6. THE HORROCKS LEGACY

6.1 INTRODUCTION

During 1752 a major change took place in Mayer's thinking about modelling of the moon's motion, affecting the procedure to be followed when computing a lunar position from the tables. Initially the tables were applied in the single-stepped fashion (using the terminology explained in section 3.5), but during 1752 Mayer switched to a multistepped procedure. He would basically adhere to the multistepped procedure ever after, changing only some of the details later on.

The purpose of the present chapter is to investigate the origin of this multistepped procedure. I will show that Mayer developed it out of a lunar theory of Newton's, which first appeared in print in 1702. This dependence of Mayer's tables on Newton's 1702 lunar theory, which at best has a very troublesome relation to the theory of gravitation, has never been noted before. On the contrary, it is usually said that Mayer's tables were in some way based on Euler's lunar theory, and that their advent made those founded on Newton's 1702 lunar theory obsolete.[1] It will here be shown that Newton's theory has exercised, through Mayer's tables, a much more profound impact on 18th century positional astronomy than has hitherto been thought.

We will see in the next chapter how to transform any multistepped scheme into a single-stepped scheme that is equivalent for all practical purposes, The implication is that none of the possible schemes has a real advantage over any other as far as achievable accuracy is concerned. Nonetheless Mayer attained a most striking improvement in accuracy as soon as he adopted the multistep procedure. The manuscripts that witness the emergence of his new scheme suggest that he was at that time unable to provide a complete and coherent lunar theory to back up his new tables. The development of the multistepped scheme happened apparently for pragmatic rather than theoretical reasons, and it is highly unlikely that Mayer had a valid theoretical justification for the multistep over the single-step procedure.

The influence of Newton's lunar theory on Mayer provides yet another argument against the commonly held belief that Mayer's tables are merely Euler's equations (either in the astronomical or the mathematical sense) with their coefficients

1 The relation between Mayer's and Euler's lunar theories has been discussed in the previous chapter. Statements effectively equivalent to 'Newton out, Mayer in' can be found in e.g., [Wilson and Taton, 1989, p. 162], [Kollerstrom, 2000, pp. 233–4], [Whiteside, 1976, p. 324]. Petrus Frisius came very close to recognizing Mayer's dependence on Newton's lunar theory, only to conclude that Mayer had fitted Euler's theory to observations , [Frisius, 1768, pp. 272–3, 357].

adjusted to observations: that belief leaves the multistep procedure unexplained. We also see the role of the calculus in the development of Mayer's 1753 `kil` tables reduced to a subsidiary one, at most. But a final judgement of the relative merits of the calculus, Newton's theory, and observations is more delicate and will be postponed until the final chapter.

The current chapter is organized as follows. In section 6.2 we will start with a study of the 1702 'theory' of Isaac Newton, which is, as we will see, rather a set of rules to construct lunar tables than a theory in the modern sense of the word. Those rules are presented in section 6.3. Then follow two sections on a crucial ingredient of Newton's lunar theory that had developed out of an idea of Jeremiah Horrocks (1618–1641): a variable eccentricity and direction of the apsidal line of the lunar orbit. The kinematics of Horrocks's idea is explained in section 6.4, and in 6.5 it is translated into the language of trigonometric functions. The results will go slightly beyond what has already been published on the subject.

Section 6.6 is devoted to the lunar tables of Pierre Charles Lemonnier. His tables were probably the widest available implementation of Newton's prescriptions, and they were instrumental in the transmission of Newton's rules to Mayer. Mayer's assimilation of the Newtonian Lemonnier tables is the subject of section 6.7, where we study the relevant manuscripts. It will become clear that Mayer's interest in Newton's theory arose when he was apparently in doubt of how to proceed further in lunar theory.

Then follows an investigation of the measure of success of some of Mayer's table versions. In particular, we compare the accuracy of his tables before and after the multistep reform. The accuracy of Lemonnier's tables is included too. We will discover that the tables based on a multistepped scheme result in higher accuracy than Mayer's initial single-stepped scheme. Some insight will also be gained in the further improvements that Mayer attained. Finally, will be looked upon with fresh eyes we will take a look at Mayer's own preface to his printed tables.

6.2 NEWTON ON LUNAR MOTION, 1702

In 1702 Isaac Newton's *Theory of the Moon's Motion* (*NTM*) appeared, a pamphlet containing a set of rules for the production of tables for the computation of the position of the moon. Four nearly identical editions, three in English and one in Latin, have appeared of that text; the Latin version was first published in David Gregory's *Astronomiae Physicae & Geometricae Elementa*, 1702. All four have been reproduced in facsimile with a general introduction by I. Bernard Cohen.[2]

2 [Newton, 1975]. Cohen makes a distinction between (i) the 1702 pamphlet, (ii) its contents without reference to a specific edition, and (iii) Newton's work on lunar theory in general. The abbreviation *NTM* for Newton's Theory of the Moon is in line with Cohen's indication of the second category. An impression of Newton's work on lunar theory is provided in [Whiteside, 1976]; also see *A guide to Newton's Principia*, in [Newton et al., 1999], particularly §8.14, *The Motion of the Moon*, by I. B. Cohen, and §8.15, *Newton and the Problem of the Moon's Motion*, by George E. Smith. [Kollerstrom, 2000] discusses the procedures of *NTM*,

NTM is not a theory as one might perhaps expect. In fact, quite different from the modern scientific idea of a theory, *lunar theory* in that time denoted, in the words of Francis Baily, 'rules or formulae for constructing diagrams and tables that would represent the celestial motions and observations with accuracy'.[3] Nothing else could have been expected before the formulation of a causal physical theory of movement of celestial bodies. Even after he had provided precisely such a physical theory, Newton continued to use the word in its traditional sense.

The first edition of Newton's *Principia*, of 1687, contained a significantly more rudimentary lunar theory than the 1702 pamphlet. Somewhat modified and condensed forms of the *NTM* prescriptions found their way into the second (1713) and third (1726) editions of the *Principia*, where they can be found in the Scholium to Proposition 35, Book III, following a *quantitative* examination (ignoring eccentricity) of the variability of the inclination of the lunar orbit, the motion of the nodes, and the variation. In addition, Proposition 25 of Book III called upon the many corollaries of Proposition 66 of Book I for a *qualitative* explanation of all known lunar equAtions, and some new ones.[4]

A very characteristic feature of *NTM*, which we will discuss at length in the following two sections, was adapted from an older, kinematic, lunar theory of Jeremiah Horrocks. This suggests that Newton's lunar theory was a mix of his own dynamical research and of Horrocks's kinematic model.

Surprisingly, gravitation was only mentioned once in *NTM*, and then only in the preface, perhaps written by Halley:

> This Irregularity of the Moon's Motion depends (as is now well known, since Mr. Newton hath demonstrated the Law of Universal Gravitation) on the Attraction of the Sun, which perturbs the Motion of the Moon [...]. But this being *now* to be accounted for, and reduced to a Rule; by this Theory such Allowances are made for it, as that the Place of the Planet shall be truly Equated.[5]

The text ascribes the cause of the perturbations to the attraction of the sun, and stresses that it is now time to lay down rules to compute the perturbed motion of the moon, but it avoids to aver that the rules are purely deduced from the law of gravitation. Indeed, the main text of *NTM* supplied these rules, as a true theory in the pre-Newtonian sense of that word.

In *Principia* however, Newton repeatedly averred that he had obtained all his results from application of the law of gravitation to the Sun-Earth-Moon system,

assesses its accuracy using computer simulations, and points to various places where *NTM* was used in the 18th century, missing—like every other researcher before him—its influence on Mayer. [Cook, 2000] contains a well-balanced and illuminating view of Newton's work on lunar motion, providing physical insight while avoiding mathematical detail.

3 [Baily, 1835, p. 690], quoted in [Newton, 1975, p. 3]. The on-line edition of the Merriam-Webster dictionary (`http://www.m-w.com`) defines theory as 'a plausible or scientifically acceptable general principle or body of principles offered to explain phenomena'. Indeed, not everything that Newton wrote was Newtonian!

4 The notorious attempt in Book I Proposition 45 on the mean motion of the apsidal line is of no concern to us now; see [Waff, 1975] for that.

5 [Newton, 1975, pp. 94–95]. Cohen discusses the allusion to Halley on pp. 31–32.

but he omitted his derivations in all but the above-mentioned three cases: variation, nodal movement, and inclination. Michael Nauenberg has pointed to certain manuscripts in the Portsmouth collection where Newton apparently applied a perturbation technique that might have allowed him to obtain the results of *NTM*.[6] Nauenberg's point of view that Newton indeed had provided a dynamical and gravitational basis for the Horrocksian part of his theory is not accepted by everybody. Probably, though, Newton was able to obtain the form of some equAtions of lunar motion theoretically, whereafter he adjusted the coefficients to observations; at least, he explained to Flamsteed that this was his procedure. Although some of the equAtions were new discoveries disclosed by the law of gravitation, Whiteside and Kollerstrom maintain that Newton went back to the Horrocksian model in despair, after failing to account for *all* the lunar equAtions by gravitation.[7] It is of interest that Newton adapted a similar model in an attempt on the Jupiter-Saturn inequAlity, as Wilson noted.[8]

Surely, the moon had played a crucial role in Newton's discovery of the law of gravitation; yet the intricacies of lunar motion made his head ache, as Newton confessed to Machin. Although we regard Newton also as an inventor of the differential- and integral calculus, the tools at Newton's disposal were quite different from the tools that were developed later in the 18th century. Wilson explained it lucidly thus:

> Newton worked out the motions of celestial bodies while thinking predominantly geometrically, and at every step he had to give full account of the dynamics of the problem. In the eighteenth-century approaches, differential equations were formed on geometrical and dynamical grounds, whereafter the solution lay in the realm of analysis, having to find successive approximations to an analytical function.[9]

It is safe to say that the theoretical basis of *NTM* is still unclear today.[10]

6.3 THE EQUATIONS OF *NTM*

NTM first specified the epochs and mean motions of the lunar longitude, apogee, and node, and the solar longitude and apogee, and then proceeded to describe the several equAtions. We will now have a more detailed look at these equAtions, representing them in modern, analytical form.[11] We will need the amount of detail included here in this section to appreciate the impact of *NTM* on Mayer later on in the chapter.

6 For pointers, see [Nauenberg, 1998], [Nauenberg, 2000], [Nauenberg, 2001].

7 [Whiteside, 1976], [Kollerstrom, 2000].

8 [Wilson, 1985, p. 17].

9 [Wilson, 2001, p. 178]; also see [Wilson, 1985, p. 69+]

10 [Baily, 1835, pp. 139–140], [Newton, 1975, p. 39], [Whiteside, 1976], [Wilson, 1995a, p. 50], [Kollerstrom, 2000]. The source of the anecdote of Newton's headache is a notebook of John Conduitt [Whiteside, 1976, p. 324].

11 The research is based mainly on Cohen's edition of *NTM* [Newton, 1975, pp. 91–119], with [Kollerstrom, 2000] as a secondary source.

NTM's procedure is of the kind where each equAtion affects the arguments before the next equAtion is computed: it can be regarded as a multistep procedure with only one equAtion per step. *NTM* has seven steps for longitude, some very simple, and some more complicated. Several equAtions are subject to seasonal variations due to the varying distance of the earth-moon system from the sun. The central fourth step embodies both the equAtion of centre and the evection combined via a geometrical construction, to be discussed in the next two sections. We will now go through Newton's seven steps one by one.

[1] The first lunar equAtions specified in *NTM* are the annual equAtions to lunar longitude, apogee, and node. Newton stated the maximum values of these equAtions as $+11'49''$, $-20'$, and $+9'30''$ respectively. He specified further that they be proportional to the solar equAtion of centre with argument solar mean anomaly, here denoted by ς. From this proportionality it follows that the equAtions encompass terms proportional not only to $\sin\varsigma$, but also to $\sin 2\varsigma$.[12] The annual equAtion is the only step in Newton's prescriptions that affects the node. In all the following steps, the equAtions apply only to lunar longitude, except the fourth one which also affects the apogee.

[2] In order to convey the character of Newton's text, I quote here his prescription for the second equAtion:

> There is also an *Equation of the Moon's mean Motion* depending on the Situation of her Apogee in respect of the Sun; which is *greatest* when the Moon's Apogee is in an Octant with the Sun, and is nothing at all when it is in the Quadratures or Syzygys.[13] This Equation, when greatest, and the Sun *in Perigaeo*, is $3'56''$. But if the Sun be *in Apogaeo*, it will never be above $3'34''$. At other distances of the Sun from the Earth, this Equation, when greatest, is reciprocally as the cube of such Distance. But when the Moon's Apogee is any where but in the *Octants*, this Equation grows less, and is mostly at the same distance between the Earth and Sun, as the Sine of the double Distance of the Moon's Apogee from the next Quadrature or Syzygy, to the Radius.
>
> This is to be *added* to the Moon's Motion, while her Apogee passes from a Quadrature with the Sun to a Syzygy; but it is to be *subtracted* from it, while the Apogee moves from the Syzygy to the Quadrature. [Newton, 1975, pp. 105–6]

In other words, the second equAtion depends on the sine of twice the distance of the lunar apogee from the sun, that is $\sin(2\varpi - 2p)$ in Mayer's notation. Its coefficient, says Newton, varies annually between $3'56''$ and $3'34''$ reciprocally as the cube of the distance of the sun from the earth. When we express the eccentricity of the earth's orbit by $\varepsilon \approx 0.0168$, then the cube of the earth-sun distance reciprocally is very nearly $\left(\frac{1}{1+\varepsilon\cos\varsigma}\right)^3 \approx 1 - 3\varepsilon\cos\varsigma$. Hence the seasonal fluctuation amounts to approximately $\frac{1}{20}$ of the coefficient at the mean distance. We take the mean coefficient as the arithmetical mean of the annual extremes, or $3'45''$. Hence we deduce that Newton's second equAtion is very nearly $3'45''(1-3\varepsilon\cos\varsigma)\sin(2\varpi - 2p)$, that is, $(3'45'' - 11''\cos\varsigma)\sin(2\varpi - 2p)$.

12 I indeed found such terms, which were overlooked by Kollerstrom, in Lemonnier's tables to be discussed below. Higher order terms, proportional to $\sin kp$ for $k \geq 3$, contribute less than an arc-second and are therefore undetectable at the precision of Lemonnier's tables.

13 Syzygy and quadrature: cf. fn. 19 on p. 22.

[3] The third equAtion, represented analytically, is $47'' \sin(2\omega - 2\delta)$. Newton's description is in the same vein as the quote above, but simpler, because the seasonal change is too small to warrant mention.

[4] The middle of Newton's steps encompasses a kinematic construction borrowed from Horrocks's lunar theory, periodically modifying both the eccentricity of the lunar orbit and the orientation of its apsidal line before the equAtion of centre is computed using this modified eccentricity and anomaly. I postpone further discussion to the next section.

[5] Next comes the variation, which, like the second equAtion, has a seasonal component depending on the earth-sun distance. It is approximated analytically by $(35'32'' - 1'53'' \cos \varsigma) \sin 2\omega$.

[6] Newton's next equAtion amounts to $2'10'' \sin(2\omega + \varsigma - p)$. Originally, Newton had specified this equAtion in *NTM* with the wrong sign. The second and third *Principia* editions corrected the mistake, increased the coefficient to $2'25''$, and gave it a treatment conjointly with the equAtions of the fourth step, whence Newton called it his Second equAtion of centre. In the single-stepped computational procedure commonly used today, this equAtion indeed has a negative coefficient.[14]

[7] EquAtion seven is equivalent to $\big(2'20'' + 54'' \cos(\omega + \varsigma - p)\big) \sin \omega$. This equAtion was omitted in the third *Principia* edition. Kollerstrom noticed that the sign of the $54''$ annual coefficient is negative in some of the *NTM* versions.[15]

So far for the longitude equAtons of *NTM*; the text continues with equAtions for parallax, latitude, and reduction to the ecliptic, which are of no interest for our current discussion.

In the number of equAtions dealt with, *NTM* surpassed every other lunar theory extant at its time of publication. The annual equAtions to apogee and node, prescribed in *NTM*'s first step, were new inventions of Newton.[16] He introduced four other new equAtions: the 2nd, 3d, 6th, and 7th. The annual equAtion to longitude, present in step 1, and the variation of step 5, had both been discovered by Tycho Brahe. Ptolemy had modelled evection, while the equAtion of centre had been known throughout antiquity.

Newton's precepts were the best means of computing lunar positions in the first half of the 18th century. They were indeed used to construct tables, although it took about thirty years before those were widely available. Flamsteed made tables already in 1702, Halley did so too in about 1720, but although Halley's tables were printed, neither his nor Flamsteed's were ever published. Tables based on *NTM* were also produced by Wright (1732), Leadbetter (1735) and several others. Perhaps the first Newtonian tables to be published were those of Peder Horrebow,

14 See chapter 7 and in particular display 7.1 for the effect of the multistepped procedure on the coefficients; see e.g. [Meeus, 1998, p. 339] for modern values of coefficients.

15 [Kollerstrom, 2000, p. 106].

16 Newton discussed them in [Newton et al., 1999, Bk. III, Scholion after Prop. 35], and [Newton, 1975, pp. 103–105]. Wilson [Wilson, 1989b, p. 265] refers to Newton's *Principia* bk. III Prop. 22, where Newton alludes that 'there are other inequalities not observed by former astronomers' mentioning a.o. the inequAlities that these annual equAtions are to correct.

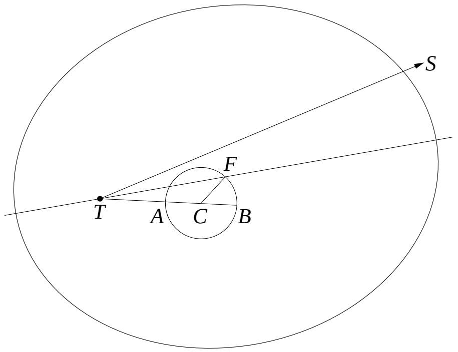

Figure 6.1: The variable lunar orbit

1718, but they lacked a widespread distribution. Lemonnier's handbook *Institutions Astronomiques* was very instrumental in spreading Newton's theory in the form of tables.[17]

6.4 HORROCKS'S VARIABLE ORBIT

Gravitational motion in an elliptic orbit respects the area law, hence it is not uniform. As is well known among astronomers, this brings about the equAtion of centre, a correction of the mean motion whose instantaneous value depends on the anomaly (i.e., the angular distance of the orbiting body from an apside) and the eccentricity. The fourth step in Newton's *NTM* sets up an equAtion of centre for a lunar orbit which is subjected to a variable apsidal line orientation and a variable eccentricity. Thus the form of the approximate elliptical orbit of the moon is supposed to change in *NTM*. The form change is effectuated by moving the centre of the elliptical lunar orbit in a small circle about its mean position, as follows.

In figure 6.1, let T be the earth, TS the direction of the sun from the earth, and TC the direction of the lunar apogee corrected by the annual equAtion of Newton's first step. C is the mean position of the centre of the lunar orbit. Its actual centre is in F, which is taken to revolve on the circle BFA around C. With respect to the (once equAted) lunar apogee, F revolves twice as fast as the sun, so $\angle FCB = 2\angle STC$. The length TF between the focus and centre of the elliptical lunar orbit represents the eccentricity, because the semi-major axis of the orbit is taken constant. As F revolves around C, the eccentricity varies between its minimum $TF = TA$ and maximum $TF = TB$. Furthermore, T and F lie on the apsidal line, the direction of which is therefore that of TF. As F rotates, the apogee rocks back and forth around its mean position on the line $TACB$ extended. The angle $\angle FTC$ between the actual and mean apse line is a second equAtion to the apogee position. The equAtion of

17 Cf. [Kollerstrom, 2000, Ch. 14].

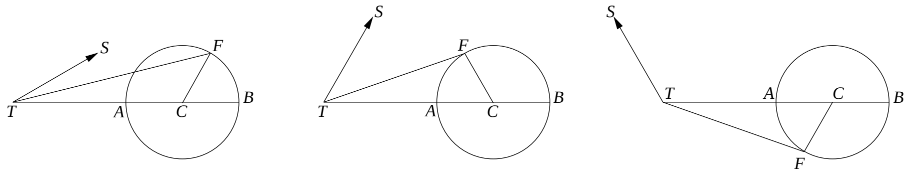

Figure 6.2: The eccentricity TF and second apse equAtion angle FTC of the variable orbit in three different states. The lunar orbit itself is not shown.

centre, being a function of eccentricity and anomaly, must then be computed with this modified eccentricity and apogee position. Figure 6.2 shows three diagrams for three different orientations of the sun with respect to the mean lunar apsidal line.

The fourth step in *NTM* provides then (1) a variable eccentricity TF of the lunar orbit; and (2) a (second) equAtion to the lunar apogee in the angle FTC; both affecting (3) the equAtion of centre.

Newton provided the following values in his *NTM*: the eccentricity ranged from $TA = 0.043319$ to $TB = 0.066782$ (expressed as fractions of the constant semi-major axis of the orbit); the equAtion of centre would then (as stated in *NTM*) attain its extremes at $4°57'56''$ and $7°39'30''$, respectively. These values imply a mean eccentricity of

$$TC = \tfrac{1}{2}(TA + TB) = 0.0550505$$

and radius of

$$FC = \tfrac{1}{2}(TB - TA) = 0.0117315.$$

For future reference we note the ratio $FC : TC = 0.213104$. The maximum second equAtion of the apogee is then $\angle FTC = \arcsin \frac{FC}{TC} = 12°18'15''$, whereas, surprisingly, Newton in *NTM* asserts the maximum to be $12°15'4''$. In the second edition of *Principia* all values were revised and brought in agreement to each other, the eccentricity ranging over 0.05505 ± 0.0117275 with a maximum apogee correction of $12°18'$.

Newton included a qualitative description of the variable eccentricity and rocking apsidal line in *Principia* book I, Proposition 66, corollaries 7–9, without explicitly referring to this mechanism of the rotating centre of the moon's orbit. The idea of an ellipse of variable eccentricity and rocking apse had been introduced into the lunar theory by Jeremiah Horrocks, who developed his theory, from 1638 until his early death two years later, from a combination of Keplerian ellipses and Lansbergen's periodical variations of a (circular) orbit.[18] The theory of Horrocks had remained unknown until Wallis published Horrocks's manuscripts in 1673. To this posthumous edition were attached Flamsteed's tables of Horrocks's lunar theory;

18 On Horrocks, see [Wilson and Taton, 1989, Ch. 10] and [Chapman, 1982]. On his lunar theory, see particularly [Wilson, 1987]. His idea of the lunar orbit!less as a form-changing ellipse was seminal to Euler's later work concerning variation of constants.

and it was Flamsteed who later, in 1694, convinced Newton of the quality of Horrocks's theory. Such happened perhaps at a time when Newton's own researches on lunar motion were coming to a grinding halt.

Curtis Wilson considered Horrocks's theory the chief improvement in lunar theory during the 17th and early 18th centuries, thereby implicitly giving it more weight than Newton's theory of gravitation and its—currently unclarified—influence on lunar theory: 'The [Newton's] approach [to lunar theory] failed, and as a predictive model, Horrocks's own theory remained as good as any available down to the publication of Tobias Mayer's first lunar tables in 1753.'[19] Wilson's view is endorsed by George Smith: 'Newton himself did not significantly advance the problem of the moon's motion beyond Horrocks.'[20] On the other hand, Smith *does* venerate Newton's significant contribution of providing the study of the moon's motion with a gravitational basis. We must also not forget that more than half of the longitude equAtions in *NTM* were original and at least qualitatively, if not quantitatively, derived from the law of gravitation. We should not, therefore, interpret Newton's lunar theory as only an improved version of Horrocks's lunar theory. One might say that Newton's lunar theory was the first to contain equAtions *predicted* to be of any relevance, without first having been *observed*.[21]

6.5 OLD WINE IN NEW BOTTLES

We will now translate the kinematic Horrocksian model of the previous section into the language of trigonometric functions. It will become clear that quite a number of other equAtions are involved. Computations similar to those below were made by Mayer, which shows his interest in Newton's *NTM*. A brief comment on his calculations will be made near the end of this section. Later in the chapter, his calculations will also be seen to play a crucial role in the conception of the `kil` tables.

What needs to be done, therefore, is to analyse how the Horrocksian variable orbit affects the equAtion of centre. For an unperturbed elliptical motion, the equAtion of centre $\mathscr{C}$ may be written to the third order in the eccentricity e as

$$\mathscr{C} = -\left(2 - \tfrac{1}{4}e^2\right)e\sin v + \tfrac{5}{4}e^2\sin 2v - \tfrac{13}{12}e^3\sin 3v, \tag{6.1}$$

where v denotes the mean anomaly measured from the apogee. The key concept of Horrocks's variable orbit is to periodically change the eccentricity and the apse line orientation of the lunar orbit, and to take the equAtion of centre pertaining to the form-changing instantaneous ellipse. Referring to either figure 6.1 or 6.2, we put the mean eccentricity $TC = a$, the actual eccentricity $TF = e$, and the fixed ratio $FC : TC = \beta$. The kinematic model specifies that $\angle FCB = 2\angle STC = 2(\omega - p)$, where p equals the lunar mean anomaly (once equAted for the annual equAtion of

19 [Wilson, 1987, p. 77]; also see [Wilson and Taton, 1989, p. 194].

20 [Newton et al., 1999, p. 256].

21 [Wilson, 1987, p. 77], [Newton et al., 1999, p. 256]; also see [Wilson and Taton, 1989, p. 194].

the apogee), and ω the mean lunar longitude minus mean solar longitude. Hence we express the eccentricity via

$$\begin{aligned} e^2 = TF^2 &= TC^2 + FC^2 + 2\,TC\,FC\cos\angle FCB \\ &= a^2 + a^2\beta^2 + 2a^2\beta\cos(2\omega - 2p). \end{aligned} \tag{6.2}$$

For the modified anomaly in (6.1) we write $v = p + \delta$, where, deviating from our usual meaning of the symbol δ for the duration of this section, $\delta = \angle FTC$, the second equAtion of the apogee. Let D (not in figure) be the foot of the perpendicular from F to TC. The following relations hold for δ:

$$\begin{aligned} \sin\delta &= \frac{FD}{TF}, \qquad \text{whence} \qquad e\sin\delta = a\beta\sin(2\omega - 2p), \\ \cos\delta &= \frac{TD}{TF}, \qquad \text{whence} \qquad e\cos\delta = a\bigl(1 + \beta\cos(2\omega - 2p)\bigr). \end{aligned} \tag{6.3}$$

The equations (6.2) and (6.3), which provide the moon's variable eccentricity as well as the second equAtion of the apogee, will be of use again in the next section. Presently, we use them to write the first-order term in the modified equAtion of centre as

$$\begin{aligned} e\sin(p+\delta) &= e\cos\delta\sin p + e\sin\delta\cos p \\ &= a\bigl(1 + \beta\cos(2\omega - 2p)\bigr)\sin p + a\beta\sin(2\omega - 2p)\cos p \\ &= a\sin p + a\beta\sin(2\omega - p). \end{aligned} \tag{6.4}$$

This is beautiful. The term $a\sin p$ on the right-hand side is the first-order term of the equAtion of centre for the mean eccentricity a and mean anomaly p. The second term has argument $2\omega - p$ and we recognize it as the prime evection term. Thus we learn the important fact that the Horrocksian-modified equAtion of centre equals the sum of the unperturbed equAtion of centre and the evection, to the first order in the eccentricity. Next, we combine (6.2), (6.4) and the familiar relation

$$\sin x\cos y = \tfrac{1}{2}\bigl(\sin(x+y) + \sin(x-y)\bigr), \tag{6.5}$$

in order to expand the first term of (6.1) into

$$\begin{aligned} \bigl(2 - \tfrac{1}{4}e^2\bigr)\,e\sin(p+\delta) = {} & \bigl(2a - \tfrac{1}{4}a^3 - \tfrac{1}{2}a^3\beta^2\bigr)\sin p \\ & + \bigl(2a\beta - \tfrac{1}{2}a^3\beta - \tfrac{1}{4}a^3\beta^3\bigr)\sin(2\omega - p) \\ & - \tfrac{1}{4}a^3\beta^2\sin(4\omega - 3p) + \tfrac{1}{4}a^3\beta\sin(2\omega - 3p). \end{aligned} \tag{6.6}$$

Continuing now to the second term in (6.1), we first observe that

$$\begin{aligned} e^2\sin 2\delta &= 2e^2\sin\delta\cos\delta \\ &= 2a^2\beta\sin(2\omega - 2p)\bigl(1 + \beta\cos(2\omega - 2p)\bigr) \\ &= 2a^2\beta\sin(2\omega - 2p) + a^2\beta^2\sin(4\omega - 4p); \\ e^2\cos 2\delta &= e^2(\cos^2\delta - \sin^2\delta) \\ &= a^2\bigl(1 + \beta\cos(2\omega - 2p)\bigr)^2 - a^2\beta^2\sin^2(2\omega - 2p) \\ &= a^2 + 2a^2\beta\cos(2\omega - 2p) + a^2\beta^2\cos(4\omega - 4p); \end{aligned}$$

hence

$$\begin{aligned}
e^2\sin(2p+2\delta) &= e^2\sin 2p\cos 2\delta + e^2\cos 2p\sin 2\delta \\
&= \sin 2p\big(a^2+2a^2\beta\cos(2\omega-2p)+a^2\beta^2\cos(4\omega-4p)\big) \\
&\qquad +\cos 2p\big(2a^2\beta\sin(2\omega-2p)+a^2\beta^2\sin(4\omega-4p)\big) \\
&= a^2\sin 2p+2a^2\beta\sin 2\omega+a^2\beta^2\sin(4\omega-2p).
\end{aligned} \tag{6.7}$$

We see that the second order term of the equAtion of centre contributes to the variation and also (in the second order) to the evection. Repeating the same procedure for the third term in (6.1) yields

$$e^3\sin(3p+3\delta) = a^3\sin 3p+3a^3\beta\sin(2\omega+p) \tag{6.8}$$

$$+3a^3\beta^2\sin(4\omega-p)+a^3\beta^3\sin(6\omega-3p). \tag{6.9}$$

Finally then, we find that the equAtion of centre (6.1) expands into

$$\begin{aligned}
\mathscr{C} = &\big(-2a+\tfrac{1}{4}a^3+\tfrac{1}{2}a^3\beta^2\big)\sin p+\tfrac{5}{4}a^2\sin 2p-\tfrac{13}{12}a^3\sin 3p+\tfrac{5}{2}a^2\beta\sin 2\omega \\
&+\big(-2a\beta+\tfrac{1}{2}a^3\beta+\tfrac{1}{4}a^3\beta^3\big)\sin(2\omega-p) \\
&+\tfrac{5}{4}a^2\beta^2\sin(4\omega-2p)-\tfrac{13}{12}a^3\beta^3\sin(6\omega-3p) \\
&-\tfrac{1}{4}a^3\beta\sin(2\omega-3p)+\tfrac{1}{4}a^3\beta^2\sin(4\omega-3p) \\
&-\tfrac{13}{4}a^3\beta^2\sin(4\omega-p)-\tfrac{13}{4}a^3\beta\sin(2\omega+p).
\end{aligned} \tag{6.10}$$

Upon substitution of the typical Newtonian values $a = 0.05505$ and $\beta = 0.2131$, and rounding to arc-seconds, we find that the equAtion of centre in the form-changing ellipse equals

$$\begin{aligned}
\mathscr{C} = &-6^\circ 18'20''\sin p+13'1''\sin 2p-37''\sin 3p+5'33''\sin 2\omega \\
&-1^\circ 20'36''\sin(2\omega-p)+35''\sin(4\omega-2p) \\
&-2''\sin(2\omega-3p) \\
&-5''\sin(4\omega-p)-24''\sin(2\omega+p).
\end{aligned} \tag{6.11}$$

Several authors have presented a similar but less extensive analysis.[22] In particular Gaythorpe and Wilson showed that the Horrocksian variable orbit is equivalent to the combined equAtion of centre and evection. But Gaythorpe missed the contribution to variation of $\approx 5'\sin 2\omega$ in (6.10). Because of this contribution, every Horrocksian lunar theory (i.e., with variable eccentricity and apsidal line plugged into the equAtion of centre) necessarily has a variation coefficient of $\approx 35'$ instead of the total $\approx 40'$.[23]

22 These include [d'Alembert, 1756, I pp. 91–93], [Godfray, 1852, p. 60], [Gaythorpe, 1957], [Brown, 1896], and [Wilson and Taton, 1989, p. 198].

23 Gaythorpe's oversight was corrected by Jørgensen [Jørgensen, 1974]; the point had however already been noted by d'Alembert [d'Alembert, 1756, I p. 93, 253].

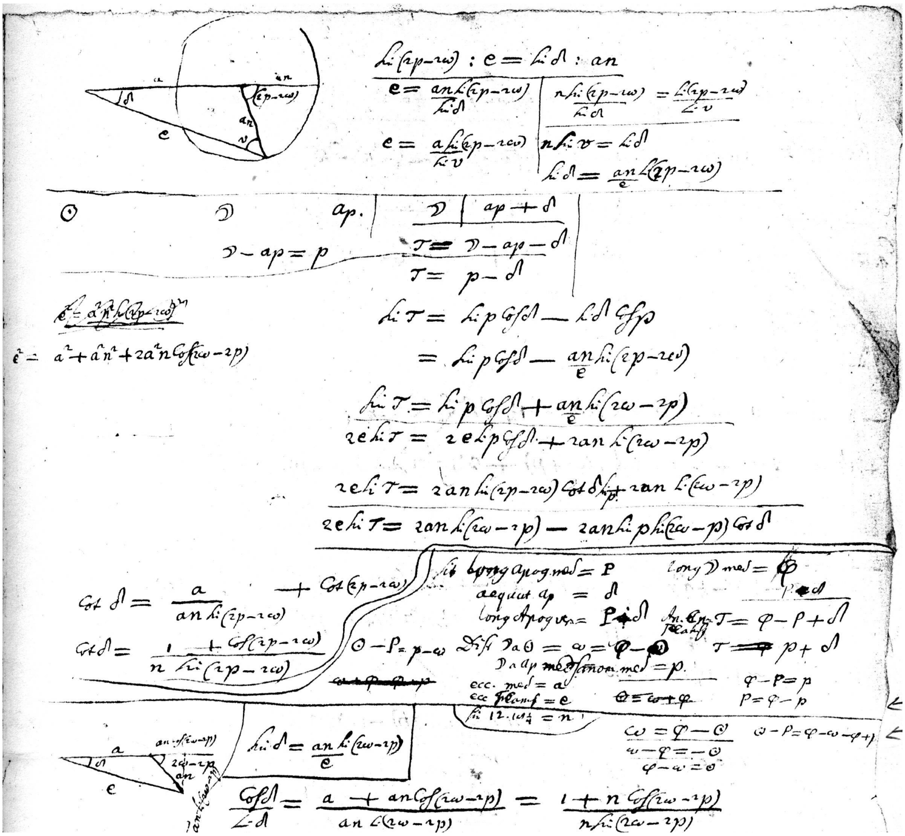

Figure 6.3: Part of an unnumbered folio of Cod. $\mu^{\sharp}_{28}$, showing Mayer's work on the Horrocksian mechanism. The top half of the folio contains an error, which is corrected in the bottom half (below the double line). Reproduced with the kind permission of SUB, Göttingen.

Mayer too made computations such as those above, as I discovered on various loose folio sheets among his work on lunar theory (cf. figure 6.3). The same sheets carry several other computations that are manifestly connected with his endeavours to understand the successive steps of Newton's lunar theory in the language of trigonometric quantities. In these computations, Mayer kept terms involving $a^3\beta^2 \approx 2''$, and rejected $a^3\beta^3 \approx 0.3''$ and a^4. Because $\beta^2 \approx a$, he might have rejected $a^3\beta^2$ just as well. He treated the trigonometric quantities in a distinctly algebraical style, as developed by Euler since 1739,[24] and very similar to our treatment on the preceding pages of this section.

24 Euler expounded the calculus of the trigonometric functions in his treatise on the great inequAlity of Jupiter and Saturn [Euler, 1749a], well known to Mayer. See [Katz, 1987, p. 322], [Golland and Golland, 1993].

The contrast between the geometrical style of Newton and Mayer's reformulation in trigonometric quantities is telling of the change in perception of trigonometry which Euler had brought about. This contrast is perhaps even more clearly perceived when we turn to the lunar tables: first to those of Lemonnier, which are fully based on Newton's *NTM*, then to the twist that Mayer gave to Lemonnier's tables.

6.6 LEMONNIER'S VERSION OF *NTM*

Although Flamsteed did not publish his *NTM*-based lunar tables, some form of a copy of his manuscript must have reached France, where Lemonnier adapted it for publication in his *Institutions Astronomiques*,[25] an enlarged translation of John Keill's *Introductiones ad veram physicam et veram astronomiam*. Mayer was evidently well acquainted with both books.[26]

Because *Institutions Astronomiques* was an important source to Mayer, we will now undertake to appraise the conformity of Lemonnier's tables with Newton's precepts. Our general approach is as follows. Because Lemonnier acknowledged Flamsteed as his source, we may assume as a working hypothesis that the form of the equAtions on which his tables are based, agrees to the Newtonian prescriptions as discussed above. We then deduce the coefficients in these equAtions from the values in Lemonnier's tables. This is straightforward for some equAtions, but more elaborate for others. Finally, we check if these equAtions and coefficients indeed reproduce Lemonnier's tables sufficiently accurate, i.e., to within $1''$ or $2''$.

We now turn to the details, referring to section 6.3 for Newton's seven steps, and to display 6.1 for the results of the analysis, and, as usual, using the argument notation of Mayer.

Newton's first step comprises three annual equAtions, proportional to the solar equAtion of centre. The latter is a formula of the form $c_1 \sin\varsigma + c_2 \sin 2\varsigma +$

25 [Lemonnier and Keill, 1746], also see comments in [Kollerstrom, 2000, pp. 205–14].

26 [Forbes, 1971a, p. 83], [Forbes, 1980, p. 120]. Rob van Gent suggested (private communication) that there might have been an interesting alternative source of information of Flamsteed's tables to Mayer, via Johann Gabriel Doppelmayr (1671–1750). Both Germans were working together in the Homann building during the later 1740's. Doppelmayr visited England in 1701, where he met Gregory for certain, and Flamsteed and Newton very probably [Gaab, 2001]. Newton had then already written his *NTM* [Kollerstrom, 2000, p. 43]. Gregory had not yet published it but at least he might have known it. In 1705, after returning to Nuremberg, Doppelmayr published his Latin translation of Streete's *Astronomia Carolina*. Although Streete's work was surely a long-standing classic in astronomy, one might expect Doppelmayr to include the Flamsteed lunar tables instead of Streete's—which he did not, suggesting that the newer Newtonian tables made at most no lasting impression on him. Moreover, Flamsteed calculated his manuscript lunar tables of the Newton variety only after Doppelmayr had left England, and he had made apparently only two manuscript copies of them [Baily, 1835, p. 695, 704]. The text on the lunar plate of Doppelmayr's *Atlas Coelestis* (1742) suggests that the author did not fully grasp the details of *NTM*. Strikingly, though, Mayer consistently referred to the lunar tables of *Flamsteed*, never to those of *Lemonnier*, in his manuscripts. Lemonnier had dutifully acknowledged Flamsteed as the provenance of his tables.

1	Annual equAtions	
	to longitude	$+11'49'' \sin\varsigma - 7\frac{1}{2}'' \sin 2\varsigma$
	to apogee	$-20'\,0'' \sin\varsigma + 13'' \sin 2\varsigma$
	to node	$+\,9'30'' \sin\varsigma - \,6'' \sin 2\varsigma$
2		$(+3'45'' - 11'' \cos\varsigma) \sin(2\omega - 2p)$
3		$+47'' \sin(2\omega - 2\delta)$
4	2nd apogee equAtion	$\arctan \frac{\beta \sin(2\omega-2p)}{1+\beta\cos(2\omega-2p)}$
	eccentricity e	$a\sqrt{1+\beta^2+2\beta\cos(2\omega-2p)}$
	equAtion of centre	$\left(2e - \frac{1}{4}e^3\right)\sin p - \left(\frac{5}{4}e^2 - \frac{11}{24}e^4\right)\sin 2p$ $+\frac{13}{12}e^3 \sin 3p - \frac{103}{96}e^4 \sin 4p$
5	Variation	$(35'15'' - 2'11'' \cos\varsigma) \sin 2\omega$
6		$+2'10'' \sin(2\omega - p + \varsigma)$
7		$-2'20'' \sin\omega$
	2nd node equAtion	$\arctan \frac{\sin(2\delta-2\omega)}{\gamma+\cos(2\delta-2\omega)}$
	Eqn. of Inclination	$5^\circ 8'30'' + 9' \cos(2\delta - 2\omega)$

Display 6.1: EquAtions of lunar longitude, node, and inclination of Lemonnier's lunar tables. The leftmost numbers correspond to the steps in Newton's *NTM*, described in section 6.3.

$c_3 \sin 3\varsigma + \ldots$ where ς is the solar mean anomaly, and the coefficients stand in the ratio $c_1 : c_2 : c_3 = 95 : 1 : \frac{1}{69}$, roughly. The three annual equAtions should then take the same form, with the same ratio between the coefficients, while the coefficients follow from the extreme values of the equAtions as specified by Newton. This yields the annual equAtions listed in display 6.1; the third coefficients of each are negligible. Lemonnier's tables satisfy these equAtions.

Lemonnier's second equAtion is equivalent to Newton's second step, and may be represented as $(3'45'' - 11'' \cos\varsigma)\sin(2\omega - 2p)$. As explained in section 6.3, this equAtion has a seasonal fluctuation in its coefficient. Lemonnier has two tables to represent this equAtion: one table gives $3'45'' - 11'' \cos\varsigma$ as a function of ς, the other tabulates $3'45'' \sin(2\omega - 2p)$. The user of these tables has to perform an additional calculation to establish the magnitude of the second equAtion. If, for given values of the arguments ς and $2\omega - 2p$, the first table yields x and the second table returns y, then $\frac{x}{3'45''}y$ is the value of the second equAtion.

A similar seasonal fluctuation is also found in Newton's step 5, his variation, which we have represented previously as $(35'32'' - 1'53'' \cos\varsigma)\sin(2\omega)$. Lemonnier's fifth equAtion takes two tables again, but his coefficients differ slightly from Newton's values: the table for the seasonal part matches $35'15'' - 2'11'' \cos\varsigma$, the other table lists $35'15'' \sin(2\omega)$.[27]

27 Lemonnier combined the seasonal tables of equAtions 2 and 5 in one table, for practical reasons.

The third and sixth steps of Newton are easy to implement in tables. For these equAtions, a straightforward check confirms that Lemonnier apparently adhered to Newton's coefficients of *NTM*. He also tabulated Newton's seventh equAtion except that he dropped its seasonal term.

Finally we come to Newton's fourth step, with its variable eccentricity and oscillating apsidal line in the Horrocksian way. Lemonnier implements the motion of the apsidal line as a second equAtion of the apogee, but the eccentricity has a more involved rendering. We will illuminate the apogee equAtion first.

In figure 6.1 and equation (6.3), $\angle FTC = \delta$ is the second apogee equAtion, and $FC : TC = \beta$. It follows from (6.3) that

$$\delta = \arctan \frac{\beta \sin(2\omega - 2p)}{1 + \beta \cos(2\omega - 2p)}. \tag{6.12}$$

Presuming that Lemonnier tabulated this relation, with $\omega - p$ as the argument, we proceed as follows to derive β from his tabulated values: after some manipulation of the formula, we find that

$$\frac{1}{\beta} = \frac{\sin 2(\omega - p)}{\tan \delta} - \cos 2(\omega - p). \tag{6.13}$$

Now we fill in all pairs of arguments and tabulated values to obtain as many values for β, which when averaged give $\beta = 0.213104$. This β substituted back in (6.12) suffices to recreate Lemonnier's table with no differences larger than $2''$. Besides, this value for β is the same as the ratio for $FC : TC = \beta$ that we have computed on page 91 from data in *NTM*. Thus, Lemonnier's second apogee equAtion is in perfect agreement with *NTM*.

To account for the variable eccentricity, Lemonnier again included two tables. The first of these listed the extremal values of the equAtion of centre for different values of the argument $\omega - p = \angle STC = \frac{1}{2}\angle FCB$ in figure 6.1. The second table listed four different equAtions of centre, with, respectively, extremal values of $-5°$, $-6°$, $-7°$, and $-7°39.5'$. The user was then supposed to interpolate (or even extrapolate) between two columns of this table, depending on the extremal value produced by the first table. Confusingly, the headings of the second table contained not these extremal values, but the eccentricities at which they occur. The link to the extremes was only explained in a commentary.[28]

Lemonnier's table for maximum equAtion of centre can be reproduced to within $3''$, as follows: compute the variable eccentricity e according to equation (6.2), with a mean eccentricity of $a = 0.05506$ instead of Newton's value 0.05505; then substitute this eccentricity in the equAtion of centre, and compute the equAtion for argument $p \approx 94°$.[29]

28 [Lemonnier and Keill, 1746, p. 629]. The link between argument $\omega - p$ and eccentricity e is provided by equation (6.2) above.

29 The equAtion of centre attains its maximum when $\cos p = \frac{1}{5e} - \sqrt{\frac{1}{(5e)^2} + \frac{1}{2}}$; for the small eccentricities concerned this maximum is reached when p is approximately $94°$.

With the well-known formula for the equAtion of centre, Lemonnier's fourfold table of that function is then reproducible to $2''$ or better. Interestingly, Lemonnier had apparently included terms of the fourth order in the eccentricity (see formula in the middle of display 6.1).

D'Alembert published a similar analysis, but less detailed, missing for example the second terms in the annual equAtions.[30] For completeness, display 6.1 also shows Lemonnier's equAtions for node and inclination; these will not be further discussed here.[31] Kollerstrom rightly remarked that Lemonnier's tables form a true representation of *NTM*, with a few small changes to parameters, without the seasonal modulation of the 7th equAtion, and with the sign error of *NTM*'s sixth equAtion corrected.[32]

We have seen that Lemonnier's tables necessitate multiplications and divisions to accommodate the seasonally varying equAtions, and that they incorporate a somewhat complicated scheme to accommodate the variable eccentricity. A user of Lemonnier's tables has to make more involved calculations than a user of Mayer's tables, as exemplified in chapter 4.

6.7 *'MONDTAFELN (WAHRSCHEINLICH ÄLTERER ENTWURF)'*

Among the Mayer manuscripts in Göttingen is the quire Cod. $\mu_{15}^{\sharp}$, to which Lichtenberg added the title *'Mondtafeln (wahrscheinlich älterer Entwurf)'* ('Lunar tables, probably of older design'). It is of interest here because it is a witness of the introduction of multiple steps in Mayer's lunar tables. Most of the quire was composed during 1752. It marked the transition from Mayer's single-stepped `zand` theory, contained in a letter to Euler of 1752 January 6, to the multistepped precursors of the `kil` tables published in the spring of 1753.[33] Part of its contents depended on a design considerably older than Lichtenberg might have perceived, as we will see.

The items in the manuscript that are currently of interest, are: (1) a comparison of the coefficients in several lunar theories on pp. 8v and 9r; (2) tables of lunar equAtions on pp. 9v–16v; (3) a page (p. 17r) with the superscript *Entwurf neuer* ☽ *Tafeln*; followed by (4) again tables of lunar equAtions. Each of these are discussed below, whereafter I provide an interpretation of the manuscript.[34]

6.7.1 Peering at the peers

The facing folios 8v and 9r of Cod. $\mu_{15}^{\sharp}$ are laid out in the form of an array (see figure 6.4). The first column on the left side of the array lists trigonometric expressions: successively $\sin p$, $\sin 2p$, $\sin 3p$, $\sin\varsigma$, …, $\sin(p-\varsigma)$, $\sin(p+\varsigma)$, ….

30 [d'Alembert, 1756, I, Ch. 13].

31 The second node equAtion takes $\gamma = 38.3341$. These equAtions were demonstrated by Newton in the *Principia*. They do not affect lunar longitude.

32 [Kollerstrom, 2000, p. 212].

33 Letter to Euler: [Forbes, 1971a] p. 48.

34 Aliases of the versions treated here are all listed in display A.1 in appendix A.

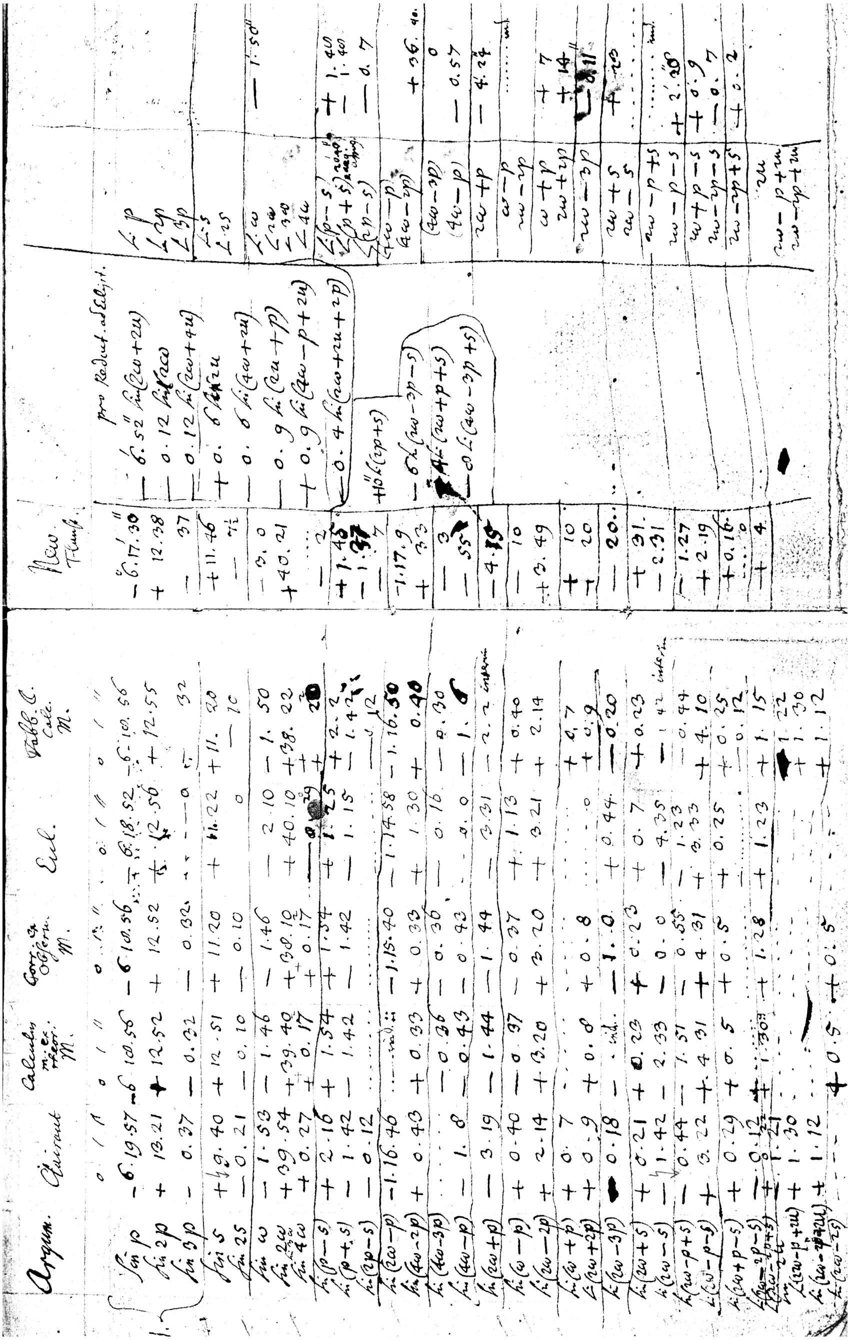

Figure 6.4: Comparison of coefficients from various astronomers, on fol. 8v and 9r in Cod. $\mu^{\sharp}_{15}$. Reproduced with the kind permission of SUB, Göttingen.

The other columns bear the following superscripts (numbers added in square brackets for ease of reference): [1] *Clairaut*, [2] *Calculus m. ex theor. M.*, [3] *Corr. ex observ. M.*, [4] *Eul.*, [5] *Tabb ☽. Calc. M.*, and [6] *New. Flamst.* Under these headings, the columns are filled with numbers, which are clearly coefficients, forming equAtions together with the trigonometric quantities on the left side.

The coefficients in column [3] agree exactly to `zand`, i.e., to those that Mayer transmitted to Euler in his letter as mentioned above. The letter provides an excellent backdrop against which Cod. $\mu_{15}^{\sharp}$ falls into perspective. In it, Mayer explained to Euler why he chose to base his tables on the *mean* arguments, contrary to Euler's preference for the *eccentric* arguments:

> The angles ω, p, and ς invariably denote the mean motion, which in fact brings several advantages not in the solution, but in practice, because in such a manner the arguments of the inequAlities can be calculated more simply.[35]

Next in the same letter he referred to the problem of the motion of the lunar apogee. Although the famous problem of its *mean* motion had recently been solved by Clairaut, Mayer's remark makes more sense when interpreted in relation to the *variable* part of the motion of the apogee, $\angle FTC$ in figure 6.1:

> I have indeed always supposed, yet could never be certain, that the inequality associated with the angle $2\omega - p$ [i.e., evection], which is without doubt difficult to determine and which was explained by Newton as due to the variation of the eccentricity of the Moon's orbit, strongly affects the motion of the apogee. Now, I want particularly to make new attempts to determine the motion of the apogee, and to see whether I do not arrive at it if instead of the abovementioned inequality I take the eccentricity as being really variable.[36]

In fact, this is intelligible only in the context of Newton's lunar theory and the Horrocksian variable orbit. This remark of Mayer's presages the multistep tables. In the same letter, Mayer had indicated his desire to see Euler's and Clairaut's lunar theories:

> Meanwhile, I eagerly await the treatises of yourself and Mr. Clairaut. You would greatly oblige me if I could obtain through your assistance a copy as soon as they are published. . . .[37]

Euler had been an adjudicator for the 1751 prize contest of the Academy of Saint Petersburg on lunar theory, which was won by Clairaut's contribution, therefore Euler was an eminent source of information. He responded on March 18th, 1752 that he was unable to compare the magnitudes of the equAtions of his own lunar theory with Mayer's, because of the differences in the arguments already alluded to: Euler's tables made use of the eccentric anomaly, but Mayer's used the mean anomaly throughout. Like Mayer's, Clairaut's equAtions used the mean arguments, and therefore Euler included a complete list of the equAtions in Clairaut's theory. Euler kept the equAtions in the same sequence as Mayer had in his letter, inserting a few that were present in Clairaut's, but absent from Mayer's theory (most notably several equAtions that involved the angular distance of the sun from the nodes of the

35 [Forbes, 1971a, p. 49].
36 [Forbes, 1971a, p. 49].
37 [Forbes, 1971a, p. 49].

lunar orbit, until then overlooked by Mayer). Mayer copied precisely this list out of Euler's letter into column [1] of the array on fol. 8v and 9r of Cod. $\mu^{\sharp}_{15}$. Therefore we can be sure that Mayer started this array after he received Euler's response; moreover, we may speculate that he did not waste much time before doing so.

Mayer's own coefficients in columns [2] and [3] are mostly identical: apparently, he adjusted only a few of the theoretically derived coefficients to observations. The superscript of column [4] suggests that Mayer included coefficients from Euler's tables. I have not investigated how Mayer obtained these; due to the difference in arguments it was probably a non-trivial exercise.

Column [5] is peculiar in that the coefficients in it sometimes follow Mayer's values, sometimes those of Clairaut (particularly for those equAtions where Mayer did not have a value of his own), and sometimes they fall in between. Only one coefficient was drawn from Euler's column, and two are marked as preliminary ('*interim*'). It seems as if Mayer was, in a sense, interpolating between his own, Clairaut's, and (to a lesser extent) Euler's lunar theories.

Column [6] with its superscript '*New. Flamst.*' suggests that it is linked to Newton's lunar theory. Indeed it is: Mayer calculated these coefficients on the same folios of Cod. $\mu^{\sharp}_{28}$, mentioned on page 94 above, where he expressed Horrocks's variable eccentricity and apsidal line as trigonometric quantities. This column was added perhaps a little while after the other columns had been completed.

6.7.2 The first set of lunar tables

The next item of interest in Cod. $\mu^{\sharp}_{15}$ is a set of lunar tables aliased `grond`. It contains mean motion tables equivalent to Lemonnier's, except that the node epoch was adjusted. The mean motions are followed by 19 tables of equAtions catering for exactly the equAtions with their coefficients as listed in column [5], fol. 8v. The superscript *Tabb* ☽. *Calc. M.* of column [5], just discussed, refers to these tables.

6.7.3 '*Entwurf neuer* ☽ *Tafeln*'

Apparently Mayer abandoned those `grond` lunar tables; they are immediately followed by a page so revealing that I have transcribed it in display 6.2. The following observations apply.

A heading *Entwurf neuer* ☽ *Tafeln* is written along the top edge of the page: 'design of new lunar tables'. The design is specified below the heading. There are 11 numbered arguments, which are given symbolically as well as in descriptive language. For each argument, an equAtion is specified, with one or more terms, and appropriate coefficients in sexagesimal notation. The majority of the equAtions apply to lunar longitude, but there are a few that apply to the apogee or node.

The plain-language descriptions of the arguments in the third column clearly reveal that some of them are to be corrected in steps. The elongation of the moon is to be corrected using the first equAtions, depending on argument ς, the mean solar anomaly. Mayer indicates this by his addition '*I corr.*' in the description of arguments II and V. Similarly, argument IV is the anomaly of a 'corrected' moon; Mayer uses the symbol q to distinguish it from the mean anomaly p. There is a similar distinction between the elongations ω and $\tilde{\omega}$, although less consistently applied: argument VI is lunar elongation after application of the Vth equAtion (and, I suppose, equAtions I to IV as well).

To summarize, Mayer introduces the multistep procedure here in this 'new design'. By the doubled lines between III and IV and between V and VI he clearly distinguished the three steps.

The basic structure of this design has some striking resemblances to Newton's *NTM*, including the following. The first argument, solar mean anomaly ς, is not only used to equAte the lunar *longitude*, but also its apogee and node positions. This is a significant influence of Newton on Mayer. At least as significant, we discover the equAtion of centre and the evection close together as equAtions IV and V in the middle step. These are followed by the variation in VI and several subsequent smaller equAtions. Among those, numbers VIII and IX take up Newton's seasonally modified variation; they were most likely derived (at least in form) by application of equation (6.5). Differing from *NTM*, Mayer's design collects several equAtions together into one step, and his last two equAtions are new. Apparently Mayer dropped these two almost immediately afterwards. Also most (but not all) of the coefficients differ from Newton's.

6.7.4 The second set of lunar tables

On the following pages of the manuscript we find lunar tables in an untidy handwriting, as if they were a preliminary or intermediate version. However, these tables differ from the new design just discussed. Apart from mean motion tables and an unnumbered equAtion of centre answering to $-6°18'56'' \sin p + 12'55'' \sin 2p - 32'' \sin 3p$, we find:

II	$+40'' \sin(\omega - p) + 3'20'' \sin(2\omega - 2p)$
III	$-58'' \sin(2\omega + \varsigma)$
IV	$-58'' \sin(2\omega - \varsigma)$
V	$-58'' \sin(2\omega - 2\delta)$
VI	$+1'30'' \sin(2\omega - p + \varsigma)$
VII	$+1'30'' \sin(2\omega - p - \varsigma)$
VIII	$+58'' \sin(2\delta - p)$
IX	$+30'' \sin(2\delta - 2p)$
X	$+1'30'' \sin(2\omega + p)$

At first sight, this does not seem to tally with the multistepped development of the *Entwurf*. However, using techniques described in section 8.4, I analysed certain position calculations in Cod. $\mu_{41}^{\sharp}$, and discovered that they used the equAtions embodied in these tables, and moreover that these calculations adhered to a compu-

Entwurf neuer ☽ tafeln.

			Aeq. ☽	Aeq. apog.	Aeq.☊
1 Arg.I.	ς	Anomalia media Solis	$+11'40''\sin\varsigma$ $-\quad 10''\sin 2\varsigma$	$-20'0''\sin\varsigma$ $+\ 20''\sin 2\varsigma$	&c
Arg.II.	$\omega - p$	Dist ☽ a ⊙ I corr. − Anom ☽ med	$+3'20''\sin(2\omega-2p)$ $\pm\ 20\sin(\omega-p)$		
2 Arg.III.	u	Long ⊙ − Long ☊ med.	$-1'0''\sin 2u$		
Arg.IV.	$q\ldots$	Long ☽ corr. −Ap.☾ corr.	$-6°18'27''\sin q$ $+\ 12.38\sin 2q$ $-\quad 37\sin 3q$		
Arg.V.	$2\omega - q$	ad arg II add. Dist.☽ a ⊙ I.corr.	$-1°21'0''\sin(2\omega-p)$ $+\quad 36\sin(4\omega-2p)$		
Arg.VI.	$\tilde{\omega}$	Dist.☽ a ⊙ post V aequationem	$-\ 2'0''\sin\omega$ $+40.21\sin 2\omega$ $+\quad 2\sin 3\omega$ $+\quad 22\sin 4\omega$		
Arg.VII.	$2\tilde{\omega} - q + \varsigma$	ad arg V add. anom ⊙ med	$+[?].10\sin(2\omega-q+\varsigma)$		
Arg.VIII.	$2\omega + \varsigma$	ad arg VII add anom☽ add arg IV	$-1.10\sin(2\omega+\varsigma)$		
Arg.IX.	$2\omega - \varsigma$	ad dupl dist ☽ a ⊙ subt an ⊙	$-0.30\sin(2\omega-\varsigma)$ vel 1.0 vel 0.0		
Arg.X.	$2\omega - q + 2u$	ad Arg V add dupl.arg III.	$+1.\ 0\sin(2\omega-q+2u)$		
Arg.XI	$2\omega - 2q + 2u$	ab Arg X subtr. Arg IV	$+0.30\sin(2\omega-2q+2u)$		

Display 6.2: *Design of new lunar tables*, transcription of fol. 17r in Cod. $\mu^{\sharp}_{15}$, alias `geer`. The small numbers 1 and 2 in the left margin and the dots after q in Arg. IV are as in Mayer's original. The illegible coefficient in eqn. VII should read $2'10''$.

tational scheme with multiple steps.[38] Unlike what the numbering suggests, their first equAtion depends on ς; the second step is an equAtion of centre and evection, and the third step is variation. Thus, the computations show a procedure identical to the `zwin` version which Mayer wrote to Euler on the 7th of January, 1753, and to the `kil` tables which were published in the spring of that year.[39]

In short, the manuscript Cod. $\mu^{\sharp}_{15}$ contains a list of the coefficients of Clairaut's lunar theory, compared to coefficients of Mayer's own theory at that time (1752), as well as to their fitted counterparts of `gors-zand`, and also to coefficients extracted from theories of Euler and ultimately (perhaps added a while later) of Newton-Flamsteed-Lemonnier. This is followed first by tables based on a kind of mediated coefficients, then by a sketch of the new design that adopted Newton's multistepped scheme, and finally by tables of a multistepped nature, which can be regarded as predecessors of the `kil` tables.

This may be interpreted as follows. Prior to 1752, Mayer had a lunar theory, and he had fitted the coefficients to observations. Not being satisfied with the result, he discussed various aspects of lunar theory with Euler, including a variable eccentricity and the uneven motion of the apsides. Euler kindly transmitted the coefficients of Clairaut. Mayer looked closely at the results of his fellow mathematicians and mingled some of it with his own coefficients, but soon he turned to a Newtonian-like scheme characterized by multiple steps, annual equAtions of node and apogee, and an analytical equivalent of Horrocks's variable ellipse. This marked the start of a new branch of development.

6.8 ACCURACY OF THEORIES COMPARED

In order to assess if Mayer reached an improved accuracy by adopting the Newtonian steps, I conducted the following numerical experiments. I loaded several sets of (Mayer's and Lemonnier's) coefficients into a computer program. Each set corresponded to a different version of tables; and with each set the program computed 1000 lunar longitudes at three-day intervals starting Jan. 1, 1740, covering a little less than half a Saros. These longitudes were subtracted from ones obtained by a modern theory, and the standard deviation of the differences was taken. The results are listed in display 6.3. All the non-modern lunar longitudes were computed using a modern solar theory[40], and all computations were conducted with one and

38 Those calculations on pp. 1–29, 74, 90, 91 apparently used equAtions much like the ones at hand, in a context of Mayer developing his tables. His work included improvement of the mean motions, from those of the first table set mentioned on page 102 above, to those of the published `kil` tables. Mayer compared the outcome of his table computations with observations of James Bradley that Euler had sent him in the fall or winter of 1752 [Forbes, 1971a].

39 In fact, it appears that the development went from the `geer` *Entwurf* via `gat` and `put` to `zwin`, which is almost identical to its successor `kil`. Improvements of `put` and `kil` are discussed in chapter 8.

40 [Meeus, 1998, pp. 163–5].

alias	`zand`	`grond`	`geer`	`gat`	`zwin`	`kil`	`rede`	Lemonnier
σ [$''$]	395	404	96	62	45	43	30	111

Display 6.3: Standard deviations of the longitude terms in some table versions, computed for $n = 1000$, starting date 1740 Jan. 1, step size 3 days, Xephem computer program taken as modern reference. Computations done with the same (`zwin`) lunar mean motions and with modern solar mean motions. `Geer` is the *Entwurf* version.

the same version of Mayer's lunar mean motion parameters. Therefore the computations usually do not reproduce the same lunar longitudes as Mayer would have obtained himself. This is perfectly reasonable because our goal here is to get an impression of the error distribution of positions generated from various of Mayer's periodic equAtions. As it turns out, the adopted mean-motion practice has an effect primarily on the mean of the computed differences, and hardly on their standard deviations. By using the same solar theory and lunar mean motions with various equAtion versions, we concentrate on the quality of those equAtions.

Kollerstrom reported that Newton's lunar theory, with the sign error in the sixth equAtion corrected, had a standard deviation of $\sigma = 1.88' \approx 113''$ on 40 samples at 4-day intervals after 1681.0 using his full computer implementation of *NTM*. I found $\sigma = 123''$ using a somewhat different implementation that had the variable eccentricity and apsidal line represented by trigonometric quantities obtained as in section 6.5. I could discern that small variations to the mean motion parameters had an impact on the pattern of the position errors, but not on the standard deviation of the errors. Kollerstrom reported $\sigma = 1.9' = 114''$ for Lemonnier's tables, which agrees very well with my determination of $111''$. These correspondences provide some sense of feasibility of my approach. In various other sources, the accuracy of *NTM* has been quoted as anything ranging from $2'$–$3'$ (by Halley) to $8'$–$10'$ (by Flamsteed) but usually it was not exactly specified to what these numbers referred.[41]

Clairaut, in his lunar theory, included a list of 100 observations of the moon compared to positions predicted by his own tables, which also had been slightly adjusted to observations.[42] Interestingly, their standard deviation comes out at $110''$, suggesting that Clairaut's tables were no better than Lemonnier's. However, this conclusion must be treated with considerable care, because the data sets are completely incomparable. A test of the tables that was included in the *Connoissance des Temps* for the year 1783, suggests significantly smaller standard deviations.[43]

Display 6.3 shows that Mayer's `grond` version, being presumably an attempt to improve `zand` by mixing in his peer's results, did not meet its objective. But Mayer attained a dramatic fourfold increase in performance as soon as he adapted

41 [Baily, 1835, p. 695]. Kollerstrom's numerical results are taken from [Kollerstrom, 2000, pp. 143–4, 227].

42 [Clairaut, 1752b, p. 91].

43 [Lémery, 1780]; part of the difference can be explained by the circumstance that Lémery performed the test with the—fitted—tables of Clairaut's later (1765) edition of his theory.

the Newton/Lemonnier theory. We can imagine that Mayer felt he had discovered something to stick to, even when he lacked our modern statistical concepts.

When his `kil` tables went to the press less than a year later, he had again doubled the accuracy. In the decade between the 1752 `zand` variant and the final `rede` tables, Mayer gained a factor 13 in the accuracy of his equAtions, measured by their standard deviations. His changes in mean motion parameters are not taken into account here.

6.9 THE PREFACE TO THE KIL TABLES OF 1753

Because I have conjectured that the `kil` tables published in the Göttingen *Commentarii*[44] developed out of Mayer's embracing of *NTM*, it will be interesting to hold Mayer's own comments to those tables up against that conjecture, in particular with regard to their kinematic vs. dynamical background.

In his introduction to those tables, Mayer asserted that he had deduced the inequAlities out of 'that most famous theory of the great Newton', which Euler had first reduced to 'analytical equations', and which Mayer had himself—after several fruitless attempts—solved 'by a singular and sufficiently elegant method', although it would be too lengthy for him to explain how. Instead, he elaborated on the cause of the equAtions.[45]

From the vantage point of our current conjecture we begin to wonder *which* of Newton's 'theories' he addressed: the theory of gravitation, or *NTM*? At first sight, his reference to Euler is with regard to the casting of Newtonian physics in the language of differential equations. But couldn't Mayer have had Euler's codification of trigonometric functions in mind, as a prerequisite to translate *NTM* into that formalism? Could his 'singular and elegant method' refer to the translation of the Horrocksian mechanism into the language of trigonometric functions? If so, that could explain why Mayer was silent about his method, for Newton's *NTM* might have been regarded as old-fashioned compared to the analytical advancement of

44 The tables were published, with an introduction, as [Mayer, 1753b]. In the *Gentleman's Magazine* for August 1754 appeared an almost littoral translation into English of their preface, which Forbes in turn included in [Forbes, 1980, pp. 143–146].

45 'Yet I have deduced these tables, so far as the inequAlities of motion are concerned, from that most famous theory of the great Newton; which the celebrated Mr. Euler has firstly reduced most elegantly to general analytical equations. In solving these equations, after trying other ways in vain, I have used a particular and quite elegant method, but to exhibit it here would take too long. Therefore I have resolved to disclose only those things that make it possible to see through the origin and causes of the inequAlities presented in the tables, so far as one can without calculation' (*'Deduxi autem has tabulas, quoad inaequalitates motuum, ex famosissima illa magni Newtoni theoria; quam Vir Celeberrimus* EULERUS *primus ad aequationes analyticas generales elegantissime reduxit. Usus sum in hisce aequationibus resolvendis post frustra tentatas alias vias methodo singulari satisque concinna, sed quam hic exponere nimis longum foret. Quapropter ea tantum indicare decrevi, quae ad originem causasque inaequalitatum in tabulis exhibitarum, quantum quidem sine calculo licet, perspiciendas facere possunt'*) [Mayer, 1753b]. D'Alembert criticized that it was not Euler, but he himself and Clairaut, who had first produced analytical lunar theories [d'Alembert, 1756, I p. 252].

Euler and Clairaut. But Mayer's remark of finally attaining a solution after fruitless attempts seems to refer rather to gravitation and differential equations again. His message to the reader is of having deduced the tables from the Newtonian theory of gravitation. Yet, the *NTM*-like structure of the tables and the inadequate state of his theory at that time suggest otherwise.

Instead of supplying the reader with an account or even a sketch of his theory, Mayer chose rather to construe the nature of the various equAtions. After he had expounded both the equAtion of centre and the evection, he related these two equAtions to the variable eccentricity and apse movement as explained by Newton:

> By those XIth and XIIth equAtions [i.e., the equAtion of centre and evection], the same inequAlities are saved that Newton and those who have closely followed him have explained by a variable eccentricity of the lunar orbit and an unequal motion of the apsides. And although that method [of Newton] answers exactly to the theory, as indeed I can demonstrate, yet it is somewhat more difficult and almost useless for table calculation. The astronomy of the moon owes therefore much to the celebrated Euler. He has been the first who elegantly put a constant equAtion of centre linked with what we have called the evection, in the place of a variable eccentricity, and in this way he has furnished the theory of the moon, otherwise extremely intricate, with distinguished profit.[46]

Thus, Mayer showed in this text that he was well acquainted with *NTM*. He said that its variable orbit was not suitable for lunar tables, and he praised Euler for substituting a fixed orbit with a constant equAtion of centre—not a word about differential equations or forces! Strictly, the passage does *not* imply that there was no coherent theory supporting Mayer's tables. One wonders, though, in what way he saw Newton's variable eccentricity and apse as justified *exactly by theory*.

Continuing his tale, Mayer recognized that the variable distances between earth and sun, and also between moon and earth, give rise to periodic alterations of the evection and variation. For the evection, these alterations are expressed by equAtions depending on arguments $2\omega \pm \varsigma$, $2\omega - 2p$, and a part of the variation (argument 2ω). For the variation, they bring about equAtions depending on $2\omega \pm \varsigma$ and $2\omega \pm p$; the $2\omega - p$ effect being part of the evection. We have seen that some of these equAtions are implicitly present in *NTM*, in the form of coefficients varying annually with the mean anomaly of the sun. Mayer had no trouble to translate these into the form of his own tables.

In conclusion, Mayer offered very little in this preface to support his alleged dynamical theory. He carefully avoided to assert that he indeed had such a theory: his comments about it are evasive. On the other hand, the kinematics of *NTM* clearly show through.

46 *'Duabus istis aequationibus XI nempe & XII eaedem inaequalitates salvantur, quas Newtonus, & qui stricte eum secuti sunt, per variabilem eccentricitatem orbitae lunae & inaequalem apsidum motum explicuerunt. Et quamquam ista methodus theoriae, quod equidem demontrare possum, exacte respondet, paulo tamen est difficilior & ad calculum tabularem fere inepta. Multum igitur debet astronomia lunaris Cel[ebri]* EVLERO. *Is enim primus fuit, qui constantem aequationem centri iunctam cum ea quam evectionem diximus, loco eccentricitatis variabilis concinne substituit, eoque theoriam lunae alias maxime intricatam insigni compendio ornavit'* [Mayer, 1753b, p. 385]; also see [Forbes, 1980, p. 143].

6.10 CONCLUSION

I have shown that Mayer had a strong interest in *NTM* at a time when his own lunar theory was still very imperfect and incomplete. He compared his own results (i.e., `gors`) with those of others: first with Clairaut and Euler, leading to the soon-to-be-abandoned `griend`/`grond` variety of tables; and then with Newton's 'theory', which inspired him to the `geer` version modelled after Newton's theory. Several folios in Cod. $\mu_{28}^{\sharp}$ attest Mayer's engagement with Newton's lunar theory and his ability to translate its equAtions into the analytical language of trigonometric quantities. Although the dating of those folios is uncertain, their results are largely in accord with the list in Cod. $\mu_{15}^{\sharp}$, fol. 9r, in the column under the heading *New. Flamst.* There Mayer started a line of development characterized by the multistep scheme of application, with the middle step equivalent to the Horrocksian kinematics. Manuscript Cod. $\mu_{15}^{\sharp}$ in combination with the Mayer-Euler correspondence show how Mayer embraced the essence of the computational scheme of Newton's lunar theory during 1752. Mayer's remarks to Euler in a letter of January 1752, quoted above on page 101, presage this development.

Mayer's change of strategy seems to be made for pragmatic rather than theoretical reasons. It happened when his results remained inadequate, when he was looking around for inspiration, and when he had shared with Euler his intentions to employ a variable eccentricity as in Newton's theory. The standard deviations listed in display 6.3 show that the new plan, embodied in the *Entwurf neuer ☽ Tafeln*, brought about a dramatic fourfold increase in accuracy. The reason for the surprising increase in accuracy after adoption of Newton's scheme should be the subject of further research.

I am not aware of any position computations that immediately showed to Mayer the success of his new scheme, neither have I seen indications that his change was theoretically motivated. In the next chapter I will show that all the multistepped computational schemes can be transformed into a single-stepped scheme; this implies that—as far as accuracy is concerned—there is no real advantage of any scheme over any other. In other words, the multistepped scheme is not really necessary, and eventually we have to address the question why Mayer adhered to it once he had the technical tools to get rid of it.

Mayer introduced the multistepped procedure in this stage of his work as a consequence of the assimilation of Lemonnier's lunar tables, which were rooted in Flamsteed's rendering of *NTM*, while *NTM* in turn incorporated Horrocks's variable ellipse. Lemonnier's tables necessitated the user to perform quite involved additional computations, associated with the variable eccentricity and the seasonal fluctuations of some equAtions. In contrast, Mayer's tables were much more straightforward to apply. In his preface to the `kil` tables, Mayer credited Euler for providing the means to make the transition. We must understand his allusions to Euler not in connection with a dynamical lunar theory, but rather in connection with the changing perception of trigonometry which the latter had brought about. It is hard to imagine that the computations in section 6.5 would have been performed if every sine and cosine were regarded as half a chord in a circle. Among the advances in celestial mechanics taking place in the 18th century, the changing perception of trigonometry may well have been as important a development as the advent of the differential calculus.

The new *Entwurf* soon developed into the `kil` tables, printed in 1753. Mayer was so proud of the latter that he boasted to Euler:

> So much is certain, that the tables give the Moon's position as accurately as could hitherto have been obtained through observations, and that therefore no greater reliability in these can be supplied or hoped for until more diligence is first of all applied to the method of observing.[47]

The accuracy of `kil` was widely recognized and highly appreciated; the tables were even accurate enough to bring the application of the lunar distance method of longitude determination within reach. Thus, Mayer was encouraged to enter the quest for the Longitude Prize. Therefore these tables play a key role in Mayer's work on lunar motion. In their introductory comments Mayer showed that he had a firm understanding of the physical meaning of each of its equAtions. Yet, contrasting with Mayer's claim, in the light of their *NTM*-like way of application it is unlikely that the `kil` tables of 1753 were backed by a coherent lunar theory on a dynamical basis. There may have been partial results derived from such a theory, e.g., his new equAtions of arguments $p - \varsigma$ and $2\omega - p - \varsigma$. Such results may perhaps provide the reason why Mayer was able to quickly improve upon Lemonnier's tables.

At least one contemporary astronomer might possibly have recognized the link between Mayer's `kil` tables and the Newton/Lemonnier theory, had he been searching for it. This was d'Alembert, who went to great lengths to compare his own lunar theory to Lemonnier's tables. He even preferred to present his own results in the format of those tables. `Kil` reached d'Alembert while the last pages of his own theory were being printed, and he had just enough time left to wedge in a contemptuous comment on them. He did make an effort to compare several theories including Mayer's, even recognizing that Mayer's tables came closest to his favourite tables of Lemonnier; but apparently he did not recognize that they relied on the same kinematic model. Instead, he wondered whether Mayer had adopted Euler's theory, or whether he had a theory of his own.[48]

Petrus Frisius compared the theories of Newton, Clairaut, Euler, d'Alembert and Mayer. He was able to convert the Horrocksian kinematic mechanism of Newton's *NTM* into trigonometric expressions and he also recognized that the position of evection had changed between Mayer's `kil` and `rede` tables. Yet he too was

47 Mayer to Euler, May 7, 1753, [Forbes, 1971a, pp. 65–66].

48 [d'Alembert, 1756, I pp. xxvi, 250–252]. The quantitative statements of d'Alembert do not all seem to make sense; e.g., he asserted that Mayer's evection is effectively very different from Lemonnier's, while in truth both are almost the same after conversion into comparable form. Later, d'Alembert recognized that Mayer's `kil` tables were the most accurate available and also the most practical to use, whereupon he constructed new lunar tables with a multistep format [d'Alembert, 1780, II, p. 271–312].

unable to discover the link between *NTM* and `kil`.[49] The form in which *NTM* was presented, be it in Newton's descriptive pamphlet or in Lemonnier's tables, was apparently just too different from Mayer's `kil` tables.

The final implication of all this seems to be that the influence of the differential calculus on the success of Mayer's lunar tables of 1753 is much less significant, and the impact of Newton's 1702 theory (more specifically, its implementation by Flamsteed as published by Lemonnier) much more significant, than has hitherto been assumed. The success of Mayer's 1753 lunar tables depends on a hybrid mix of these two ingredients, with a firm dash of coefficient fitting as a third component.

49 [Frisius, 1768, pp. 272–3, 357].