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Electric and magnetic charges in $N = 2$ conformal supergravity theories

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Abstract: General Lagrangians are constructed for $N = 2$ conformal supergravity theories in four space-time dimensions involving gauge groups with abelian and/or non-abelian electric and magnetic charges. The charges are encoded in the gauge group embedding tensor. The scalar potential induced by the gauge interactions is quadratic in this tensor, and, when the embedding tensor is treated as a spurionic quantity, it is formally covariant with respect to electric/magnetic duality. This work establishes a general framework for studying any deformation induced by gauge interactions of matter-coupled $N = 2$ supergravity theories. As an application, full and residual supersymmetry realizations in maximally symmetric space-times are reviewed. Furthermore, a general classification is presented of supersymmetric solutions in $\mathrm{AdS}_2 \times S^2$ space-times. As it turns out, these solutions allow either eight or four supersymmetries. With four supersymmetries, the spinorial parameters are Killing spinors of AdS_2 that are constant on S^2, so that they carry no spin, while the bosonic background is rotationally invariant.

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1 Introduction

In four space-time dimensions, Lagrangians with abelian gauge fields have generically less symmetry than their corresponding equations of motion. The full invariance group of the combined field equations and Bianchi identities in principle involves a subgroup of the electric/magnetic duality group, $\mathrm{Sp}(2n, \mathbb{R})$ for n vector fields, suitably combined with transformations of the matter fields. Subgroups of the symmetry group of the Lagrangian can be gauged in the conventional way by introducing covariant derivatives and covariant field strengths. Introducing gauge groups which involve elements of the electric/magnetic duality group that do not belong to the symmetry group of the Lagrangian, are not possible in this way.

To circumvent this problem, one may therefore first convert the Lagrangian by an electric/magnetic equivalence transformation to a different, but equivalent, Lagrangian that has the desired gauge group as a symmetry. However, this procedure is cumbersome. One reason for this is that the gauge fields in the old and in the new electric/magnetic duality frame are not generically related by local field redefinitions. The effect of changing the duality frame is therefore not straightforward, and it is by no means trivial to explicitly obtain the new Lagrangian (see e.g. [1]). A related aspect is that, when the gauge fields belong to supermultiplets, their relation with other fields of the multiplet will be affected by changes of the duality frame, unless one simultaneously performs corresponding redefinitions of these fields as well.[1] The modern embedding tensor approach circumvents all these problems by introducing, from the start, both electric and magnetic gauge fields as well as tensor gauge fields. In this approach the gauge group is not restricted to a subgroup of the invariance group of the Lagrangian, but it must only be a subgroup of the symmetry group of field equations and Bianchi identities. The formalism is straightforwardly applicable to any given Lagrangian, and the gauge group is only restricted by two group-theoretical constraints on the embedding tensor [3].

In this paper we study general gaugings of $N = 2$ supergravity theories based on vector supermultiplets and hypermultiplets. Because these theories can generally be studied by means of the superconformal multiplet calculus [4–6], it suffices to understand the embedding tensor framework in the context of conformal supergravity. This study is facilitated by the fact that the embedding tensor framework has already been considered for rigid $N = 2$ supersymmetric gauge theories [7], without paying particular attention to the class of superconformally invariant models. The purpose of the present paper is to fill this gap by presenting a comprehensive treatment of the embedding tensor method in the context of locally supersymmetric $N = 2$ theories.

Theories with $N = 2$ supersymmetry are special with respect to electric/magnetic duality. For $N = 1$ supersymmetry the transformations of the matter fields under electric/magnetic duality, and thus under the gauge group, are not a priori defined, and will depend on the details of the model. On the other hand, in theories with $N > 2$ supersymmetries all of the matter fields are closely linked to the vector fields, because they belong to common supermultiplets. Theories with $N = 2$ supersymmetries are exceptional in that they exhibit both of these characteristic features. The complex scalars belonging to the vector multiplets transform in a well-defined way under electric/magnetic duality so that the Lagrangian will retain its standard form expressed in terms of a holomorphic function, while the scalars of the hypermultiplets have no a priori defined transformations under electric/magnetic duality. Prior to switching on the gauging, the hypermultiplets are invariant under some rigid symmetry group that is independent of the electric/magnetic duality group. Once the gauge group has been embedded in the latter group, then one has to separately specify its embedding into the symmetry group associated with the hypermultiplets.

The embedding tensor approach of [3] makes use of both electric and magnetic charges and their corresponding gauge fields. The charges are encoded in terms of an embedding tensor, which specifies the embedding of the gauge group into the full rigid invariance group. This embedding tensor is treated as a spurionic object (a quantity that is treated as a dynamical field, but that is frozen to a constant at the end of the calculation), so that the electric/magnetic duality structure of the ungauged theory is preserved when the charges are turned on. Besides introducing a set of dual magnetic gauge fields, also tensor gauge fields are required transforming in the adjoint representation of the rigid invariance group. These extra fields carry additional off-shell degrees of freedom, but the number of

[1]One way to circumvent this is by describing the scalar fields in terms of sections whose parametrization is linked to a specific frame (see, for instance, [2]).

physical degrees of freedom remains the same owing to extra gauge transformations. Prior to [3] it had already been discovered that magnetic charges tend to be accompanied by tensor fields. An early example of this was presented in [8], and subsequently more theories with magnetic charges and tensor fields were constructed, for instance, in [9–11], mostly in the context of abelian gauge groups. The embedding tensor approach has already been explored for many supersymmetric theories in four space-time dimensions. For instance, it was successfully applied to $N = 4$ supergravity [12] and to $N = 8$ supergravity [13]. More recently it has also been discussed for $N = 1$ supergravity [14]. In [7] some applications to $N = 2$ supergravity were already presented, under the assumption that the conformal multiplet calculus [4–6] is applicable. As it turned out, the results of the embedding tensor approach confirm and/or clarify various previous results in the literature, especially for abelian gaugings [15, 16]. The embedding tensor is ideally suited for the study of flux compactifications in string theory (for a review, see [17]). Recently it was successfully employed in a study of partial breaking of $N = 2$ to $N = 1$ supersymmetry [18, 19].

The supersymmetric Lagrangians derived in this paper incorporate gaugings in both the vector and hypermultiplet sectors. The vector multiplets are initially defined as off-shell multiplets, but the presence of the magnetic charges causes a breakdown of off-shell supersymmetry. Of course, conventional hypermultiplets based on a finite number of fields will not constitute an off-shell representation of the supersymmetry algebra irrespective of the presence of charges. We refer to a more in-depth discussion of the off-shell aspects of the embedding tensor method in [7], where a construction was presented in which the tensor fields associated with the magnetic charges were contained in a tensor supermultiplet.

Besides giving a comprehensive treatment of the embedding tensor formalism in the context of local $N = 2$ supersymmetric theories, we also present two applications to illustrate how the embedding tensor formalism can be used to obtain rather general results about realizations of $N = 2$ gauged supergravities. One concerns the supersymmetric realizations in maximally symmetric spaces. In flat Minkowski space, it was established that residual supersymmetry is only possible in the presence of magnetic charges [20–24]. Here, we therefore briefly review the situation in the context of the embedding tensor approach, where it is natural to have both electric and magnetic charges.

A second application deals with supersymmetric solutions in $\mathrm{AdS}_2 \times S^2$ space-times. Here we establish that there exist only two classes of supersymmetric solutions. One concerns fully supersymmetric solutions. It contains the solutions described in [25] as well as the near-horizon solution of ungauged supergravity that appears for BPS black holes. The other class exhibits four supersymmetries and these solutions may appear as near-horizon geometries of BPS black holes in $N = 2$ gauged supergravity. Interestingly enough, solutions in $\mathrm{AdS}_2 \times S^2$ with only two supersymmetries are excluded. The spinor parameters associated with the four supersymmetries are AdS_2 Killing spinors that are constant on S^2, so that they carry no spin. Nevertheless the bosonic background is rotationally invariant. The spin assignments change in this background, because the spin rotations associated with the S^2 isometries become entangled with R-symmetry transformations, a phenomenon that is somewhat similar to what happens for magnetic monopole solutions where the rotational symmetry becomes entangled with gauge transformations [26]. In the superconformal per-

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spective, these solutions have R-symmetry connections living on S^2, and this explains the geometric origin of the entanglement. It is to be expected that the near-horizon geometry of a recently presented static, spherically symmetric, black hole solution [27, 28] will coincide with one of the solutions described in this paper. The results of this paper then imply that this black hole solution must exhibit supersymmetry enhancement at the horizon.

This paper is organized as follows. In section 2 we recall the relevant features of $N = 2$ vector multiplets and electric/magnetic duality in the context of conformal supergravity, and we introduce the electric and magnetic gauge fields. Hypermultiplets, hyperkähler cones and their isometries are introduced in a superconformal setting in section 3. In section 4 we present the relevant Lagrangians for matter fields coupled to conformal supergravity. Section 5 contains a discussion of the possible gauge transformations, the electric and magnetic charges, and the embedding tensor. In section 6 we describe the introduction of tensor fields, needed in the presence of general charge assignments. Section 7 deals with the algebra of superconformal transformations in the presence of a gauging. It presents the extra masslike terms and the scalar potential in the vector multiplet and hypermultiplet Lagrangians that are induced by these gaugings. Finally, in section 8 we summarize our results and review two applications. Readers who are not primarily interested in the more technical details of the embedding tensor formalism, can proceed directly to this section. We have refrained from collecting additional information in an appendix and refer instead to the appendices presented in [29].

2 Superconformal vector multiplets and electric/magnetic duality

Vector supermultiplets in four space-time dimensions with $N = 2$ supersymmetry can be defined in a superconformal background. Consider $n + 1$ of these multiplets, labeled by indices $\Lambda = 0, 1, \ldots, n$. Vector supermultiplets comprise complex scalar fields X^Λ, gauge fields $W_\mu{}^\Lambda$, and Majorana spinors which are conveniently decomposed into chiral and anti-chiral components: spinors $\Omega_i{}^\Lambda$ have positive, and spinors $\Omega^{i\Lambda}$ have negative chirality (so that $\gamma^5\Omega_i{}^\Lambda = \Omega_i{}^\Lambda$ and $\gamma^5\Omega^{i\Lambda} = -\Omega^{i\Lambda}$). The spinors carry indices $i = 1, 2$, and transform as doublets under the R-symmetry group SU(2). This group is realized locally with gauge fields belonging to the superconformal background, as we shall discuss below. Furthermore there are auxiliary fields $Y_{ij}{}^\Lambda$, which satisfy the pseudo-reality constraint $(Y_{ij}{}^\Lambda)^* = \varepsilon^{ik}\varepsilon^{jl}Y_{kl}{}^\Lambda$, so that they transform as real vectors under SU(2). The tensors $F_{\mu\nu}^{\pm\,\Lambda}$ are the (anti-)selfdual (complex) components of the field strengths, which will be expressed in terms of vector fields $W_\mu{}^\Lambda$. The supersymmetry transformations of these fields will depend on the superconformal background.

Before presenting the supersymmetry transformations of the vector multiplets, we first specify the superconformal background fields, which comprise the so-called Weyl supermultiplet, and their relation to the superconformal transformations. The latter contains the generators of general-coordinate, local Lorentz, dilatation, special conformal, chiral SU(2) and U(1), supersymmetry (Q) and special supersymmetry (S) transformations. The gauge fields associated with general-coordinate transformations ($e_\mu{}^a$), dilatations (b_μ), chiral symmetry ($\mathcal{V}_\mu{}^i{}_j$ and A_μ) and Q-supersymmetry ($\psi_\mu{}^i$) are independent fields. The

field	X^M	$\Omega_i{}^M$	$W_\mu{}^M$	$Y_{ij}{}^\Lambda$
w	1	$\frac{3}{2}$	0	2
c	-1	$-\frac{1}{2}$	0	0

Table 1. Weyl and chiral weights of the vector multiplet fields.

remaining gauge fields associated with the Lorentz ($\omega_\mu{}^{ab}$), special conformal ($f_\mu{}^a$) and S-supersymmetry transformations ($\phi_\mu{}^i$) are dependent fields. They are composite objects, which depend on the independent fields of the multiplet [4–6]. The corresponding super-covariant curvatures and covariant fields are contained in a tensor chiral multiplet, which comprises 24 + 24 off-shell degrees of freedom. In addition to the independent superconformal gauge fields, it contains three other fields: a Majorana spinor doublet χ^i, a scalar D, and a selfdual Lorentz tensor T_{abij}, which is anti-symmetric in $[ab]$ and $[ij]$. We refer to the appendices in [29] for an extended summary of the superconformal transformations of the Weyl multiplet fields, the expressions for the curvatures and other useful details.

The transformations of the vector multiplet fields under dilatations and chiral transformations are given in table 1. Under local Q- and S-supersymmetry they are as follows [4],

$$\begin{aligned}
\delta X^\Lambda &= \bar\epsilon^i\Omega_i{}^\Lambda\,,\\
\delta W_\mu{}^\Lambda &= \varepsilon^{ij}\bar\epsilon_i(\gamma_\mu\Omega_j{}^\Lambda + 2\,\psi_{\mu j}X^\Lambda) + \varepsilon_{ij}\bar\epsilon^i(\gamma_\mu\Omega^{j\,\Lambda} + 2\,\psi_\mu{}^j\bar X^\Lambda)\,,\\
\delta\Omega_i{}^\Lambda &= 2\not{D}X^\Lambda\epsilon_i + \frac{1}{2}\gamma^{\mu\nu}\hat F_{\mu\nu}^{-\,\Lambda}\varepsilon_{ij}\epsilon^j + Y_{ij}{}^\Lambda\epsilon^j + 2\,X^\Lambda\eta_i\,,\\
\delta Y_{ij}{}^\Lambda &= 2\,\bar\epsilon_{(i}\not{D}\Omega_{j)}{}^\Lambda + 2\,\varepsilon_{ik}\varepsilon_{jl}\,\bar\epsilon^{(k}\not{D}\Omega^{l)\Lambda}\,.
\end{aligned} \tag{2.1}$$

Here ϵ^i and ϵ_i denote the spinorial parameters of Q-supersymmetry and η^i and η_i those of S-supersymmetry. The field strengths $F_{\mu\nu}{}^\Lambda = 2\,\partial_{[\mu}W_{\nu]}{}^\Lambda$ are contained in the supercovariant combination,

$$\begin{aligned}
\hat F_{\mu\nu}{}^\Lambda = {}& F_{\mu\nu}^{+\,\Lambda} + F_{\mu\nu}^{-\,\Lambda} - \varepsilon^{ij}\bar\psi_{[\mu\,i}(\gamma_{\nu]}\Omega_j{}^\Lambda + \psi_{\nu]j}X^\Lambda) - \varepsilon_{ij}\bar\psi_{[\mu}{}^i(\gamma_{\nu]}\Omega^{j\,\Lambda} + \psi_{\nu]}{}^j\bar X^\Lambda)\\
& - \frac{1}{4}(X^\Lambda\,T_{\mu\nu ij}\,\varepsilon^{ij} + \bar X^\Lambda\,T_{\mu\nu}{}^{ij}\,\varepsilon_{ij})\,.
\end{aligned} \tag{2.2}$$

The full superconformally covariant derivatives are denoted by D_μ, while $\mathcal{D}_\mu$ will denote a covariant derivative with respect to Lorentz, dilatation, chiral U(1), and SU(2) transformations. As an example of the latter, we note the definitions,

$$\begin{aligned}
\mathcal{D}_\mu X^\Lambda &= (\partial_\mu - b_\mu + \mathrm{i}A_\mu)X^\Lambda\,,\\
\mathcal{D}_\mu\Omega_i{}^\Lambda &= \Big(\partial_\mu - \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - \frac{3}{2}b_\mu + \frac{1}{2}\mathrm{i}A_\mu\Big)\Omega_i{}^\Lambda - \frac{1}{2}\mathcal{V}_\mu{}^j{}_i\,\Omega_j{}^\Lambda\,.
\end{aligned} \tag{2.3}$$

We now assume an holomorphic function $F(X)$ of the fields X^Λ, which is homogeneous of second degree, i.e. $F(\lambda X) = \lambda^2 F(X)$, for any complex parameter λ. As is well known [5,

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30], such a function can be used to write down a consistent action for the vector multiplets in the superconformal background provided by the Weyl multiplet fields. Rather than to determine this action, we first consider an extension of the field representation that will facilitate the treatment of electric/magnetic duality in the presence of non-zero gauge charges. Since this duality ultimately involves the equations of motion, it will be essential that the action exists, but for the purpose of this section it is not necessary to display its precise form.

In the absence of charged fields, abelian gauge fields $W_\mu{}^\Lambda$ appear exclusively through the field strengths, $F_{\mu\nu}{}^\Lambda = 2\,\partial_{[\mu} W_{\nu]}{}^\Lambda$. The field equations for these fields and the Bianchi identities for the field strengths comprise $2(n+1)$ equations,

$$\partial_{[\mu} F_{\nu\rho]}{}^\Lambda = 0 = \partial_{[\mu} G_{\nu\rho]\,\Lambda}\,, \tag{2.4}$$

where

$$G_{\mu\nu\,\Lambda} = \mathrm{i}e\,\varepsilon_{\mu\nu\rho\sigma}\,\frac{\partial\mathcal{L}}{\partial F_{\rho\sigma}{}^\Lambda}\,. \tag{2.5}$$

At this point we cannot give the form of $G_{\mu\nu\Lambda}$, because we have not yet specified the action. Instead, we will extract its definition below by using supersymmetry.

It is convenient to combine the tensors $F_{\mu\nu}{}^\Lambda$ and $G_{\mu\nu\Lambda}$ into a $(2n+2)$-dimensional vector,

$$G_{\mu\nu}{}^M = \begin{pmatrix} F_{\mu\nu}{}^\Lambda \\ G_{\mu\nu\Lambda} \end{pmatrix}\,, \tag{2.6}$$

so that (2.4) reads $\partial_{[\mu} G_{\nu\rho]}{}^M = 0$. Obviously these $2(n+1)$ equations are invariant under real $2(n+1)$-dimensional rotations of the tensors $G_{\mu\nu}{}^M$,

$$\begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix} \longrightarrow \begin{pmatrix} U^\Lambda{}_\Sigma & Z^{\Lambda\Sigma} \\ W_{\Lambda\Sigma} & V_\Lambda{}^\Sigma \end{pmatrix} \begin{pmatrix} F^\Sigma \\ G_\Sigma \end{pmatrix}\,. \tag{2.7}$$

Half of the rotated tensors can be adopted as new field strengths defined in terms of new gauge fields, and the Bianchi identities on the remaining tensors can then be interpreted as field equations belonging to some new Lagrangian expressed in terms of the new field strengths. In order that such a Lagrangian exists, the real matrix in (2.7) must belong to the group $\mathrm{Sp}(2n+2;\mathbb{R})$. This group consists of real matrices that leave the skew-symmetric tensor Ω_{MN} invariant,

$$\Omega = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}\,. \tag{2.8}$$

The conjugate matrix Ω^{MN} is defined by $\Omega^{MN}\Omega_{NP} = -\delta^M{}_P$. Here we employ an $\mathrm{Sp}(2n+2;\mathbb{R})$ covariant notation for the $2(n+1)$-dimensional symplectic indices $M, N, \ldots$, such that $Z^M = (Z^\Lambda, Z_\Sigma)$. Likewise we use vectors with lower indices according to $Y_M = (Y_\Lambda, Y^\Sigma)$, transforming according to the conjugate representation so that $Z^M\,Y_M$ is invariant.

The $\mathrm{Sp}(2n+2;\mathbb{R})$ transformations are known as electric/magnetic dualities, which also act on electric and magnetic charges (for a review of electric/magnetic duality, see [1]). The Lagrangian depends on the electric/magnetic duality frame and is therefore not unique.

Different Lagrangians related by electric/magnetic duality lead to equivalent field equations and thus belong to the same equivalence class. These alternative Lagrangians remain supersymmetric but because the field strengths (and thus the underlying gauge fields) have been redefined, the standard relation between the various fields belonging to the vector supermultiplet, encoded in (2.1), is lost. However, upon a suitable redefinition of the other vector multiplet fields (possibly up to terms that will vanish subject to equations of motion) this relation can be preserved. It is to be expected that the new Lagrangian is again encoded in terms of a holomorphic homogeneous function, expressed in terms of the redefined scalar fields. Just as the Lagrangian changes, this function will change as well. Hence, different functions $F(X)$ can belong to the same equivalence class. The new function is such that the vector $X^M=(X^\Lambda,F_\Lambda)$ transforms under electric/magnetic duality according to

$$\begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde X^\Lambda \\ \tilde F_\Lambda \end{pmatrix} = \begin{pmatrix} U^\Lambda{}_\Sigma & Z^{\Lambda\Sigma} \\ W_{\Lambda\Sigma} & V_\Lambda{}^\Sigma \end{pmatrix} \begin{pmatrix} X^\Sigma \\ F_\Sigma \end{pmatrix} . \tag{2.9}$$

The new function $\tilde F(\tilde X)$ of the new scalars $\tilde X^\Lambda$ follows from integration of (2.9) and takes the form

$$\begin{aligned} \tilde F(\tilde X) = {}& F(X) - \frac{1}{2} X^\Lambda F_\Lambda(X) + \frac{1}{2}(U^{\rm T}W)_{\Lambda\Sigma} X^\Lambda X^\Sigma \\ & + \frac{1}{2}(U^{\rm T}V + W^{\rm T}Z)_\Lambda{}^\Sigma X^\Lambda F_\Sigma(X) + \frac{1}{2}(Z^{\rm T}V)^{\Lambda\Sigma} F_\Lambda(X) F_\Sigma(X) . \end{aligned} \tag{2.10}$$

There are no integration constants in this case because the function must remain homogeneous of second degree.

In general it is not easy to determine $\tilde F(\tilde X)$ from (2.10) as it involves the inversion of $\tilde X^\Lambda = U^\Lambda{}_\Sigma X^\Sigma + Z^{\Lambda\Sigma} F_\Sigma(X)$. As we emphasized in section 1, this is the reason why one prefers to avoid changing the electric/magnetic duality frame. The duality transformations on higher derivatives of $F(X)$ follow by differentiation and we note the results,

$$\begin{aligned} \tilde F_{\Lambda\Sigma}(\tilde X) &= (V_\Lambda{}^\Gamma F_{\Gamma\Xi} + W_{\Lambda\Xi})\, [\mathcal{S}^{-1}]^\Xi{}_\Sigma , \\ \tilde F_{\Lambda\Sigma\Gamma}(\tilde X) &= F_{\Xi\Delta\Omega}\, [\mathcal{S}^{-1}]^\Xi{}_\Lambda\, [\mathcal{S}^{-1}]^\Delta{}_\Sigma\, [\mathcal{S}^{-1}]^\Omega{}_\Gamma , \end{aligned} \tag{2.11}$$

where

$$\mathcal{S}^\Lambda{}_\Sigma = \frac{\partial \tilde X^\Lambda}{\partial X^\Sigma} = U^\Lambda{}_\Sigma + Z^{\Lambda\Gamma} F_{\Gamma\Sigma} . \tag{2.12}$$

It is also convenient to introduce the symmetric real matrix,

$$N_{\Lambda\Sigma} = -\mathrm{i} F_{\Lambda\Sigma} + \mathrm{i}\bar F_{\Lambda\Sigma} , \tag{2.13}$$

whose inverse will be denoted by $N^{\Lambda\Sigma}$, and which transforms under electric/magnetic duality according to

$$\tilde N_{\Lambda\Sigma}(\tilde X, \bar{\tilde X}) = N_{\Gamma\Delta}\, [\mathcal{S}^{-1}]^\Gamma{}_\Lambda\, [\bar{\mathcal{S}}^{-1}]^\Delta{}_\Sigma . \tag{2.14}$$

To determine the action of the dualities on the fermion fields, we consider supersymmetry transformations of the symplectic vector $X^M=(X^\Lambda,F_\Lambda)$, which can be written as

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$\delta X^M = \bar{\epsilon}^i \Omega_i{}^M$, thus defining an $\mathrm{Sp}(2n+2;\mathbb{R})$ covariant fermionic vector, $\Omega_i{}^M$,

$$\Omega_i{}^M = \begin{pmatrix} \Omega_i{}^\Lambda \\ F_{\Lambda\Sigma}\,\Omega_i{}^\Sigma \end{pmatrix} . \tag{2.15}$$

Complex conjugation leads to a second vector, Ω^{iM}, of opposite chirality. From (2.15) one derives that, under electric/magnetic duality,

$$\tilde{\Omega}_i{}^\Lambda = \mathcal{S}^\Lambda{}_\Sigma\,\Omega_i{}^\Sigma . \tag{2.16}$$

Note the identity

$$\Omega_{MN}\,X^M \Omega_i{}^N = 0 , \tag{2.17}$$

which also implies that supersymmetry variations of $\Omega_i{}^M$ are subject to $\Omega_{MN}\,X^M\,\delta\Omega_i{}^N = 0$ up to terms quadratic in the vector multiplet spinors. This observation explains some of the identities that we will encounter in due course.

The supersymmetry transformation of $\Omega_i{}^M$ follows from (2.1), and we decompose it into the following form,

$$\delta\Omega_i{}^M = 2\not{D} X^M \epsilon_i + \frac{1}{2}\gamma^{\mu\nu}\hat{G}^{-}_{\mu\nu}{}^M \varepsilon_{ij}\,\epsilon^j + Z_{ij}{}^M \epsilon^j + 2\,X^M \eta_i . \tag{2.18}$$

From this the existence follows of a symplectic vector of anti-selfdual supercovariant field strengths,

$$\hat{G}^{-}_{\mu\nu}{}^M = \begin{pmatrix} \hat{G}^{-}_{\mu\nu}{}^\Lambda \\ \hat{G}^{-}_{\mu\nu\Lambda} \end{pmatrix} . \tag{2.19}$$

where $\hat{G}^{-}_{\mu\nu}{}^\Lambda = \hat{F}^{-}_{\mu\nu}{}^\Lambda$, with $\hat{F}^{-}_{\mu\nu}{}^\Lambda$ defined in (2.2), and $\hat{G}^{-}_{\mu\nu\Lambda}$ is defined by,

$$\hat{G}^{-}_{\mu\nu\Lambda} = F_{\Lambda\Sigma}\hat{F}^{-}_{\mu\nu}{}^\Sigma - \frac{1}{8} F_{\Lambda\Sigma\Gamma}\,\bar{\Omega}_i{}^\Sigma \gamma_{\mu\nu}\Omega_j{}^\Gamma\,\varepsilon^{ij} . \tag{2.20}$$

We can also define a second symplectic array of anti-selfdual field strengths,

$$G^{-}_{\mu\nu}{}^M = \begin{pmatrix} G^{-}_{\mu\nu}{}^\Lambda \\ G^{-}_{\mu\nu\Lambda} \end{pmatrix} , \tag{2.21}$$

with $G_{\mu\nu}{}^\Lambda = F_{\mu\nu}{}^\Lambda$. The second component, $G_{\mu\nu\Lambda}$, then follows from the identification (compare to the decomposition (2.2)),

$$\begin{aligned} \hat{G}_{\mu\nu}{}^M = {} & G^{+}_{\mu\nu}{}^M + G^{-}_{\mu\nu}{}^M - \varepsilon^{ij}\bar{\psi}_{[\mu\,i}(\gamma_{\nu]}\Omega_j{}^M + \psi_{\nu]j}X^M) - \varepsilon_{ij}\bar{\psi}_{[\mu}{}^i(\gamma_{\nu]}\Omega^{j\,M} + \psi_{\nu]}{}^j\bar{X}^M) \\ & - \frac{1}{4}(X^M\,T_{\mu\nu ij}\,\varepsilon^{ij} + \bar{X}^M\,T_{\mu\nu}{}^{ij}\,\varepsilon_{ij}) . \end{aligned} \tag{2.22}$$

This implies the following decomposition for $G^{-}_{\mu\nu\Lambda}$ (and likewise for $G^{+}_{\mu\nu\Lambda}$),

$$G^{-}_{\mu\nu\Lambda} = F_{\Lambda\Sigma}F^{-}_{\mu\nu}{}^\Sigma - 2\mathrm{i}\mathcal{O}^{-}_{\mu\nu\Lambda} , \tag{2.23}$$

with

$$\mathcal{O}^-_{\mu\nu\Lambda} = -\frac{1}{16}\mathrm{i}F_{\Lambda\Sigma\Gamma}\,\bar\Omega_i{}^{\Sigma}\gamma_{\mu\nu}\Omega_j{}^{\Gamma}\,\varepsilon^{ij} - \frac{1}{8}N_{\Lambda\Sigma}\varepsilon_{ij}\bar\psi_\rho{}^i\gamma_{\mu\nu}\gamma^\rho\Omega^{j\Sigma}$$
$$-\frac{1}{8}N_{\Lambda\Sigma}\bar X^{\Sigma}\,\varepsilon_{ij}\bar\psi_\rho{}^i\gamma^{\rho\sigma}\gamma_{\mu\nu}\psi_\sigma{}^j + \frac{1}{8}N_{\Lambda\Sigma}\bar X^{\Sigma}\,T_{\mu\nu}{}^{ij}\varepsilon_{ij}\,. \tag{2.24}$$

Note that the homogeneity of $F(X)$ is crucial for deriving these results. The definition (2.22) shows that also $(F_{\mu\nu}{}^\Lambda, G_{\mu\nu\Sigma})$ transforms as a symplectic vector under electric/magnetic duality.

Consistency requires that the field strengths $G_{\mu\nu}{}^M$ satisfy a Bianchi identity. While $G_{\mu\nu}{}^\Lambda$ clearly does, it is not obvious for the field strengths $G_{\mu\nu\Lambda}$. The latter Bianchi identity can, however, be provided by the field equation for the vector fields following from some supersymmetric action. In that case $G_{\mu\nu\Lambda}$ will coincide with (2.5). We shall verify in section 4 that this is indeed the case for the action encoded in the holomorphic function $F(X)$. It should be obvious that also the field strengths $\hat G_{\mu\nu}{}^M$ satisfy a Bianchi-type identity, but of a more complicated form. Identities of this type have been presented in [4] for $\hat G_{\mu\nu}{}^\Lambda$.

To summarize, both the fields strengths $\hat G_{\mu\nu}{}^M$ and $G_{\mu\nu}{}^M$ transform as a symplectic vector under duality, and they differ in their fermionic terms and in terms proportional to the selfdual and anti-selfdual tensor fields T_{abij} and $T_{ab}{}^{ij}$, respectively. The supercovariant field strengths $\hat G_{\mu\nu}{}^M$ appear in the supersymmetry transformation rules of the fermions, while the field strengths $G_{\mu\nu}{}^M$, when constrained by the standard Bianchi identities, imply that $F_{\mu\nu}{}^\Lambda$ can be expressed in terms of a vector potential $W_\mu{}^\Lambda$, and is subject to corresponding field equations.

Regarding the quantities $Z_{ij}{}^M$, that also follow from (2.18), we have a similar situation. They are defined by

$$Z_{ij}{}^M = \begin{pmatrix} Y_{ij}{}^\Lambda \\ F_{\Lambda\Sigma}\,Y_{ij}{}^\Sigma - \tfrac{1}{2}F_{\Lambda\Sigma\Gamma}\,\bar\Omega_i{}^\Sigma\Omega_j{}^\Gamma \end{pmatrix}\,, \tag{2.25}$$

which suggests that $Z_{ij}{}^M$ transforms under electric/magnetic duality as a symplectic vector. However, this is only possible provided we impose a pseudo-reality condition on $Z_{ij\Lambda}$. This constraint can also be understood as the result of field equations associated with a supersymmetric action, whose Lagrangian will be presented in the next section 4.

From the fact that the field strengths $G_{\mu\nu\Lambda}$ are subject to a Bianchi identity, it follows that they can be expressed in terms of magnetic duals $W_{\mu\Lambda}$. Hence we introduce these magnetic gauge fields, whose role will eventually become clear in the context of the embedding tensor formalism which will be introduced in due course. Together with the electric gauge fields $W_\mu{}^\Lambda$, the magnetic duals constitute a symplectic vector, $W_\mu{}^M = (W_\mu{}^\Lambda, W_{\mu\Lambda})$, where $G_{\mu\nu}{}^M = 2\,\partial_{[\mu}W_{\nu]}{}^M$. As we shall see, this relationship is, however, not exact and the identification is subject to certain equations of motion. The supersymmetry transformations of $W_\mu{}^M$ are conjectured to take a duality covariant form,

$$\delta W_\mu{}^M = \varepsilon^{ij}\bar\epsilon_i(\gamma_\mu\Omega_j{}^M + 2\,\psi_{\mu j}X^M) + \varepsilon_{ij}\bar\epsilon^i(\gamma_\mu\Omega^{j\,M} + 2\,\psi_\mu{}^j\bar X^M)\,. \tag{2.26}$$

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Observe that, with this transformation rule, the field strengths $\hat{G}_{\mu\nu}{}^M$ are supercovariant. As mentioned above, $G_{\mu\nu\Lambda}$ and $2\,\partial_{[\mu}W_{\nu]\Lambda}$ are not identical! This can be seen by calculating the supersymmetry variation of $2\,\partial_{[\mu}W_{\nu]\Lambda}$ and showing that it only coincides with the supersymmetry variation of (2.23) up to equations of motion. In the presence of gauge charges in the context of embedding tensor formalism, the Lagrangian can depend simultaneously on electric and magnetic gauge fields, as is described in later sections.

The consistency, up to equations of motion, of introducing dual gauge fields $W_{\mu\Lambda}$ is also confirmed when considering the closure of the supersymmetry algebra, based on (2.26). Although we started with an off-shell definition of the vector multiplets, so that all superconformal transformations will close under commutation without imposing the equations of motion, this is not necessarily the case for the newly introduced gauge field $W_{\mu\Lambda}$. Before discussing this in detail we present the decomposition of the commutator of two infinitesimal Q-supersymmetry transformations, with parameters ϵ_1 and ϵ_2,

$$[\delta(\epsilon_1),\delta(\epsilon_2)] = \xi^\mu D_\mu + \delta_M(\varepsilon) + \delta_K(\Lambda_K) + \delta_S(\eta) + \delta_{\text{gauge}}(\Lambda^M)\,, \tag{2.27}$$

where the parameters of the various infinitesimal transformations on the right-hand side are given by

$$\begin{aligned}
\xi^\mu &= 2\,\bar{\epsilon}_2{}^i\gamma^\mu\epsilon_{1i} + \text{h.c.}\,,\\
\varepsilon^{ab} &= \bar{\epsilon}_1{}^i\epsilon_2{}^j\,T^{ab}{}_{ij} + \text{h.c.}\,,\\
\Lambda_K^a &= \bar{\epsilon}_1{}^i\epsilon_2{}^j\,D_bT^{ba}{}_{ij} - \frac{3}{2}\,\bar{\epsilon}_2{}^i\gamma^a\epsilon_{1i}\,D + \text{h.c.}\,,\\
\eta^i &= 6\,\bar{\epsilon}_{[1}{}^i\epsilon_{2]}{}^j\,\chi_j\,,\\
\Lambda^M &= 4\,\bar{X}^M\,\bar{\epsilon}_2{}^i\epsilon_1{}^j\,\varepsilon_{ij} + \text{h.c.}\,,
\end{aligned} \tag{2.28}$$

where the first term proportional to ξ^μ denotes a supercovariant translation, i.e. a general coordinate transformation with parameter ξ^μ, suitably combined with field-dependent gauge transformations so that the result is supercovariant. The terms proportional to Λ^M denote the abelian gauge transformation acting on both the electric and the magnetic gauge fields $W_\mu{}^M$. This result was already known for all the fields [4], except for $W_{\mu\Lambda}$. The validity of (2.27) on $W_{\mu\Lambda}$ can be derived in direct analogy with the calculation of the commutation relation on $W_\mu{}^\Lambda$, upon replacing $G_{\mu\nu\Lambda}$ by $2\,\partial_{[\mu}W_{\nu]\Lambda}$.

The electric/magnetic duality transformations define equivalence classes of Lagrangians. A subgroup thereof may constitute an invariance of the theory, meaning that the Lagrangian and its underlying function $F(X)$ do not change [5, 31]. More specifically, an invariance implies

$$\tilde{F}(\tilde{X}) = F(\tilde{X})\,, \tag{2.29}$$

so that the result of the duality leads to a Lagrangian based on $\tilde{F}(\tilde{X})$ which is identical to the original Lagrangian. Because $\tilde{F}(\tilde{X}) \neq F(X)$, as is obvious from (2.10), $F(X)$ is not an invariant *function*. Instead the above equation implies that the substitution $X^\Lambda \to \tilde{X}^\Lambda$

into the function $F(X)$ and its derivatives, induces precisely the duality transformations. For example, we obtain,

$$\begin{aligned} F_\Lambda(\tilde X) &= V_\Lambda{}^\Sigma F_\Sigma(X) + W_{\Lambda\Sigma} X^\Sigma\,, \\ F_{\Lambda\Sigma}(\tilde X) &= (V_\Lambda{}^\Gamma F_{\Gamma\Xi} + W_{\Lambda\Xi})\,[\mathcal{S}^{-1}]^\Xi{}_\Sigma\,, \\ F_{\Lambda\Sigma\Gamma}(\tilde X) &= F_{\Xi\Delta\Omega}\,[\mathcal{S}^{-1}]^\Xi{}_\Lambda\,[\mathcal{S}^{-1}]^\Delta{}_\Sigma\,[\mathcal{S}^{-1}]^\Omega{}_\Gamma\,. \end{aligned} \tag{2.30}$$

Another useful transformation rule is,

$$\tilde{\mathcal{O}}^-_{\mu\nu\Lambda} = \mathcal{O}^-_{\mu\nu\Sigma}\,[\mathcal{S}^{-1}]^\Sigma{}_\Lambda\,. \tag{2.31}$$

In section 5 we are precisely interested in this subclass of electric/magnetic duality transformations, as these are the ones that can be gauged.

3 Superconformal hypermultiplets

In this section we give a brief description of hypermultiplets and their isometries, following the framework of [32]. The $n_{\rm H}+1$ hypermultiplets are described by $4(n_{\rm H}+1)$ real scalars ϕ^A, $2(n_{\rm H}+1)$ positive-chirality spinors $\zeta^{\bar\alpha}$ and $2(n_{\rm H}+1)$ negative-chirality spinors ζ^α. Hence target-space indices $A, B, \ldots$ take values $1, 2, \ldots, 4(n_{\rm H}+1)$, and the indices $\alpha, \beta, \ldots$ and $\bar\alpha, \bar\beta, \ldots$ run from 1 to $2(n_{\rm H}+1)$. The chiral and anti-chiral spinors are related by complex conjugation (as we are dealing with $2(n_{\rm H}+1)$ Majorana spinors) under which indices are converted according to $\alpha \leftrightarrow \bar\alpha$.

For superconformally invariant Lagrangians, the scalar fields of the hypermultiplets parametrize a $4(n_{\rm H}+1)$-dimensional hyperkähler cone [32–35]. Such a cone has a homothetic conformal Killing vector χ^A,

$$D_A\chi^B = \delta_A{}^B\,, \tag{3.1}$$

which, locally, can be expressed in terms of a hyperkähler potential χ (in later sections denoted by $\chi_{\rm hyper}$),

$$\chi_A = \partial_A\chi\,. \tag{3.2}$$

The cone metric can thus be written as $g_{AB} = D_A\partial_B\chi$. This relation does not define the metric directly, because of the presence of the covariant derivative which contains the Christoffel connection. We also note the relation

$$\chi = \frac{1}{2} g_{AB}\,\chi^A\chi^B\,. \tag{3.3}$$

Hyperkähler spaces have three hermitian, covariantly constant complex structures $J_{ij} = J_{ji}$, satisfying the algebra of quaternions,

$$J_{ij\,AB} \equiv (J^{ij}{}_{AB})^* = \varepsilon_{ik}\varepsilon_{jl} J^{kl}{}_{AB}\,, \qquad J^{ij}{}_A{}^C\,J^{kl}{}_{CB} = \frac{1}{2}\varepsilon^{i(k}\varepsilon^{l)j}\,g_{AB} + \varepsilon^{(i(k}\,J^{l)j)}{}_{AB}\,. \tag{3.4}$$

As it turns out, the hyperkähler potential serves as a Kähler potential for each of the complex structures.

Hyperkähler cones have SU(2) isometries; the corresponding Killing vectors are expressed in terms of the complex structures and the homothetic Killing vector,

$$k_{ij}{}^{A} = J_{ij}{}^{AB}\,\chi_B\,, \tag{3.5}$$

from which it follows that

$$D_A k^{ij}{}_B = -J^{ij}{}_{AB}\,. \tag{3.6}$$

From the above results, it follows that the homothetic Killing vector χ^A and the three SU(2) Killing vectors k^{ijA} are mutually orthogonal,

$$\chi^A\chi_A = 2\chi\,, \qquad k_{ij}{}^{A}\,k^{kl}{}_A = \delta_{(i}{}^{k}\,\delta_{j)}{}^{l}\,\chi\,, \qquad \chi^A\,k^{ij}{}_A = 0\,. \tag{3.7}$$

The hypermultiplet fields transform under dilations, associated with the homothetic Killing vector, and the $\mathrm{SU}(2)\times\mathrm{U}(1)$ transformations of the superconformal group, with parameters Λ_{D}, $\Lambda_{\mathrm{SU}(2)}$ and $\Lambda_{\mathrm{U}(1)}$, respectively,

$$\begin{aligned} \delta\phi^A &= \Lambda_{\mathrm{D}}\,\chi^A + \Lambda_{\mathrm{SU}(2)}{}^{i}{}_{k}\,\varepsilon^{jk}\,k_{ij}{}^{A}\,,\\ \delta\zeta^\alpha + \delta\phi^A\,\Gamma_A{}^{\alpha}{}_{\beta}\,\zeta^\beta &= \Big(\frac{3}{2}\Lambda_{\mathrm{D}} - \frac{1}{2}\mathrm{i}\Lambda_{\mathrm{U}(1)}\Big)\zeta^\alpha\,. \end{aligned} \tag{3.8}$$

Here $\Gamma_A{}^{\alpha}{}_{\beta}$ denote the connections associated with field-dependent reparametrizations of the fermions of the form $\zeta^\alpha \to S^\alpha{}_\beta(\phi)\,\zeta^\beta$. Naturally the conjugate connections $\bar\Gamma_A{}^{\bar\alpha}{}_{\bar\beta}$ are associated with the reparametrizations $\zeta^{\bar\alpha} \to \bar S^{\bar\alpha}{}_{\bar\beta}(\phi)\,\zeta^{\bar\beta}$. These tangent-space reparametrizations act on all quantities carrying indices α and $\bar\alpha$. The corresponding curvatures $R_{AB}{}^{\alpha}{}_{\beta}$ and $\bar R_{AB}{}^{\bar\alpha}{}_{\bar\beta}$ take their values in $\mathrm{sp}(n_{\mathrm{H}}+1)\cong \mathrm{usp}(2n_{\mathrm{H}}+2;\mathbb{C})$. These curvatures are linearly related to the Riemann curvature $R_{ABC}{}^{D}$ of the target space, as we shall see later.

Before turning to the supersymmetry transformations, it is of interest to discuss possible additional isometries of hyperkähler cones that commute with supersymmetry. They are characterized by Killing vectors $k^A{}_{\mathsf{m}}(\phi)$, labeled by indices $\mathsf{m},\mathsf{n},\mathsf{p}$, etcetera. They generate a group of motions, denoted by $\mathrm{G}_{\mathrm{hyper}}$, that leaves the complex structures invariant so that they are called tri-holomorphic. Furthermore, they commute with SU(2) and dilatations. These three properties are reflected in the following equations,

$$\begin{aligned} k^C{}_{\mathsf{m}}\,\partial_C J^{ij}{}_{AB} - 2\partial_{[A}k^C{}_{\mathsf{m}}\,J^{ij}{}_{B]C} &= 0\,,\\ k_{ij}{}^{B}\,D_B k^A{}_{\mathsf{m}} = D_B k_{ij}{}^{A}\,k^B{}_{\mathsf{m}} &= J_{ij}{}^{A}{}_{B}\,k^B{}_{\mathsf{m}}\\ \chi_A\,k^A{}_{\mathsf{m}} &= 0\,. \end{aligned} \tag{3.9}$$

Such tri-holomorphic isometries can be gauged by coupling to the (electric and/or magnetic) gauge fields belonging to the vector multiplets, as we shall discuss in due course.[2] The total isometry group of the hyperkähler space is thus the product of SU(2) times the

[2] As always, the dilatations and the $\mathrm{SU}(2)\times\mathrm{U}(1)$ symmetries will be gauged when coupling to the corresponding gauge fields of conformal supergravity.

group $G_{\rm hyper}$ generated by the Killing vectors $k^A{}_{\sf m}$. The structure constants of the latter are denoted by $f_{\sf mn}{}^{\sf p}$, and follow from the Lie bracket relation,[3]

$$k^B{}_{\sf m}\,\partial_B k^A{}_{\sf n} - k^B{}_{\sf n}\,\partial_B k^A{}_{\sf m} = -f_{\sf mn}{}^{\sf p}\,k^A{}_{\sf p}\,. \tag{3.11}$$

The infinitesimal transformations act on the hypermultiplet fields according to

$$\begin{aligned}\delta\phi^A &= g\,\Lambda^{\sf m}\,k^A{}_{\sf m}(\phi)\,,\\ \delta\zeta^\alpha + \delta\phi^A\Gamma_A{}^\alpha{}_\beta\,\zeta^\beta &= g\,\Lambda^{\sf m}\,t_{\sf m}{}^\alpha{}_\beta(\phi)\,\zeta^\beta\,,\end{aligned} \tag{3.12}$$

where we introduced a generic coupling constant g and ϕ-dependent matrices $t_{\sf m}{}^\alpha{}_\beta(\phi)$ which take values in $\mathrm{sp}(n_{\rm H}+1)$, and are proportional to $D_A k^B{}_{\sf m}$. Explicit definitions will be given later, but we already note that they satisfy the following relations,

$$\begin{aligned}D_A t_{\sf m}{}^\alpha{}_\beta &= R_{AB}{}^\alpha{}_\beta\,k^B{}_{\sf m}\,,\\ [\,t_{\sf m},\,t_{\sf n}\,]^\alpha{}_\beta &= f_{\sf mn}{}^{\sf p}\,(t_{\sf p})^\alpha{}_\beta + k^A{}_{\sf m}\,k^B{}_{\sf n}\,R_{AB}{}^\alpha{}_\beta\,.\end{aligned} \tag{3.13}$$

This result is consistent with the Jacobi identity. The above results can be summarized by noting that the linear combinations, $X_{\sf m}{}^\alpha{}_\beta = \delta^\alpha{}_\beta\,k^A{}_{\sf m}D_A - t_{\sf m}{}^\alpha{}_\beta$, close under commutation according to[4]

$$[X_{\sf m}, X_{\sf n}]^\alpha{}_\beta = -f_{\sf mn}{}^{\sf p}\,X_{\sf p}{}^\alpha{}_\beta\,. \tag{3.14}$$

One can show that the curl of $J^{ij}{}_{AB}\,k^B{}_{\sf m}$ vanishes, so that these vectors can be solved in terms of the derivative of the so-called Killing potentials, or moment maps, denoted by $\mu^{ij}{}_{\sf m}$. On the hyperkähler cone there are no integration constants, and one can explicitly determine these potentials,

$$\mu^{ij}{}_{\sf m} = -\frac{1}{2}k^{ij}{}_A\,k^A{}_{\sf m}\,. \tag{3.15}$$

This can easily be verified by showing that $\partial_A\mu^{ij}{}_{\sf m} = J^{ij}{}_{AB}\,k^B{}_{\sf m}$, making use of (3.9) and the Killing equation given in (3.10). Using also (3.11) one derives the so-called equivariance condition,

$$J^{ij}{}_{AB}\,k^A{}_{\sf m}\,k^B{}_{\sf n} = -f_{\sf mn}{}^{\sf p}\,\mu^{ij}{}_{\sf p}\,. \tag{3.16}$$

The Killing potentials scale with weight $w=2$ under dilatations and transform covariantly under the isometries and SU(2) transformations,

$$\begin{aligned}\delta\mu^{ij}{}_{\sf m} &= \big(g\,\Lambda^{\sf n}\,k^A{}_{\sf n} + \Lambda_{\rm SU(2)}{}^k{}_m\,\varepsilon^{lm}\,k_{kl}{}^A\big)\,\partial_A\mu^{ij}{}_{\sf m}\\ &= \big(-g\,\Lambda^{\sf n}\,f_{\sf nm}{}^{\sf p}\,\mu^{ij}{}_{\sf p} + 2\,\Lambda_{\rm SU(2)}{}^{(i}{}_k\,\mu^{j)k}{}_{\sf m}\big)\,.\end{aligned} \tag{3.17}$$

[3]We note that derivatives of Killing vectors are constrained by the Killing equation, which induces constraints on multiple derivatives, as is shown below,

$$D_A k_B + D_B k_A = 0\,, \qquad D_A D_B k_C = R_{BCAE}\,k^E\,. \tag{3.10}$$

[4]To be precise, the $X_{\sf m}$ are the generators acting on ϕ-dependent tangent-space tensors (provided the matrix $t_{\sf m}$ is replaced by the appropriate generator for the corresponding tensor representation).

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So far, supersymmetry played a central role, as most of the above results are implied by the superconformal algebra imposed on the hypermultiplet fields. We refer the reader to [32] for a full derivation along these lines. To define the supersymmetry transformations one needs the notion of quaternionic vielbeine, which can convert the $4(n_{\rm H}+1)$ target-space indices $A, B, \ldots$ to the tangent-space indices $\alpha, \beta, \ldots, \bar\alpha, \bar\beta \ldots$ carried by the fermions. All quantities of interest can be expressed in terms of these vielbeine. For instance, the scalar fields transform as follows under supersymmetry,

$$\delta\phi^A = 2(\gamma^A_{i\bar\alpha}\,\bar\epsilon^i\zeta^{\bar\alpha} + \bar\gamma^{Ai}_{\alpha}\,\bar\epsilon_i\zeta^\alpha)\,, \tag{3.18}$$

where the pseudoreal quantity $\gamma^A_{i\bar\alpha}(\phi)$ corresponds to the $(4n_{\rm H}+4)\times(4n_{\rm H}+4)$ inverse quaternionic vielbein. Its inverse is the vielbein denoted by $\bar V^{i\bar\alpha}_A$, which is needed for writing down the supersymmetry transformation of the fermions. So we have,

$$\begin{aligned} \bar V^{i\bar\alpha}_A\,\gamma^A_{j\bar\beta} &= \delta^i{}_j\,\delta^{\bar\alpha}{}_{\bar\beta}\,,\\ \gamma^A_{i\bar\alpha}\bar V^{j\bar\alpha}_B + \bar\gamma^{Aj}_{\alpha}V^\alpha_{Bi} &= \delta_i{}^j\,\delta^A{}_B\,. \end{aligned} \tag{3.19}$$

Here we emphasize that we use a notation (as elsewhere in this paper) where SU(2) indices are raised and lowered by complex conjugation. The quaternionic vielbeine are covariantly constant, e.g.,

$$D_A\gamma^B_{i\bar\alpha} = \partial_A\gamma^B_{i\bar\alpha} + \Gamma_{AC}{}^B\gamma^C_{i\bar\alpha} - \bar\Gamma_A{}^{\bar\beta}{}_{\bar\alpha}\,\gamma^B_{i\bar\beta} = 0\,. \tag{3.20}$$

Observe that it is not necessary to introduce a SU(2) connection here. When coupling to the superconformal fields, the SU(2) symmetry will be realized locally and a connection will be provided by the gauge field $\mathcal{V}_\mu{}^i{}_j$ of the Weyl multiplet. The fact that the vielbeine are covariantly constant provides a relation between the Riemann curvature $R_{ABC}{}^D$ and the tangent-space curvature $\bar R_{AB}{}^{\bar\alpha}{}_{\bar\beta}$,

$$R_{ABC}{}^D\,\gamma^C_{i\bar\alpha} - \bar R_{AB}{}^{\bar\beta}{}_{\bar\alpha}\,\gamma^D_{i\bar\beta} = 0\,. \tag{3.21}$$

Both curvatures can actually be written in terms of

$$W_{\bar\alpha\beta\bar\gamma\delta} = \frac{1}{2}R_{ABCD}\,\gamma^A_{i\bar\alpha}\,\bar\gamma^{iB}_{\beta}\,\gamma^C_{j\bar\gamma}\,\bar\gamma^{jD}_{\delta}\,, \tag{3.22}$$

which appears as the coefficient of the four-spinor term in the supersymmetric Lagrangian (cf. (4.7)).

A typical feature of the superconformal hypermultiplets is that they can be formulated in terms of local sections $A_i{}^\alpha(\phi)$ of an $\mathrm{Sp}(n_{\rm H}+1)\times\mathrm{Sp}(1)$ bundle.[5] This section is provided by

$$A_i{}^\alpha(\phi) \equiv \chi^B(\phi)\,V^\alpha_{Bi}(\phi)\,. \tag{3.23}$$

Obviously the vielbeine can be re-obtained from these sections, as we easily derive,

$$D_B A_i{}^\alpha = V^\alpha_{Bi}\,. \tag{3.24}$$

[5] The existence of such an associated quaternionic bundle was established based on a general analysis of quaternion-Kähler manifolds [36]. Here $\mathrm{Sp}(1)\cong\mathrm{SU}(2)$ denotes the corresponding R-symmetry subgroup of the $N=2$ superconformal group.

We note a few relevant equations,

$$
\begin{aligned}
g^{AB}\, D_A A_i{}^\alpha\, D_B A_j{}^\beta &= \varepsilon_{ij}\, \Omega^{\alpha\beta}\,, \\
g^{AB}\, D_A A_i{}^\alpha\, D_B A^{j\bar\beta} &= \delta_i{}^j\, G^{\alpha\bar\beta}\,,
\end{aligned} \tag{3.25}
$$

which defines two tensors, $\Omega^{\alpha\beta}$ and $G^{\alpha\bar\beta}$, which are skew symmetric and hermitian, respectively. Obviously both tensors are covariantly constant. We also note the following relations,

$$
\begin{aligned}
G_{\bar\alpha\beta}\, V_{A\,i}^{\beta} &= \varepsilon_{ij}\, \Omega_{\bar\alpha\bar\beta}\, \bar V_A^{j\bar\beta} = g_{AB}\, \gamma^B_{i\bar\alpha}\,, \\
G_{\bar\gamma\alpha} \bar\Omega^{\bar\gamma\bar\delta} G_{\bar\delta\beta} &= \bar\Omega_{\alpha\beta}\,, \\
\Omega_{\bar\alpha\bar\beta} \bar\Omega^{\bar\beta\bar\gamma} &= -\,\delta_{\bar\alpha}{}^{\bar\gamma}\,, \\
\bar\Omega_{\alpha\beta}\, A_i{}^\alpha A_j{}^\beta &= \varepsilon_{ij}\chi\,.
\end{aligned} \tag{3.26}
$$

The first one establishes the fact that the quaternionic vielbein V^α_{Ai} is pseudoreal. Furthermore we note

$$
\begin{aligned}
\bar\Omega_{\alpha\beta} A_i{}^\alpha\, D_B A_j{}^\beta &= \frac{1}{2}\varepsilon_{ij}\chi_B + k_{ijB}\,, \\
\bar\Omega_{\alpha\beta}\, D_A A_i{}^\alpha\, D_B A_j{}^\beta &= \frac{1}{2}\varepsilon_{ij}\, g_{AB} - J_{ij\,AB}\,, \\
A^{i\bar\alpha} \equiv (A_i{}^\alpha)^* &= \varepsilon^{ij}\, \bar\Omega^{\bar\alpha\bar\beta}\, G_{\bar\beta\gamma}\, A_j{}^\gamma\,.
\end{aligned} \tag{3.27}
$$

Let us now introduce the local Q- and S-supersymmetry transformations of the hypermultiplet fields, employing the sections $A_i{}^\alpha$

$$
\begin{aligned}
\delta A_i{}^\alpha + \delta\phi^B \Gamma_B{}^\alpha{}_\beta A_i{}^\beta &= 2\,\bar\epsilon_i \zeta^\alpha + 2\,\varepsilon_{ij} G^{\alpha\bar\beta} \Omega_{\bar\beta\bar\gamma}\, \bar\epsilon^j \zeta^{\bar\gamma}\,, \\
\delta\zeta^\alpha + \delta\phi^A\, \Gamma_A{}^\alpha{}_\beta\, \zeta^\beta &= \not{D} A_i{}^\alpha\, \epsilon^i + A_i{}^\alpha\, \eta^i\,, \\
\delta\zeta^{\bar\alpha} + \delta\phi^A\, \bar\Gamma_A{}^{\bar\alpha}{}_{\bar\beta}\, \zeta^{\bar\beta} &= \not{D} A^{i\bar\alpha}\, \epsilon_i + A^{i\bar\alpha}\, \eta_i\,.
\end{aligned} \tag{3.28}
$$

The Weyl and chiral weights of these sections and the fermion fields are listed in table 2. The reader can easily verify that these weight assignments are consistent with the above supersymmetry transformations. The bosonic parts of the covariant derivatives on the scalar and fermion fields is given by,

$$
\begin{aligned}
\mathcal{D}_\mu \phi^A &= \partial_\mu \phi^A - b_\mu\, \chi^A + \frac{1}{2}\mathcal{V}_\mu{}^i{}_k\, \varepsilon^{jk}\, k^A_{ij}\,, \\
\mathcal{D}_\mu A_i{}^\alpha &= \partial_\mu A_i{}^\alpha - b_\mu A_i{}^\alpha + \frac{1}{2}\mathcal{V}_{\mu i}{}^j A_j{}^\alpha + \partial_\mu\phi^A \Gamma_A{}^\alpha{}_\beta A_i{}^\beta\,, \\
\mathcal{D}_\mu \zeta^\alpha &= \partial_\mu \zeta^\alpha - \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\, \zeta^\alpha - \frac{3}{2} b_\mu \zeta^\alpha + \frac{1}{2}\mathrm{i} A_\mu \zeta^\alpha + \partial_\mu \phi^A\, \Gamma_A{}^\alpha{}_\beta\, \zeta^\beta\,,
\end{aligned} \tag{3.29}
$$

where we have now introduced the superconformal gauge fields, in addition to the target-space connections. The covariantization of the above derivatives with respect to Q- and S-supersymmetry follows immediately from (3.28).

field	$A_i{}^\alpha$	ζ^α
w	1	$\frac{3}{2}$
c	0	$-\frac{1}{2}$

Table 2. Weyl and chiral weights of the hypermultiplet fields.

An expression for the generators t_{m} associated with the tri-holomorphic Killing vectors follows from requiring the invariance of the quaternionic vielbeine V_{Ai}^α up to a target-space rotation,

$$(t_{\mathsf{m}})^\alpha{}_\beta = \frac{1}{2} V_{Ai}^\alpha \, \bar\gamma_\beta^{Bi} \; D_B k^A{}_{\mathsf{m}} \,. \tag{3.30}$$

The invariance implies that target-space scalars satisfy algebraic identities such as

$$\bar t_{\mathsf{m}}{}^{\bar\gamma}{}_{\bar\alpha} \, G_{\bar\gamma\beta} + t_{\mathsf{m}}{}^\gamma{}_\beta \, G_{\bar\alpha\gamma} = 0 = \bar t_{\mathsf{m}}{}^{\bar\gamma}{}_{[\bar\alpha} \, \Omega_{\bar\beta]\bar\gamma} \,, \tag{3.31}$$

which confirm that the matrices $t_{\mathsf{m}}{}^\alpha{}_\beta$ take values in $\mathrm{sp}(n_{\mathrm{H}}+1)$. Furthermore we note the relations,

$$\begin{aligned} k^A{}_{\mathsf{m}} \, V_{Ai}^\alpha &= k^A{}_{\mathsf{m}} \, D_A A_i{}^\alpha = t_{\mathsf{m}}{}^\alpha{}_\beta \, A_i{}^\beta \,, \\ \mu_{ij\mathsf{m}} &= -\frac{1}{2} k_{Aij} \, k^A{}_{\mathsf{m}} = -\frac{1}{2} \bar\Omega_{\alpha\beta} \, A_i{}^\alpha \, t_{\mathsf{m}}{}^\beta{}_\gamma A_j{}^\gamma \,. \end{aligned} \tag{3.32}$$

For a more complete list of identities we refer to [32].

4 Lagrangians

In this section we consider the various matter Lagrangians that are superconformally invariant. All these Lagrangians can be found in the literature (see, e.g., [4–6, 32]), including some of the terms quartic in the fermions. We have not eliminated any auxiliary fields, so that the results pertain to fully off-shell couplings, with the exception of the hypermultiplets. In the formula below, we have substituted the explicit expressions for the dependent gauge fields associated with Lorentz transformations, conformal boosts and S-supersymmetry. For these expressions we refer to the appendices in [29].

All Lagrangians given below can be viewed as matter Lagrangians in a given superconformal supergravity background. However, the conformal supergravity background represents dynamical degrees of freedom which will mix with the matter degrees of freedom. For the Lagrangian of the vector multiplets, physical fields can be identified that are invariant under scale transformations and S-supersymmetry, so that we will be dealing with supergravity coupled to only n vector supermultiplets. The remaining vector multiplet acts as a compensating field: its scalar and spinor degrees of freedom are not physical and only the vector field and the corresponding triplet of auxiliary fields remain. For the hypermultiplet Lagrangians, a similar rearrangement of degrees of freedom will take place. One of the hypermultiplets will play the role of a compensator with respect to the local SU(2).

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The precise choice of the compensator multiplets is irrelevant, and the resulting theories remain gauge equivalent.[6] Therefore it is best to not make any particular choice for the compensating multiplets at this stage and keep the formulae in their most symmetric form. At the end one may then select fields that are invariant under certain local superconformal transformations, so that the compensating fields decouple from the Lagrangian, or one may simply adopt a convenient gauge choice.

The Lagrangian for the vector multiplets is decomposed into four separate parts,

$$\mathcal{L}_{\text{vector}} = \mathcal{L}^{(1)}_{\text{kin}} + \mathcal{L}^{(2)}_{\text{kin}} + \mathcal{L}_{\text{aux}} + \mathcal{L}_{\text{conf}} \,, \tag{4.1}$$

which are each separately consistent with electric/magnetic duality. We stress that this is not a invariance property. Under generic electric/magnetic duality, one obtains in general a different Lagrangian based on a function $\tilde{F}(\tilde{X})$ that is not identical to the original function. Only the subgroup that satisfies (2.29) constitutes an invariance. The only terms that have been suppressed in (4.1) are quartic in the fermion fields and separately consistent with respect to electric/magnetic duality.

The first term in (4.1) contains the kinetic terms of the scalar and spinor fields,

$$\begin{aligned} e^{-1}\mathcal{L}^{(1)}_{\text{kin}} =& -\mathrm{i}\Omega_{MN}\, \mathcal{D}_\mu X^M\, \mathcal{D}^\mu \bar{X}^N + \frac{1}{4}\mathrm{i}\Omega_{MN}\left[\bar{\Omega}^{iM} \not{D}\Omega_i{}^N - \bar{\Omega}_i{}^M \not{D}\Omega^{iN}\right] \\ & -\frac{1}{2}\mathrm{i}\Omega_{MN}\left[\bar{\psi}_\mu{}^i \not{D}\bar{X}^M \gamma^\mu\, \Omega_i{}^N - \bar{\psi}_{\mu i}\not{D}X^M\gamma^\mu\, \Omega^{iN}\right] . \end{aligned} \tag{4.2}$$

The kinetic terms for the vector fields and their moment couplings to the tensor and fermion fields are contained in $\mathcal{L}^{(2)}_{\text{kin}}$,

$$\begin{aligned} e^{-1}\mathcal{L}^{(2)}_{\text{kin}} =& \frac{1}{4}\mathrm{i}\left[F_{\Lambda\Sigma}\, F^{-\Lambda}_{\mu\nu} F^{-\mu\nu\Sigma} - \bar{F}_{\Lambda\Sigma}\, F^{+\Lambda}_{\mu\nu} F^{+\mu\nu\Sigma}\right] \\ & + \left[\mathcal{O}^-_{\mu\nu\Lambda} F^{-\mu\nu\Lambda} - N^{\Lambda\Sigma}\, \mathcal{O}^-_{\mu\nu\Lambda}\mathcal{O}^{-\mu\nu}{}_\Sigma + \text{h.c.}\right] , \end{aligned} \tag{4.3}$$

with $\mathcal{O}^-_{\mu\nu\Lambda}$ as defined in (2.24). Here we included a term quadratic in the tensors $\mathcal{O}$, such that the resulting expression is consistent with respect to electric/magnetic duality.[7] Note that one can explicitly construct the field strength tensors $G_{\mu\nu\Lambda}$ from (4.1), according to definition (2.5). The result coincides precisely with the expression given by (2.23), as was claimed previously.

The terms associated with the auxiliary fields $Y_{ij}{}^\Lambda$ are given in $\mathcal{L}_{\text{aux}}$ [7],

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{aux}} =& \frac{1}{8}N^{\Lambda\Sigma}\left(N_{\Lambda\Gamma}Y_{ij}{}^\Gamma + \frac{1}{2}\mathrm{i}(F_{\Lambda\Gamma\Omega}\,\bar{\Omega}_i{}^\Gamma\Omega_j{}^\Omega - \bar{F}_{\Lambda\Gamma\Omega}\,\bar{\Omega}^{k\Gamma}\Omega^{l\Omega}\varepsilon_{ik}\varepsilon_{jl})\right) \\ & \times\left(N_{\Sigma\Xi}Y^{ij\Xi} + \frac{1}{2}\mathrm{i}(F_{\Sigma\Xi\Delta}\,\bar{\Omega}_m{}^\Xi\Omega_n{}^\Delta\varepsilon^{im}\varepsilon^{jn} - \bar{F}_{\Sigma\Xi\Delta}\,\bar{\Omega}^{i\Xi}\Omega^{j\Delta})\right) . \end{aligned} \tag{4.5}$$

[6]The hypermultiplet compensator can be replaced by a tensor multiplet, but this option will not be considered here.

[7]To appreciate the presence of this term, we note that (4.3) can be written as

$$e^{-1}\mathcal{L}^{(2)}_{\text{kin}} = \frac{1}{4}\mathrm{i}\left[F^{-\Lambda}_{\mu\nu}\, G^{-\mu\nu}{}_\Lambda + \text{h.c.}\right] - \mathrm{i}\left[\mathcal{O}^{-\mu\nu}{}_\Sigma\, N^{\Sigma\Lambda}\,(G^-_{\mu\nu\Lambda} - \bar{F}_{\Lambda\Gamma}\, F^{-}_{\mu\nu}{}^\Lambda) + \text{h.c.}\right] . \tag{4.4}$$

Modulo the field equation of the vector fields, the first term can be written as a total derivative, whereas the second term is manifestly consistent with electric/magnetic duality as follows from (2.14), (2.30) and (2.31).

Note that the field equations for the auxiliary fields $Y_{ij}{}^{\Lambda}$ indeed imply the pseudo-reality of $Z_{ij\Lambda}$, as was claimed below (2.25). The last part of the Lagrangian describes the remaining couplings of the vector multiplet fields to conformal supergravity,

$$\begin{aligned}
e^{-1}\mathcal{L}_{\text{conf}} =& \frac{1}{6}\chi_{\text{vector}}\left[R + (e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_{\mu}{}^{i}\gamma_{\nu}\mathcal{D}_{\rho}\psi_{\sigma i} - \bar{\psi}_{\mu}{}^{i}\psi_{\nu}{}^{j}\,T^{\mu\nu}{}_{ij} + \text{h.c.})\right] \\
& - \chi_{\text{vector}}\left[D + \frac{1}{2}\bar{\psi}_{\mu}{}^{i}\gamma^{\mu}\chi_i + \frac{1}{2}\bar{\psi}_{\mu i}\gamma^{\mu}\chi^{i}\right] \\
& - \left(\frac{\partial\chi_{\text{vector}}}{\partial X^{\Lambda}}\left[\frac{1}{3}\bar{\Omega}_i{}^{\Lambda}\gamma^{\mu\nu}\mathcal{D}_{\mu}\psi_{\nu}{}^{i} - \bar{\Omega}_i{}^{\Lambda}\chi^{i}\right] + \text{h.c.}\right) \\
& - \left(\frac{\partial\chi_{\text{vector}}}{\partial X^{\Lambda}}\left[\frac{1}{4}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_{\mu i}\gamma_{\nu}\psi_{\rho}{}^{i}\,\mathcal{D}_{\sigma}X^{\Lambda} + \frac{1}{48}\,\bar{\psi}_{i\mu}\gamma^{\mu}\gamma_{\rho\sigma}\Omega_j{}^{\Lambda}\,T^{ij\rho\sigma}\right] + \text{h.c.}\right),
\end{aligned} \tag{4.6}$$

where $\chi_{\text{vector}} = \mathrm{i}(X^{\Lambda}\bar{F}_{\Lambda} - \bar{X}^{\Lambda}F_{\Lambda}) = N_{\Lambda\Sigma}X^{\Lambda}\bar{X}^{\Sigma} = \mathrm{i}\Omega_{MN}X^{M}\bar{X}^{N}$. Note that $\partial\chi_{\text{vector}}/\partial X^{\Lambda} = N_{\Lambda\Sigma}\bar{X}^{\Sigma}$.

We now exhibit the superconformal Lagrangian for hypermultiplets [32, 35],

$$\begin{aligned}
e^{-1}\mathcal{L}_{\text{hyper}} =& \frac{1}{6}\,\chi_{\text{hyper}}\left[R + (e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_{\mu}{}^{i}\gamma_{\nu}\mathcal{D}_{\rho}\psi_{\sigma i} - \frac{1}{4}\bar{\psi}_{\mu}{}^{i}\psi_{\nu}{}^{j}\,T^{\mu\nu}{}_{ij} + \text{h.c.})\right] \\
& + \frac{1}{2}\,\chi_{\text{hyper}}\left[D + \frac{1}{2}\bar{\psi}_{\mu}{}^{i}\gamma^{\mu}\chi_i + \frac{1}{2}\bar{\psi}_{\mu i}\gamma^{\mu}\chi^{i}\right], \\
& - \frac{1}{2}G_{\bar{\alpha}\beta}\,\mathcal{D}_{\mu}A_i{}^{\beta}\,\mathcal{D}^{\mu}A^{i\bar{\alpha}} - G_{\bar{\alpha}\beta}(\bar{\zeta}^{\bar{\alpha}}\not{\mathcal{D}}\zeta^{\beta} + \bar{\zeta}^{\beta}\not{\mathcal{D}}\zeta^{\bar{\alpha}}) - \frac{1}{4}W_{\bar{\alpha}\beta\bar{\gamma}\delta}\,\bar{\zeta}^{\bar{\alpha}}\gamma_{\mu}\zeta^{\beta}\,\bar{\zeta}^{\bar{\gamma}}\gamma^{\mu}\zeta^{\delta} \\
& - \frac{\partial\chi_{\text{hyper}}}{\partial\phi^{A}}\left(\gamma^{A}{}_{i\bar{\alpha}}\left[\frac{2}{3}\bar{\zeta}^{\bar{\alpha}}\gamma^{\mu\nu}\mathcal{D}_{\mu}\psi_{\nu}{}^{i} + \bar{\zeta}^{\bar{\alpha}}\chi^{i} - \frac{1}{6}\,\bar{\zeta}^{\bar{\alpha}}\gamma_{\mu}\psi_{\nu j}\,T^{\mu\nu ij}\right] + \text{h.c.}\right) \\
& + \left[\frac{1}{16}\,\bar{\Omega}_{\alpha\beta}\,\bar{\zeta}^{\alpha}\gamma^{\mu\nu}T_{\mu\nu ij}\varepsilon^{ij}\zeta^{\beta} - \frac{1}{2}\,\bar{\zeta}^{\alpha}\gamma^{\mu}\gamma^{\nu}\psi_{\mu i}\left(\bar{\psi}_{\nu}{}^{i}\,G_{\alpha\bar{\beta}}\,\zeta^{\bar{\beta}} + \varepsilon^{ij}\,\bar{\Omega}_{\alpha\beta}\,\bar{\psi}_{\nu j}\zeta^{\beta}\right)\right. \\
& \left. + G_{\bar{\alpha}\beta}\,\bar{\zeta}^{\beta}\gamma^{\mu}\not{\mathcal{D}}A^{i\bar{\alpha}}\psi_{\mu i} - \frac{1}{4}e^{-1}\epsilon^{\mu\nu\rho\sigma}G_{\bar{\alpha}\beta}\,\bar{\psi}_{\mu}{}^{i}\gamma_{\nu}\psi_{\rho j}\,A_i{}^{\beta}\mathcal{D}_{\sigma}A^{j\bar{\alpha}} + \text{h.c.}\right],
\end{aligned} \tag{4.7}$$

where $W_{\bar{\alpha}\beta\bar{\gamma}\delta}$ was defined in (3.22), and the hyperkähler potential was introduced in section 3. Since this Lagrangian is superconformally invariant, the target-space geometry is that of a hyperkähler cone, which is a cone over a so-called tri-Sasakian manifold. The latter is a fibration of $\mathrm{Sp}(1)$ over a $4n_{\mathrm{H}}$-dimensional quaternion-Kähler manifold $\mathbb{Q}^{4n_{\mathrm{H}}}$. Hence the hyperkähler cone can be written as $R^{+} \times (\mathrm{Sp}(1) \times \mathbb{Q}^{4n_{\mathrm{H}}})$.

Also tensor multiplets can be coupled to conformal supergravity (see, e.g. [37]), but since those multiplets are not involved in the gaugings they will not be considered here.

5 Gauge invariance, electric and magnetic charges, and the embedding tensor

Possible gauge groups must be embedded into the rigid invariance group $\mathrm{G}_{\text{rigid}}$ of the theory. In the context of this paper, we are in principle dealing with a product group,

$G_{\rm rigid} = G_{\rm symp} \times G_{\rm hyper}$, where $G_{\rm symp}$ refers to the invariance group of the electric/magnetic dualities, which acts exclusively on the vector multiplets, and $G_{\rm hyper}$ refers to the possible invariance group of the hypermultiplet sector generated by the tri-holomorphic Killing vectors.[8] Here we first concentrate on the gauge group embedded into $G_{\rm symp}$, which constitutes a subgroup of the electric/magnetic duality group $\mathrm{Sp}(2n+2;\mathbb{R})$ related to the matrices considered in (2.7). The corresponding gauge group generators thus take the form of $(2n+2)$-by-$(2n+2)$ matrices T_M. Since we are assuming the presence of both electric and magnetic gauge fields, these generators decompose according to $T_M = (T_\Lambda, T^\Lambda)$. Obviously the gauge-group generators $T_{MN}{}^P$ must generate a subalgebra of the Lie algebra associated with $\mathrm{Sp}(2n+2;\mathbb{R})$, which implies,

$$T_{M[N}{}^Q\,\Omega_{P]Q} = 0\,, \tag{5.1}$$

or, in components,

$$T_{M\Lambda}{}^\Sigma = -T_M{}^\Sigma{}_\Lambda\,, \qquad T_{M[\Lambda\Sigma]} = 0 = T_M{}^{[\Lambda\Sigma]}\,. \tag{5.2}$$

Denoting the gauge group parameters by Λ^M, infinitesimal variations of generic $2(n+1)$-dimensional $\mathrm{Sp}(2n+2;\mathbb{R})$ vectors Y^M and Z_M thus take the form

$$\delta Y^M = -g\Lambda^N\, T_{NP}{}^M\, Y^P\,, \qquad \delta Z_M = g\Lambda^N\, T_{NM}{}^P\, Z_P\,, \tag{5.3}$$

where g denotes a universal gauge coupling constant.[9] Covariant derivatives can easily be constructed, and read,[10]

$$\begin{aligned} \mathcal{D}_\mu Y^M &= \partial_\mu Y^M + gW_\mu{}^N\, T_{NP}{}^M\, Y^P \\ &= \partial_\mu Y^M + gW_\mu{}^\Lambda\, T_{\Lambda P}{}^M\, Y^P + gW_{\mu\Lambda}\, T^\Lambda{}_P{}^M\, Y^P\,, \end{aligned} \tag{5.4}$$

and similarly for $\mathcal{D}_\mu Z_M$. The gauge fields then transform according to

$$\delta W_\mu{}^M = \mathcal{D}_\mu \Lambda^M = \partial_\mu \Lambda^M + g\, T_{PQ}{}^M W_\mu{}^P\, \Lambda^Q\,. \tag{5.5}$$

Note that, for constant parameters Λ^M, (5.5) will only be consistent with (5.3) provided that $T_{MN}{}^P$ is antisymmetric in $[MN]$. Nevertheless, as we shall see, antisymmetry of $T_{MN}{}^P$ is not necessary in the general case. Rather, it is sufficient that the $T_{MN}{}^P$ are subject to the so-called representation constraint [3],

$$T_{(MN}{}^Q\,\Omega_{P)Q} = 0 \Longrightarrow \begin{cases} T^{(\Lambda\Sigma\Gamma)} = 0\,, \\ 2\,T^{(\Gamma\Lambda)}{}_\Sigma = T_\Sigma{}^{\Lambda\Gamma}\,, \\ T_{(\Lambda\Sigma\Gamma)} = 0\,, \\ 2\,T_{(\Gamma\Lambda)}{}^\Sigma = T^\Sigma{}_{\Lambda\Gamma}\,. \end{cases} \tag{5.6}$$

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[8] Observe that the R-symmetry group, $\mathrm{SU}(2)\times\mathrm{U}(1)$, does not play a role here, as this group is already realized locally in the coupling to the superconformal background.

[9] The generators follow by expanding the symplectic matrix appearing in (2.7) and (2.9) about the identity. Comparing with (5.3), one establishes the correspondence, $U^\Lambda{}_\Sigma \approx \delta^\Lambda{}_\Sigma - g\Lambda^M T_{M\Sigma}{}^\Lambda$, $V_\Lambda{}^\Sigma \approx \delta_\Lambda{}^\Sigma + g\Lambda^M T_{M\Lambda}{}^\Sigma$, $Z^{\Lambda\Sigma} \approx -g\Lambda^M T_M{}^{\Lambda\Sigma}$, $W_{\Lambda\Sigma} \approx -g\Lambda^M T_{M\Lambda\Sigma}$.

[10] In this section and in section 6, we suppress the covariantization with respect to superconformal symmetries. Starting with section 7 the derivative $\mathcal{D}_\mu$ will indicate covariantization with respect to Lorentz, dilatation, and chiral symmetries, and with the newly introduced gauge symmetries associated with the fields $W_\mu{}^M$.

which does not imply antisymmetry of $T_{MN}{}^P$ in $[M,N]$. However, for the conventional electric gaugings, where the magnetic gauge fields $A_{\mu\Lambda}$ decouple and where $T^{\Lambda}{}_N{}^P=0$ and $T_\Lambda{}^{\Sigma\Gamma}=0$, (5.6) does imply that $T_{\Gamma\Sigma}{}^\Lambda$ is antisymmetric in $[\Gamma\Sigma]$.

Note that full covariance of the derivative defined in (5.4) has not yet been established to order g^2, since we have not discussed the closure of the gauge group generators. This point will be addressed later in this section.

Let us first consider some generic features of the infinitesimal transformations (5.3). Combining the two equations (2.10) and (2.29) leads to an expression for $F(\tilde{X})-F(X)$, which, for an infinitesimal symmetry transformation $\delta X^\Lambda=-g\,\Lambda^M T_{MN}{}^\Lambda\,X^N$, yields

$$F_\Lambda\,\delta X^\Lambda=-\frac{1}{2}g\Lambda^M\Big(T_{M\Lambda\Sigma}X^\Lambda X^\Sigma+T_M{}^{\Lambda\Sigma}F_\Lambda F_\Sigma\Big)\,. \tag{5.7}$$

Substituting the expression for δX^Λ then leads to the condition [5],

$$T_{MN}{}^Q\Omega_{PQ}\,X^NX^P=T_{M\Lambda\Sigma}X^\Lambda X^\Sigma-2T_{M\Lambda}{}^\Sigma X^\Lambda F_\Sigma-T_M{}^{\Lambda\Sigma}F_\Lambda F_\Sigma=0\,. \tag{5.8}$$

which must hold for general X^Λ. The solution of this condition will specify all continuous symmetries of the Lagrangian. There are two more useful identities that follow from it. First one takes the derivative of (5.8) with respect to X^Λ,

$$T_{MN\Lambda}X^N=F_{\Lambda\Sigma}\,T_{MN}{}^\Sigma X^N\,, \tag{5.9}$$

and subsequently applies a supersymmetry transformation leading to,

$$T_{MN\Lambda}\Omega_i{}^N=F_{\Lambda\Sigma}\,T_{MN}{}^\Sigma\Omega_i{}^N+F_{\Lambda\Sigma\Gamma}\,\Omega_i{}^\Sigma\,T_{MN}{}^\Gamma X^N\,. \tag{5.10}$$

The latter two identities show that the gauge covariantization of the kinetic term for the scalars and spinors in (4.2) will not involve $T_{M\Lambda\Sigma}$. We refer to [7] for further details about these covariant derivatives.

By introducing a vector $U^M=(U^\Lambda,F_{\Lambda\Sigma}U^\Sigma)$, it is possible to cast (5.9) in the symplectically covariant form, $T_{MN}{}^Q\,\Omega_{PQ}\,X^NU^P=0$. This equation can be rewritten by making use of the representation constraint (5.6). Note, for instance, the following identities,

$$\begin{aligned}
T_{(MN)}{}^P\,X^M\,U^N&=0\,,\\
T_{MN}{}^Q\,\Omega_{PQ}\,\bar{X}^MX^NX^P&=T_{MN}{}^Q\,\Omega_{PQ}\,\bar{X}^MX^N\bar{X}^P=0\,,\\
T_{MN}{}^\Lambda\,X^M\bar{X}^N\,N_{\Lambda\Sigma}\,X^\Sigma&=0\,.
\end{aligned} \tag{5.11}$$

As a side remark we note that the Killing potential (or moment map) associated with the isometries considered above, is related to

$$\nu_M=g\,T_{MN}{}^Q\Omega_{PQ}\bar{X}^NX^P\,. \tag{5.12}$$

Its derivative takes the form $\partial_\Lambda\nu_M=\mathrm{i}N_{\Lambda\Sigma}\,\delta\bar{X}^\Sigma$, as follows from making use of (5.9).

Finally we return to the gauge transformations of the auxiliary fields $Y_{ij}{}^\Lambda$, which can be derived by requiring that $\mathcal{L}_{\rm aux}$ written in (4.5) is gauge invariant. A straightforward calculation leads to the following result,

$$\delta Y_{ij}{}^\Lambda=-\frac{1}{2}g\Lambda^MT_{MN}{}^\Lambda(Z_{ij}{}^N+\varepsilon_{ik}\varepsilon_{jl}\,Z^{klN})\,, \tag{5.13}$$

where ${Z_{ij}}^M$ was defined in (2.25). Note that this result is in accord with the electric/magnetic dualities suggested for ${Z_{ij}}^M$.

In the remainder of this section we consider the gauge group embedding in more detail. The embedding into the rigid invariance group $\mathrm{G}_{\text{rigid}} = \mathrm{G}_{\text{symp}} \times \mathrm{G}_{\text{hyper}}$ is encoded in a so-called embedding tensor. This tensor must be specified separately for the vector multiplet and for the hypermultiplet sector, so that we have the following definitions,

$$\begin{aligned} T_{MN}{}^P &= \Theta_M{}^{\mathsf{a}}\, t_{\mathsf{a}N}{}^P \,, \\ k^A{}_M &= \Theta_M{}^{\mathsf{m}}\, k^A{}_{\mathsf{m}} \,, \qquad T_M{}^\alpha{}_\beta = \Theta_M{}^{\mathsf{m}}\, t_{\mathsf{m}}{}^\alpha{}_\beta \,, \end{aligned} \tag{5.14}$$

where the t_{a} denote the generators of G_{symp}, and $k^A{}_{\mathsf{m}}$ and t_{m} the tri-holomorphic Killing vectors and the corresponding matrices of the group $\mathrm{G}_{\text{hyper}}$. Because these generators belong to different groups and act on different multiplets, they carry different indices (namely, indices $M, N, \ldots$ for the vector multiplets and indices $\alpha, \beta, \ldots$ for the hypermultiplets). The embedding tensor can be further decomposed into electric and magnetic components, according to $\Theta_M{}^{\mathsf{a}} = (\Theta_\Lambda{}^{\mathsf{a}}, \Theta^{\Lambda\,\mathsf{a}})$, and $\Theta_M{}^{\mathsf{m}} = (\Theta_\Lambda{}^{\mathsf{m}}, \Theta^{\Lambda\,\mathsf{m}})$. With these definitions, we can now also present the gauge-covariant derivatives on the hypermultiplet fields (we remind the reader that in this section and in the next one, we suppress the covariantization with respect to the superconformal symmetries),

$$\begin{aligned} \mathcal{D}_\mu \phi^A &= \partial_\mu \phi^A - g W_\mu{}^M\, k^A{}_M \,, \\ \mathcal{D}_\mu A_i{}^\alpha &= \partial_\mu A_i{}^\alpha + \partial_\mu \phi^A\, \Gamma_A{}^\alpha{}_\beta\, A_i{}^\beta - g W_\mu{}^M\, T_M{}^\alpha{}_\beta A_i{}^\beta \,, \\ \mathcal{D}_\mu \zeta^\alpha &= \partial_\mu \zeta^\alpha + \partial_\mu \phi^A\, \Gamma_A{}^\alpha{}_\beta\, \zeta^\beta - g W_\mu{}^M T_M{}^\alpha{}_\beta\, \zeta^\beta \,. \end{aligned} \tag{5.15}$$

In particular the covariant derivative of the spinor field is not entirely straightforward, in view of the fact that matrices $t_{\mathsf{m}}{}^\alpha{}_\beta$ depend on the fields ϕ^A. However, because the Jacobi identity is satisfied on these matrices, there are no further complications associated with this feature (see (3.13)).

The gauge group generators T_M should close under commutation for both representations. This leads to two equations that depend quadratically on the embedding tensor [38],

$$\begin{aligned} f_{\mathsf{ab}}{}^{\mathsf{c}}\, \Theta_M{}^{\mathsf{a}}\, \Theta_N{}^{\mathsf{b}} + (t_{\mathsf{a}})_N{}^P\, \Theta_M{}^{\mathsf{a}} \Theta_P{}^{\mathsf{c}} &= 0 \,, \\ f_{\mathsf{mn}}{}^{\mathsf{p}}\, \Theta_M{}^{\mathsf{m}}\, \Theta_N{}^{\mathsf{n}} + (t_{\mathsf{a}})_N{}^P\, \Theta_M{}^{\mathsf{a}} \Theta_P{}^{\mathsf{p}} &= 0 \,, \end{aligned} \tag{5.16}$$

where $f_{\mathsf{ab}}{}^{\mathsf{c}}$ and $f_{\mathsf{mn}}{}^{\mathsf{p}}$ are the structure constants of G_{symp} and $\mathrm{G}_{\text{hyper}}$, respectively.[11] The above equations imply that the gauge algebra generators close according to

$$[T_M, T_N] = -T_{MN}{}^P\, T_P \,, \qquad k^B{}_M \partial_B k^A{}_N - k^B{}_N \partial_B k^A{}_M = T_{MN}{}^P\, k^A{}_P \,, \tag{5.17}$$

so that the structure constants of the gauge group are contained in $-T_{MN}{}^P \equiv -\Theta_M{}^{\mathsf{a}}\, (t_{\mathsf{a}})_N{}^P$, as is required by the gauge group embedding in G_{symp}. This observation was in fact used as input when deriving (5.16). Note, however, that the gauge group structure

[11] For convenience we have ignored that the matrices t_{m} depend on the scalar fields (see, (3.14), and the preceding text).

constants are not necessarily identical to $-T_{MN}{}^{P}$, as they may differ by terms that vanish upon contraction with the embedding tensor $\Theta_P{}^{\mathsf{a}}$ or $\Theta_P{}^{\mathsf{m}}$. This explains why the $T_{MN}{}^{P}$ are not necessarily antisymmetric in M, N.

Here and henceforth, the embedding tensor will be regarded as a spurionic object which we allow to transform under the rigid invariance group $\mathrm{G}_{\text{rigid}}$, so that the Lagrangian and transformation rules will remain formally invariant. Therefore the embedding tensor can be assigned to a (not necessarily irreducible) representation of $\mathrm{G}_{\text{rigid}}$. Eventually the embedding tensor will be frozen to a constant, so that the invariance under $\mathrm{G}_{\text{rigid}}$ will be broken. In this context, it is relevant to note that (5.16) implies that the embedding tensor is invariant under the gauge group. The gauge group is thus contained in the corresponding stability subgroup of $\mathrm{G}_{\text{rigid}}$. From symmetrizing the first constraint (5.16) in (MN) and making use of the linear conditions (5.6) and (5.1), one further derives that $\Omega^{MN}\,\Theta_M{}^{\mathsf{a}}\Theta_N{}^{\mathsf{b}}\,(t_{\mathsf{b}})_P{}^{Q}$ must vanish. Hence,

$$\Omega^{MN}\,\Theta_M{}^{\mathsf{a}}\Theta_N{}^{\mathsf{b}} \;=\; 0 \quad\Longleftrightarrow\quad \Theta^{\Lambda\,[\mathsf{a}}\Theta_\Lambda{}^{\mathsf{b}]} = 0\,, \tag{5.18}$$

which implies that the charges in the vector multiplet sector are mutually local, so that an electric/magnetic duality must exist that converts all the charges to electric ones. Likewise, one derives from the second constraint (5.16),

$$\Omega^{MN}\,\Theta_M{}^{\mathsf{a}}\Theta_N{}^{\mathsf{m}} \;=\; 0 \quad\Longleftrightarrow\quad \Theta^{\Lambda\,[\mathsf{a}}\Theta_\Lambda{}^{\mathsf{m}]} = 0\,, \tag{5.19}$$

which implies that the charges in the hypermultiplet sector are mutually local with the vector multiplet charges. It is clear that gauge fields that couple exclusively to charges associated to hypermultiplets are not restricted by (5.18) and (5.19). Their corresponding gauge groups are necessarily abelian. To ensure that those charges are also mutually local, we must impose an additional constraint,

$$\Omega^{MN}\,\Theta_M{}^{\mathsf{m}}\Theta_N{}^{\mathsf{n}} \;=\; 0 \quad\Longleftrightarrow\quad \Theta^{\Lambda\,[\mathsf{m}}\Theta_\Lambda{}^{\mathsf{n}]} = 0\,, \tag{5.20}$$

which is obviously not related to the closure of the gauge algebra. As it turns out, the relations (5.18), (5.19) and (5.20) play an crucial role when discussing the Lagrangian.

Generically only a subset of the gauge fields will be involved in the gauging, so that the embedding tensor will project out a restricted set of (linear combinations of) gauge fields; the rank of the tensor determines the dimension of the gauge group, up to possible central extensions associated with abelian factors.

As stressed before, the generators $T_{MN}{}^{P}$ are not required to be antisymmetric in M, N. The symmetric part can be written as follows,

$$T_{(MN)}{}^{P} = Z^{P,\mathsf{a}}\,d_{\mathsf{a}\,MN}\;, \tag{5.21}$$

with

$$\begin{aligned} d_{\mathsf{a}\,MN} &\equiv (t_{\mathsf{a}})_M{}^{P}\,\Omega_{NP}\,,\\ Z^{M,\mathsf{a}} &\equiv \frac{1}{2}\Omega^{MN}\Theta_N{}^{\mathsf{a}} \quad\Longrightarrow\quad \begin{cases} Z^{\Lambda\mathsf{a}} = \tfrac{1}{2}\Theta^{\Lambda\mathsf{a}}\,,\\ Z_\Lambda{}^{\mathsf{a}} = -\tfrac{1}{2}\Theta_\Lambda{}^{\mathsf{a}}\,, \end{cases} \end{aligned} \tag{5.22}$$

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so that $d_{\mathsf{a}\,MN}$ defines an $\mathrm{Sp}(2n+2,\mathbb{R})$-invariant tensor symmetric in (MN). Likewise one can introduce a similar tensor $Z^{M,\mathsf{m}}$, relevant for the hypermultiplets, by

$$Z^{M,\mathsf{m}} \equiv \frac{1}{2}\Omega^{MN}\Theta_N{}^{\mathsf{m}} \quad\Longrightarrow\quad \begin{cases} Z^{\Lambda\mathsf{m}} = \;\;\tfrac{1}{2}\Theta^{\Lambda\mathsf{m}}\,, \\ Z_\Lambda{}^{\mathsf{m}} = -\tfrac{1}{2}\Theta_\Lambda{}^{\mathsf{m}}\,, \end{cases} \tag{5.23}$$

Subsequently we note that the constraints (5.18), (5.19) and (5.20) can now be written as,

$$Z^{M,\mathsf{a}}\,\Theta_M{}^{\mathsf{b}} = 0 = Z^{M,\mathsf{a}}\,\Theta_M{}^{\mathsf{m}}\,, \qquad Z^{M,\mathsf{m}}\,\Theta_M{}^{\mathsf{a}} = 0 = Z^{M,\mathsf{m}}\,\Theta_M{}^{\mathsf{n}}\,. \tag{5.24}$$

The latter implies that $Z^{M,\mathsf{a}}$ and $Z^{M,\mathsf{m}}$ vanish when contracted with the gauge-group generators T_M. Because of these constraints, only the antisymmetric part of $T_{MN}{}^P$ will appear in the commutation relation (5.17). What remains is to consider the Jacobi identity on the generators T_M. Explicit calculation based on (5.17) leads to

$$T_{[NP}{}^R\,T_{Q]R}{}^M = \frac{2}{3}Z^{M,\mathsf{a}}\,d_{\mathsf{a}R[N}\,T_{PQ]}{}^R\,, \tag{5.25}$$

which shows that the Jacobi identity holds up to terms that vanish upon contraction with the embedding tensor. In the following section we will describe how to introduce a consistent gauging in this non-standard situation.

6 The gauge hierarchy

To compensate for the lack of closure noted in the previous section, and, at the same time, to avoid unwanted degrees of freedom, the strategy is to introduce an extra gauge invariance for the gauge fields, in addition to the usual nonabelian gauge transformations,

$$\delta W_\mu{}^M = \mathcal{D}_\mu\Lambda^M - g\big[Z^{M,\mathsf{a}}\,\Xi_{\mu\,\mathsf{a}} + Z^{M,\mathsf{m}}\,\Xi_{\mu\,\mathsf{m}}\big]\,, \tag{6.1}$$

where the Λ^M are the gauge transformation parameters and the covariant derivative reads, $\mathcal{D}_\mu\Lambda^M = \partial_\mu\Lambda^M + g\,T_{PQ}{}^M\,W_\mu{}^P\Lambda^Q$. The transformations proportional to $\Xi_{\mu\,\mathsf{a}}$ and $\Xi_{\mu\,\mathsf{m}}$ enable one to gauge away those vector fields that are in the sector where the Jacobi identity is not satisfied (this sector is perpendicular to the embedding tensor by virtue of (5.24)). Note that the covariant derivative is invariant under the transformations parametrized by $\Xi_{\mu\,\mathsf{a}}$ and $\Xi_{\mu\,\mathsf{m}}$, because of the contraction of the gauge fields $W_\mu{}^M$ with the generators T_M. However, gauge transformations do no longer form a group by themselves, as is reflected in the commutation relation,

$$[\delta(\Lambda_1),\delta(\Lambda_2)] = \delta(\Lambda_3) + \delta(\Xi_{\mathsf{a}\,3})\,, \tag{6.2}$$

where

$$\begin{aligned} \Lambda_3{}^M &= g\,T_{[NP]}{}^M\,\Lambda_1^N\Lambda_2^P\,, \\ \Xi_{3\mu\,\mathsf{a}} &= d_{\mathsf{a}NP}(\Lambda_1^N\mathcal{D}_\mu\Lambda_2^P - \Lambda_2^N\mathcal{D}_\mu\Lambda_1^P)\,, \end{aligned} \tag{6.3}$$

with $T_{M\mathsf{a}}{}^{\mathsf{b}} = -\Theta_M{}^{\mathsf{c}}f_{\mathsf{ca}}{}^{\mathsf{b}}$ the gauge group generators in the adjoint representation of $\mathrm{G}_{\mathrm{symp}}$. As it turns out, this commutation relation forms the beginning of a full hierarchy of vector and tensor gauge fields that form a closed algebra [39, 40]. Other commutators involving $\delta(\Lambda)$, $\delta(\Xi_{\mathsf{a}})$ and $\delta(\Xi_{\mathsf{m}})$ vanish on the gauge fields $W_\mu{}^\Lambda$, so that those can only be uncovered for the higher-rank tensor gauge fields that we will introduce shortly.

Non-abelian field strengths associated with the gauge fields $W_\mu{}^M$ follow from the Ricci identity, $[D_\mu, D_\nu] = -g\mathcal{F}_{\mu\nu}{}^M\, T_M$, and depend only on the antisymmetric part of $T_{MN}{}^P$,

$$\mathcal{F}_{\mu\nu}{}^M = \partial_\mu W_\nu{}^M - \partial_\nu W_\mu{}^M + g\, T_{[NP]}{}^M\, W_\mu{}^N W_\nu{}^P\,. \tag{6.4}$$

Because of the lack of closure expressed by (5.25), these field strengths do not satisfy the Palatini identity,

$$\delta\mathcal{F}_{\mu\nu}{}^M = 2\,\mathcal{D}_{[\mu}\delta W_{\nu]}{}^M - 2g\, T_{(PQ)}{}^M\, W_{[\mu}{}^P\, \delta W_{\nu]}{}^Q\,, \tag{6.5}$$

under arbitrary variations $\delta W_\mu{}^M$, because of the last term, which cancels upon multiplication with the generators T_M. The result (6.5) shows in particular that $\mathcal{F}_{\mu\nu}{}^M$ transforms under the combined gauge transformations (6.1) as

$$\delta\mathcal{F}_{\mu\nu}{}^M = g\,\Lambda^P T_{NP}{}^M\,\mathcal{F}_{\mu\nu}{}^N - 2g\, Z^{M,\mathsf{a}}\big(\mathcal{D}_{[\mu}\Xi_{\nu]\mathsf{a}} + d_{\mathsf{a}PQ}\, W_{[\mu}{}^P\,\delta W_{\nu]}{}^Q\big) - 2g\, Z^{M,\mathsf{m}}\,\mathcal{D}_{[\mu}\Xi_{\nu]\mathsf{m}}\,, \tag{6.6}$$

and is therefore not covariant. In deriving this one makes use of the fact that the tensors $Z^{M,\mathsf{a}}$ and $Z^{M,\mathsf{m}}$ are invariant under the gauge group. The covariant derivative on $\Xi_{\nu\mathsf{a}}$ is defined by $\mathcal{D}_\mu\Xi_{\nu\mathsf{a}} = \partial_\mu\Xi_{\nu\mathsf{a}} - gW_\mu{}^M T_{M\mathsf{a}}{}^{\mathsf{b}}\Xi_{\nu\mathsf{b}}$, and similarly for $\Xi_{\nu\mathsf{m}}$. These tensor fields belong to the adjoint representation of the group $G_{\rm symp}$.

The standard strategy is therefore to define modified field strengths,

$$\mathcal{H}_{\mu\nu}{}^M = \mathcal{F}_{\mu\nu}{}^M + g\big[Z^{M,\mathsf{a}}\, B_{\mu\nu\,\mathsf{a}} + Z^{M,\mathsf{m}}\, B_{\mu\nu\,\mathsf{m}}\big]\,, \tag{6.7}$$

by introducing new tensor fields $B_{\mu\nu\,\mathsf{a}}$ and $B_{\mu\nu\,\mathsf{m}}$ with suitably chosen gauge transformation rules, so that covariant results are obtained. This implies that the variation of the tensor fields should in any case absorb the unwanted non-covariant terms in (6.6). At this point we recall that the invariance transformations in the ungauged case transform on the field strengths $G_{\mu\nu}{}^M$, defined in (2.6), according to a subgroup of $\mathrm{Sp}(2n+2,\mathbb{R})$ (cf. (2.7)). The field strengths $G_{\mu\nu}{}^M$ consist of the abelian field strengths $F_{\mu\nu}{}^\Lambda$ and the dual field strengths $G_{\mu\nu\Lambda}$. The latter were decomposed in (2.23) in the form $G^-_{\mu\nu\Lambda} = F_{\Lambda\Sigma}\, F^{-\,\Sigma}_{\mu\nu} - 2\mathrm{i}\mathcal{O}^-_{\mu\nu\Lambda}$. Obviously, in the presence of the non-abelian gauge interactions, the abelian field strengths $F_{\mu\nu}{}^\Lambda$ should now be replaced by (6.7). Hence it is natural to define new covariant field strengths according to

$$\mathcal{G}_{\mu\nu}{}^M = \begin{pmatrix} \mathcal{H}_{\mu\nu}{}^\Lambda \\ \mathcal{G}_{\mu\nu\Lambda} \end{pmatrix} \tag{6.8}$$

with

$$\begin{aligned} \mathcal{G}^{-\,\Lambda}_{\mu\nu} &= \mathcal{H}^{-\,\Lambda}_{\mu\nu}\,, \\ \mathcal{G}^-_{\mu\nu\Lambda} &= F_{\Lambda\Sigma}\,\mathcal{H}^{-\,\Sigma}_{\mu\nu} - 2\mathrm{i}\mathcal{O}^-_{\mu\nu\Lambda}\,. \end{aligned} \tag{6.9}$$

Just as in section 2, there exist corresponding supercovariant field strengths $\hat{\mathcal{G}}_{\mu\nu}{}^M$ that will appear in the supersymmetry transformations of the vector multiplet fermion fields.

Those will be discussed in the next section. Just as before, the field strengths $\hat{\mathcal{G}}_{\mu\nu}{}^M$ and $\mathcal{G}_{\mu\nu}{}^M$ will only differ by fermionic bilinears and by terms proportional to the tensor field of the Weyl multiplet.

Following [3] we subsequently introduce the following transformation rule for $B_{\mu\nu\mathsf{a}}$ and $B_{\mu\nu\mathsf{m}}$ (contracted with $Z^{M,\mathsf{a}}$ and $Z^{M,\mathsf{m}}$, respectively, because only these combinations will appear in the Lagrangian),

$$\begin{aligned} Z^{M,\mathsf{a}}\,\delta B_{\mu\nu\,\mathsf{a}} &= 2\,Z^{M,\mathsf{a}}\big(\mathcal{D}_{[\mu}\Xi_{\nu]\mathsf{a}} + d_{\mathsf{a}\,NP}W_{[\mu}{}^N\delta W_{\nu]}{}^P\big) - 2\,T_{(NP)}{}^M\Lambda^P\mathcal{G}_{\mu\nu}{}^N\,,\\ Z^{M,\mathsf{m}}\,\delta B_{\mu\nu\,\mathsf{m}} &= 2\,Z^{M,\mathsf{m}}\,\mathcal{D}_{[\mu}\Xi_{\nu]\mathsf{m}}\,. \end{aligned} \tag{6.10}$$

Note that $B_{\mu\nu\,\mathsf{a}}$ has variations proportional to $\Xi_{\mu\mathsf{m}}$ through the term $\delta W_\mu{}^M$ (cf. (6.1)). As a result of (6.10) the modified field strengths (6.7) are invariant under tensor gauge transformations. Under the vector gauge transformations we derive the following result,

$$\begin{aligned} \delta\mathcal{G}_{\mu\nu}^{-}{}^{\Lambda} &= -\,g\,\Lambda^P T_{PN}{}^{\Lambda}\,\mathcal{G}_{\mu\nu}^{-}{}^{N} - g\,\Lambda^P T^{\Gamma}{}_{P}{}^{\Lambda}\,(\mathcal{G}_{\mu\nu}^{-} - \mathcal{H}_{\mu\nu}^{-})_{\Gamma}\,,\\ \delta\mathcal{G}_{\mu\nu\Lambda}^{-} &= -\,g\,\Lambda^P T_{PN\Lambda}\,\mathcal{G}_{\mu\nu}^{-}{}^{N} - g\,F_{\Lambda\Sigma}\,\Lambda^P T^{\Gamma}{}_{P}{}^{\Sigma}\,(\mathcal{G}_{\mu\nu}^{-} - \mathcal{H}_{\mu\nu}^{-})_{\Gamma}\,,\\ \delta(\mathcal{G}_{\mu\nu}^{-} - \mathcal{H}_{\mu\nu}^{-})_{\Lambda} &= g\,\Lambda^P(T^{\Gamma}{}_{P\Lambda} - T^{\Gamma}{}_{P}{}^{\Sigma}\,F_{\Sigma\Lambda})\,(\mathcal{G}_{\mu\nu}^{-} - \mathcal{H}_{\mu\nu}^{-})_{\Gamma}\,. \end{aligned} \tag{6.11}$$

Hence $\delta\mathcal{G}_{\mu\nu}{}^M = -g\,\Lambda^P T_{PN}{}^M\,\mathcal{G}_{\mu\nu}{}^N$, just as the variation of the abelian field strengths $G_{\mu\nu}{}^M$ in the absence of charges, up to terms proportional to $\Theta^{\Lambda,\mathsf{a}}(\mathcal{G}_{\mu\nu} - \mathcal{H}_{\mu\nu})_\Lambda$. According to [3], the latter terms represent a set of field equations, and so the last equation of (6.11) expresses the well-known fact that, under a symmetry, field equations transform into field equations. As a result the gauge algebra on the tensors $\mathcal{G}{\mu\nu}^M$ closes according to (6.2), up to the same field equations.

In order that the Lagrangian corresponding to (4.1) becomes invariant under vector and tensor gauge transformations, we have to make a number of changes. First of all, we replace the covariant derivatives on the scalars and spinors by gauge-covariant derivatives. This ensures the invariance of $\mathcal{L}^{(1)}_{\text{kin}}$, $\mathcal{L}_{\text{conf}}$ and $\mathcal{L}_{\text{hyper}}$, given in (4.2), (4.6) and (4.7), respectively. The Lagrangian for the auxiliary fields (4.5) is already gauge-invariant. In the following we therefore concentrate on $\mathcal{L}^{(2)}_{\text{kin}}$ (4.3) which depends on the abelian field strengths $F_{\mu\nu}{}^\Lambda$. These abelian field-strengths are now replaced by $\mathcal{H}_{\mu\nu}{}^\Lambda$, so that

$$\mathcal{G}_{\mu\nu\,\Lambda} = \mathrm{i}e\,\varepsilon_{\mu\nu\rho\sigma}\,\frac{\partial\mathcal{L}_{\text{vector}}}{\partial\mathcal{H}_{\rho\sigma}{}^{\Lambda}}\,. \tag{6.12}$$

The Lagrangian $\mathcal{L}^{(2)}_{\text{kin}}$ therefore reads,

$$\begin{aligned} e^{-1}\mathcal{L}^{(2)}_{\text{kin}} &= \frac{1}{4}\mathrm{i}\,\big[F_{\Lambda\Sigma}\,\mathcal{H}^{-\Lambda}_{\mu\nu}\mathcal{H}^{-\Sigma\,\mu\nu} - \bar{F}_{\Lambda\Sigma}\,\mathcal{H}^{+\Lambda}_{\mu\nu}\mathcal{H}^{+\mu\nu\Sigma}\big]\\ &\quad + \big[\mathcal{O}^{-}_{\mu\nu\Lambda}\mathcal{H}^{-\mu\nu\Lambda} - N^{\Lambda\Sigma}\mathcal{O}^{-}_{\mu\nu\Lambda}\mathcal{O}^{-\mu\nu}{}_{\Sigma} + \text{h.c.}\big]\,. \end{aligned} \tag{6.13}$$

It is separately invariant under the tensor gauge transformations, because the tensors $\mathcal{H}$ are invariant.

However, the Lagrangian (4.1) is not invariant under the vector gauge transformations. To establish this one has to take into account that also the other fields of the vector multiplets transform under the gauge group. For instance, there are contributions from infinitesimal gauge transformations of $F_{\Lambda\Sigma}$ and $\mathcal{O}_{\mu\nu\Lambda}$, which follow from (2.30) and (2.31),

$$\begin{aligned}\delta F_{\Lambda\Sigma} &= g\Lambda^M\big(-T_{M\Lambda\Sigma}+2\,T_{M(\Lambda}{}^{\Gamma}F_{\Sigma)\Gamma}+F_{\Lambda\Gamma}T_M{}^{\Gamma\Xi}F_{\Xi\Sigma}\big)\,,\\ \delta\mathcal{O}^-_{\mu\nu\Lambda} &= g\Lambda^M\mathcal{O}^-_{\mu\nu\Sigma}\,\big(T_{M\Lambda}{}^{\Sigma}+T_M{}^{\Sigma\Gamma}F_{\Gamma\Lambda}\big)\,.\end{aligned} \tag{6.14}$$

Nevertheless, it was shown in [3] that this is still not sufficient for gauge invariance, and it is necessary to introduce an additional, universal, term to the Lagrangian, equal to,

$$\begin{aligned}\mathcal{L}_{\rm top} = {}&\frac{1}{8}\mathrm{i}g\,\varepsilon^{\mu\nu\rho\sigma}\,\big(\Theta^{\Lambda\mathsf{a}}\,B_{\mu\nu\,\mathsf{a}}+\Theta^{\Lambda\mathsf{m}}\,B_{\mu\nu\,\mathsf{m}}\big)\\ &\qquad\times\Big(2\,\partial_\rho W_{\sigma\,\Lambda}+gT_{MN\,\Lambda}\,W_\rho{}^MW_\sigma{}^N-\frac{1}{4}g\Theta_\Lambda{}^{\mathsf{b}}B_{\rho\sigma\,\mathsf{b}}-\frac{1}{4}g\Theta_\Lambda{}^{\mathsf{n}}B_{\rho\sigma\,\mathsf{n}}\Big)\\ &+\frac{1}{3}\mathrm{i}g\,\varepsilon^{\mu\nu\rho\sigma}T_{MN\,\Lambda}\,W_\mu{}^MW_\nu{}^N\Big(\partial_\rho W_\sigma{}^\Lambda+\frac{1}{4}gT_{PQ}{}^\Lambda W_\rho{}^PW_\sigma{}^Q\Big)\\ &+\frac{1}{6}\mathrm{i}g\,\varepsilon^{\mu\nu\rho\sigma}T_{MN}{}^\Lambda\,W_\mu{}^MW_\nu{}^N\Big(\partial_\rho W_{\sigma\Lambda}+\frac{1}{4}gT_{PQ\Lambda}W_\rho{}^PW_\sigma{}^Q\Big)\,.\end{aligned} \tag{6.15}$$

The first term represents a topological coupling of the anti-symmetric tensor fields with the magnetic gauge fields; the last two terms are a generalization of the Chern-Simons-like terms that were first found in [6].

Under arbitrary variations of the vector and tensor fields, (6.13) and (6.15) yield (up to total derivative terms),

$$\begin{aligned}e^{-1}\Big(\delta\mathcal{L}^{(2)}_{\rm kin}+\delta\mathcal{L}_{\rm top}\Big) = {}&-\frac{1}{4}\mathrm{i}g\,\big(\mathcal{G}^{+\mu\nu M}-\mathcal{H}^{+\mu\nu M}\big)\,\Theta_M{}^{\mathsf{a}}(\delta B_{\mu\nu\mathsf{a}}-2d_{\mathsf{a}PQ}W_\mu{}^P\delta W_\nu{}^Q)\\ &-\frac{1}{4}\mathrm{i}g\,\big(\mathcal{G}^{+\mu\nu M}-\mathcal{H}^{+\mu\nu M}\big)\,\Theta_M{}^{\mathsf{m}}\,\delta B_{\mu\nu\mathsf{m}}\\ &+\mathrm{i}\mathcal{G}^{+\mu\nu M}\Omega_{MN}\,\mathcal{D}_\mu\delta W_\nu{}^N+\text{h.c.}\,.\end{aligned} \tag{6.16}$$

Under the tensor gauge transformations this variation becomes equal to,

$$e^{-1}\big(\delta\mathcal{L}^{(2)}_{\rm kin}+\delta\mathcal{L}_{\rm top}\big) = \mathrm{i}g\mathcal{H}^{+\mu\nu M}\big[\Theta_M{}^{\mathsf{a}}\mathcal{D}_\mu\Xi_{\nu\mathsf{a}}+\Theta_M{}^{\mathsf{m}}\mathcal{D}_\mu\Xi_{\nu\mathsf{m}}\big]+\text{h.c.}\,. \tag{6.17}$$

We already demonstrated that $\mathcal{L}^{(2)}_{\rm kin}$ is separately invariant under tensor gauge transformations, so that the above terms originate exclusively from the variation of $\mathcal{L}_{\rm top}$. The expression (6.17) turns out to be equal to a total derivative because there exists a Bianchi identity,

$$\mathcal{D}_{[\mu}\mathcal{H}_{\nu\rho]}{}^M = \frac{1}{3}g\big[Z^{M,\mathsf{a}}\,\mathcal{H}_{\mu\nu\rho\,\mathsf{a}}+Z^{M,\mathsf{m}}\,\mathcal{H}_{\mu\nu\rho\,\mathsf{m}}\big]\,, \tag{6.18}$$

and because the embedding tensor is gauge invariant. Here the gauge-covariant field strengths of the tensor fields are defined as,

$$\begin{aligned}\mathcal{H}_{\mu\nu\rho\,\mathsf{a}} &= 3\,\mathcal{D}_{[\mu}B_{\nu\rho]\,\mathsf{a}}+6\,d_{\mathsf{a}\,NP}\,W_{[\mu}{}^N\Big(\partial_\nu W_{\rho]}{}^P+\frac{1}{3}gT_{[RS]}{}^PW_\nu{}^RW_{\rho]}{}^S+(\mathcal{G}-\mathcal{H})_{\nu\rho]}{}^P\Big)\,,\\ \mathcal{H}_{\mu\nu\rho\,\mathsf{m}} &= 3\,\mathcal{D}_{[\mu}B_{\nu\rho]\,\mathsf{m}}\,,\end{aligned} \tag{6.19}$$

where $\mathcal{D}_\mu B_{\nu\rho\mathsf{a}} = \partial_\mu B_{\nu\rho\mathsf{a}} - gW_\mu{}^M T_{M\mathsf{a}}{}^{\mathsf{b}} B_{\nu\rho\mathsf{b}}$, and likewise for $\mathcal{D}_\mu B_{\nu\rho\mathsf{m}}$. The fully gauge-covariant derivative of $\mathcal{H}_{\mu\nu}{}^M$ takes the form,

$$\begin{aligned}\mathcal{D}_\rho \mathcal{H}_{\mu\nu}{}^M &= \partial_\rho \mathcal{H}_{\mu\nu}{}^M + gW_\rho{}^P\, T_{PN}{}^M\, \mathcal{G}_{\mu\nu}{}^N + gW_\rho{}^P\, T_{NP}{}^M\, (\mathcal{G}-\mathcal{H})_{\mu\nu}{}^N \\ &= \partial_\rho \mathcal{H}_{\mu\nu}{}^M + gW_\rho{}^P\, T_{PN}{}^M\, \mathcal{H}_{\mu\nu}{}^N + 2\, gW_\rho{}^P\, Z^{M,\mathsf{a}} d_{\mathsf{a}PN}\, (\mathcal{G}-\mathcal{H})_{\mu\nu}{}^N\,, \end{aligned} \tag{6.20}$$

Observe that the covariantization proportional to $(\mathcal{G}-\mathcal{H})_{\mu\nu}{}^N$ is not generated by partially integrating the right-hand side of (6.17), but it vanishes upon contraction with the embedding tensor. So does the right-hand side of (6.18), so that (6.17) is indeed a total derivative.

As was mentioned before, the combined gauge invariance of the vector and tensor gauge fields are important to ensure that the number of physical degrees of freedom will not change by the introduction of the magnetic vector gauge fields and the tensor gauge fields [3]. The combined gauge algebra is consistent for the tensor fields upon projection with the embedding tensor, which is sufficient because the action depends only on these projected fields. If this were not the case, new tensor fields of higher rank would have been required [39]. The projection with the embedding tensor will determine in which fields the physical degrees of freedom can reside. The precise way in which the number of physical degrees of freedom are accounted for is therefore rather subtle. From (6.16) it is indeed clear that the components of the tensor fields that are projected to zero by multiplication with $\Theta^{\Lambda\mathsf{a}}$ or $\Theta^{\Lambda\mathsf{m}}$, are simply not present in the action. Their absence can be regarded as the result of an additional gauge invariance. In addition, there are transformations of the tensor fields linear in $(\mathcal{G}-\mathcal{H})_{\mu\nu\Lambda}$ that leave the Lagrangian invariant [7, 13],

$$\begin{aligned}\Theta^{\Lambda\mathsf{a}}\delta B_{\mu\nu\mathsf{a}} &= \Delta_1^{[\Lambda\Sigma]}\, (\mathcal{G}-\mathcal{H})^+_{\mu\nu\Sigma} + \text{h.c.}\,, \\ \Theta^{\Lambda\mathsf{a}}\delta B_{\mu\nu\mathsf{a}} &= \Delta_2^{(\Lambda\Sigma)\rho}{}_{[\mu}\, (\mathcal{G}-\mathcal{H})_{\nu]\rho\Sigma}\,, \end{aligned} \tag{6.21}$$

where $\Delta_1^{\Lambda\Sigma}$ is an arbitrary complex parameter, and $\Delta_2^{\Lambda\Sigma\rho}{}_\mu$ is real and traceless. Similar transformations exist for variations contracted with $\Theta^{\Lambda\mathsf{m}}$. Often these transformations emerge when verifying the validity of the supersymmetry algebra, something that we will discuss in section 7.

A similar situation arises with the magnetic gauge fields $W_{\mu\Lambda}$. Under variations of the gauge fields $W_\mu{}^M$ one derives,

$$\delta\mathcal{L}^{(2)}_{\text{kin}} + \delta\mathcal{L}_{\text{top}} = \frac{1}{2}\mathrm{i}\,\varepsilon^{\mu\nu\rho\sigma}\, \mathcal{D}_\nu \mathcal{G}_{\rho\sigma}{}^M \Omega_{MN}\delta W_\mu{}^N\,, \tag{6.22}$$

where $\mathcal{L}^{(2)}_{\text{kin}}$ was defined in (6.13), up to a total derivative and up to terms that vanish as a result of the field equation for $B_{\mu\nu\,\mathsf{a}}$. Substituting (6.18) we can rewrite (6.22) as follows,

$$\delta\mathcal{L}^{(2)}_{\text{kin}} + \delta\mathcal{L}_{\text{top}} = \frac{1}{2}\mathrm{i}\,\varepsilon^{\mu\nu\rho\sigma}\left[-\mathcal{D}_\nu\mathcal{G}_{\rho\sigma\Lambda}\,\delta W_\mu{}^\Lambda + \frac{1}{6}g\big(\mathcal{H}_{\nu\rho\sigma\mathsf{a}}\,\Theta^{\Lambda\mathsf{a}} + \mathcal{H}_{\nu\rho\sigma\mathsf{m}}\,\Theta^{\Lambda\mathsf{m}}\big)\delta W_{\mu\Lambda}\right]. \tag{6.23}$$

Because the minimal coupling of the gauge fields to matter fields is always proportional to the embedding tensor, the full Lagrangian does not change under variations of the magnetic gauge fields that are projected to zero by the embedding tensor components $\Theta^{\Lambda \mathsf{a}}$ or $\Theta^{\Lambda \mathsf{m}}$, up to terms that are generated by the variations of the tensor fields through the 'universal' variation, $\delta B_{\mu\nu\mathsf{a}} = 2\, d_{\mathsf{a}PQ} W_{[\mu}{}^P \delta W_{\nu]}{}^Q$.

All these gauge symmetries have a role to play in balancing the degrees of freedom. In [3] a precise accounting of all gauge symmetries was bypassed in the analysis. Observe that not all these symmetries have a bearing on the dynamical modes of the theory as they also act on fields that only play an auxiliary role.

7 General gaugings: the superconformal algebra and the Lagrangian

When switching on a gauging there are several qualitative changes that are of interest. First of all, the superconformal algebra will no longer be realized off shell (i.e. without using the equations of motion) in the vector multiplet sector, at least for gaugings with magnetic charges. Only for the Weyl multiplet the closure remains realized off shell. Naturally a generic gauging induces the presence of vector multiplet fields into the hypermultiplet supersymmetry transformations. It is therefore not surprising that also the vector multiplet transformations will generically acquire terms proportional to the hypermultiplet fields. In this section we will present the full transformation rules that include new terms of order g, and subsequently we will re-establish the closure for general gaugings. As it turns out, additional symmetries such as (6.21), are relevant for the closure. This feature is well known from previous applications of the embedding tensor formalism.

A second, not unrelated, feature is that the Lagrangian must be modified by including masslike terms for the fermions proportional to g, and a scalar potential proportional to g^2. The explicit expressions for these terms, which are relevant for many applications, will be presented at the end of this section. These modifications are familiar from $N = 2$ supergravity theories with purely electric charges [4, 6, 32].

Rigid $N = 2$ supersymmetric theories with both electric and magnetic charges, have been presented in [7], and it remains to complete these results in a fully superconformal setting. It is clear that the modification of the results derived in [7] must be relatively minor. The supersymmetry transformations of the matter fields will now become covariant with respect to the superconformal symmetries, while at the same time they should remain in accord with the known results for rigid theories. Modifications that supersede previous work will therefore mainly involve terms proportional to the gravitino fields. The most conspicuous ones are those appearing in the supersymmetry transformations of the tensor fields $B_{\mu\nu\mathsf{a}}$ and $B_{\mu\nu\mathsf{m}}$.

To exhibit this in more detail, let us first present the full Q- and S-supersymmetry transformations for the hypermultiplet fields. They follow straightforwardly upon super-covariantizing the rules presented in section 3, including the terms of order g that were already found in [7],

$$\begin{aligned}
\delta\phi^A &= 2\,(\gamma^A_{i\bar\alpha}\,\bar\epsilon^i\zeta^{\bar\alpha} + \bar\gamma^{Ai}_{\alpha}\,\bar\epsilon_i\zeta^\alpha)\,,\\
\delta A_i{}^\alpha + \delta\phi\Gamma_A{}^\alpha{}_\beta A_i{}^\beta &= 2\,\bar\epsilon_i\zeta^\alpha + 2\,\varepsilon_{ij} G^{\alpha\bar\beta}\Omega_{\bar\beta\bar\gamma}\,\bar\epsilon^j\zeta^{\bar\gamma}\,,\\
\delta\zeta^\alpha + \delta\phi^A\,\Gamma_A{}^\alpha{}_\beta\,\zeta^\beta &= \not{D} A_i{}^\alpha\,\epsilon^i + 2gX^M\,T_M{}^\alpha{}_\beta A_i{}^\beta\,\varepsilon^{ij}\epsilon_j + A_i{}^\alpha\,\eta^i\,.
\end{aligned} \tag{7.1}$$

where D_μ denotes the derivative fully covariantized with respect to all the superconformal transformations and the gauge symmetries. Likewise we present the full Q- and S-supersymmetry transformations for the vector multiplet fields,

$$
\begin{aligned}
\delta X^M &= \bar\epsilon^i \Omega_i{}^M\,, \\
\delta\Omega_i{}^M &= 2\not{D} X^M \epsilon_i + \hat Z_{ij}{}^M \epsilon^j + \frac{1}{2}\gamma^{\mu\nu}\hat{\mathcal{G}}_{\mu\nu}^{-\,M}\varepsilon_{ij}\epsilon^j \\
&\qquad - 2g\, T_{PN}{}^M \bar X^P X^N \varepsilon_{ij}\epsilon^j + 2\,\mathrm{i}g\Omega^{MN}\mu_{ijN}\epsilon^j + 2X^M\eta_i\,, \\
\delta W_\mu{}^M &= \varepsilon^{ij}\bar\epsilon_i(\gamma_\mu\Omega_j{}^M + 2\,\psi_{\mu j}X^M) + \varepsilon_{ij}\bar\epsilon^i(\gamma_\mu\Omega^{j\,M} + 2\,\psi_\mu{}^j\bar X^M)\,, \\
\delta Y_{ij}{}^\Lambda &= 2\,\bar\epsilon_{(i}\not{D}\Omega_{j)}{}^\Lambda + 2\,\varepsilon_{ik}\varepsilon_{jl}\bar\epsilon^{(k}\not{D}\Omega^{l)\Lambda} \\
&\qquad - 4g\, T_{MN}{}^\Lambda\big[\bar\Omega_{(i}{}^M\epsilon^k\varepsilon_{j)k}\,\bar X^N - \bar\Omega^{kM}\epsilon_{(i}\varepsilon_{j)k}\,X^N\big] \\
&\qquad + 4\,\mathrm{i}g\,k^{A\Lambda}\big[\varepsilon_{k(i}\,\gamma_{j)\bar\alpha A}\bar\epsilon^k\zeta^{\bar\alpha} + \varepsilon_{k(i}\,\bar\epsilon_{j)}\zeta^\alpha\,\bar\gamma^k_{\alpha A}\big]\,.
\end{aligned}
\tag{7.2}
$$

Here the moment maps are defined by,

$$
\mu_{ijM} = \Theta_M{}^{\mathsf{m}}\mu_{ij\mathsf{m}}\,, \tag{7.3}
$$

and the symplectic vector $\hat Z_{ij}{}^M$ appearing in $\delta\Omega_i{}^M$ is given by,

$$
\hat Z_{ij}{}^M = \begin{pmatrix} Y_{ij}{}^\Lambda \\ F_{\Lambda\Sigma}\,Y_{ij}{}^\Sigma - \frac{1}{2}F_{\Lambda\Sigma\Gamma}\,\bar\Omega_i{}^\Sigma\Omega_j{}^\Gamma + 2\,\mathrm{i}g[\mu_{ij\Lambda} + F_{\Lambda\Sigma}\,\mu_{ij}{}^\Sigma] \end{pmatrix}. \tag{7.4}
$$

This expression differs from the previous one for the ungauged theory, given in (2.25), by the presence of the moment maps originating from the hypermultiplet sector. This implies that the original pseudo-reality condition on $Z_{ij\Lambda}$ must be replaced by a pseudo-reality condition on $\hat Z_{ij\Lambda}$. As this condition was previously imposed by invoking the field equations for the auxiliary fields, it follows that those field equations must now receive modifications proportional to the moment maps, as we shall confirm later in this section. Note that, in (7.2), we refrained from giving the supersymmetry transformation of $\hat Z_{ij\Lambda}$, which is not an independent field.

Another tensor appearing in $\delta\Omega_i{}^M$, a modification of the tensor (2.21), is the supercovariant field strength $\hat{\mathcal{G}}_{\mu\nu}{}^M$, which coincides with the field strengths (6.8) up to fermion bilinears and terms proportional to the tensor field of the Weyl multiplet. These supercovariant field strengths are defined by,

$$
\begin{aligned}
\hat{\mathcal{G}}_{\mu\nu}^{-\,\Lambda} &= \hat{\mathcal{H}}_{\mu\nu}^{-\,\Lambda}\,, \\
\hat{\mathcal{G}}_{\mu\nu\Lambda}^{-} &= F_{\Lambda\Sigma}\,\hat{\mathcal{H}}_{\mu\nu}^{-\,\Sigma} - \frac{1}{8}F_{\Lambda\Sigma\Gamma}\,\bar\Omega_i{}^\Sigma\gamma_{\mu\nu}\Omega_j{}^\Gamma\,\varepsilon^{ij}\,.
\end{aligned}
\tag{7.5}
$$

where $\hat{\mathcal{H}}_{\mu\nu}{}^\Lambda$ is the supercovariant extension of (6.7). In view of (2.2), we expect the following decomposition for $\hat{\mathcal{H}}_{\mu\nu}{}^\Lambda$,

$$
\begin{aligned}
\hat{\mathcal{H}}_{\mu\nu}{}^\Lambda &= \mathcal{H}_{\mu\nu}{}^\Lambda - \varepsilon^{ij}\bar\psi_{[\mu\,i}(\gamma_{\nu]}\Omega_j{}^\Lambda + \psi_{\nu]j}X^\Lambda) - \varepsilon_{ij}\bar\psi_{[\mu}{}^i(\gamma_{\nu]}\Omega^{j\,\Lambda} + \psi_{\nu]}{}^j\bar X^\Lambda) \\
&\qquad - \frac{1}{4}(X^\Lambda\,T_{\mu\nu ij}\,\varepsilon^{ij} + \bar X^\Lambda\,T_{\mu\nu}{}^{ij}\,\varepsilon_{ij})\,.
\end{aligned}
\tag{7.6}
$$

However, in the presence of a gauging, this expression leads to supersymmetry variations proportional to the gravitini fields induced by the terms in $\delta\Omega_i{}^\Lambda$ of order g. As it turns out, by suitably adjusting the supersymmetry transformations of the tensor fields, $\delta B_{\mu\nu\mathsf{a}}$ and $\delta B_{\mu\nu\mathsf{m}}$, one can ensure that the $\hat{\mathcal{H}}_{ab}{}^\Lambda$ will still transform covariantly under Q- and S-supersymmetry,

$$\begin{aligned}\delta\hat{\mathcal{H}}_{ab}{}^\Lambda =&\, -2\,\varepsilon_{ij}\,\bar\epsilon^i\gamma_{[a}D_{b]}\Omega^{j\Lambda} - 2g\,T_{(NP)}{}^\Lambda\bar X^N\,\bar\Omega_i{}^P\gamma_{ab}\epsilon^i\\&-2\mathrm{i}g\,k^{A\Lambda}\,\gamma_{Ai\bar\alpha}\,\bar\zeta^{\bar\alpha}\gamma_{ab}\epsilon^i - \varepsilon^{ij}\bar\eta_i\gamma_{ab}\Omega_j{}^\Lambda + \text{h.c.}\,.\end{aligned}\tag{7.7}$$

As a result the combined transformations of the tensor fields, $B_{\mu\nu\mathsf{a}}$ and $B_{\mu\nu\mathsf{m}}$, under tensor and vector gauge transformations and Q- and S-supersymmetry, now read as follows,

$$\begin{aligned}Z^{M,\mathsf{a}}\,\delta B_{\mu\nu\,\mathsf{a}} =&\, 2\,Z^{M,\mathsf{a}}\mathcal{D}_{[\mu}\Xi_{\nu]\mathsf{a}} + 2\,T_{(NP)}{}^M\big[W_{[\mu}{}^N\delta W_{\nu]}{}^P - \Lambda^N\mathcal{G}_{\mu\nu}{}^P\big]\\&-2\,T_{(NP)}{}^M\big[\bar X^N\bar\Omega_i{}^P\gamma_{\mu\nu}\epsilon^i + X^N\bar\Omega^{iP}\gamma_{\mu\nu}\epsilon_i + 2\,\bar X^N X^P\big(\bar\epsilon^i\gamma_{[\mu}\psi_{\nu]i} + \bar\epsilon_i\gamma_{[\mu}\psi_{\nu]}{}^i\big)\big]\,,\\ Z^{M,\mathsf{m}}\,\delta B_{\mu\nu\,\mathsf{m}} =&\, 2\,Z^{M,\mathsf{m}}\,\mathcal{D}_{[\mu}\Xi_{\nu]\mathsf{m}} - 2\mathrm{i}\Omega^{MN}k^A{}_N\,\big[\gamma_{Ai\bar\alpha}\,\bar\zeta^{\bar\alpha}\gamma_{\mu\nu}\epsilon^i - \bar\gamma^i_{A\alpha}\,\bar\zeta^\alpha\gamma_{\mu\nu}\epsilon_i\big]\\&+4\,\mathrm{i}\,\Omega^{MN}\mu_{jkN}\,\varepsilon^{ij}\big[\bar\psi_{i[\mu}\gamma_{\nu]}\epsilon^k + \bar\psi^k{}_{[\mu}\gamma_{\nu]}\epsilon_i\big]\,.\end{aligned}\tag{7.8}$$

Note that the tensors transform covariantly under diffeomorphisms, and are scale invariant. As was already alluded to, the moment maps μ_{ijM} enter the transformation rules of the vector multiplet fields. In fact, only the magnetic moment maps $\mu_{ij}{}^\Lambda$ appear in these transformation rules.[12] For purely electric charges and corresponding moment maps $\mu_{ij\Lambda}$, the supersymmetry transformations (7.1) and (7.2) reduce to the transformations presented in [6] and [32]. The latter transformations still realize the supersymmetry algebra for the vector multiplet fields (but not for the hypermultiplet fields) without the need for imposing equations of motion.

Now that the full supersymmetry transformations have been established, we consider the superconformal algebra. Its most non-trivial commutation relation is the one of two Q-supersymmetries. This commutation relation, which was already specified in (2.27), must now be extended with tensor gauge transformations. Hence

$$\begin{aligned}[\delta(\epsilon_1),\delta(\epsilon_2)] =&\, \xi^\mu D_\mu + \delta_M(\varepsilon) + \delta_K(\Lambda_K) + \delta_S(\eta) + \delta_{\text{gauge}}(\Lambda^M)\\&+\delta_{\text{tensor}}(\Xi_{\mu\,\mathsf{a}}) + \delta_{\text{tensor}}(\Xi_{\mu\,\mathsf{m}})\,,\end{aligned}\tag{7.9}$$

and it should hold modulo field equations and some of the spurious symmetries that we discussed in the previous section. The various parameters in (7.9) have already been specified in (2.28), except for the parameters of the tensor gauge transformations, which read,

$$\begin{aligned}\Xi_{\mu\,\mathsf{a}} =&\, -2\,d_{\mathsf{a}\,NP}\bar X^N X^P\xi_\mu\,,\\ \Xi_{\mu\mathsf{m}} =&\, -8\,\mathrm{i}\,\varepsilon^{ij}\mu_{jk\mathsf{m}}\big(\bar\epsilon_{2i}\gamma_\mu\epsilon_1{}^k + \bar\epsilon_2{}^k\gamma_\mu\epsilon_{1i}\big)\,,\end{aligned}\tag{7.10}$$

up to terms that vanish upon contraction with the embedding tensor.[13] The combination $\xi^\mu D_\mu$ denotes an infinitesimal covariant general coordinate transformation, which includes

[12]The reader may verify that the contribution to $\Omega_i{}^M$ proportional to $\mu_{ij\Lambda}$ vanishes against a similar contribution contained in $\hat Z_{ij}{}^M$.

[13]The result for $\Xi_{\mu\mathsf{m}}$ given in (7.10) is new compared to previous work. It is determined by verifying the commutator (7.9) on the vector and tensor gauge fields, as will be discussed in some detail below.

contributions from *all* the field-dependent gauge transformations such as a Q- and S-supersymmetry transformation with parameters $-\frac{1}{2}\xi^\rho\psi_\rho{}^i$ and $-\frac{1}{2}\xi^\rho\phi_\rho{}^i$, or vector gauge transformations with parameters $\Lambda^M = -\xi^\rho W_\rho{}^M$, such that the combined result takes a supercovariant form. For the corresponding field-dependent tensor gauge transformations, the parameters take a slightly more complicated form [7, 13],

$$\begin{aligned} \Xi_{\mu\,\mathsf{a}} &= -\,\xi^\rho\left(B_{\rho\mu\,\mathsf{a}} + d_{\mathsf{a}\,NP} W_\rho{}^N W_\mu{}^P\right)\,, \\ \Xi_{\mu\,\mathsf{m}} &= -\,\xi^\rho B_{\rho\mu\,\mathsf{m}}\,. \end{aligned} \tag{7.11}$$

In what follows we will verify the validity of (7.9) on the auxiliary fields $Y_{ij}{}^\Lambda$, $W_\mu{}^M$ and the tensor fields $B_{\mu\nu\mathsf{a}}$ and $B_{\mu\nu\mathsf{m}}$, as these are most susceptible to the presence of the new gauge transformations, thereby exhibiting a variety of subtleties that play a role. Many aspects of this evaluation have their counterpart in a similar evaluation of $N=8$ supergravity, which appeared in [13]. At this point we mention two general identities that are relevant in the present calculations. They follow from (5.9), (5.10) and (5.11),

$$\begin{aligned} T_{(MN)}{}^P X^M \hat{Z}_{ij}{}^N &= \frac{1}{2}\,T_{(MN)}{}^P\,\bar{\Omega}_i{}^M\Omega_j{}^N - 2\mathrm{i}g T_{(MN)}{}^P X^M\Omega^{NQ}\mu_{ijQ}\,, \\ T_{(MN)}{}^P X^M \hat{\mathcal{G}}^-_{\mu\nu}{}^N &= \frac{1}{8}\,T_{(MN)}{}^P\,\varepsilon^{ij}\,\bar{\Omega}_i{}^M\gamma_{\mu\nu}\Omega_j{}^N\,. \end{aligned} \tag{7.12}$$

Of course, in the calculations we must also take into account that the superconformal gauge fields, $\omega_\mu{}^{ab}$, $f_\mu{}^a$ and $\phi_\mu{}^i$, depend on the other superconformal fields.

Let us first consider the supersymmetry commutator (7.9) on the auxiliary fields $Y_{ij}{}^\Lambda$. As it turns out, its validity requires to impose the field equations associated with the tensor fields, which take the following form,

$$\Theta^{\Lambda\mathsf{a}}\,\mathcal{G}_{\mu\nu\Lambda} = \Theta^{\Lambda\mathsf{a}}\,\mathcal{H}_{\mu\nu\Lambda}\,, \qquad \Theta^{\Lambda\mathsf{m}}\,\mathcal{G}_{\mu\nu\Lambda} = \Theta^{\Lambda\mathsf{m}}\,\mathcal{H}_{\mu\nu\Lambda}\,, \tag{7.13}$$

and the field equations associated with the magnetic gauge fields,

$$\begin{aligned} 0 = {}& \frac{1}{6}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\left(Z^{\Lambda,\mathsf{a}}\mathcal{H}_{\nu\rho\sigma\,\mathsf{a}} + Z^{\Lambda,\mathsf{m}}\mathcal{H}_{\nu\rho\sigma\,\mathsf{m}}\right) + T_{(MN)}{}^\Lambda\Big(-2\,\bar{X}^M\,\overleftrightarrow{\mathcal{D}}^\mu X^N \\ & + \bar{\Omega}^{iM}\gamma^\mu\Omega_i{}^N + \bar{X}^M\bar{\psi}_\nu{}^i\gamma^\mu\gamma^\nu\Omega_i{}^N - X^M\bar{\psi}_{\nu i}\gamma^\mu\gamma^\nu\Omega^{iN} - \frac{1}{2}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_{\nu i}\gamma_\rho\psi_\sigma{}^i\,\bar{X}^M X^N\Big) \\ & + \mathrm{i}G_{\bar{\alpha}\beta}T^{\Lambda\beta}{}_\gamma\Big(\frac{1}{2}A^{i\bar{\alpha}}\,\overleftrightarrow{\mathcal{D}}^\mu A_i{}^\gamma - 2\bar{\zeta}^{\bar{\alpha}}\gamma^\mu\zeta^\gamma + \bar{\psi}_\nu{}^i\gamma^\mu\gamma^\nu\zeta^{\bar{\alpha}}A_i{}^\gamma - \bar{\psi}_{\nu i}\gamma^\mu\gamma^\nu\zeta^\gamma A^{i\bar{\alpha}}\Big) \\ & - \mathrm{i}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\nu{}^i\gamma_\rho\psi_{\sigma j}\varepsilon^{jk}\mu_{ik}{}^\Lambda\,, \end{aligned} \tag{7.14}$$

where we made use of the Bianchi identity (6.18).

Secondly we evaluate the supersymmetry commutator on the vector fields $W_\mu{}^M$,

$$\begin{aligned} [\delta(\epsilon_1),\delta(\epsilon_2)]W_\mu{}^M = {}& \xi^\rho\mathcal{G}_{\rho\mu}{}^M + \mathcal{D}_\mu\Lambda^M - g\,Z^{M,\mathsf{a}}\,\Xi_{\mu\,\mathsf{a}} - g\,Z^{M,\mathsf{m}}\,\Xi_{\mu\,\mathsf{m}} \\ & -\xi^\rho\left(\frac{1}{2}\,\varepsilon_{ij}\,\bar{\psi}_\rho{}^i\gamma_\mu\Omega^{jM} + \varepsilon_{ij}\bar{X}^M\bar{\psi}_\rho{}^i\psi_\mu{}^j + \text{h.c.}\right)\,, \end{aligned} \tag{7.15}$$

where the parameters ξ^μ, Λ^M, $\Xi_{\mu\,\mathsf{a}}$ and $\Xi_{\mu\,\mathsf{m}}$ are as in (7.9). In this result one can replace ${\mathcal{G}_{\mu\nu}}^M$ by ${\mathcal{H}_{\mu\nu}}^M$. For the electric gauge fields this is trivial as ${\mathcal{G}_{\mu\nu}}^\Lambda$ and ${\mathcal{H}_{\mu\nu}}^\Lambda$ are identical. For the magnetic gauge fields the replacement is effectively allowed because $W_{\mu\Lambda}$ appear in the Lagrangian contracted with the embedding tensor, as can be seen from (6.23). Therefore, without loss of generality, one can safely contract (7.15) for the magnetic gauge fields with the embedding tensors, $\Theta^{\Lambda\mathsf{a}}$ or $\Theta^{\Lambda\mathsf{m}}$, upon which one can replace $\mathcal{G}_{\mu\nu\Lambda}$ with $\mathcal{H}_{\mu\nu\Lambda}$ by virtue of (7.13). Finally one uses the following equality,

$$\begin{aligned}\xi^\rho {\mathcal{H}_{\rho\mu}}^M =&\, \xi^\rho\partial_\rho {W_\mu}^M + \partial_\mu\xi^\rho {W_\rho}^M - \mathcal{D}_\mu\left(\xi^\rho {W_\rho}^M\right)\\ &+ gZ^{M,\mathsf{a}}\xi^\rho\left(B_{\rho\mu\,\mathsf{a}} + d_{\mathsf{a}\,NP}{W_\rho}^N{W_\mu}^P\right) + gZ^{M,\mathsf{m}}\xi^\rho B_{\rho\mu\,\mathsf{m}}\,. \end{aligned}\tag{7.16}$$

Substituting this identity into (7.15) shows that the ξ^μ-dependent terms decompose into a general coordinate transformation with parameter ξ^μ, a non-abelian gauge transformation with parameter $-\xi^\mu {W_\mu}^M$, tensor gauge transformations with parameters $-\xi^\rho\left(B_{\rho\mu\,\mathsf{a}} + d_{\mathsf{a}\,NP}W_\rho^N W_\mu^P\right)$ and $-\xi^\rho B_{\rho\mu\,\mathsf{m}}$ and a supersymmetry transformation with parameter $-\frac{1}{2}\xi^\mu\psi_{\mu i}$. Together they constitute a covariant general coordinate transformation with parameter ξ^μ. Consequently the supersymmetry commutator closes according to (7.9).

Subsequently we turn to the supersymmetry commutator on the tensor fields $B_{\mu\nu\,\mathsf{a}}$. Here it suffices to consider those fields contracted with $Z^{\Lambda,\mathsf{a}}$ because no other components of the tensor field appear in the Lagrangian according to (6.16). Hence, we first evaluate

$$\begin{aligned}
&Z^{\Lambda,\mathsf{a}}\left[\delta(\epsilon_1),\delta(\epsilon_2)\right]B_{\mu\nu\,\mathsf{a}}\\
&\quad= 2\,Z^{\Lambda,\mathsf{a}}\mathcal{D}_{[\mu}\Xi_{\nu]\mathsf{a}} - 2\,{T_{(MN)}}^\Lambda\Lambda^M{\mathcal{G}_{\mu\nu}}^N\\
&\qquad + 2\,{T_{(MN)}}^\Lambda\,{W_{[\mu}}^M\left[\delta(\epsilon_1),\delta(\epsilon_2)\right]{W_{\nu]}}^N\\
&\qquad + {T_{(MN)}}^\Lambda\xi^\rho\left(\bar X^M\bar\Omega_i{}^N\gamma_{\mu\nu}\psi_\rho{}^i - 2\bar\psi_\rho{}^i\gamma_{[\mu}\psi_{\nu]i}\,\bar X^M X^N + \text{h.c.}\right)\\
&\qquad + e\,\varepsilon_{\mu\nu\rho\sigma}\,{T_{(MN)}}^\Lambda\xi^\rho\Big(-2\,\bar X^M\,\overleftrightarrow{\mathcal{D}}^\sigma X^N + \bar\Omega^{iM}\gamma^\sigma\Omega_i{}^N\\
&\qquad + \bar X^M\bar\psi_\lambda{}^i\gamma^\sigma\gamma^\lambda\Omega_i{}^N - X^M\bar\psi_{\lambda i}\gamma^\sigma\gamma^\lambda\Omega^{iN} - \frac{1}{2}e^{-1}\varepsilon^{\sigma\lambda\tau\omega}\bar\psi_{\lambda i}\gamma_\tau\psi_\omega{}^i\,\bar X^M X^N\Big)\\
&\qquad + 16\,\mathrm{i}\,g\,{T_{(MN)}}^\Lambda\Omega^{MP}\big(X^N\,\mu^{ij}{}_P\,\bar\epsilon_{2i}\gamma_{\mu\nu}\epsilon_{1j} - \bar X^N\,\mu_{ijP}\,\bar\epsilon_2^i\gamma_{\mu\nu}\epsilon_1^j\big)\,,
\end{aligned}\tag{7.17}$$

with the parameters ξ^μ, Λ^M and $\Xi_{\mu\,\mathsf{a}}$ as in (7.9). The first four terms can straightforwardly be compared to the variation of $B_{\mu\nu\mathsf{a}}$ given in the first formula of (7.8). However, there is a subtlety regarding the commutator on ${W_\nu}^N$ in the third term, because this supersymmetry commutator only closes on the gauge fields, up to a term $\xi^\rho(\mathcal{G}-\mathcal{H})_{\rho\nu}{}^N$. Therefore the commutator yields the transformations indicated on the right-hand side of (7.9) plus this extra term.[14] Obviously the commutator on ${W_\nu}^N$ generates also a diffeomorphism, which

[14] Upon contraction with $Z^{M\,\mathsf{a}}$ this term vanishes and we have argued that it could therefore be suppressed in the commutator on the gauge fields on ${W_\nu}^N$. See the text preceding (7.16). However, in the case at hand the extra term has to be retained.

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will play a role later on in the calculation. Finally the fourth term represents precisely a supersymmetry transformation with parameter $\epsilon^i = -\frac{1}{2}\xi^\rho\psi_\rho{}^i$.

The remaining terms in (7.17), however, do not seem to have a role to play. At this point we note that the Lagrangian does not depend separately on $Z^{\Lambda,\mathsf{a}}B_{\mu\nu\,\mathsf{a}}$ and $Z^{\Lambda,\mathsf{m}}B_{\mu\nu\,\mathsf{m}}$, but depends only on the linear combination $Z^{\Lambda,\mathsf{a}}\,B_{\mu\nu\,\mathsf{a}} + Z^{\Lambda,\mathsf{m}}\,B_{\mu\nu\,\mathsf{m}}$. Consequently, the algebra is required to close only on this linear combination. Therefore we also evaluate the commutator on $Z^{\Lambda,\mathsf{m}}\,B_{\mu\nu\,\mathsf{m}}$,

$$
\begin{aligned}
Z^{\Lambda,\mathsf{m}}\,[\delta(\epsilon_1),\delta(\epsilon_2)]B_{\mu\nu\,\mathsf{m}} = {}& 2\,Z^{\Lambda,\mathsf{m}}\mathcal{D}_{[\mu}\Xi_{\nu]\mathsf{m}} \\
& + \mathrm{i}\,\xi^\rho\big(k^{A\Lambda}\,\gamma_{Ai\bar\alpha}\,\bar\zeta^{\bar\alpha}\gamma_{\mu\nu}\psi_\rho{}^i - 2\,\varepsilon^{ij}\,\mu_{jk}{}^\Lambda\bar\psi_{i[\mu}\gamma_{\nu]}\psi_\rho{}^k - \text{h.c.}\big) \\
& - 16\,\mathrm{i}gT_{(MN)}{}^\Lambda\Omega^{MP}\big(X^N\,\mu^{ij}{}_P\,\bar\epsilon_{2i}\gamma_{\mu\nu}\epsilon_{1j} - \bar X^N\,\mu_{ijP}\,\bar\epsilon_2^i\gamma_{\mu\nu}\epsilon_1^j\big) \\
& + \mathrm{i}e\,\varepsilon_{\mu\nu\rho\sigma}\xi^\rho\bigg[G_{\bar\alpha\beta}T^{\Lambda\beta}{}_\gamma\,\bigg(\frac{1}{2}A^{i\bar\alpha}\,\overleftrightarrow{\mathcal{D}}{}^\sigma A_i{}^\gamma - 2\,\bar\zeta^{\bar\alpha}\gamma^\sigma\zeta^\gamma \\
& + \bar\psi_\lambda{}^i\gamma^\sigma\gamma^\lambda\zeta^{\bar\alpha}A_i{}^\gamma - \bar\psi_{\lambda i}\gamma^\sigma\gamma^\lambda\zeta^\gamma A^{i\bar\alpha}\bigg) \\
& - e^{-1}\varepsilon^{\sigma\lambda\tau\omega}\,\bar\psi_\lambda{}^i\gamma_\tau\psi_{\omega j}\varepsilon^{jk}\mu_{ik}{}^\Lambda\bigg]\,,
\end{aligned}
\tag{7.18}
$$

with the parameters ξ^μ and $\Xi_{\mu\,\mathsf{m}}$ as in (7.9). The first line establishes closure with respect to $\Xi_{\mu\,\mathsf{m}}$. Furthermore, the next line correctly reproduces a supersymmetry transformation with parameter $\epsilon^i = -\frac{1}{2}\xi^\rho\psi_\rho{}^i$.

When considering the sum of the two variations (7.17) and (7.18) there are some cancelations, and on the remaining terms we can impose the field equation (7.14). This leaves the following terms,

$$
\begin{aligned}
[\delta(\epsilon_1),\delta(\epsilon_2)]\big(Z^{\Lambda,\mathsf{a}}\,B_{\mu\nu\,\mathsf{a}} + Z^{\Lambda,\mathsf{m}}\,B_{\mu\nu\,\mathsf{m}}\big) = {}& Z^{\Lambda,\mathsf{a}}\,\xi^\rho\mathcal{H}_{\mu\nu\rho\,\mathsf{a}} + Z^{\Lambda,\mathsf{m}}\,\xi^\rho\mathcal{H}_{\mu\nu\rho\,\mathsf{m}} \\
& - 2\,T_{(MN)}{}^\Lambda W_{[\mu}{}^M\,\xi^\rho(\mathcal{G}-\mathcal{H})_{\nu]\rho}{}^N + \cdots\,,
\end{aligned}
\tag{7.19}
$$

where the dots refer to terms that have already been accounted for in the context of (7.9). The explicit terms in (7.19) contribute to the (covariant) general coordinate transformation, as follows from the following identities, which can be derived straightforwardly from (6.19),

$$
\begin{aligned}
Z^{\Lambda,\mathsf{a}}\,\xi^\rho\,\mathcal{H}_{\rho\mu\nu\,\mathsf{a}} = {}& Z^{\Lambda,\mathsf{a}}\,\big(\xi^\rho\partial_\rho B_{\mu\nu\,\mathsf{a}} - 2\,\partial_{[\mu}\xi^\rho B_{\nu]\rho\,\mathsf{a}}\big) \\
& + 2\,Z^{\Lambda,\mathsf{a}}\mathcal{D}_{[\mu}\,\big(\xi^\rho B_{\nu]\rho\,\mathsf{a}} - \xi^\rho d_{\mathsf{a}\,MN}W_{\nu]}{}^M W_\rho{}^N\big) \\
& + 2\,T_{(MN)}{}^\Lambda\xi^\rho W_\rho{}^M\mathcal{G}_{\mu\nu}{}^N \\
& - 2\,T_{(MN)}{}^\Lambda W_{[\mu}{}^M\,\big(\xi^\rho\partial_{|\rho|}W_{\nu]}{}^N + \partial_{\nu]}\xi^\rho W_\rho{}^N - 2\,\xi^\rho(\mathcal{G}-\mathcal{H})_{\nu]\rho}{}^N\big) \\
& - 2\,g\,T_{(MN)}{}^\Lambda Z^{M,\mathsf{m}}\,\xi^\rho\,W_\rho{}^N B_{\mu\nu\,\mathsf{m}}\,, \\
Z^{\Lambda,\mathsf{m}}\,\xi^\rho\,\mathcal{H}_{\rho\mu\nu\,\mathsf{m}} = {}& Z^{\Lambda,\mathsf{m}}\,\big(\xi^\rho\partial_\rho B_{\mu\nu\,\mathsf{m}} - 2\,\partial_{[\mu}\xi^\rho B_{\nu]\rho\,\mathsf{m}}\big) \\
& + 2\,Z^{\Lambda,\mathsf{m}}\mathcal{D}_{[\mu}(\xi^\rho B_{\nu]\rho\,\mathsf{m}}) \\
& + 2\,g\,T_{(MN)}{}^\Lambda Z^{M,\mathsf{m}}\,\xi^\rho\,W_\rho{}^N B_{\mu\nu\,\mathsf{m}}\,.
\end{aligned}
\tag{7.20}
$$

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The first two terms in the equations (7.20) denote the expected general coordinate transformation, and the tensor gauge transformations with parameters given in (7.11). The third term in the first equations represents the appropriate gauge transformation. The last terms in the two equations cancel directly, so that the only terms in (7.19) that are still unaccounted for, are given by

$$\begin{aligned}[\delta(\epsilon_1),\delta(\epsilon_2)]\big(Z^{\Lambda,\mathsf{a}}\,B_{\mu\nu\,\mathsf{a}}+Z^{\Lambda,\mathsf{m}}\,B_{\mu\nu\,\mathsf{m}}\big) = &-2\,T_{(MN)}{}^{\Lambda}W_{[\mu}{}^{M}\big(\xi^\rho\partial_{|\rho|}W_{\nu]}{}^{N}+\partial_{\nu]}\xi^\rho W_\rho{}^{N}\big)\\ &+2\,T_{(MN)}{}^{\Lambda}W_{[\mu}{}^{M}\xi^\rho(\mathcal{G}-\mathcal{H})_{\nu]\rho}{}^{N}+\cdots\,. \end{aligned} \tag{7.21}$$

The first of these terms cancels against the general coordinate transformation induced by the supersymmetry commutator on $W_\nu{}^N$ in (7.17), which we already referred to earlier, and which is not required on the tensor fields in view of the fact that the above equations (7.20) already account for the general coordinate transformation. The second term can be suppressed by virtue of the special invariance noted in (6.21). To see this, we note that, up to the first equation of motion (7.13), we can write the induced variation of $B_{\mu\nu\mathsf{a}}$ as,

$$\begin{aligned}Z^{\Lambda,\mathsf{a}}\,\delta B_{\mu\nu\mathsf{a}} \propto T^{(\Lambda}{}_{M}{}^{\Sigma)}\,[4\,\xi^\rho W_{[\mu}{}^{M}-\xi^\sigma W_\sigma{}^{M}\,\delta^\rho_{[\mu}](\mathcal{G}-\mathcal{H})_{\nu]\rho\Sigma}\\ -T^{[\Lambda}{}_{M}{}^{\Sigma]}\,\xi^\sigma W_\sigma{}^{M}\,(\mathcal{G}-\mathcal{H})_{\mu\nu\Sigma}\,. \end{aligned} \tag{7.22}$$

This completes our discussion of the supersymmetry algebra.

Finally we summarize the modifications to the Lagrangian that are required by the general gaugings. As usual these concern both masslike terms for the fermions, which are proportional to the gauge coupling g, and a scalar potential proportional to g^2. The masslike terms independent of the gravitini follow directly from the rigid theory in the presence of both electric and magnetic charges [7]. The terms that involve gravitini are generalizations of the known results for the superconformal theory in the presence of electric charges [4, 6, 32]. The result includes also a non-fermionic term which describes the coupling of the auxiliary fields $Y_{ij}{}^{\Lambda}$ to the moments μ_{ijM},

$$\begin{aligned}e^{-1}\mathcal{L}_g = &-\tfrac{1}{2}\mathrm{i}g\,\Omega_{MQ}T_{PN}{}^{Q}\,\varepsilon^{ij}\,\bar X^N\bar\Omega_i{}^M\big(\Omega_j{}^P+\gamma^\mu\psi_{\mu j}X^P\big)+\text{h.c.}\\ &+2g\,k_{AM}\gamma^A_{i\bar\alpha}\varepsilon^{ij}\,\bar\zeta^{\bar\alpha}\big(\Omega_j{}^M+\gamma^\mu\psi_{\mu j}X^M\big)+\text{h.c.}\\ &+g\,\mu^{ij}{}_M\,\bar\psi_{\mu i}\,\big(\gamma^\mu\Omega_j{}^M+\gamma^{\mu\nu}\psi_{\nu j}X^M\big)+\text{h.c.}\\ &+2g\left[\bar X^M T_M{}^{\gamma}{}_{\alpha}\,\bar\Omega_{\beta\gamma}\,\bar\zeta^\alpha\zeta^\beta+X^M T_M{}^{\bar\gamma}{}_{\bar\alpha}\,\Omega_{\bar\beta\bar\gamma}\,\bar\zeta^{\bar\alpha}\zeta^{\bar\beta}\right]\\ &-\tfrac{1}{4}g\,\left[F_{\Lambda\Sigma\Gamma}\,\mu^{ij\Lambda}\,\bar\Omega_i{}^{\Sigma}\Omega_j{}^{\Gamma}+\bar F_{\Lambda\Sigma\Gamma}\,\mu_{ij}{}^{\Lambda}\,\bar\Omega^{i\Sigma}\Omega^{j\Gamma}\right]\\ &+g\,Y^{ij\Lambda}\left[\mu_{ij\Lambda}+\tfrac{1}{2}(F_{\Lambda\Sigma}+\bar F_{\Lambda\Sigma})\,\mu_{ij}{}^{\Sigma}\right]. \end{aligned} \tag{7.23}$$

Upon solving the auxiliary fields $Y_{ij}{}^I$ one obtains an additional contribution to the scalar potential of order g^2. Without this contribution the scalar potential reads,

$$\begin{aligned}e^{-1}\mathcal{L}_{g^2} = &\,\mathrm{i}g^2\,\Omega_{MN}\,T_{PQ}{}^{M}X^P\bar X^Q\,T_{RS}{}^{N}\bar X^R X^S\\ &-2g^2k^A{}_M\,k^B{}_N\,g_{AB}\,X^M\bar X^N-\tfrac{1}{2}g^2\,N_{\Lambda\Sigma}\,\mu_{ij}{}^{\Lambda}\,\mu^{ij\Sigma}\,. \end{aligned} \tag{7.24}$$

Upon eliminating the auxiliary fields, the last term in this expression changes into

$$-\frac{1}{2}g^2\,N_{\Lambda\Sigma}\,\mu_{ij}{}^{\Lambda}\,\mu^{ij\Sigma}\longrightarrow -2\,g^2\left[\mu^{ij}{}_{\Lambda}+F_{\Lambda\Gamma}\,\mu^{ij\Gamma}\right]\,N^{\Lambda\Sigma}\left[\mu_{ij\Sigma}+\bar F_{\Sigma\Xi}\,\mu_{ij}{}^{\Xi}\right]\,. \tag{7.25}$$

The above expressions are not of definite sign. From the Lagrangians in section 4 one can deduce that χ_{vector}, χ_{hyper} and the metrics that appear in the kinetic terms of the physical scalar fields should be negative. The latter metrics are proportional to two matrices, $M_{\Lambda\Sigma}$ and G_{AB}, that should therefore be negative definite. They are defined by

$$\begin{aligned}M_{\Lambda\bar\Sigma}&=\chi_{\text{vector}}^{-2}\,(N_{\Lambda\Sigma}N_{\Gamma\Xi}-N_{\Lambda\Gamma}N_{\Sigma\Xi})\,\bar X^{\Gamma}X^{\Xi}\,,\\ G_{AB}&=\chi_{\text{hyper}}^{-1}\left(g_{AB}-\chi_{\text{hyper}}^{-1}\left(\frac{1}{2}\chi_A\chi_B+k_{Aij}k_B{}^{ij}\right)\right)\,.\end{aligned} \tag{7.26}$$

With these observations we can separate the terms in the potential in positive and negative ones,

$$\begin{aligned}e^{-1}\mathcal{L}_{g^2}=&-g^2\,\chi_{\text{vector}}\,M_{\bar\Lambda\Sigma}\,(T_{PQ}{}^{\Lambda}X^P\bar X^Q)\,(T_{RS}{}^{\Sigma}\bar X^RX^S)\\&-4\,g^2\chi_{\text{vector}}\,k^A{}_M\,k^B{}_N\,G_{AB}\,X^M\bar X^N\\&-2\,g^2\,\chi_{\text{vector}}\,M_{\bar\Lambda\Sigma}\,N^{\Lambda\Gamma}\left[\mu^{ij}{}_{\Gamma}+F_{\Gamma\Omega}\,\mu^{ij\Omega}\right]\,N^{\Sigma\Xi}\left[\mu_{ij\Xi}+\bar F_{\Xi\Delta}\,\mu_{ij}{}^{\Delta}\right]\\&-6\,g^2\chi_{\text{vector}}^{-1}\,X^M\bar X^N\,\mu_{ijM}\,\mu^{ij}{}_N\,,\end{aligned} \tag{7.27}$$

where we used that $\chi_{\text{hyper}}=2\,\chi_{\text{vector}}$, as is implied by the field equation associated with the field D. It then follows that all contributions to $\mathcal{L}_{g^2}$ are negative, with the exception of the last term which is positive. This decomposition generalizes a similar decomposition known for purely electric charges.

8 Summary and some applications

In this paper we presented Lagrangians and supersymmetry transformations for general superconformal systems of vector multiplets and hypermultiplets in the presence of both electric and magnetic charges. The results were verified to all orders and are consistent with results known in the literature based on both rigidly supersymmetric theories and on superconformal systems without magnetic charges. In the presence of magnetic charges the off-shell closure of the superconformal algebra is only realized on the Weyl multiplet. The results of this paper establish a general framework for studying gauge interactions in matter-coupled $N=2$ supergravity.

In the remainder of this last section we discuss two specific applications to demonstrate the consequences of this general framework. The first one discusses full and partial supersymmetric solutions in maximally symmetric space-times, and the second one deals with full or partial supersymmetric solutions in $\mathrm{AdS}_2\times S^2$ space-times.

8.1 Maximally symmetric space-times and supersymmetry

In this application we briefly consider the question of full or partial supersymmetry in a maximally symmetric space-time. Hence one evaluates the supersymmetry variations of the

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fermion fields in the maximally symmetric background, where only $g_{\mu\nu}$, $A_i{}^\alpha$, X^Λ and $Y_{ij}{}^\Lambda$ can take non-zero values, taking into account that the fermion fields transform under both Q- and S-supersymmetry. In this particular background, it turns out that the gravitino field strength, $R(Q)_{\mu\nu}{}^i$ (and the related spinor χ^i) is S-invariant. Since its Q-supersymmetry variation is proportional to the field D, it immediately follows that $D = 0$, so that the special conformal gauge field takes the value (we assume the gauge choice $b_\mu = 0$, which leaves a residual invariance under constant scale transformations),

$$f_\mu{}^a = \frac{1}{2}R(e,\omega)_\mu{}^a - \frac{1}{12}e_\mu{}^a\,R(e,\omega)\,, \tag{8.1}$$

where $R(e,\omega)_{\mu\nu}{}^{ab}$ denotes the space-time curvature.

In what follows it thus suffices to concentrate on the fermions belonging to the vector multiplets and the hypermultiplets. We first present their variations in the background, which follow directly from (7.1) and (7.2),

$$\begin{aligned}\delta\zeta^\alpha &= 2gX^M\,T_M{}^\alpha{}_\beta A_i{}^\beta\,\varepsilon^{ij}\epsilon_j + A_i{}^\alpha\,\eta^i\,,\\ \delta\Omega_i{}^M &= \hat{Z}_{ij}{}^M\epsilon^j - 2g\,T_{PN}{}^M\bar{X}^P X^N\varepsilon_{ij}\epsilon^j + 2\,\mathrm{i}g\Omega^{MN}\mu_{ijN}\epsilon^j + 2X^M\eta_i\,.\end{aligned} \tag{8.2}$$

Substituting the equations of motion for the auxiliary fields $Y_{ij}{}^\Lambda$, the variation of the independent fermion fields $\delta\Omega_i{}^\Lambda$ takes the following form,

$$\delta\Omega_i{}^\Lambda = -2g\,T_{NP}{}^\Lambda\,\bar{X}^N X^P\,\varepsilon_{ij}\,\epsilon^j - 4\,gN^{\Lambda\Sigma}\big(\mu_{ij\Sigma} + \bar{F}_{\Sigma\Gamma}\,\mu_{ij}{}^\Gamma\big)\epsilon^j + 2\,X^\Lambda\eta_i\,, \tag{8.3}$$

Following the strategy adopted by [42], we consider only combinations of fermion fields that are invariant under S-supersymmetry. To construct S-invariant combinations of these fermions, it is convenient to define the following two spinor fields,

$$\begin{aligned}\zeta_i^{\rm H} &= \chi_{\rm hyper}^{-1}\bar{\Omega}_{\alpha\beta}A_i{}^\alpha\,\zeta^\beta\\ \Omega_i^{\rm V} &= -\frac{1}{2}\mathrm{i}\chi_{\rm vector}^{-1}\Omega_{MN}\bar{X}^M\Omega_i{}^N = \frac{1}{2}\chi_{\rm vector}^{-1}\,\bar{X}^\Lambda N_{\Lambda\Sigma}\Omega_i{}^\Sigma\,,\end{aligned} \tag{8.4}$$

which are both formally invariant under duality when treating the embedding tensor as a spurion. Under supersymmetry these two spinors transform equivalently in this background, provided we also use the field equation of the field D, which yields $\chi_{\rm hyper} = 2\,\chi_{\rm vector}$. Indeed one easily derives,

$$\delta\Omega_i^{\rm V} = A_{ij}\,\epsilon^j + \eta_i = -\varepsilon_{ij}\,\delta\zeta^{{\rm H}\,j}\,, \tag{8.5}$$

where the symmetric matrix A_{ij} is given by,

$$A_{ij} = -2\,g\,\chi_{\rm vector}^{-1}\,\bar{X}^M\mu_{ijM}\,. \tag{8.6}$$

Here we made use of equations (5.11).

To make contact with the terms appearing in the potential (7.27), we consider the variations of three other spinors, which are S-supersymmetry invariant and consistent with duality. As it turns out, considering such variations gives important information regarding the possible supersymmetric realizations, although it will not yet fully determine whether the corresponding solutions will actually be realized. The first two variations are,

$$
\begin{aligned}
g\big(\mu^{ij}{}_\Lambda + F_{\Lambda\Sigma}\,\mu^{ij\Sigma}\big)\,\delta[\Omega_j{}^\Lambda - 2\,X^\Lambda\Omega_j^{\rm V}] ={}& -2\,g^2\,\bar X^M X^N T_{MN}{}^P \mu^{ij}{}_P\,\varepsilon_{jk}\,\epsilon^k \\
& -2\,g^2(\mu^{kl}{}_\Lambda + F_{\Lambda\Sigma}\,\mu^{kl\Sigma})N^{\Lambda\Gamma}(\mu_{kl\Gamma} + \bar F_{\Gamma\Xi}\,\mu_{kl}{}^\Xi)\epsilon^i \\
& +\chi_{\rm vector}\,A^{ij}A_{jk}\epsilon^k\,, \\
gN_{\Lambda\Sigma}\,T_{MN}{}^\Sigma X^M \bar X^N\,\delta[\Omega_i{}^\Lambda - 2\,X^\Lambda\Omega_i^{\rm V}] ={}& 2{\rm i}\,g^2\Omega_{MN}(T_{PQ}{}^M X^P\bar X^Q)\,(T_{RS}{}^N\bar X^R X^S)\,\varepsilon_{ij}\epsilon^j \\
& -4\,g^2\,X^M\bar X^N T_{MN}{}^P\mu_{ijP}\,\epsilon^j\,.
\end{aligned}
\tag{8.7}
$$

In deriving this result we made use of identities such as (5.9) and (5.11). Furthermore we used $\Omega^{MN}\mu_{ijM}\,\mu_{klN} = \mu_{ij\Lambda}\,\mu_{kl}{}^\Lambda - \mu_{ij}{}^\Lambda\,\mu_{kl\Lambda} = 0$, which follows directly from (5.20). The third spinor variation is based on hypermultiplets,

$$
\begin{aligned}
g\,\bar X^M T_M{}^\alpha{}_\beta A_i{}^\beta\,\bar\Omega_{\alpha\gamma}\,\delta\big[\zeta^\gamma + \varepsilon^{jk}A_j{}^\gamma\,\zeta_k^{\rm H}\big] ={}& -g^2\bar X^M X^N\,k^A{}_M\,k^B{}_N\,g_{AB}\,\epsilon_i \\
& -2\,g^2\bar X^M X^N\,T_{MN}{}^P\,\mu_{ijP}\,\varepsilon^{jk}\,\epsilon_k \\
& +\chi_{\rm vector}\,A_{ij}A^{jk}\,\epsilon_k\,.
\end{aligned}
\tag{8.8}
$$

Here we made use of the identity,

$$
T_M{}^\alpha{}_\beta A_i{}^\beta\,\bar\Omega_{\alpha\gamma}\,T_N{}^\gamma{}_\delta A_j{}^\delta = \frac{1}{2}\varepsilon_{ij}\,k^A{}_M\,k_{AN} + T_{MN}{}^P\,\mu_{ijP}\,,
\tag{8.9}
$$

which follows from (3.16), (3.27), (3.32) and (5.17). Combining (8.8) with the two previous identities gives,

$$
\big[e^{-1}\mathcal{L}_{g^2}\,\delta^i{}_j + 3\,\chi_{\rm vector}A^{ik}A_{kj}\big]\,\epsilon^j = 0\,.
\tag{8.10}
$$

This relation requires $e^{-1}\mathcal{L}_{g^2}$ to be non-negative, confirming the known result that de Sitter space-times cannot be supersymmetric.

According to [42] one must also consider the symmetry variation of the supercovariant derivative of at least one of these spinor fields. Let us, for instance, consider $D_\mu\Omega_i^{\rm V}$, which transforms also under S-supersymmetry. The following combination is then again S-invariant, and changes under Q-symmetry according to,

$$
\delta\Big[D_\mu\Omega_i^{\rm V} - \frac{1}{2}A_{ij}\gamma_\mu\Omega^{{\rm V}\,j}\Big] = f_\mu{}^a\gamma_a\epsilon_i - \frac{1}{2}A_{ij}A^{jk}\,\gamma_\mu\epsilon_k\,.
\tag{8.11}
$$

Therefore we must require that the supersymmetry parameters are subject to the eigenvalue condition,

$$
\Big[\delta^i{}_j\,\big(R(e,\omega)_\mu{}^a - \frac{1}{6}e_\mu{}^a R(e,\omega)\big) - e_\mu{}^a\,A^{ik}A_{kj}\Big]\epsilon^j = 0\,.
\tag{8.12}
$$

Combining this result with (8.10) reproduces the Einstein equation for the maximally symmetric space-time, irrespective of whether supersymmetry is realized fully or partially. Observe that full supersymmetry requires that $A^{ik}A_{kj} \propto \delta^i{}_j$.

The result (8.10) can also be written as

$$
\Big[A^{ik}A_{kj} - \frac{1}{2}A^{kl}A_{kl}\,\delta^i{}_j\Big]\epsilon^j = -\frac{e^{-1}\mathcal{L}^-_{g^2}}{3\,\chi_{\rm vector}}\,\epsilon^i\,,
\tag{8.13}
$$

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where $\mathcal{L}^-_{g^2}$ pertains to the negative terms in $\mathcal{L}_{g^2}$. For full supersymmetry we thus find that $\mathcal{L}^-_{g^2}$ must vanish, while partial supersymmetry is associated with the smallest eigenvalue of $A^{ik}A_{kj}$ and $\mathcal{L}^-_{g^2} \neq 0$. We refrain from giving more explicit details here, but we briefly consider the special case of Minkowski space-time. For partial supersymmetry, the unbroken supersymmetry parameter is subject to the condition $A_{ij}\epsilon^j = 0$. In this context one can consider the variation of yet another spinor, which is invariant under S-supersymmetry, but no longer under duality,

$$\begin{aligned} X^\Lambda N_{\Lambda\Sigma}\, \delta[\Omega_i{}^\Sigma - 2\, X^\Sigma\, \Omega_i^{\rm V}] = & - 2\, g X^\Lambda N_{\Lambda\Sigma}\, \big[T_{MN}{}^\Sigma\, \bar X^M X^N\, \varepsilon_{ij} - 2\mathrm{i}\mu_{ij}{}^\Sigma\big]\epsilon^j \\ & + 2\, X^\Lambda N_{\Lambda\Sigma}\, \big[\bar X^\Sigma\, \varepsilon_{ik}\varepsilon_{jl}\, A^{kl} - X^\Sigma\, A_{ij}\big]\epsilon^j\,. \end{aligned} \tag{8.14}$$

In the absence of magnetic charges, the first term on the right-hand side vanishes because $T_{MN}{}^\Sigma \bar X^M X^N$ can be replaced by $T_{(MN)}{}^\Sigma \bar X^M X^N$ by virtue of the third equation of (5.11), which vanishes without magnetic charges, and so does the moment map $\mu_{ij}{}^\Sigma$. Therefore both $A_{ij}\epsilon^j$ and $A^{ij}\varepsilon_{jk}\epsilon^k$ vanish, which implies that A_{ij} vanishes so that supersymmetry must be fully realized. This is in accord with a known theorem according to which $N = 2$ supersymmetry can only be broken to $N = 1$ supersymmetry in Minkowski space in the presence of magnetic charges [18, 20–24]. For the abelian gaugings the situation simplifies, and one can show that Minkowski solutions with residual $N = 1$ supersymmetry are possible provided that,

$$\begin{aligned} \bar X^M\, T_M{}^\alpha{}_\beta\, A_i{}^\beta \epsilon^i &= 0\,, \\ (\mu_{ij\Lambda} + \bar F_{\Lambda\Sigma}\, \mu_{ij}{}^\Sigma)\, \epsilon^j &= 0\,, \end{aligned} \tag{8.15}$$

with the two terms of the abelian potential vanishing separately (this follows from the first equation of (8.7) and from (8.8)),

$$\begin{aligned} \bar X^M X^N\, k^A{}_M\, k^B{}_N\, g_{AB} &= 0\,, \\ (\mu^{kl}{}_\Lambda + F_{\Lambda\Sigma}\, \mu^{kl\Sigma}) N^{\Lambda\Gamma} (\mu_{kl\Gamma} + \bar F_{\Gamma\Xi}\, \mu_{kl}{}^\Xi) &= 0\,. \end{aligned} \tag{8.16}$$

Without magnetic charges, one can easily verify that residual $N = 1$ supersymmetric solutions are not possible.

Apart from this latter result, the above analysis only indicates which supersymmetric solutions can, in principle, exist. To confirm that they are actually realized, one has to also examine the supersymmetry variations of the remaining fermion fields. This can be done, but we prefer not to demonstrate this here. Instead we will discuss this explicitly in the application presented in the next subsection, which is less straightforward, and where we will follow the same set-up as in this subsection.

8.2 Supersymmetry in $\mathrm{AdS}_2 \times S^2$

In this second application we consider an $\mathrm{AdS}_2 \times S^2$ space-time background and analyze possible supersymmetric solutions. Hence the space-time metric can be chosen equal to,

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = v_1\Big(- r^2\, \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2}\Big) + v_2\Big(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2\Big)\,, \tag{8.17}$$

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whose non-vanishing Riemann curvature components are equal to

$$R_{\underline{ab}}{}^{\underline{cd}} = 2\, v_1^{-1} \delta_{\underline{ab}}{}^{\underline{cd}}\,, \qquad R_{\hat{a}\hat{b}}{}^{\hat{c}\hat{d}} = -2\, v_2^{-1} \delta_{\hat{a}\hat{b}}{}^{\hat{c}\hat{d}}\,, \tag{8.18}$$

so that the four-dimensional Ricci scalar equals $R = 2(v_1^{-1} - v_2^{-1})$. Observe that we used tangent-space indices above, where $\underline{a}, \underline{b}, \ldots$ label the flat AdS_2 indices $(0,1)$ associated with (t,r), and $\hat{a}, \hat{b}, \ldots$ label the flat S^2 indices $(2,3)$ associated with (θ, φ). Furthermore the non-vanishing components of the auxiliary tensor field are parametrized by a complex scalar w,

$$-\,T_{01}{}^{ij}\varepsilon_{ij} = -\mathrm{i}\, T_{23}{}^{ij}\varepsilon_{ij} = w\,. \tag{8.19}$$

Using the previous results one finds the following expressions for the bosonic part of the special conformal gauge field $f_a{}^b$,

$$\begin{aligned} f_{\underline{a}}{}^{\underline{b}} &= \Big(\frac{1}{6}(2\, v_1^{-1} + v_2^{-1}) - \frac{1}{4}D - \frac{1}{32}|w|^2\Big)\delta_{\underline{a}}{}^{\underline{b}} + \frac{1}{2}R(A)_{23}\, \varepsilon_{\underline{a}}{}^{\underline{b}}\,, \\ f_{\hat{a}}{}^{\hat{b}} &= \Big(-\frac{1}{6}(v_1^{-1} + 2\, v_2^{-1}) - \frac{1}{4}D + \frac{1}{32}|w|^2\Big)\delta_{\hat{a}}{}^{\hat{b}} + \frac{1}{2}R(A)_{01}\, \varepsilon_{\hat{a}}{}^{\hat{b}}\,, \end{aligned} \tag{8.20}$$

where the two-dimensional Levi-Civita symbols are normalized by $\varepsilon^{01} = \varepsilon^{23} = 1$. The non-zero components of the modified curvature $\mathcal{R}(M)_{ab}{}^{cd}$ are given by,

$$\begin{aligned} \mathcal{R}(M)_{\underline{ab}}{}^{\underline{cd}} &= \Big(D + \frac{1}{3}R\Big)\,\delta_{\underline{ab}}{}^{\underline{cd}}\,, \\ \mathcal{R}(M)_{\hat{a}\hat{b}}{}^{\hat{c}\hat{d}} &= \Big(D + \frac{1}{3}R\Big)\,\delta_{\hat{a}\hat{b}}{}^{\hat{c}\hat{d}}\,, \\ \mathcal{R}(M)_{\underline{a}\hat{b}}{}^{\underline{c}\hat{d}} &= \frac{1}{2}\Big(D - \frac{1}{6}R\Big)\,\delta_{\underline{a}}{}^{\underline{c}}\,\delta_{\hat{b}}{}^{\hat{d}} - \frac{1}{2}R(A)_{23}\,\varepsilon_{\underline{a}}{}^{\underline{c}}\,\delta_{\hat{b}}{}^{\hat{d}} - \frac{1}{2}R(A)_{01}\,\delta_{\underline{a}}{}^{\underline{c}}\,\varepsilon_{\hat{b}}{}^{\hat{d}}\,. \end{aligned} \tag{8.21}$$

We refer to the appendices presented in [29] for the general definitions of these quantities, which appear in the superconformal transformation rules of the Weyl multiplet fields and are therefore needed below.

Motivated by the maximal symmetry of the two two-dimensional subspaces, we expect the various fields to be invariant under the same symmetry. Therefore we will assume that the scalars X^M and $A_i{}^\alpha$ are covariantly constant (for other fields the covariant constancy will be discussed in due course). The corresponding integrability condition then requires that the U(1) and SU(2) R-symmetry curvatures are not necessarily vanishing, and are related to the curvatures of the vector multiplet gauge fields. This result is consistent with the field equations for the R-symmetry gauge fields, A_μ and $\mathcal{V}_\mu{}^i{}_j$, which lead to the expressions (we again choose the gauge $b_\mu = 0$),

$$\begin{aligned} R(A)_{\mu\nu} &= g\, \chi_{\text{vector}}^{-1} \mathcal{H}_{\mu\nu}{}^M T_{MQ}{}^N \Omega_{PN} \bar{X}^Q X^P\,, \\ R(\mathcal{V})_{\mu\nu}{}^i{}_j &= -\,4g\chi_{\text{hyper}}^{-1} \mathcal{H}_{\mu\nu}{}^M \mu^{ik}{}_M\, \varepsilon_{kj}\,. \end{aligned} \tag{8.22}$$

Observe that the above equations only contribute for $\mu,\nu = t,r$, or $\mu,\nu = \theta,\varphi$, in view of the space-time symmetry. We can rewrite these equations in a different form, which is convenient later on,

$$
\begin{aligned}
R(A)^-_{\mu\nu} &= g\,\chi^{-1}_{\text{vector}}\hat{\mathcal{H}}^{-}_{\mu\nu}{}^{\Lambda}\big[T_{\Lambda Q}{}^{N} + F_{\Lambda\Sigma}\,T^{\Sigma}{}_{Q}{}^{N}\big]\,\Omega_{PN}\bar{X}^{Q}X^{P}\,,\\
R(\mathcal{V})^-_{\mu\nu}{}^{i}{}_{j} &= -\,4g\chi^{-1}_{\text{hyper}}\hat{\mathcal{H}}^{-}_{\mu\nu}{}^{\Lambda}\big[\mu^{ik}{}_{\Lambda} + F_{\Lambda\Sigma}\,\mu^{ik\Sigma}\big]\,\varepsilon_{kj} + \frac{1}{4}\,\varepsilon^{ik}A_{kj}\,T_{\mu\nu}{}^{mn}\varepsilon_{mn}\,,
\end{aligned}
\tag{8.23}
$$

where we suppressed all the fermionic terms which vanish in the background and made use of the field equations (7.13) of the tensor fields $B_{\mu\nu\,\mathsf{a}}$ and $B_{\mu\nu\,\mathsf{m}}$, and of (5.11).

To study supersymmetry in this background, we present the non-vanishing terms in the supersymmetry transformations of the spinors $\Omega_i{}^{\Lambda}$ and ζ^{α},

$$
\begin{aligned}
\delta\Omega_i{}^{\Lambda} &= \frac{1}{2}\gamma^{\mu\nu}\hat{\mathcal{H}}^{-}_{\mu\nu}{}^{\Lambda}\,\varepsilon_{ij}\epsilon^{j} - 2g\,T_{NP}{}^{\Lambda}\,\bar{X}^{N}X^{P}\,\varepsilon_{ij}\,\epsilon^{j} - 4\,gN^{\Lambda\Sigma}\big(\mu_{ij\Sigma} + \bar{F}_{\Sigma\Gamma}\,\mu_{ij}{}^{\Gamma}\big)\epsilon^{j} + 2\,X^{\Lambda}\eta_i\,,\\
\delta\zeta^{\alpha} &= 2gX^{M}\,T_{M}{}^{\alpha}{}_{\beta}A_i{}^{\beta}\,\varepsilon^{ij}\epsilon_j + A_i{}^{\alpha}\,\eta^{i}\,.
\end{aligned}
\tag{8.24}
$$

Note that $\delta\Omega_i{}^{\Lambda}$ has changed as compared to (8.3) by the presence of the field strength (7.6) (suppressing the fermionic terms, so that $\hat{\mathcal{H}}^{-}_{\mu\nu}{}^{\Lambda} = \mathcal{H}^{-}_{\mu\nu}{}^{\Lambda} - \frac{1}{4}\bar{X}^{\Lambda}T_{\mu\nu}{}^{ij}\varepsilon_{ij}$), while the expression for $\delta\zeta^{\alpha}$ is identical to the one given in (8.2). Just as before, we make use of the two spinors Ω_i^{V} and ζ_i^{H} defined in (8.4). The supersymmetry variation of these fields in the given background are,

$$
\begin{aligned}
\delta\Omega_i^{\text{V}} &= \frac{1}{4}\chi^{-1}_{\text{vector}}\,\bar{X}^{\Lambda}N_{\Lambda\Sigma}\hat{\mathcal{H}}^{-}_{\mu\nu}\gamma^{\mu\nu}\varepsilon_{ij}\epsilon^{j} + A_{ij}\epsilon^{j} + \eta_i\,,\\
\delta\zeta_i^{\text{H}} &= \varepsilon_{ij}\big(A^{jk}\epsilon_k + \eta^{j}\big)\,,
\end{aligned}
\tag{8.25}
$$

where A_{ij} was defined in (8.6). Supersymmetry therefore implies that the terms proportional to $\gamma^{\mu\nu}$ must vanish. As it turns out, this condition is just the field equation for $T_{ab}{}^{ij}$,

$$
\bar{X}^{\Lambda}N_{\Lambda\Sigma}\,\hat{\mathcal{H}}^{-\Sigma}_{ab} = 0\,.
\tag{8.26}
$$

Two additional fermionic variations are,

$$
\begin{aligned}
\delta\Big[R(Q)_{ab}{}^{i} - \frac{1}{8}T_{cd}{}^{ij}\gamma^{cd}\gamma_{ab}\Omega_j^{\text{V}}\Big] &= R(\mathcal{V})^-_{ab}{}^{i}{}_{j}\epsilon^{j} - \frac{1}{2}\mathcal{R}(M)_{ab}{}^{cd}\gamma_{cd}\epsilon^{i} - \frac{1}{8}T_{cd}{}^{ij}\,\gamma^{cd}\gamma_{ab}A_{jk}\,\epsilon^{k}\,,\\
\delta\Big[D_a\Omega_i^{\text{V}} - \frac{1}{2}A_{ij}\gamma_a\Omega^{\text{V}\,j}\Big] &= f_a{}^{b}\gamma_b\epsilon_i + \frac{1}{4}\mathrm{i}R(A)^-_{cd}\gamma^{cd}\gamma_a\epsilon_i - \frac{1}{8}R(\mathcal{V})^-_{bci}{}^{j}\gamma^{bc}\gamma_a\epsilon_j\\
&\quad + \frac{1}{16}A_{ij}T_{bc}{}^{jk}\gamma^{bc}\gamma_a\epsilon_k - \frac{1}{2}A_{ij}A^{jk}\,\gamma_a\,\epsilon_k\,,
\end{aligned}
\tag{8.27}
$$

where we refer again to the appendices presented in [29] for more details. Observe that we have assumed, motivated by the maximal symmetry of the two-dimensional subspaces, that also $T_{ab}{}^{ij}$ and A_{ij} are covariantly constant.

The consequences of (8.27) can be expressed as follows,[15]

$$
\begin{aligned}
\Big(D+\frac{1}{12}R\Big)\epsilon^i+\big[R(\mathcal{V})_{23}^{-}{}^i{}_j-\mathrm{i}R(A)_{23}^{-}\,\delta^i{}_j\big]\gamma^{23}\,\epsilon^j&=0\,,\\
\Big(D-\frac{1}{6}R\Big)\epsilon^i-\Big[2\mathrm{i}R(A)_{23}^{-}\,\delta^i{}_j+\frac{1}{2}\mathrm{i}w\,\varepsilon^{ik}A_{kj}\Big]\gamma^{23}\,\epsilon^j&=0\,,\\
\Big[A^{ik}A_{kj}\epsilon^j+\frac{1}{4}\mathrm{i}w\,\varepsilon^{ik}A_{kj}\,\gamma^{23}\Big]\,\epsilon^j&=0\,,\\
\Big(v_1^{-1}+v_2^{-1}-\frac{1}{8}|w|^2\Big)\epsilon^i-\Big[\frac{1}{2}\mathrm{i}\bar{w}A^{ik}\varepsilon_{kj}+2\,R(\mathcal{V})_{23}^{+}{}^i{}_j+2\mathrm{i}R(A)_{23}^{+}\,\delta^i{}_j\Big]\gamma^{23}\epsilon^j&=0\,. \qquad (8.29)
\end{aligned}
$$

Furthermore we note that the covariant constancy of $T_{ab}{}^{ij}$ and A_{ij} implies the conditions,

$$
w\,R(A)_{\mu\nu}=0\,,\qquad R(\mathcal{V})_{\mu\nu}{}^k{}_{(i}\,A_{j)k}=-\mathrm{i}R(A)_{\mu\nu}\,A_{ij}\,. \qquad (8.30)
$$

An important observation is that both $\mathrm{i}R(\mathcal{V})_{\mu\nu}{}^i{}_j$ (for any μ,ν) and $\varepsilon^{ik}A_{kj}$ are 2×2 matrices that take their value in the Lie algebra of SU(2). However, while the matrices $\mathrm{i}R(\mathcal{V})_{\mu\nu}{}^i{}_j$ are necessarily hermitian, this is not the case with $\varepsilon^{ik}A_{kj}$, which is in general complex-valued.

We now turn to possible supersymmetric solutions for this background. We proceed in two steps. First we analyze the conditions for supersymmetry, ignoring the transformations (8.25). This will reveal the possible existence of three distinct classes of supersymmetric solutions, with four or eight supersymmetries, depending on the values of $R(\mathcal{V})_{\mu\nu}{}^i{}_j$ and A_{ij}. The corresponding information is summarized in table 3. As a last step we then analyze the transformations (8.25), which lead to additional constraints. It then follows that one of the classes listed in table 3 is actually not realized. In what follows we will decompose the equations (8.29) in eigenstates of $\mathrm{i}\gamma^{23}$, denoted by $\epsilon^i_\pm=\frac{1}{2}(1\pm\mathrm{i}\gamma^{23})\epsilon^i$. Observe that these spinors transform as a product representation of the SU(2) isometry group associated with S^2 and the SU(2) R-symmetry. This observation will be relevant shortly. Note also that the spinors transform according to $\epsilon_\pm{}^i\to\epsilon_{i\mp}$ under charge conjugation.

We start by noting that $w=0$ will only lead to a supersymmetric solution provided $v_1^{-1}=0$. Discarding this singular solution, we thus assume $R(A)_{\mu\nu}=0$. Then we consider two classes of solutions, denoted by A and B in table 3, depending on whether $D-\frac{1}{6}R$ vanishes or not.

For $R(A)_{\mu\nu}=0$ and $D-\frac{1}{6}R=0$, the equations (8.29) imply,

$$
\begin{aligned}
w\,A_{ij}\,\epsilon^j_\pm&=0\,,\\
\mathrm{i}R(\mathcal{V})_{23}^{-}{}^i{}_j\,\epsilon^j_\pm&=\pm\,\frac{1}{4}R\,\epsilon^i_\pm\,,\\
\Big[\mathrm{i}R(\mathcal{V})_{23}^{+}{}^i{}_j-\frac{1}{4}\bar{w}A^{ik}\varepsilon_{kj}\Big]\,\epsilon^j_\pm&=\mp\,\frac{1}{2}\Big(v_1^{-1}+v_2^{-2}-\frac{1}{8}|w|^2\Big)\,\epsilon^i_\pm\,. \qquad (8.31)
\end{aligned}
$$

[15] There are also charge conjugated equations. For instance, the first equation reads,

$$
(D+\frac{1}{12}R)\epsilon_i+\big[R(\mathcal{V})_{23}^{+}{}_i{}^j+\mathrm{i}R(A)_{23}^{+}\,\delta_i{}^j\big]\gamma^{23}\,\epsilon_j=0\,. \qquad (8.28)
$$

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Let us now assume that $A_{ij} \neq 0$. In that case $\varepsilon^{ik} A_{kj}$ must have a single null vector in order that a supersymmetric solution exists. On the other hand, it must commute with the SU(2) curvatures, which in this case implies that the $R(\mathcal{V})_{\mu\nu}{}^i{}_j$ must vanish. Supersymmetry then requires that $v_1 = v_2$ and

$$w\, A_{ij}\, \epsilon^j_\pm = 0\,, \qquad \bar{w} A^{ik} \varepsilon_{kj}\, \epsilon^j_\pm = \pm\Big(4\, v_1^{-1} - \frac{1}{4}|w|^2\Big)\, \epsilon^i_\pm\,. \tag{8.32}$$

These equations have no solution unless $A_{ij} = 0$. When $A_{ij} = 0$ and the SU(2) curvatures are non-vanishing, one can show that (8.31) implies,

$$\mathrm{i}R(\mathcal{V})_{23}{}^i{}_j\, \epsilon^j_\pm = \pm\frac{1}{2} R\, \epsilon^i_\pm\,, \qquad v_1^{-1} = \frac{1}{16}|w|^2\,. \tag{8.33}$$

This solution, denoted by $A_{[2]}$, has generically four supersymmetries, two associated with two of the spinor parameters $\epsilon^i_\pm$, and two related with the charge-conjugated spinors $\epsilon_{i\mp}$. The two spinors of the $\epsilon^i_\pm$ must be eigenspinors of both $\mathrm{i}\gamma^{23}$ and $\mathrm{i}R(\mathcal{V})_{23}{}^i{}_j$ with related eigenvalues. Therefore the supersymmetries of class $A_{[2]}$ (and also of class B, as we shall see later) cannot transform consistently under the SU(2) isometry group. We will return to this aspect shortly.

In the special case where both A_{ij} and the SU(2) curvatures vanish, we have $v_1^{-1} = v_2^{-1} = \frac{1}{16}|w|^2$. Generically we then have eight supersymmetries. This class is denoted by $A_{[1]}$. Here the supersymmetries act consistently under the action of both SU(2) groups. This completes the discussion of the type-A solutions.

Subsequently we turn to the solutions of class B, where $D - \frac{1}{6}R \neq 0$ and $R(A)_{\mu\nu} = 0$. This class is denoted by B. In that case the first two equations (8.29) imply,

$$\begin{aligned}
\mathrm{i}R(\mathcal{V})^-_{23}{}^i{}_j \epsilon^j_\pm &= \pm\,\Big(D + \frac{1}{12}R\Big)\,\epsilon^i_\pm\,,\\
\frac{1}{2} w\, \varepsilon^{ik} A_{kj}\, \epsilon^j_\pm &= \pm\,\Big(D - \frac{1}{6}R\Big)\,\epsilon^i_\pm\,.
\end{aligned} \tag{8.34}$$

With this result, the last two equations then yield the eigenvalue equations,

$$\begin{aligned}
\mathrm{i}R(\mathcal{V})^+_{23}{}^i{}_j \epsilon^j_\pm &= \mp\,\frac{1}{2}\Big(v_1^{-1} + v_2^{-1} - \frac{1}{4}|w|^2\Big)\,\epsilon^i_\pm\,,\\
\frac{1}{2}\bar{w}\, A^{ik}\varepsilon_{kj}\, \epsilon^j_\pm &= \pm\,\frac{1}{8}|w|^2\,\epsilon^i_\pm\,.
\end{aligned} \tag{8.35}$$

Combining these equations leads to,

$$\begin{aligned}
\bar{w}\, A^{ij} &= -\, w\, \varepsilon^{ik}\, \varepsilon^{jl}\, A_{kl}\,,\\
R(\mathcal{V})^-_{23}{}^i{}_j = R(\mathcal{V})^+_{23}{}^i{}_j &= \frac{1}{2} R(\mathcal{V})_{23}{}^i{}_j = -\frac{2\mathrm{i}}{v_2\, \bar{w}}\, \varepsilon^{ik}\, A_{kj}\,,\\
\mathrm{i}R(\mathcal{V})_{23}{}^i{}_j \epsilon^j_\pm &= \mp\, v_2^{-1}\, \epsilon^i_\pm\,,\\
D &= -\,\frac{1}{6}\big(v_1^{-1} + 2v_2^{-1}\big)\,,\\
v_1^{-1} &= \frac{1}{4}|w|^2\,.
\end{aligned} \tag{8.36}$$

class	$R(\mathcal{V})$	A_{ij}	v_1, v_2	susy
$A_{[1]}$	$R(\mathcal{V}) = 0$	$A_{ij} = 0$	$v_1^{-1} = v_2^{-1} = \frac{1}{16}\vert w\vert^2$	$\mathbf{4} + \bar{\mathbf{4}}$
$A_{[2]}$	$R(\mathcal{V})_{23} = \mathcal{O}(v_1^{-1} - v_2^{-1})$	$A_{ij} = 0$	$v_1^{-1} = \frac{1}{16}\vert w\vert^2 \neq v_2^{-1}$	$\mathbf{2} + \bar{\mathbf{2}}$
B	$R(\mathcal{V})_{23}{}^i{}_j = -\frac{4\mathrm{i}}{v_2\,\bar{w}}\,\varepsilon^{ik} A_{kj} = \mathcal{O}(v_2^{-1})$		$v_1^{-1} = \frac{1}{4}\vert w\vert^2$	$\mathbf{2} + \bar{\mathbf{2}}$

Table 3. Three classes of supersymmetric solutions. As shown in due course, only the classes $A_{[1]}$ and B are actually realized.

Just as in class $A_{[2]}$, these solution have generically four supersymmetries, which cannot transform consistently under the action of the SU(2) isometry group. Furthermore, note that the solutions become singular in the limit where $\mathcal{V}_{\mu\nu}{}^i{}_j$ and A_{ij} vanish, so that this class is really distinct from the type-A class.

In view of the fact that the supersymmetry spinors do not always seem to transform consistently under the action of the SU(2) transformations associated with the S^2 isometries, let us now first clarify this issue and turn to a discussion of the Killing spinor equations (in gauge $b_\mu = 0$) for each of the three classes. These equations take the following form,

$$\delta\big(\psi_\mu{}^i + \gamma_\mu\,\Omega^{\mathrm{V}i}\big) = 2\mathring{\nabla}_\mu\epsilon^i + \mathrm{i}A_\mu\,\epsilon^i + \mathcal{V}_\mu{}^i{}_j\,\epsilon^j - \varepsilon^{ik}\left[\frac{1}{4}\mathrm{i}w\,\gamma^{23}\delta_k{}^j + \varepsilon_{kl}A^{lj}\right]\gamma_\mu\,\epsilon_j\,. \qquad (8.37)$$

where $\mathring{\nabla}_\mu$ denotes the $\mathrm{AdS}_2 \times S^2$ covariant derivative. Obviously we may set A_μ and $\mathcal{V}_{\underline{a}} = 0$.

For class-A solutions (8.37) leads to,

$$\begin{aligned} \mathring{\nabla}_{\underline{a}}\epsilon^i_\pm \mp \frac{1}{8}w\,\varepsilon^{ij}\gamma_{\underline{a}}\,\epsilon_{j\pm} &= 0\,, \\ \mathring{\nabla}_{\hat{a}}\epsilon^i_\pm + \frac{1}{2}\mathcal{V}_{\hat{a}}{}^i{}_j\,\epsilon^j_\pm \mp \frac{1}{8}w\,\varepsilon^{ij}\gamma_{\hat{a}}\,\epsilon_{j\mp} &= 0\,, \end{aligned} \qquad (8.38)$$

where $v_1^{-1} = \frac{1}{16}|w|^2$. For the solution of class $A_{[1]}$, we may take $\mathcal{V}_{\hat{a}}{}^i{}_j = 0$, so that we obtain the standard Killing spinor equations for $\mathrm{AdS}_2 \times S^2$. For the $A_{[2]}$ solutions, the Killing spinor equation on S^2 is somewhat unusual, because of the presence of the R-symmetry connection whose strength is not related to the size of the S^2. Since we will show later that the type-$A_{[2]}$ solutions are in fact not realized, we refrain from further discussion concerning these solutions.

Hence we proceed to the class-B solutions. In this case, the Killing spinor equation (8.37) decomposes into,

$$\begin{aligned} \mathring{\nabla}_{\underline{a}}\epsilon^i_\pm \mp \frac{1}{4}w\,\varepsilon^{ij}\gamma_{\underline{a}}\,\epsilon_{j\pm} &= 0\,, \\ \mathring{\nabla}_{\hat{a}}\epsilon^i_\pm + \frac{1}{2}\mathcal{V}_{\hat{a}}{}^i{}_j\,\epsilon^j_\pm &= 0\,. \end{aligned} \qquad (8.39)$$

Because $v_1^{-1} = \frac{1}{4}|w|^2$, the first equation is the standard AdS_2 Killing spinor equation. However, the second equation does not coincide with the standard Killing spinor equation

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on S^2. We note that the strength of the R-symmetry connection is proportional to v_2^{-1}, and is therefore also determined by the S^2 radius. To elucidate the situation, let us briefly discuss the relevant equations for the unit sphere ($v_2 = 1$).

We use the standard coordinates θ and φ on S^2, with zweibeine $e^2 = \mathrm{d}\theta$ and $e^3 = \sin\theta\,\mathrm{d}\varphi$, and gamma matrices γ_2 and γ_3 that satisfy the standard Clifford algebra relation with positive signature. The spin connection field in our convention equals $\omega = \omega^{23} = -\omega^{32} = \cos\theta\,\mathrm{d}\varphi$. Consequently we have that $\overset{\circ}{\nabla}_\theta = \partial_\theta$ and $\overset{\circ}{\nabla}_\varphi = \big(\partial_\varphi - \frac{1}{2}\cos\theta\,\gamma^{23}\big)$. Now we adopt an R-symmetry transformation to bring $R(\mathcal{V})_{23}{}^i{}_j$ in diagonal form. In that case we can assume $\mathcal{V}^i{}_j = -\mathrm{i}\lambda\,(\sigma_3)^i{}_j\,\cos\theta\,\mathrm{d}\varphi$ with λ some real constant and σ_3 the diagonal Pauli matrix. This leads to the corresponding field strength $R(\mathcal{V})_{23}{}^i{}_j = \mathrm{i}\lambda\,(\sigma_3)^i{}_j$. From the third equation of (8.36) we conclude that $|\lambda| = 1$ and by an additional R-symmetry transformation we can ensure that $\lambda = 1$. In that case (remember that we put $v_2 = 1$) the supersymmetries are parametrized by the parameters ϵ^1_+ and ϵ^2_-. It is now straightforward to verify that these spinors do not depend on the S^2 coordinates as a result of the second equation (8.39).

Consequently the supersymmetries do not transform under the isometries of S^2, which implies that they carry no spin! Along the same lines one expects that also the fields in this background will change their spin assignment. The reason that the spin assignments change in this background, is that the spin rotations associated with the isometries of S^2 become entangled with R-symmetry transformations, in a similar way as in magnetic monopole solutions, where the rotational symmetry becomes entangled with gauge transformations [26]. In the superconformal context, where one has R-symmetry connections (which in this solution live on S^2), the geometric origin of the entanglement is clear. While such conditions on the supersymmetry spinor have been obtained previously in the literature for a variety of four- and five-dimensional supersymmetric solutions (see, e.g. [27, 28, 43–45], this phenomenon seems not to have received special attention.

Finally we must investigate the remaining variations based on (8.24). Consider first the variation for the fields $\Omega_i{}^\Lambda$, which we parametrize as $\delta\Omega_i{}^\Lambda = A_{ij}{}^\Lambda\epsilon^j - 2X^\Lambda\eta_i$, so that

$$A_{ij}{}^\Lambda = 2\,\hat{\mathcal{H}}^{-}_{23}{}^\Lambda\varepsilon_{ij}\gamma^{23} - 2g\,T_{NP}{}^\Lambda\,\bar{X}^N X^P\,\varepsilon_{ij} - 4\,gN^{\Lambda\Sigma}\big(\mu_{ij\Sigma} + \bar{F}_{\Sigma\Gamma}\,\mu_{ij}{}^\Gamma\big)\,. \tag{8.40}$$

Then we consider the variation of two S-invariant combinations, $\Omega_i{}^\Lambda - 2X^\Lambda\Omega_i^{\mathrm{V}}$, and $D_a(\Omega^{i\,\Lambda} - 2\bar{X}^\Lambda\Omega^{i\mathrm{V}}) - \frac{1}{2}(A^{ij\,\Lambda} - 2\,\bar{X}^\Lambda A^{ij})\gamma_a\Omega_j^{\mathrm{V}}$, whose vanishing under supersymmetry imply the following identities,

$$\begin{aligned}
\big[A_{ij}{}^\Lambda - 2\,X^\Lambda A_{ij}\big]\,\epsilon^j &= 0\,,\\
\Big(A^{ik\Lambda} - 2\,\bar{X}^\Lambda A^{ik}\big)\,\Big(A_{kj} - \frac{1}{8}T_{bckj}\,\gamma^{bc}\Big)\,\gamma_a\epsilon^j &= 0\,,
\end{aligned} \tag{8.41}$$

where we assumed that $\mathcal{D}_\mu A^\Lambda = 0$ in line with our earlier ansätze. Likewise we obtain two equations for the hypermultiplets,

$$\begin{aligned}
\big[2\,g\bar{X}^M\bar{T}_M{}^{\bar\alpha}{}_{\bar\beta}\,A^{i\bar\beta}\,\varepsilon_{ij} - A^{i\bar\alpha}A_{ij}\big]\,\epsilon^j &= 0\,,\\
\big(2\,gX^M T_M{}^\alpha{}_\beta\,A_i{}^\beta\,\varepsilon^{ik} - A_i{}^\alpha A^{ik}\big)\,\Big(A_{kj} - \frac{1}{8}T_{bckj}\,\gamma^{bc}\Big)\,\gamma_a\epsilon^j &= 0\,.
\end{aligned} \tag{8.42}$$

We note the presence of a universal factor on the right-hand side of the equation in (8.41) and (8.42), which is proportional to

$$A_{kj} - \frac{1}{8} T_{bckj}\, \gamma^{bc} = -\varepsilon_{kl} \big(\varepsilon^{lm} A_{mj} - \frac{1}{4} \mathrm{i}\bar{w}\, \gamma^{23}\, \delta^l{}_j \big)\,, \tag{8.43}$$

which is the hermitian conjugate of the term that appears at the right-hand side of (8.37).

The equations (8.41) and (8.42) lead to the following six conditions,

$$\begin{aligned}
\big[g\, T_{NP}{}^\Lambda\, \bar{X}^N X^P\, \delta^i{}_j - 2\, g N^{\Lambda\Sigma}\, \varepsilon^{ik} \big(\mu_{kj\Sigma} + \bar{F}_{\Sigma\Gamma}\, \mu_{kj}{}^\Gamma \big) - X^\Lambda\, \varepsilon^{ik} A_{kj} \big]\, \epsilon^j_\pm &= \mp\, \mathrm{i} \hat{\mathcal{H}}^-_{23}{}^\Lambda\, \epsilon^i_\pm\,, \\
\big[g\, T_{NP}{}^\Lambda\, X^N \bar{X}^P\, \varepsilon^{ik} + 2\, g N^{\Lambda\Sigma} \big(\mu^{ik}{}_\Sigma + F_{\Sigma\Gamma}\, \mu^{ik\Gamma} \big) + \bar{X}^\Lambda\, A^{ik} \big]\, A_{kj}\, \epsilon^j_\pm &= \frac{1}{4} \mathrm{i}\bar{w}\, \hat{\mathcal{H}}^+_{23}{}^\Lambda\, \epsilon^i_\pm\,, \\
\bar{w}\big[g\, T_{NP}{}^\Lambda\, X^N \bar{X}^P\, \delta^i{}_j - 2\, g N^{\Lambda\Sigma} \varepsilon^{ik} \big(\mu_{kj\Sigma} + F_{\Sigma\Gamma}\, \mu_{kj}{}^\Gamma \big) - \bar{X}^\Lambda\, A^{ik} \varepsilon_{kj} \big]\, \epsilon^j_\pm &= 4\mathrm{i}\, \hat{\mathcal{H}}^+_{23}{}^\Lambda\, \varepsilon^{ik} A_{kj}\, \epsilon^j_\pm\,, \\
\big[2\, g \bar{X}^M \bar{T}_M{}^{\bar\alpha}{}_{\bar\beta}\, A^{i\bar\beta}\, \varepsilon_{ij} - A^{i\bar\alpha} A_{ij} \big]\, \epsilon^j_\pm &= 0\,, \\
\big[2\, g X^M T_M{}^\alpha{}_\beta\, A_i{}^\beta\, \varepsilon^{ik} - A_i{}^\alpha A^{ik} \big]\, A_{kj}\, \epsilon^j_\pm &= 0\,, \\
\big[2\, g X^M T_M{}^\alpha{}_\beta\, A_i{}^\beta\, \varepsilon^{ik} - A_i{}^\alpha A^{ik} \big]\, \varepsilon_{kj}\, \epsilon^j_\pm &= 0\,.
\end{aligned} \tag{8.44}$$

Let us now consider the various classes of solutions shown in table 3. First of all the solutions of type A, characterized by $A_{ij} = 0$. From the second equation of (8.44) it then follows that $\hat{\mathcal{H}}_{\mu\nu}{}^\Lambda = 0$. Combining this result with the equations (8.23) shows that both $R(A)_{\mu\nu}$ and $R(\mathcal{V})_{\mu\nu}{}^i{}_j$ must vanish. This implies that solution $A_{[2]}$ is not realized. Hence we are left with the fully supersymmetric solution $A_{[1]}$. Therefore we proceed by determining the additional restrictions for this solution.

The first, third, fourth and sixth equations of (8.44) can be written as follows,

$$\begin{aligned}
\mathrm{i}\varepsilon^{ik} \mu_{kj}{}^\Lambda\, \epsilon^j_\pm &= -\frac{1}{2} T_{NP}{}^\Lambda (\bar{X}^N X^P - X^N \bar{X}^P)\, \epsilon^i_\pm\,, \\
\mathrm{i} N^{\Lambda\Sigma} \varepsilon^{ik} \big(2\, \mu_{kj\Sigma} + (F_{\Sigma\Gamma} + \bar{F}_{\Sigma\Gamma}) \mu_{kj}{}^\Gamma \big)\, \epsilon^j_\pm &= \frac{1}{2} \mathrm{i} T_{NP}{}^\Lambda (\bar{X}^N X^P + X^N \bar{X}^P)\, \epsilon^i_\pm\,, \\
\bar{X}^M \bar{T}_M{}^{\bar\alpha}{}_{\bar\beta}\, A^{i\bar\beta}\, \varepsilon_{ij}\, \epsilon^j_\pm &= 0\,, \\
X^M T_M{}^\alpha{}_\beta\, A_i{}^\beta\, \epsilon^i_\pm &= 0\,.
\end{aligned} \tag{8.45}$$

Since a hermitian matrix must have real eigenvalues, it follows that both sides of the first two equations should vanish. Also the factors in the last two equations should vanish, so that

$$\begin{aligned}
\mu_{ij\Lambda} = \mu_{ij}{}^\Lambda &= 0\,, \\
T_{NP}{}^\Lambda X^N \bar{X}^P &= 0\,, \\
X^M T_M{}^\alpha{}_\beta\, A_i{}^\beta &= 0 = \bar{X}^M T_M{}^\alpha{}_\beta\, A_i{}^\beta\,.
\end{aligned} \tag{8.46}$$

Note that $\mathcal{L}_{g^2}$ is now vanishing. For electric charges these solutions have already been identified in [25]. Without charges this is the well-known solution that arises as a near-horizon geometry of BPS black holes. The fact that the moment maps and certain combinations of Killing vectors are vanishing does not warrant the conclusion that there is no gauging. One can only conclude that the field equations require some of these quantities to vanish for these solutions.

Now consider the type-B solution where A_{ij} is non-vanishing. In that case the first three equations of (8.44) lead to two independent equations,

$$\begin{aligned}
&\left[g\,T_{NP}{}^{\Lambda}\,\bar X^N X^P\,\delta^i{}_j - 2\,gN^{\Lambda\Sigma}\,\varepsilon^{ik}\big(\mu_{kj\Sigma}+\bar F_{\Sigma\Gamma}\,\mu_{kj}{}^{\Gamma}\big)\right]\epsilon^j_{\pm} = \mp\left(\mathrm{i}\hat{\mathcal{H}}_{23}^{-}{}^{\Lambda}+\frac{1}{4}\bar w X^{\Lambda}\right)\epsilon^i_{\pm}\,,\\
&\left[g\,T_{NP}{}^{\Lambda}\,X^N \bar X^P\,\delta^i{}_j - 2\,gN^{\Lambda\Sigma}\varepsilon^{ik}\big(\mu_{kj\Sigma}+F_{\Sigma\Gamma}\,\mu_{kj}{}^{\Gamma}\big)\right]\epsilon^j_{\pm} = \mp\left(\mathrm{i}\,\hat{\mathcal{H}}_{23}^{+}{}^{\Lambda}-\frac{1}{4}w\bar X^{\Lambda}\right)\epsilon^i_{\pm}\,.
\end{aligned} \tag{8.47}$$

These equations can be analyzed in a similar way as the corresponding equations in (8.45). The results are as follows,

$$\begin{aligned}
&T_{NP}{}^{\Lambda}\bar X^N X^P = 0\,,\\
&\quad g\varepsilon^{ik}\mu_{kj}{}^{\Lambda}\,\epsilon^j_{\pm} = \mp\frac{1}{2}\left[\left(\hat{\mathcal{H}}_{23}^{-\Lambda}-\frac{1}{4}\mathrm{i}\bar w X^{\Lambda}\right)-\left(\hat{\mathcal{H}}_{23}^{+\Lambda}+\frac{1}{4}\mathrm{i}w\bar X^{\Lambda}\right)\right]\epsilon^i_{\pm}\,,\\
&\quad g\varepsilon^{ik}\mu_{kj\Lambda}\,\epsilon^j_{\pm} = \pm\frac{1}{2}\left[F_{\Lambda\Sigma}\left(\hat{\mathcal{H}}_{23}^{-\Sigma}-\frac{1}{4}\mathrm{i}\bar w X^{\Sigma}\right)-\bar F_{\Lambda\Sigma}\left(\hat{\mathcal{H}}_{23}^{+\Sigma}+\frac{1}{4}\mathrm{i}w\bar X^{\Sigma}\right)\right]\epsilon^i_{\pm}\,.
\end{aligned} \tag{8.48}$$

From (5.9), it follows that the first constraint of (8.48) can be generalized to $T_{MN}{}^{P}\bar X^M X^N = 0$. Using also the representation constraint (5.6), one reconfirms that $R(A)_{\mu\nu}$, as given in (8.23), vanishes. The same argument applies to solutions of type $A_{[1]}$. Furthermore, as a check one may also reconstruct the eigenvalue equation for A_{ij} which shows once more that (8.26) must be valid.

One can use the same strategy and determine $R(\mathcal{V})_{23}{}^{i}{}_{j}$ from (8.23), making use of (8.48) with $T_{MN}{}^{P}\bar X^M X^N = 0$. Evaluating this curvature on the supersymmetry parameters, making use of the eigenvalue condition for this curvature presented in (8.36) as well as of (8.26), it follows that

$$v_2^{-1} = -2\chi_{\mathrm{vector}}^{-1}N_{\Lambda\Sigma}\hat{\mathcal{H}}_{23}^{-\Lambda}\hat{\mathcal{H}}_{23}^{+\Sigma}-\frac{1}{8}|w|^2\,. \tag{8.49}$$

In the first expression on the right-hand side, one can verify, replacing $N_{\Lambda\Sigma}$ by the negative definite metric $M_{\Lambda\bar\Sigma}$ defined in (7.26) and using (8.26), that this expression must be positive, which yields an upper bound on $|w|^2$ for given field strengths $\hat{\mathcal{H}}_{23}{}^{\Lambda}$.

The last three equations of (8.44) lead to two equations,

$$\begin{aligned}
X^M\left[T_M{}^{\alpha}{}_{\beta}A_i{}^{\beta}+\chi_{\mathrm{vector}}^{-1}\varepsilon_{ij}\,\mu^{jk}{}_M\,A_k{}^{\alpha}\right] &= 0\,,\\
\bar X^M\left[T_M{}^{\alpha}{}_{\beta}A_i{}^{\beta}+\chi_{\mathrm{vector}}^{-1}\varepsilon_{ij}\,\mu^{jk}{}_M\,A_k{}^{\alpha}\right] &= 0
\end{aligned} \tag{8.50}$$

From these equations, one derives, upon using (8.9),

$$g^2\bar X^M X^N\,k^A{}_M\,k_{AN} = \frac{1}{16}\chi_{\mathrm{vector}}\,|w|^2\,. \tag{8.51}$$

The scalar potential in the type-B solutions thus takes the form,

$$e^{-1}\mathcal{L}_{g^2} = -\,2\,g^2\,\chi_{\rm vector}\,M_{\bar\Lambda\Sigma}\,N^{\Lambda\Gamma}\left[\mu^{ij}{}_{\Gamma} + F_{\Gamma\Omega}\,\mu^{ij\Omega}\right]\,N^{\Sigma\Xi}\left[\mu_{ij\Xi} + \bar F_{\Xi\Delta}\,\mu_{ij}{}^{\Delta}\right] \\ -\frac{3}{16}\chi_{\rm vector}\,|w|^2\,, \tag{8.52}$$

where the first term is negative and the second one positive. We refrain from giving further results.

For a single (compensating) hypermultiplet, which can only have abelian gaugings, we expect that one of these type-B solutions describes the near-horizon geometry of the spherically symmetric static black hole solution presented in [27, 28]. The result of this paper then ensures that this black hole solution has supersymmetry enhancement at the horizon.

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