

# Termination of term rewriting by semantic labelling

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## Abstract

A new kind of transformation of term rewriting systems (TRS) is proposed, depending on a choice for a model for the TRS. The labelled TRS is obtained from the original one by labelling operation symbols, possibly creating extra copies of some rules. This construction has the remarkable property that the labelled TRS is terminating if and only if the original TRS is terminating. Although the labelled version has more operation symbols and may have more rules (sometimes infinitely many), termination is often easier to prove for the labelled TRS than for the original one. This provides a new technique for proving termination, making classical techniques like path orders and polynomial interpretations applicable even for non-simplifying TRS's. The requirement of having a model can slightly be weakened, yielding a remarkably simple termination proof of the system SUBST of [11] describing explicit substitution in  $\lambda$ -calculus.

## 1 Introduction

The functional program computing the factorial can be described as a TRS as follows:

$$\begin{aligned} fact(s(x)) &\rightarrow fact(p(s(x))) * s(x) \\ p(s(0)) &\rightarrow 0 \\ p(s(s(x))) &\rightarrow s(p(s(x))). \end{aligned}$$

Termination of this program is not difficult to see: for each recursive call of *fact* the value of the argument strictly decreases. However, if we forget about the semantics of the terms representing numbers, then proving termination of the TRS is not that easy any more. The left hand side of the first rule can be embedded in the corresponding right hand side, hence the system is not simply terminating and standard techniques like recursive path order (RPO) fail. We should like to have a technique for proving termination of a TRS making use of the semantics of the TRS. One technique doing so is semantic path order ([12, 6]). It can be seen as a generalization of RPO and is discussed in section 8.

In this paper we describe another technique: given a TRS having some semantics, we introduce a labelling of the operation symbols in the TRS depending on the semantics of their arguments. We do this in such a way that termination of the original TRS is equivalent to termination of the labelled TRS. The labelled TRS has more operation symbols than the original TRS, and often more rules, sometimes even infinitely many. The original TRS can be obtained from the labelled TRS by removing all labels and removing multiple copies of rules. Although the labelled TRS is greater in some sense than the original one, in many cases termination of the labelled version is easier to prove than termination of the original one. We propose proving termination of a TRS by proving termination of a particular labelled version as a new method. This method we call *semantic labelling*.

For instance, in the factorial system we can label every symbol ‘*fact*’ by the value of its argument. We obtain infinitely many distinct operation symbols ‘*fact<sub>i</sub>*’ instead of one symbol ‘*fact*’; the other operation symbols do not change. The labelled TRS is obtained from the original one by replacing the first rule by infinitely many rules

$$fact_{i+1}(s(x)) \rightarrow fact_i(p(s(x))) * s(x),$$

one for every natural number  $i$ . It is easy to prove termination of this infinite labelled system by RPO or by an interpretation in the naturals, hence proving termination of the original factorial system.

Globally we distinguish two ways of using this technique. In the first way we choose a model which reflects the original semantics of the TRS, as we did for the factorial example. In the second way we choose an artificial model reflecting syntactic properties that are recognized in the rewrite rules, making the technique purely syntactical. In this way we obtain termination proofs of systems like  $f(f(x)) \rightarrow f(g(f(x)))$  and  $f(0, 1, x) \rightarrow f(x, x, x)$ . This approach is closely related to typing the operation symbols and proving termination of the resulting order-sorted system as discussed in [10]. Other approaches of proving termination of non-simply terminating systems in a syntactic way can be found in [18, 17, 3, 15, 22].

The technique of semantic labelling does not restrict to plain TRS’s. In section 4 we show that the same construction and the preservation of termination behaviour also holds for term rewriting modulo equations. Further semantic labelling serves well for completion of an equational specification: if the original equations hold in the model we want to use, the same holds for all critical pairs emerging during the completion process, and all these critical pairs can be labelled and oriented using a termination order we have for labelled terms.

In section 5 we present an extension of the theory in which the requirement of having a model is weakened. In a model the left hand side of any rule has to be equal to the corresponding right hand side; in this extension the left hand side is allowed to be greater than the corresponding right hand side.

Recent applications of semantic labelling outside the scope of pure term rewriting are in process algebra ([8]), logic programming ([2]) and in explicit substitution in

$\lambda$ -calculus as described by the system SUBST. Two papers ([11, 5]) were devoted exclusively to termination of SUBST. In [21, 22] we gave a simpler proof even proving simple termination of SUBST, using the technique of distribution elimination. In section 6 we give an even more simpler proof of simple termination of SUBST using semantic labelling.

Semantic labelling does not only provide termination proofs; it can also be used for proving bounds on reduction lengths. By labelling the length of a reduction does not change. So if we have a bound on the reduction lengths in the labelled version, such a bound can be used to prove a bound for the unlabelled version. Semantic labelling also holds for other properties like confluence, in the sense that confluence of a TRS follows from confluence of its labelled version. However, we do not know examples of confluence proofs that are simplified by this observation.

In section 7 we sketch an alternative proof of our main theorem based on the characterization of termination by monotone algebras. In section 8 we compare semantic labelling with semantic path order. In section 9 we sketch how labelling leads to a generalization of Kruskal's theorem, and can be a starting point for purely syntactic RPO-like orderings having the power to prove termination of systems that are not simply terminating.

## 2 The basic theorem

Let  $\mathcal{F}$  be a set of operation symbols, each having a fixed arity  $\geq 0$ . We define an  $\mathcal{F}$ -algebra  $\mathcal{M}$  to consist of a set  $M$  (the carrier set) and for every  $f \in \mathcal{F}$  of arity  $n$  a function  $f_{\mathcal{M}} : M^n \rightarrow M$ . In the following we fix an  $\mathcal{F}$ -algebra  $\mathcal{M}$ .

Let  $\mathcal{X}$  be a set of variable symbols. For  $\sigma : \mathcal{X} \rightarrow M$  we define the term evaluation  $[\sigma] : \mathcal{T}(\mathcal{F}, \mathcal{X}) \rightarrow M$  inductively by

$$\begin{aligned} [\sigma](x) &= x^\sigma, \\ [\sigma](f(t_1, \dots, t_n)) &= f_{\mathcal{M}}([\sigma](t_1), \dots, [\sigma](t_n)) \end{aligned}$$

for  $x \in \mathcal{X}, f \in \mathcal{F}, t_1, \dots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ .

**Lemma 1** *Let  $\sigma : \mathcal{X} \rightarrow M$ , let  $\tau : \mathcal{X} \rightarrow \mathcal{T}(\mathcal{F}, \mathcal{X})$  and let  $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ . Then*

$$[\sigma](t^\tau) = [[\sigma] \circ \tau](t).$$

**Proof:** By induction on the structure of  $t$ .  $\square$

Next we introduce labelling of operation symbols: choose for every  $f \in \mathcal{F}$  a corresponding non-empty set  $S_f$  of labels. Now the new signature  $\overline{\mathcal{F}}$  is defined by

$$\overline{\mathcal{F}} = \{f_s | f \in \mathcal{F}, s \in S_f\},$$

where the arity of  $f_s$  is defined to be the arity of  $f$ . An operation symbol  $f$  is called *labelled* if  $S_f$  contains more than one element. For unlabelled  $f$  the set  $S_f$  containing only one element can be left implicit; in that case we shall often write  $f$  instead of  $f_s$ .

Choose for every  $f \in \mathcal{F}$  a map  $\pi_f : M^n \rightarrow S_f$ , where  $n$  is the arity of  $f$ . This map describes how a function symbol is labelled depending on the values of its arguments as interpreted in  $\mathcal{M}$ . For unlabelled  $f$  this function  $\pi_f$  can be left implicit. We extend the labelling of operation symbols to a labelling of terms by defining  $\text{lab} : \mathcal{T}(\mathcal{F}, \mathcal{X}) \times M^{\mathcal{X}} \rightarrow \mathcal{T}(\overline{\mathcal{F}}, \mathcal{X})$  inductively by

$$\begin{aligned} \text{lab}(x, \sigma) &= x, \\ \text{lab}(f(t_1, \dots, t_n), \sigma) &= f_{\pi_f([\sigma](t_1), \dots, [\sigma](t_n))}(\text{lab}(t_1, \sigma), \dots, \text{lab}(t_n, \sigma)) \end{aligned}$$

for  $x \in \mathcal{X}, \sigma : \mathcal{X} \rightarrow M, f \in \mathcal{F}, t_1, \dots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ . This labelling of terms satisfies the following property.

**Lemma 2** *Let  $\sigma : \mathcal{X} \rightarrow M$  and let  $\tau : \mathcal{X} \rightarrow \mathcal{T}(\mathcal{F}, \mathcal{X})$ . Define  $\overline{\tau} : \mathcal{X} \rightarrow \mathcal{T}(\overline{\mathcal{F}}, \mathcal{X})$  by  $\overline{\tau}(x) = \text{lab}(\tau(x), \sigma)$ . Then*

$$\text{lab}(t^\tau, \sigma) = \text{lab}(t, [\sigma] \circ \tau)^\overline{\tau}.$$

**Proof:** By induction on the structure of  $t$ . If  $t$  is a variable the lemma follows from the definition of  $\overline{\tau}$ . If  $t = f(t_1, \dots, t_n)$  we obtain

$$\text{lab}(t^\tau, \sigma) = \text{lab}(f(t_1^\tau, \dots, t_n^\tau), \sigma) = f_{\pi_f([\sigma](t_1^\tau), \dots, [\sigma](t_n^\tau))}(\text{lab}(t_1^\tau, \sigma), \dots, \text{lab}(t_n^\tau, \sigma))$$

and

$$\begin{aligned} \text{lab}(t, [\sigma] \circ \tau)^\overline{\tau} &= \text{lab}(f(t_1, \dots, t_n), [\sigma] \circ \tau)^\overline{\tau} = \\ &= f_{\pi_f([\sigma \circ \tau](t_1), \dots, [\sigma \circ \tau](t_n))}(\text{lab}(t_1, [\sigma] \circ \tau)^\overline{\tau}, \dots, \text{lab}(t_n, [\sigma] \circ \tau)^\overline{\tau}). \end{aligned}$$

The labels of  $f$  are equal due to lemma 1 and the arguments are equal due to the induction hypothesis. Hence both terms are equal.  $\square$

Let  $R$  be a TRS over  $\mathcal{F}$ . We say that an  $\mathcal{F}$ -algebra  $\mathcal{M}$  is a *model* for  $R$  if  $[\sigma](l) = [\sigma](r)$  for all  $\sigma : \mathcal{X} \rightarrow M$  and all rules  $l \rightarrow r$  of  $R$ . It follows from the definition of  $[\cdot]$  and lemma 1 that  $[\sigma](t) = [\sigma](t')$  in a model for  $R$  for all rewrite steps  $t \rightarrow_R t'$  and all  $\sigma : \mathcal{X} \rightarrow M$ .

Fix an  $\mathcal{F}$ -algebra  $\mathcal{M}$  together with corresponding sets  $S_f$  and functions  $\pi_f$ . For any TRS  $R$  over  $\mathcal{F}$  we define  $\overline{R}$  to be the TRS over  $\overline{\mathcal{F}}$  consisting of the rules

$$\text{lab}(l, \sigma) \rightarrow \text{lab}(r, \sigma)$$

for all  $\sigma : \mathcal{X} \rightarrow M$  and all rules  $l \rightarrow r$  of  $R$ . Note that if  $R$  and all  $S_f$  are finite, then  $\overline{R}$  is finite too. The following lemma states how reduction over  $R$  can be transformed to reduction over  $\overline{R}$ .

**Lemma 3** Let  $\mathcal{M}$  be a model for  $R$ . Let  $t, t' \in \mathcal{T}(\mathcal{F}, \mathcal{X})$  satisfy  $t \rightarrow_R t'$ . Then

$$\text{lab}(t, \sigma) \rightarrow_{\overline{R}} \text{lab}(t', \sigma)$$

for all  $\sigma : \mathcal{X} \rightarrow M$ .

**Proof:** If  $t = l^\tau$  and  $t' = r^\tau$  for some rule  $l \rightarrow r$  of  $R$  and some  $\tau : \mathcal{X} \rightarrow \mathcal{T}(\mathcal{F}, \mathcal{X})$  we obtain from lemma 2

$$\text{lab}(t, \sigma) = \text{lab}(l, [\sigma] \circ \tau)^\tau \rightarrow_{\overline{R}} \text{lab}(r, [\sigma] \circ \tau)^\tau = \text{lab}(t', \sigma),$$

since  $\text{lab}(l, [\sigma] \circ \tau) \rightarrow \text{lab}(r, [\sigma] \circ \tau)$  is a rule of  $\overline{R}$ .

Let  $t \rightarrow_R t'$  and  $\text{lab}(t, \sigma) \rightarrow_{\overline{R}} \text{lab}(t', \sigma)$ . We still have to prove that

$$\text{lab}(f(\dots, t, \dots), \sigma) \rightarrow_{\overline{R}} \text{lab}(f(\dots, t', \dots), \sigma).$$

Since  $\mathcal{M}$  is a model for  $R$  we know that  $[\sigma](t) = [\sigma](t')$ . We obtain

$$\begin{aligned} \text{lab}(f(\dots, t, \dots), \sigma) &= f_{\pi_f(\dots, [\sigma](t), \dots)}(\dots, \text{lab}(t, \sigma), \dots) \\ &= f_{\pi_f(\dots, [\sigma](t'), \dots)}(\dots, \text{lab}(t, \sigma), \dots) \\ &\rightarrow_{\overline{R}} f_{\pi_f(\dots, [\sigma](t'), \dots)}(\dots, \text{lab}(t', \sigma), \dots) \\ &= \text{lab}(f(\dots, t', \dots), \sigma). \end{aligned}$$

□

As usual, a TRS  $R$  is defined to be *terminating* if it does not admit infinite reductions

$$t_1 \rightarrow_R t_2 \rightarrow_R t_3 \rightarrow_R \dots$$

In the literature a terminating TRS is also called *strongly normalizing* or *noetherian*. Now we arrive at the main theorem of this paper.

**Theorem 4** Let  $\mathcal{M}$  be a model for a TRS  $R$  over  $\mathcal{F}$ . Choose for every  $f \in \mathcal{F}$  a non-empty set  $S_f$  of labels and a map  $\pi_f : M^n \rightarrow S_f$ , where  $n$  is the arity of  $f$ . Define  $\overline{R}$  as above. Then  $R$  is terminating if and only if  $\overline{R}$  is terminating.

**Proof:** Assume  $\overline{R}$  allows an infinite reduction. Then removing all labels yields an infinite reduction in  $R$ .

On the other hand assume  $R$  allows an infinite reduction

$$t_1 \rightarrow_R t_2 \rightarrow_R t_3 \rightarrow_R \dots$$

Choose  $\sigma : \mathcal{X} \rightarrow M$  arbitrarily. Then according to lemma 3  $\overline{R}$  allows an infinite reduction

$$\text{lab}(t_1, \sigma) \rightarrow_{\overline{R}} \text{lab}(t_2, \sigma) \rightarrow_{\overline{R}} \text{lab}(t_3, \sigma) \rightarrow_{\overline{R}} \dots$$

□

In section 7 an alternative proof of this theorem is proposed. One can wonder whether similar theorems hold for other interesting properties like confluence, weak confluence and weak normalization. Due to lemma 3 and the trivial counterpart (removing labels in an  $\overline{R}$ -reduction yields an  $R$ -reduction) it is not difficult to prove that if  $\overline{R}$  is confluent, weakly confluent or weakly normalizing, then  $R$  satisfies the same property. However, we do not know examples in which these observations are helpful for proving these properties; in the typical case the proof obligations for  $\overline{R}$  are similar or more complicated than for  $R$ .

Before giving a list of examples of termination proofs using theorem 4 we briefly discuss the notion of *simple termination*. For a set  $\mathcal{F}$  of operation symbols define  $Emb(\mathcal{F})$  to be the TRS consisting of all the rules

$$f(x_1, \dots, x_n) \rightarrow x_i$$

with  $f \in \mathcal{F}$  and  $i \in \{1, \dots, n\}$ . A TRS  $R$  over  $\mathcal{F}$  is defined to be *simply terminating* if  $R \cup Emb(\mathcal{F})$  is terminating. In the literature ([14, 21]) some other equivalent definitions appear. If  $\mathcal{F}$  is finite it is also equivalent to the notion of a *simplifying* TRS ([13]). If  $\mathcal{F}$  is infinite then it is natural to change these definitions slightly ([16]). However, for the scope of this paper it suffices to see that some terminating TRS's are *not* simply terminating using our definition, and to know that standard techniques like RPO and Knuth-Bendix order, both with status (see e.g. [19]), and polynomial interpretations, all fail for TRS's that are not simply terminating.

### 3 Examples

We start with three examples in which the (finite) model is based on syntactical observations. A typical syntactical observation is that in a rule

$$\dots f(g(\dots)) \dots \rightarrow \dots f(h(\dots)) \dots$$

the  $f$ 's can be forced to obtain distinct labels by choosing the images of  $g$  and  $h$  in the model to be distinct.

**Example 1.** The simplest example  $R$  of a terminating TRS that is not simply terminating is

$$f(f(x)) \rightarrow f(g(f(x))).$$

Intuitively termination of this system is not difficult: at every step the number of operation symbols  $f$  of which the argument is again a term with head symbol  $f$  decreases. This idea can be transformed directly to a semantic labelling: define the model  $\mathcal{M}$  with  $M = \{1, 2\}$ , and  $f_{\mathcal{M}}(x) = 2$  and  $g_{\mathcal{M}}(x) = 1$  for  $x = 1, 2$ ; note that  $\mathcal{M}$  is indeed a model since the interpretations of both the left hand side and the right hand side are always equal 2. Choose  $S_f = \{1, 2\}$  and  $\pi_f$  is the identity; choose

$g$  to be unlabelled. Then  $\overline{R}$  is

$$\begin{aligned} f_2(f_1(x)) &\rightarrow f_1(g(f_1(x))) \\ f_2(f_2(x)) &\rightarrow f_1(g(f_2(x))); \end{aligned}$$

the first rule is obtained by choosing  $\sigma(x) = 1$ , the second by choosing  $\sigma(x) = 2$ . Termination of  $\overline{R}$  is easily proved by counting the number of  $f_2$  symbols. Also recursive path order and polynomial interpretations ( $[f_1](x) = [g](x) = x$ ,  $[f_2](x) = x + 1$ ) suffice for proving termination. Using theorem 4 we conclude that the original system  $R$  is terminating too.

**Example 2.** Consider the TRS

$$f(0, 1, x) \rightarrow f(x, x, x)$$

from [20]. This system is not simply terminating. For proving termination we want to use the observation that in the left hand side the first and the second argument of  $f$  are distinct while in the right hand side they are equal. This distinction is made by choosing  $S_f = \{1, 2\}$  and  $\pi_f(x, y, z) = 1$  if  $x = y$  and  $\pi_f(x, y, z) = 2$  if  $x \neq y$ . We still need any model in which 0 and 1 are indeed distinct; a simple one is  $M = \{0, 1\}$  with  $0_{\mathcal{M}} = 0$ ,  $1_{\mathcal{M}} = 1$ , and  $f_{\mathcal{M}}(x, y, z) = 0$  for  $x, y, z = 0, 1$ . Now we obtain the labelled system  $f_2(0, 1, x) \rightarrow f_1(x, x, x)$  which is easily proved to be terminating by any standard technique.

**Example 3.** In the system

$$\begin{aligned} (x * y) * z &\rightarrow x * (y * z) \\ (x + y) * z &\rightarrow (x * z) + (y * z) \\ x * (y + f(z)) &\rightarrow g(x, z) * (y + a) \end{aligned}$$

from [6] we can force that the symbols ‘ $*$ ’ in the last rule get distinct labels by choosing the model  $\{1, 2\}$  and defining  $a_{\mathcal{M}} = 1$ ,  $f_{\mathcal{M}}(x) = 2$ ,  $\pi_*(x, y) = x +_{\mathcal{M}} y = y$ ,  $x *_{\mathcal{M}} y = 1$  for all  $x, y = 1, 2$ . The labelled system is

$$\begin{aligned} (x *_{\mathbf{1}} y) *_{\mathbf{1}} z &\rightarrow x *_{\mathbf{1}} (y *_{\mathbf{1}} z) \\ (x *_{\mathbf{1}} y) *_{\mathbf{2}} z &\rightarrow x *_{\mathbf{1}} (y *_{\mathbf{2}} z) \\ (x *_{\mathbf{2}} y) *_{\mathbf{1}} z &\rightarrow x *_{\mathbf{1}} (y *_{\mathbf{1}} z) \\ (x *_{\mathbf{2}} y) *_{\mathbf{2}} z &\rightarrow x *_{\mathbf{1}} (y *_{\mathbf{2}} z) \\ (x + y) *_{\mathbf{1}} z &\rightarrow (x *_{\mathbf{1}} z) + (y *_{\mathbf{1}} z) \\ (x + y) *_{\mathbf{2}} z &\rightarrow (x *_{\mathbf{2}} z) + (y *_{\mathbf{2}} z) \\ x *_{\mathbf{2}} (y + f(z)) &\rightarrow g(x, z) *_{\mathbf{1}} (y + a) \end{aligned}$$

and is proved terminating using RPO: give  $*_{\mathbf{1}}$  a lexicographic status, choose  $*_{\mathbf{2}}$  to be greater than all the other symbols and choose  $*_{\mathbf{1}} > +$ .

In the next examples the model corresponds to the natural semantics of the rewrite system.

**Example 4.** In the factorial system from the introduction choose  $M = \mathbb{N}$ ,  $0_{\mathcal{M}} = 0$ ,  $s_{\mathcal{M}}(x) = x+1$ ,  $p_{\mathcal{M}}(0) = 0$ , and  $p_{\mathcal{M}}(x) = x-1$  for  $x > 0$ . Further choose  $x *_{\mathcal{M}} y = x * y$  and  $fact_{\mathcal{M}}(x) = x!$ . Clearly  $\mathcal{M}$  is a model for the system; by labelling  $fact$  with the naturals and choosing  $\pi_{fact}(x) = x$  we get the labelled version

$$\begin{aligned} fact_{i+1}(s(x)) &\rightarrow fact_i(p(s(x))) * s(x) \\ p(s(0)) &\rightarrow 0 \\ p(s(s(x))) &\rightarrow s(p(s(x))) \end{aligned}$$

in which the first line stands for infinitely many rules, one for every  $i \in \mathbb{N}$ . An interpretation in  $\mathbb{N}$  proving termination is  $[0] = 0$ ,  $[s](x) = x+1$ ,  $[p](x) = 2x$ ,  $x[*]y = x + y$ ,  $[fact_i](x) = 4^i * x$ .

**Example 5.** A valid definition of the function  $max$  to compute the maximum of two natural numbers is the following: if  $x \geq y$  then  $max(x, y) = x$ , otherwise  $max(x, y) = max(y, x)$ . This definition can be transformed to the following TRS  $MAX$ :

$$\begin{aligned} max(x, y) &\rightarrow c(x, y, x \geq y) \\ x \geq 0 &\rightarrow true \\ 0 \geq s(x) &\rightarrow false \\ s(x) \geq s(y) &\rightarrow x \geq y \\ c(x, y, true) &\rightarrow x \\ c(x, y, false) &\rightarrow max(y, x). \end{aligned}$$

This system is not simply terminating since by adding the rule  $x \geq y \rightarrow x$  which is in  $Emb(\mathcal{F})$  we obtain the infinite reduction

$$\begin{aligned} max(false, false) &\rightarrow c(false, false, false \geq false) \\ &\rightarrow c(false, false, false) \rightarrow max(false, false) \rightarrow \dots \end{aligned}$$

However,  $MAX$  can be proved to be terminating by semantic labelling. As a model  $\mathcal{M}$  we choose the natural numbers in which we identify  $true$  and  $false$  by 1 and 0, respectively. More precisely:  $M = \mathbb{N}$ ,  $max_{\mathcal{M}}(x, y) = \max(x, y)$ ,  $true_{\mathcal{M}} = 1$ ,  $false_{\mathcal{M}} = 0$ ,  $0_{\mathcal{M}} = 0$ ,  $s_{\mathcal{M}}(x) = x + 1$ ,

$$c_{\mathcal{M}}(x, y, z) = \begin{cases} x & \text{if } z > 0 \\ \max(x, y) & \text{if } z = 0 \end{cases}, \quad x \geq_{\mathcal{M}} y = \begin{cases} 1 & \text{if } x \geq y \\ 0 & \text{if } x < y. \end{cases}$$

One easily checks that  $\mathcal{M}$  is indeed a model for  $MAX$ . We still have to find an appropriate labelling. The labelling will be motivated by the intuition that switches from  $c$  to  $max$  and vice versa cannot go on forever. The maximal number of switches is in the reduction

$$c(s(0), 0, false) \rightarrow max(0, s(0)) \rightarrow^+ c(0, s(0), false) \rightarrow max(s(0), 0) \rightarrow^+ c(s(0), 0, true).$$



We shall label  $max$  and  $c$  in such a way that the three occurrences of  $c$  and the two occurrences of  $max$  in this sequence get distinct labels. A possible choice is  $S_{max} = \{1, 2\}$  and  $S_c = \{1, 2, 3\}$  and

$$\pi_{max}(x, y) = \begin{cases} 1 & \text{if } x \geq y \\ 2 & \text{if } x < y \end{cases} \quad \pi_c(x, y, z) = \begin{cases} 1 & \text{if } z > 0 \\ 2 & \text{if } z = 0 \wedge x < y \\ 3 & \text{if } z = 0 \wedge x \geq y. \end{cases}$$

Now  $\overline{MAX}$  is

$$\begin{aligned} max_1(x, y) &\rightarrow c_1(x, y, x \geq y) \\ max_2(x, y) &\rightarrow c_2(x, y, x \geq y) \\ x \geq 0 &\rightarrow true \\ 0 \geq s(x) &\rightarrow false \\ s(x) \geq s(y) &\rightarrow x \geq y \\ c_1(x, y, true) &\rightarrow x \\ c_2(x, y, false) &\rightarrow max_1(y, x) \\ c_3(x, y, false) &\rightarrow max_1(y, x) \\ c_3(x, y, false) &\rightarrow max_2(y, x) \end{aligned}$$

and can be proved to be terminating by RPO using the precedence

$$c_3 > max_2 > c_2 > max_1 > c_1 > \geq > true > false.$$

## 4 Rewriting modulo equations

In this section we show how theorem 4 extends to rewriting modulo equations.

**Theorem 5** *Let  $\mathcal{M}$  be a model for a TRS  $R$  over  $\mathcal{F}$ . Choose for every  $f \in \mathcal{F}$  a non-empty set  $S_f$  of labels and a map  $\pi_f : M^n \rightarrow S_f$ , where  $n$  is the arity of  $f$ . Define  $\overline{R}$  as in section 2. Let  $\mathcal{F}_u = \{f \in \mathcal{F} \mid \#S_f = 1\}$ . Let  $\mathcal{E}$  be any set of equations over  $\mathcal{F}_u$  that hold in  $\mathcal{M}$ . Then  $R$  is terminating modulo  $\mathcal{E}$  if and only if  $\overline{R}$  is terminating modulo  $\mathcal{E}$ .*

**Proof:** Assume  $\overline{R}$  allows an infinite reduction modulo  $\mathcal{E}$ :

$$t_1 \rightarrow_{\overline{R}} t_2 \equiv_{\mathcal{E}} t_3 \rightarrow_{\overline{R}} t_4 \equiv_{\mathcal{E}} t_5 \rightarrow_{\overline{R}} t_6 \cdots.$$

Then removing all labels yields an infinite reduction in  $R$  modulo  $\mathcal{E}$ .

On the other hand assume  $R$  allows an infinite reduction modulo  $\mathcal{E}$ :

$$t_1 \rightarrow_R t_2 \equiv_{\mathcal{E}} t_3 \rightarrow_R t_4 \equiv_{\mathcal{E}} t_5 \rightarrow_R t_6 \cdots.$$

Choose  $\sigma : \mathcal{X} \rightarrow M$  arbitrarily. Similar to the proof of lemma 3 one proves that

$$\text{lab}(t, \sigma) \equiv_{\mathcal{E}} \text{lab}(t', \sigma)$$

for any  $t, t'$  satisfying  $t \equiv_{\mathcal{E}} t'$ . From this observation and lemma 3 we conclude that  $\overline{R}$  allows an infinite reduction modulo  $\mathcal{E}$ :

$$\text{lab}(t_1, \sigma) \rightarrow_{\overline{R}} \text{lab}(t_2, \sigma) \equiv_{\mathcal{E}} \text{lab}(t_3, \sigma) \rightarrow_{\overline{R}} \text{lab}(t_4, \sigma) \equiv_{\mathcal{E}} \text{lab}(t_5, \sigma) \rightarrow_{\overline{R}} \dots$$

□

In section 8 we present an application of this theorem. Note that all operation symbols in  $\mathcal{E}$  are required to be unlabelled. This restriction is essential: otherwise the theorem does not hold without introducing extra restrictions. For instance, for the system

$$(x + y) + z \rightarrow x + (y + z)$$

we can choose the model of positive integers in which  $+$  is interpreted as addition, which is commutative. If we choose  $\pi_+(x, y) = x$ , then the infinite labelled system is easily proved to be terminating modulo commutativity by the polynomial interpretation

$x[+_i]y = x + y + i$ . However, the original system is not terminating modulo commutativity.

Theorem 5 can be extended to allow  $\mathcal{E}$  to contain commutativity of labelled symbols if  $\pi_f$  is required to be symmetric for these symbols. For other equations on labelled symbols it is not clear how it can be extended.

## 5 Quasi-models

In this section we give an extension of theorem 4 in the sense that  $\mathcal{M}$  is not required to be a model for  $R$  any more. As a motivation consider the following TRS introduced in [7] for showing that completeness is not a modular property:

$$\begin{array}{ll} f(a, b, x) & \rightarrow f(x, x, x) \\ f(x, y, z) & \rightarrow c \\ a & \rightarrow c \\ b & \rightarrow c. \end{array}$$

Clearly this system is closely related to example 2 of section 3. However, it does not allow any non-trivial model since in all models any term has the same interpretation as  $c$ . So theorem 4 is not helpful for proving termination of this system; using the extension presented in this section it is easily proved.

Until now the model  $\mathcal{M}$  and label sets  $S_f$  were sets. Here we require them to be (well-founded) posets. The maps  $f_{\mathcal{M}}$  and  $\pi_f$  have to be weakly monotone in all coordinates. Until now  $\mathcal{M}$  was required to be a model for the TRS, meaning that the interpretation of a left hand side of a rule is always equal to the interpretation of the corresponding right hand side. Here  $\mathcal{M}$  is only required to be a quasi-model for the TRS, meaning that the interpretation of a left hand side of a rule is  $\geq$  the

interpretation of the corresponding right hand side. Before presenting the theorem we give some definitions and lemmas.

Let  $\mathcal{M}$  be an  $\mathcal{F}$ -algebra provided with a partial order  $\geq$  for which each algebra operation is weakly monotone in all coordinates <sup>1</sup>, more precisely: for all operation symbols  $f \in \mathcal{F}$  and all  $a_1, \dots, a_n, b_1, \dots, b_n \in M$  satisfying  $a_i \geq b_i$  for all  $i$ , we have  $f_{\mathcal{M}}(a_1, \dots, a_n) \geq f_{\mathcal{M}}(b_1, \dots, b_n)$ . For all  $f \in \mathcal{F}$  let  $S_f$  be any set, provided with a well-founded partial order  $\geq$ . For all  $f \in \mathcal{F}$  of arity  $n$  let  $\pi_f : M^n \rightarrow S_f$  any map that is weakly monotone in all coordinates. Define  $[\cdot]$ ,  $\mathbf{lab}$  and  $\overline{\mathcal{F}}$  as in section 2. Let  $R$  be a TRS over  $\mathcal{F}$ . We say that the  $\mathcal{F}$ -algebra  $\mathcal{M}$  is a *quasi-model* for  $R$  if

$$[\sigma](l) \geq [\sigma](r)$$

for all  $\sigma : \mathcal{X} \rightarrow M$  and all rules  $l \rightarrow r$  of  $R$ . As in section 2 we define  $\overline{R}$  to be the TRS over  $\overline{\mathcal{F}}$  consisting of the rules

$$\mathbf{lab}(l, \sigma) \rightarrow \mathbf{lab}(r, \sigma)$$

for all  $\sigma : \mathcal{X} \rightarrow M$  and all rules  $l \rightarrow r$  of  $R$ . Further the TRS **Decr** over  $\overline{\mathcal{F}}$  is defined to consist of the rules

$$f_s(x_1, \dots, x_n) \rightarrow f_{s'}(x_1, \dots, x_n)$$

for all  $f \in \mathcal{F}$  and all  $s, s' \in S_f$  satisfying  $s > s'$ . Here  $>$  denotes the strict part of  $\geq$ .

**Lemma 6** *Let  $\mathcal{M}$  be a quasi-model for  $R$ . Let  $t, t' \in \mathcal{T}(\mathcal{F}, \mathcal{X})$  satisfy  $t \rightarrow_R t'$ . Then  $[\sigma](t) \geq [\sigma](t')$  for all  $\sigma : \mathcal{X} \rightarrow M$ .*

**Proof:** If  $t = l^\tau$  and  $t' = r^\tau$  for some rule  $l \rightarrow r$  of  $R$  and some  $\tau : \mathcal{X} \rightarrow \mathcal{T}(\mathcal{F}, \mathcal{X})$  the assertion follows from lemma 1 and the definition of quasi-model.

Let  $t \rightarrow_R t'$  and  $[\sigma](t) \geq [\sigma](t')$ ; we still have to prove that

$$[\sigma](f(\dots, t, \dots)) \geq [\sigma](f(\dots, t', \dots))$$

for all  $f \in \mathcal{F}$  and all  $\sigma : \mathcal{X} \rightarrow M$ . This follows from the definition of  $[\cdot]$  and the fact that  $f_{\mathcal{M}}$  is weakly monotone in all coordinates.  $\square$

**Lemma 7** *Let  $\mathcal{M}$  be a quasi-model for  $R$ . Let  $t, t' \in \mathcal{T}(\mathcal{F}, \mathcal{X})$  satisfy  $t \rightarrow_R t'$ . Then for all  $\sigma : \mathcal{X} \rightarrow M$  there is a term  $u$  over  $\overline{\mathcal{F}}$  such that*

$$\mathbf{lab}(t, \sigma) \rightarrow_{\mathbf{Decr}}^* u \rightarrow_{\overline{R}} \mathbf{lab}(t', \sigma).$$

---

<sup>1</sup>It was remarked by Aart Middeldorp that this order is not necessarily well-founded

**Proof:** If  $t = l^\tau$  and  $t' = r^\tau$  for some rule  $l \rightarrow r$  of  $R$  and some  $\tau : \mathcal{X} \rightarrow \mathcal{T}(\mathcal{F}, \mathcal{X})$  we obtain from lemma 2

$$\mathbf{lab}(t, \sigma) = \mathbf{lab}(l, [\sigma] \circ \tau)^{\bar{\tau}} \rightarrow_{\bar{R}} \mathbf{lab}(r, [\sigma] \circ \tau)^{\bar{\tau}} = \mathbf{lab}(t', \sigma),$$

hence the assertion holds.

Write  $\leadsto$  for the composition of  $\rightarrow_{\mathbf{Decr}}^*$  and  $\rightarrow_{\bar{R}}$ . Let  $t \rightarrow_R t'$  and  $\mathbf{lab}(t, \sigma) \leadsto \mathbf{lab}(t', \sigma)$ . We still have to prove that

$$\mathbf{lab}(f(\dots, t, \dots), \sigma) \leadsto \mathbf{lab}(f(\dots, t', \dots), \sigma).$$

According to lemma 6 and the fact that  $\pi_f$  is weakly monotone in all coordinates, we obtain  $\pi_f(\dots, [\sigma](t), \dots) \geq \pi_f(\dots, [\sigma](t'), \dots)$ . Hence

$$\begin{aligned} \mathbf{lab}(f(\dots, t, \dots), \sigma) &= f_{\pi_f(\dots, [\sigma](t), \dots)}(\dots, \mathbf{lab}(t, \sigma), \dots) \\ &\rightarrow_{\mathbf{Decr}}^* f_{\pi_f(\dots, [\sigma](t'), \dots)}(\dots, \mathbf{lab}(t, \sigma), \dots) \\ &\leadsto f_{\pi_f(\dots, [\sigma](t'), \dots)}(\dots, \mathbf{lab}(t', \sigma), \dots) \\ &= \mathbf{lab}(f(\dots, t', \dots), \sigma). \end{aligned}$$

□

**Theorem 8** *Let  $\mathcal{M}$  be a quasi-model for a TRS  $R$  over  $\mathcal{F}$ . Let  $\bar{R}$  and  $\mathbf{Decr}$  be as above for any choice of  $S_f$  and  $\pi_f$ . Then  $R$  is terminating if and only if  $\bar{R} \cup \mathbf{Decr}$  is terminating.*

**Proof:** Assume  $\bar{R} \cup \mathbf{Decr}$  allows an infinite reduction. Since the order on  $S_f$  is well-founded for all  $f \in \mathcal{F}$ , the system  $\mathbf{Decr}$  is terminating. So the infinite reduction of  $\bar{R} \cup \mathbf{Decr}$  contains infinitely many  $\bar{R}$ -steps. Then removing all labels yields an infinite reduction of  $R$ .

On the other hand assume that  $R$  allows an infinite reduction. Then applying  $\mathbf{lab}$  for a fixed substitution on this infinite reduction yields an infinite reduction of  $\bar{R} \cup \mathbf{Decr}$  according to lemma 7. □

This proof is very similar to the proof of theorem 4. In fact theorem 4 can be considered as a special case of theorem 8 by choosing the discrete order (i.e.,  $x \geq y$  if and only if  $x = y$ ) on both  $\mathcal{M}$  and  $S_f$ . In this special case the requirements of weak monotonicity are trivially fulfilled, the notions of model and quasi-model coincide, and the TRS  $\mathbf{Decr}$  is empty.

Again consider the TRS introduced at the beginning of this section. The constant  $c$  serves as a bottom element: anything can be rewritten to  $c$ , but not the other way around. The elements  $a$  and  $b$  are essentially distinct. So choose the model  $\mathcal{M}$  to consist of three elements  $a, b$  and  $c$  with  $a > c$  and  $b > c$ ;  $a$  and  $b$  are incomparable.

By choosing  $a_{\mathcal{M}} = a$ ,  $b_{\mathcal{M}} = b$ ,  $c_{\mathcal{M}} = c$  and  $f_{\mathcal{M}}(x, y, z) = c$  for all  $x, y, z$  we have a quasi-model. Define  $S_f = \{0, 1\}$  with  $1 > 0$ , and

$$\pi_f(x, y, z) = \begin{cases} 1 & \text{if } x = a \wedge y = b \\ 0 & \text{otherwise.} \end{cases}$$

One easily checks that  $\pi_f$  is weakly monotone in all three coordinates. Now  $\overline{R}$  consists of the rules

$$\begin{array}{lcl} f_1(a, b, x) & \rightarrow & f_0(x, x, x) \\ f_0(x, y, z) & \rightarrow & c \\ f_1(x, y, z) & \rightarrow & c \\ a & \rightarrow & c \\ b & \rightarrow & c \end{array}$$

and **Decr** consists of the rule

$$f_1(x, y, z) \rightarrow f_0(x, y, z).$$

The system  $\overline{R} \cup \mathbf{Decr}$  is easily proved to be terminating by choosing the interpretation

$$[a] = [b] = 2, [c] = 1, [f_0](x, y, z) = x + y + z, [f_1](x, y, z) = x + y + 3z$$

over the positive integers. Hence according to theorem 8 the original system is terminating.

In Appendix A of [4] termination of the TRS describing an algebra of communicating processes was proved by first transforming it to another TRS. This transformation is a particular case of our construction, and the proof of preservation of termination is a particular case of theorem 8.

One can wonder whether it is essential in theorem 8 to add the system **Decr** to the labelled system. It is indeed; consider the following example:  $R$  consists of one rule

$$f(g(x)) \rightarrow g(g(f(f(x)))).$$

Choose  $M = S_f = \{0, 1\}$  with  $0 < 1$ , let  $f_{\mathcal{M}}(x) = 1$  and  $g_{\mathcal{M}}(x) = 0$  for all  $x$ . Clearly  $\mathcal{M}$  is a quasi-model for  $R$ . Choose  $\pi_f$  to be the identity which is clearly monotone. Then the system  $\overline{R}$  consists of the two rules

$$\begin{array}{lcl} f_0(g(x)) & \rightarrow & g(g(f_1(f_0(x)))) \\ f_0(g(x)) & \rightarrow & g(g(f_1(f_1(x)))) \end{array}$$

and is terminating: choose the interpretation  $[f_0](x) = 3x$ ,  $[f_1](x) = x$ ,  $[g](x) = x + 1$  over the positive integers. However, both  $R$  and  $\overline{R} \cup \mathbf{Decr}$  are not terminating since  $R$  allows the infinite reduction

$$f(f(g(x))) \rightarrow f(g(g(f(f(x)))))) \rightarrow g(g(\underbrace{f(f(g(f(f(x))))))}_{\text{...}})) \rightarrow \dots$$

By similar examples one can show that weak monotonicity of both  $f_{\mathcal{M}}$  and  $\pi_f$  are essential.

## 6 Termination of SUBST

In this section we give an application of theorem 8. Let  $\circ$  and  $\cdot$  be binary symbols,  $\lambda$  a unary symbol, and  $1$ ,  $id$  and  $\uparrow$  constants. Consider the TRS

$$\begin{aligned} \lambda(x) \circ y &\rightarrow \lambda(x \circ (1 \cdot (y \circ \uparrow))) \\ (x \cdot y) \circ z &\rightarrow (x \circ z) \cdot (y \circ z) \\ (x \circ y) \circ z &\rightarrow x \circ (y \circ z) \\ id \circ x &\rightarrow x \\ 1 \circ id &\rightarrow 1 \\ \uparrow \circ id &\rightarrow \uparrow \\ 1 \circ (x \cdot y) &\rightarrow x \\ \uparrow \circ (x \cdot y) &\rightarrow y, \end{aligned}$$

named  $\sigma_0$  in [5], which is essentially the same as the system SUBST in [11]. This system describes the process of substitution in combinatory categorical logic. Here ‘ $\lambda$ ’ corresponds to currying, ‘ $\circ$ ’ to composition, ‘ $id$ ’ to the identity, ‘ $\cdot$ ’ to pairing and ‘ $1$ ’ and ‘ $\uparrow$ ’ to projections. The original termination proof of SUBST in [11] is very complicated; the same holds for the newer proof by [5]. Both papers are devoted only to the termination proof of this particular system. The result implies termination of the process of explicit substitution in untyped  $\lambda$ -calculus; an overview of this approach to explicit substitution is given in [1]. In [21, 22] the technique of distribution elimination was developed to prove simple termination of  $\sigma_0$ . Define the TRS  $R$  to consist of the first three rules of  $\sigma_0$  and the embedding rules

$$\lambda(x) \rightarrow x, \quad x \circ y \rightarrow x, \quad x \circ y \rightarrow y, \quad x \cdot y \rightarrow x, \quad x \cdot y \rightarrow y.$$

Clearly simple termination of  $\sigma_0$  is equivalent to termination of  $R$ . Here we prove termination of  $R$  by means of theorem 8. As the quasi-model we choose the natural numbers (including 0) and

$$\lambda_{\mathcal{M}}(x) = x + 1, \quad x \circ_{\mathcal{M}} y = x + y, \quad x \cdot_{\mathcal{M}} y = \max(x, y), \quad 1_{\mathcal{M}} = \uparrow_{\mathcal{M}} = 0.$$

One easily checks that this is indeed a quasi-model for  $R$ . Only the symbol  $\circ$  is labelled; it is labelled by its own value. More precisely, we choose  $S_{\circ}$  to be the natural numbers and  $\pi_{\circ}(x, y) = x + y$ . Now the system  $\overline{R} \cup \mathbf{Decr}$  reads

$$\begin{aligned} \lambda(x) \circ_i y &\rightarrow \lambda(x \circ_j (1 \cdot (y \circ_k \uparrow))) && \text{for values } i > j \text{ and } i > k \\ (x \cdot y) \circ_i z &\rightarrow (x \circ_j z) \cdot (y \circ_k z) && \text{for values } i \geq j \text{ and } i \geq k \\ (x \circ_j y) \circ_i z &\rightarrow x \circ_i (y \circ_k z) && \text{for values } i \geq j \text{ and } i \geq k \\ \lambda(x) &\rightarrow x \\ x \circ_i y &\rightarrow x && \text{for all values } i \\ x \circ_i y &\rightarrow y && \text{for all values } i \\ x \cdot y &\rightarrow x \\ x \cdot y &\rightarrow y \\ x \circ_i y &\rightarrow x \circ_j y && \text{for all values } i > j. \end{aligned}$$

By choosing the well-founded precedence

$$\circ_i > \circ_j \text{ for } i > j, \quad \circ_i > \lambda, \circ_i > \cdot, \circ_i > 1, \circ_i > \uparrow \text{ for all } i$$

termination is easily proved by the lexicographic path order. Now theorem 8 yields termination of  $R$ , and hence simple termination of  $\sigma_0$ .

## 7 Monotone algebras

In this section we describe alternative proofs of our theorems based on the characterization of termination from [21, 22]; in fact this was the line along which semantic labelling was discovered.

A *well-founded monotone  $\mathcal{F}$ -algebra*  $(\mathcal{A}, >)$  is defined to be an  $\mathcal{F}$ -algebra  $\mathcal{A}$  for which the underlying set is provided with a well-founded strict partial order  $>$  and each algebra operation is strictly monotone in all of its coordinates, more precisely: for each operation symbol  $f \in \mathcal{F}$  and all  $a_1, \dots, a_n, b_1, \dots, b_n \in A$  for which  $a_i > b_i$  for some  $i$  and  $a_j = b_j$  for all  $j \neq i$  we have

$$f_{\mathcal{A}}(a_1, \dots, a_n) > f_{\mathcal{A}}(b_1, \dots, b_n).$$

Note the difference with the partial orders as they occurred in section 5: there operations were weakly monotone and here they are strictly monotone.

We define the partial order  $>_{\mathcal{A}}$  on  $\mathcal{T}(\mathcal{F}, \mathcal{X})$  as follows:

$$t >_{\mathcal{A}} t' \iff \forall \alpha \in A^{\mathcal{X}} : [\alpha](t) > [\alpha](t'),$$

where  $[\cdot]$  is the term evaluation in the algebra  $\mathcal{A}$  as defined in section 2. Intuitively:  $t >_{\mathcal{A}} t'$  means that for each interpretation of the variables in  $A$  the interpreted value of  $t$  is greater than that of  $t'$ .

In [21, 22] the following characterization of termination was given.

**Theorem 9** *A TRS  $R$  over  $\mathcal{F}$  is terminating if and only if there is a non-empty well-founded monotone  $\mathcal{F}$ -algebra  $(\mathcal{A}, >)$  for which  $l >_{\mathcal{A}} r$  for every rule  $l \rightarrow r$  of  $R$ .*

If  $l >_{\mathcal{A}} r$  for every rule  $l \rightarrow r$  of  $R$  we say that  $(\mathcal{A}, >)$  is *compatible* with  $R$ . Using this characterization we now sketch alternative proofs of theorems 4 and 8; in fact this was the line along which semantic labelling was discovered. Since theorem 4 is a special case of theorem 8 we concentrate on theorem 8. The interesting direction of the theorem is proving termination of  $R$  from termination of  $\overline{R} \cup \text{Decr}$ . So assume that  $\overline{R} \cup \text{Decr}$  is terminating. Then it admits a compatible well-founded monotone  $\overline{\mathcal{F}}$ -algebra  $(\overline{\mathcal{A}}, >)$ . We define the well-founded monotone  $\mathcal{F}$ -algebra  $(\mathcal{A}, >)$  by choosing  $A = M \times \overline{A}$  as the carrier set, where  $M$  is the carrier set of the model  $\mathcal{M}$  and  $\overline{A}$  is the carrier set of  $(\overline{\mathcal{A}}, >)$ . As the order we define

$$(m, a) > (m', a') \iff m \geq m' \wedge a > a';$$

clearly it is well-founded. As operations we choose

$$f_A((m_1, a_1), \dots, (m_n, a_n)) = (f_{\mathcal{M}}(m_1, \dots, m_n), f_{s, \overline{\mathcal{A}}}(a_1, \dots, a_n)),$$

where  $s = \pi_f(m_1, \dots, m_n)$ . It can be checked straightforwardly that  $(\mathcal{A}, >)$  is compatible with  $R$ , so  $R$  is terminating.

A similar proof of theorem 5 using theorem 9 can be given, even of a "quasi-model" version of theorem 5, generalizing both theorem 8 and theorem 5.

## 8 Semantic path order

In this section we argue that typical applications of semantic path order can be treated simpler and more powerful by semantic labelling. Let  $\succeq$  be any quasi-ordering on terms, i.e.,  $\succeq$  is reflexive and transitive. Write  $t \succ u$  for  $t \succeq u$  and not  $u \succeq t$ , and write  $t \approx u$  for  $t \succeq u$  and  $u \succeq t$ . The quasi-ordering  $\succeq$  is called well-founded if the strict partial order  $\succ$  is well-founded. The *semantic path order*  $\succeq_{spo}$  on terms is defined recursively as follows:  $s = f(s_1, \dots, s_m) \succeq_{spo} g(t_1, \dots, t_n) = t$  if and only if one of the following conditions holds

- $s_i \succeq_{spo} t$  for some  $i = 1, \dots, m$ ,
- $s \succ t$  and  $s \succ_{spo} t_j$  for all  $j = 1, \dots, n$ ,
- $s \approx t$  and  $\{s_1, \dots, s_m\} \succeq_{M, spo} \{t_1, \dots, t_n\}$ ,

where  $u \succ_{spo} u'$  means  $u \succeq_{spo} u'$  and not  $u' \succeq_{spo} u$ , and  $\succeq_{M, spo}$  is the multiset ordering induced by  $\succeq_{spo}$ . The basic theorem ([12, 6, 9]) motivating this order is the following:

**Theorem 10** *A TRS  $R$  is terminating if and only if there is a well-founded quasi-ordering  $\succeq$  on terms such that  $t \rightarrow_R u \Rightarrow f(\dots, t, \dots) \succeq f(\dots, u, \dots)$  holds for all terms and  $l^\sigma \succ_{spo} r^\sigma$  holds for all rules  $l \rightarrow r$  in  $R$  and all substitutions  $\sigma$ .*

If  $\geq$  is a well-founded quasi-ordering on the set  $\mathcal{F}$  of operation symbols and  $\succeq$  is defined by

$$f(s_1, \dots, s_m) \succeq g(t_1, \dots, t_n) \iff f \geq g$$

then the corresponding semantic path order is called *recursive path order* (RPO).

For practical applications the following observations are useful. Define the subterm relation  $\subseteq$  recursively by  $s \subseteq t = f(t_1, \dots, t_n)$  if and only if  $s = t$  or  $\exists i : s \subseteq t_i$ . Write  $s \subset t$  for  $s \subseteq t \wedge s \neq t$ . If  $t \subset s$  then we may conclude  $s \succ_{spo} t$ . Further if for all  $u \subseteq t$  we have either  $s \succ u$  or  $u \subset s$  we also may conclude that  $s \succ_{spo} t$ . The 'only if' part of the theorem easily follows from this observation by defining

$$s \succeq t \iff \exists u : s \rightarrow^* u \wedge t \subseteq u.$$



A typical example of a termination proof by semantic path order is found in [6]:

$$\begin{array}{ll} x * (y + 1) & \rightarrow (x * (y + (1 * 0))) + x \\ x * 1 & \rightarrow x \\ x + 0 & \rightarrow x \\ x * 0 & \rightarrow 0 \end{array}$$

which is not simply terminating. The semantic path order is defined as follows. First choose the obvious model  $\mathcal{M}$  in which  $M$  consists of the natural numbers and  $0, 1, +, *$  are interpreted as  $0, 1, +, *$ . Next define  $s \succeq t$  if and only if either the head symbol of  $t$  is not  $*$ , or

$$s = s_1 * s_2 \wedge t = t_1 * t_2 \wedge \forall \sigma : [\sigma](s_2) \geq [\sigma](t_2).$$

Here  $[\cdot]$  is defined as in section 2. Now one can check all proof obligations of theorem 10, concluding that the system is terminating.

Using similar ingredients we can give a termination proof of the same system by semantic labelling: choose the same  $\mathcal{M}$ , label  $*$  by the naturals and define  $\pi_*(x, y) = y$ . The resulting labelled system is

$$\begin{array}{ll} x *_{i+1} (y + 1) & \rightarrow (x *_{i+1} (y + (1 *_{i+1} 0))) + x \\ x *_{i+1} 1 & \rightarrow x \\ x + 0 & \rightarrow x \\ x *_{i+1} 0 & \rightarrow 0 \end{array}$$

for all  $i \geq 0$ . We can give the termination proof of this labelled system by RPO. Then the structure of the complete termination proof is essentially the same as that of Dershowitz; labelling is only used to split up the definition of  $\succeq$  in two layers.

However, we are not forced to use a path order like approach to prove termination of the labelled system, for example the interpretation in the naturals  $\geq 2$  defined by  $[0] = [1] = 2, x[+]y = x + y, x[*_i]y = x * (y + 4i)$  provides another termination proof. In this latter approach the symbol  $+$  is interpreted by a commutative and associative operation, so the labelled system is even terminating modulo commutativity and associativity of  $+$ . Also in the model  $\mathcal{M}$  the operation  $+$  is commutative and associative. According to theorem 5 we conclude that the original system is terminating modulo commutativity and associativity of  $+$ .

Finally, using the latter approach one easily proves by induction on the depth that a term of depth  $d$  can not have reductions of length greater then  $2^{2^{C*d}}$  for some constant  $C$ . Semantic path order does not provide tools for deriving such bounds.

## 9 Conclusions and further research

We introduced semantic labelling as a new technique for proving termination of term rewriting systems. The starting point is a model for a TRS, i.e., a model in which

each left hand side of a rewrite rule has the same value as the corresponding right hand side. An operation symbol in a term can now be labelled in a way depending on the interpretation of its arguments in the model. This is applied to all rewrite rules. We proved that the labelled TRS is terminating if and only if the original TRS is terminating. We illustrated this new technique for proving termination by several examples. In the typical case the TRS whose termination has to be proved is not simply terminating, while the labelled TRS is proved terminating by RPO or by an interpretation in the natural numbers.

Globally we distinguish two ways of using this technique: semantical and syntactical. In section 5 we saw that the requirement of having a model for the TRS can essentially be weakened. This technique also works for termination modulo equations.

The technique of semantic labelling is hard to automate since it depends on either the knowledge of a semantic model or on heuristics for choosing a model in a syntactic way. A promising approach of using labelling without any model to avoid this drawback is the following. Choose the labelling in which every operation symbol in a term is labelled by the head symbols of its direct subterms<sup>2</sup>. If the original signature is finite then the labelled signature is still finite. By applying the basic version of Kruskal's theorem to this labelled signature, the following generalization of Kruskal's theorem over finite signatures can be derived:

Let  $E$  consist of all rewrite rules

$$f(y_1, \dots, y_{k-1}, C[f(x_1, \dots, x_n)], y_{k+1}, \dots, y_n) \rightarrow f(x_1, \dots, x_n)$$

for all operation symbols  $f$  and all contexts  $C$ . Then  $\rightarrow_E^*$  is a well-quasi order.

If we replace  $E$  by the system  $Emb(\mathcal{F})$  as introduced in section 2 we obtain the basic version of Kruskal's theorem. However,  $E$  is more restrictive than  $Emb(\mathcal{F})$ , so this theorem is more powerful than the basic version. For example, it succeeds in ordering  $f(f(x)) > f(g(f(x)))$  (as in the approach of [18, 17]) and even  $f(0, 1, x) > f(x, x, x)$ .

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<sup>2</sup>Essentially this labelling was independently proposed by Isabelle Gnaedig; she calls it *typing*

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