

Unexpected Spin-Off from Quantum Gravity

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Abstract

We propose a novel way of investigating the universal properties of spin systems by coupling them to an ensemble of causal dynamically triangulated lattices, instead of studying them on a fixed regular or random lattice. Somewhat surprisingly, graph-counting methods to extract high- or low-temperature series expansions can be adapted to this case. For the two-dimensional Ising model, we present evidence that this ameliorates the singularity structure of thermodynamic functions in the complex plane, and improves the convergence of the power series.

Introduction

Quantum gravity is not usually regarded a subject of particular relevance to physics outside the exotic realm of Planck scale phenomena. Nevertheless, progress in science can sometimes come from an unexpected direction. In this letter, we will give a concrete example of how a specific way of constructing a theory of quantum gravity may lead to a new method for understanding the critical behaviour of certain spin and matter systems.

Apart from a handful of well-known exceptions, most thermodynamic properties of statistical mechanical models are not known to us in exact, closed form. Consequently, we must rely on a variety of approximation and numerical methods to study their behaviour. Lattice structures appearing in such models can either reflect the actual microscopic composition of a particular magnetic material, say, or play the role of a convenient discrete regulator for a continuous system whose symmetry properties have little to do with those of the lattice approximation.

Our focus will be on instances of the latter, where one is only interested in universal properties of the lattice models, which pertain to the system in a suitable scaling limit and are largely independent of discretization details, including those of the geometry of the underlying lattice. This is a point of view also encountered in the nonperturbative lattice formulations of both quantum gauge theories and gravity in high-energy physics.

In such a “utilitarian” view of lattice systems one is naturally led to ask how one should set up the lattice discretization to extract the desired continuum information in the quickest and most reliable way. Part of this quest is a systematic study of the influence of the lattice geometry on results, in an effort to separate as cleanly as possible ‘universal behaviour’ from ‘lattice artefacts’. An example are the investigations of two-dimensional Ising models [1] on different regular lattices and their thermodynamic functions in the complex-temperature plane [2], which give clues on how to improve the convergence behaviour of approximation methods applied to high- and low-temperature expansions.

Going beyond regular lattices, and staying within the same philosophy, the use of *random lattices* has been advocated [3], in the hope that the absence of discrete lattice symmetries and therefore of distinguished lattice directions may accelerate the restoration of continuous rotational and translational invariances, and thus the approach to the continuum limit. However, we are not aware of any applications in lattice field theory where this would have led to a practical or conceptual breakthrough.

Taking yet a further step toward randomizing the spaces underlying the statistical models, one may consider an *additional averaging over random lattices*. This is inspired by quantum gravity, where the “path integral” (a nonperturbative quantum superposition of all spacetime geometries, which is central to the quan-

tum dynamics) can be defined via a statistical, weighted sum over triangulated random geometries. The original approach of (*Euclidean*) *Dynamical Triangulations* (EDT – see [4, 5] for reviews) turns out to be unsuitable for our purposes, because the contributing triangulated lattices are highly curved, with an effectively fractal structure, for any dimension $d \geq 2$. Their geometry is so radically different from the usual flat lattices that it alters the universal behaviour of matter systems defined on them, as has been well documented for two-dimensional spin systems (see [6] and references therein).

In fact, these geometries are so “wild” that they are not even suited for modelling the quantum behaviour of four-dimensional gravity. However, a promising new avenue has opened up recently with the advent of *Causal Dynamical Triangulations* (CDT), which were exactly invented to fix the extreme geometric degeneracies of the previous, Euclidean approach (see [7] for a review). One still works with an ensemble of lattices with large local curvature fluctuations, but one where a partial order has been imposed in one of the lattice directions (“time”). In “pure” gravity (i.e. without matter coupling) this is sufficient to produce geometries whose effective (or Hausdorff) large-scale dimension d_H – in the sense of ensemble averages – equals the dimension d of their microscopic triangular building blocks, for $d = 2$ [8], $d = 3$ [9] and $d = 4$ [10], which was *not* the case for the corresponding Euclidean models, and is an indication that the geometries are much better behaved.

Are causal dynamical triangulations the “better lattices”?

We will in what follows concentrate on the model of causal dynamical triangulations in dimension $d = 2$, where the partition function of pure quantum gravity has been computed exactly [8]. There is no known exact solution in the presence of Ising spins, but CDT coupled to one [11] and eight [12, 13] Ising models has been studied using Monte Carlo simulations.¹ A rather surprising outcome² of this analysis was that the behaviour of the matter is very robust: to high accuracy, the critical matter exponents coincide in both cases with those of the Onsager solution, despite large fluctuations of the underlying lattice geometry, and – in the case of the eight Ising copies – a shift in the geometry’s Hausdorff dimension d_H from 2 to 3, due to the presence of the matter. This provides strong evidence (though not a proof) that the universal properties of any spin or matter system coupled to two-dimensional causal dynamical triangulations will be identical to those on a fixed, flat two-dimensional lattice. Since in terms of the degree of

¹Regarding the lattice fluctuations as a type of disorder, the CDT-plus-matter models are examples of models with so-called *annealed* disorder, i.e. there is a genuine backreaction of the matter on the geometry.

²especially for those closely familiar with the analogous Euclidean EDT results [14]

geometric disorder our model presumably lies in between the highly disordered EDT models and the more mildly disordered random Voronoi-Delaunay lattices, this would also lend credence to findings that the three-state Potts model on the latter exhibits a behaviour unchanged from the flat-lattice case [15].

Assuming that the universal flat-space properties of the matter systems do remain unaffected when coupled to causal dynamical triangulations, we want to advocate the CDT ensemble of triangulated geometries as a “background lattice” for studying their continuum behaviour. In view of the fact that these geometries are presumably the “maximally disordered” lattices with this property, we will investigate the hypothesis that this will improve maximally the approach to the scaling limit.

One way to test this hypothesis would be a systematic numerical Monte Carlo study, comparing CDT-coupled matter models to the same models on fixed regular or random lattices. However, for the time being we will take a different route, and try to understand how far the issue can be pushed (semi-)analytically, by exploiting another remarkable property of these models. Namely, despite the absence of a *single* lattice of known geometry, series expansions for thermodynamic functions *can* be evaluated diagrammatically on the CDT ensemble, both at high and low temperature. We have solved the associated combinatorial problem by devising algorithms for counting graphs, both for spin variables at the vertices and at the centres of the triangles. Since this constitutes to our knowledge a qualitatively new way of setting up an asymptotic power series expansion, this is of interest in its own right. It may also yield information on the analytic structure of the magnetic susceptibility, say, in a statistical model that is still relatively unexplored. This in turn could provide a new reference point for the study of spin models on a variety of regular and random lattices (see [16] for a review of some current problems for the latter).

After recalling some necessary ingredients from the original CDT model [8, 11], we will give a sketch of the combinatorial algorithms (details of which can be found in our forthcoming publication [17]), and then present our results. For Ising spins located at vertices, we have evaluated the high- T series expansion for the magnetic susceptibility to order six. Evaluating the series by straightforward ratio method gives a remarkably good result for the critical susceptibility exponent γ , compared with that of the corresponding fixed, regular lattice. We have also computed the same expansion for the Ising model on dual CDT lattices, up to order 12. As expected, the data are less good, but also demonstrate clearly the simplified singularity structure of the susceptibility function.

We are very encouraged by these results. Though not conclusive with regard to our hypothesis, they provide sufficient evidence for the viability of the method, and warrant a more extensive investigation of the critical properties of the underlying triangulated spin model. In view of the computational effort involved, the

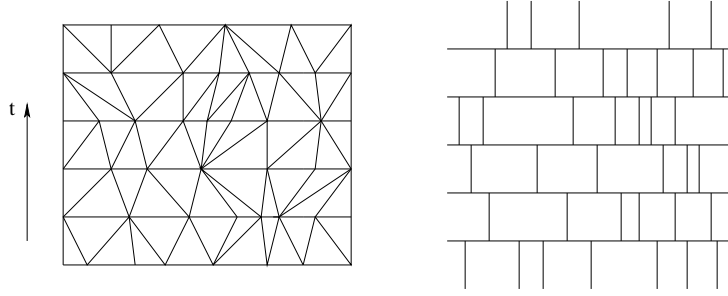


Figure 1: A piece with five time steps of a triangulation contributing to the CDT ensemble of geometries (left), and its corresponding dual lattice (right).

diagram counting can then no longer be done by hand, but will at least in part have to be computerized. Given the random nature of the graphs, this certainly presents a non-trivial task, but one that we believe is feasible.

Method

In the approach of causal dynamical triangulations, the gravitational path integral over two-dimensional causal spacetimes is represented by the discrete sum

$$G(N, t) = \sum_{T \in \mathcal{T}_{N,t}} 1 \quad (1)$$

over triangulations $T \in \mathcal{T}_{N,t}$ of a fixed number N of triangles and t of time steps. A piece of such a spacetime is depicted in Fig. 1 (left). Coupling the pure gravity model to Ising spins $\sigma_i = \pm 1$ in a magnetic field H , one obtains the partition function

$$\tilde{Z}(N, t, \beta, H) = \sum_{T \in \mathcal{T}_{N,t}} \sum_{\{\sigma_i(T)\}} \exp \left(\beta \sum_{\langle ij \rangle \in T} \sigma_i \sigma_j + H \sum_i \sigma_i \right), \quad (2)$$

with $\beta = J/k_B T$, and $\langle ij \rangle$ denoting a pair of nearest neighbours, corresponding to an edge of the triangulation or one of the trivalent graph dual to the triangulation (Fig. 1 right), depending on whether the spins reside on the vertices or at the centres of the triangles. The ferromagnetic Ising coupling has been set to $J = 1$.

Turning now our interest to the high- T expansion, we reexpress (2) as power series in the customary variables $u = \tanh \beta$ and $\tau = \tanh H$, obtaining

$$\tilde{Z}(N, t, \beta, H) = (\cosh \beta)^{3N/2} (2 \cosh H)^{N/2} \sum_{T \in \mathcal{T}_{N,t}} \left(1 + \sum_{s=0}^{\infty} \sum_{n=1}^{\infty} D_{n,2s}^{(N)}(T) u^n \tau^{2s} \right), \quad (3)$$

$$\begin{aligned}
\text{(a): } \langle \text{---} \rangle &= 1 & \text{(b): } \langle \text{↗} \rangle &= 2 \\
\text{(c): } \langle \text{^} \rangle &= 4 & \text{(d): } \langle \text{↘} \rangle &= 4
\end{aligned}$$

Figure 2: Expectation values for the counting of some simple two-odd susceptibility graphs on CDT lattices.

where $D_{n,2s}^{(N)}(\mathbf{T})$ counts graphs with n edges and $2s$ odd vertices on the triangulation \mathbf{T} . It follows that the magnetic susceptibility at vanishing H is (up to a constant term) given by [11, 17]

$$\chi(u) = \frac{1}{N} \frac{\partial^2 \ln \tilde{Z}}{\partial H^2} \Big|_{H=0} = \frac{2}{N} \left(\sum_{\mathbf{T} \in \mathcal{T}_{N,t}} 1 \right)^{-1} \sum_{n=1}^{\infty} \left(\sum_{\mathbf{T} \in \mathcal{T}_{N,t}} \tilde{D}_{n,2}^{(N)}(\mathbf{T}) \right) u^n, \quad (4)$$

where the tilde on \tilde{D} indicates that we count only graph contributions of order N^m with $m = 1$. The crucial observation of [11] was that the expectation values per lattice vertex

$$\langle \tilde{D}_{n,2} \rangle := \frac{1}{N} \left(\sum_{\mathbf{T} \in \mathcal{T}_{N,t}} 1 \right)^{-1} \sum_{\mathbf{T} \in \mathcal{T}_{N,t}} \tilde{D}_{n,2}^{(N)}(\mathbf{T}) \quad (5)$$

can in the continuum limit $N \rightarrow \infty$ be computed from the known probability distribution of time-like edges (the non-horizontal edges of Fig. 1, left, emanating from a given vertex forward and backward in time) for the *pure* gravity theory (which is the distribution relevant at both $\beta = 0$ and $\beta = \infty$). The characteristic features of this probability distribution are that (i) for a given time step $[t, t+1]$, we have

$$p_k(v) = \frac{1}{2^k} \quad (6)$$

for the probability that $k \geq 1$ time-like links emanate from a given vertex v on the lower time-slice t toward the “future” (increasing time), *independently* for all vertices on the slice, and (ii) there are no correlations between time-like edges from different time steps $[t, t+1]$. Note that the absence of a similar structure of the local probabilities would prevent the direct application of this method to the case of Euclidean dynamical triangulations.

Results and discussion

Some illustrative examples of expectation values for particular two-odd graphs contributing to the calculation of the high- T susceptibility for Ising spins located

at vertices are shown in Fig.2. They already include appropriate multiplicities in case the graph is not identical with its mirror image under left-right and/or up-down reflection. The systematic algorithm we have devised to calculate the contributions to (4) at order n relies on a “strip decomposition” of a given graph into a sequence of time strips $\Delta t = 1$ [17], thus exploiting the independence of the probability distributions for distinct strips. As can be seen from Table 1, the number of graphs rises quite rapidly with n . To give an idea of the computational effort involved, there are 387 distinct open graphs which contribute at order 6 and whose computation requires a careful use of the strip decomposition and the associated algebra. Our results confirm and extend those cited in [11].

Table 1: Order- n susceptibility graphs per vertex, CDT lattice.						
n	1	2	3	4	5	6
$\langle \tilde{D}_{n,2} \rangle$	3	17	87	$423\frac{2}{3}$	1995	$9192\frac{8}{27}$

To obtain more data on the behaviour of asymptotic power series in the CDT framework, we next considered the high- T expansion of the susceptibility on the ensemble of random lattices *dual* to CDT, corresponding to Ising spins placed at the centres of the triangles of the original lattices. By virtue of the fact that the dual graphs are trivalent, one can – despite their randomness – apply a powerful counting theorem [18], which relates contributions of different orders n recursively. We refer to [17] for details of the counting algorithm. Another reason for why in this case we could push the evaluation to order 12 is that there are considerably fewer distinct graphs on the dual trivalent random lattice than there are on the original lattice, where *any* number of edges (larger than 3) can meet at a vertex. The results of the counting (per vertex of the *original* lattice) are summarized in Table 2.

Table 2: Order- n susceptibility graphs per vertex, dual CDT lattice.												
n	1	2	3	4	5	6	7	8	9	10	11	12
$\langle \tilde{D}_{n,2} \rangle$	3	6	12	23	$42\frac{3}{4}$	$78\frac{1}{2}$	$142\frac{3}{4}$	258	$461\frac{13}{16}$	$820\frac{1}{8}$	$1446\frac{13}{32}$	$2532\frac{11}{16}$

In order to get an idea of the quality of these data, we have compared them with the corresponding regular triangular and honeycomb lattices for which similar expansions (to order 16 and 32 respectively) are available [19]. In line with

our conjecture that the singularity structure with the use of CDT lattices should be maximally simplified, we make the simplest possible ansatz for χ , namely,

$$\chi(u) \sim A(u) \left(1 - \frac{u}{u_c}\right)^{-\gamma} + B(u) \quad (7)$$

near the critical point u_c , with analytical functions A and B . Using the ratio method, which is known to work best in the absence of interfering unphysical singularities [20], we have fitted the ratios of subsequent susceptibility coefficients to

$$r_n = \frac{\langle \tilde{D}_{n,2} \rangle}{\langle \tilde{D}_{n-1,2} \rangle} = \frac{1}{u_c} \left(1 + \frac{\gamma - 1}{n}\right). \quad (8)$$

By plotting the r_n linearly against $1/n$, we have extracted simultaneous estimates for the critical point u_c and the critical exponent γ , the latter of which is illustrated in Fig. 3 as function of n_{\max} . The convergence to the known true value $\gamma = 7/4$ is rather striking, especially when one removes the lowest-order ratio (which is maximally affected by omitting higher-order terms in $1/n$). Further

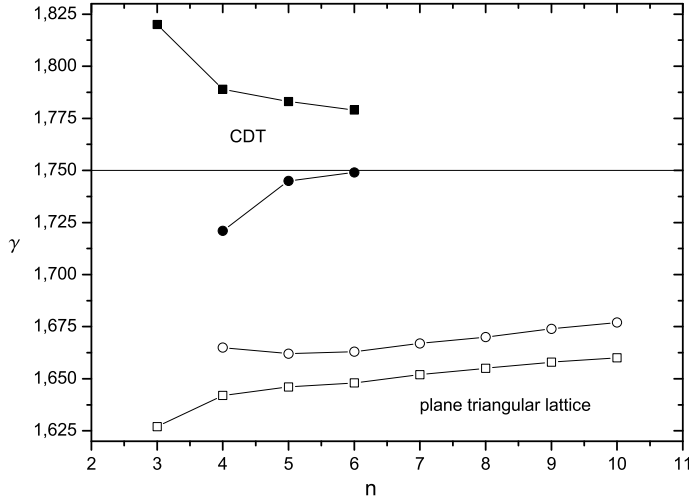


Figure 3: The critical susceptibility exponent γ , obtained from an order-6 high- T expansion on CDT lattices (solid dots). The shorter curve has the lowest-order ratio omitted. The analogous curves for the regular triangular lattice (up to order 10) are given for comparison.

evidence of how the use of dynamical lattices seems to simplify the singularity structure is the model with Ising spins at the vertices of the dual, trivalent CDT

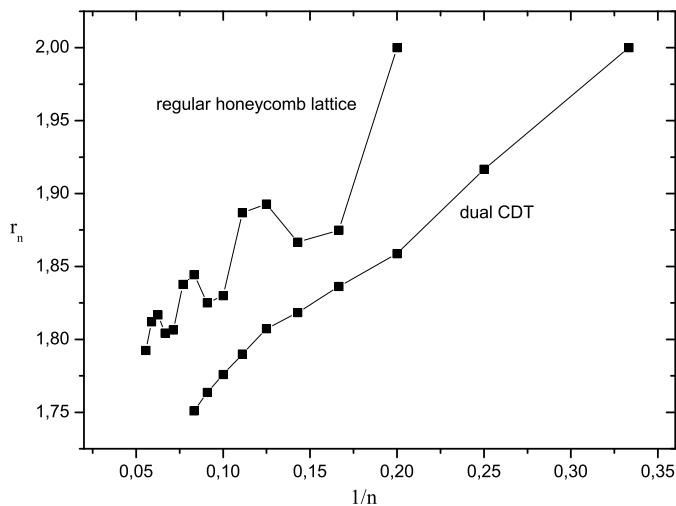


Figure 4: A comparative plot of the ratios r_n of the regular honeycomb lattice versus those of the dual CDT model, as function of $1/n$.

lattices. A priori one would not expect the data to be equally good, because triangles (or, equivalently, lattices of high coordination number) tend to explore the configuration space more effectively. Nevertheless, a plot of the ratios r_n as function of $1/n$ (Fig. 4) reveals a *much* smoother curve than that for the corresponding regular, trivalent honeycomb lattice, justifying again the use of the straightforward ratio method (see [17] for details). This is likely due to the absence of an antiferromagnetic phase of the model³, implying the absence of the interfering equidistant singularity on the negative real axis of the complex u -plane of the regular lattice, as well as to the possible absence of the equidistant “segments” along the imaginary axis [2], an issue that deserves further investigation.

Note that we are *not* advocating here the use of the ratio method for analyzing the susceptibility coefficients for the *regular* honeycomb lattice; if one wants to stick to a regular lattice, there clearly are better ways of trying to take into account the influence of the unphysical singularities on the negative real axis and in the complex plane evident in Fig. 4 (see, for example, [19, 22]). Rather, we are advocating the use of dynamical CDT lattices instead of fixed, regular lattices as a general method for determining the critical behaviour of spin and matter systems, for cases which are not already known by other methods. The strategy we suggest here is to first test our main conjecture more thoroughly for other two-dimensional spin systems (e.g. higher- q Potts and $O(n)$ -models) for which our

³A similar effect was already noted in an EDT-Ising model in [21].

exact counting method can be applied, and then move on to higher-dimensional CDT-matter models, about whose analytic structure and exact critical properties much less is known. By eliminating maximally the influence of spurious singularities present for regular lattices, they will (hopefully) lend themselves to straightforward approximation methods without the need for any *lattice-specific* subtraction schemes.

The results obtained so far for the well-known “test case” of the Ising susceptibility are interesting and encouraging. We believe it is not accidental that the high- T expansions give good results with the simple ratio method and for relatively few terms in the expansion, but we ultimately do not know why this is so without a more detailed understanding of the critical structure of these coupled CDT-spin models. We have also computed part of the low- T expansion [17] (which in the case of regular lattices is notorious for the interference of unphysical complex- T singularities [23, 2]), but the series we obtain are simply too short to draw any conclusions one way or the other. The same seems to be true when we try to use more elaborate methods to extract information about the critical properties from these finite series, such as the Dlog Padé and differential approximants [17]. With the counting algorithms for two-dimensional CDT lattices in place, the method now needs to be computerized, so that longer expansions can be obtained, evaluated, and compared to known results on regular lattices. Besides paving the way for a possible application to higher-dimensional matter systems, the two-dimensional analysis is likely to shed further light on recent attempts to formulate quantitative criteria for when geometric randomness of an underlying lattice is relevant to the critical behaviour of a matter system defined on it. For example, if the three-state Potts model coupled to CDT exhibited the critical exponents of the flat-lattice model, and if the local fluctuations of geometry could be shown to be sufficiently uncorrelated (both of which we expect to be true), it would provide a counterexample to the so-called Harris-Luck criterion [24, 16]. Complementary to the use of approximation methods based on series expansions, one can also explore different matter types and higher-dimensional models purely numerically. A study of the physically most interesting case of four dimensions is currently under way [25].

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