Introduction

In this thesis we introduce the new concept of dendroidal set which is an extension of simplicial set. This notion is particularly useful in the study of operads and their algebras in the context of homotopy theory. We hope to convince the reader that the theory presented below provides new tools to handle some of the difficulties arising in the theory of up-to-homotopy algebras and supplies a uniform setting for the weakening of algebraic structures in many contexts of abstract homotopy. The thesis is based on, and expands [**37, 38**].

Background

An operad is a an algebraic gadget that can be used to describe sometimes very involved algebraic structures on objects in various categories. The notion was developed by May [**36**] in the theory of loop spaces. Indeed the complexity of the algebraic structure present on a loop space necessitates some machinery to effectively handle that complexity, and operads do the job. The theory of operads experienced, in the mid 90's, the so called renaissance period [**32**], and consequently the importance of operad theory in many areas of mathematics became established.

Let us quickly explain in some more detail how operads are used in the context of up-to-homotopy algebras. For simplicity let us only consider topological operads. Loosely speaking, an up-to-homotopy algebraic structure is the structure present on a space Y that is weakly equivalent to a space X endowed with a certain algebraic structure. So if X is a topological monoid then Y will have the structure of an A_∞-space. We say then that an A_∞-space is the up-to-homotopy (or weak) version of a topological monoid. The way operads come into the picture is explained by the work [**5**] of Boardman and Vogt who construct for each operad $\mathcal{P}$ another operad $W\mathcal{P}$ such that $W\mathcal{P}$-algebras correspond to weak $\mathcal{P}$-algebras and the same construction can also be used to produce a notion of weak maps between $W\mathcal{P}$-algebras.

It would appear that the problem of weak algebras is fully solved by operads and by the Boardman-Vogt W construction. However, there is one difficulty that arises, namely that the collection of weak algebras and their weak maps rarely forms a category. The reason is that the composition of weak maps, if at all defined, is in general not associative. Boardman and Vogt offer the following solution. Using their W construction they produce for any operad $\mathcal{P}$ a simplicial set X in which X_0 is the set of weak $\mathcal{P}$-algebras and X_1 is the set of weak maps of weak $\mathcal{P}$-algebras. An element of X_2 consists of three weak $\mathcal{P}$-algebras A_1, A_2, A_3 and three weak maps $f : A_1 \to A_2$, $g : A_2 \to A_3$, and $h : A_1 \to A_3$ together with some extra structure that can be thought of as exhibiting h as a possible composition of g with f. Similarly, X_n consists of chains of n weak maps and possible compositions of these maps, compositions of the compositions and so on. They show that this simplicial set satisfies what they call the restricted Kan condition. Joyal is studying such simplicial sets under the name quasi-categories, emphasizing that a quasi-category is a weakened notion of a category. Let us briefly outline some of the concepts of quasi-categories.

Recall that a Kan complex is a simplicial set X such that every horn $\Lambda^k[n] \to X$ has a filler $\Delta[n] \to X$. A simplicial set is a quasi-category if it is required that every horn $\Lambda^k[n] \to X$ with $0 < k < n$ has a filler. Such horns are called inner horns. Recall also the nerve functor $N : Cat \to sSet$ given by

$$N(\mathcal{C})_n = Hom_{Cat}([n], \mathcal{C}).$$

It is easy to see that $N(\mathcal{C})$ is a quasi-category for any category $\mathcal{C}$ and that the nerve functor N is fully-faithful. Moreover, those simplicial sets that are nerves of categories can be characterized as follows. Call a quasi-category X a strict quasi-category if every inner horn $\Lambda^k[n] \to X$ has a unique filler. It can then be shown that a simplicial set is a strict quasi-category if, and only if, it is the nerve of a category. In this way quasi-categories can be seen to extend categories. A quasi-category can be thought of as a special case of an ω-category, one in which all cells of dimension bigger than 1 are invertible. As it turns out, much of the theory of categories can be extended to quasi-categories. Thus the notion of a quasi-category is a good replacement for categories particularly in cases such as weak algebras when the objects we wish to study do not form a category but do form a quasi-category. For instance in **[23]** Joyal lays the foundations of the theory of limits and colimits in a quasi-category, so that it becomes meaningful for example to talk about limits and colimits of weak P-algebras inside the quasi-category of such algebras.

Another, somewhat less common, approach to operads is as a generalization of categories. In a category every arrow has an object as its domain and an object as its codomain. If instead of having just one object as domain we allow an arrow to have an ordered tuple of objects as domain (including the empty tuple) then we obtain the notion of an operad. We should immediately emphasize that from now on by an operad we mean a symmetric coloured operad in Set, which is also known as a symmetric multicategory. A category is then precisely an operad in which the only operations present are of arity 1 (i.e, they only have 1-tuples as domains). The objects of the category are the colours of the operad and the arrows in the category are the operations in the operad. In this way the category of all small categories embeds in the category of all small operads (an operad is small if its colours and its operations form a set). Similarly, for a symmetric monoidal category $\mathcal{E}$, the category of categories enriched in $\mathcal{E}$ embeds in the category of symmetric coloured operads in $\mathcal{E}$.

While this point of view is almost trivial, the development of operad theory made it quite obscure. The reason, we suspect, is that originally operads were defined in topological spaces and had just one object. Such operads were already complicated enough and, more importantly, they did the job they were designed for (see **[36]**). One can say that early research of operad theory concerned itself with the sub-category of symmetric topological coloured operads spanned by those operads with just one object. On the other hand, category theory was from the outset concerned with categories with all possible objects and not just one-object categories (i.e., monoids) and enriched categories came later.

Content and results

The main aim of this thesis is to introduce the theory of dendroidal sets as one that extends and complements the theory of operads and their algebras, and more specifically the theory of up-to-homotopy algebras of operads.

Chapter one is an unorthodox introduction to operads. The basic notions of operad theory are presented as a generalization of category theory rather than the classical operadic approach. This serves to fix notation but also, and perhaps more importantly, to present a certain point of view on operads quite different than the usual one. One consequence of this approach is that some new results about operads become apparent. Thus, while the chapter is expository, it contains a few new results all of which extend known results from category theory and are easy to prove. We would like to mention one such result that exhibits the importance of considering all coloured operads rather than one-colour operads, namely that the category of all coloured operads is very naturally a closed monoidal category (non-cartesian) in a way that extends the cartesian closed structure on categories. While the tensor product of operads in this monoidal structure is not new (it is essentially the Boardman-Vogt tensor product of operads [**7**]) the fact that it actually is a closed monoidal structure is new. The proof is very easy and serves to show that considering all coloured operads gives a more complete picture.

Chapter two contains the construction of the new category of dendroidal sets. A dendroidal set will be defined as a functor $\Omega^{op} \to Set$ on a certain category Ω, which we call the dendroidal category, whose objects are non-planar rooted trees. This category extends the simplicial category Δ. One way to define an embedding $\Delta \to \Omega$ is to view the objects of Δ as trees all of whose vertices are of valence 1. Two approaches to the definition of the dendroidal category are given, one of which is quite straightforward and the other more technically involved. The two approaches are shown to produce the same category and the basic terminology of dendroidal sets is introduced. Bearing in mind the point of view of operads as extension of categories one can say that the notion of dendroidal set extends that of simplicial set along similar lines.

Chapter three deals with the relation between operads and dendroidal sets. The functor relating the two notions is the dendroidal nerve functor $N_d : Operad \to dSet$ which associates to an operad a dendroidal set, its nerve, in a way that extends the nerve construction for categories. The notion of an inner Kan complex is then introduced. This concept is a generalization of the notion of a quasi-category discussed above. We present a detailed discussion of homotopy inside an inner Kan complex which results in a proof that a dendroidal set is the nerve of an operad if, and only if, it is a strict inner Kan complex. This is analogous to the characterization of nerves of categories as strict quasi-categories. Another important result in this chapter is that if X is a dendroidal set satisfying a certain normality condition and K is an inner Kan complex then the internal hom $\underline{Hom}_{dSet}(X, K)$ is again an inner Kan complex. This result, specialized to simplicial sets, proves that if X is any simplicial set and K is a quasi-category then $\underline{Hom}_{sSet}(X, K)$ is again a quasi-category. This result is proved by Joyal [**24**] though the proof is different from ours.

Chapter four presents applications of dendroidal sets to the theory of operads. It is shown that for an operad $\mathcal{P}$ in a suitable monoidal model category $\mathcal{E}$, the nerve construction can be refined to incorporate the homotopy information in $\mathcal{E}$. The resulting dendroidal set is called the homotopy coherent dendroidal nerve of $\mathcal{P}$ and is denoted by $hcN_d(\mathcal{P})$. We prove that for a locally fibrant operad $\mathcal{P}$ the dendroidal set $hcN_d(\mathcal{P})$ is an inner Kan complex. This approach allows for a new method to tackle up-to-homotopy $\mathcal{P}$-algebras as follows. The ambient monoidal model category $\mathcal{E}$ can itself be seen as an operad in $\mathcal{E}$ and it thus has a homotopy coherent nerve $hcN_d(\mathcal{E})$. We extend the whole notion of algebras of operads to a notion of an X algebra in E where X and E are dendroidal sets. Given a discrete operad $\mathcal{P}$ we show that an $N_d(\mathcal{P})$-algebra in $hcN_d(\mathcal{E})$ is the same as a weak $\mathcal{P}$-algebra. Thus the approach to weak algebras suggested by the theory of dendroidal sets is orthogonal to the approach given by the W construction (at least for discrete operad) in the following sense. The classical approach converts the operad describing a certain algebraic structure τ to a usually much more complicated operad whose algebras are weak τ structures. If we think of a $\mathcal{P}$-algebra as a map $\mathcal{P} \to \mathcal{E}$ then this approach replaces the domain of the map. In the context of dendroidal sets an algebra is a map $X \to E$ and then a weak $\mathcal{P}$ algebra is a map $N_d(\mathcal{P}) \to hcN_d(\mathcal{E})$, thus replacing the codomain and not the domain.

This chapter continues with the introduction of the notion of a category enriched in a dendroidal set. This enrichment generalizes ordinary enrichment of categories and formalizes the notion of a category weakly enriched in a monoidal model category. Examples of objects that are actually such enrichments are A_∞-spaces, A_∞-algebras, A_∞-categories, monoidal categories and bicategories and thus our approach provides a uniform environment for these structures. Using our notion of weak enrichment we obtain a new definition of weak n-categories that because of the general theory of dendroidal sets comes equipped with notions of weak functors (of several variables) of weak n-categories, and a homotopy theory of such functors. The closing section conjectures about a possible Quillen model structure on the category of dendroidal sets and explores one consequence of this model structure for the transferability of algebraic structures along weak equivalences.