emphasizing two peculiarities of AdS_2 that are significant [6, 7] in this context.

- For $d > 2$, the constant mode of the gauge field corresponding to the electric potential is dominant near the boundary and is hence kept fixed, while the r-dependent mode corresponding to the electric field falls off at the boundary, and hence is allowed to fluctuate in the quantum theory. This corresponds to the grand-canonical ensemble where the chemical potential is held fixed[2]. For $d = 1$, the r-dependent mode of the gauge field corresponding to the electric field grows linearly and must be kept fixed, while the constant mode is allowed to fluctuate. Fixing electric fields fixes all charges by Gauss law. This corresponds to the microcanonical ensemble.
- For $d > 1$, the CFT_d of massless fields obtained by focusing on modes below a mass gap in the worldvolume still allows for a continuum of long wavelength, low energy excitations. For $d = 1$, there are no spatial directions. The boundary CFT_1 obtained by taking a low energy limit simply consists of the ground states in the charge sector (q, p) and has a degenerate and finite-dimensional Hilbert space with zero Hamiltonian. The partition function of the CFT_1 is then simply the *number* $d(q, p)$ of these states. Put another way, for any general d, conformal invariance allows all excitations with traceless stress tensor. In the special case of $d = 1$, traceless stress tensor implies that the Hamiltonian is zero and there is no dynamics. This is consistent with the fact that Lorentzian AdS_2 cannot support any finite energy fluctuations without disturbing the asymptotic boundary conditions because of the large gravitational backreaction in low dimensions.

The AdS_2/CFT_1 correspondence thus provides us with a satisfactory definition of quantum entropy as well as a simple and yet nontrivial example of holography. It implies

$$d(q, p) = W(q, p) \tag{2.6}$$

The main challenge in the subsequent sections will be to find a context where these formal definitions can be used for concrete calculations to compute both sides of this equation.

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[1] Regularizations corresponding with more general cut-offs lead to the same renormalized action [20].

[2] This is also true in $d = 2$ where the analysis is more subtle because of the Chern-Simons terms in AdS_3 [21, 22].

3. Microscopic Quantum Partition Function

Consider Type-II string compactified on a 6-torus T^6. The resulting four-dimensional theory has $\mathcal{N} = 8$ supersymmetry with 28 massless $U(1)$ gauge fields. A charged state is therefore characterized by 28 electric and 28 magnetic charges which combine into the **56** representation of the U-duality group $E_{7,7}(\mathbb{Z})$. Under the $SO(6,6;\mathbb{Z})$ T-duality group, the 28 gauge fields decompose as

$$\mathbf{28} = \mathbf{12} + \mathbf{16} \tag{3.1}$$

where the fields in the vector representation **12** come from the NS-NS sector, while the fields in the spinor representation **16** come from the R-R sector. We obtain an $\mathcal{N} = 4$ reduction of this theory by dropping four gravitini multiplets. Since each graivitini multiplet of $\mathcal{N} = 4$ contains four gauge fields, this amounts to dropping sixteen gauge fields which we take to be the R-R fields in the above decomposition.The U-duality group of the reduced theory is

$$SO(6,6;\mathbb{Z}) \times SL(2,\mathbb{Z}) \tag{3.2}$$

where $SL(2,\mathbb{Z})$ is the electric-magnetic S-duality group.

3.1 Charge Configuration

We will be interested in one-eighth BPS dyonic states in this theory which perserve four of the thirty-two supersymmetries. To simplify things, we consider the 6-torus to be the product $T^4 \times S^1 \times \widetilde{S}^1$ of a 4-torus and two circles. Let n and w be the momentum and winding along the circle S^1, and K and W be the corresponding Kaluza-Klein monopole and NS5-brane charges. Let $\widetilde{n}, \widetilde{w}, \widetilde{K}, \widetilde{W}$ be the corresponding charges associated with the circle $\widetilde{S}^1$. A general charge vector with these charges can be written as a doublet of $SL(2,\mathbb{Z})$

$$\Gamma = \begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} \widetilde{n} & n & \widetilde{w} & w \\ \widetilde{W} & W & \widetilde{K} & K \end{bmatrix}_{B'}, \tag{3.3}$$

where the subscript B' denotes a particular Type-IIB duality frame. The T-duality invariants for this configuration are [23]

$$Q^2 = 2(nw + \widetilde{n}\widetilde{w}), \qquad P^2 = 2(KW + \widetilde{K}\widetilde{W}), \qquad Q \cdot P = nK + \widetilde{n}\widetilde{K} + wW + \widetilde{w}\widetilde{W}, \tag{3.4}$$

and the quartic U-duality invariant can be written as

$$\Delta = Q^2 P^2 - (Q \cdot P)^2 . \tag{3.5}$$

For our purposes it will suffice to excite only five charges

$$\Gamma = \begin{bmatrix} 0 & n & 0 & w \\ \widetilde{W} & W & \widetilde{K} & 0 \end{bmatrix}_{B'} \tag{3.6}$$

so that the T-duality invariants are all nonzero. There are three other duality frames that are of interest.

- Frame B: In this frame the charge configuration becomes

$$\Gamma = \begin{bmatrix} 0 & n & 0 & \widetilde{K} \\ Q_1 & \widetilde{n} & Q_5 & 0 \end{bmatrix}_B , \tag{3.7}$$

where Q_1 is the number of D1-branes wrapping S^1 and Q_5 is the number of D5-branes wrapping $T^4 \times S^1$. This frame is particularly useful for the microscopic derivation of the degeneracies described in §3.2. With $\widetilde{K} = 1$, the Kaluza-Klein monopole interpolates between $\mathbb{R}^3 \times \widetilde{S}^1$ at asymptotic infinity and $\mathbb{R}^4$ at the center. The momentum $\widetilde{n}$ at infinity becomes angular momentum at the center. This allows for a 4d-5d lift [24, 25] to relate the degeneracies of the four-dimensional state to those of five-dimentional D1-D5 system carrying momentum n and angular momentum $\widetilde{n}$.

- Frame A: In this frame the charge configuration becomes

$$\Gamma = \begin{bmatrix} 0 & q_0 & 0 & -p^1 \\ p^2 & q_2 & p^3 & 0 \end{bmatrix}_A , \tag{3.8}$$

where q_0 is the number of D0-branes, q_2 is the number of D2-branes wrapping $S^1 \times \widetilde{S}^1$, p^1 is a D4-brane wrapping T^4, p^2 is a D4-brane wrapping $\Sigma_{67} \times S^1 \times \widetilde{S}^1$ and p^3 is a D4-brane wrapping $\Sigma_{89} \times S^1 \times \widetilde{S}^1$ where Σ_{ij} is a 2-cycle in T^4 along the directions ij. We will use this frame for localization in §4 and §5.

- Frame B'': In this frame the charge configuration becomes

$$\Gamma = \begin{bmatrix} 0 & n & 0 & Q_5 \\ Q_3 & Q_1 & Q_3 & 0 \end{bmatrix}_{B''} , \tag{3.9}$$

where all D-branes wrap the circle S^1 and an appropriate cycle in the T^4.

We can choose a charge configuration which is even simpler:

$$\Gamma = \begin{bmatrix} 0 & n & 0 & 1 \\ 1 & \nu & 1 & 0 \end{bmatrix} \tag{3.10}$$

where n is a positive integer and ν takes values 0 or 1. The U-duality invariant is

$$\Delta = 4n - \nu^2 . \tag{3.11}$$

It is clear that $\nu = \Delta$ modulo 2, and so these states are completely specified by Δ. The states preserve four of the thirty-two supersymmetries. We will henceforth denote the degeneracies of these one-eighth BPS-states with charges (3.10) by $d(\Delta)$ instead of $d(q, p)$.

We should emphasize that a large class of states with the same value of Δ can be mapped by U-duality to the state (3.10) considered here but that does not exhaust all states. Note that the invariant Δ is the unique quartic invariant of the continous duality group $E_{7,7}(\mathbb{R})$ but in general there are additional arithmetic duality invariants of the arithmetic group $G(\mathbb{Z})$ that cannot be written as invariants of $G(\mathbb{R})$. As a result, not all states with the same value of Δ are related by duality. Classification of arithmetic invariants of $G(\mathbb{Z})$ is a subtle number-theoretic problem. For example, for the $\mathcal{N}=4$ compactification where the duality group $O(22,6;\mathbb{Z}) \times SL(2,\mathbb{Z})$, essentially the only relevant arithmetic invariant is given by $I = \gcd(Q \wedge P)$; and the degeneracies are known for all values of I [26, 27, 28, 29]. To our knowledge a similar complete classification of $E_{7,7}(\mathbb{Z})$ invariants is not known at present. This would be a problem if one wishes to use canonical or a mixed ensemble. For our purposes, since we will working in the microcanonical ensemble, it will suffice to know the degeneracies for the states in the duality orbit of (3.10).

3.2 Microscopic Counting

The degeneracies of the 1/8-BPS dyonic states in the type II string theory on a T^6 are given in terms of the Fourier coefficients of the following counting function [30, 31, 32]:

$$F(\tau, z) = \frac{\vartheta_1^2(\tau, z)}{\eta^6(\tau)} \,. \tag{3.12}$$

where ϑ_1 is the Jacobi theta function and η is the Dedekind function. With $q := e^{2\pi i\tau}$ and $y := e^{2\pi i z}$, they have the product representations

$$\begin{aligned} \vartheta_1(\tau, z) &= q^{\frac{1}{8}}(y^{\frac{1}{2}} - y^{-\frac{1}{2}}) \prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^n)\,, \\ \eta(\tau) &= q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)\;. \end{aligned} \tag{3.13}$$

The derivation of the counting function is simplest in the B frame (3.7) where we have a D1-D5 system in the background of a single Kaluza-Klein monopole. By the 4d-5d lift, the momentum ν can be interpreted as 5d angular momentum. The counting problem essentially reduces to counting bound states in *five* dimensions of a single D1-brane bound to a single D5-brane carrying n units of momentum and ν units of angular momentum. Since the D1-brane can move inside the $D5$ anywhere on the T^4, the moduli space of this motion is T^4. The function F is the generalized elliptic genus of the corresponding superconformal field theory with target space T^4. This is evident from the product representation which can be seen as coming from four bosons and four fermions.

Analysis of the Fourier coefficients of F simplifies enormously by the fact that F is a *weak Jacobi form*. We recall below a few relevant facts about Jacobi forms [33].

1. *Definition:* A Jacobi form of weight k and index m is a holomorphic function $\varphi(\tau, z)$ from $\mathbb{H}\times\mathbb{C}$ to $\mathbb{C}$ which is "modular in τ and elliptic in z" in the sense that it transforms under the modular group as

$$\varphi\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) = (c\tau+d)^k\, e^{\frac{2\pi i m c z^2}{c\tau+d}}\, \varphi(\tau, z) \qquad \forall \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{Z}) \tag{3.14}$$

and under the translations of z by $\mathbb{Z}\tau + \mathbb{Z}$ as

$$\varphi(\tau, z + \lambda\tau + \mu) \;=\; e^{-2\pi i m(\lambda^2\tau + 2\lambda z)}\varphi(\tau, z) \qquad \forall \quad \lambda,\, \mu \in \mathbb{Z}\,, \tag{3.15}$$

where k is an integer and m is a positive integer.

2. *Fourier expansion:* Equations (3.14) include the periodicities $\varphi(\tau+1, z) = \varphi(\tau, z)$ and $\varphi(\tau, z+1) = \varphi(\tau, z)$, so φ has a Fourier expansion

$$\varphi(\tau, z) \;=\; \sum_{n,r} c(n,r)\, q^n\, y^r\,, \qquad\qquad (q := e^{2\pi i\tau},\; y := e^{2\pi i z})\;. \tag{3.16}$$

Equation (3.15) is then equivalent to the periodicity property

$$c(n,r) \;=\; C_r(4nm - r^2)\;, \qquad \text{where } C_r(D) \text{ depends only on } r \bmod 2m\;. \tag{3.17}$$

The function is called a *weak* Jacobi form if it satisfies the condition

$$c(n,r) \;=\; 0 \qquad \text{unless} \qquad n \geq 0\,. \tag{3.18}$$

3. *Theta expansion:* The transformation property (3.15) implies a Fourier expansion of the form

$$\varphi(\tau, z) \;=\; \sum_{\ell\in\mathbb{Z}} q^{\ell^2/4m}\, h_\ell(\tau)\, e^{2\pi i \ell z} \tag{3.19}$$

where $h_\ell(\tau)$ is periodic in ℓ with period $2m$. In terms of the coefficients (3.17) we have

$$h_\ell(\tau) \;=\; \sum_D C_\ell(D)\, q^{D/4m} \qquad\qquad (\ell \,\in\, \mathbb{Z}/2m\mathbb{Z})\;. \tag{3.20}$$

Because of the periodicity property of h_ℓ, equation (3.19) can be rewritten in the form

$$\varphi(\tau, z) = \sum_{\ell\in\mathbb{Z}/2m\mathbb{Z}} h_\ell(\tau)\, \vartheta_{m,\ell}(\tau, z)\,, \tag{3.21}$$

where $\vartheta_{m,\ell}(\tau, z)$ denotes the standard index m theta function

$$\vartheta_{m,\ell}(\tau, z) \;:=\; \sum_{\substack{\lambda\in\mathbb{Z}\\ \lambda = \ell\, (\mathrm{mod}\, 2m)}} q^{\lambda^2/4m}\, y^\lambda \;=\; \sum_{n\in\mathbb{Z}} q^{m(n+\ell/2m)^2}\, y^{\ell+2mn} \tag{3.22}$$

This is the theta expansion of φ. The vector $h := (h_1, \ldots, h_{2m})$ transforms like a modular form of weight $k - \frac{1}{2}$ under $SL(2,\mathbb{Z})$.

With these definitions, $F(\tau, z)$ is a weak Jacobi form of weight -2 and index 1. The indexed degeneracies for a state carrying n units of momentum and r units of angular momentum is then given by $c(n,r)$ in the Fourier expansion (3.16) of F. Using (3.17) for $m = 1$, we see that $c(n,r)$ depend only on $D = 4n - r^2$ and r mod 2 which in this case equals D mod 2. Hence, all information about the Fourier coefficients $c(n,r)$ of F is contained in a single function of D alone which we denote by $C(D)$. Our task is thus reduced to determining $C(D)$ given (3.12).

To read off $C(D)$ more systematically we use the theta expansion

$$F(\tau,z) = h_0(\tau)\,\vartheta_{1,0}(\tau,z) + h_1(\tau)\,\vartheta_{1,1}(\tau,z)\,. \tag{3.23}$$

The functions $h_\ell(\tau)$ in this case are given explicitly by:

$$h_0(\tau) = -\frac{\vartheta_{1,1}(\tau,0)}{\eta^6(\tau)} = -2 - 12q - 56q^2 - 208q^3 \dots \tag{3.24}$$

$$h_1(\tau) = \frac{\vartheta_{1,0}(\tau,0)}{\eta^6(\tau)} = q^{-\frac{1}{4}}\big(1 + 8q + 39q^2 + \dots\big) \tag{3.25}$$

For even and odd D, the coefficients $C(D)$ can be read off from these expansions of h_0 and h_1 respectively using (3.20).

It is clear that D can be identified with the duality invariant Δ in (3.11). The degeneracies are then given in terms of $C(D)$ by

$$d(\Delta) = (-1)^{\Delta+1} C(\Delta)\,. \tag{3.26}$$

The factor of $(-1)^\Delta$ arises because the state in five dimensional spacetime is fermionic for odd Δ and contributes to the index with a minus sign. The overall minus sign arises in relating the 4d degeneracies to the 5d degeneracies using the 4d-5d lift [31, 32].

3.3 Index, Degeneracy, and Fermions

The first few terms in the Fourier expansion of F are given by

$$F(\tau,z) = \frac{(y-1)^2}{y} - 2\,\frac{(y-1)^4}{y^2}\,q + \frac{(y-1)^4(y^2-8y+1)}{y^3}\,q^2 + \cdots\,, \tag{3.27}$$

In Table (1) we tabulate the coefficients $C(\Delta)$ for the first few values of Δ.

Table 1: Some Fourier coefficients

Δ	-1	0	3	4	7	8	11	12	15
$C(\Delta)$	1	-2	8	-12	39	-56	152	-208	513

It is striking that the sign of $C(\Delta)$ is alternating. This implies from (3.26) that the degeneracies $d(\Delta)$ are always positive. This is, in fact, true not only for the first leading coefficients but for all Fourier coefficients, as can be seen from the equations (3.23)–(3.25). Mathematically, the alternating sign of the Fourier coefficients is a somewhat nontrivial property of the specific Jacobi form (3.12) under consideration [34]. Physically, the positivity of $d(\Delta)$ is even more surprising. After all, these are *indexed* degeneracies corresponding to a spacetime helicity supertrace for a complicated bound states of branes. There is no *a priori* microscopic reason why these should be all positive.

Holography gives a simple physical explanation of the positivity [20, 35]. The near-horizon AdS_2 geometry has an $SU(1,1)$ symmetry. If the black hole geometry leaves at least four supersymmetries unbroken, then closure of the supersymmetry algebra requires that the near horizon symmetry must contain the supergroup $SU(1,1|2)$. This implies that that such a supersymmetric horizon must have $SU(2)$ symmetry which can be identified with spatial rotations. If J is a Cartan generator of this $SU(2)$, then for a classical black hole with spherical symmetry, this could mean (depending on the ensemble) that either J is zero or the chemical potential conjugate to J is zero. As explained earlier, the AdS_2 path integral naturally fixes the charges and not the chemical potentials and hence $J = 0$. Together, this implies

$$\mathrm{Tr}(1) = \mathrm{Tr}(-1)^J \, , \tag{3.28}$$

that is, index equals degeneracy and must be positive. For a more detailed discussion see [36].

Note the the index equals degeneracy only for the horizon degrees of freedom, but usually one does not compute the index of the horizon degrees of freedom directly. It is easier to compute the index of the asymptotic states as a spacetime helicity supertrace which receives contribution also from the degrees of freedom external to the horizon. It is crucial that the contribution of these external modes is removed from the helicity supertrace before checking the equality (3.28). Typically, modes localized outside the horizon come from fluctuations of supergravity fields and can carry NS-NS charges such as the momentum but not D-brane charges [37, 38]. In a given frame such as the A frame where all charges come from D-branes, one expects that the Fourier coefficients of $F(\tau, z)$ will give the degeneracies of only the horizon degrees of freedom.

For the Wilson line expectation value (2.5) the equality (3.28) implies that the functional integral with periodic boundary conditions for the fermions must equal the functional integral with antiperiodic boundary conditions. This is possible for the following reason. All fermionic fields have nonzero J and couple to the Kaluza-Klein gauge field coming from the dimensional reduction on the S^2. As discussed above, the microcanonical boundary conditions (2.2) for the functional integral instructs us to integrate over all the fluctuations of the constant mode. By a change of variables in the functional integral, one can change the origin of the constant mode of the gauge field, and therefore the periodic and antiperiodic boundary conditions for the fermionic fields are equivalent.

3.4 Rademacher Expansion

One can make very good estimates of Fourier coefficients of a modular form using an expansion due to Hardy and Ramanujan. The leading term of this expansion gives the Cardy formula. A generalization due to Rademacher [39] in fact gives an exact convergent expansion for these coefficients in terms of the coefficients of the polar terms *i.e.* terms with $D < 0$.

One can apply these methods to the Fourier coefficients of the vector valued modular form $\{h_l\}$ $(l = 0, \dots 2m-1)$ of negative weight $-w$ to obtain [22, 40] a Rademacher expansion for the coefficients $C_\ell(D)$ (3.20)

$$C_\ell(D) = (2\pi)^{2-w} \sum_{c=1}^{\infty} c^{w-2} \sum_{\widetilde{\ell}\in\mathbb{Z}/2m\mathbb{Z}} \sum_{\widetilde{D}<0} C_{\widetilde{\ell}}(\widetilde{D})\, K(D,\ell,\widetilde{D},\widetilde{\ell};c) \left|\frac{\widetilde{D}}{4m}\right|^{1-w} \widetilde{I}_{1-w}\left[\frac{\pi}{c}\sqrt{|\widetilde{D}|D}\right],$$

where

$$\widetilde{I}_\rho(z) = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\sigma}{\sigma^{\rho+1}} \exp[\sigma + \frac{z^2}{4\sigma}] \tag{3.29}$$

is called the modified Bessel function of index ρ. This is related to the standard Bessel function of the first kind $I_\rho(z)$ by

$$\widetilde{I}_\rho(z) = \big(\frac{z}{2}\big)^{-\rho} I_\rho(z)\,. \tag{3.30}$$

The sum over $(\widetilde{\ell},\,\widetilde{D})$ picks up a contribution $C_{\widetilde{\ell}}(\widetilde{D})$ from every non-zero term $q^{\widetilde{D}}$ with $\widetilde{D}<0$ in $h_{\widetilde{\ell}}(\tau)$ (3.20). The coefficients $K\ell(D,\ell,\widetilde{D},\widetilde{\ell};c)$ are generalized Kloosterman sums. For $c>1$ it is defined as

$$K(D,\ell;\widetilde{D},\widetilde{\ell};c) := e^{-\pi i w/2} \sum_{\substack{-c\le d<0\\(d,c)=1}} e^{2\pi i \frac{d}{c}(D/4m)}\ M(\gamma_{c,d})^{-1}_{\ell\widetilde{\ell}}\ e^{2\pi i\frac{a}{c}(\widetilde{D}/4m)}\,, \tag{3.31}$$

where

$$\gamma_{c,d} = \begin{pmatrix} a & (ad-1)/c \\ c & d\end{pmatrix} \tag{3.32}$$

is an element of $Sl(2,\mathbb{Z})$ and $M(\gamma)$ is the matrix representation of γ on the vector space spanned by the $\{h_l\}$. Note that it follows from (3.32) that $ad = 1 \bmod c$.

The Jacobi form $F(\tau,z)$ has weight -2 and index $m=1$, so its theta expansion gives a two-component vector $\{h_0,h_1\}$ of modular forms of weight $w=-5/2$. Since there is only a single polar term $(\widetilde{\ell}=1,\widetilde{D}=-1)$, the Rademacher expansion takes the form:

$$C(D) = 2\pi\,\big(\frac{\pi}{2}\big)^{7/2} \sum_{c=1}^{\infty} c^{-9/2}\, K_c(D)\, \widetilde{I}_{7/2}\big(\frac{\pi\sqrt{D}}{c}\big)\,, \tag{3.33}$$

where the Kloosterman sum $K_c(D)$ is defined by

$$K_c(D) := e^{5\pi i/4} \sum_{\substack{-c\le d<0;\\(d,c)=1}} e^{2\pi i\frac{d}{c}(D/4)}\ M(\gamma_{c,d})^{-1}_{\ell 1}\ e^{2\pi i \frac{a}{c}(-1/4)} \tag{3.34}$$

with $\ell = D \bmod 2$ and $ad = 1 \bmod c$.

Under the $SL(2,\mathbb{Z})$ generators, the modular form $h_\ell(\tau)$ transform as

$$h_0(\tau+1) = h_0(\tau)\,, \qquad h_0(-1/\tau) = \frac{1+i}{2}\,\tau^{-5/2}\big(h_0(\tau)+h_1(\tau)\big)\,; \tag{3.35}$$

$$h_1(\tau+1) = -i\,h_1(\tau)\,, \qquad h_1(-1/\tau) = \frac{1+i}{2}\,\tau^{-5/2}\big(h_0(\tau)-h_1(\tau)\big)\,. \tag{3.36}$$

From these transformations, we can read off the matrices $M(\gamma)$ for the generators S and T

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} , \qquad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.37}$$

to be

$$M(T) = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} , \qquad M(S) = \frac{e^{\pi i/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} . \tag{3.38}$$

Using the expression for a general $SL(2,\mathbb{Z})$ matrix γ in terms of the generators S and T, and the representation (3.38), we can obtain the representation $M(\gamma)$.

We see from (3.33) that the microscopic degeneracy is an infinite sum of the form

$$d(\Delta) = \sum_{c=1}^{\infty} d_c(\Delta) . \tag{3.39}$$

where each term is given by

$$d_c(\Delta) = (-1)^{\Delta+1}\, 2\pi \Big(\frac{\pi}{\Delta}\Big)^{7/2} I_{\frac{7}{2}}\Big(\frac{\pi\sqrt{\Delta}}{c}\Big) \frac{1}{c^{9/2}} K_c(\Delta) . \tag{3.40}$$

It is easy to check that

$$K_1 = (-1)^{\Delta+1} \frac{1}{\sqrt{2}} . \tag{3.41}$$

We will see that the Wilson line from the macroscopic side also naturally has the same expansion

$$W(\Delta) = \sum_{c=1}^{\infty} W_c(\Delta) , \tag{3.42}$$

coming from $\mathbb{Z}_c$ orbifolds of AdS_2. Our objective then is to compute each of these terms exactly using localization. We compute the leading term $W_1(\Delta)$ in §5 and the subleading terms corresponding to $c > 1$ in §5.4.

4. Localization of Functional Integral in Supergravity

Evaluating the formal functional integral (2.5) over string fields for $W(q,p)$ is of course highly nontrivial. To proceed further, we first integrate out the infinite tower of massive string modes and massive Kaluza-Klein modes to obtain a *local* Wilsonian effective action for the massless supergravity fields keeping all higher derivative terms. We can regard the ultraviolet finite string theory as providing a supersymmetric and consistent cutoff at the string scale. Our task is then reduced to evaluating a functional integral in supergravity. The near horizon geometry preserves eight superconformal symmetries and the action, measure, operator insertion, boundary conditions of the functional integral (2.5) are all supersymmetric[3]. The formal supersymmetry of the functional integral makes it possible to apply localization techniques [13, 41] to evaluate it.

To apply localization to our system, we drop two gravitini multiplets to obtain a $\mathcal{N} = 2$ theory and also drop the hypermultiplets to consider a reduced theory. This theory contains a supergravity multiplet coupled to eight vector multiplets with a duality group

$$SO(6, 2; \mathbb{Z}) \times SL(2, \mathbb{Z})\,. \tag{4.1}$$

In the effective action for these fields we will further ignore the D-type terms. For a partial justification for this reduction in this context and for further discussion see §6 and [13]. We will denote the functional integral (2.5) restricted to this reduced theory by $\widehat{W}(q, p)$ which is what we compute in the subsequent sections. We find that $\widehat{W}(q, p)$ itself agrees perfectly with (3.33) for $d(q, p)$. This rather nontrivial agreement can be regarded as post-facto evidence that the reduced theory correctly captures the relevant physics.

4.1 Functional Integral in $\mathcal{N} = 2$ Off-shell Supergravity

Localization of the supergravity functional integral is considerably simplified in the off-shell formalism. The main advantage of the off-shell formalism is that the supersymmetry transformations are specified once and for all, and do not need to be modified as one modifies the action with higher derivative terms. Consequently, the localizing instantons that we describe below do not depend upon the form of the physical action. The problem of finding the model-independent localizing instantons is then cleanly separated from the problem of evaluating the renormalized action for a specific physical action.

In the off-shell formalism for $\mathcal{N} = 2$ supergravity developed in [42, 43, 44] the vielbein and its superpartners reside in the Weyl multiplet. In addition, we consider $n_v + 1$ vector multiplets with the field content

$$\mathbf{X}^I = \left(X^I, \Omega_i^I, A_\mu^I, Y_{ij}^I\right)\,, \quad I = 0, \ldots, n_v\,. \tag{4.2}$$

For each I, the multiplet contains eight bosonic and eight fermionic degrees of freedom: X^I is a complex scalar, the gaugini Ω_i^I are an $SU(2)$ doublet of chiral fermions, A_μ^I is a vector field, and Y_{ij}^I are an $SU(2)$ triplet of auxiliary scalars. The auxiliary fields Y_{ij}^I play a very important role in localization.

Localization is a general technique for evaluating superintegrals of the form

$$I = \int_{\mathcal{M}} d\mu\, h\, e^{-\mathcal{S}}\,. \tag{4.3}$$

Here $\mathcal{M}$ is the supermanifold with integration measure $d\mu$, which has an odd (fermionic) vector field Q which squares to a compact bosonic vector field H; h, S, and the measure are all

[3] Supersymmetry of the Wilson line and the action is discussed in the appendix of [13].

invariant under Q. To evaluate this integral one first deforms it to

$$I(\lambda) = \int_{\mathcal{M}} d\mu\, h\, e^{-\mathcal{S}-\lambda QV} \,, \tag{4.4}$$

where V is a fermionic, H-invariant function which means $Q^2V = 0$ and QV is Q-exact. One has

$$\frac{d}{d\lambda}\int_{\mathcal{M}} d\mu\, h\, e^{-\mathcal{S}-\lambda QV} = \int_{\mathcal{M}} d\mu\, h\, QV\, e^{-\mathcal{S}-\lambda QV} = \int_{\mathcal{M}} d\mu\, Q(h\, e^{-\mathcal{S}-\lambda QV}) = 0 \,, \tag{4.5}$$

and hence $I(\lambda)$ is independent of λ. This implies that one can perform the integral $I(\lambda)$ for any value of λ and in particular for $\lambda \to \infty$. In this limit, the functional integral localizes onto the critical points of the functional $S^Q := QV$ which we refer to as the localizing instanton solutions. One can choose in particular,

$$V = (Q\Psi, \Psi) \tag{4.6}$$

where Ψ are the fermionic coordinates with some positive definite inner product defined on the fermions. In this case, the bosonic part of S^Q can be written as a perfect square $(Q\Psi, Q\Psi)$, and hence critical points of S^Q are the same as the zeros of Q. Let us denote the set of zeros of Q by $\mathcal{M}_Q$. The reasoning above shows that the integral over the supermanifold $\mathcal{M}$ localizes to an integral over the submanifold $\mathcal{M}_Q$. In the large λ limit, the integration for directions transverse can be performed exactly in the saddle point evaluation. One is then left with an integral over the submanifold $\mathcal{M}_Q$ with a measure $d\mu_Q$ induced on the submanifold.

In our case, $\mathcal{M}$ is the field space of off-shell supergravity, $\mathcal{S}$ is the off-shell supergravity action with appropriate boundary terms, h is the supersymmetric Wilson line. To localize, we will choose the fermionic symmetry generated by the supersymmetry generator Q which squares to $4(L - J)$, where L is the generator of rotations of the Poincaré disk and J is the generator of rotations of S^2. With this choice for Q, the localizing Lagrangian is then defined by

$$\mathcal{L}^Q := QV \quad \text{with} \quad V := (Q\Psi, \Psi) \,, \tag{4.7}$$

where Ψ refers to all fermions in the theory. The localizing action is then defined by

$$\mathcal{S}^Q = \int d^4x \sqrt{g}\, \mathcal{L}^Q \,. \tag{4.8}$$

The localization equations that follow from this action are

$$Q\Psi = 0 \,. \tag{4.9}$$

These are the equations that we needs to solve subject to the AdS_2 boundary conditions. The scalar fields are fixed to their attractor values

$$X^I_* = \frac{1}{2}(e^I_* + ip^I) \tag{4.10}$$

where e^I_* are the attractor value of the electric fields determined in terms of the charges (q, p).

The solution to this system of differential equations subject to the AdS_2 boundary condition turns out to be surprisingly simple and can be given in a closed form [13]. The most general solution parametrized by (n_v+1) real parameters $\{C^I\},\ I=1,\dots,n_v+1$, and is given by the field configurations

$$X^I = X^I_* + \frac{C^I}{r}\ , \qquad \overline{X}^I = \overline{X}^I_* + \frac{C^I}{r}\ , \qquad Y_1^{1I} = -Y_2^{2I} = \frac{2C^I}{r^2}\ , \tag{4.11}$$

with other fields fixed to their attractor values[4]. The real parameters $\{C^I\}$ can be thought of as the collective coordinates of the localizing instantons. The functional integral of supergravity thus localizes onto a finite number of ordinary bosonic integrals over $\{C^I\}$ which enormously simplifies the evaluation of the Wilson line [13]. So far we have not assumed any particular form of the physical action. As emphasized in [13], these localizing instanton solutions are *universal* in that they follow simply from the off-shell supersymmetry transformation laws of the vector multiplet fermions and hence are independent of the physical action.

When the action contains only F-type terms, it is governed by a single prepotential $F(X^I, \hat{A})$ which is a meromorphic function of its arguments and obeys the homogeneity condition:

$$F(\lambda X, \lambda^2\hat{A}) = \lambda^2 F(X, \hat{A})\,. \tag{4.12}$$

where $\hat{A}$ is an auxiliary field from the supergravity multiplet. Terms depending on $\hat{A}$ lead to higher derivative terms in the action [45].

To obtain the integrand over this localizing integral, one must substitute the solution (4.11) into the physical action and extract the finite part as a function of the collective coordinates $\{C^I\}$ following the prescription in §2. One obtains [13] a remarkably simple expression for the the renormalized action for the localizing instantons as a function of the collective coordinates $\{C^I\}$:

$$\mathcal{S}_{\text{ren}} = -\pi q_I e^I_* - 2\pi q_I C^I - 2\pi i\big(F(X^I_* + C^I) - \overline{F}(X^I_* + C^I)\big)\ . \tag{4.13}$$

Using the scalar attractor values (4.10) and the new variable

$$\phi^I := e^I_* + 2C^I\ , \tag{4.14}$$

we can express the renormalized action as

$$\mathcal{S}_{ren}(\phi, q, p) = -\pi q_I \phi^I + \mathcal{F}(\phi, p)\,. \tag{4.15}$$

with

$$\mathcal{F}(\phi, p) = -2\pi i\left[F\Big(\frac{\phi^I + ip^I}{2}\Big) - \overline{F}\Big(\frac{\phi^I - ip^I}{2}\Big)\right]\ . \tag{4.16}$$

[4]It was shown in [13] that the gauge fields in the vector multiplets are not excited for the localizing solutions. A similar analysis remains to be done to show that there are no other more general localizing solutions exciting fields in the supergravity multiplet. In what follows we will assume this to be true.

Written this way, note that the prepotential is evaluated precisely for values of the scalar fields at the origin of the AdS_2 and not at the boundary of the AdS_2. At the boundary, the fields remain pinned to their attractor values and in particular the electric field remains fixed as required by the microcanonical boundary conditions of the functional integral. The collective coordinates ϕ^I in (4.14) still fluctuate because C^I take values over the real line.

The renormalized action $S_{ren}(\phi)$ has the same functional form as the classical entropy function. In particular, its extrema $\phi = \phi_*$ correspond to the attractor values of the scalar fields and its value at the extremum $S_{ren}(\phi^*)$ equals the Wald entropy for the local Lagrangian described with a prepotential $\mathcal{F}$. However, the physics behind the renormalized action is completely different. Unlike the classical entropy function which is essentially a classical on-shell object, the renormalized action is a quantum object obtained after a complicated holographic renormalization procedure using an off-shell localizing field configuration (4.11). Even though the scalar fields in the localizing solution asymptote to the attractor values at the boundary of the AdS_2, they have a nontrivial coordinate dependence in the bulk and they take the value $X^I_* + C^I$ at the center of AdS_2. In particular, they are excited away from their attractor values and are no longer at the minimum of S_{ren}. Even though the scalar fields thus 'climb up the potential' away from the minimum of the entropy function, the localizing solution remains Q-supersymmetric (in the Euclidean theory) because the auxiliary fields Y^I_{ij} get excited appropriately to satisfy the Killing spinor equations. This is what enables us to integrate over ϕ for values in field space far away from the on-shell values.

The infinite dimensional functional integral (2.5) for the Wilson line in the reduced theory can thus be written as a finite integral

$$\widehat{W}(q,p) = \int_{\mathcal{M}_Q} e^{-\pi\phi^I q_I}\, e^{\mathcal{F}(\phi,p)}\, |Z_{inst}|^2\, Z_{det}\, [d\phi]_\mu \tag{4.17}$$

The measure of integration $[d\phi]_\mu$ is computable from the original measure μ of the functional integral of massless fields of string theory by standard collective coordinate methods. The factor Z_{det} is the one-loop determinant of the quadratic fluctuation operator around the localizing instanton solution. Such one-loop determinant factors in closely related problems have been computed in [46, 47]. We have included $|Z_{inst}|^2$ to include possible contributions from brane instantons which is partially captured by the topological string for a class of branes.

Note that the exponential of the integrand is in the spirit of the conjecture by Ooguri, Strominger, and Vafa [48]. Our treatment differs from [48] in that the natural ensemble in our analysis is the microcanonical one. Moreover, we will be able determine the measure factor from first principles and the determine the subleading orbifolded localizing instantons that contribute to the functional integral. For earlier related work see [49, 50].

To compute $\widehat{W}(q,p)$, it is necessary to evaluate all these factors explicitly and then perform the finite dimensional integral over ϕ. This is what we will do for our system in §5. For the $\mathcal{N}=2$ reduction of the $\mathcal{N}=8$ theory that we consider, $n_v = 7$ and the prepotential is given by

$$F(X) = -\frac{1}{2}\frac{X^1 C_{ab} X^a X^b}{X^0}\,, \qquad a,b = 2,\ldots,7\,. \tag{4.18}$$

where C_{ab} is the intersection matrix of the six 2-cycles of T^4. In our normalization, it is given by

$$C_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{1}_{3\times 3} \tag{4.19}$$

where $\mathbf{1}_{3\times 3}$ is a 3×3 identity matrix. This prepotential describes the classical two-derivative supergravity action. Note that this does not depend the field $\hat{A}$ because there are no higher-derivative quantum corrections to the prepotential.

4.2 Integration Measure

The measure $[d\phi]_\mu$ is inherited from the standard measure on field space in the original functional integral. The collective coordinates $\{\phi^I\}$ of the localizing instanton solutions correspond to the values of the scalar fields $\{X^I\}$ at the center of the AdS_2. The functional integration measure for the scalar fields is a pointwise product of integration measure over the scalar manifold. The metric and hence the measure on the scalar manifold can be read off from the kinetic term of the scalar fields [45, 51]. The scalar kinetic action is

$$8\pi\mathcal{L} = \sqrt{|g|}g^{\mu\nu}\left[i(\partial_\mu F_I + i\mathcal{A}_\mu F_I)(\partial^\mu \overline{X}^I - i\mathcal{A}^\mu\overline{X}^I) + h.c.\right] , \tag{4.20}$$

where $\mathcal{A}_\mu$ is the gauge field for the $U(1)$ gauge symmetry of the off-shell supergravity theory. This field does not have a kinetic term and it is therefore determined by its equation of motion to be

$$\mathcal{A}^*_\mu = \frac{1}{2}\frac{\overline{F}_I\vec{\partial}_\mu X^I - \overline{X}^I\vec{\partial}_\mu F_I}{-i(\overline{F}_I X^I - F_I\overline{X}^I)} . \tag{4.21}$$

The Lagrangian $8\pi\mathcal{L}^*$ computed by substituting $\mathcal{A}^*_\mu$ in (4.20) becomes

$$-\sqrt{|g|}g^{\mu\nu}\left[N_{IJ}\partial_\mu X^I\partial_\nu\overline{X}^J - \frac{e^{-K}}{4}(K_I\partial_\mu X^I - \overline{K}_I\partial_\mu\overline{X}^I)(K_I\partial_\nu X^I - \overline{K}_I\partial_\nu\overline{X}^I)\right] , \tag{4.22}$$

with

$$N_{IJ} := -i(F_{IJ} - \overline{F}_{IJ}) = 2\,\mathrm{Im}(F_{IJ}) , \tag{4.23}$$
$$e^{-K} := -i(X^I\overline{F}_I - \overline{X}^I F_I) , \tag{4.24}$$
$$K_I := \frac{\partial K}{\partial X^I} = ie^K\left(\overline{F}_I - F_{IJ}\overline{X}^J\right) . \tag{4.25}$$

The metric $g_{\mu\nu}$ is not the physical metric of Poincaré supergravity because it does not come with the canonical kinetic term. It is related to the dilatation-invariant physical metric G as

$$G_{\mu\nu} = e^{-K}g_{\mu\nu} , \tag{4.26}$$

whose kinetic term is given by the standard Einstein-Hilbert action. We have

$$\sqrt{|g|}g^{\mu\nu} = e^K\sqrt{|G|}G^{\mu\nu} . \tag{4.27}$$

It is natural to define the scalar functional integral measure using the physical metric $G_{\mu\nu}$. The measure can be determined by the metric induced by the inner product in field space:

$$(\delta X, \delta X) = \int d^4x \sqrt{|G|}\, \delta X\, \delta X\,. \tag{4.28}$$

Substituting $X^I = (\phi^I + ip^I)/2$ in (4.22), and using (4.26), (4.27), we obtain the induced metric on the localizing submanifold in the field space

$$d\Sigma^2 = M_{IJ}\, \delta\phi^I \delta\phi^J\,, \tag{4.29}$$

with

$$M_{IJ} = e^K \left[N_{IJ} - \frac{e^K}{4}(K_I - \overline{K}_I)(K_J - \overline{K}_J) \right]. \tag{4.30}$$

It is possible to write the metric on the localizing manifold entirely in terms of the Kähler potential[5] K (4.24). It is easy to check that

$$N_{IJ} = -\frac{\partial^2 e^{-K}}{\partial X^I \overline{X}^J} = e^{-K} \left(\frac{\partial^2 K}{\partial X^I \partial \overline{X}^J} - \frac{\partial K}{\partial X^I} \frac{\partial K}{\partial \overline{X}^J} \right). \tag{4.31}$$

Defining the metric K_{IJ} in terms of the Kähler potential in the usual way

$$K_{IJ} := \frac{\partial^2 K}{\partial X^I \partial \overline{X}^J}\,, \tag{4.32}$$

and using (4.31), we can write the Lagrangian (4.22) entirely in terms of the Kähler potential:

$$8\pi \mathcal{L} = -\sqrt{|g|} g^{\mu\nu} e^{-K} \left[K_{IJ} \partial_\mu X^I \partial_\nu \overline{X}^J - \frac{1}{4} \partial_\mu K \partial_\nu K \right]. \tag{4.33}$$

Substituting $X^I = (\phi^I + ip^I)/2$ in (4.33), we can rewrite the moduli space metric (4.29) as

$$M_{IJ} = K_{IJ} - \frac{1}{4} \frac{\partial K}{\partial \phi^I} \frac{\partial K}{\partial \phi^J}\,. \tag{4.34}$$

Since the metric K_{IJ} is given in terms of the Kähler potential (4.32), this expresses the moduli space metric M_{IJ} entirely in terms of the Kähler potential. The measure on the localizing manifold is simply the measure induced by this metric and is given by

$$\prod_{I=0}^{n_v} d\phi^I \sqrt{\det(M)}\,. \tag{4.35}$$

[5]Upon gauge-fixing, on the space of projective coordinates, K becomes the Kähler potential. We will refer to K as the Kähler potential even though we do not fix any gauge here.

5. Macroscopic Quantum Partition Function

The two-derivative action of $\mathcal{N} = 8$ is invariant under the continuous duality group $E_{7,7}(\mathbb{R})$. We therefore expect to be able to write the macroscopic answer in terms of Δ which is the unique quartic invariant of $E_{7,7}(\mathbb{R})$. For this purpose, we will first write the renormalized action in new variables so that it depends only on the invariant Δ and then work out the measure in the same variables to obtain a manifestly duality invariant expression for the Wilson line.

5.1 Renormalized Action and Duality-invariant Variables

As discussed in §3.1 the electric and magnetic charge vectors Q and P respectively are related to the charges in the Type-IIA frame (3.8) by

$$Q = (q_0, -p^1; q_a) \qquad P = (q_1, p^0; p^a) \quad . \tag{5.1}$$

The inner product is defined for example by

$$P \cdot P = 2\, q^1 p^0 + p^a\, C_{ab}\, p^b \,, \tag{5.2}$$

The charge configuration (3.10) has only five nonzero charges $q_0 = n$, $q_1 = l$, $p^1 = -w$, and p^2, p^3. Hence, the three T-dualiy invariants all have nonzero values given by

$$Q^2 = 2\, nw \,, \quad P^2 = 2\, p^2 p^3 \,, \quad Q \cdot P = w\, l \,. \tag{5.3}$$

The natural variables to start with are the projective coordinates

$$S := X^1/X^0 \,, \qquad T^a := X^a/X^0 \quad a = 2, \ldots, n_v \,, \tag{5.4}$$

with real and imaginary parts defined by

$$S := a + is \,, \quad T^a := t^a + ir^a \,. \tag{5.5}$$

For our localizing instanton solutions we obtain

$$a = \phi^1/\phi^0 \,, \qquad s = -w/\phi^0 \tag{5.6}$$

$$t^a = \phi^a/\phi^0 \,, \qquad r^a = p^a/\phi^0 \,. \tag{5.7}$$

The renormalized action (4.13) for this charge configuration and prepotential (4.18) is

$$S_{ren} = -\frac{\pi}{2\phi^0} \left[-w(\phi^2 - P^2) + 2\, \phi^1 (\phi \cdot P) \right] - \pi n \phi^0 - \pi l \phi^1 \,, \tag{5.8}$$

where $\phi^2 = \phi^a\, C_{ab}\, \phi^b$ and $\phi \cdot P = \phi^a\, C_{ab}\, P^b$. Using the parametrization (5.4) and (5.5) and the T-duality invariants (5.3) it can be written as

$$S_{ren} = \frac{\pi}{2} \left[P^2 s + \frac{Q^2}{s} + \frac{2\, Q \cdot P\, a}{s} \right] - \frac{\pi w^2\, t^2}{2s} + \frac{\pi a w\, t \cdot P}{s} \quad . \tag{5.9}$$

Our next goal will be to define integration variables to write the action entirely in terms of the U-duality invariant Δ. Since the action is quadratic in the t^a variables, it is useful to complete the squares by defining

$$\tau^a = \frac{w}{\sqrt{s}} \left(t^a - \frac{a\, p^a}{w} \right) \tag{5.10}$$

so that

$$S_{ren} = \frac{\pi}{2} \left[P^2 s + \frac{Q^2}{s} + \frac{P^2\, a^2}{s} + \frac{2\, Q \cdot P\, a}{s} \right] - \frac{\pi\, \tau^2}{2} \, . \tag{5.11}$$

Note that the parenthesis is a manifestly S-duality invariant combination which is quadratic in the axion variable a. So we complete the square again by defining

$$\sigma = \frac{\pi P^2 s}{2} \, , \qquad \alpha = \frac{1}{\sqrt{\sigma}} \left(P^2 a + Q \cdot P \right) \tag{5.12}$$

The renormalized action then becomes

$$S_{ren} = \left(\sigma + \frac{z^2}{4\sigma} \right) - \frac{\pi\, \tau^2}{2} + \frac{\pi\, \alpha^2}{2} \, . \tag{5.13}$$

with

$$z^2 = \pi^2 (Q^2 P^2 - (Q.P)^2) \; = \pi^2 \Delta \, . \tag{5.14}$$

The variables (σ, α, τ^a) can be regarded as the duality invariant variables.

5.2 Conformal compensator, Gauge-fixing, and Analytic Continuation

The constants C^I which characterize the localizing instanton solution (4.11) are all real. Hence, the contour of integration for the variables s and t would appear to be along the real axis. The quadratic terms in t in the action (5.11) would lead to divergent Gaussian integrals. We will see below that this is nothing but the divergence of Euclidean quantum gravity arising from the integration over the conformal factor that has a wrong sign kinetic term.

We recall that the scalar kinetic term (4.33) can be written as

$$-\sqrt{-g} g^{\mu\nu} \left[e^{-K} K_{IJ} \partial_\mu X^I \partial_\nu \overline{X}^J - \frac{1}{4} e^{-K} \partial_\mu K \partial_\nu K \right] \, . \tag{5.15}$$

The kinetic term for the spacetime metric $g_{\mu\nu}$ is of the form[6]

$$-\frac{1}{6} \sqrt{-g} e^{-K} R_g \, , \tag{5.16}$$

[6]We suppress an overall factor of $1/8\pi$ that is irrelevant for the discussion here but is important for the normalization of the renormalized action in §5.

We can thus identify $e^{-K/2}$ as a conformal compensator Ω which is often used to extend the gauge principle to include scale invariance in addition to diffeomorphism invariance. The Einstein-Hilbert action is then replaced by

$$\sqrt{-g}\left[-\frac{1}{6}\Omega^2\, R_g - g^{\mu\nu}\,\partial_\mu\Omega\,\partial_\nu\Omega\right] , \tag{5.17}$$

which is now invariant under both diffeomorphisms and Weyl rescalings. As can be seen from (5.15), the kinetic term for Ω has a wrong sign compared to a physical scalar, as is usual for the conformal compensator field. In D-gauge [45] Ω is gauge-fixed to a constant and one recovers the Einstein-Hilbert action. Our localizing solution is however in a different gauge in which the metric g is gauge-fixed so that AdS_2 has fixed volume and hence Ω is effectively a fluctuating field. This also explains why we have $n_v + 1$ scalar moduli $\{\phi^I\}$ even though there are only n_v physical scalars. Essentially, our choice of gauge enables us to borrow the conformal factor Ω as an additional scalar degree of freedom. The advantage is that the symplectic symmetry acts linearly on the fields $\{\phi^I\}$.

Since the kinetic term for conformal compensator Ω has a wrong sign, to make the Euclidean functional integral well defined, it is necessary to analytically continue the contour of integration in field space [52]. For our prepotential (4.18), the Kähler potential is given by

$$\exp[-K] = 4\,|X^0|^2\,\mathrm{Im}(S)\,C_{ab}\,\mathrm{Im}(T^a)\,\mathrm{Im}(T^b)\,. \tag{5.18}$$

For S and T^a fixed, we see that Ω is proportional to X^0 up to a phase that can gauge-fixed by using the additional $U(1)$ gauge symmetry. Thus, the analytic continuation in the Ω space can be achieved by analytically continuing in the X^0 space. For the localizing solution, $X^0 = \phi^0$. Thus, analytic continuation in Ω space can be achieved by analytically continuing in the ϕ^0 space. Correspondingly, we take the contour of integration of ϕ^0 or equivalently of σ along the imaginary axis rather than along the real axis[7].

A familar example of such analytic continuation is the functional integral for the worldsheet metric in first-quantized string theory. The conformal factor of the metric is the Liouville mode which can be thought of as a conformal compensator. Critical bosonic string with $c = 26$ can be regarded as a noncritical string theory with $c = 25$ coupled to this Liouville mode. The Liouville mode plays the role of time coordinate in target space [54] and has a wrong-sign kinetic term on the worldsheet. The corresponding functional integral then has to be defined by a similar analytic continuation [55].

5.3 Evaluation of the Localized Integral

The localizing action QV with abelian gauge fields is purely quadratic. Hence, the quadratic fluctuation operator around the localizing instantons does not depend on the collective coordinates $\{C^I\}$. As a result, Z_{det} is independent of $\{\phi^I\}$ and charges can be absorbed in the

[7]In general there can subtleties in such analytic continuation, see for example [53]. These will not be important in the present context.

overall normalization constant. Another simplification for the $\mathcal{N}=8$ theory is that $|Z_{inst}|^2=1$ because the classical prepotential (4.18) that we have used is quantum exact.

Thus, all that remains is to compute the determinant of the matrix M_{IJ} introduced in (4.30). Since there are no terms that depend on $\hat{A}$ for our prepotential, it is homogenous of degree 2 in the variables X. As a result, $F_{IJ}X^J = F_I$, and it follows from (4.25) that

$$K_I = e^K N_{IJ}\overline{X}^J\,, \qquad \overline{K}_I = e^K N_{IJ} X^J\,. \tag{5.19}$$

This allows us to write (4.30) as

$$M_{IJ} = e^K\left(N_{IJ} + \frac{1}{4}e^K N_{IK}\,p^K N_{JL}\,p^L\right)\,. \tag{5.20}$$

We have

$$\det(M) = \exp\left[\frac{(n_v+1)}{2}K\right]\det(N)\det(1+\Lambda)\,, \tag{5.21}$$

where the matrix Λ is defined by

$$\Lambda^I_J = \frac{1}{4}e^K\,p^I N_{JL}\,p^L\,. \tag{5.22}$$

Some elements of this measure such as the matrix N_{IJ} were anticipated in the work of [56, 57, 58] based on considerations of sympletic invariance. Our derivation follows from the analysis of the induced metric on the localizing manifold and has additional terms depending on K_I and $\exp(K)$ which are also sympletic invariant. Unlike in the $\mathcal{N}=4$ theory, in the $\mathcal{N}=8$ theory the higher-derivative corrections are zero, and do not provide a useful guide for the determination of nonholomorphic terms of the measure such as the powers of $\exp(K)$.

It is easy to see that for our system $\mathrm{Tr}(\Lambda^n)=\lambda^n$ where λ is a numerical constant independent of charges. As a result,

$$\det(1+\Lambda) = \exp(\mathrm{Tr}\log(1+\Lambda)) = \exp(\log(1+\lambda)) \tag{5.23}$$

is a field-independent and charge-independent numerical constant. In what follows, we will ignore all such numerical constants in the evaluation of the measure and determine the overall normalization of the functional integral in the end.

Hence, up to a constant, $\det(M)$ is determined by $\det(N)$ and $\exp(K)$. For our prepotential, evaluating on the localizing instanton solution we obtain

$$\exp[-K] = 4\,P^2 s = 8\sigma/\pi \tag{5.24}$$

which is manifestly duality invariant. Similarly,

$$\det(N) = \frac{s^{n_v-3}\det(C_{ab})}{4|X^0|^4}e^{-2K} = s^{n_v+3}\,\Big(\frac{P^2}{w^2}\Big)^2 \tag{5.25}$$

as can be checked using Mathematica. In terms of the duality invariant variables defined earlier, we see that the measure is given by

$$\prod_{I=0}^{n_v} d\phi^a \sqrt{\det(N)} = \frac{1}{\sqrt{\sigma}}\, d\sigma\, d\alpha \prod_{2}^{n_v} d\tau^a \tag{5.26}$$

up to an overall constant that is independent of charges and fields. The total measure is thus given by

$$\prod_{I=0}^{n_v} d\phi^I \sqrt{\det(M)} = \frac{d\sigma}{\sigma^{\rho+1}} d\alpha \prod_{2}^{n_v} d\tau^a \tag{5.27}$$

with $\rho = n_v/2$. Our total integral is hence manifestly duality invariant.

Performing the Gaussian integrals over α and τ^a we obtain

$$\int \frac{d\sigma}{\sigma^{\rho+1}} \exp\left(\sigma + \frac{z^2}{4\sigma}\right) \tag{5.28}$$

which gives exactly the integral representation of the Bessel function $\widetilde{I}_{7/2}(z)$ for $n_v = 7$. The overall numerical normalization needs to be fixed by hand but once it is fixed for one value of Δ, one obtains a nontrivial a function for all other values of Δ given by

$$W_1(\Delta) = \sqrt{2}\,\pi \left(\frac{\pi}{\Delta}\right)^{7/2} I_{7/2}(\pi\sqrt{\Delta})\,. \tag{5.29}$$

This macroscopic calculation thus precisely reproduces the first term with $c = 1$ in (3.42) and matches beautifully with the first term in (3.39) from the Rademacher expansion (3.33) for of the microscopic degeneracy $d(\Delta)$.

For large z, the Bessel function has an expansion

$$I_\rho(z) \sim \frac{e^z}{\sqrt{2\pi z}}\left[1 - \frac{(\mu-1)}{8z} + \frac{(\mu-1)(\mu-3^2)}{2!(8z)^3} - \frac{(\mu-1)(\mu-3^2)(\mu-5^2)}{3!(8z)^5} + \ldots\right], \tag{5.30}$$

with $\mu = 4\rho^2$. The exponential term $\exp(\pi\sqrt{\Delta})$ gives the Cardy formula and $\pi\sqrt{\Delta}$ can be identified with the Wald entropy of the black hole. Higher terms in the series give power-law suppressed finite size corrections to the Wald entropy. This is however not a convergent expansion but only an asymptotic expansion. This means that for any given z only the first few terms are useful for making an accurate estimate. Beyond a certain number of terms that depends on a positive power of z, including more terms actually makes the estimate worse rather than improve it. For larger and larger z one can include more or more terms to improve the approximation but this is never convergent for a fixed z.

It should be emphasized that our computation of $W_1(\Delta)$ gives an exact integral representation (5.28) of the Bessel function $I_{7/2}(z)$ and not merely the asymptotic expansion (5.30). This is made possible because localization gives an exact evaluation of the functional integrals and allows one to access large regions in the field space far away from the classical saddle point of the entropy function used to derive the Cardy formula.

Table 2: Comparison of the microscopic degeneracy $d(\Delta)$ with the functional integral $W_1(\Delta)$ and the exponential of the Wald entropy. The last three rows in the table equal each other asymptotically.

Δ	-1	0	3	4	7	8	11	12	15
$d(\Delta)$	1	2	8	12	39	56	152	208	513
$W_1(\Delta)$	1.040	1.855	7.972	12.201	38.986	55.721	152.041	208.455	512.958
$\exp(\pi\sqrt{\Delta})$	-	1	230.765	535.492	4071.93	7228.35	33506	53252	192401

It is instructive to compare the integers $d(\Delta)$ with the $W_1(\Delta)$ and the exponential of the Wald entropy. We tabulate these numbers in Table (2) for the first few values of Δ. Note that the area of the horizon goes as $4\pi\sqrt{\Delta}$ in Planck units. Already for $\Delta = 12$ this area would be much larger than one, and one might expect that the Bekenstein-Hawking-Wald entropy would be a good approximation to the logarithm of the quantum degeneracy. However, we see from the table that these two differ quite substantially. Indeed, in this example, there are no relevant higher-derivative local terms which arise from integrating out the massive fields. Thus, the Wald entropy equals the Bekenstein-Hawking entropy. The discrepancy between the degeneracy and the exponential of the Wald entropy arises entirely from integration over massless fields. Localization enables an exact evaluation of these quantum effects. The resulting $W_1(\Delta)$ is in spectacular agreement with $d(\Delta)$ and in fact comes very close to the actual integer even for small values of Δ.

We see from the asymptotic expansion (5.30) that the subleading logarithmic correction to the Bekenstein-Hawking entropy goes as $-2\log(\Delta)$. This in agreement with the results in [59, 60, 61] where the logarithmic correction was computed by evaluating one-loop determinants of various massless fields around the classical background. Using localization, this logarithmic correction follows essentially from the analysis of the induced measure on the localizing manifold without the need for any laborious evaluation of one-loop determinants. Moreover, since localization accesses regions in field space very off-shell from the classical background the entire series of power-law suppressed terms in (5.30) follows with equal ease.

5.4 Nonperturbative Corrections, Orbifolds, and Localization

We have seen that localization correctly reproduces the first term in the Rademacher expansion. This term already captures all power-law and logarithmic corrections to the leading Bekenstein-Hawking-Wald entropy exactly to all orders. We turn next to the computation of the higher terms in the Rademacher expansion (3.33) with $c > 1$. These terms are nonperturbative because they are exponentially suppressed with respect to the terms in (5.30).

It was proposed in [62, 20, 63, 41] that such non-perturbative corrections could arise from $\mathbb{Z}_c$ orbifolds for all positive integers c because such orbifolds respect the same boundary conditions (2.2) on the fields. In general, it is difficult to justify keeping such subleading exponentials if the power-law suppressed terms are evaluated only in an asymptotic expansion. However, as we have seen, localization gives an exact integral representation of the leading Bessel function in §5.3. Since the power-law suppressed contributions are computed exactly, it is justified to systematically take into account the exponentially suppressed contributions.

The $\mathbb{Z}_c$ orbifold configurations that contribute to the localization integral are obtained as follows. We mod out with a symmetry $R_c T_c$ which combines a supersymmetric order c twist R_c on $AdS_2 \times S^2$ with an order c shift T_c along the T^6. The orbifold twist is required to be supersymmetric because to preserve the Q supercharge used for localization, the orbifold action must commute with $L - J$ [41]. At the center of AdS_2 and at the poles of S^2 the twist looks like a generator of the supersymmetric $C^2/\mathbb{Z}_c$ orbifold. With an appropriate shift, this action is freely acting and can be used to get smooth solutions [63].

To illustrate how this works together with localization let us first discuss the case when $T_c(\delta)$ is a simple shift of $2\pi\delta/c$ along the circle S^1. It acts on the momentum modes by

$$T_c(\delta)\,|m\rangle = e^{\frac{2\pi i \delta m}{c}}\,|m\rangle\,. \tag{5.31}$$

Let ϕ be the azimuthal angle along the S^2 and y be the coordinate of the circle S^1 with 2π periodicities. We will denote the orbifolded coordinates with a tilde. The orbifold operaton $R_c T_c$ identifies points in $AdS_2 \times S^2 \times S^1$ with the identification

$$(\widetilde{\theta}, \widetilde{\phi}\,, \widetilde{y}) \equiv (\widetilde{\theta} + \frac{2\pi}{c}\,, \widetilde{\phi} - \frac{2\pi}{c}\,, \widetilde{y} + \frac{2\pi\delta}{c}) \tag{5.32}$$

The combined action $R_c T_c(\delta)$ means that as we go around the boundary of AdS_2 the momentum modes pick up a phase as in (5.31). This corresponds to turning on a Wilson line of the Kaluza-Klein gauge field $\mathcal{A}$ that couples to the momentum n by modifying the gauge field as

$$\mathcal{A} = -ie_*(\widetilde{r} - 1)d\widetilde{\theta} + \delta\, d\widetilde{\theta} \tag{5.33}$$

The metric on the orbifolded AdS_2 factor has the same form

$$ds^2 = v_*\left[(\widetilde{r}^2 - 1)d\widetilde{\theta}^2 + \frac{d\widetilde{r}^2}{(\widetilde{r}^2 - 1)}\right] \qquad 1 \leq \widetilde{r} < \widetilde{r}_0; \qquad 0 \leq \widetilde{\theta} < \frac{2\pi}{c} \tag{5.34}$$

as the original unorbifolded metric (2.1) but the $\widetilde{\theta}$ variable now has a different periodicity and we have cutoff at $\widetilde{r} = \widetilde{r}_0$. Thus, it is not immediately obvious that asymptotic conditions on the fields are the same as for the unorbifolded theory. To see this, we change coordinates

$$\widetilde{\theta} = \frac{\theta}{c}\,, \quad \widetilde{\phi} = \phi - \frac{\theta}{c}\,, \quad \widetilde{y} = y + \frac{\theta}{c}\,, \quad \widetilde{r} = cr, \tag{5.35}$$

so that in the new coordinates, the fields have the same asymptotics (2.2) as before:

$$ds_2^2 \sim v_*\left[r^2 d\theta^2 + \frac{dr^2}{r^2}\right], \qquad \mathcal{A} \sim -ie_* r d\theta\ . \tag{5.36}$$

Moreover, the new coordinates have the same identification

$$(\theta, \phi, y) \equiv (\theta + 2\pi, \phi, y) \equiv (\theta, \phi + 2\pi, y) \equiv (\theta, \phi, y + 2\pi) \tag{5.37}$$

as in the unorbifolded theory. Such orbifolded field configurations with the same asymptotic behavior will therefore contibute to the functional integral.

The orbifold action is freely acting if δ and c are relatively prime. Therefore, the localizing equations, which are local differential equations, remain the same as before and one obtains the same localizing instantons (4.11) as before. To compute the renormalized action it is convenient to use the tilde coordinates. If we put a cutoff at r_0, the range of r is $1/c \leq r \leq r_0$ and that of $\widetilde{r}$ is $1 \leq \widetilde{r} \leq c r_0$. The physical action is an integral of the same local Lagrangian density as the unorbifolded theory but now the ranges of integration are different. Since the localizing instantons do not depend on the angular coordinates, the nontrivial integration is over the coordinate $\widetilde{r}$. The r_0 dependent contribution from this integral is therefore c times larger than before but the r_0 independent constant piece is the same as before. On the other hand, from the angular integrations one gets an overall factor of $1/c$ because the range of these coordinates is divided by c by the identification (5.32). Altogether, the renormalized action obtained by removing the r_0 dependent divergence is smaller by a factor of c. Moreover, with the modified gauge field (5.33) the Wilson line contributes an additional phase. In summary, instead of (4.15) we obtain

$$\exp\left[\frac{\mathcal{S}_{ren}(\phi)}{c} + \frac{2\pi i\, n\delta}{c}\right] , \tag{5.38}$$

where $\mathcal{S}_{ren}$ is the unorbifolded renormalized action for the localizing instantons given by (5.8).

Since the phase factor in (5.38) does not depend on ϕ we can first integrate over ϕ as before and then sum over all phases. Thus W_c factorizes as

$$W_c(\Delta) = A_c(\Delta) B_c(\Delta) \tag{5.39}$$

where A_c comes from integration over ϕ and B_c comes from the sum over phases. Since the renormalized action is now smaller by a factor of c, it is easy to see that the integral A_c gives precisely the modified Bessel function but with an argument $z_c = z/c$ with possible powers of c coming from the measure which we absorb for now in $B_c(\Delta)$. The final answer thus has the form

$$W_c(\Delta) = \sqrt{2}\,\pi \left(\frac{\pi}{\Delta}\right)^{7/2} I_{\frac{7}{2}}\left(\frac{\pi\sqrt{\Delta}}{c}\right) B_c(\Delta)\,. \tag{5.40}$$

This is very close to the c-th term in the Rademacher expansion. To obtain agreement we would need to show

$$B_c(\Delta) = c^{-9/2} K_c(\Delta)\,. \tag{5.41}$$

We see from (3.34) that the Kloosterman sum is also a rather intricate sum over various c and Δ dependent phases. This suggests that by summing over the phases for various allowed orbifolds and properly fixing their relative normalization with respect to the $c = 1$ term, it may be possible to compute $B_c(\Delta)$ to reproduce the desired expression (5.41) in terms of the Kloosterman sum [64].

6. Open Questions and Speculations

It is remarkable that a functional integral of string theory in AdS_2 precisely reproduces the first term in the Rademacher expansion that already captures all power-law suppressed corrections to the Bekenstein-Hawking-Wald formula as described in §5.3. As we have seen in §5.4, the functional integral has all the ingredients to reproduce even the subleading nonperturbative corrections in the Rademacher expansion. It would be insteresting to see how the intricate number theoretic details of the Kloosterman sum (3.34) will arise from the string theory functional integral [64]. Since $d(\Delta)$ is an integer, $W(\Delta)$ would also have to be an integer. This suggests an underlying integral structure in quantum gravity at a deeper level.

Our computation suggests that the bulk AdS string theory is every bit as fundamental as the boundary CFT. Even though one sometimes refers to the AdS computation as macroscopic and thermodynamic, quantum gravity in AdS_2 does not appear to be an emergent, coarse-grained description of the more microscopic boundary theory. Each theory has its own rules of computation. It seems more natural to regard AdS/CFT holography as an exact strong-weak coupling duality.

So far we have used holography in its original sense to mean a complete accounting of the degrees of freedom associated with the AdS_2 black hole horizon in terms of the states of a CFT_1 in one lower dimension. The AdS_2/CFT_1 correspondence actually extends this idea further to apply correlation functions as well. The boundary CFT_1 has a $GL(d)$ symmetry that acts upon $d(q,p)$ zero energy states. The observables of the theory are thus simply $d \times d$ matrices $\{M_i\}$. A precise state-operator correspondence has been suggested [65] that allows one to define, at least formally, the corresponding correlation functions for some of the observables in the bulk theory. In the boundary theory it is easy to define correlation functions of observables as traces of strings of operators such as

$$\mathrm{Tr}(M_1 M_2 \ldots M_k) \ . \tag{6.1}$$

We have seen that localization techniques can be successfully applied for computing the partition function to compute the integer d. A natural question is if localization can be useful for computing the correlation functions such as above. Such a computation would allow us to recover the discrete information about the microstates of a black hole from observables living in the bulk near the horizon. This of course goes to heart of the problem of information retrieval from black holes. It is likely that one would need to extend the localization analysis beyond the massless fields to higher string modes to access this information.

The content of the boundary CFT_1 is essentially completely determined by the integer d. The bulk theory has an elaborate field content and action that depends on the compactification K and the charges of the black hole. Imagine two different bulk theories $AdS_2 \times K$ and $AdS_2 \times K'$ but with the same black hole degeneracy d. This would suggest that the two string theories near the horizon of two very different black holes in very different compactifications are dual to the same CFT_1. By transitivity of duality, this would imply that the two string theories themselves are dual to each other. This conclusion seems inescapable from the perspective of the CFT_1. Note that it is not easy to arrange the situation when the degeneracies of two different black holes are given by the same integer. For example, if the degeneracy is given by the Fourier coefficients of some modular form, it would be rare, but not impossible, that two such Fourier coefficients are precisely equal.

Our analysis uses an $\mathcal{N} = 2$ reduction of the full $\mathcal{N} = 8$ theory by dropping six gravitini multiplets of $\mathcal{N} = 2$ and the hypermultiplets. This is partially motivated by the fact that the hypermultiplets are flat directions of the classical entropy function and our black hole is not charged under the gauge fields that belong to the gravitini multiplets. We have also ignored D-terms. This is partially justified by the fact that the black hole horizon is supersymmetric and a large class of D-terms are known not to contribute to the Wald entropy as a consequence of this supersymmetry [66]. Our final answer strongly suggests that these assumptions are justified and our reduced theory fully captures the physics. A technical obstacle in analyzing the validity of these assumptions stems from the fact that the incorporation of the hypermultiplets and the gravitini multiplets would require infinite number of auxiliary fields if all $\mathcal{N} = 8$ supersymmetries are realized off-shell. It may be possible to make progress in this direction perhaps by using a formulation where only the Q-supersymmetry used for localization is realized off-shell but on all fields of $\mathcal{N} = 8$ supergravity. Alternatively, it may be possible to repeat the localization analysis in a different off-shell formalism such as the harmonic superspace [67] where all $\mathcal{N} = 8$ supersymmetries are realized off-shell with infinite number of auxiliary fields; but perhaps only a small number of auxiliary fields get excited for the localizing solution.

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