5. *THEORIA LUNAE*

5.1 INTRODUCTION

The subject of this chapter is *Theoria Lunae iuxta Systema Newtonianum*, a booklet published in London in 1767 by order of the Commissioners of Longitude. Its author was Tobias Mayer, and the text had been prepared for the press by Nevil Maskelyne, the Astronomer Royal.[1] We will undertake a study of Mayer's theory (*Theoria Lunae* for short) as well as the circumstances under which it was conceived, in order to re-evaluate, in the final chapter of this thesis, its place in Mayer's work, particularly its relation to his lunar tables. As usual, our main focus will be directed towards the lunar longitude, taking latitude and parallax into account only where necessary.

Theoria Lunae has been regarded, to varying degrees, as essentially Euler's lunar theory adjusted to observations.[2] Indeed there are some similarities between the two, at least when Euler's theory is taken to mean his *Theoria Motus Lunae* of 1753.[3] A comparison of the two theories, however, shows that the similarity goes only as far as the construction of the differential equations of the moon's motion and the application of trigonometric series to find a solution.[4] The manner in which the solutions are reached is quite different in the two theories.

There are also specific points where Clairaut's essay *Théorie de la Lune*[5] may have been a source of inspiration to Mayer. Clairaut's theory won the prize contest of the Academy of Saint Petersburg of 1750, which posed to the participants the problem whether the inequAlities in lunar motion could be explained by Newton's theory of gravitation. Euler, who as an adjudicator was exempted from participation, drew inspiration from Clairaut's contribution for his own theory, and Mayer eventually became acquainted with both.

1 [Mayer, 1767]. Mayer's manuscript is in RGO 4/108 (see appendix of consulted manuscripts).

2 Examples are: Dictionary of Scientific Biography, Euler lemma, p. 481; [Lalande, 1764, 2nd ed., Vol. II, p. 224]. [Moulton, 1902, p. 364]; [Pannekoek, 1951, p. 251]; [Sadler, 1977, p. 8]; [Cook, 1998, p. 398]; More circumspect are the viewpoints expressed in [Brown, 1896, p. 246], [Forbes and Wilson, 1995], and (less explicit) [Linton, 2004, p. 304]. Waters makes the following curious remark: '. . . only in the 1760s did it [i.e., the lunar distance method] become feasible, when it was based on Newton's theory of the Moon, published in Gregory's *Astronomiae Physicae* of 1702' [Waters, 1990, fn. 34 on p. 202]. This is unlikely to be based on research, yet in chapter 6 I argue that there is nevertheless quite some truth in his statement.

3 [Euler, 1753]. Several other lunar theories of Euler exist as well, including the tables [Euler, 1746], and [Euler, 1772] in which he employed a rotating rectangular frame of coordinates.

4 Every solution of the differential equations of the sun-earth-moon three-body problem is, of course, only approximative.

5 [Clairaut, 1752b].

Apparently, Mayer's lunar theory did not raise a tremendous interest, judging by the paucity of literature in which *Theoria Lunae* is truly examined. The French astronomer-historian Delambre, although full of admiration for Mayer, is rather short on the contents of *Theoria Lunae*, asserting that

> We will not delve any deeper into his analysis; he warns us himself that one can not see the exactness therein other than by committing to calculations much longer than those that he has made himself,[6]

after which Delambre gives an eight-line summary of Mayer's theory. Forbes, who masterly framed the coherence of Mayer's oeuvre, never ventured too far into the intricacies of lunar motion, and scratched only the surface when it comes to the more technical aspects of his lunar theory. The only in-depth review of *Theoria Lunae* that I have been able to find is Gautier's,[7] who is of the opinion, writing in 1817, that Mayer's theory is one of the most elegant and exact that have appeared. He, too, asserted that Mayer took Euler's lunar theory of 1753 as a base. Gautier referred to Clairaut as his source of information, and indeed the latter conjectured in 1765 that

> One cannot know if it is by a principle similar to mine, that that skilful Astronomer firstly has reached the simple procedure that he gives for the calculation of his elements, because he has said nothing about his Lunar theory from which he has started out, nor the manner in which he has employed it. I suspect that it is of Monsieur Euler's that he has adjusted the equAtions through observations, & of which he has made particular good use by thinking of correcting the mean place by the smallest equAtions before making use of the large ones.[8]

At that time Mayer's *Theoria Lunae* was still lying in cache, and unfortunately Clairaut did not live to see it. Had he had a chance to see it, he would almost surely have recognized certain aspects of his own lunar theory. Mayer himself, on the other hand, was keen to stress that his theory had nothing of either Clairaut's or d'Alembert's, as appears for instance from his own account:

> In fact, having worked on the lunar theory since 1749, long before I could have seen the works of those two clever geometers, I managed to obtain rather precise lunar tables, as evidenced by a letter that I have written to Mr. De L'isle around 1749 or 1750. Also the way that I have chosen to resolve the general equations of Mr. Euler is very different from those that Mr. Clairaut and Mr. d'Alembert have incidentally judged to follow. An example of this difference can be seen in my printed tables concerning the calculation of the latitude although I have disregarded there some small corrections which I then judged useless. All geometers have only explored the motion of the Nodes and the inclination of the orbit of the Moon separately. Instead, I have

6 *«Nous n'entreprendrons pas davantage l'extrait de son analyse; il nous avertit lui-même qu'on ne peut en voir l'exactitude qu'en se livrant à des calculs plus longs que ceux qu'il a faits lui-même»* [Delambre, 1827, pp. 443–4]. Jean Plana, in a notice that clearly demonstrates that he knew *Theoria Lunae* very well, warned that Delambre was overestimating Mayer [Plana, 1856].

7 [Gautier, 1817, pp. 65–73].

8 *«On ne peut pas savoir si c'est par un principe pareil au mien, que cet habile Astronome est parvenu le premier au procédé simple qu'il donne pour le calcul de ces élémens, parce qu'il n'a point dit quelle étoit la théorie de la Lune d'où il étoit parti, ni la maniere dont il l'avoit employée. J'ai lieu de croire que c'est de celle de M. Euler dont il a rectifié les équations par les observations, & dont il a tiré un parti singulier en pensant à ne faire usage des grandes équations qu'après avoir corrigé le lieu moyen par les plus petites.»* The quote is from the second, enlarged, edition of his *Théorie de la Lune* [Clairaut, 1752b, 2nd ed., p. 102].

> extracted the true latitude of the Moon directly from the theory, without requiring either the true place of the Node or the inclination. My complete calculation which I have touched up several times is explained in a manuscript that I have sent with the tables to London 3 years ago. I still possess lunar tables that were constructed before the publication of my printed tables, in which all the 22 arguments are specified by the mean motions as in those of Mr. Clairaut; but my equAtions are always additive. The many equAtions in these tables have led me to change the form, without losing anything regarding the exactness, in which I succeeded perfectly. When time will allow me to publish all my researches on the theory of the Moon, it will evidently appear that I have taken nothing from others.[9]

Mayer wrote this in 1758, at a time when the Board of Longitude had still to decide on the awards, a circumstance that may have slightly coloured his remarks. Several points that he raised in the letter are recognizable: for instance, he stressed his characteristic treatment of the latitude equation on several occasions, including *Theoria Lunae* which he had indeed written three years before and which we are about to study. The older lunar tables with their always additive equAtions must be the `kreek` version. The form change that Mayer mentioned, refers to his introduction of the multistep format, which will be further investigated in the next two chapters.

5.2 CIRCUMSTANCES OF COMPOSITION

In the preface of *Theoria Lunae*, Mayer explained that his goal was not to show that the motion of the moon can be accurately derived from Newton's law of gravitation, but rather that nothing *against* his tables could be launched from that side. He mentioned that his theoretical equAtions in *Theoria Lunae* differ hardly more than half a minute from the tabulated coefficients of his most recent lunar tables at the time (alias `wijd`), which had been fitted to observed positions of the moon.

9 *«En effet, ayant travaillé sur la theorie de la Lune des l'an 1749; longtems avant que j'aye pu voir les ouvrages de ces deux habiles geometres, je fus delors parvenu à des tables de Lune assez exactes, temoin une lettre que j'ai ecrite a M. De L'isle vers l'an 1749 ou 1750. Aussi la route que j'ai tenue pour resoudre les Equations generales de Mr. Euler est elle très differente de celles que M^s. Clairaut & D'alembert ont jugé à propos de suivre. On peut voir un example de cette difference dans mes tables imprimées au sujet du calcul de la latitude quoique j'y aye negligé quelques petites corrections, que je jugai alors inutiles. Tous les geometres ont uniquement cherché separement le mouv. des Noeuds et l'inclinaison de l'orbite de La Lune. Au lieu que j'ai directement tiré de la theorie la latitude vraye de la Lune, sans avoir besoin ni du lieu vrai du Noeud ni de l'inclinaison. Mon calcul entier au quel j'ai retouché plusieur fois est expliqué dans un ecrit que j'ai envoyé avec des tables à Londres il y a 3 ans. Je possede encore des tables de Lune construites avant la publication de mes tables imprimees, dans les quelles tous les argumens au nombre de* 22*. sont determiné par les mouvemens moyens com̃e dans celles de M^r. Clairaut; mais mes aequations sont toujours additives. Le grand nombre des aequations dans ces tables m'a determine à en changer la forme, sans perdre quelque chose du cote de l'exactitude; ce qui m'a reussi parfaitement. Lorsque le tems me permettra de publier toutes mes recherches sur la theorie de La Lune, il paroitra evidem̃ent que je n'ai rien emprunté des autres.»* Mayer to Lacaille, post scriptum, 31 Oct. 1758 [Forbes and Gapaillard, 1996, pp. 519–20]. Forbes identified the letter to De l'Isle that Mayer referred to as the one written on 14 Jan. 1751, from which we quoted earlier (see fn. 9 on p. 26).

Mayer explained that his lunar tables are more to be trusted than his lunar theory, since the latter leaves some equAtions rather inaccurately determined. Moreover, he warned that his lunar theory rests for a part on the solar theory, which still had some uncertainties.[10]

One may wonder: why these excuses, why did he write up the theory anyway? The reason has partly already been mentioned earlier: Mayer wrote the theory at the request of James Bradley, the Astronomer Royal at the time when he applied for the Longitude Prize.[11] Since Bradley would be the prime adviser to the Government, his request could not be dismissed lightly. But though this provides the reason why Mayer wrote *Theoria Lunae*, it leaves the prudent tone unexplained.

Bradley's request for a theory backing the submitted tables reached Mayer through the usual agents William Best and Johann David Michaelis in November 1754. At that time Mayer was industriously improving the accuracy of his tables by comparison to observations. The tables he was working with were modelled after `kil`, the tables printed in the spring of 1753. As we will see in chapter 6, Mayer had no firm theoretical foundation for those tables. We will now take a closer look at the sequence of events of that crucial period.

In the summer of 1753, Mayer had received Euler's *Theoria Motus Lunae*.[12] Mayer had already thanked Euler for his excellent contributions to celestial mechanics, in particular the essay on the great inequAlity of Jupiter and Saturn, which had shown Mayer how to proceed. Mayer gathered new inspiration from Euler's lunar theory, and in the winter of 1753/4 he tackled the problem of the lunar latitude, which his tables had up till then represented unsatisfactorily.[13] But because his tables for lunar longitude were still at most partially backed up by a theory, Bradley's request for their foundations may have embarrassed Mayer. A confession that his tables had developed out of Newton's lunar theory of more than half a century earlier (for indeed this was the case, as I will show in the next chapter) would hardly buttress his claim to the Longitude Prize. A sound theory had to be produced, and Mayer set out to his chore. He composed substantial portions of *Theoria Lunae* in January and February of 1755, as is revealed by the title *'Theoria ☽æ Jan & Feb 1755'* that he penned down on the cover of a draft of it.[14] Mayer was then still optimistic about his progress. He wrote to Euler on February 23d:

> I shall send to [the printer Nourse in London] the method which I have used for deriving the inequalities of the moon's motion from theory.[15]

He said that his method was simple, correct, and in excellent agreement with what he had reported in his earlier letter on the problem of the lunar latitude.

10 [Mayer, 1767, p. 50–51].

11 [Forbes, 1980, p. 168]. The request was quoted above on p. 32.

12 [Euler, 1753].

13 [Forbes, 1971a, pp. 69–72, 78, 81–83].

14 The draft design of *Theoria Lunae* is in Cod. $\mu^{\sharp}_{28}$, which holds three folders of folio papers related to lunar theory. The third of the folders is the item indicated in the text. The second folder is related to *Theoria Lunae* too, the disordered first folder consists of various older attempts at lunar theories.

15 [Forbes, 1971a, p. 96].

But apparently prospects changed soon after. Around that time Mayer's friend and former colleague from Nuremberg, Georg Moritz Lowitz, moved to Göttingen to take up a professorship in Cosmology. We can be sure that the two men discussed the progress of the project. The pressure from London was kept high but Mayer sought excuses to gain more time. The exact reasons for his delay are not known to us. In April, he wrote to Michaelis:

> What pertains to the theory of the tables, and the way in which I have derived them from the law of attraction: I will certainly take trouble in order that I will send it to you at a fitting time when it is brought in convenient order. But I reckon it must be stressed again and again, that this theory, if you perceive something properly, has nothing of any weight to confirm the excellence of the tables and the agreement with heaven itself (although it must be admitted that it has been impossible without the help of the theory to bring the lunar tables to such a degree of perfection as they now rejoice); this will be abundantly clear from that treatise in which I shall shortly expound this theory.[16]

Here, Mayer clearly expressed that the theory had been of help to perfect his tables, but no more than that. How far he had proceeded to bring it in convenient order can be inferred from the fact that in May he was checking the lengthy computations for the theoretical coefficients of the equAtions, treated below in section 5.4.9. It is plausible that the difficulty of connecting the theory with the multistepped format of the tables still loomed over the project at that time and that it was the cause of further delay.

Somehow finally Mayer found a way to fulfil the expectations, for he finished *Theoria Lunae*, and in November 1755 the manuscript arrived in the hands of Best and Bradley. Best confessed to Michaelis, through whom the manuscript had been dispatched, that

> The parcel of Prof. Mayer that you sent to me in the second letter, and which concerns the Longitude at Sea, has arrived not exactly at the right moment, because Mylord Anson is making a journey to Bath, and one is presently very engaged at the Admiralty.[17]

Not that Anson would have been able to review *Theoria Lunae*, but certainly his verdict on Mayer's claim was needed, since the Admiralty was considered as hav-

16 *'Quod ad theoriam tabularum attinet, et modum quo eas ex lege attractionis derivavi, dabo quidem operam, ut eum in convenientem ordinem redactum suo tempore mittam. Verum iterum iterumque monendum duco, eam theoriam, si quid recte sentis, nihil ponderis habere, ad confirmandum tabularum praestantiam et consensum cum ipso coelo; quamvis negari non possit, impossibile fuisse absque eius auxilio tabulas lunares ad hunc perfectionis gradum, quo nunc gaudent evehere; id quod liquebit abunde ex ipsa tractatione, qua hanc theoriam sum expositurus.'* Mayer to Michaelis, 14th April 1755, published in [Mayer, 1770, pp. 43–4]. Undoubtably the letter was part of the correspondence over the Longitude Prize. The changing temper after February is also apparent from the contemporary correspondence between Michaelis in Göttingen and his cousin William Best in London, [Michaelis, 1796]. Of the correspondence between Mayer and Bradley's assistant Gael Morris, only two letters written by Morris in May 1755 and March 1756 could be found (Göttingen, Philos. 159, fol. 26–28).

17 *„Das in Dero 2tem Schreiben mir zugesandte Päckchen von dem Hrn. Prof. Mayer, die Longitudinem Maris betreffend, ist zwar zu nicht recht gelegener Zeit angelaufen, da Mylord Anson nach Bath verreiset, und man gegenwärtig bey der Admiralität sehr geschäftig ist“,* Best to Michaelis, 13 Sep. 1754, see Cod. Michael. 320 p. 555.

ing the greatest interest in this matter. Bradley dutifully wrote a positive report soon after he received the manuscript. However, his report addressed the utility of Mayer's lunar *tables* for finding the longitude at sea; regarding the *theory* he wrote only that he had received it.[18]

The Admiralty was indeed busy embarking upon the Seven Years' War (1756–1763), therefore a decision about Mayer's (and Harrison's) contribution to the longitude problem was not reached before 1765, as has been more fully recounted in section 3.8. The theory was prepared for the press by Nevil Maskelyne and appeared in 1767, twelve years after its conception.

By that time new methods were tried in celestial mechanics, particularly by Lagrange, and Mayer's theory was already nearly obsolete. Lagrange, primarily concerned with planetary theory, advanced the study of the development of orbital elements over time. In this so-called variation of constants approach, the planet is assumed to move in an elliptic orbit, while the form and orientation of the ellipse slowly change due to the perturbing forces. Were these perturbing forces to stop suddenly, then the planet would continue in an elliptical orbit, defined by the instantaneous values of the orbital elements. Euler had pioneered the study of the variation of orbital elements, inspired by a particularly effective device in the lunar theory of Jeremiah Horrocks (which we will study extensively in the next chapter).

The prime role left for *Theoria Lunae* upon publication, when it was already twelve years old, would be to show to the world the foundations of Mayer's lunar tables; a role which it performed particularly poorly, as Lagrange pointed out to d'Alembert. He thought that Mayer's tables would lose their good reputation,

> ...if the astronomers, for whom they are destined, were able to judge the theory that serves as their foundation. It is remarkable that their author, after having found a certain number of equAtions, rejects some and changes the value of others without reason, and one must notice that he makes perpetual changes, because the equAtions of the Tables are not at all the same as the equAtions corrected by the theory.[19]

The prudent tone of Mayer's preface, alluded to at the beginning of this section, is a defense against precisely these and similar castigations. To Mayer, the ostensible accuracy of the tables mattered more than an *a posteriori* theoretical support of them.

18 The report is contained in a letter of Bradley to Clevland, Secretary of the Admiralty, dated Feb. 10 1756; printed in [Mayer, 1770, p. cix–cx] and [Rigaud, 1832, pp. 84–85].

19 *«...si les astronomes, pour qui elles sont destinées, étaient bien en état de juger de la théorie qui leur sert de fondement. Ce qu'il y a de singulier, c'est que l'auteur, après avoir trouvé un certain nombre d'équations, en rejette les unes et change la valeur des autres sans raison, et remarquez qu'il y a fait des changements continuels, car les équations des Tables sont pas tout à fait les mêmes que les équations corrigées de la théorie»* Lagrange to d'Alembert, 4 April 1771 [Lagrange, 1882, p. 196]. D'Alembert, characteristically critical of just about anyone's work, phrased his verdict as *«Il me semble que comme théorie c'est assez peu de chose»* ('According to me, this theory is not much') without even having seen it yet (p. 193 *ibid.*). Not long afterwards, Mayer's theory must have fallen into his hands, because d'Alembert commented on it in 1773, restricting himself to the differences between the theory and the fitted tables [d'Alembert, 1780, VI, pp. 43–44].

5.3 OUTLINE OF *THEORIA LUNAE*

The problem that Mayer addresses in his theory is now known as the 'main problem' of lunar motion: he considers the sun-earth-moon system of three bodies, taken to be point masses. Effects due to the oblateness of the bodies are not included, although Mayer is aware of their possible impact on the lunar orbit.[20] He assumes that the motion of the sun relative to the earth is known *a priori* with sufficient accuracy.[21] He takes account of the inclination of the lunar orbit (i.e., the angle between the lunar orbit and the ecliptic) and makes use of the small size of this angle. Not included in Mayer's lunar theory are the small perturbations caused by other planets. The theory does not account for secular acceleration either.

Now follows a brief general outline of Mayer's lunar theory, intended to provide a skeleton to the more extensive study in the next section, up to the point where Mayer introduced the multistepped format. The numbers between square brackets correspond in both sections, so that the reader may always return here to regain a lost thread.

[1] Mayer sets up a spherical coordinate system (figure 5.1) with the origin in the centre of the earth, and the plane of the ecliptic as the plane of reference.

[2] He finds expressions for the gravitational accelerations that the moon, earth, and sun exert on each other. With Newton's second law on the accelerations caused by forces, he arrives at three differential equations expressing the lunar acceleration in the radial, tangential, and axial directions.

[3] These equations have time as independent variable, for which Mayer substitutes the lunar mean motion. This eliminates the mean distance of the moon from the earth.

[4] He expresses the sun-moon distance in the coordinates of those bodies.

[5] Mayer integrates the equation for the tangential component of the acceleration once. In the simpler two-body problem, this equation has a zero right-hand side, whence it induces the area law. But in the three-body case an unevaluated integral of the gravitational accelerations remains after the integration, to which I will here provisionally refer as the 'angular momentum integral'.

Up to this point Mayer's theory is very similar to Euler's lunar theory of 1753, and partly also to his essay on the great inequAlity in the movement of Jupiter and Saturn. Then Mayer makes three more substitutions.

20 See letters Mayer to Euler, Nov. 15, 1751 and Jan. 6, 1752 [Forbes, 1971a, pp. 43, 47], and my remarks on Forbes' misinterpretation of these letters (section 3.4).

21 This is a reasonable assumption; from observations it was known that the apparent orbit of the sun around the earth is not sensibly perturbed by the moon. The largest perturbation in the Keplerian apparent orbit of the sun (larger than any of the planetary perturbations), is caused by the circumstance that the barycentre of the earth-moon system, rather than the centre of the earth, takes its place in the focus of the ellipse. But at any time the barycentre is located well below the earth's surface. Therefore, the perturbation in the apparent motion of the sun is smaller than the angle subtended by the earth's radius as seen from the sun, i.e., smaller than the solar parallax. The magnitude of the latter quantity was not well known. By the beginning of the 18th century its value was believed to be about $10''$–$12''$ (now $8.8''$), well below the threshold of observational accuracy.

[6] Two of the substitutions can already be found in Euler's theory so they were not new.

[7] The third substitution signifies a break with Euler's line of development. It introduces a new independent variable, which is somewhat related to the true anomaly in an osculating orbit. Besides, Mayer manipulates the equation with the latitude coordinate, which Euler had preferred to split into one for inclination and one for node position, into a form nearly identical to the equation for the moon's longitude, so that he can solve it later with the same technique.

[8] After these manipulations, latitude occurs still in one place outside the latitude equation. But because the inclination of the lunar orbit is only about 5°, Mayer contends to take an estimation of latitude in that place, so that he makes solution of the latitude equation an independent problem.

[9] At this stage Mayer has formulated the lunar motion problem as five ordinary differential equations. Three of these equations specify the response of the moon to gravitational accelerations. Two of these three equations are of second order and of similar structure; one equation is of first order. The remaining two equations are of the first order: one of them relates the independent variable to the lunar *true* longitude, the other relates it to the lunar *mean* longitude.

[10] Remarking that it is apparently hopeless to try to solve these equations exactly, Mayer then introduces trigonometric series with indeterminate coefficients to approximate a solution. He does not specify that the periods of the arguments of the series have rational ratios to each other, and in fact the frequencies of the terms could be incommensurable. Therefore, a periodic solution is neither intended nor implied.

[11] A good deal of rather dull work is then necessary to express every term that occurs in the differential equations as trigonometric series too.

[12] His next task is to determine the coefficients of the series. At this stage Mayer draws quite heavily on his own earlier results of mixed (theoretical as well as observational) provenance.

[13] When these coefficients have been found, he has the motion of the moon expressed as a function of an independent variable that is inappropriate for practical work. Therefore Mayer reformulates the solution in terms of the lunar mean longitude by means of a series inversion.

This solution is in the single-stepped format, while Mayer's tables employ a multistepped scheme. How Mayer accommodates the difference between the two in *Theoria Lunae* is the subject of chapter 7. Mayer also imparted that his tables had been adjusted to observations, but his methods to do so were not disclosed in *Theoria Lunae*; they are the subject of chapter 8. Mayer's evaluation of the *solar* parallax based on the adjusted coefficient of the so-called parallactic equAtion of the moon, though interesting in itself, is outside the scope of the current research.[22]

22 See, e.g., [Mayer, 1767, p. 53], [Laplace, 1802, p. 657], [Godfray, 1852, p. 79–81].

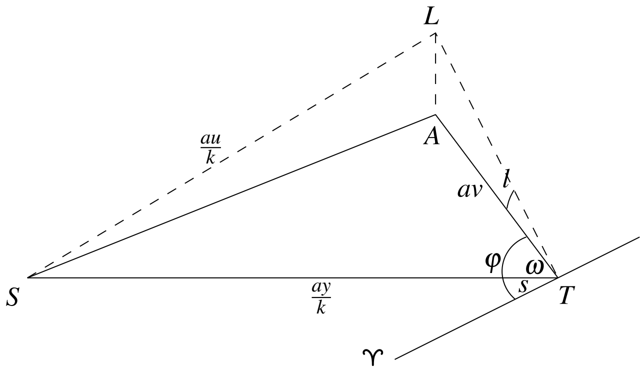

Figure 5.1: Parameters in the lunar problem

5.4 EXTENDED SURVEY OF *THEORIA LUNAE*

5.4.1 Notation

In order to write understandable mathematical formulae, one has to adhere to established conventions, but the conventions may change over time. A historian of mathematics, being an intermediary between past and present times, tries to strike a balance (depending on the specific needs of his message) between form and content of the mathematics that he passes on. My current aim is to convey the contents of Mayer's lunar theory; therefore I have deviated from Mayer's choice of symbols where confusion was likely to arise. Particularly the letters π and e now have a definite mathematical connotation which was virtually absent in the mid-eighteenth century. I have also modernised Mayer's notation in other respects. For example, the formula $dp = ex^2\,dq$ in *Theoria Lunae* is rendered here as $\frac{dq}{dp} = \frac{1}{hx^2}$, avoiding loose differentials and the symbol e. Displays 5.1 and 5.2 list the most important constants and variables of the theory. The numbers in square brackets in the following text correspond with the same numbers in the preceding outline section.

[1] We start our discussion of *Theoria Lunae* with the geometry of the sun-earth-moon system as depicted in figure 5.1. Let T be the earth, S the sun, and L the moon, all three considered as point masses. L is projected perpendicularly onto A in the plane of the ecliptic ΥST; this is in the plane of the drawing, and dotted lines are outside it. $T\Upsilon$ is a fixed direction in this plane. Mayer remarks that the plane of the ecliptic is not exactly fixed, but that its motion is negligible for his purpose.[23] Likewise, Mayer uses the *symbol* Υ, but he is careful *not* to call it the vernal equinox, which is not in a fixed direction due to precession and nutation.

Notwithstanding these details, he takes the angle $\varphi = \angle \Upsilon TA$ as the longitude

23 This is even more the case since the lunar orbit tends to remain fixed with respect to the moving ecliptic, as Laplace pointed out [Laplace, 1802, p. 374]. Laplace's proof takes several pages.

constant	value	meaning
a		mean lunar distance, to be eliminated in [4]
A	0.0545400	mean lunar eccentricity
ε	0.0168350	solar eccentricity
k	0.0031505	ratio of (mean) solar and lunar parallaxes
n	0.0748017	solar mean motion
α	0.9915478	lunar mean motion relative to the apogee
r	0.9251983	mean motion relative to the sun ($r = 1 - n$)
i	1.0040194	mean motion relative to the node
g	1.018408	see page 64
h	1.0047450	see page 74
τ	0.9999183	
f^2	0.9958270	

Display 5.1: Constants with the values that Mayer assumed. The mean motions are stated as ratios to the lunar mean motion. h, τ and f^2 are composite (see text).

of the moon; $l = \angle LTA$ its latitude; and $av = AT$ its curtate distance,[24] where the constant a denotes the mean distance of the moon from the earth and $v \approx 1$ is a variable. We may interpret the triple (v, φ, l) as the spherical coordinates of the moon relative to the earth (albeit in a somewhat modified form because the scaled curtate distance takes the place of the radius coordinate).

To represent the sun in the three-body geometry, let k be the ratio of the mean moon-earth distance a to the mean sun-earth distance. Mayer's value for k agrees to his initial estimate of $10.8''$ for the solar parallax. The mean sun-earth distance is then $\frac{a}{k}$. Let the true sun-earth distance be given by $ST = \frac{ay}{k}$, which defines y, and let the true sun-moon distance be $SL = \frac{au}{k}$, which defines u. Like v, the variable y is rather close to unity. Moreover, u is close to unity, too, because the moon is always close to the earth when compared to the diameter of the terrestrial orbit. Further, let the longitude of the sun be denoted by $s = \angle \Upsilon TS$, and the difference of the lunar and solar longitudes by $\omega = \varphi - s$.

5.4.2 Differential equations in spherical coordinates

[2] Euler had been the first to formulate Newton's second law as differential equations, equivalent to

$$\frac{d^2x}{dt^2} = \frac{F_x}{2M}, \qquad \frac{d^2y}{dt^2} = \frac{F_y}{2M}, \qquad \frac{d^2z}{dt^2} = \frac{F_z}{2M}, \tag{5.1}$$

for each of three rectangular coordinates x, y, z of a body, with forces F_x, F_y, F_z and mass of body M. Quite some work is needed to transform these equations from

24 The *curtate distance* is the distance projected perpendicularly onto the plane of the ecliptic.

variable	meaning
φ	true longitude of moon
l	true latitude of moon
v	scaled curtate distance of moon
x	$x = \frac{1}{v}$
y	scaled true earth-sun distance
u	scaled true moon-sun distance
s	true longitude of sun
σ	true anomaly of sun
ω	$\omega = \varphi - s$
q	mean longitude of moon
ψ	mean longitude of sun
X,Y,Z	forces in the right hand sides of eqns. (I),(II),(III) (p. 65)
P	$P = \int Y/x\,dq$
p	$\frac{dp}{dq} = hx^2$, $p(0) = q(0)$

Display 5.2: Variables.

rectangular to spherical coordinates. Euler had shown how to do it,[25] and Mayer felt apparently no need to repeat Euler's calculations in *Theoria Lunae*. Mayer formulated (5.1) in rectangular and spherical coordinates and referred to Euler's essay on Jupiter and Saturn, and also to his lunar theory, for the details.

Thus Mayer arrives at the differential equations of motion of the moon in spherical coordinates, subject to yet unspecified accelerations F_r parallel to AT, F_t perpendicular to AT and parallel to the ecliptic plane, and F_a perpendicular to the ecliptic. In the radial, tangential, and axial directions, these differential equations are, respectively:

$$\begin{aligned} &\frac{d^2v}{dt^2} - v\left(\frac{d\varphi}{dt}\right)^2 = -\frac{1}{2a}F_r, \\ &2\frac{dv}{dt}\frac{d\varphi}{dt} + v\frac{d^2\varphi}{dt^2} = -\frac{1}{2a}F_t, \\ &\frac{d^2(v\tan l)}{dt^2} = -\frac{1}{2a}F_a. \end{aligned} \tag{5.2}$$

No approximations have been made in arriving at these equations. Mayer followes Euler's practice in the use of the factor $\frac{1}{2}$, which makes for a simpler relation $v^2 = h$ between speed v and distance traversed h for a freely falling body.[26]

Next, Mayer specifies the accelerations, although he calls them forces. Let the masses of sun, earth, and moon be $M_\odot$, $M_\oplus$, $M_☽$. Using Newton's law of gravitation Mayer finds that:

25 [Euler, 1749b, pp. 9–12], [Euler, 1749a, p. 54], [Euler, 1753, §3–6].

26 See [Euler, 1749b, §19] (and the foreword by Max Schürer of the reprint), also [Volk, 1983, p. 346], [Wilson, 1985, p. 72]. A note in Nevil Maskelyne's papers, RGO 4/187[21], reveals that the latter was for a time somewhat confused by this practice.

along LT, the acceleration equals $\dfrac{(M_{☽}+M_{\oplus})\cos^2 l}{a^2v^2}$,

along LS, it is $\dfrac{M_{\odot}k^2}{a^2u^2}$, and

along ST, $\dfrac{M_{\odot}k^2}{a^2y^2}$.

We observe that the lunar and terrestrial masses should be added to the solar mass in the 2nd and 3d expressions, respectively. The simplification made by omitting them leads to a negligible error.

By decomposition, the forces in the right hand sides of equations (5.2) become:

$$\begin{aligned}
F_r &= \frac{(M_{☽}+M_{\oplus})\cos^3 l}{a^2v^2} + \frac{M_{\odot}k^3v}{a^2u^3} + \left(\frac{M_{\odot}k^2}{a^2y^2} - \frac{M_{\odot}k^2y}{a^2u^3}\right)\cos\omega, \\
F_t &= -\left(\frac{M_{\odot}k^2}{a^2y^2} - \frac{M_{\odot}k^2y}{a^2u^3}\right)\sin\omega, \\
F_a &= \frac{(M_{☽}+M_{\oplus})\cos^3 l\tan l}{a^2v^2} + \frac{M_{\odot}k^3v\tan l}{a^2u^3}.
\end{aligned} \tag{5.3}$$

Analogous expressions can be found in Euler's treatises.[27]

5.4.3 Two eliminations

Next Mayer rescales his time variable and he eliminates the distance u from those equations (5.2). The benefit of the time rescaling is that it eliminates the earth-moon distance a, which was not yet very accurately known in the 18th century. The rescaling is effectuated as follows.

[3] In the autonomous differential equations (5.2), time appears only in the form of the differential dt and its square. Probably inspired by Euler,[28] Mayer replaces the time by the mean ecliptic longitude of the moon. Let therefore q be the mean longitude of the moon at any moment t, and let ψ be the mean longitude of the sun at that time. Their derivatives $\frac{dq}{dt}$ and $\frac{d\psi}{dt}$ are the mean motions of moon and sun, which are constants very accurately known from observations. Kepler's third law in analytical form for the motion of the earth around the sun reads

$$\left(\frac{d\psi}{dt}\right)^2 = \frac{M_{\odot}+M_{\oplus}}{2(a/k)^3}. \tag{5.4}$$

27 E.g., [Euler, 1749a, p. 56], [Euler, 1753, §§15–20].
28 See [Euler, 1749b, §26].

Putting n for the ratio $\frac{d\psi}{dt} : \frac{dq}{dt}$ of the solar and lunar mean motions,[29] he obtained

$$\left(\frac{dq}{dt}\right)^2 = \frac{k^3(M_\odot + M_\oplus)}{2n^2a^3}, \tag{5.5}$$

which suffices to replace the differential of the time variable t by that of the new time-like variable q. When subsequently the equations (5.2) are rewritten, the factor a^3 will be seen to cancel in the right-hand-sides. The factor 2 in the numerator arises from Euler's practice referred to above.

For later reference we note that Mayer puts the constant[30] $g = \frac{n^2(M_{\leftmoon}+M_\oplus)}{k^3(M_\odot+M_\oplus)}$. We also remark that differential equation (5.5) relates q to t, but that an initial condition has not been given. Therefore (5.5) defines q only partially. A solution of the differential equation would be a function $q(t) = c_1t + c_2$ (modulo 360°), and implicitly Mayer chooses the boundary condition in such a way that $c_2 = 0$.

[4] Now we turn to the elimination of u, the scaled distance between sun and moon. In triangle ATS we have $AT = av$, $TS = \frac{ay}{k}$, and $\angle ATS = \omega$,[31] so Mayer can express SA by

$$SA^2 = a^2v^2 + \frac{a^2y^2}{k^2} - \frac{2a^2vy}{k}\cos\omega.$$

In right-angled triangle SAL we have $SL^2 = SA^2 + a^2v^2\tan^2 l$; therefore, applying $1+\tan^2 l = \frac{1}{\cos^2 l}$,

$$\left(\frac{au}{k}\right)^2 = SL^2 = \frac{a^2v^2}{\cos^2 l} + \frac{a^2y^2}{k^2} - \frac{2a^2vy}{k}\cos\omega,$$

so that

$$u = \sqrt{y^2 - 2yvk\cos\omega + \frac{v^2k^2}{\cos^2 l}}.$$

Using the binomial theorem he obtains the expansion

$$\frac{1}{u^3} = \frac{1}{y^3} + \frac{3kv\cos\omega}{y^4} + \frac{15k^2v^2(1+\cos 2\omega)}{4y^5} - \frac{3k^2v^2}{2y^5\cos^2 l}, \tag{5.6}$$

up to the second order in k. Mayer then substitutes (5.6) in equations (5.3). In the end result, Mayer keeps terms up to the first order in k. Apparently he has put $\cos l = 1$ here, equivalent to neglecting the difference of SA and SL right from the beginning (the difference is less than $1 : 4\cdot 10^7$).

29 Actually Mayer defines this as $\bar{n}$, and he defines $n^2 = \frac{M_\odot \bar{n}^2}{M_\odot + M_\oplus}$; but the mass $M_\oplus$ is very small compared to the mass $M_\odot$, so $\bar{n} \approx n$, and Mayer mixes the two at will. I will not distinguish between $\bar{n}$ and n.

30 In §8 of both the manuscript and the printed edition of *Theoria Lunae*, the square on the n is missing.

31 Mayer's text (§9) has SAT instead of $\angle ATS$.

After these substitutions and rewriting, Mayer arrives at the following equations to describe the motion of the moon:[32]

$$\begin{aligned}\frac{d^2v}{dq^2} - v\Big(\frac{d\varphi}{dq}\Big)^2 = -\frac{g\cos^3 l}{v^2} + \frac{n^2 v}{2y^3} + \frac{3n^2 v\cos 2\omega}{2y^3}& \\ + \frac{9kn^2v^2\cos\omega}{8y^4} + \frac{15kn^2v^2\cos 3\omega}{8y^4}&,\end{aligned} \tag{I}$$

$$2\frac{dv}{dq}\frac{d\varphi}{dq} + v\frac{d^2\varphi}{dq^2} = -\frac{3n^2v\sin 2\omega}{2y^3} - \frac{3kn^2v^2\sin\omega}{8y^4} - \frac{15kn^2v^2\sin 3\omega}{8y^4}, \tag{II}$$

$$\frac{d^2(v\tan l)}{dq^2} = -\Big(\frac{g\cos l^3}{v^2} + \frac{n^2v}{y^3} + \frac{3kn^2v^2\cos\omega}{y^4}\Big)\tan l. \tag{III}$$

For brevity, the right hand sides of equations (I), (II), and (III) are denoted by $-X$, $-Y$, and $-Z\tan l$, respectively. These are the gravitational accelerations up to and including the first order in k.

It is interesting to note in what respect these equations differ from the easier two-body problem. When two bodies revolve around each other, the motion is contained in a plane, and coordinates can be chosen to reflect this, obviating the need for the third differential equation for latitude. Apart from that, we may note that the forces due to the perturbing influence of the sun are present in the terms involving y. Were the sun's influence to stop, then only the terrestrial influence present through the two terms $\frac{g}{v^2}$ in equations (I) and (III) would remain in the right hand sides. In particular, the right hand side of the second equation is zero in the case of two bodies. This property induces the law of equal areas, also known as Kepler's second law, which is apparently lost in the three-body case.

The terms through which the solar perturbation is expressed all have n^2 or kn^2 as a factor. Considering that $n^2 \approx 0.005$ and $k \approx 0.003$, we note that k and n^2 are of the same order of magnitude. It is useful in lunar theory to distinguish between quantities $\approx \frac{1}{20}$ of the *first order*, those $\approx (\frac{1}{20})^2$ of the *second order*, etc.[33] Following this terminology, n is a quantity of the first order, while k and the perturbing solar accelerations ($\sim n^2$) are quantities of the second order. We see therefore that the terms involving kn^2 included in Mayer's equations are of the fourth order, comparable to the square of the perturbing acceleration.

5.4.4 An inventive substitution

Mayer says that a solution of these equations is best found by means of approximation (it is now known to be the only way). He proceeds as follows.

32 These equations occur twice in §10 of *Theoria Lunae*, with some typographical errors which do not propagate into the sequel.

33 [Laplace, 1802, pp. 356, 387], [Godfray, 1852, pp. 19, 23]. According to Gautier, this distinction is due to d'Alembert [Gautier, 1817, p. 40].

[5] He multiplies the second equation by v, and then he integrates once:

$$\frac{v^2\,d\varphi}{dq} = h - \int Y v\,dq, \tag{5.7}$$

where h denotes a constant to be determined later, absorbing the constant part of the unevaluated integral; h is twice the mean rate at which the earth-moon radius sweeps out area. Clairaut had first applied this integration in about 1745.[34]

[6] Mayer then puts

$$x = \frac{1}{v}, \qquad \text{and} \qquad P = \int \frac{Y}{x}\,dq. \tag{5.8}$$

The first of these is a rather natural substitution to make, because in the two-body problem it renders the differential equations in a linear form. The second is little more than a shorthand notation for an unevaluated integral, similar to practices of Euler and Clairaut.

[7] The next substitution, introducing Mayer's new independent variable p, is

$$\frac{dq}{dp} = \frac{1}{hx^2}. \tag{5.9}$$

This substitution is a characteristic feature of Mayer's lunar theory.[35] We note that p is not uniquely defined because an initial value is not given, just as was the case in the introduction of q; again, it is implied that $p = 0$ when $q = 0$.

With these three substitutions, equation (5.7) turns into

$$\frac{d\varphi}{dp} = 1 - \frac{P}{h}, \tag{5.10}$$

while differentiation of P yields

$$\frac{dP}{dp} = \frac{Y}{hx^3}. \tag{II$'$}$$

These two equations together replace equation (II); subsequently Mayer rewrites equations (I) and (III) also in the new variables. The first equation becomes

$$\frac{d^2x}{dp^2} + x - \frac{X}{h^2x^2} - \frac{2Px}{h} + \frac{P^2x}{h^2} = 0. \tag{I$'$}$$

34 [Wilson, 1985, p. 79].

35 It might seem more logical to represent the substitution as $\frac{dp}{dq} = hx^2$, since p is the new variable; however, this would obfuscate the further description of Mayer's theory, in particular in 5.4.8 below. Mayer wrote the three substitutions as $v = \frac{1}{x}$, $\int \frac{Y\,dq}{x} = P$, and $hx^2\,dq = dp$, adding an alternative form $dq = \frac{dp}{hx^2}$ for the latter [Mayer, 1767, §14].

He transforms the remaining third equation as follows. After applying the substitutions, he adds the transformed first equation (I′) multiplied by $\tan l$, divides the sum by x, and arrives at

$$\frac{d^2\tan l}{dp^2}+\tan l+\frac{(Z-X)\tan l}{h^2x^3}-\frac{2P\tan l}{h}+\frac{P^2\tan l}{h^2}=0. \tag{III$'$}$$

[8] The latitude l occurs not at all in (II′), and it occurs only once in (I′) where it is present in a term $-gh^{-2}\cos^3 l$ included in X. But because $|l|<5\frac{1}{2}^\circ$, this term is not very sensitive to errors in l. Therefore Mayer feels justified to use an independently obtained estimate of l in equation (I′). Consequently, he treats the system of three differential equations as two independent subsystems: one comprising equations (I′) and (II′), which will be solved first, the other with the transformed latitude equation (III′). The structure of (III′) is very much alike (I′), and he will ultimately solve it by the same techniques.

In contrast, Euler's and Clairaut's policy was to split the latitude equation into two equations of first order: one describes the motion of the node, the other the inclination of the orbit. Mayer stresses that their policy not only leads to more work, but also to tables that are less useful to astronomers. Indeed, Euler and Clairaut failed to equal the economy of method that Mayer accomplished. Although the node and inclination of the moon certainly do have their merit for astronomers, especially in the prediction of eclipses, they would more often be just clumsy intermediates in the computation of the lunar latitude. We will not be concerned with the latitude equation from here on.

[9] Now we may summarize Mayer's position as follows. He has the motion of the moon described by the three differential equations (I′), (II′), and (III′); the first and last of these are of second order and of similar form, the middle one is of first order. These equations have an independent variable that is linked to both true and mean longitude of the moon by two subsidiary equations (5.9) and (5.10). The original equations (I), (II) and (III) each contained derivatives of more than one coordinate, but the substitutions (primarily (5.9)) have separated them. Although the equations have not been decoupled, they do look less complicated this way.

5.4.5 Interpretation of p

The two subsidiary equations (5.9) and (5.10) link the new independent variable p to the true longitude φ and mean longitude q of the moon. This independent variable is a characteristic feature of Mayer's theory and it is worth to meditate over its meaning for a while. For that purpose we first turn to the two-body case again; the discussion of *Theoria Lunae* will be continued in section 5.4.7.

It is a standard result in the study of the two-body problem that, with θ for true anomaly, $v^2\frac{d\theta}{dt}=$ constant. This result is already comprised in equation (5.7). To see this, observe that (i) $\frac{dq}{dt}$ is constant (it is the mean motion of the moon); (ii) $Y=0$ in absence of perturbing masses; and (iii) in absence of perturbing masses, the moon would move in a stationary elliptic orbit, so that its true longitude φ and true anomaly θ differ only by a constant, the longitude of apogee. Still with perturbing masses absent, the variable v (or equally $\frac{1}{x}$) would be the scaled earth-moon radius in an elliptic lunar orbit. The new variable p is introduced by the differential equation (5.9), which may also be written as $v^2 \frac{dp}{dq} = h$; therefore p would be true anomaly, except perhaps for a constant phase difference.

Now take the solar perturbation into account again, and observe that v is actually the *perturbed* earth-moon distance. Clearly, then, p can no longer be identified as the true anomaly in an *un*perturbed elliptic orbit.

A somewhat (but not entirely) similar idea is incorporated in a memoir of Euler's whose title began *Nouvelle méthode de déterminer les dérangemens dans le mouvement des corps célestes* and which appeared in print in 1770, but was already presented for the Academy of Berlin in 1763. There, Euler arrived at a relation equivalent to, in our notation,

$$\frac{dq}{d\varphi} = \frac{1}{hx^2}$$

with the understanding that h is variable instead of a constant.[36] In this particular memoir, Euler was in search of ways to describe small perturbations of elliptical orbits in general. The same relation appeared again in another of his articles where he addressed the lunar theory in particular; there he discussed what is now known as the *osculating orbit*, i.e., the orbit that the moon would follow if the sun's action were suddenly stopped at any moment.[37]

This notion of the osculating orbit is closely associated with the method of the variation of constants, to which Lagrange contributed much. In an unperturbed elliptic orbit the six orbital elements are constants; if perturbations are present, it is often feasible to study the motion in an orbit that is thought of as a slowly form-changing ellipse, considering the orbital elements no longer as constants but as functions of time. When for any desired time the instantaneous elements of the orbit are known, then these may serve to compute the position of the body in the usual way for unperturbed motion.

The difference between Mayer's approach and the approach of Euler in the articles just mentioned, is primarily that Euler intended to study perturbations through the slow changes in the elements, whereas Mayer continued straightforwardly to the computation of the moon's coordinates. Also, contrary to Mayer, Euler regarded h as a variable quantity which expressed the instantaneous angular momentum in the osculating orbit, so that his φ can be termed the osculating true anomaly. Mayer did not enunciate any speculations about the nature of his independent variable, and the sequel of his theory shows that he did not consider it in connection with variation

36 Euler's expression reads $vv\,d\varphi = r\,dt$; the equivalence follows from $x = \frac{1}{v}$ and $q = \text{constant} \cdot t$ [Euler, 1770a, p. 173].

37 *«si la force perturbatrice du Soleil venoit à évanouir subitement»* [Euler, 1770b, p. 185]; two pages further on Euler substitutes $vv\,d\varphi = s\,dt$.

of constants. The reason for him to introduce p was, presumably, that it rendered the differential equations in a particularly pretty form.[38]

Returning now to a comparison of the lunar theories: Euler's and Mayer's theories run largely in parallel up to the point where Mayer introduces this new independent variable. Euler continued by rewriting his differential equations in a form which takes advantage of the relatively small deviations of lunar motion with respect to an elliptical orbit, with true anomaly in the unperturbed orbit as the independent variable.[39] From here onwards the theories of Euler and Mayer develop rather dissimilarly.

5.4.6 Intermezzo: a link to Clairaut?

A strong but not immediately obvious relation exists between Mayer's variable p and certain expressions in Clairaut's theory, as I will now demonstrate. In Clairaut's theory,[40] (5.7) takes the form

$$v^2\frac{d\varphi}{dt} = f + \int \Pi v\,dt. \tag{5.11}$$

Here, f and Π are virtually the same as Mayer's h and Y except for a constant factor relating time and mean longitude. Clairaut multiplies both sides by $\Pi v\,dt$ and integrates again to get

$$\int \Pi v^3\,d\varphi = f\int \Pi v\,dt + \tfrac{1}{2}\Big(\int \Pi v\,dt\Big)^2.$$

Completing the square on the right hand side and extracting roots, he obtains

$$f + \int \Pi v\,dt = \sqrt{f^2 + 2\int \Pi v^3\,d\varphi}.$$

He substitutes this result in (5.11) and arrives at

$$v^2\frac{d\varphi}{dt} = f\sqrt{1+2\rho}, \tag{5.12}$$

38 p is certainly not eccentric anomaly. Proof (see any textbook on elementary celestial mechanics for the formulae): assume elliptical orbit, let E be the eccentric anomaly, and ε the eccentricity. We have $E - \varepsilon\sin E = q$; differentiate with respect to q to get

$$\frac{dE}{dq}(1-\varepsilon\cos E) = 1.$$

But $1-\varepsilon\cos E = \frac{1}{x}$, the ratio of the true to the mean distance of the moon from the earth. Therefore, for the eccentric anomaly we have $\frac{dE}{dq} = x$, while p obeys $\frac{dp}{dq} = hx^2$, so $p \neq E$. [Forbes and Wilson, 1995, p. 64] assert that Mayer relates both true and mean anomaly to eccentric anomaly; and that he finally eliminates the latter. Their statement can only be interpreted to imply that p is eccentric anomaly, but I have just proved that this is false.

39 [Euler, 1753, Ch. 3].

40 [Clairaut, 1752b]; substantial portions of it also appeared in [Clairaut, 1752a].

where

$$\rho = \frac{1}{f^2} \int \Pi v^3 \, d\varphi.$$

Clairaut used this result to substitute true longitude for time, where Mayer had substituted mean longitude using (5.5). But, more interestingly, if we express (5.7) partly in Mayer's new variables and then combine with (5.10), we obtain

$$\frac{v^2 \, d\varphi}{dq} = h - P = h \frac{d\varphi}{dp}.$$

This implies that Mayer's $\frac{d\varphi}{dp}$ is proportional to Clairaut's $\sqrt{1+2\rho}$.

It is intriguing that Mayer has chosen an independent variable with an unfamiliar astronomical interpretation, which is yet so strongly related to certain aspects of Clairaut's theory. There is some evidence, inconclusive however, that Mayer conceived the substitution (5.9) quite late, perhaps as late as the beginning of 1755. He possessed Clairaut's theory already at least half a year then, so it is possible that Mayer's choice of independent variable was inspired by it.

5.4.7 Trigonometric series

[10] We now continue our investigation of Mayer's *Theoria Lunae*. From here on, the manipulation of the differential equations makes place for the handling of trigonometric series. With p as the independent variable, Mayer needs to express every term in the differential equations (particularly x, y, l, ω, and P) in this p, developing in series expansions where necessary. To this end, he asserts that x and P can be expressed by trigonometric series:

$$x = 1 - A\cos\alpha p - B\cos\beta p - C\cos\gamma p - \ldots, \tag{5.13}$$

$$\frac{dP}{dp} = a\sin\alpha p + b\sin\beta p + c\sin\gamma p + \ldots. \tag{5.14}$$

Trigonometric series were appealing because it was known from experience that the motion of the moon is nearly periodic.[41] We pause to ascertain whether the series as Mayer proposed them are indeed appropriate. The constant term 1 in (5.13) is in accord with the distance scaling as explained in section 5.4.1. It can be readily seen that Y in (II), (II′) consists of sine terms; Y and $\frac{Y}{x^3}$ are thus both odd functions. Therefore the null constant in (5.14) is in harmony with equation (II′). Besides, X in equations (I), (I′) contains cosine terms. Thus, after plugging in the above series, all terms that appear in (II′) are odd, and all terms in (I′) are even. The series that Mayer proposed are therefore appropriate. Moreover, we can see that they extend

41 Cf. fn. 24 on p. 95. Besides, note that the letter a, which originally denoted the mean distance of the moon, is reused here with a different meaning.

the two-body case (where $x = 1 - A\cos p$ and $\frac{dP}{dp} = 0$) with additional perturbational terms.

Now let us see how those series will serve him. Equation (5.14) integrated once yields a cosine series for P. Substitution of this series in equation (5.10), followed by integration, yields a sine series development for φ. This series, (5.13), and the solution of the latitude equation (III′) to be derived separately, together express the true position (longitude and latitude) of the moon for any value of p. Unfortunately, p is a rather awkward quantity for practical purposes. Therefore Mayer re-expresses the series using q as the independent variable, after expressing p in terms of q by integration of equation (5.9) and inversion of series.

The programme will unfold as follows. Section 5.4.8 shows how Mayer expresses the remaining variables y, l, ω in equations (I′) and (II′) through p. Section 5.4.9 then explains how he derives the values of of the coefficients B, C, … from equation (I′), and the values of b, c, … from (II′), and those of α, β, γ, … from both. The latter are clearly related to the periods of the perturbations. Special considerations apply to A and α, as will now be explained.

We know already (see section 5.4.3) that q is the lunar mean motion, and that $q = p$ if solar perturbation ceased: in that case (5.13) would reduce to $x = 1 - A\cos\alpha q$, describing the radius vector in an elliptic orbit. Hence it follows that A is equal to the (mean) eccentricity of the lunar orbit, which is not obtainable from the differential equations and must take a value derived from observations. Also, αq is lunar mean anomaly and $(1-\alpha)q$ is the mean motion of lunar apogee: it would be nil in absence of perturbations. Mayer considers (undoubtedly encouraged by the recent success of Clairaut in his derivation of the apogee motion) that if his theory yields a value for α close to the value known from observations, then this is a confirmation of the Newtonian law of gravitation. Yet, as we will see, α appears in many more places in the theory. In those other places it is always multiplied by a small constant, and thus much less significant, therefore Mayer allows to save himself a lot of labour by taking an observed value for α in those cases.[42]

42 The full text, which was paraphrased above, is: 'Thus it has to be observed that the letter α is as much as unknown and indeterminate, at least in this first term; in the rest however, where it is combined with others, it will be possible to suppose its true value from observations, in order that the determination of the remaining quantities will come out so much more easy and exact. Now when, after the troubles to determine this are resolved, the same value for the letter α comes out of the first term of equation I that the observations show, then this will be evidence that the Newtonian theory of gravity, where the forces of the Sun, the Earth and the Moon are fixed proportional to the squares of the distances reciprocally, is the truth and according to observations; if not, while the calculation is nevertheless accurately performed, it will inform us that it [i.e., Newton's theory] needs a correction ('*Spectari ergo debet haec litera* α *tanquam incognita aut indeterminata, saltem in hoc primo termino; in reliquis tamen, ubi ea cum aliis [...] combinatur, licebit eius valorem verum ex observationibus supponere, eo fine, ut tanto facilior exactiorque evadat determinatio reliquarum quantitatum. Quodsi vero absoluto hoc determinandi negotio ex primo termino aequationis I. idem valor pro litera* α *prodeat, quem observationes ostendunt; indicio id erit, theoriam gravitatis Newtonianam, qua vires Solis, Terrae, atque Lunae, quadratis distantiarum reciproce statuuntur propertionales, veritati atque observationibus esse consentaneam; sin minus hoc accidat, calculo ceteroquin*

We should note some particulars about the series (5.13) and (5.14). First, Mayer accomplished a surprising efficiency in the number of terms that he explicitly wrote down. The indeterminates β and γ will prove to be necessary and sufficient to explore a myriad of trigonometric terms that arise in the sequel. With just three explicit terms, Mayer keeps his formulae manageable.[43] This is a characteristic feature of his theory. Euler, also applying trigonometric series in his lunar theory,[44] consumed alphabets at high speed: upper and lower case, primed and unprimed, Roman and Fraktur, in want of identifiers for his terms.

In the second place, regarding trigonometric series and periodicity, we remark that Mayer was very well aware that the moon's motion was almost, but not exactly, repeated after a Saros of 223 lunations. The non-periodicity of lunar motion implies that the argument parameters α, β, γ, ... do not all have rational proportions. The questions of rationality and periodicity were not considered by Mayer.

Lastly, it needs to be remarked that Mayer integrated several series term by term. In a modern theoretical context, an investigation into the validity of the integrations would be in order. But in the 18th century such was highly exceptional, and it is also neglected in many modern practical contexts. But convergence of series is not a trivial matter in the planetary and lunar theories, as small divisors may cause coefficients to blow up in the integrations, as we will see later.

5.4.8 Working out the expansions

5.4.8.1 Integrations

[11] As was explained in the previous section, Mayer has to integrate three equations: (5.14), (5.10), and (5.9), which will here be performed in that order.

Integration of (5.14) and division by h yields

$$-\frac{P}{h} = \frac{a}{\alpha h}\cos\alpha p + \frac{b}{\beta h}\cos\beta p + \frac{c}{\gamma h}\cos\gamma p + \dots.$$

There is no need for an integration constant here because the integral in equation (5.7) does not have one.

Next, integration of (5.10) yields

$$\varphi = p + \frac{a}{\alpha^2 h}\sin\alpha p + \frac{b}{\beta^2 h}\sin\beta p + \frac{c}{\gamma^2 h}\sin\gamma p + \dots. \tag{5.15}$$

An integration constant might have been expected here. Its absence from Mayer's theory implies that $\varphi = 0$ when $p = q = t = 0$. This condition is extremely desirable: a non-zero integration constant would mean that $\varphi = 0$ at an instant $p = p_0 \neq 0$, and

accurate peracto, correctione eam indigere docebit') [Mayer, 1767, p. 33].

43 This advantage was signalled by Gautier [Gautier, 1817, p. 63–71].

44 [Euler, 1753].

in that case either p_0 or the integration constant would necessarily depend on the number of periodic terms that are considered in the series approximation of (5.15).[45]

To integrate (5.9), Mayer first put

$$z = 1 - x = -A\cos\alpha p - B\cos\beta p - C\cos\gamma p - \cdots,$$

so, provided that $z < 1$ (which can be taken to be true, for we know from experience that the earth-moon distance is finite over a long period):

$$x^{-2} = (1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + 5z^4 + \cdots.$$

Termwise backsubstitution and reordering, keeping of the fourth order only the terms involving A^4, gives[46]

$$\begin{aligned} x^{-2} = {} & 1 + \tfrac{15}{8}A^4 \\ & + \tfrac{3}{2}A^2 + \left(2A + 3A^3 + 6AB^2 + 6AC^2\right)\cos\alpha p + \left(\tfrac{3}{2}A^2 + \tfrac{5}{2}A^4\right)\cos 2\alpha p \\ & + \tfrac{3}{2}B^2 + \left(2B + 3B^3 + 6BA^2 + 6BC^2\right)\cos\beta p + \tfrac{3}{2}B^2\cos 2\beta p \\ & + \tfrac{3}{2}C^2 + \left(2C + 3C^3 + 6CA^2 + 6CB^2\right)\cos\gamma p + \tfrac{3}{2}C^2\cos 2\gamma p \\ & + 3AB\cos(\alpha\pm\beta)p + 3AC\cos(\alpha\pm\gamma)p + 3BC\cos(\beta\pm\gamma)p \\ & + A^3\cos 3\alpha p + 3A^2B\cos(2\alpha\pm\beta)p + 3A^2C\cos(2\alpha\pm\gamma)p \\ & + B^3\cos 3\beta p + 3B^2A\cos(2\beta\pm\alpha)p + 3B^2C\cos(2\beta\pm\gamma)p \\ & + C^3\cos 3\gamma p + 3C^2A\cos(2\gamma\pm\alpha)p + 3C^2B\cos(2\gamma\pm\beta)p \\ & + 6ABC\cos(\alpha\pm\beta\pm\gamma)p + \tfrac{5}{8}A^4\cos 4\alpha p + \ldots. \end{aligned} \tag{5.16}$$

Mayer said that he kept the fourth order in A only as a check on the accuracy of his results, in the following sense. It is evident from experience that the lunar motion is nearly elliptical (the sun causing only small perturbations), hence A is the dominant coefficient in the series (5.13). The terms involving $A^4 \approx \frac{1}{20^4}$, which are of the fourth order, should be negligible; Mayer keeps them in order to check if this is really true. If so, then he expects that other terms are justly neglected, because A is the dominant coefficient.[47]

45 See also the earlier remarks on integration constants on pp. 64 and 66, and [Godfray, 1852, p. 30].

46 An expression such as $\cos(\alpha+\beta)p$ means $\cos((\alpha+\beta)p)$, and $\cos(\alpha\pm\beta)p$ indicates that there are two such terms: $\cos((\alpha+\beta)p)$ and $\cos((\alpha-\beta)p)$.

47 Mayer expressed this a bit further on in his tract as follows: 'In order that yet no scruples arise with this omission, we will everywhere retain the largest of those terms that are rejected. For, after this has been done, if the terms that flow on and on from this source into the longitude of the moon are found to be very small, then it will be safe to conclude from that fact, that the remaining terms too, which might have come up from the omitted ones, are of no importance (*'Ne tamen ullus scrupulus propter hanc obmissionem oriatur, retinebimus ubique maximum eorum terminorum, qui rejiciuntur. Quodsi enim, re peracta, termini, qui hinc in longitudinem Lunae redundant, minimi inveniantur, tutissime inde colligi poterit, neque reliquos, qui ex obmissis oriri potuissent, alicuius momenti esse'*) [Mayer, 1767, p. 12–13].

Expression (5.16) is useful in two ways. Mayer uses the constant part of it to fix h: considering that *on average* $\frac{dq}{dp} = 1$, it follows from (5.9) that

$$h = 1 + \tfrac{3}{2}A^2 + \tfrac{3}{2}B^2 + \tfrac{3}{2}C^2 + \tfrac{15}{8}A^4, \tag{5.17}$$

to the fourth order in A. This expression for h serves him later when he needs a numerical value for it: he will then substitute values for A, B, C, ... derived from prior results. Second, he substitutes the variable part of (5.16) in (5.9) whereafter termwise integration yields

$$\begin{aligned} q = p &+ \frac{2A+3A^3+6AB^2+6AC^2}{\alpha h}\sin\alpha p + \frac{3A^2+5A^4}{4\alpha h}\sin 2\alpha p \\ &+ \frac{2B+3B^3+6BA^2+6BC^2}{\beta h}\sin\beta p + \frac{3B^2}{4\beta h}\sin 2\beta p \\ &+ \frac{2C+3C^3+6CA^2+6CB^2}{\gamma h}\sin\gamma p + \frac{3C^2}{4\gamma h}\sin 2\gamma p \\ &+ \frac{3AB}{\alpha\pm\beta}\sin(\alpha\pm\beta)p + \frac{3AC}{\alpha\pm\gamma h}\sin(\alpha\pm\gamma)p + \frac{3BC}{\beta\pm\gamma}\sin(\beta\pm\gamma)p \\ &+ \frac{A^3}{3\alpha h}\sin 3\alpha p + \frac{3A^2B}{(2\alpha\pm\beta)h}\sin(2\alpha\pm\beta)p + \frac{3A^2C}{(2\alpha\pm\gamma)h}\sin(2\alpha\pm\gamma)p \\ &+ \frac{B^3}{3\beta h}\sin 3\beta p + \frac{3B^2A}{(2\beta\pm\alpha)h}\sin(2\beta\pm\alpha)p + \frac{3B^2C}{(2\beta\pm\gamma)h}\sin(2\beta\pm\gamma)p \\ &+ \frac{C^3}{3\gamma h}\sin 3\gamma p + \frac{3C^2A}{(2\gamma\pm\alpha)h}\sin(2\gamma\pm\alpha)p + \frac{3C^2B}{(2\gamma\pm\beta)h}\sin(2\gamma\pm\beta)p \\ &+ \frac{6ABC}{(\alpha\pm\beta\pm\gamma)h}\sin(\alpha\pm\beta\pm\gamma)p + \frac{5A^4}{32\alpha h}\sin 4\alpha p + \ldots. \end{aligned} \tag{5.18}$$

Mayer has to invert this series further on in the theory, in order to free the solution from p and express it in terms of the mean motion, q. He cannot work around the series inversion by integrating $\frac{dp}{dq} = hx^2$, because p is his independent variable and x is expressed as a function of this p (see also footnote 35).

5.4.8.2 *Evolution of terms* $\frac{1}{x^i y^j}$, $\cos(j\omega)$, $\sin(j\omega)$, *etc.*

We turn our attention to the equations (I′) and (II′), with X and Y as in the righthand sides of (I) and (II). In particular, we note those terms in them that have not yet been expressed as a function of the independent variable p. They will be expressed as such, in series, now.

Mayer develops the factors x^{-j} $(3 \leq j \leq 5)$ in the same manner as our example x^{-2} above, whereafter he simplifies the end result slightly by putting $1 + 3(A^2 + B^2 + C^2 + \ldots) \approx h^2$. In order to work to fourth order, Mayer needs to keep only the terms of second order in the indeterminates A, a, ... when expanding terms that become multiplied by the small factor n^2 (which is of second order). As a check on accuracy[48] he keeps the largest of the third-order terms.

To develop y^{-3} (and similarly y^{-4}) he proceeds as follows. He assumes (correctly) that the apparent solar motion is undisturbed by the gravitational pull of the moon, so that the apparent solar orbit around the earth is elliptical. Let σ be the true solar anomaly, and let ε be the eccentricity of the solar orbit. Then $y^{-1} = \frac{1-\varepsilon\cos\sigma}{1-\varepsilon^2} = 1+\varepsilon^2 - \varepsilon\cos\sigma + O(\varepsilon^3)$, therefore

$$y^{-3} = 1 + \tfrac{9}{2}\varepsilon^2 - 3\varepsilon\cos\sigma + \tfrac{3}{2}\varepsilon^2\cos 2\sigma + O(\varepsilon^3).$$

Approximation to the second order in ε is sufficiently accurate when working to fourth order, because all terms involving y^{-3} get multiplied by a factor n^2, which is a quantity of the second order, and $\varepsilon \approx 0.0168 \approx \frac{1}{60}$ is a quantity of the first order.

Next, Mayer expresses σ in the mean solar anomaly $\varsigma = nq$ using the solar equAtion of centre, disregarding the very slight motion of the solar apogee:

$$\sigma = nq - 2\varepsilon\sin nq + \tfrac{5}{4}\varepsilon^2\sin 2nq + O(\varepsilon^3),$$

and hence

$$\begin{aligned}\cos\sigma &= \varepsilon + \cos nq - \varepsilon\cos 2nq + O(\varepsilon^2),\\ \cos 2\sigma &= \cos 2nq + O(\varepsilon).\end{aligned}$$

Using only the largest terms of (5.18) to express the arguments of the cosine terms as functions of p, Mayer arrives at

$$\begin{aligned}y^{-3} =& 1 + \frac{3}{2}\varepsilon^2 - 3\varepsilon\cos np + \frac{9\varepsilon^2}{2}\cos 2np\\ &\mp \frac{3An\varepsilon}{\alpha}\cos(\alpha\pm n)p \mp \frac{3Bn\varepsilon}{\beta}\cos(\beta\pm n)p \mp \frac{3Cn\varepsilon}{\gamma}\cos(\gamma\pm n)p.\end{aligned}$$

By similar procedures, Mayer finds expansions first for ω, which denotes the difference of longitudes of the true moon and true sun, and then for $\sin\omega$ and other trigonometric functions of ω. These expansions are expedited by the definition of $r := 1-n$, so that $rq = q - nq$ is the difference of longitudes of the mean moon and the mean sun. For instance, $\sin\omega$ expands in

$$\begin{aligned}\sin\omega =& \sin rp + \varepsilon\sin(r+n)p - \varepsilon\sin(r-n)p\\ &\pm \frac{1}{2}\Big(\frac{a}{\alpha^2 h} - \frac{2An - 3A^3 n}{\alpha h}\Big)\sin(r\pm\alpha)p \pm \frac{3A^2 n}{8\alpha}\sin(r\pm 2\alpha)p\\ &\pm \frac{1}{2}\Big(\frac{b}{\beta^2 h} - \frac{2Bn}{\beta h}\Big)\sin(r\pm\beta)p\ \ldots.\end{aligned}$$

48 See fn. 47 and the text leading up to it.

Finally, Mayer has to face the fact that X in his equation (I′) depends crucially on the unknown lunar latitude l, through the term $\frac{g\cos^3 l}{h^2}$. There is no chance to solve the latitude equation (III′) first because it depends on x being solved. He gets out of the snag by assuming a previously obtained[49] solution for l. This brings in another period-related number i, the ratio of the lengths of the draconic and tropical months, such that iq is the distance of the mean moon from the mean ascending node.

5.4.9 Filling in the numbers

[12] We now come to the coefficients B, b, C, c, ... in the series (5.13) and (5.14). To compute their values, Mayer substitutes the expansions that have been found so far into (I′) and (II′), in order that every term therein is expressed as a function of his independent variable p. After he has collected terms with common arguments together, two lengthy formulae remain, having 122 and 68 terms respectively: one of these will provide conditions on B, C, ..., the other on b, c, ... For convenience I reproduce them here in a symbolic and much shorter form:

$$\sum_{j=1}^{122} K_j \cos\lambda_j p = 0, \tag{I$''$}$$

$$\sum_{j=1}^{68} L_j \sin\lambda_j p = 0. \tag{II$''$}$$

In this modern notation, the λ_j stand for expressions such as $r+2\alpha$ and $\beta-n$, related to the periods of the perturbations of lunar motion. They involve the known constants α, n, $r=1-n$, and i, and the indeterminates β, γ of which we are to speak shortly. The factors K_j and L_j stand for combinations of the constants h, k, n, ε, A, α, which are known from observations, and the unknowns a, B, b, β, C, c, γ. By way of illustration, the factors for $j=10$ are:

$$K_{10} = \tfrac{3}{2}n^2\big((\tfrac{2Cn}{\gamma h} - \tfrac{c}{\gamma^2 h})(1+\tfrac{3}{2}\varepsilon^2) + \tfrac{3C\tau}{2h^2}(1-\tfrac{5}{2}\varepsilon^2)\big), \tag{5.19}$$

$$L_{10} = \tfrac{3}{2}n^2\big((\tfrac{2Cn}{\gamma h} - \tfrac{c}{\gamma^2 h})(1+\tfrac{3}{2}\varepsilon^2)h^{7/3} + \tfrac{2C\tau}{h}(1-\tfrac{5}{2}\varepsilon^2)\big), \text{ and} \tag{5.20}$$

$$\lambda_{10} = 2r-\gamma,$$

where $\tau \approx 1$ is Mayer's shorthand for a specific complicated expression involving all the unknown constants.

49 '...from my earlier computations' ('*ex calculis meis prioribus*') [Mayer, 1767, p. 22]. Mayer may here be addressing computations made in the winter of 1753/4, of which he gave an extensive report in a letter to Euler on March 6, 1754 [Forbes, 1971a, p. 81–83]. His original computations of that period are in Cod. μ_4, fol. 6 onwards. These computations were clearly extended during the preparation of *Theoria Lunae*. The same manuscript contains on fol. 21 nearly the same expansion for $\cos^3 l$ as is found in *Theoria Lunae*.

j	K_j	L_j	λ_j
1	$+A(\alpha^2-(f')^2)+\frac{2a}{\alpha h}-86g$	$-a$	α
2	$+B(\beta^2-f^2)+\frac{2b}{\beta h}$	$-b$	β
3	$+C(\gamma^2-f^2)+\frac{2c}{\gamma h}$	$-c$	γ
4	$+e_5-3516g$	$+e_5h^{7/3}$	$2r$
5	$+t_1+(\frac{3}{2}A+\frac{15}{4}A^3)\frac{e_5}{h^2}+36g$	$+t_1h^{7/3}+(2A+\frac{15}{2}A^3)\frac{e_5}{h}$	$2r+\alpha$
6	$-t_1+(\frac{3}{2}A+\frac{15}{4}A^3)\frac{e_5}{h^2}-126g$	$-t_1h^{7/3}+(2A+\frac{15}{2}A^3)\frac{e_5}{h}$	$2r-\alpha$
7	$+t_2+\frac{3e_5}{2h^2}B$	$+t_2h^{7/3}+\frac{2e_5}{h}B$	$2r+\beta$
8	$-t_2+\frac{3e_5}{2h^2}B$	$-t_2h^{7/3}+\frac{2e_5}{h}B$	$2r-\beta$
9	$+t_3+\frac{3e_5}{2h^2}C$	$+t_3h^{7/3}+\frac{2e_5}{h}C$	$2r+\gamma$
10	$-t_3+\frac{3e_5}{2h^2}C$	$-t_3h^{7/3}+\frac{2e_5}{h}C$	$2r-\gamma$
11	$+e_2-e_4-72g$	$(+e_2-e_4)h^{7/3}$	$2r+n$
12	$-e_2-e_4+186g$	$(-e_2-e_4)h^{7/3}$	$2r-n$
13	$\ldots$	$\ldots$	$2r+2\alpha$

Display 5.3: The first twelve expressions for K_j, L_j and λ_j in simplified form.

Mayer is silent in his theory about how to determine the coefficients. He merely asserts that it would be a tedious and prolix job to explain, but that his method contains nothing that is not already known to anyone who is versed in the modern methods of analysis.[50] His remark is somewhat comparable to Clairaut's *Avertissement* occurring in logically the same place of his lunar theory.[51] Gautier pitied that Mayer did not describe more fully how he worked out the coefficients, and subsequently conjectured how it might have been done.[52] I have been able to glean Mayer's procedure, to be discussed next, from his manuscripts;[53] Gautier's reconstruction is generally in accord with it but considerably less detailed.

Display 5.3 and some examples will help to gain an understanding of the principle of Mayer's procedure. The display lists simplified expressions for λ_j, K_j, and L_j for $j = 1\ldots 12$ in equations (I″) and (II″). These expressions have been simplified in order to make it easier to see how they serve to determine the unknowns B, C, b, c. All the factors of lesser importance have to a large extent been gathered together in the subscripted letters e_i and t_i, which do not appear in Mayer's work. The definitions of e_i and t_i are unimportant here; to get the idea, one may compare equations (5.19) and (5.20) with their simplified forms K_{10} and L_{10} in display 5.3.

50 [Mayer, 1767, p. 37].
51 [Clairaut, 1752b, p. 53].
52 [Gautier, 1817, p. 70–72].
53 Cod. μ_4, fol. 14v onwards.

As far as possible Mayer substitutes numbers, as listed in display 5.1, for the constants occurring in (I$''$) and (II$''$). These numbers are deduced from observations. In part, they may be interpreted as the boundary conditions to the differential equations, such as the eccentricities of the orbits (A, ε) and the mean distance ratio of sun and moon (k). Additionally, Mayer uses his 'prior results' to estimate the composite quantities h, τ, f^2, e_i and t_i, which depend in part on the very coefficients B, b, etc. which are to be determined.

This practice invalidates, in principle, the independence of his theory, because he inserts data that the theory is supposed to yield. Mayer argues that the end result will be sufficiently accurate with these data put in; e.g., h, τ, and f^2 are close to unity, making their exact values less critical for the theory, since they appear as factors. Perhaps his practice could be formally justified in an iterative procedure.

Then, all quantities in display 5.3 are known numerically except B, b, β, C, c, γ; in addition α is considered known except when $j = 1$ (cf. p. 71). In particular, every λ_j which is free of β and γ is known, again exempting $\lambda_1 = \alpha$. Now look at the right hand sides of equations (I$''$) and (II$''$): they are both nil. For all practical purposes, this implies conditions on the coefficients:

$$\sum_{j\in J} K_j = 0, \qquad\qquad \sum_{j\in J} L_j = 0 \tag{5.21}$$

whenever J is a maximal set satisfying that $\lambda_i = \lambda_j$ for all $i, j \in J$. These will hereafter be referred to as the vanishing conditions.

Mayer applies these conditions as follows. He assigns to β or γ consecutively every one of the known values $2r$, $2r \pm \alpha$, $2r \pm n$, $\ldots$, in other words, precisely those values of λ_j which do not contain β and γ. For each value assigned to, say, β, there is an individual term

$$K_2 \cos \lambda_2 p = \left(B(\beta^2 - f^2) - \tfrac{2b}{\beta h}\right) \cos \beta p$$

in (I$''$) and an individual term

$$L_2 \sin \lambda_2 p = -b \sin \beta p$$

in (II$''$). The only unknowns in the expressions for K_2 and L_2 are B and b, and their values can consequently be determined with the help of the vanishing conditions (5.21). When their values are known for all consecutive assignments to β and γ, equations (5.13) and (5.15) will yield the position of the moon.

Mayer now has a price to pay for the efficiency of his notation which we signalled before, namely that there will be many different instances of β, γ and B, b, C, c. For this reason we will append subscript indices where appropriate. In his private papers Mayer sporadically employed a superscript notation with the same effect as these subscripts.

We will now illustrate the procedure with an example for the calculation of the 11th terms in (I$''$) and (II$''$). We see in display 5.3 that $\lambda_{11} = 2r + n$, therefore we assign $\beta = 2r+n$. In order to find values for $B_{(2r+n)}$ and $b_{(2r+n)}$, consider the vanishing condition

$$(K_2+K_{11})\cos(2r+n)p = 0 \quad \text{for all } p$$

With K_2 and K_{11} as in display 5.3, the required condition is that

$$B_{(2r+n)}\left((2r+n)^2 - f^2\right) + \frac{2b_{(2r+n)}}{(2r+n)h} + e_2 - e_4 - 72g = 0$$

or equivalently

$$B_{(2r+n)} = \frac{e_2 - e_4 - 72g + \frac{2b_{(2r+n)}}{(2r+n)h}}{f^2 - (2r+n)^2}.$$

The expression on the right hand side cannot be computed, because the coefficient $b_{(2r+n)}$ is not yet known. In this particular example it can be easily computed using the vanishing condition for L_{11} and L_2, but in other cases Mayer resolves the issue by filling in its value guided by previous experience. He borrowed its value probably from a lunar table coefficient which he had previously corrected to observations. In specific cases, to be dealt with below, drawing upon such previous experience provides a considerable simplification of the computations.

With the right-hand side established, a value for coefficient $B_{(2r+n)}$ rolls out. Thus the series (5.13) is required to contain a term

$$\frac{e_2 - e_4 - 72g + \frac{2b_{(2r+n)}}{(2r+n)h}}{f^2 - (2r+n)^2}\cos(2r+n)p$$

in order to cancel in (I$''$) the term $(e_2 - e_4 - 72g)\cos(2r+n)p$ produced by the gravitational accelerations.

Similarly, to find coefficients $B_{(2r-n)}$ and $b_{(2r-n)}$, the process is repeated with $\beta = 2r-n$, to yield terms that cancel against $(-e_2 - e_4 + 186g)\cos(2r-n)p$ and $(-e_2 - e_4)h^{7/3}\sin(2r-n)p$; then it is repeated again for $\beta = 2r+2\alpha$ and other vaules of β, until all terms have been canceled.

5.4.10 Odds and ends

With the basic principle of coefficient determination now mastered, we turn to some of the details. To begin with, we have used the specific choice of $\beta = 2r+n$ to cancel exactly one term in (I$''$) and one in (II$''$). Yet β occurs in many more terms in these series (such as the 7th and 8th) and we have to give due consideration to those occurances as well. For instance, this particular choice of β renders $\lambda_7 = 4r+n$, and the seventh terms with arguments $(4r+n)p$ have to be annihilated by a similar calculation as in the example above. This entails first the calculation of the coefficients K_7 and L_7 with $B = B_{(2r+n)}$ and $b = b_{(2r+n)}$, and then the calculation of $B_{(4r+n)}$ and $b_{(4r+n)}$ in K_2 and L_2 using $\beta = 4r+n$ and the vanishing condition. In most instances these contributions are luckily negligible and of no concern, because the coefficients $B_{(2r+n)}$ and $b_{(2r+n)}$ which enter the expressions for K_7 and L_7 are usually small.

However, in some instances the contribution *is* significant. This happens when β, γ are assigned values of $2r$ and $2r-\alpha$, associated with the largest perturbations of lunar motion: evection and variation, which come with large coefficients.

Those values $2r$ and $2r-\alpha$ are so important that Mayer goes once through the series (I$''$) and (II$''$) with $\beta = 2r$ and $\gamma = 2r-\alpha$ simultanuously. The single reason to include β- *and* γ-terms in the series was to compute the terms with $\lambda_j = 2r \pm \beta \pm \gamma$ or $\lambda_j = \alpha \pm \beta \pm \gamma$, which may be sizeable. They are most likely of some considerable size when $\beta = 2r$ and $\gamma = 2r-\alpha$.[54]

An other point of interest is that the assignment $\beta = 2r$ makes $\lambda_8 = 2r - \beta = 0$, and hence $\cos\lambda_8 p = 1$, $\sin\lambda_8 p = 0$, for all p. Mayer seems silently to disregard these constant terms. The rationale, as I see it, comes in three parts. In the first place, if $\lambda_j = 0$ then $K_j \sin\lambda_j p = 0$ for all p, therefore such a term has no contribution to lunar longitude. Secondly, at the same time $L_j \cos\lambda_j p = L_j$, which would give a constant contribution to the radial distance; yet this contribution must have been included in the mean distance taken from observations, therefore it is safe to disregard this term too. Thirdly, the computation of the coefficients B, b, … will yield values that may prove inaccurate, especially when the computations involve small divisors. These occur when either $\beta \approx 0$ (giving bad B and b) or $\beta^2 \approx f^2$ (giving bad B only). The former condition applies when $\beta = 2(\alpha - i) \approx -0.03$, the latter when $\beta = r+n = 1$ and $\beta = 2i - \alpha \approx 1.02$. Their coefficients are best determined from observations.[55]

To make a third remark we return to the motion of the apsidal line. Mayer computes the value of α from the factor K_1 in basically the same way as outlined in the previous section, except that the roles of A and α are reversed: A is known and α is the unknown quantity. Mayer finds $\alpha = 0.9915965$,[56] which is in excellent agreement to the observational value quoted in display 5.1. He computes a lunar apogee advance of $6'38.6''$ per day, which compares favourably to the value from observations of $6'41''$ per day. He had apparently kept the right number of higher order terms to include the notorious half of the apogee movement which had plagued Newton, Clairaut, and others so much. This might perhaps be another of the lessons that he learnt from Clairaut, although some doubt arises because Mayer

54 The arguments $2rq$ and $(2r-\alpha)q$ are 2ω and $2\omega - p$ in the standard notation of section 1.3.

55 In the paragraph devoted to the adjustment of theory to observations, Mayer remarks that 'several terms appear […] which the theory, even if treated with the utmost care, cannot furnish accurately, for reasons well known to anyone who has exercised his vigour and patience in this matter' [Mayer, 1767, p. 50]; also quoted below in fn. 2 on p. 133. This problem of small divisors was first signalled by Euler in his treatise on the great inequAlity of Jupiter and Saturn, cf. [Wilson, 1985, p. 105].

56 [Mayer, 1767, p. 41].

neither mentioned that the success is due to those higher order terms, nor did he write them down in his theory.

5.5 THE SOLUTION

[13] With the coefficients in (I′) and (II′) determined, Mayer expresses x, q, and φ as trigonometric series in the independent variable p. We may consider them as functions of p, thus: $x = x(p)$, $q = q(p)$, and $\varphi = \varphi(p)$. By an inversion of series he then expresses $p = p(q)$ as a series in q, whereupon both $x = x(p(q))$ and $\varphi = \varphi(p(q))$ may be expressed as series in q. Clairaut, in his lunar theory, explained in considerable detail how to invert series.[57] The lengthy and not very inspiring computations result in expressions of the reciprocal earth-moon distance x and of the lunar true longitude φ in terms of the lunar mean longitude q. Together with the separately obtained solution of the latitude equation, Mayer then has expressions for each of three coordinates of the moon's position. It is what we call a single-stepped solution of the problem of lunar motion, meaning that no intermediate changes are made to the arguments of the equAtions: every single term in the solution is the sine or cosine of a mean motion argument. Mayer's single-stepped solution for the longitude coordinate is reprinted in the left hand column of display 7.1 in chapter 7. In that chapter, we will see how this single-stepped solution relates to the multistepped format of the tables. In chapter 8 we turn to the fitting of coefficients to observations, which was only briefly mentioned in *Theoria Lunae*.

5.6 CONCLUDING REMARKS

Similar to his contemporaries Euler, Clairaut, and d'Alembert, Mayer developed an attack on the 'main problem' of the moon's motion from dynamical principles and in spherical coordinates. The characteristic features of Mayer's lunar theory are:

- the choice of independent variable, by which the derivatives of the coordinates are each expressed in their own differential equation;
- a way to manage the multitude of terms in the trigonometric series, by which he prevents getting impaired by them;
- one equation for latitude, instead of separate equations for node and inclination.

Mayer was probably the first to treat the latitude equation in a way similar to the treatment of the longitude equation.[58] At the same time, his deprecating the separation of latitude into node and inclination entailed a disregard of the method of variation of constants with which Euler was experimenting at the time, an approach that proved to be very successful later in the century and that is regarded as fundamental today. This makes it even more unlikely that he had an osculating orbit in mind with his choice of the osculating anomaly as independent variable.

57 There are three lemmata and a problem to that extent in [Clairaut, 1752b, p. 55–59].

58 Cf. [Gaythorpe, 1957, p. 143], [Godfray, 1852, p. iv].

In some respects, Mayer's lunar theory is not completely self-supporting. He assumed the mean motion of the apogee unknown in some places, but known (for simplicity) in other places. Likewise, he took knowledge of the lunar latitude for granted in solving the differential equation for lunar longitude. He also used previously obtained magnitudes of many equAtions when he had to find the coefficients of the series numerically. So, even before the adjustment of theory to observations, Mayer took much more from observationally obtained knowledge than the necessary minimum of six integration constants plus the ratios of masses and mean distances of sun and moon. But whenever Mayer reverted to assuming values based on observations or previous results, he made an effort to explain that an approximate value would suffice, so that his theory would not depend crucially on it.[59]

Why was Mayer's theory more accurate than others? In the introduction to *Theoria Lunae*, Mayer advertised that the coefficients in his theory differ little from those in his tables. The latter, being adjusted to fit observations, were much more accurate than those of his rivals Euler, Clairaut, and d'Alembert. Mayer's theory, up to the point where the coefficients are actually computed, was not of a higher order of accuracy than the most important contemporary theories (they were all of the second order in the lunar eccentricity), nor did Mayer include physical causes that were neglected by others. Did Mayer have better values for the fundamental orbital constants? The four constants with a fundamental role in Clairaut's theory are lunar and solar eccentricity, the ratio of lunar and solar mean motions, and the ratio of their parallaxes. Of these, only the latter differs markedly from Mayer's value: Mayer supposed the solar parallax $10.8''$, Clairaut $12''$.[60] It seems unlikely that the differences between Mayer's and Clairaut's final coefficients (and, similarly, those of other theories) are due mainly to the small differences in such fundamental constants. Probably the only reason why Mayer's theory was so much more accurate, is its greater dependence on empirical data, which went directly into the coefficients. One wonders then whether the difference between his theory and fitted tables might be so slight *because* the coefficients of the theory depend so much on empirical results.

When we compared Mayer's *Theoria Lunae* to Euler's *Theoria Motus Lunae*, we found that the two run largely in parallel up to the formulation of the differential equations and the forces. From there on the two theories start to deviate: Euler chooses true anomaly as his independent variable, Mayer chooses an independent variable which somewhat resembles osculating true anomaly. Although both start from the same basic equations and share some of the same techniques, such as the application of trigonometric series, the theories proceed differently: they deviate in the ways that the game of approximation is played. Euler considered various physical effects present in the equations individually, which may provide a better understanding of the nature of the various perturbations. Mayer had an efficient way to deal with all effects at once but he needed to feed in many more *a priori* values

59 Examples are in §§35, 39, 41, 42 of *Theoria Lunae*.
60 [Clairaut, 1752b, p. 55].

for constants. Unlike Euler, Mayer saw no reason to divide the latitude equation into separate differential equations of the first order for node and inclination. This means that Mayer did not appreciate Euler's approach that went in the direction of the method of variation of constants.

It is harder to say to what extent *Theoria Lunae* was influenced by Clairaut's *Theorie de la Lune*. Mayer had decided to model the beginning of his theory after Euler's, thereby leaving less occasion for Clairaut's influence. Any extant similarities are further obscured because Euler himself had drawn inspiration from Clairaut's work. Yet all three theories employ a different independent variable. The use of an integrating factor to integrate equation (II), and the ensuing substitution for the integral on its right hand side, were common in all, as was the application of trigonometric series. Those were the tricks of the trade. Clairaut had possibly two direct influences on Mayer: the method of computation of the coefficients, as outlined in section 5.4.9, and the inversion of series: both were treated to some extent in Clairaut's theory. Conceivably, Clairaut also inspired Mayer's choice of independent variable.