Available online at www.sciencedirect.com
SCIENCE DIRECT®
Computational Statistics & Data Analysis 48 (2005) 765–778
COMPUTATIONAL STATISTICS & DATA ANALYSIS
www.elsevier.com/locate/csda

Robustness properties of A-, D-, and E-optimal designs for polynomial growth models with autocorrelated errors

M. Moerbeek*

Department of Methodology and Statistics, Utrecht University, P.O.Box 80140, 3508 TC Utrecht, The Netherlands

Received 6 November 2003; received in revised form 9 March 2004; accepted 25 March 2004

Abstract

Optimal designs for polynomial growth models for longitudinal data with autocorrelated errors determine the optimal allocation and number of time points. A-, D-, and E-optimal designs for linear, quadratic and cubic growth models with four or less time points on the time interval $[0, 2]$ are given and four robustness properties of these designs are examined. The first considers the robustness against using too many time points. It is shown that the design with the number of time points equal to the number of regression coefficients is optimal, and that the efficiency of a design decreases when the number of time points increases. The second robustness property deals with the consequences of using an incorrect order of the polynomial. The efficiency of a design is shown to be generally higher if the assumed order of the polynomial is closer to the true order. The third robustness property deals with the robustness against an incorrect value of the autocorrelation coefficient. Results show that the optimal designs are very robust. The fourth robustness property considers the robustness of optimal designs with respect to other optimality criteria. The optimal designs are shown to be very robust to the other two optimality criteria.

Keywords: Optimal designs; Robustness; Efficiency; Autocorrelation; Polynomial

1. Introduction

Longitudinal studies are carried out in many fields of science to study the change of a particular outcome variable across time or age. Examples can be found in

* Tel.: +31-30-2531450; fax: +31-30-2535797.
E-mail address: m.moerbeek@fss.uu.nl (M. Moerbeek).

0167-9473/$ - see front matter
doi:10.1016/j.csda.2004.03.014

various fields of study. A small literature search resulted in the following examples. In life stock production science, linear, quadratic and cubic growth models were used to model the weight of cattle of less than 20 months (De Behr et al., 2001). In a study on child development quadratic growth models were used to describe the total number of phonological processes of children between the age of 2 and 9 (Burchinal and Appelbaum, 1991). Quadratic growth models were also used to model the lung volumes of foetuses between 20 and 40 weeks (Chang et al., 2003). Cubic growth models were used to fit the body mass of beetle larvae up to 200 days (Greenberg and Ar, 1996). A sixth-order polynomial was found to best describe the logarithm of dry plant weight of a plant species called Blackstonia perfoliata (Elias and Causton, 1976).

A researcher designing a longitudinal study has to decide on the number and allocation of time points. In practice, the choice of a design of a longitudinal study is often driven by non-statistical criteria and preconditions such as a limited budget or a limited number of measurements per subject, and the design with equally spaced time points is frequently selected because of its feasibility. A number of recent papers has focussed on the optimal allocation of time points in a longitudinal study from a statistical point of view. Abt et al. (1997) derived optimal designs for slope parameter estimation and growth prediction in a linear growth curve with intraclass (compound symmetry) and autocorrelated error structures. In a follow-up paper Abt et al. (1998) studied A-, D-, and E-optimal designs for quadratic growth curves. Tan and Berger (1999) derived D-optimal designs for linear, quadratic, and cubic growth curves as a function of the autocorrelation coefficient and compared the optimal designs with the designs with equally spaced time points. D-optimal designs for polynomial regression models with order less than seven and uncorrelated errors were also presented by Atkinson and Donev (1996). Chang and Lay (2002) studied the optimal allocation of time points for the growth curve model $E(y \mid x) = \beta_0 + \beta_1 x + \beta_2 x^{\alpha}$ with α known and assuming uncorrelated errors.

Many optimal designs depend on the regression model that generates the data. Longitudinal data are often modelled using a polynomial regression model and the order of the polynomial needs to be known to derive the optimal design. This may cause problems since the order is usually unknown in the design stage and selecting an incorrect order may lead to a less efficient design with too few or many time points and a less efficient allocation of these time points. Moreover, the degree of correlation between two repeated measurements needs to be known in the design stage since it also affects the optimal design, see the papers mentioned above. Furthermore, many optimality criteria exist and each of these may lead to a different optimal design. A researcher seldom has just one optimality criterion in mind when planning a longitudinal study and it is therefore important to study the robustness of an optimal design with respect to different optimality criteria.

The purpose of the present paper is to study robustness properties of optimal designs for first-, second- and third-order polynomials with autocorrelated errors, and to give guidelines for the planning of longitudinal studies. Three optimality criteria are used: A-, D-, and E-optimality. Four robustness properties are studied: robustness against too many time points, against an incorrect order of the polynomial, against an incorrect value of the autocorrelation coefficient, and with respect to other optimality criteria.

A few papers on this topic have been published previously, but these either restricted to one or two optimality criteria, ignored autocorrelation, or did not investigate all four robustness properties. So far, no thorough and systematic investigation of these four robustness properties of A-, D-, and E-optimal designs has been published. This paper summarizes and builds upon results that have been published so far, and fills up some gaps in knowledge. Wong (1994) studied the robustness of A-, D-, E-, and G-optimal designs for polynomial models with uncorrelated errors against an incorrect order of the polynomial and against other optimality criteria. He showed that D-optimal designs are more robust to an incorrect order of the polynomial than A-optimal designs, which in their turn are more robust than E-optimal designs. Furthermore, he showed that D-optimal designs are robust in terms of A-optimality but not in terms of E-optimality, while the robustness of A-, and E-optimal designs with respect to A-, D-, and E-optimality was shown to be high. Abt et al. (1997) showed that the two-points design with equal load at the beginning and end of the study turns out to be the best for the linear growth model in almost all situations. In their follow-up Abt et al. (1998) showed that, with only one exception, A-, D-, and E-optimal designs for quadratic growth models are independent of the value of the autocorrelation coefficient. Tan and Berger (1999) studied the robustness of D-optimal designs against an incorrect order of the polynomial. They concluded that the best choice is to use the highest possible relevant order of the polynomial and to have the number of time points equal to the number of regression coefficients. Moreover, they showed that the equally spaced design is almost as efficient as the D-optimal design.

The remainder of this paper is as follows. In the next section the polynomial growth model with autocorrelated errors is presented and optimality criteria and their rationale are given. Since optimal designs are not easily calculated analytically, they are obtained numerically. This section also shows how the robustness of a design can be expressed in terms of a relative efficiency. The next section studies four robustness properties: robustness against using too many time points, against using an incorrect order of the polynomial, against using an incorrect value of the autocorrelation coefficient, and with respect to other optimality criteria. An application in the next section shows how the results can be used in a practical setting. Finally some conclusions are given.

2. Optimal designs and relative efficiencies

A polynomial regression model of order p is used to model the response y_{ij} of subject j at time point i

$$y_{ij} = \beta' f(t_i)_{ij} + e_{ij}, \tag{1}$$

where $\beta = (\beta_0, \beta_1, \ldots, \beta_p)'$ is the vector of regression coefficients, $f(t_i) = (1, t_i, \ldots, t_i^p)'$ is the vector of polynomial terms, and e_{ij} is the random error term. Note that the regression coefficients are assumed constant across subjects so that the model is a fixed effects model. In this paper linear ($p = 1$), quadratic ($p = 2$) and cubic ($p = 3$) growth curves are studied. The number of time points per subject is denoted q, and each subject is measured once at each time point. It is assumed that all subjects are measured at the same time points, hence the time points t_i are not subscribed j. Consequently, a unique estimate of the regression coefficients can only be obtained when $q \geqslant p+1$.

For each subject j model (1) can be expressed in a more general matrix–vector notation

$$\mathbf{y}_j = \mathbf{X}_j \beta + \mathbf{e}_j, \tag{2}$$

where $\boldsymbol{y}_j$ is the vector of length q with responses, $\boldsymbol{X}_j$ is the q by $(p+1)$ design matrix, and β is the vector of length $p+1$ with regression coefficients. The vector $\boldsymbol{e}_j$ of random errors has zero mean and $q \times q$ covariance matrix $\sigma^2 \mathbf{V}_j$. The element (i, i') of the matrix $\mathbf{V}_j$ is equal to $\rho^{|t_i - t_{i'}|}$, where $\rho \in [0, 1]$ is the autocorrelation coefficient that gives the correlation between two time points that are one unit of time apart. The maximum likelihood estimator of the vector of regression coefficient is

$$\hat{\beta} = \left(\sum_j \mathbf{X}_j' \mathbf{V}_j^{-1} \mathbf{X}_j \right)^{-1} \sum_j \mathbf{X}_j' \mathbf{V}_j^{-1} \mathbf{y}_j, \tag{3}$$

with covariance matrix

$$\mathrm{var}(\hat{\beta}) = \left(\sum_j \mathbf{X}_j' \mathbf{V}_j^{-1} \mathbf{X}_j \right)^{-1}. \tag{4}$$

The design ξ is an element of the design space Ω and gives the allocation of time points. The design ξ is affected by the number of time points q, and is henceforward denoted ξ_q. The amount of information in a design is measured by the information matrix $M_p(\xi_q)$, which is equal to the inverse of the covariance matrix of the parameter estimates. Note that the information matrix is subscribed p to stress the dependency on the order of the polynomial.

The optimal design ξ_q^* is the design among all possible designs ξ_q in the design space Ω for which the inverse information matrix is minimized. Since matrices cannot be ordered in a unique way, various functions $\Psi_p(\xi_q)$ have been proposed as optimality criteria, which at least have to be convex and differentiable. Three optimality criteria are used in this paper, namely A-, D-, and E-optimality. An A-optimal design minimizes $A_p(\xi) = \mathrm{tr}(M_p^{-1}(\xi_q))$, where $\mathrm{tr}(B)$ denotes the trace of matrix B. This is equal to minimizing the average variance of the parameter estimates. So, A in the name of this criterion stands for average. A D-optimal design minimizes $D_p(\xi) = \det(M_p^{-1}(\xi_q))$, where $\det(B)$ denotes the determinant of matrix B, and thus the D in the name of the criterion stands for determinant. This is equal to minimizing the volume of the confidence ellipsoid of the parameter estimates. An E-optimal design minimizes $E_p(\xi) = \lambda_{\max}(M_p^{-1}(\xi_q))$, where $\lambda_{\max}(B)$ denotes the maximum eigenvalue of matrix B. This is equal to minimizing the variance of the least well-estimated contrast $a'\beta$, subject to the constraint $a'a = 1$. So E in the name of the criterion stands for extreme. See Kiefer (1975) for a more extensive discussion of these optimality criteria.

A nice feature of D-optimal designs is that they do not depend on the chosen time interval $[a, b]$. A linear transformation of the time interval leaves the D-optimum design unchanged. Unfortunately, this is not generally the case for A- and E-optimal designs. As an illustration optimal designs on the time interval $[0, 2]$ are derived and it is shown how robustness properties of these designs can be studied. An application in Section 4 shows how the results can be used in a practical setting. The A- and E-optimal designs and their robustness properties may change when another time interval is used. However, the methods to derive the optimal designs and to study their robustness properties remain unchanged.

Only solutions for which $t_1 = 0$ and $t_q = 2$ were derived, since designing a longitudinal study often requires fixing the first and last time point. The optimal designs were derived numerically using the S-PLUS 2000 (S-PLUS, 1999) function NLMINB, which minimizes non-linear functions subject to bound-constrained parameters using a quasi-Newton method. To avoid finding a local minimum, 10 different sets of random starting values were used. When different sets of starting values resulted in different solutions, the design with the smallest value of the optimality criterion $\Psi_p(\xi_q)$ was selected. This random search algorithm seems to work well and is much faster than evaluating each possible combination of time points (Ouwens, 2002). Equivalence theorems can be used for construction and checking of optimal designs for models with uncorrelated data, see for instance Atkinson and Donev (1996).

The optimal allocations of time points for $p+1 \leqslant q \leqslant 4$ and $\rho \in [0.0001, 0.995]$ are given in Fig. 1. For large ρ the optimal time points for linear and quadratic growth are almost equally spaced. For small ρ the optimal time points converge to the optimal time points in a model with uncorrelated errors as given in Table 1. For linear growth with two time points the optimal time points are 0 and 2, irrespective of the optimality criterion and value of the autocorrelation coefficient. Most D-optimal designs in Fig. 1 are symmetric around the time point 1. The only exception are the D-optimal designs for linear growth with three time points, the $D_1(\xi_3)$-optimal designs. As is obvious, if the design with time points 0, t, and 2 is D-optimal, then the design with time points 0, 2-t, and 2 is also D-optimal.

Robustness properties of designs are expressed in terms of relative efficiencies. Note that designs ξ_q may vary in their number of time points per subject. To fairly compare the designs with different numbers of time points per subject, only designs with the same total number of measurements are compared. The total number of measurements is given by $n = q \times m_q$ where q is the number of time points per subject and m_q is the number of subjects. Then the A-, D-, and E-efficiencies of a design ξ_q relative to the optimal design $\xi^*_{q^*}$ are given by

$$A_p\text{-Eff}(\xi_q, \xi^*_q) = \frac{q^*}{q}\,\frac{\operatorname{tr}(M_p^{-1}(\xi^*_{q^*}))}{\operatorname{tr}(M_p^{-1}(\xi_q))}, \tag{5}$$

$$D_p\text{-Eff}(\xi_q, \xi^*_q) = \frac{q^*}{q}\left\{\frac{\det(M_p^{-1}(\xi^*_{q^*}))}{\det(M_p^{-1}(\xi_q))}\right\}^{1/(p+1)} \tag{6}$$

and

$$E_p\text{-Eff}(\xi_q, \xi^*_q) = \frac{q^*}{q}\,\frac{\lambda_{\max}(M_p^{-1}(\xi^*_{q^*}))}{\lambda_{\max}(M_p^{-1}(\xi_q))}, \tag{7}$$

respectively. q^* and q are the number of time points per subject under design $\xi^*_{q^*}$ and ξ_q, respectively. The inverse relative efficiency gives the number of times the design

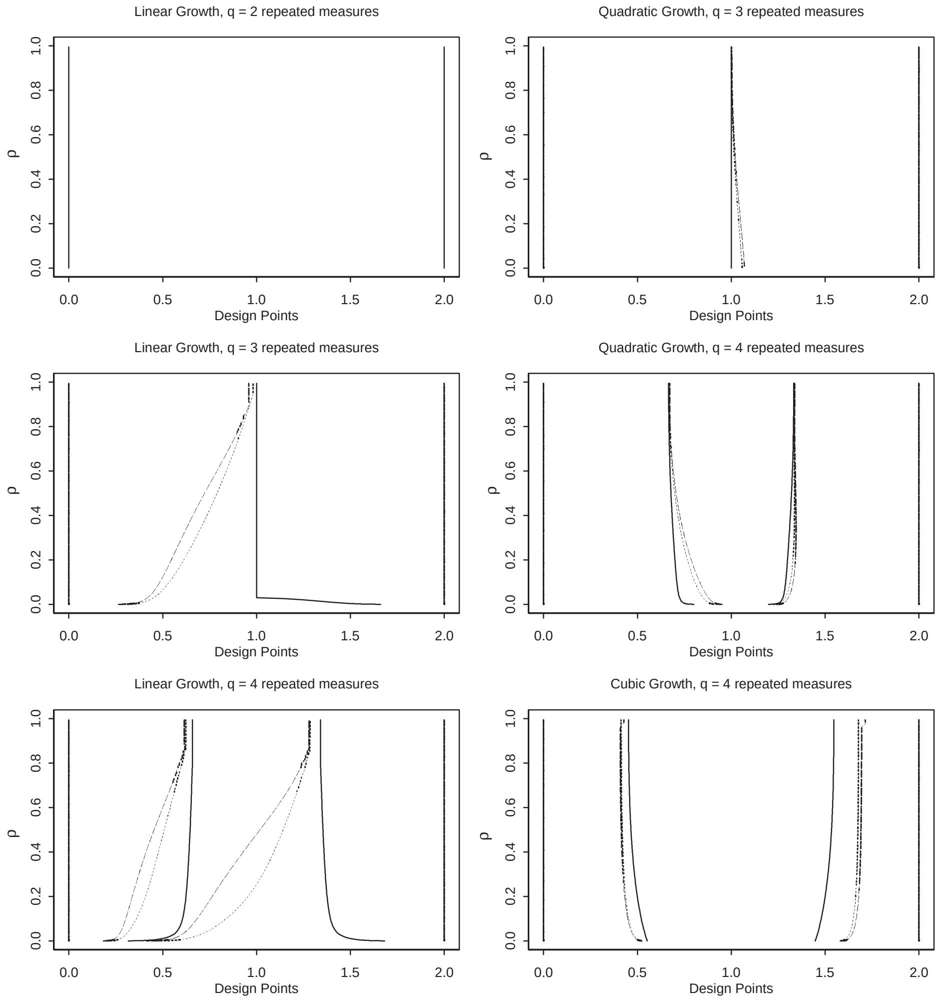

Fig. 1. Optimal allocation of time points as a function of ρ. Dotted lines: A-optimal design. Solid lines: D-optimal design. Dashed lines: E-optimal design.

Table 1
Optimal time points for polynomial growth models in the time interval $[0, 2]$ with uncorrelated errors and $q = p + 1$

p	A-optimal design	D-optimal design	E-optimal design
1	0, 2	0, 2	0, 2
2	0, 1.057, 2	0, 1, 2	0, 1.069, 2
3	0, 0.528, 1.572, 2	0, 0.553, 1.447, 2	0, 0.526, 1.582, 2

ξ_q needs to be replicated to have the same value of the optimality criterion $\Psi_p(\xi^*_{q^*})$ as the optimal design $\xi^*_{q^*}$ (Atkinson and Donev, 1996). So, efficiencies of, say, 0.8 or 0.9 or higher are preferred. As is obvious, the efficiency of an optimal design is equal to 1.

3. Robustness properties

3.1. Robustness against too many time points

Fig. 2 shows the efficiency of optimal designs with $p+1 \leqslant q \leqslant 6$ time points relative to the efficiency of the optimal design with $q^* = p + 1$ time points as a function of $\rho \in [0.0001, 0.995]$. The efficiency decreases to $q^*/q = (p + 1)/q$ if $\rho \to 1$. This is obvious, since no information is added to the information matrix if ρ is close to 1. Hence, the values of the optimality criteria remain unchanged if an additional time point is added. So, the second factor at the right-hand side of Eqs. (5)–(7) is equal to 1, and the efficiency goes to the first factor at the right-hand side of these equations, which is equal to q^*/q. For instance, the efficiency of the optimal design with four time points for linear growth is as low as 0.5, which means that it has to be replicated twice to do as well as the optimal design with two time points.

Tan and Berger (1999) argue that the efficiency of D-optimal designs goes to 1 when $\rho \to 0$. As follows from Fig. 2, this is also the case for A- and E-optimal designs. Thus, a design with $q = p + 1$ time points is most efficient and the robustness of only these designs against an incorrect order of the polynomial, against an incorrect ρ, and against other optimality criteria will be studied in the next three sub-sections.

3.2. Robustness against an incorrect order of the polynomial

In the design stage the order of the underlying polynomial that generates the data is often unknown. Using an incorrect order of the polynomial may result in a design that is less efficient. For instance, consider the case where a researcher assumes that the data are generated by a second order polynomial (quadratic growth). Then, a design with three time points as given in the upper right plot of Fig. 1 is the optimal design. If the true model were a linear growth model, a design with two time points located at 0 and 2 would be the optimal design. Eqs. (5)–(7) can be used to calculate the efficiency of the optimal design ξ^*_q with $q = p + 1$ time points for the incorrect assumed order p against the optimal design $\xi^*_{q^*}$ with $q^* = p^* + 1$ time points for the true order p^*. The plots at the left side of Fig. 3 show these efficiencies if the assumed model is quadratic and the true model is linear; those at the right side show the efficiencies if the assumed model is cubic and the true model is quadratic or linear. As argued in the previous section, these efficiencies go to q^*/q when $\rho \to 1$. So for large ρ the efficiencies may be very low. For instance, the efficiency of an optimal design for a cubic growth model goes to 0.5 when the true model is linear and $\rho \to 1$. In other words, such a design would have to be replicated twice to do as well in terms of the optimality criterion as the optimal design with two time points for the linear model.

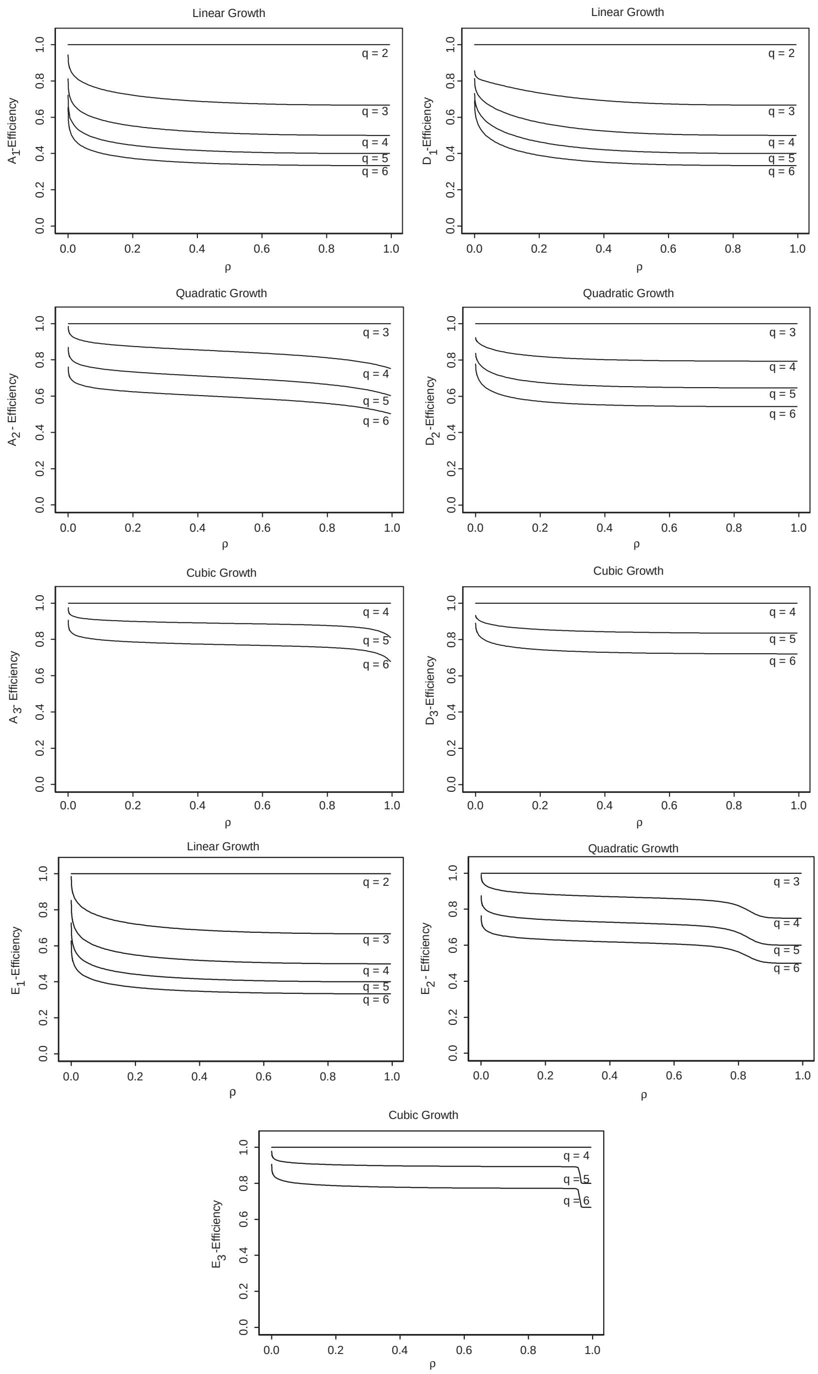

Fig. 2. Efficiency of optimal designs with $p+1 \leqslant q \leqslant 6$ time points relative to the optimal design with $p+1$ time points and as a function of ρ.

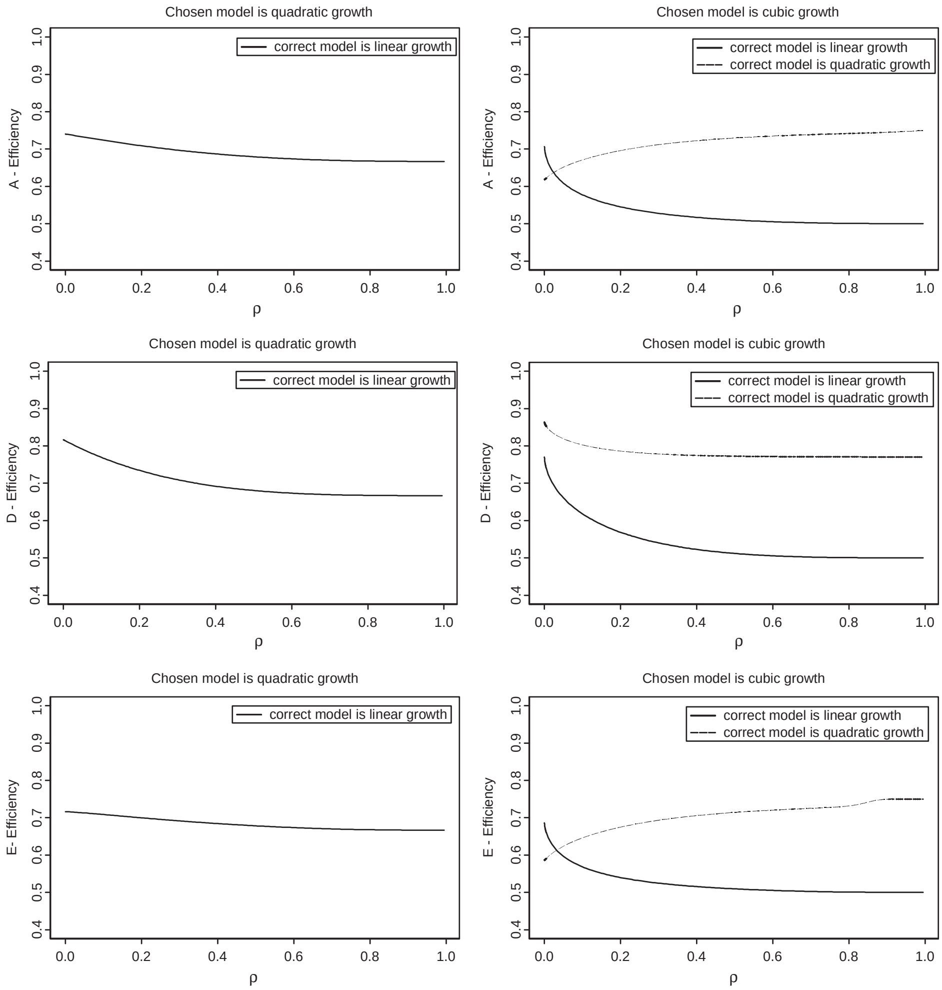

Fig. 3. Robustness of optimal designs against an incorrect order of the polynomial.

Fig. 3 also shows that the efficiency of a design is in general higher if the assumed order is closer to the true order. So, when planning a longitudinal study one should choose the highest relevant order of the polynomial. This guideline, however, does not apply to A- and E-optimal designs when ρ is small. In such cases, the efficiency of the optimal design for a cubic growth model may be larger if the true underlying model is linear than when it is quadratic.

3.3. *Robustness against an incorrect chosen value of the autocorrelation coefficient*

As follows from Fig. 1 the optimal allocation of time points is often locally optimal in the sense that it depends on the true value of the autocorrelation coefficient ρ. To plan a study, a realistic value of ρ must be chosen in order to calculate the optimal allocation of time points. In this section the robustness against an incorrect chosen value of ρ is studied for designs with $q = p + 1$ time points. As follows from the upper left plot in Fig. 1 the optimal time points of $\Psi_1(\xi_2)$-optimal designs are globally optimal since they do not depend on the value of ρ. This is not the case for $\Psi_2(\xi_3)$- and $\Psi_3(\xi_4)$-optimal designs, although the optimal allocation of time points of these designs only slightly depends on ρ, see the upper and lower plots at the right side of Fig. 1. Fig. 4 shows the efficiencies of $\Psi_3(\xi_4)$-optimal designs for the chosen ρ=0.005, 0.25, and 0.50 as a function of the true ρ. The efficiencies of these and also of all other possible values of the chosen ρ are very high (> 0.9), and thus the $\Psi_3(\xi_4)$-optimal designs are very robust against an incorrect chosen value of ρ. As is obvious, the efficiency is equal to 1 if the chosen ρ is equal to the true ρ. The efficiencies of $\Psi_2(\xi_3)$-optimal designs against an incorrect chosen ρ are larger than 0.99 for A- and E-optimal designs and equal to 1 for D-optimal designs, since the optimal allocation of time points for the latter optimality criterion does not depend on the true ρ. Plots of these efficiencies are not shown.

3.4. Robustness with respect to other optimality criteria

In this section the robustness of optimal designs with respect to other optimality criteria is studied for designs with $q = p + 1$ time points. As already follows from the top left plot of Fig. 1, the $\Psi_1(\xi_2)$-optimal designs do not depend on the chosen optimality criterion. This is not true for the $\Psi_2(\xi_3)$- and the $\Psi_3(\xi_4)$-optimal designs, see the upper and lower plots at the right side of Fig. 1. The optimal allocations of time points for the A- and E-optimal designs are rather similar. As a consequence, the A-efficiency of E-optimal designs and the E-efficiency of A-optimal designs are almost equal to 1, see Fig. 5. As follows from this figure, the efficiencies of $\Psi_2(\xi_3)$-optimal designs with respect to other optimality criteria is at least 0.99, and the efficiencies of $\Psi_3(\xi_4)$-optimal designs with respect to other optimality criteria is at least 0.91. So, the $\Psi_2(\xi_3)$- and the $\Psi_3(\xi_4)$-optimal designs perform very well in terms of the other optimality criteria.

4. An application

In livestock species, growth is generally an economically important trait. It is therefore self-evident that much attention has been paid to modelling weight as a function of age. Many different growth curves have been proposed and evaluated in various species. Examples are the class of polynomial curves, step-wise linear curves, and non-linear curves, such as power, sigmoid and exponential growth. A study on the relationship between age and weight may be very time-consuming and expensive. It is therefore necessary to carefully design the study.

This section discusses the issues that arise when designing a study on polynomial growth. Suppose, for instance, that a researcher is interested in the relation between age and body weight of male cattle of a particular breed up to the age of two years.

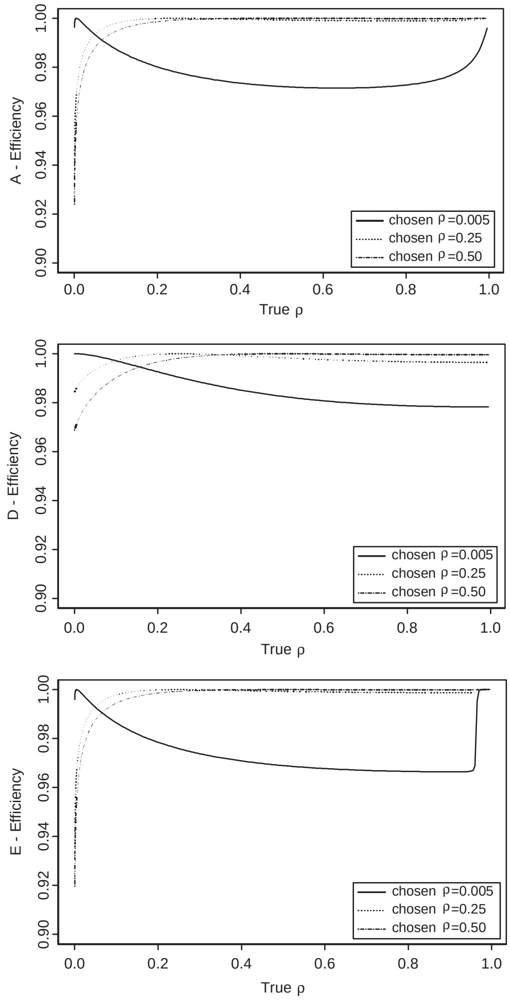

Fig. 4. Robustness of $\Psi_3(\xi_4)$-optimal designs against an incorrect chosen value of ρ.

Resources are limited such that a total of 300 measurements can be taken. So, the first question is whether a design with a few animals and many observations per animal is preferred above a design with many animals and only a few observations per animal. As was shown in Section 3.1 of this paper the optimal number of time points per animal is determined by the order of the true underlying polynomial, which is often unknown in the design stage. As was shown in Section 3.2 the best choice is to select the highest relevant order. Moreover, results from similar studies in the past may be used to decide on the polynomial order. De Behr et al. (2001) showed that a cubic model describes the growth of Belgian Blue males up to 1.67 years (20 months) best in

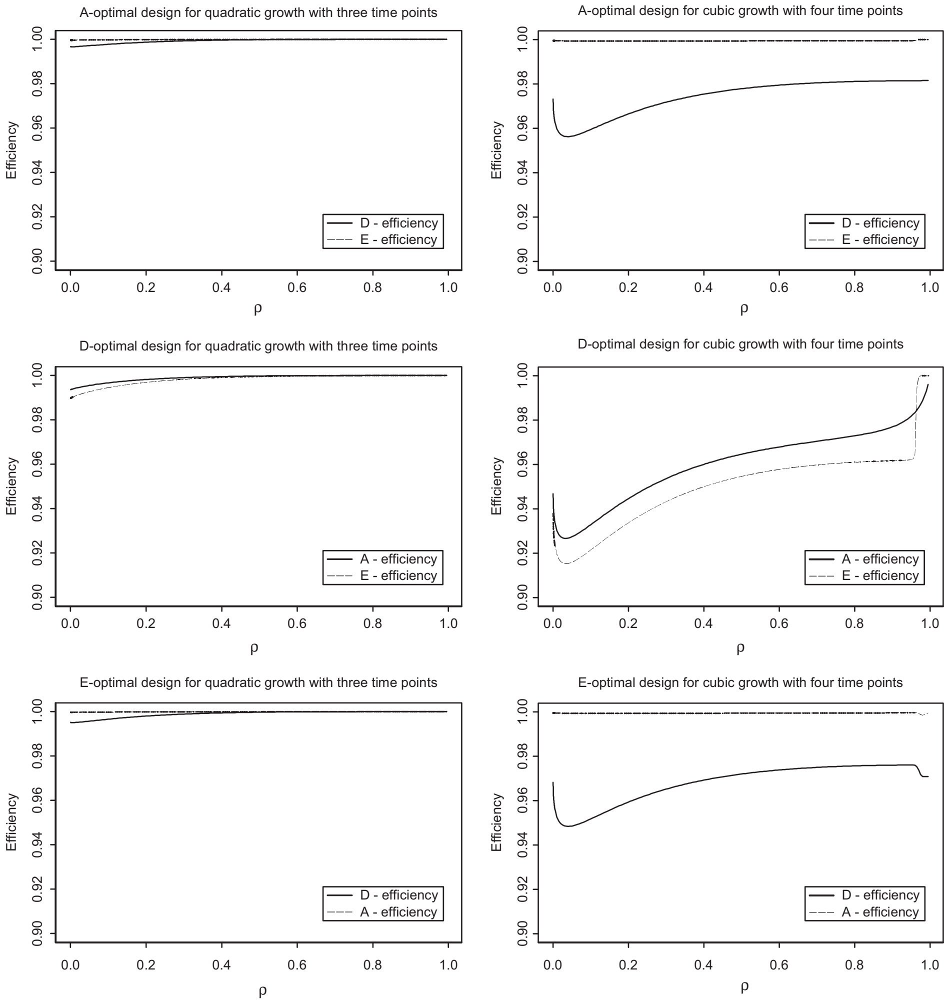

Fig. 5. Robustness of $\Psi_2(\xi_3)$- and $\Psi_3(\xi_4)$-optimal designs with respect to other optimality criteria.

terms of squared multiple correlation. This information may convince the experimenter to use this order of the polynomial to model growth of the breed under his study up to the age of two years. Then, a design with 75 animals and four measurements per animal is most efficient in terms of *A*-, *D*-, and *E*-optimality.

The second question is the optimal location of the time points. The optimal allocation is determined by the degree of autocorrelation and the selected optimality criterion. As followed from Section 3.4, the optimal designs on the time interval $[0, 2]$ are very robust with respect to the other optimality criteria. Moreover, an incorrectly chosen value of the autocorrelation coefficient leads to only a very small loss in efficiency, see Section 3.3. So, a design with measurements at time points 0, 0.5, 1.5, and 2 years does well for all three optimality criteria. Note that the points in this design are not equally spaced.

This optimal design based on a cubic growth model is less efficient when the true model is linear or quadratic. As follows from the plots at the right-hand side of Fig. 3 the efficiency of this design may be as low as 0.66 when the true response model is quadratic, and as low as 0.5 when the true response is linear. Consequently, the optimal design has to be replicated 1.5 times or twice when the true response model is quadratic or linear and the chosen response model is cubic. Of course, models with a higher order than the cubic growth model cannot be fitted to the data since all animals are measured at the same four time points.

5. Conclusions

D-optimal designs are invariant under linear transformations of the time interval. For this optimality criterion the optimal design has number of time points equal to the number of regression coefficients $p+1$. Using more time points q leads to a minimum efficiency equal to $(p+1)/q$. The second conclusion for *D*-optimal designs is that the chosen order of the polynomial should be as close as possible to the true order. The efficiency of using an incorrect order may be as small as $(p^*+1)/(p+1)$, where the numerator and denominator of this ratio are equal to the number of time points in the true and chosen order of the polynomial. The third conclusion is that *D*-optimal designs are very robust against an incorrect chosen value of the autocorrelation coefficient. Finally, the *D*-optimal designs on the time interval $[0,2]$ are very robust with respect to the other optimality criteria.

Robustness properties of *A*- and *E*-optimal designs for models on the time interval $[0,2]$ were also derived in Section 3. The conclusions with respect to the four robustness properties were generally similar to those for a *D*-optimal design. However, the *A*- and *E*-optimal designs depend on the chosen time interval, and so do their robustness properties. The conclusions of such optimal designs with respect to the four robustness properties as given in Section 3 may therefore change if another time interval is chosen. In order to study the robustness properties of designs on a chosen time interval, the optimal time points and the value of the optimality criterion have to be computed using a computer program that minimizes non-linear functions subject to bound constrained parameters, such as the S-PLUS 2000 function `NLMINB`. The robustness properties of such designs in terms of relative efficiencies can then be calculated from Eqs. (5)–(7).

Acknowledgements

I wish to thank an anonymous referee for helpful comments and suggestions. This research was supported by a grant from the Netherlands Organization for Scientific Research (NWO).

References

Abt, M., Liski, E.P., Mandal, N.K., Sinha, B.K., 1997. Optimal designs in growth curve models. Part I. Correlated model for linear growth: optimal designs for slope parameter estimation and growth prediction. J. Statist. Plann. Inference 64 (1), 141–150.

Abt, M., Gaffke, N., Liski, E.P., Sinha, B.K., 1998. Optimal designs in growth curve models—II. Correlated model for quadratic growth: optimal designs for parameter estimation and growth prediction. J. Statist. Plann. Inference 67 (2), 287–296.

Atkinson, A.C., Donev, A.N., 1996. Optimum Experimental Designs. Clarendon Press, Oxford.

Burchinal, M., Appelbaum, M.I., 1991. Estimating individual developmental functions: methods and their assumptions. Child Develop. 62 (1), 23–43.

Chang, F.-C., Lay, C.-F., 2002. Optimal designs for a growth curve model. J. Statist. Plann. Inference 104 (2), 427–438.

Chang, C.-H., Yu, C.-H., Chang, F.-M., Ko, H.-C., Chen, H.-Y., 2003. Volumetric assessment of normal fetal lungs using three-dimensional ultrasound. Ultrasound Med. Biol. 29 (7), 935–942.

De Behr, V., Hornick, J.L., Cabaraux, J.F., Alvarez, A., Istasse, L., 2001. Growth patterns of Belgian Blue replacement heifers and growing males in commercial farms. Livestock Product. Sci. 71 (2–3), 121–130.

Elias, C.O., Causton, D.R., 1976. Studies on data variability and the use of polynomials to describe plant growth. New Phytol. 77 (2), 421–430.

Greenberg, S., Ar, A., 1996. Effects of chronic hypoxia, normoxia and hyperoxia on larval development in the beetle tenebrio molitor. J. Insect Physiol. 42 (11–12), 991–996.

Kiefer, J., 1975. Optimal design: variation in structure and performance under change of criterion. Biometrika 62 (2), 277–288.

Ouwens, M.J.N., 2002. Improving Parameter Estimates in Generalized Linear Mixed Models. Maastricht University, Maastricht.

S-PLUS, 1999. S-PLUS 2000 User's Guide. MathSoft, Seattle.

Tan, F.E.S., Berger, M.P.F., 1999. Optimal allocation of time points for the random effects model. Comm. Statist. Simulation Comput. 28 (2), 517–540.

Wong, W.K., 1994. Comparing robust properties of A, D, E and G-optimal designs. Comput. Statist. Data Anal. 18 (4), 441–448.