Dendroidal Sets

Boomachtige Verzamelingen

(met een samenvatting in het Nederlands)

Proefschrift

ter verkrijging van de graad van doctor aan de Universiteit Utrecht op gezag van de rector magnificus, prof.dr. W.H. Gispen, ingevolge het besluit van het college voor promoties in het openbaar te verdedigen op dinsdag 18 september 2007 des middags te 2.30 uur

door

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geboren op 24 januari 1977 te Jeruzalem, Israël

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ISBN 978-90-3934629-7
2000 Mathematics Subject Classification: 55P48, 55U10, 55U40

It is a miracle that curiosity survives formal education

(Albert Einstein)

To Rahel

Contents

Introduction

In this thesis we introduce the new concept of dendroidal set which is an extension of simplicial set. This notion is particularly useful in the study of operads and their algebras in the context of homotopy theory. We hope to convince the reader that the theory presented below provides new tools to handle some of the difficulties arising in the theory of up-to-homotopy algebras and supplies a uniform setting for the weakening of algebraic structures in many contexts of abstract homotopy. The thesis is based on, and expands [**37, 38**].

Background

An operad is a an algebraic gadget that can be used to describe sometimes very involved algebraic structures on objects in various categories. The notion was developed by May [**36**] in the theory of loop spaces. Indeed the complexity of the algebraic structure present on a loop space necessitates some machinery to effectively handle that complexity, and operads do the job. The theory of operads experienced, in the mid 90's, the so called renaissance period [**32**], and consequently the importance of operad theory in many areas of mathematics became established.

Let us quickly explain in some more detail how operads are used in the context of up-to-homotopy algebras. For simplicity let us only consider topological operads. Loosely speaking, an up-to-homotopy algebraic structure is the structure present on a space Y that is weakly equivalent to a space X endowed with a certain algebraic structure. So if X is a topological monoid then Y will have the structure of an A_∞-space. We say then that an A_∞-space is the up-to-homotopy (or weak) version of a topological monoid. The way operads come into the picture is explained by the work [**5**] of Boardman and Vogt who construct for each operad $\mathcal{P}$ another operad $W\mathcal{P}$ such that $W\mathcal{P}$-algebras correspond to weak $\mathcal{P}$-algebras and the same construction can also be used to produce a notion of weak maps between $W\mathcal{P}$-algebras.

It would appear that the problem of weak algebras is fully solved by operads and by the Boardman-Vogt W construction. However, there is one difficulty that arises, namely that the collection of weak algebras and their weak maps rarely forms a category. The reason is that the composition of weak maps, if at all defined, is in general not associative. Boardman and Vogt offer the following solution. Using their W construction they produce for any operad $\mathcal{P}$ a simplicial set X in which X_0 is the set of weak $\mathcal{P}$-algebras and X_1 is the set of weak maps of weak $\mathcal{P}$-algebras. An element of X_2 consists of three weak $\mathcal{P}$-algebras A_1, A_2, A_3 and three weak maps $f : A_1 \to A_2$, $g : A_2 \to A_3$, and $h : A_1 \to A_3$ together with some extra structure that can be thought of as exhibiting h as a possible composition of g with f. Similarly, X_n consists of chains of n weak maps and possible compositions of these maps, compositions of the compositions and so on. They show that this simplicial set satisfies what they call the restricted Kan condition. Joyal is studying such simplicial sets under the name quasi-categories, emphasizing that a quasi-category is a weakened notion of a category. Let us briefly outline some of the concepts of quasi-categories.

Recall that a Kan complex is a simplicial set X such that every horn $\Lambda^k[n] \to X$ has a filler $\Delta[n] \to X$. A simplicial set is a quasi-category if it is required that every horn $\Lambda^k[n] \to X$ with $0 < k < n$ has a filler. Such horns are called inner horns. Recall also the nerve functor $N : Cat \to sSet$ given by

$$N(\mathcal{C})_n = Hom_{Cat}([n], \mathcal{C}).$$

It is easy to see that $N(\mathcal{C})$ is a quasi-category for any category $\mathcal{C}$ and that the nerve functor N is fully-faithful. Moreover, those simplicial sets that are nerves of categories can be characterized as follows. Call a quasi-category X a strict quasi-category if every inner horn $\Lambda^k[n] \to X$ has a unique filler. It can then be shown that a simplicial set is a strict quasi-category if, and only if, it is the nerve of a category. In this way quasi-categories can be seen to extend categories. A quasi-category can be thought of as a special case of an ω-category, one in which all cells of dimension bigger than 1 are invertible. As it turns out, much of the theory of categories can be extended to quasi-categories. Thus the notion of a quasi-category is a good replacement for categories particularly in cases such as weak algebras when the objects we wish to study do not form a category but do form a quasi-category. For instance in **[23]** Joyal lays the foundations of the theory of limits and colimits in a quasi-category, so that it becomes meaningful for example to talk about limits and colimits of weak P-algebras inside the quasi-category of such algebras.

Another, somewhat less common, approach to operads is as a generalization of categories. In a category every arrow has an object as its domain and an object as its codomain. If instead of having just one object as domain we allow an arrow to have an ordered tuple of objects as domain (including the empty tuple) then we obtain the notion of an operad. We should immediately emphasize that from now on by an operad we mean a symmetric coloured operad in Set, which is also known as a symmetric multicategory. A category is then precisely an operad in which the only operations present are of arity 1 (i.e, they only have 1-tuples as domains). The objects of the category are the colours of the operad and the arrows in the category are the operations in the operad. In this way the category of all small categories embeds in the category of all small operads (an operad is small if its colours and its operations form a set). Similarly, for a symmetric monoidal category $\mathcal{E}$, the category of categories enriched in $\mathcal{E}$ embeds in the category of symmetric coloured operads in $\mathcal{E}$.

While this point of view is almost trivial, the development of operad theory made it quite obscure. The reason, we suspect, is that originally operads were defined in topological spaces and had just one object. Such operads were already complicated enough and, more importantly, they did the job they were designed for (see **[36]**). One can say that early research of operad theory concerned itself with the sub-category of symmetric topological coloured operads spanned by those operads with just one object. On the other hand, category theory was from the outset concerned with categories with all possible objects and not just one-object categories (i.e., monoids) and enriched categories came later.

Content and results

The main aim of this thesis is to introduce the theory of dendroidal sets as one that extends and complements the theory of operads and their algebras, and more specifically the theory of up-to-homotopy algebras of operads.

Chapter one is an unorthodox introduction to operads. The basic notions of operad theory are presented as a generalization of category theory rather than the classical operadic approach. This serves to fix notation but also, and perhaps more importantly, to present a certain point of view on operads quite different than the usual one. One consequence of this approach is that some new results about operads become apparent. Thus, while the chapter is expository, it contains a few new results all of which extend known results from category theory and are easy to prove. We would like to mention one such result that exhibits the importance of considering all coloured operads rather than one-colour operads, namely that the category of all coloured operads is very naturally a closed monoidal category (non-cartesian) in a way that extends the cartesian closed structure on categories. While the tensor product of operads in this monoidal structure is not new (it is essentially the Boardman-Vogt tensor product of operads [**7**]) the fact that it actually is a closed monoidal structure is new. The proof is very easy and serves to show that considering all coloured operads gives a more complete picture.

Chapter two contains the construction of the new category of dendroidal sets. A dendroidal set will be defined as a functor $\Omega^{op} \to Set$ on a certain category Ω, which we call the dendroidal category, whose objects are non-planar rooted trees. This category extends the simplicial category Δ. One way to define an embedding $\Delta \to \Omega$ is to view the objects of Δ as trees all of whose vertices are of valence 1. Two approaches to the definition of the dendroidal category are given, one of which is quite straightforward and the other more technically involved. The two approaches are shown to produce the same category and the basic terminology of dendroidal sets is introduced. Bearing in mind the point of view of operads as extension of categories one can say that the notion of dendroidal set extends that of simplicial set along similar lines.

Chapter three deals with the relation between operads and dendroidal sets. The functor relating the two notions is the dendroidal nerve functor $N_d : Operad \to dSet$ which associates to an operad a dendroidal set, its nerve, in a way that extends the nerve construction for categories. The notion of an inner Kan complex is then introduced. This concept is a generalization of the notion of a quasi-category discussed above. We present a detailed discussion of homotopy inside an inner Kan complex which results in a proof that a dendroidal set is the nerve of an operad if, and only if, it is a strict inner Kan complex. This is analogous to the characterization of nerves of categories as strict quasi-categories. Another important result in this chapter is that if X is a dendroidal set satisfying a certain normality condition and K is an inner Kan complex then the internal hom $\underline{Hom}_{dSet}(X, K)$ is again an inner Kan complex. This result, specialized to simplicial sets, proves that if X is any simplicial set and K is a quasi-category then $\underline{Hom}_{sSet}(X, K)$ is again a quasi-category. This result is proved by Joyal [**24**] though the proof is different from ours.

Chapter four presents applications of dendroidal sets to the theory of operads. It is shown that for an operad $\mathcal{P}$ in a suitable monoidal model category $\mathcal{E}$, the nerve construction can be refined to incorporate the homotopy information in $\mathcal{E}$. The resulting dendroidal set is called the homotopy coherent dendroidal nerve of $\mathcal{P}$ and is denoted by $hcN_d(\mathcal{P})$. We prove that for a locally fibrant operad $\mathcal{P}$ the dendroidal set $hcN_d(\mathcal{P})$ is an inner Kan complex. This approach allows for a new method to tackle up-to-homotopy $\mathcal{P}$-algebras as follows. The ambient monoidal model category $\mathcal{E}$ can itself be seen as an operad in $\mathcal{E}$ and it thus has a homotopy coherent nerve $hcN_d(\mathcal{E})$. We extend the whole notion of algebras of operads to a notion of an X algebra in E where X and E are dendroidal sets. Given a discrete operad $\mathcal{P}$ we show that an $N_d(\mathcal{P})$-algebra in $hcN_d(\mathcal{E})$ is the same as a weak $\mathcal{P}$-algebra. Thus the approach to weak algebras suggested by the theory of dendroidal sets is orthogonal to the approach given by the W construction (at least for discrete operad) in the following sense. The classical approach converts the operad describing a certain algebraic structure τ to a usually much more complicated operad whose algebras are weak τ structures. If we think of a $\mathcal{P}$-algebra as a map $\mathcal{P} \to \mathcal{E}$ then this approach replaces the domain of the map. In the context of dendroidal sets an algebra is a map $X \to E$ and then a weak $\mathcal{P}$ algebra is a map $N_d(\mathcal{P}) \to hcN_d(\mathcal{E})$, thus replacing the codomain and not the domain.

This chapter continues with the introduction of the notion of a category enriched in a dendroidal set. This enrichment generalizes ordinary enrichment of categories and formalizes the notion of a category weakly enriched in a monoidal model category. Examples of objects that are actually such enrichments are A_∞-spaces, A_∞-algebras, A_∞-categories, monoidal categories and bicategories and thus our approach provides a uniform environment for these structures. Using our notion of weak enrichment we obtain a new definition of weak n-categories that because of the general theory of dendroidal sets comes equipped with notions of weak functors (of several variables) of weak n-categories, and a homotopy theory of such functors. The closing section conjectures about a possible Quillen model structure on the category of dendroidal sets and explores one consequence of this model structure for the transferability of algebraic structures along weak equivalences.

Preliminaries

0.1. Category theory

Throughout this work we will largely ignore size issues regarding such problems as the existence of the category of all categories and similar questions. We will assume a suitable setting of universes and content ourselves with the construction of the category of all small sets, all small categories, all small spaces and so on, where 'small' is meant with respect to some universe of discourse. We will not always explicitly mention the word 'small', always assuming it is meant when necessary.

We start by setting the notation of category theory used in this work. Most of the category theory used herein is rather elementary and almost all of it can be found in [**34**], to which the reader is referred to for more details if needed. Our choice of notation is somewhat different than the one presented in [**34**] but is still (largely) the standard one. Given a category $\mathcal{C}$ we denote its set of objects by $ob(\mathcal{C})$. For every two objects $a, b \in ob(\mathcal{C})$ we denote by $\mathcal{C}(a, b)$ the set of arrows with domain a and codomain b.

Given a symmetric monoidal category $\mathcal{C}$, it is said to be a *closed* monoidal category if for each $b \in ob(\mathcal{C})$ the functor $- \otimes b : \mathcal{C} \to \mathcal{C}$, that sends an object a to $a \otimes b$, has a right adjoint. Usually this right adjoint is called the *internal Hom* in $\mathcal{C}$ and is denoted by $\underline{Hom}_{\mathcal{C}}(b, -)$. We deviate from this notation and introduce the notation $\underline{\mathcal{C}}(b, -)$ instead. By definition then, there is an isomorphism

$$\mathcal{C}(a \otimes b, c) \cong \mathcal{C}(a, \underline{\mathcal{C}}(b, c))$$

natural in a, b, and c.

The monoidal functors that would interest us would always be the strong monoidal ones, i.e., those in which the components of the coherence natural transformations are isomorphisms. If the monoidal structure on a category is given by categorical products then we will call it a *cartesian* category. If that monoidal category is closed we will call it a *cartesian closed*.

Example 0.1.1. There are many examples of closed monoidal categories such as Cat with the cartesian product of categories, $Vect$ with the usual tensor product of vector spaces, etc. With our notation, given two categories $\mathcal{C}$ and $\mathcal{D}$ in $ob(Cat)$, the *set* $Cat(\mathcal{C}, \mathcal{D})$ is the set of all functors $F : \mathcal{C} \to \mathcal{D}$ while $\underline{Cat}(\mathcal{C}, \mathcal{D})$ is the *category* whose objects are all functors $F : \mathcal{C} \to \mathcal{D}$ and whose arrows are natural transformations between such functors.

0.1.1. Presheaf categories. We now introduce presheaf categories and mention some of their basic properties. Given a category Γ, the *presheaf* category on Γ is the category $Set^{\Gamma^{op}}$ of contravariant functors $\Gamma^{op} \to Set$ and their natural transformations. Given such a presheaf $X : \Gamma^{op} \to Set$ we write, for each object

$\gamma \in ob(\Gamma)$,

$$X_\gamma = X(\gamma).$$

For our purposes it will be convenient to think of presheaf categories as follows. Consider the category Γ as a category whose objects are shapes and the arrows express the way different shapes relate to one another. A presheaf on Γ should then be thought of as a set each of whose elements has a shape. In more detail, the presheaf $X : \Gamma^{op} \to Set$ should be thought of as a set where an element $x \in X_\gamma$ has shape γ. The contravariance of X as a functor means that an element $x \in X_\gamma$ of shape γ has other elements of other shapes associated to it. For example, if $\gamma' \to \gamma$ is a monomorphism in Γ (which can be interpreted as saying that γ' is a sub-shape of γ) then there is a function $X_\gamma \to X_{\gamma'}$ (which can be interpreted as mapping $x \in X_\gamma$ to an element $x' \in X_{\gamma'}$ where x' is the 'part' of x 'shaped' like γ').

A fundamental property of presheaf categories is that every presheaf is the canonical colimit of representable presheaves. In more detail, given a category Γ, each $\gamma \in ob(\Gamma)$ induces a *representable presheaf* denoted by $\Gamma[\gamma] = \Gamma(-, \gamma)$ and defined by

$$\Gamma[\gamma]_{\gamma'} = \Gamma(\gamma', \gamma).$$

Given an arbitrary presheaf $X : \Gamma^{op} \to Set$, consider all maps $\Gamma[\gamma] \to X$ for all possible $\gamma \in ob(\Gamma)$. By the Yoneda Lemma these correspond exactly to all $x \in X_\gamma$. Assigning $\Gamma[\gamma]$ to each $\Gamma[\gamma] \to X$ we obtain a diagram in $Set^{\Gamma^{op}}$. The colimit of this diagram is the original presheaf X. We write this shortly as

$$X = \varinjlim_{\Gamma[\gamma] \to X} \Gamma[\gamma].$$

Lastly we introduce a categorical construction (a special case of a Kan extension) which will repeatedly be used in this work. Given two categories $\mathcal{C}$ and $\mathcal{D}$ we denote by $adj(\mathcal{D}, \mathcal{C})$ the category of adjunctions between $\mathcal{D}$ and $\mathcal{C}$.

Proposition 0.1.2. *Let $\mathcal{C}$ be a cocomplete category and Γ an arbitrary category. There is an equivalence of categories between the category of functors from Γ to $\mathcal{C}$ and the category of adjunctions between the categories $Set^{\Gamma^{op}}$ and $\mathcal{C}$, i.e,*

$$\underline{Cat}(\Gamma, \mathcal{C}) \simeq adj(Set^{\Gamma^{op}}, \mathcal{C}).$$

Proof. We describe the effect on objects of two functors

$$\underline{Cat}(\Gamma, \mathcal{C}) \rightleftarrows adj(Set^{\Gamma^{op}}, \mathcal{C})$$

which together constitute the desired equivalence. We omit most of the details, which are just simple verifications.

Given a functor $F : \Gamma \to \mathcal{C}$ we need to construct an adjunction

$$Set^{\Gamma^{op}} \underset{N_F}{\overset{|-|_F}{\rightleftarrows}} \mathcal{C}\ .$$

The functor $N_F : \mathcal{C} \to Set^{\Gamma^{op}}$ is defined for an object $C \in ob(\mathcal{C})$ to be the presheaf $N_F(C)$ whose elements of shape γ form the set $N_F(C)_\gamma = \mathcal{C}(F(\gamma), C)$. To define the functor $|-|_F$ for an arbitrary presheaf $X : \Gamma^{op} \to Set$, recall the canonical presentation of X as a colimit of representables

$$X = \varinjlim_{\Gamma[\gamma] \to X} \Gamma[\gamma].$$

We then define

$$|X|_F = \varinjlim_{\Gamma[\gamma]\to X} F(\gamma),$$

which exists since $\mathcal{C}$ was assumed cocomplete. We now prove that $|-|_F$ is indeed the left adjoint of N_F. Given $X \in ob(Set^{\Gamma^{op}})$ and $C \in ob(\mathcal{C})$ we need to establish a natural isomorphism

$$Set^{\Gamma^{op}}(X, N_F(C)) \cong \mathcal{C}(|X|_F, C).$$

To simplify the notation we neglect the subscript F. We now obtain the desired natural isomorphism by the following calculation (all colimits and limits are taken over the same diagram as above):

$$\begin{aligned} Set^{\Gamma^{op}}(X, N(C)) &\cong \\ Set^{\Gamma^{op}}(\varinjlim \Gamma[\gamma], N(C)) &\cong \\ \varprojlim Set^{\Gamma^{op}}(\Gamma[\gamma], N(C)) &\cong \\ \varprojlim N(C)_\gamma &\cong \\ \varprojlim \mathcal{C}(F\gamma, C) &\cong \\ \mathcal{C}(\varinjlim F\gamma, C) &\cong \\ \mathcal{C}(|X|, C). \end{aligned}$$

To construct a functor $\Gamma \to C$ from a given adjunction $Set^{\Gamma^{op}} \underset{U}{\overset{G}{\rightleftarrows}} \mathcal{C}$ we simply define $F(\gamma) = G(\Gamma[\gamma])$. This completes the (sketch of the) proof. □

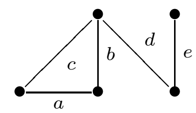
In the setting of the above construction we will refer to a functor $F : \Gamma \to \mathcal{C}$ as a *probe*. This is to be thought of as mapping each shape $\gamma \in ob(\Gamma)$ to an object $F(\gamma) \in ob(\mathcal{C})$ in a functorial way, thus specifying certain objects in $\mathcal{C}$ which behave somewhat like the shapes γ. By 'probing' an object $C \in ob(\mathcal{C})$, by mapping into it from the various objects $F(\gamma)$, we then obtain the presheaf $N_F(C)$. We call N_F the *nerve* functor induced by the probe F and the left adjoint $|-|_F$ the *realisation* in $\mathcal{C}$ of a presheaf $X : \Gamma^{op} \to Set$ induced by F. The realisation process uses the information in the presheaf X as instructions on how to 'glue' the various objects $F(\gamma)$ in $\mathcal{C}$.

0.2. A formalism of trees

Trees play a fundamental role in the theory of operads in general and in this work as well. We present here a formalism of trees which is somewhat different from the standard ones (see **[18, 35]** for two approaches to trees). Our trees are based on the notion of a graph. However, usually a graph is given by specifying a set of vertices V, and the edges are then a certain subset of $V \times V$. We will find it more convenient to have a definition of a graph where the basic ingredient is the set of edges, and the vertices are then defined in terms of those.

DEFINITION 0.2.1. A *graph* G consists of a non-empty set E of *edges* and a set $V \subseteq P(E)$ of *vertices* such that every edge belongs to at most two different vertices. Those edges that belong to two distinct vertices are called *inner* while those that belong to less then two vertices are called *outer.*

We will draw graphs in the usual way. For example, the picture



denotes the graph whose set of edges is $\{a, b, c, d, e\}$ with the following vertices

$$\{\{a, c\}, \{a, b\}, \{c, b, d\}, \{d, e\}, \{e\}\}.$$

Notice that our definition excludes graphs with edges from a vertex to itself and also graphs with two vertices and several parallel edges between these two vertices, and similar graphs. This will not concern us since our main interest is trees, in which such graphs do not occur.

If two distinct edges e, e' in a graph belong to the same vertex we say that e and e' are *linked*. For a given edge e, if e belongs to two distinct vertices u and v then we say that u and v are *adjacent* and that e *connects* u and v.

DEFINITION 0.2.2. A *path* of length $n \geq 1$ in a graph is a sequence of edges

$$e_1, \cdots, e_n$$

such that e_i is linked to e_{i+1} for each $1 \leq i < n$. We say that two edges e, e' are *connected* if there is a path as above with $e_1 = e$ and $e_n = e'$. A *loop* is then a path $e_1, \cdots, e_n$ of length of at least 2 such that $e_1 = e_n$. A graph is said to be *connected* if any two edges in it are connected.

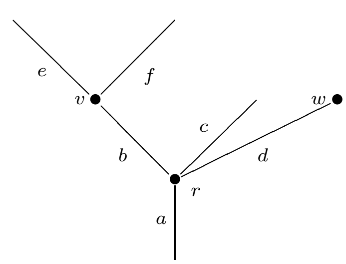
In a connected graph G we distinguish two kinds of vertices:

DEFINITION 0.2.3. Let G be a connected graph. A vertex v that consists of just one edge is called an *outer* vertex. The other vertices are called *inner* vertices.

0.2.1. Trees.

DEFINITION 0.2.4. A *tree* is a finite connected graph with no loops and a chosen outer edge called the *root*. The rest (if any) of the outer edges are called *leaves*.

We will draw trees with their root at the bottom and directed from the leaves to the root. We will use a $\bullet$ for vertices. The direction in the tree defines for each vertex v a unique outgoing edge denoted by *out*(v) (called the *output* of the vertex) and a (possibly empty) set of incoming edges denoted by *in*(v) (called the *input* of the vertex). The number of incoming edges into v is called the *valence* of v. For example in the tree:



there are three vertices of valence 2,3, and 0 and three leaves (at the outer sides of the edges e, f and c). The outer edges are e, f, c, and a, where a is the root. The inner edges are then b and d.

Thus a tree T is given by $(E(T), V(T), out(T))$ where $(E(T), V(T))$ is a finite, connected, loop-free graph and $out(T)$ is the chosen root. We will use the notation $in(T)$ to refer to the set of leaves of the tree.

The tree

consisting of just one edge and no vertices is called the *unit* tree We denote this tree by η, or η_e if we wish to explicitly name the unique edge. In this tree, its only edge is both the root and a leaf.

Definition 0.2.5. Let T and S be two trees whose only common edge is the root r of S which is also one of the leaves of T. The *grafting*, $T \circ S$, of S on T along r is the graph

$$(E(T) \cup E(S), V(T) \cup V(S), out(T)).$$

That this indeed defines a tree is easily checked. Pictorially the grafted tree $T \circ S$ is obtained by putting the tree S on top of the tree T by identifying the output edge of S with the input edge r of T. By repeatedly grafting, one can define a full grafting operation $T \circ (S_1, \cdots, S_n)$ whenever the set of the roots of the trees S_i is equal to the set of leaves of T, the sets $E(S_i)$ are pairwise disjoint, and $E(S_i)$ meets $E(T)$ at a leaf of T which is also the root of S_i (for each $1 \leq i \leq n$).

We now state a useful decomposition of trees that allows for inductive proofs on trees. The proof is trivial.

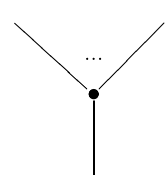
Proposition 0.2.6. *(Fundamental decomposition of trees) Let T be a tree. Suppose T has root r and $\{r, e_1, \cdots, e_n\}$ is the vertex containing r. Let T_{e_i} be the tree that contains the edge e_i as root and everything above it in T. Then*

$$T = T_{root} \circ (T_{e_1}, \cdots, T_{e_n})$$

where T_{root} is the tree consisting of r as root and $\{e_1, \cdots, e_n\}$ as the set of leaves.

Below, certain trees will appear often. For easy reference we define them here.

Definition 0.2.7. A tree C_n of the form:



that has just one vertex and n leaves will be called an *n-corolla* (or a *corolla* if we do not wish to specify the number of leaves). A tree of the form

with one leaf and only unary vertices will be called a *linear tree*. If the edges are numbered $0, \cdots, n$ from the root up, we denote it by L_n.

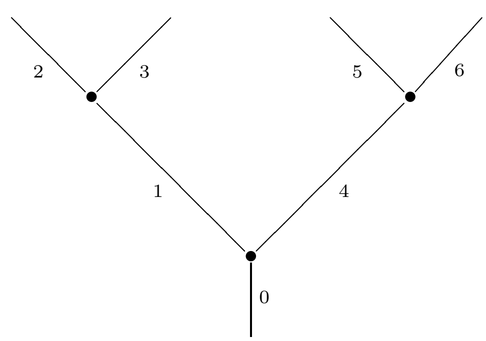
0.2.2. Planar trees.

DEFINITION 0.2.8. A *planar tree* $\bar{T}$, is a tree T together with a linear ordering of $in(v)$ for each vertex v.

The ordering of $in(v)$ for each vertex is equivalent to drawing the graph on the plane. It is evident that a single tree can become a planar tree in (usually) many different ways.

DEFINITION 0.2.9. Let T be a planar tree with n edges. We call T a *standard* planar tree if its set of edges is $E = \{0, 1, 2, \cdots, n\}$ and if when the tree is traversed left-first from the root up then the edges appear in the natural order on E.

EXAMPLE 0.2.10. The following is a standard planar tree:



It is obvious that any planar rooted tree is, up to a renaming of the edges, the same as precisely one of the standard planar trees.

The grafting of planar trees is defined just as that of non-planar ones and it is evident that the same fundamental decomposition property still holds for planar trees. We now define the grafting of standard planar trees. Let T and S be two standard planar trees. The leaves of T can be numbered from left to right. Let i be the i-th leaf. The grafting $T \circ_i S$ is given as follows. Rename the edges of S such that the root of S is equal to l, the i-th leaf of T, and such that $E(S') \cap E(T) = \{l\}$. Call the new tree S'. With this notation we have the following definition.

Definition 0.2.11. The *grafting* of the standard planar trees S and T along the i-th leaf is denoted by $T \circ_i S$ and is the standard planar tree that has the same shape as the tree $T \circ_l S'$, obtained by ordinary grafting.

CHAPTER 1

Operads

The theory of operads is a rich and well established one as seen through the works of May [**36**], Ginzburg and Kapranov [**18**], Boardman and Vogt [**7**], and Getzler and Jones [**17**], just to name a few. It is the aim of this chapter to introduce operads and their basic theory and is thus expository in nature. However, our approach is vastly different than the classical one, in which operads are introduced as algebraic structures modelled after the endomorphism operad [**36**]. Instead, our approach is categorical in the sense that we view operads as a direct generalization of categories. In fact, what we call operads are usually named symmetric multicategories [**31**] or coloured operads [**6**]. Our approach is very close to Leinster's [**31**]. Our decision to use the term 'operad' throughout is a mix of personal preference and arbitrariness, simply since a choice must be made. We attempt no justification for our choice nor do we claim that it is better than any other terminology. For the sake of clarity then we emphasize again that by an operad we mean a symmetric multicategory or, equivalently, a symmetric coloured operad (in the category of sets). While the treatment of operads presented here is very elementary and contains a lot of known results, it is sprinkled with new simple results that arise naturally by taking the categorical approach. Most notably, Sections 4-7 contain new results all of which relate to either known results in operad theory or in category theory.

The chapter starts by giving the definition of operads, maps of operads (functors), and natural transformations where an attempt to parallel the development of the theory to that of category theory is made. The construction of free operads is then introduced which facilitates the definition of operads using generators and relations, followed by an examination of limits and colimits of operads. The Boardman-Vogt tensor product of operads is presented together with a proof that this makes the category of operads into a symmetric closed monoidal category. Then the 'folk' Quillen model structure on the category of small categories is extended to operads, and it is shown that with the Boardman-Vogt tensor product the category of operads is a symmetric closed monoidal model category. Following is a presentation of a Grothendieck construction for operads. The chapter ends with a consideration of enriched operads and a comparison of our notation with the classical one.

1.1. Operads, functors, and natural transformations

In [**14**] the authors explain that categories are defined in order to be able to define functors, which in turn are defined to facilitate the definition of natural transformations. We develop the basic definitions of operad theory along the same lines.

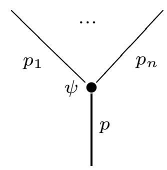
DEFINITION 1.1.1. A *planar operad* $\mathcal{P}$ is given by specifying objects and operations and supplying a composition function on the operations, which satisfies unit and associativity axioms. In detail:

<u>Objects and operations</u>

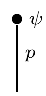
There is a specified set of objects $ob(\mathcal{P})$. For each sequence of objects

$$p_1, \cdots, p_n, p,$$

also called a *signature*, there is a set $\mathcal{P}(p_1, \cdots, p_n; p)$ of *operations* or *arrows.* Such an operation ψ will be depicted as



and will be said to have $(p_1, \cdots, p_n)$ as *input* and p as *output,* and to be of *arity* n. It is assumed that each operation has a well defined input and output, in other words if $\mathcal{P}(p_1, \cdots, p_n; p_0) \cap \mathcal{P}(q_1, \cdots, q_m; q_0) \neq \emptyset$ then $m = n$ and $p_i = q_i$ for $0 \leq i \leq n$. For each object p there is an operation $id_p \in \mathcal{P}(p; p)$ called the *identity* on p. We allow n to be 0, in which case ψ will be denoted by



<u>Composition function</u>

The operations can be composed in the following way. Given a signature $p_1, \cdots, p_n, p$, and for each $1 \leq i \leq n$, another sequence of objects $p^i_1, \cdots p^i_{m_i}$, there is a *composition* function

$$\begin{array}{c} \mathcal{P}(p_1, \cdots, p_n; p) \times \mathcal{P}(p^1_1, \cdots, p^1_{m_1}; p_1) \times \cdots \times \mathcal{P}(p^n_1, \cdots, p^n_{m_n}; p_n) \\ \big\downarrow \\ \mathcal{P}(p^1_1, \cdots, p^1_{m_1}, \cdots, p^n_1, \cdots, p^n_{m_n}; p) \end{array}$$

If $\psi \in \mathcal{P}(p_1, \cdots, p_n; p)$ and $\psi_i \in \mathcal{P}(p^i_1, \cdots, p^i_{m_i}; p_i)$, we denote by $\psi \circ (\psi_1, \cdots, \psi_n)$ (or simply by $\psi(\psi_1, \cdots, \psi_n)$) the image of $(\psi, \psi_1, \cdots, \psi_n)$ under the composition function.

<u>Axioms</u>

The identities are required to satisfy

$$id_p(\psi) = \psi$$

and

$$\varphi(id_{p_1}, \cdots, id_{p_n}) = \varphi$$

whenever the compositions are defined. Furthermore, the composition is required to be *associative* in the sense that given

$$\psi \in \mathcal{P}(p_1, \cdots, p_n; p),$$

for each $1 \leq i \leq n$ an operation

$$\psi_i \in \mathcal{P}(p^i_1, \cdots, p^i_{m_i}; p_i),$$

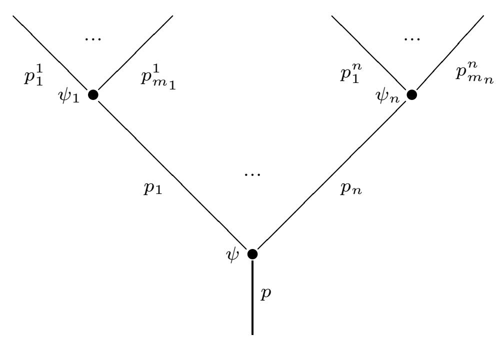
and for each pair (i, j) with $1 \leq i \leq n$ and $1 \leq j \leq m_i$ an operation ψ^i_j with output p^i_j (and an arbitrary input), the composition

$$(\psi(\psi_1, \cdots, \psi_n))(\psi^1_1, \cdots, \psi^1_{m_1}, \psi^2_1, \cdots, \psi^2_{m_2}, \cdots, \psi^n_1, \cdots, \psi^n_{m_n})$$

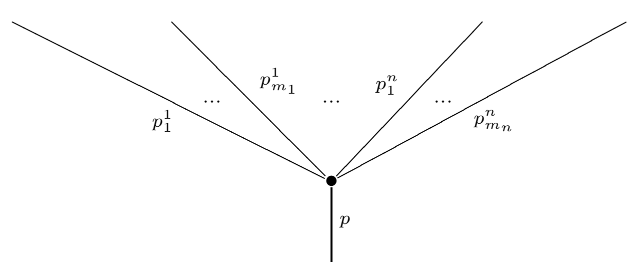
should be equal to the composition

$$\psi(\psi_1(\psi^1_1, \cdots, \psi^1_{n_1}), \cdots, \psi_n(\psi^n_1, \cdots, \psi^n_{m_n})).$$

Some light can be shed on the definition by considering certain labelled planar trees. In more detail, given an operad $\mathcal{P}$ and a planar tree T one can consider labelling the edges and vertices of the tree T respectively with objects and operations of the operad $\mathcal{P}$. We call T a *labelled* tree if each edge e is labelled by an object $p_e \in ob(\mathcal{P})$ and if each vertex v with $in(v) = (e_1, \cdots, e_n)$ and $out(v) = e$ is labelled by an operation $\psi_v \in \mathcal{P}(p_{e_1}, \cdots, p_{e_n}; p_e)$. Using this language one can interpret the composition function in the operad as follows. Given a labelled tree T



the composition function associates to it a labelled corolla (that is, just an operation) of the shape of the tree obtained from the one above by contracting all of the inner edges, where the labelling is as follows. All of the edges of the corolla are also edges in the original tree (namely the outer ones) and they retain their labels from T. The sole vertex of the corolla is then labelled by $\psi \circ (\psi_1, \cdots, \psi_n)$. Visually, the composition associated to the tree above is depicted by the following labelled corolla:



We can refine the composition a bit by introducing the so called $\circ_i$-compositions. Given $p_1, \cdots, p_n, p$ and $q_1, \cdots, q_m$ objects in $\mathcal{P}$, the $\circ_i$-composition for $1 \leq i \leq n$

is the function

$$\begin{array}{c}\mathcal{P}(p_1,\cdots,p_n;p)\times\mathcal{P}(q_1,\cdots,q_m;p_i)\\ \circ_i\big\downarrow\\ \mathcal{P}(p_1,\cdots,p_{i-1},q_1,\cdots,q_m,p_{i+1},\cdots,p_n;p)\end{array}$$

defined by

$$\psi\circ_i\varphi=\psi(id_{p_1},\cdots,id_{p_{i-1}},\varphi,id_{p_{i+1}},\cdots,id_{p_n}).$$

We can again use planar trees to get a geometric picture of the $\circ_i$-compositions. Given any labelled planar tree S and an inner edge e in it, there is a natural labelling of the tree S/e, obtained from S by contracting e. The labelling of S/e is as follows. In S/e there is just one vertex which does not appear in S, all other vertices and edges other then e occur in S as well and retain their labels. Let v be the new vertex in S/e and suppose e leads from the vertex u to w, so $in(w)=(u_1,\cdots,u_k)$ and $u_j=u$ for some $1\leq j\leq k$. The label of v in S/e is then defined to be $\psi_w\circ_j\psi_u$.

By sequentially contracting all of the inner edges in S we obtain a labelled corolla $c(S)$. It is a direct consequence of the associativity axiom that the label of the only vertex in $c(S)$ is independent of the chosen order in which edges are contracted. We will sometimes refer to a labelled tree S as a *composition scheme* in $\mathcal{P}$ and will then refer to the uniquely labelled corolla $c(S)$ (or rather to the operation labelling its unique vertex) as the *composition* of the composition scheme.

It is obvious that under the suitable associativity conditions of the various $\circ_i$-compositions, an operad can equivalently be given by a set of objects together with $\circ_i$-compositions. See **[35]** for more details on defining operads via $\circ_i$-compositions for the special case where the operad in question has just one object. The extension to the general case is trivial.

DEFINITION 1.1.2. An *operad* (or a *symmetric* operad) is a planar operad together with actions of the symmetric groups as follows. Given a permutation $\sigma\in\Sigma_n$ there is a function $\sigma^*:\mathcal{P}(p_1,\cdots,p_n;p)\to\mathcal{P}(p_{\sigma(1)},\cdots,p_{\sigma(n)};p)$. These functions are required to define a right action of Σ_n (that is, $(\sigma\tau)^*=\tau^*\sigma^*$, and for the identity permutation $id\in\Sigma_n$ we have $id^*=id$) and to respect compositions in the following sense. Given operations $\psi_0,\cdots,\psi_n$ for which the composition $\psi_0\circ(\psi_1,\cdots,\psi_n)$ is defined, and permutations $\sigma_0,\cdots,\sigma_n$ where $\sigma_i\in\Sigma_{k_i}$ (with k_i the arity of ψ_i), the equation

$$\sigma_0^*(\psi_0)\circ(\sigma_{\sigma_0(1)}^*(\psi_{\sigma_0(1)}),\cdots,\sigma_{\sigma_0(n)}^*(\psi_{\sigma_0(n)}))=[\sigma_0\circ(\sigma_1,\cdots,\sigma_n)]^*(\psi_0\circ(\psi_1,\cdots,\psi_n))$$

holds. The permutation $\sigma_0\circ(\sigma_1,\cdots,\sigma_n)$ is the block permutation product of the given permutations, which is the evident one equating inputs on both sides (see **[31]** for more details, page 77 under 'operad of symmetries', and **[35]**).

REMARK 1.1.3. It is easily seen that an operad (planar or symmetric) that has only operations of arity 1 is the same thing as a category. More precisely, given such an operad $\mathcal{P}$ we define the category $j^*(\mathcal{P})$ by setting

$$ob(j^*(\mathcal{P}))=ob(\mathcal{P})$$

and for objects $p,p'\in ob(\mathcal{P})$ we set

$$j^*(\mathcal{P})(p,p')=\mathcal{P}(p;p').$$

The units and composition are induced by those in $\mathcal{P}$ in the obvious way. The result is a category by the unit and associativity axioms for operads.

EXAMPLE 1.1.4. There are many examples of operads given in the literature (see e.g., [**18, 35, 36**]), the vast majority of which have just one object. We wish to present here a different family of operads, namely, those obtained from symmetric monoidal categories. Let $\mathcal{E}$ be any symmetric monoidal category and M a subset of $ob(\mathcal{E})$. The operad $\mathcal{P}_M$ is defined as follows. The set of objects of $\mathcal{P}_M$ is M and the set of operations with input $(A_1, \cdots, A_n)$ and output B is the set $\mathcal{E}(A_1 \otimes \cdots \otimes A_n, B)$ where some choice for the repeated tensoring was made. The composition and units in $\mathcal{P}_M$ are the evident ones, and the symmetric groups Σ_n act by permuting the variables. The operad axioms follow immediately from the usual coherence theorems for symmetric monoidal categories. We will usually write $\hat{\mathcal{E}}$ instead of $\mathcal{P}_{ob(\mathcal{E})}$, or just $\mathcal{E}$ where context will prevent confusion.

DEFINITION 1.1.5. Let $\mathcal{P}$ and $\mathcal{Q}$ be two planar operads. A *map of planar operads* $F : \mathcal{P} \to \mathcal{Q}$ is a function $F : ob(\mathcal{P}) \to ob(\mathcal{Q})$ and for any sequence of objects $p_1, \cdots, p_n, p$ in $ob(\mathcal{P})$ a function

$$F : \mathcal{P}(p_1, \cdots, p_n; p) \to \mathcal{Q}(Fp_1, \cdots, Fp_n; Fp)$$

such that

$$F(\psi(\psi_1, \cdots, \psi_n)) = F(\psi)(F\psi_1, \cdots, F\psi_n)$$

holds whenever the compositions make sense. Furthermore, for any $p \in ob(\mathcal{P})$ we demand that $F(id_p) = id_{Fp}$. If $\mathcal{P}$ and $\mathcal{Q}$ are both symmetric then a *map of symmetric operads* $F : \mathcal{P} \to \mathcal{Q}$ is the same as above with the extra condition that for any permutation $\sigma \in \Sigma_n$ one has

$$F(\sigma^*(\psi)) = \sigma^*(F(\psi))$$

for any operation ψ with input of length n.

REMARK 1.1.6. When $\mathcal{P}$ and $\mathcal{Q}$ are both categories (that is, they have only unary operations) it is immediate to see that a map of operads (planar or symmetric) $F : \mathcal{P} \to \mathcal{Q}$ is the same thing as a functor. For this reason we will also use the word *functor* to refer to maps of operads.

The category $Operad_\pi$ is the category of all planar operads and functors between them with the obvious notion of composition of functors and the evident identity functors. Likewise, *Operad* is the category of all symmetric operads and their maps. The remarks above allude to the fact that the category *Operad* can be seen as an extension of the category *Cat*. We now make this relation precise. Given a category $\mathcal{C}$ we can construct a planar operad $j_!\mathcal{C}$ by setting

$$ob(j_!\mathcal{C}) = ob(\mathcal{C})$$

and for objects $c, c' \in ob(j_!\mathcal{C})$ we define

$$j_!\mathcal{C}(c; c') = \mathcal{C}(c, c').$$

Composition and units are induced from $\mathcal{C}$ in the obvious way to make $j_!\mathcal{C}$ into an operad. Notice that since in $j_!\mathcal{C}$ all operations are unary (that is they have just one input), each symmetry group Σ_n acts trivially on the operations of the planar operad $j_!\mathcal{C}$. It follows that $j_!\mathcal{C}$ can also be considered as a symmetric operad. We thus obtain two functors (both named $j_!$):

$$\begin{array}{ccc} & Cat & \\ {\scriptstyle j_!} \swarrow & & \searrow {\scriptstyle j_!} \\ Operad_\pi & & Operad \end{array}$$

that view a category as an operad (planar or symmetric) all of which operations are unary. Clearly both of these functors are fully faithful. We will thus consider Cat to be embedded in $Operad$ and in $Operad_\pi$ via $j_!$. These two functors both have right adjoints which send a (planar or symmetric) operad $\mathcal{P}$ to the category $j^*(\mathcal{P})$ whose objects are the objects of $\mathcal{P}$, and whose arrows for any two objects $p, p' \in Ob(j^*\mathcal{P})$ are given by

$$j^*\mathcal{P}(p, p') = \mathcal{P}(p; p').$$

The identities and the compositions are as in $\mathcal{P}$. Somewhat less formally, we see that inside any operad there is a category which is the *linear* part of the operad. We will freely use category theoretic terms and notation when referring to this category. Thus for example, the meaning of "ψ is an isomorphism in the operad $\mathcal{P}$" should be interpreted as "$j^*\psi$ is an isomorphism in the category $j^*\mathcal{P}$". Again we use the same name, j^*, for both functors $Operad_\pi \to Cat$ and $Operad \to Cat$.

There is an obvious forgetful functor $U : Operad \to Operad_\pi$ which simply forgets the symmetric group actions. This functor has a left adjoint

$$Symm : Operad_\pi \to Operad,$$

called the *symmetrization* functor, which we now describe. Let $\mathcal{P}$ be a planar operad. The objects of $Symm(\mathcal{P})$ are the same as the objects of $\mathcal{P}$. To describe the operations in $Symm(\mathcal{P})$ let $p_1, \cdots, p_n, p \in ob(Symm(\mathcal{P}))$. For each $\sigma \in \Sigma_n$ let

$$\mathcal{P}_\sigma(p_1, \cdots, p_n; p) = \{\sigma\} \times \mathcal{P}(p_{\sigma^{-1}(1)}, \cdots, p_{\sigma^{-1}(n)}; p).$$

We now define

$$Symm(\mathcal{P})(p_1, \cdots, p_n; p) = \coprod_{\sigma \in \Sigma_n} \mathcal{P}_\sigma(p_1, \cdots, p_n; p).$$

The unit id_p for $p \in ob(Symm(\mathcal{P}))$ is (id, id_p) and the symmetric groups Σ_n can now freely act on the various operations in $Symm(\mathcal{P})$ as we now describe. Let $\tau \in \Sigma_n$ be a permutation, we define

$$\tau^* : Symm(\mathcal{P})(p_1, \cdots, p_n; p) \to Symm(\mathcal{P})(p_{\tau(1)}, \cdots, p_{\tau(n)}; p)$$

on $(\sigma, \psi) \in \mathcal{P}_\sigma(p_1, \cdots, p_n; p)$ by

$$\tau^*(\sigma, \psi) = (\sigma\tau, \psi) \in \mathcal{P}_{\sigma\tau}(p_{\tau(1)}, \cdots, p_{\tau(n)}; p).$$

This obviously defines a right action of the symmetric groups. To define the composition let $\psi_0 \in Symm(\mathcal{P})(p_1, \cdots, p_n; p)$ and $\psi_i \in Symm(\mathcal{P})(p^i_1, \cdots, p^i_{m_i}; p_i)$ for $1 \leq i \leq n$ be operations in $Symm(\mathcal{P})$. By definition we have then that $\psi_i = (\tau_i, \varphi_i)$ with

$$\varphi_0 \in \mathcal{P}(p_{\tau_0^{-1}(1)}, \cdots, p_{\tau_0^{-1}(n)}; p)$$

and for $1 \leq i \leq n$

$$\varphi_i \in \mathcal{P}(p^i_{\tau_i^{-1}(1)}, \cdots, p^i_{\tau_i^{-1}(m_i)}; p_i).$$

We can thus use the composition in $\mathcal{P}$ to obtain the operation

$$\varphi = \varphi_0 \circ (\varphi_{\tau_0^{-1}(1)}, \cdots, \varphi_{\tau_0^{-1}(n)}).$$

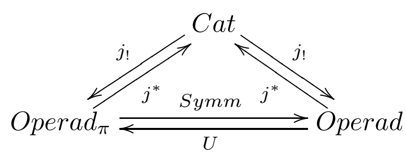
Calculating the domain of φ we see that

$$(\tau \circ (\tau_1, \cdots, \tau_n), \varphi) \in Symm(\mathcal{P})(p_1^1, \cdots, p_{n_1}^1, \cdots, p_1^n, \cdots, p_{m_n}^n; p),$$

(where $\tau \circ (\tau_1, \cdots, \tau_n)$ is again the block permutation product) and we define this operation to be the composition $\psi_0 \circ (\psi_1, \cdots, \psi_n)$ in $Symm(\mathcal{P})$. The verification of the rest of the axioms is straightforward.

We summarize the information given above relating categories, planar operads, and symmetric operads in the following theorem.

THEOREM 1.1.7. *The six functors described above fit into the triangle*

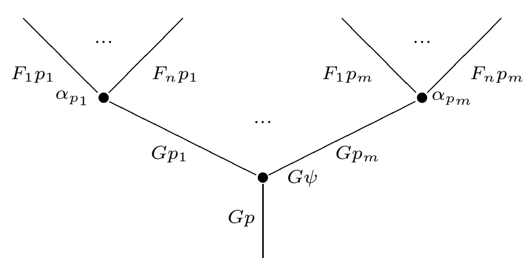


where each pair of functors is an adjunction (with the left adjoint on top), and the following equations hold:

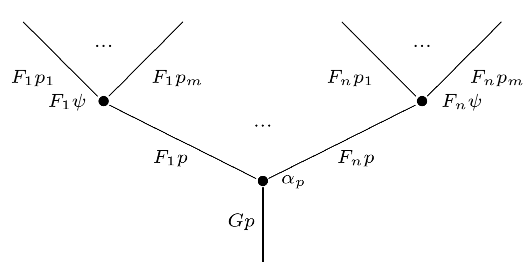
(1) $j^* j_! \cong id$ (these are actually two equalities)
(2) $j^* \circ Symm = j^*$ as functors from $Operad_\pi$ to Cat
(3) $j^* \circ U = j^*$ as functors from $Operad$ to Cat

We now turn to define natural transformations for operads.

DEFINITION 1.1.8. Let $F_i : \mathcal{P} \to \mathcal{Q}$ for $1 \leq i \leq n$ and $G : \mathcal{P} \to \mathcal{Q}$ be $n+1$ functors between symmetric operads. A *natural transformation* α from $(F_1, \cdots, F_n)$ to G is a family $\{\alpha_p\}_{p \in ob(\mathcal{P})}$, where $\alpha_p \in \mathcal{Q}(F_1 p, \cdots, F_n p; Gp)$ and is called the *component* of the natural transformation at p, satisfying the following property. Given any operation $\psi \in \mathcal{P}(p_1, \cdots, p_m; p)$ consider the following composition schemes in $\mathcal{Q}$



and



and let φ_1 and φ_2 be the compositions in $\mathcal{Q}$ of, respectively, the first and second composition schemes. We demand that $\varphi_2 = \sigma^*_{m,n}(\varphi_1)$, where $\sigma_{m,n}$ is the obvious permutation equating the inputs of both operations.

REMARK 1.1.9. It is trivial to check that when $\mathcal{P}$ and $\mathcal{Q}$ are categories, if α is a natural transformation from $(F_1, \cdots, F_n)$ to G then $n = 1$ *and* α is exactly the same thing as a natural transformation in the categorical sense. It is also immediate to verify that given a natural transformation $\alpha : F \to G$ in the operadic sense, the family $\{\alpha_p\}_{p \in ob(\mathcal{P})}$ is a natural transformation between the functors $j^*(F)$ and $j^*(G)$ in the categorical sense.

Notice as well that the symmetric actions play a vital role in the definition. One cannot define natural transformations between functors of planar operads unless the domain consists of a single functor. This is a significant difference between the category of planar operads and that of symmetric operads.

Natural transformations can be composed as follows. Fix two operads $\mathcal{P}$ and $\mathcal{Q}$. Suppose $\alpha : (F_1, \cdots, F_n) \to F$ and $\beta^i : (F^i_1, \cdots, F^i_{m_i}) \to F_i$ for $1 \le i \le n$ are natural transformations where all the functors are from $\mathcal{P}$ to $\mathcal{Q}$. The *composition* of these natural transformations is the natural transformation

$$\alpha \circ (\beta^1, \cdots, \beta^n) : (F^1_1, \cdots, F^1_{m_1}, \cdots, F^n_1, \cdots, F^n_{m_n}) \to F$$

that for each object $p \in ob(\mathcal{P})$ has the component

$$[\alpha \circ (\beta^1, \cdots, \beta^n)]_p = \alpha_p \circ (\beta^1_p, \cdots, \beta^n_p).$$

The verification of the naturality is routine.

PROPOSITION 1.1.10. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two operads. We denote the set of all functors $F : \mathcal{P} \to \mathcal{Q}$ by $ob(Func(\mathcal{P}, \mathcal{Q}))$. Given functors $F_1, \cdots, F_n, F \in ob(Func(\mathcal{P}, \mathcal{Q}))$ let $Func(\mathcal{P}, \mathcal{Q})(F_1, \cdots, F_n; F)$ be the set of all natural transformations*

$$\alpha : F_1, \cdots, F_n \to F.$$

The composition of natural transformations defined above makes $Func(\mathcal{P}, \mathcal{Q})$ into a symmetric operad (with the obvious units and Σ_n-actions).

PROOF. The proof is completely routine and thus omitted. □

REMARK 1.1.11. As noted, the symmetries in the operad play a crucial role in the definition of $Func(\mathcal{P}, \mathcal{Q})$. If $\mathcal{P}$ and $\mathcal{Q}$ were planar operads we would still be able to consider the collection of all functors between them, but in order to obtain some sensible structure on it we would have to restrict ourselves to those natural transformations that have a single functor as domain. This is done very briefly in [**31**] (page 87 under the name 'transformation') and we recount it here. Let $\mathcal{P}$ and $\mathcal{Q}$ be two planar operads. We denote by $ob(Func(\mathcal{P}, \mathcal{Q}))$ the set of all functors $F : \mathcal{P} \to \mathcal{Q}$ as above. For two functors $F, F' : \mathcal{P} \to \mathcal{Q}$ we denote by $Func(\mathcal{P}, \mathcal{Q})(F, F')$ the set of all natural transformations $\alpha : F \to F'$. Composition of such natural transformations still makes sense and makes $Func(\mathcal{P}, \mathcal{Q})$ into a category.

We can summarize the discussion so far by noticing that the above remarks imply that both *Operad* and $Operad_\pi$ are strict 2-categories. For the category $Operad_\pi$ this follows from the fact that for two planar operads we have that $Func(\mathcal{P}, \mathcal{Q})$ is a category. As for the category *Operad*, we showed that for two symmetric operads $\mathcal{P}$ and $\mathcal{Q}$, $Func(\mathcal{P},\mathcal{Q})$ is a symmetric operad. By considering the category $j^*Func(\mathcal{P},\mathcal{Q})$ we see that *Operad*, too, is a strict 2-category. We now have:

THEOREM 1.1.12. *Consider Cat, Operad, and* $Operad_\pi$ *as strict* 2*-categories. The functors* $j^* : Operad \to Cat$ *and* $j^* : Operad_\pi \to Cat$ *extend naturally to strict* 2*-functors.*

PROOF. The proof is trivial. □

EXAMPLE 1.1.13. Consider the symmetric operad *Comm* given as follows. *Comm* has one object $\star$, and for each $n \geq 0$ there is just one operation in

$$Comm(\star, \cdots, \star; \star),$$

with $\star$ in the domain repeated n times, which is denoted by m_n. There is now just one way to define an operad structure. Namely, the unit $id_\star$ is m_1, composition is given by

$$m_n \circ (m_{k_1},, \cdots, m_{k_n}) = m_{k_1+\cdots+k_n},$$

and all Σ_n actions are trivial. All of the axioms for an operad are trivially satisfied. We also consider the category *Set* of small sets, which we consider as a monoidal category via the cartesian product. Recall (see Example 1.1.4) that we then have the operad $\widehat{Set}$ which we denote by *Set* again. Suppose $F : Comm \to Set$ is a functor. Such a functor consists of a function $F : ob(Comm) \to ob(Set)$, which amounts to a choice of a set A. The functor F consists further of a function $Comm(\star, \cdots, \star; \star) \to Set(A^n, A)$, that is simply a choice of a function $F(m_n) : A^n \to A$ for each $n \geq 0$. For $n = 0$ this is a map $F(m_0) : A^0 \to A$, i.e., a map $I \to A$ where I is a one-point set, so it is just a choice of a constant $e \in A$. We have thus a constant in A and for every $n \geq 1$ an n-ary operation $F(m_n) : A^n \to A$.

Let us now examine the consequences of the functoriality of F. First of all, by definition, m_1 is mapped to the identity. Furthermore, in *Comm* we have that

$$m_2 \circ (m_1, m_2) = m_3 = m_2 \circ (m_2, m_1)$$

from which it follows that

$$F(m_2) \circ (id, F(m_2)) = F(m_3) = F(m_2) \circ (F(m_2), id),$$

which implies that $F(m_2)$ is an associative binary operation. In *Comm* we also have the relation

$$m_2 \circ (m_1, m_0) = m_1 = m_2 \circ (m_0, m_1)$$

that is

$$F(m_2) \circ (id, e) = F(m_1) = F(m_2) \circ (e, id)$$

which means that e is a two-sided inverse for the binary operation $F(m_2)$. We thus see that $(A, F(m_2), e)$ is a monoid. Lastly, since F commutes with the Σ_n-actions it follows that

$$F(m_n \cdot \sigma) = F(m_n) \cdot \sigma$$

holds for every $\sigma \in \Sigma_n$. Since σ acts trivially in *Comm* we obtain that

$$F(m_n) = F(m_n) \cdot \sigma$$

holds for each $\sigma \in \Sigma_n$. Specifically for $n = 2$ and for the twist permutation $\sigma \in \Sigma_2$, we obtain (by the fact that in *Set* the symmetric groups act by permuting the variables) that $F(m_2)$ is a commutative operation. A is thus a commutative monoid. All the other relations in the operad $Comm$ impose no new conditions on the monoid, since they all just express general associativity and commutativity for various tuples of elements of A. Conversely it is clear that given a commutative monoid A, one can construct a functor $F : Comm \to Set$, such that $F(\star) = A$, $F(m_0)$ is the unit of the monoid, and $F(m_2)$ is the binary operation.

Let $F_1, \cdots, F_n, F$ be functors from $Comm$ to Set and let $A_1, \cdots, A_n, A$ be their corresponding commutative monoids. We now examine a natural transformation $\alpha : F_1, \cdots, F_n \to F$. To start with, α consists of just one component, namely $\alpha_\star : Set(F_1(\star), \cdots, F_n(\star); F(\star))$, i.e., a function $\alpha_\star : A_1 \times \cdots \times A_n \to A$. Following the definition of a natural transformation one sees that this function $\alpha_\star$ respects the binary composition in the sense that if we endow $A_1 \times \cdots \times A_n$ with the obvious commutative monoid structure then $\alpha_\star$ is a map of commutative monoids. The converse is also true and we actually obtain the following. Let $ComMon$ be the category of commutative monoids. The usual product of monoids makes $ComMon$ into a symmetric monoidal category and we may thus consider it as a symmetric operad. We then have

$$Func(Comm, Set) \cong CommMon$$

as operads.

This example illustrates a more general phenomenon, namely that operads can be used to describe algebraic structures on objects of other operads. Thus given two operads $\mathcal{P}$ and $\mathcal{Q}$ and a functor $F : \mathcal{P} \to \mathcal{Q}$, there are two ways to think about F. One way is to think of $\mathcal{P}$ and $\mathcal{Q}$ as algebraic structures and of F as a mapping preserving this structure. The other is to think of $\mathcal{P}$ as modeling an algebraic structure and of $\mathcal{Q}$ as an operad upon whose objects we wish to define that algebraic structure. The functor $F : \mathcal{P} \to \mathcal{Q}$ can then be thought of as defining an algebraic structure in Q by realizing inside $\mathcal{Q}$ the model encoded in $\mathcal{P}$. It is useful to make a semantic distinction between these two interpretations of a functor. We thus give the second interpretation a different name.

Definition 1.1.14. Let $\mathcal{P}$ and $\mathcal{E}$ be two operads. An *algebra* for $\mathcal{P}$ in $\mathcal{E}$, or a $(\mathcal{P}, \mathcal{E})$-*algebra,* is a functor

$$A : \mathcal{P} \to \mathcal{E}.$$

For such an algebra we say that A defines an algebraic structure on the family of objects given by $F : ob(\mathcal{P}) \to ob(\mathcal{E})$.

Remark 1.1.15. The choice of the letter $\mathcal{E}$ for the codomain operad is meant to make the distinction between the two different roles of the operads clear.

We end this section by generalizing some basic properties of categories, functors, and natural transformations to our setting of operads. These results are chosen since they will be used in the sequel. Of course many other results can be generalized along the same lines.

Given a natural transformation $\alpha : F \to G$, we call α a *natural isomorphism* if each component of α is an isomorphism in $\mathcal{Q}$.

Definition 1.1.16. Let $\mathcal{P}$ and $\mathcal{Q}$ be two operads. We say that they are *equivalent* provided that there are two functors $F : \mathcal{P} \to \mathcal{Q}$ and $G : \mathcal{Q} \to \mathcal{P}$ together with two natural isomorphisms $\alpha : id_\mathcal{P} \to GF$ and $\beta : id_\mathcal{Q} \to FG$. We then call F an *equivalence* from $\mathcal{P}$ to $\mathcal{Q}$.

REMARK 1.1.17. For two categories $\mathcal{C}$ and $\mathcal{D}$, it is obvious that $\mathcal{C}$ and $\mathcal{D}$ are equivalent if, and only if, $j_!\mathcal{C}$ and $j_!\mathcal{D}$ are equivalent operads. It is also clear that if $\mathcal{P}$ and $\mathcal{Q}$ are equivalent operads then $j^*\mathcal{P}$ and $j^*\mathcal{Q}$ are equivalent categories. The converse implication is (in general) not true, as can easily be seen.

DEFINITION 1.1.18. A functor $F : \mathcal{P} \to \mathcal{Q}$ is *essentially surjective* if j^*F is. F is called *full* if for any signature $p_1, \cdots, p_n, p$ the function

$$F : \mathcal{P}(p_1, \cdots, p_n; p) \to \mathcal{Q}(Fp_1, \cdots, Fp_n; Fp)$$

is surjective. F is called *faithful* if the function above is injective. It is called *fully faithful* if that map is a bijection.

LEMMA 1.1.19. *Let $F : \mathcal{P} \to \mathcal{Q}$ be a functor between two operads. $\mathcal{F}$ is an equivalence of operads if, and only if, $\mathcal{F}$ is fully faithful and essentially surjective.*

PROOF. The proof is just like that of the corresponding result for categories ([**34**] Theorem 1, page 93). □

1.2. Free operads and operads given by generators and relations

We now turn to the construction of free operads (planar and symmetric). This will be done in terms of the standard planar trees defined in the preliminaries. (see [**31**], page 85, for a slightly different approach). We will then use this construction to define operads by generators and relations. The constructions given here are a special case of similar constructions in the theory of enriched operads (see e.g., [**6, 18**]).

DEFINITION 1.2.1. Let A be a set. A *collection* C on the set A is a family of sets $C(a_1, \cdots, a_n; a_0)$ where $a_i \in A$ and $n \geq 0$ varies over all natural numbers. An *arrow* $(A, C) \to (A', C')$ between collections is a function $f : A \to A'$ and a family of functions (all denoted f)

$$f : C(a_1, \cdots, a_n; a) \to C'(fa_1, \cdots, fa_n; fa).$$

We denote by Col the category of all collections and their arrows.

Evidently every planar operad $\mathcal{P}$ has an *underlying* collection C on the set $ob(\mathcal{P})$ given for $p_1, \cdots, p_n, p \in ob(\mathcal{P})$ simply by

$$C(p_1, \cdots, p_n; p) = \mathcal{P}(p_1, \cdots, p_n; p).$$

We thus obtain a functor $C_\pi : Operad_\pi \to Col$.

We shall now construct the left adjoint $\mathcal{F}_\pi : Col \to Operad_\pi$ of $C_\pi : Operad_\pi \to Col$.

Let C be a collection on a set A. We are going to define a planar operad $\mathcal{P}$ with $ob(\mathcal{P}) = A$. To define the arrows we consider standard planar trees labelled by the elements of A and elements from the collection C. Let T be a standard planar tree. A labelling of T is a choice of an element $a_e \in A$ for any edge $e \in E(T)$ and for any vertex v in T with $in(v) = (e_1, \cdots, e_n)$ and $out(v) = e_0$, an element $c_v \in C(a_{e_1}, \cdots, a_{e_n}; a_{e_0})$. Let LT be the set of all labelled standard planar trees. For such trees we will use the notation $in(T)$ to refer to the tuple of leaves $(l_1, \cdots, l_n)$ of T and also (so long that it is clear which one we mean) to the tuple of labels of the leaves $(a_{l_1}, \cdots, a_{l_n})$. Similarly $out(T)$ will refer both to the root of T and to the label of the root.

We can now define the arrows in $\mathcal{P}$. For objects $p_1, \cdots, p_n, p \in ob(\mathcal{P})$, we define

$$\mathcal{P}(p_1, \cdots, p_n; p) = \{T \in LT \mid in(T) = (p_1, \cdots, p_n), out(T) = p\}.$$

Composition is obtained by grafting labelled trees as follows. Given

$$\psi_0 \in \mathcal{P}(p_1, \cdots, p_n; p)$$

and

$$\psi_i \in \mathcal{P}(p_1^i, \cdots, p_{m_i}^i; p_i)$$

let φ be the standard planar tree obtained by grafting the root of each of ψ_i onto the i-th leaf of ψ_0. The labelling of φ is obtained by copying the labelling of each ψ_i onto the obvious sub-tree in φ corresponding to ψ_i. This composition is clearly associative. The unit at an object a is simply the tree η labelled by a itself. This completes the construction of $\mathcal{P}$. It is now trivial to confirm that we obtain a functor $\mathcal{F}_\pi : Col \to Operad_\pi$ which is in fact the left adjoint of $C_\pi : Operad_\pi \to Col$. A planar operad obtained in this way is called a *free* planar operad. Thus, for a collection C on a set A, a map of planar operads $G : \mathcal{F}C \to \mathcal{Q}$ is completely determined by a function $g : A \to ob(\mathcal{Q})$ and a family of functions

$$G : C(a_1, \cdots, a_n; a) \to \mathcal{Q}(fa_1, \cdots, fa_n; fa).$$

Given an arbitrary $\psi \in \mathcal{F}C(p_1, \cdots, p_n; p)$, consider the corresponding composition scheme in $\mathcal{Q}$ obtained by labelling each vertex in the planar standard planar tree representing ψ by its image under G in Q. We call this the composition scheme associated with ψ and denote it by $cs(\psi)$. It follows that $G\psi$ is the composition in $\mathcal{Q}$ of $cs(\psi)$.

We can use this construction to describe operads using generators and relations, much like the description of certain groups by generators and relations. This will become very handy when we discuss the closed monoidal structure on *Operad*. Let $\mathcal{P}$ be the free planar operad on the collection C. We refer to C as *generators*. A set of *relations* in $\mathcal{P}$ is a family of sets $R = \{R_{p_1, \cdots, p_n; p_0}\}_{p_i \in ob(\mathcal{P})}$ where $R_{p_1, \cdots, p_n; p_0}$ is a relation on the set $\mathcal{P}(p_1, \cdots, p_n; p_0)$. For two operations $\psi, \psi' \in \mathcal{P}(p_1, \cdots, p_n; p)$ we write $\psi \sim \psi'$ if $(\psi, \psi') \in R_{p_1, \cdots, p_n, p_0}$. A set of relations R is called *normal* if each $R_{p_1, \cdots, p_n, p_0}$ is an equivalence relation which is a congruence for the composition in $\mathcal{P}$ in the sense that given $\psi_0, \cdots, \psi_n$ and $\psi_0', \cdots, \psi_n'$ with $\psi_i \sim \psi_i'$ for each $0 \le i \le n$ then (whenever the composition is defined)

$$\psi_0 \circ (\psi_1, \cdots, \psi_n) \sim \psi_0' \circ (\psi_1', \cdots, \psi_n')$$

holds. Since the intersection of normal relations is again a normal relation, it follows that given any relation R in $\mathcal{P}$, there is a unique smallest normal relation R' that contains it. We call this R' the *normal* relation *generated* by R.

It is now clear that given a normal relation R' in $\mathcal{P}$, there is an operad $\mathcal{P}/R'$ given by

$$ob(\mathcal{P}/R') = ob(\mathcal{P})$$

and for objects $p_1, \cdots, p_n, p_0 \in ob(\mathcal{P}/R')$ we set

$$(\mathcal{P}/R')(p_1, \cdots, p_n; p_0) = \mathcal{P}(p_1, \cdots, p_n; p_0)/\sim$$

with the obvious operadic structure induced from $\mathcal{P}$.

Definition 1.2.2. Let C be a collection and R a set of relations in the planar operad $\mathcal{F}_\pi C$. The planar operad $\mathcal{F}_\pi C/R'$, where R' is the normal relation generated by R, is called the planar operad *generated* by the *generators* C and the *relations* R.

Obviously the same can be applied to symmetric operads. The only thing we need to do is modify the definition of a normal relation R' in a free symmetric operad $\mathcal{P}$ to involve the symmetric group actions as well. In detail, let C be a collection. A set of relations R in the symmetric operad $\mathcal{P} = Symm(\mathcal{F}_\pi C)$ is a set of relations in the planar operad underlying $\mathcal{P}$. R is called *normal* if it is normal in the planar sense and if given $\psi \sim \psi'$ and $\sigma \in \Sigma_n$ (n being the arity of ψ), then

$$\sigma^*\psi \sim \sigma^*\psi'.$$

Just as before, given any set of relations R, there exists a unique smallest normal set of relations R' containing R. Clearly, for a normal set of relations R, one can define the operad $\mathcal{P}/R$ just as above.

DEFINITION 1.2.3. Let C be a collection and R a set of relations in the symmetric operad $Symm(\mathcal{F}_\pi C)$. The symmetric operad $\mathcal{F}_\pi C/R'$, where R' is the normal set of relations generated by R, is called the symmetric operad *generated* by the *generators* C and the *relations* R.

REMARK 1.2.4. In the special case where in the collection C all sets not of the form $C(a;b)$ are empty, one may interpret C as defining a directed graph (in the traditional sense of the word). It is easily verified that in that case the operad $\mathcal{F}_\pi C$ has only unary operations and is thus (essentially) a category. This category is of course isomorphic to the free category on the graph given by C.

When we use this construction to describe operads, we will usually not define the set R of relations in the way given above. Rather, we will just give a list of the equations between various operations that we wish to force.

It is clear from our construction that if the (planar or symmetric) operad $\mathcal{P}$ is generated by C and R, then, given any operad $\mathcal{Q}$, a map of operads $F : \mathcal{P} \to \mathcal{Q}$ corresponds exactly to a function $f : ob(\mathcal{P}) \to ob(\mathcal{Q})$ and a family of functions

$$f : C(p_1, \cdots, p_n; p) \to \mathcal{Q}(fp_1, \cdots, fp_n; fp)$$

mapping the generators of $\mathcal{P}$ to operations in $\mathcal{Q}$ such that if $\psi \sim \psi'$ in $\mathcal{F}C$ then the compositions of the composition schemes $cs(\psi)$ and $cs(\psi')$ in $\mathcal{Q}$, which correspond to ψ and ψ', are equal.

REMARK 1.2.5. One can obviously define the notion of a symmetric collection and proceed to construct the free symmetric operad on a symmetric collection. This is the more usual approach in the literature (e.g., [**6**]) yet for our purposes in this work the above (slightly simpler) construction is sufficient.

1.3. Limits and colimits in the category of operads

In this section we prove that the category *Operad* is small complete and small cocomplete. We give explicit constructions for products, coproducts, equalizers, and coequalizers which of course suffice to prove that all small limits and colimits exist (see [**34**] Theorem 2, page 113). We also obtain the easy result that the functor $ob : Operad \to Set$ that sends an operad $\mathcal{P}$ to $ob(\mathcal{P})$ preserves both limits and colimits. We wish to point out that the existence of limits and colimits of operads follow from general category theory (the category of operads is defined by a finite limit theory and thus is locally finitely presentable). Our decision to give an explicit construction is motivated by two considerations. One is to stress the analogy with category theory, since the limits and colimits of operads are constructed in essentially the same way as limits and colimits of categories. The other consideration is to emphasize the difference from the construction of limits and colimits of operads in the classical sense. The common construction of limits and colimits of operads usually consist of a diagram of operads with just one object and then calculate the (co)limit inside the category of operads with just one object, which is of course very different then the (co)limit of the same diagram inside the category of all operads.

THEOREM 1.3.1. *The category Operad is small complete.*

PROOF. It is sufficient to prove that *Operad* has equalizers and small products. Let $\mathcal{P} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{Q}$ be an equalizer diagram. We construct an operad $\mathcal{Q}$ as follows. The set of objects of $\mathcal{R}$ is the equalizer

$$ob(\mathcal{R}) \xrightarrow{e} ob(\mathcal{P}) \overset{F}{\underset{G}{\rightrightarrows}} ob(\mathcal{Q})$$

in Set, which we view as a subset of $ob(\mathcal{P})$. Given objects $r_1, \cdots, r_n, r_0 \in ob(\mathcal{R})$ we have that $Fr_i = Gr_i = r_i^{'}$. Let the set of operations from $(r_1, \cdots, r_n)$ to r_0 be the equalizer

$$\mathcal{R}(r_1, \cdots, r_n; r) \xrightarrow{e} \mathcal{P}(r_1, \cdots, r_n; r) \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{Q}(r_1', \cdots, r_n'; r')$$

where again we view $\mathcal{R}(r_1, \cdots, r_n; r_0)$ as a subset of $\mathcal{P}(r_1, \cdots, r_n; r_0)$. The operadic structure on $\mathcal{R}$ is induced from that of $\mathcal{P}$ in the obvious way. This makes $\mathcal{R}$ into an operad in such a way that all of the above given equalizing maps (all called) e, form together a map of operads $e : \mathcal{R} \to \mathcal{P}$. It is easily verified that this makes $\mathcal{R}$ into an equalizer of the diagram $\mathcal{P} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{Q}$. Given a small collection $\{\mathcal{P}_i\}_{i \in I}$ of operads we can similarly construct a product for this family such that

$$ob(\prod_{i \in I} \mathcal{P}_i) = \prod_{i \in I} ob(\mathcal{P}_i).$$

We omit the details. □

THEOREM 1.3.2. *The category Operad is small cocomplete.*

PROOF. Again we just need to show that *Operad* has all coequalizers and all small coproducts. Consider a coequalizer diagram

$$\mathcal{P} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{Q}$$

of operads. Let C be the collection underlying the operad $\mathcal{Q}$. The coequalizer we are looking for is then the operad generated by C and the relations

$$F\psi = G\psi$$

for any operation ψ in $\mathcal{P}$, as well as all relations describing the composition in $\mathcal{Q}$. Coproducts are constructed similarly, and we omit the rest of the details. □

Corollary 1.3.3. *The functor* $ob : Operad \to Set$ *which sends an operad* $\mathcal{P}$ *to the set* $ob(\mathcal{P})$ *preserves all small limits and all small colimits.*

Proof. This follows either by inspection of the constructions of limits and colimits in *Operad* or from the fact that $ob : Operad \to Set$ has both a left and a right adjoint (as can easily be seen). □

1.4. Yoneda's lemma

In this section we briefly study how the Yoneda lemma extends from category theory to operad theory. To that end we introduce representable functors for operads, a construction that, from the point of view of operads as a tool to describe algebraic structures, associates with each operad $\mathcal{P}$ some canonical algebras, namely those functors that are represented by the objects of $\mathcal{P}$.

Definition 1.4.1. Let $\mathcal{P}$ be an operad and $q \in ob(\mathcal{P})$. The *representable functor* $\mathcal{P}(q^*, -) : \mathcal{P} \to Set$ is the functor of operads defined as follows. For an object $p \in ob(\mathcal{P})$ we have

$$\mathcal{P}(q^*, -)(p) = \coprod_{n=0}^{\infty} \mathcal{P}(q^n; p) = \mathcal{P}(q^*, p)$$

where q^n is the tuple $(q, \cdots, q)$ with q occurring n times. Given $\psi \in \mathcal{P}(p_1, \cdots, p_m; p)$ we define the operation $\mathcal{P}(q^*, \psi) \in Set(\mathcal{P}(q^*, p_1), \cdots, \mathcal{P}(q^*, p_m); \mathcal{P}(q^*, p))$, i.e., a function

$$\begin{array}{c} \coprod \mathcal{P}(q^n; p_1) \times \cdots \times \coprod \mathcal{P}(q^n; p_m) \\ \Big\downarrow \\ \coprod \mathcal{P}(q^n; p) \end{array}$$

as follows. For $(\psi_1, \cdots, \psi_m)$ with $\psi_i \in \mathcal{P}(q^{n_i}; p_i)$ we define $\mathcal{P}(q^*, \psi)(\psi_1, \cdots, \psi_m)$ to be $\psi \circ (\psi_1, \cdots, \psi_m)$, which has input $q^{n_1 + \cdots + n_m}$ and output p and is thus an element of $\mathcal{P}(q^*, p)$.

It is trivial to prove that $\mathcal{P}(p^*, -)$ is indeed a functor.

Remark 1.4.2. This extends the usual definition of representable functors in the theory of categories in the sense that given a category $\mathcal{C}$ and an object $C \in ob(\mathcal{C})$ the representable functor $j_!(\mathcal{C})(C^*, -)$ is naturally isomorphic to $j_!(\mathcal{C}(C, -))$. This follows since for any $D \in ob(\mathcal{C})$, by definition

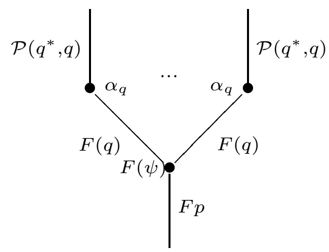
$$j_!(\mathcal{C})(C^*, D) = \coprod_{n=0}^{\infty} j_!\mathcal{C}(C^n; D)$$

however for $n \neq 1$ the set $j_!\mathcal{C}(C^n; D)$ is empty while for $n = 1$ it is exactly $\mathcal{C}(C, D)$.

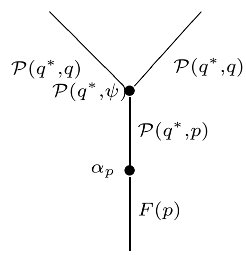
Lemma 1.4.3. *(Yoneda for operads) Let* $\mathcal{P}$ *be an operad,* $q \in ob(\mathcal{P})$*, and* $F : \mathcal{P} \to Set$ *a functor. There is a natural bijection between the set of natural transformations* $\alpha : \mathcal{P}(q^*, -) \to F$ *and the set* $F(q)$*.*

Proof. First we show that a natural transformation $\alpha : \mathcal{P}(q^*, -) \to F$ is completely determined by $\alpha_q(id_q)$, where $\alpha_q : \mathcal{P}(q^*, q) \to F(q)$ is the component of

α at q. To that end let $\psi \in \mathcal{P}(q^*, p)$, that is $\psi \in \mathcal{P}(q^k; p)$ for some $k \geq 0$. Naturality of α with respect to ψ implies that the composition of the composition scheme



is equal to that of the composition scheme



We now chase the value of $(id_q, \cdots, id_q)$ along both schemes. From the first one we obtain the value $F(\psi)(\alpha_q(id_q), \cdots, \alpha_q(id_q))$, while from the second one we obtain the value $\alpha_p(\mathcal{P}(q^*, \psi)(id_q, \cdots, id_q)) = \alpha_p(\psi \circ (id_q, \cdots, id_q)) = \alpha_p(\psi)$. Since both compositions are equal, we see that

$$\alpha_p(\psi) = F(\psi)(\alpha_q(id_q), \cdots, \alpha_q(id_q))$$

and thus that α is completely determined by $\alpha_q(id_q)$. Furthermore, a straightforward verification shows that for any fixed $a \in F(q)$, the formula

$$\alpha_p(\psi) = F(\psi)(a, \cdots, a)$$

for all p and $\psi \in \mathcal{P}(q^*, p)$, defines a natural transformation $\alpha(a) : \mathcal{P}(q^*, -) \to F$. It now follows that the assignment $\alpha \mapsto \alpha_q(id_q)$ has an inverse function, namely $a \mapsto \alpha(a)$. □

Example 1.4.4. Consider the operad *Comm* from Example 1.1.13 whose algebras are commutative monoids. There is precisely one representable functor $Comm(\star^*, -) : Comm \to Set$ since *Comm* has just one object. It is easy to verify that the commutative monoid corresponding to that representable functor is the free commutative monoid on one object. The correspondence between natural transformations $Comm(\star^*, -) \to F$ and the set $F(\star)$ is precisely the universal property of free commutative monoid on one object. Representable functors for other operads usually yield some 'free' algebras as well.

We end this section by noting that a Yoneda embedding does not exist for operads. The Yoneda embedding for categories states that the assignment $C \mapsto \mathcal{C}(C, -)$ is an embedding

$$\mathcal{C}^{op} \to \underline{Cat}(\mathcal{C}, Set).$$

Such a result is not possible for operads since for an operad $\mathcal{P}$ the opposite 'operad' $\mathcal{P}^{op}$ does not exist. Instead $\mathcal{P}^{op}$ can be defined to have the structure of an 'anti-operad' where each operation has one input and possibly many (or no) outputs. However there is then no natural definition of an arrow from an anti-operad to an operad.

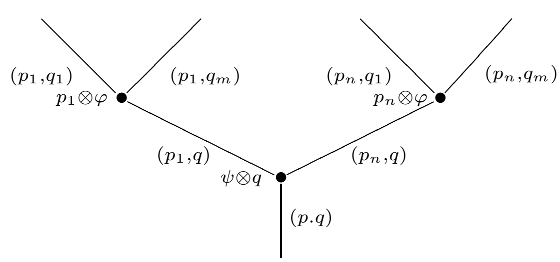
1.5. Closed monoidal structure on the category of operads

The category *Cat* is a cartesian closed category for which the internal Hom $\underline{Cat}(\mathcal{C},\mathcal{D})$ is formed by taking functors as objects, and natural transformations as arrows. In [**7**], Boardman and Vogt define a tensor product for topological operads. In this section we show that essentially the same tensor product can be defined in the context of our notion of operads and we show that it turns the category *Operad* into a symmetric closed monoidal category in an analogous way.

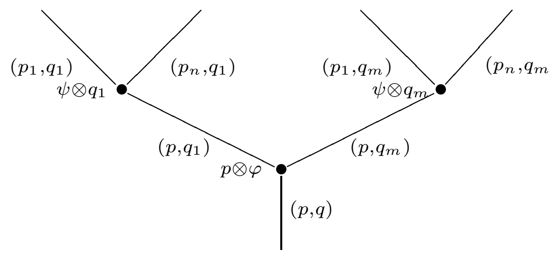
Definition 1.5.1. (The Boardman-Vogt tensor product) Let $\mathcal{P}$ and $\mathcal{Q}$ be two symmetric operads. The *Boardman-Vogt tensor product* of these operads is the symmetric operad $\mathcal{P} \otimes_{BV} \mathcal{Q}$ with $ob(\mathcal{P} \otimes_{BV} \mathcal{Q}) = ob(\mathcal{P}) \times ob(\mathcal{Q})$ given in terms of generators and relations as follows. Let C be the collection on $ob(\mathcal{P}) \times ob(\mathcal{Q})$ which contains the following generators. For each $q \in ob(\mathcal{Q})$ and each operation $\psi \in \mathcal{P}(p_1, \cdots, p_n; p)$ there is a generator $\psi \otimes_{bv} q$ in $C((p_1, q), \cdots, (p_n, q); (p, q))$ and for each $p \in ob(\mathcal{P})$ and an operation $\varphi \in \mathcal{Q}(q_1, \cdots, q_m; q)$ there is a generator $p \otimes_{bv} \varphi$ in $C((p, q_1), \cdots, (p, q_m); (p, q))$. There are five types of relations among the arrows generated by these generators:

1) $(\psi \otimes_{bv} q) \circ ((\psi_1 \otimes_{bv} q), \cdots, (\psi_n \otimes_{bv} q)) = (\psi \circ (\psi_1, \cdots, \psi_n)) \otimes_{bv} q$
2) $\sigma^*(\psi \otimes_{bv} q) = (\sigma^* \psi) \otimes_{bv} q$
3) $(p \otimes_{bv} \varphi) \circ ((p \otimes_{bv} \varphi_1), \cdots, (p \otimes_{bv} \varphi_m)) = p \otimes_{bv} (\varphi \circ (\varphi_1, \cdots, \varphi_m))$
4) $\sigma^*(p \otimes_{bv} \varphi) = p \otimes_{bv} (\sigma^* \varphi)$
5) $(\psi \otimes_{bv} q) \circ ((p_1 \otimes_{bv} \varphi), \cdots, (p_n \otimes_{bv} \varphi)) = \sigma^*_{m,n}((p \otimes_{bv} \varphi) \circ ((\psi, q_1), \cdots, (\psi, q_m)))$

By the relations above we mean every possible choice of operations for which the compositions are defined. The relations of type 1 and 2 ensure that for any $q \in ob(\mathcal{P})$, the map $p \mapsto (p, q)$ naturally extends to a map of operads $\mathcal{P} \to \mathcal{P} \otimes_{BV} \mathcal{Q}$. Similarly, the relations of type 3 and 4 guarantee that for each $p \in ob(\mathcal{P})$, the map $q \mapsto (p, q)$ naturally extends to a map of operads $\mathcal{Q} \to \mathcal{P} \otimes_{BV} \mathcal{Q}$. The relation of type 5 can be pictured as follows. The left hand side is an operation in the free operad, represented by the labelled planar tree



while the right hand side is given by the tree



before applying $\sigma^*_{m,n}$, which is the same permutation that was used in the definition of natural transformation, in order to equate the domain of the second operation to that of the first one. We call this type of relation the *interchange* relation.

THEOREM 1.5.2. *The category Operad with the Boardman-Vogt tensor product is a symmetric closed monoidal category.*

PROOF. The fact that the Boardman-Vogt tensor product makes *Operad* into a symmetric monoidal category is a straightforward verification and is omitted. We now describe the internal Hom. Let $\mathcal{Q}$ and $\mathcal{R}$ be two operads. We are going to prove that

$$\underline{Operad}(\mathcal{Q},\mathcal{R}) = Func(\mathcal{Q},\mathcal{R}),$$

that is $ob(\underline{Operad}(\mathcal{Q},\mathcal{R}))$ are all functors $\mathcal{Q}\to\mathcal{R}$ and for such functors $F_1,\cdots,F_n,G$, the operations with input $F_1,\cdots,F_n$ and output G are the natural transformations from $(F_1,\cdots,F_n)$ to G.

We need to construct a bijection

$$Operad(\mathcal{P}\otimes_{BV}\mathcal{Q},\mathcal{R}) \cong Operad(\mathcal{P},\underline{Operad}(\mathcal{Q},\mathcal{R}))$$

natural in $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$. Let $F:\mathcal{P}\otimes_{BV}\mathcal{Q}\to\mathcal{R}$ be a functor. For each $p\in ob(\mathcal{P})$ we need to construct a functor $F_p:\mathcal{Q}\to\mathcal{R}$. This functor is given on objects $q\in ob(\mathcal{Q})$ and operations φ in $\mathcal{Q}$ by

$$F_p(q) = F(p,q)$$

and

$$F_p(\varphi) = p\otimes_{bv}\varphi$$

which is obviously functorial. Actually, F_p is just the composition

$$\mathcal{Q}\longrightarrow\mathcal{P}\otimes_{BV}\mathcal{Q}\xrightarrow{F}\mathcal{R}$$

where the first functor is the one sending q to (p,q) mentioned above, right after the definition of the Boardman-Vogt tensor product. If we are now given an operation $\psi\in\mathcal{P}(p_1,\cdots,p_n;p)$, we need to construct a natural transformation

$$\alpha(\psi)=\alpha:(F_{p_1},\cdots,F_{p_n})\to F_p.$$

The component of this natural transformation at $q\in ob(\mathcal{Q})$ is the arrow

$$\alpha_q = F(\psi\otimes_{bv}q)\in\mathcal{R}(F(p_1,q),\cdots,F(p_n,q);F(p,q)) = \mathcal{R}(F_{p_1}q,\cdots,F_{p_n}q;F_pq).$$

To verify that $\alpha(p)$ is indeed a natural transformation we need to show that given an operation $\varphi\in\mathcal{Q}(q_1,\cdots,q_n;q)$ the two composition schemes from the definition of a natural transformation yield the same operation. In our case these two composition schemes are the two trees which appear in the interchange relation, with F applied to each edge and vertex. Since F is a functor, the interchange relation guarantees that $\alpha(p)$ is a natural transformation.

To go in the other direction, let $G : \mathcal{P} \to \underline{Operad}(\mathcal{Q}, \mathcal{R})$ be a functor. To construct a functor $H : \mathcal{P} \otimes_{BV} \mathcal{Q} \to \mathcal{R}$ we need to specify it on the objects and on the generators of $\mathcal{P} \otimes_{BV} \mathcal{Q}$ such that the relations are satisfied. For an object $(p, q) \in ob(\mathcal{P} \otimes_{BV} \mathcal{Q})$ let

$$H(p, q) = G(p)(q)$$

and for a generator of the form $p \otimes_{bv} \varphi$ we define

$$H(p \otimes_{bv} \varphi) = G(p)(\varphi).$$

For a generator of the form $\psi \otimes_{bv} q$ where $\psi \in \mathcal{P}(p_1, \cdots, p_n; p)$, we have the natural transformation $G(\psi) : (G(p_1), \cdots, G(p_n) \to G(p)$. We then define

$$H(\psi \otimes_{bv} q) = G(\psi)_q$$

to be the component of $G(\psi)$ at q. We omit the rest of the details. □

REMARK 1.5.3. Notice that the symmetric actions in the definition of an operad again proved to be crucial for the definition of the Boardman-Vogt tensor product. This again illustrates the significant differences between the category of planar operads and symmetric operads. One can of course define a Boardman-Vogt style tensor product for planar operads too, simply by leaving out the interchange relation, and obtain a closed monoidal structure. However, the corresponding internal Hom is not well behaved as we show below.

Recall again the operad $Comm$ (Example 1.1.13) whose algebras are commutative monoids. We have seen that $Func(Comm, Set)$ is isomorphic to the operad of commutative monoids with the cartesian product. We now make the following definition.

DEFINITION 1.5.4. Let $\mathcal{P}$ and $\mathcal{E}$ be operads. We denote by $Alg(\mathcal{P}, \mathcal{E})$ the operad $\underline{Operad}(\mathcal{P}, \mathcal{E})$ and refer to it as the operad of $\mathcal{P}$ algebras in $\mathcal{E}$, or as the operad of $(\mathcal{P}, \mathcal{E})$-operads.

This is again just a shift in focus regarding the roles that the two operads play (see Remark 1.1.15). Notice that the objects of $Alg(\mathcal{P}, \mathcal{E})$ are precisely the $\mathcal{P}$-algebras in $\mathcal{E}$.

The internal Hom captures thus the notion of $\mathcal{P}$-algebras in $\mathcal{E}$, provides a notion of operations between such algebras (namely, natural transformations), and in such a way that they form themselves an operad.

REMARK 1.5.5. Let us return now to discuss the differences between symmetric and non-symmetric operads. Assume that we consider $Operad_\pi$ as a closed monoidal category via the modified Boardman-Vogt tensor product (i.e., without the interchange relations). Let As_π be the planar operad that has just one object and one n-ary operation of each arity. If we now inspect the operad

$$\underline{Operad_\pi}(As_\pi, Set)$$

we easily see that the objects correspond to associative monoids and that unary arrows correspond to maps between the corresponding associative monoids. However, arrows of arity $n > 1$ fail to preserve the monoid structures, precisely because of the lack of symmetries. On the other hand, for the symmetric operad

$As = Symm(As_\pi)$, it is easy to confirm that

$$\underline{Operad}(As, Set) \cong Mon,$$

where we view Mon, the category of associative monoids, as an operad via the usual cartesian product of monoids.

In short, if we define $Alg_\pi(\mathcal{P}, \mathcal{E}) = \underline{Operad_\pi}(\mathcal{P}, \mathcal{E})$ for planar operads $\mathcal{P}$ and $\mathcal{E}$ then we have

$$j^*(Alg_\pi(\mathcal{P}, \mathcal{E})) \cong j^*(Alg(Symm(\mathcal{P}), Symm(\mathcal{E})))$$

but in general

$$Alg_\pi(\mathcal{P}, \mathcal{E}) \ncong Alg(Symm(\mathcal{P}), Symm(\mathcal{E})).$$

Thus planar operads fail to capture the correct notion of multi-maps of algebras.

We end this section by noting that given two operads $\mathcal{P}$ and $\mathcal{Q}$, one has the equality:

$$Alg(\mathcal{P}, Alg(\mathcal{Q}, \mathcal{E})) \cong Alg(\mathcal{P} \otimes_{BV} \mathcal{Q}, \mathcal{E}) \cong Alg(\mathcal{Q}, Alg(\mathcal{P}, \mathcal{E})).$$

This property can loosely be stated by saying that $\mathcal{P} \otimes_{BV} \mathcal{Q}$-algebras in $\mathcal{E}$ are the same as $\mathcal{P}$ algebras in $\mathcal{Q}$-algebras in $\mathcal{E}$ and, at the same time, the same as $\mathcal{Q}$-algebras in $\mathcal{P}$-algebras in $\mathcal{E}$.

1.6. Quillen model structure on the category of operads

In this section we introduce a Quillen model structure which is a direct generalization of the 'folk' Quillen model structure on Cat. Rezk, in an unpublished manuscript [**41**], gave a complete proof of this model structure, which we will now recall. More recently, Joyal and Tierney [**22**] establish the same model structure as a special case in the much more general context of internal categories in a topos. Again, the same model structure is established by Lack [**28**] as a special case in the context of model structures on 2-categories.

THEOREM 1.6.1. *The category Cat admits a cartesian closed Quillen model structure where:*

1) The weak equivalences are the categorical equivalences.

2) The cofibrations are those functors $F : \mathcal{C} \to \mathcal{D}$ that are injective on objects.

3) The fibrations are those functors $F : \mathcal{C} \to D$ such that for any $c \in Ob(\mathcal{C})$ and each isomorphism $\psi : Fc \to d$ in $\mathcal{D}$, there exists an isomorphism $\phi : c \to c'$ for which $F\phi = \psi$.

We refer to this model structure as the folk model structure on Cat. The proof itself is not at all difficult and constitutes one of the rare examples of non-trivial, interesting Quillen model structures which are easily proved by elementary means. As stated, this model structure extends, to what we call the folk model structure, to the category $Operad$ as we now prove.

THEOREM 1.6.2. *The category Operad admits a Quillen model structure where:*

1) The weak equivalences are the operadic equivalences.

2) The cofibrations are those functors $F : \mathcal{P} \to \mathcal{Q}$ that are injective on objects.

3) The fibrations are those functors $F : \mathcal{P} \to \mathcal{Q}$ such that for any $p \in Ob(\mathcal{P})$ and each isomorphism $\psi : Fp \to q$ in $\mathcal{Q}$, there exists an isomorphism $\phi : p \to p'$ for which $F\phi = \psi$.

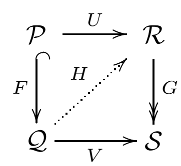
Proof. Notice that a functor $F : \mathcal{P} \to \mathcal{Q}$ is a fibration (respectively cofibration) if, and only if, j^*F is a fibration (respectively cofibration). Notice as well that a functor $F : \mathcal{P} \to \mathcal{Q}$ is a trivial fibration if, and only if, the function $ob(F) : ob(\mathcal{P}) \to ob(\mathcal{Q})$ is surjective and F is fully faithful. We now set out to prove the Quillen axioms.

M1 (Existence of limits and colimits): As discussed above, *Operad* has all small limits and small colimits (Theorem 1.3.1 and 1.3.2).

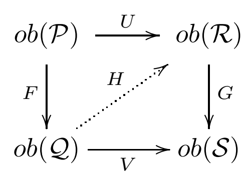
M2 (2 out of 3 property): Obviously holds.

M3 (Closed under retracts): Can easily be established.

M4 (Liftings): Consider the square



where F is a cofibration and G is a fibration. We need to prove the existence of a lift H making the diagram commute, whenever F or G is a weak equivalence. Assume first that G is a weak equivalence. Applying the object functor (that sends an operad $\mathcal{P}$ to the set $ob(\mathcal{P})$) to the lifting diagram we obtain



where F is injective and G is surjective. We can thus find a lift H. Let now $\psi \in \mathcal{Q}(q_1, \cdots, q_n; q)$, and consider $V(\psi) \in \mathcal{S}(Vq_1, \cdots, Vq_n; Vq)$. Since G is fully faithful and $HG = V$ on the level of objects, we obtain that the function

$$G : \mathcal{R}(Hq_1, \cdots, Hq_n; Hq) \to \mathcal{S}(Vq_1, \cdots, Vq_n; Vq)$$

is an isomorphism. We now define $H(\psi) = G^{-1}(V(\psi))$. It is easily checked that this (uniquely) extends H and makes it into the desired lift.

Assume now that F is a trivial cofibration. We can thus construct a functor $F' : \mathcal{Q} \to \mathcal{P}$ such that

$$F' \circ F = id_{\mathcal{P}}$$

together with a natural isomorphism $\alpha : F \circ F' \to id_{\mathcal{Q}}$. We can moreover choose α such that for each $p \in ob(\mathcal{P})$, the component at Fp is given by

$$\alpha_{Fp} = id_{Fp}.$$

To define $H : ob(\mathcal{Q}) \to ob(\mathcal{R})$ let $q \in ob(\mathcal{Q})$ and consider the object $VFF'q \in ob(\mathcal{S})$. Since

$$VFF'q = GUF'q$$

it follows from the definition of fibration that there is an object $H(q)$ and an isomorphism

$$\beta_q : UF'q \to Hq$$

in $\mathcal{R}$ such that

$$GHq = Vq$$

and

$$G\beta_q = V\alpha_q.$$

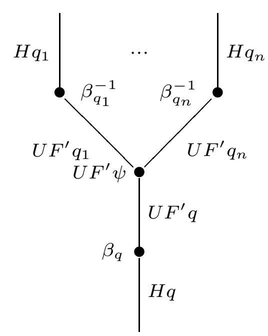
We can also choose β such that for every $p \in ob(\mathcal{P})$

$$HFp = Up$$

and

$$\beta_{Fp} = id_{Up}.$$

Let now $\psi \in \mathcal{Q}(q_1, \cdots, q_n; q)$ and define $H(\psi)$ to be the composition of the following composition scheme in $\mathcal{R}$:



The resulting H is easily seen to be a functor and the desired lift.

M5 (Factorizations): Let $F : \mathcal{P} \to \mathcal{Q}$ be a functor. We first construct a factorization of F into a trivial cofibration followed by a fibration. Construct first the following operad $\mathcal{P}'$ with

$$ob(\mathcal{P}') = \{(p, \varphi, q) \in ob(\mathcal{P}) \times \mathcal{Q}(Fp, q) \times ob(\mathcal{Q}) \mid \varphi \text{ is an isomorphism}\}$$

and, for objects $(p_1, \varphi_1, q_1), \cdots, (p_n, \varphi_n, q_n), (p, \varphi, q)$, the arrows

$$\mathcal{P}'((p_1, \varphi_1, q_1), \cdots, (p_n, \varphi_n, q_n); (p, \varphi, q)) = \mathcal{P}(p_1, \cdots, p_n; p)$$

with the obvious operadic structure. If we now define $G : \mathcal{P} \to \mathcal{P}'$ on objects $p \in ob(\mathcal{P})$ by

$$G(p) = (p, id_{Fp}, Fp)$$

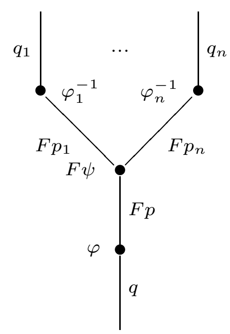
and for and arrow $\psi \in \mathcal{P}(p_1, \cdots, p_n; p)$ by

$$G(\psi) = \psi$$

we evidently get a functor, which is clearly a trivial cofibration. We now define the functor $H : \mathcal{P}' \to Q$ on objects $(p, \varphi, q) \in ob(\mathcal{P}')$ by

$$H(p, \varphi, q) = q$$

and on an arrow $\psi \in \mathcal{P}'((p_1,\varphi_1,q_1),\cdots,(p_n,\varphi_n,q_n);(p,\varphi,q))$ to be the composition of the composition scheme



Clearly, H is a fibration since if $f : H(p,\varphi,q) \to q'$ is an isomorphism in $\mathcal{Q}$ then $(p, f\varphi, q')$ is also an object of $\mathcal{Q}$ and id_p is an isomorphism in $\mathcal{P}'$ from (p,φ,q) to $(p, f\varphi, q')$ which, by definition, maps under H to $f\varphi \circ F(id_p) \circ \varphi^{-1} = f$. Since we obviously have that $F = H \circ G$ we have the desired factorization.

We now proceed to prove that F can be factored as a composition of a cofibration followed by a trivial fibration. Let $\mathcal{Q}'$ be the operad with

$$ob(\mathcal{Q}') = ob(\mathcal{P}) \coprod ob(\mathcal{Q})$$

and with arrows defined as follows. Given an object $x \in ob(\mathcal{Q}')$ let (somewhat ambiguously)

$$Fx = \begin{cases} x, & \text{if } x \in ob(\mathcal{Q}) \\ Fx, & \text{if } x \in ob(\mathcal{P}) \end{cases}$$

Now, for objects $x_1,\cdots,x_n,x \in ob(\mathcal{Q}')$ let

$$\mathcal{Q}'(x_1,\cdots,x_n;x) = \mathcal{Q}(Fx_1,\cdots,Fx_n;Fx).$$

The operad structure is the evident one. If we now define a functor $G : \mathcal{P} \to Q'$ for an object $p \in ob(\mathcal{P})$ and an arrow $\psi \in \mathcal{P}(p_1,\cdots,p_n;p)$ by

$$Gp = p$$

and

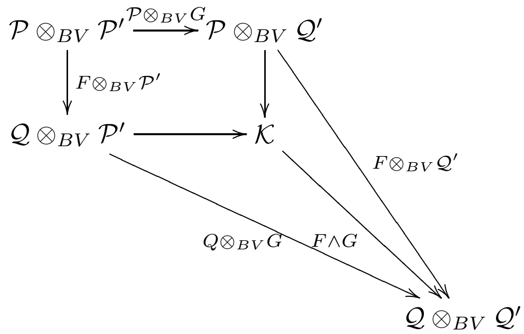
$$G\psi = F\psi$$

then we obviously obtain a cofibration. We now define $H : \mathcal{Q}' \to \mathcal{Q}$ as follows. Given an object $x \in ob(\mathcal{Q}')$, if $x \in ob(\mathcal{P})$ then we set $Hx = Fx$ and if $x \in ob(\mathcal{Q})$ then we set $Hx = x$ (thus in our slightly ambiguous notation we have that $Hx = Fx$). Given an arrow $\psi \in \mathcal{Q}'(x_1,\cdots,x_n;x)$, defining $H\psi = \psi$ makes H into a functor, clearly fully faithful. Moreover H is a fibration as can easily be seen. Since obviously $F = H \circ G$ the proof is complete. □

Note that all operads are both fibrant and cofibrant under this model structure.

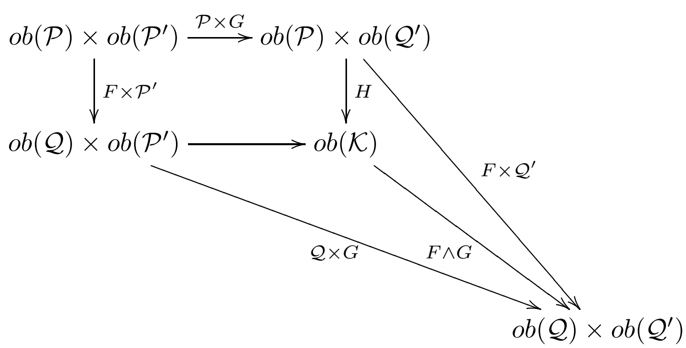
THEOREM 1.6.3. *The category Operad with the Boardman-Vogt tensor product and the model structure defined above is a monoidal model category.*

PROOF. Since all objects are cofibrant we only have to prove that given two cofibrations $F : \mathcal{P} \hookrightarrow \mathcal{Q}$ and $G : \mathcal{P}' \hookrightarrow \mathcal{Q}'$, the push-out corner map $F \wedge G$



is a cofibration which is a trivial cofibration if F is a trivial cofibration.

Since in general $ob(\mathcal{P} \otimes_{BV} \mathcal{Q}) = ob(\mathcal{P}) \times ob(\mathcal{Q})$ and since $ob : Operad \to Set$ commutes with colimits, if we apply the functor ob we obtain the following diagram



which is again a pushout. We are given that F and G are injective from which follows that $F \times \mathcal{P}'$ and $\mathcal{P} \times G$ are also injective. It is now easy to verify that $F \wedge G$ is injective as well which proves that the operad map $F \wedge G : \mathcal{K} \to \mathcal{Q} \otimes_{BV} \mathcal{Q}'$ is a cofibration.

Assume now that F in the first diagram is also a weak equivalence, i.e., an operadic equivalence. It is trivial to verify that $F \otimes_{BV} \mathcal{P}'$ is also an equivalence. Thus $F \times \mathcal{P}'$ is a trivial cofibration. Since trivial cofibrations are closed under cobase change it follows that H is a trivial cofibration. Since $F \times \mathcal{Q}'$ is too an equivalence, the two out of three property implies that $F \wedge G$ is a trivial cofibration. $\square$

REMARK 1.6.4. Considering categories as operads, it is easily seen that in the proofs above every construction applied to categories yields again a category. For this reason these proofs can be restricted to the case of categories to give a proof of the folk model structure on Cat. Such a proof is essentially identical to the one given in [**41**].

LEMMA 1.6.5. *The adjunction* $Operad \underset{j_!}{\overset{j^*}{\rightleftarrows}} Cat$ *is a Quillen adjunction.*

PROOF. It is enough to prove that $j_!$ preserves cofibrations and trivial cofibrations. Actually it is trivial to verify the much stronger property that both j^* and $j_!$ preserve fibrations, cofibrations, and weak equivalences. $\square$

We end our treatment of the model structure on *Operad* with the following:

THEOREM 1.6.6. *The operadic model structure on Operad is cofibrantly generated.*

PROOF. Let $*$ be the operad with one object and just one arrow (the identity on $*$) and let H be the free living isomorphism operad, which has two objects and, besides the necessary identities, just one isomorphism between the two objects. It is a triviality to check that a functor $F : \mathcal{P} \to \mathcal{Q}$ is a fibration if, and only if, it has the right lifting property with respect to (any one of the two possible functors) $* \to H$.

To characterize the trivial fibrations by right lifting properties we will need to consider several other operads. First of all, it is clear that if a functor $F : \mathcal{P} \to \mathcal{Q}$ has the right lifting property with respect to $\phi \to *$ then $F : ob(\mathcal{P}) \to ob(\mathcal{Q})$ is surjective (where ϕ is the initial operad with no objects). For each $n \geq 1$ consider the operad Ar_n that has $n+1$ objects $\{0, 1, \cdots, n\}$ and is generated by a single arrow from $(1, \cdots, n)$ to 0. Thus a functor $Ar_n \to \mathcal{P}$ is just a choice of an arrow in $\mathcal{P}$ of arity n. Let ∂Ar_n be the sub-operad of Ar_n that contains all the objects of Ar_n but only the identity arrows. It now easily follows that if a functor $F : \mathcal{P} \to \mathcal{Q}$ has the right lifting property with respect to the inclusion $\partial Ar_n \to Ar_n$ then for any objects $p_1, \cdots, p_n, p \in ob(\mathcal{P})$, the function

$$F : \mathcal{P}(p_1, \cdots, p_n; p) \to \mathcal{Q}(Fp_1, \cdots, Fp_n; Fp)$$

is surjective. Consider now the operad PAr_n with $n+1$ objects $\{0, 1, \cdots, n\}$ generated by two different arrows from $(1, \cdots, n)$ to 0 and the obvious map $PAr_n \to Ar_n$ which identifies those two arrows. If a functor $F : \mathcal{P} \to \mathcal{Q}$ has the right lifting property with respect to $PAr_n \to Ar_n$ then the map

$$F : \mathcal{P}(p_1, \cdots, p_n; p) \to \mathcal{Q}(Fp_1, \cdots, Fp_n; Fp)$$

is injective. Combining these results we see that if a functor $F : \mathcal{P} \to \mathcal{Q}$ has the right lifting property with respect to the set of functors

$$\{\phi \to *\} \cup \{\partial Ar_n \to Ar_n \mid n \geq 0\} \cup \{PAr_n \to Ar_n \mid n \geq 0\}$$

then F is fully faithful and $F : ob(\mathcal{P}) \to ob(\mathcal{Q})$ is surjective, which implies that F is a trivial fibration. Finally, since all the functors just mentioned are cofibrations it follows that all trivial fibrations have the right lifting property with respect to them. This then proves that the trivial fibrations are exactly those functors having the right lifting property with respect to that set. □

1.7. Grothendieck construction for operads

We now turn to the definition of a Grothendieck construction for diagrams of operads. This construction is useful if one wishes to 'glue' a suitably parametrized family of operads into one operad. We start by giving an example where such a gluing procedure is required and then proceed to the construction itself.

For a fixed set A we consider the planar operad $\mathcal{C}_{\pi A}$ whose objects are

$$ob(\mathcal{C}_{\pi A}) = A \times A$$

and for a given signature $(a_1, a_2), (a_2, a_3) \cdots, (a_{n-1}, a_n); (a_1, a_n)$ there is a single operation in

$$\mathcal{C}_{\pi A}((a_1, a_2), (a_2, a_3), \cdots, (a_{n-1}, a_n); (a_1, a_n))$$

and for every $a \in A$ there is one operation in

$$\mathcal{C}_{\pi A}(;(a,a)).$$

There are no other operations except for those just mentioned. The operadic structure is now uniquely determined. We denote

$$\mathcal{C}_A = Symm(\mathcal{C}_{\pi A})$$

the symmetrization of $\mathcal{C}_{\pi A}$.

Let $\mathcal{E}$ be a symmetric monoidal category and let us consider a functor $\mathcal{C}_A \to \mathcal{E}$ where we view $\mathcal{E}$ as a symmetric operad. By adjunction this is the same as a functor $\mathcal{B} : \mathcal{C}_{\pi A} \to \mathcal{E}$ where $\mathcal{E}$ is now considered as a planar operad. Such a functor F consists of a function $\mathcal{B} : ob(\mathcal{C}_{\pi A}) \to ob(\mathcal{E})$, that is a choice of an object $\mathcal{B}(a,a')$ for any two elements $a, a' \in A$. Further, the operations of $\mathcal{C}_{\pi A}$ are to be mapped to operations of $\mathcal{E}$, so for each $a \in A$ we have a map

$$\mathcal{C}_{\pi A}(;(a,a)) \to \mathcal{E}(;\mathcal{B}(a,a))$$

which is just a choice of an arrow

$$id_a : I \to \mathcal{B}(a,a)$$

in $\mathcal{E}$, where I is the monoidal unit. Furthermore, for any two elements $a_1, a_2 \in A$ there is a map

$$\mathcal{C}_{\pi A}((a_1,a_2),(a_2,a_3);(a_1,a_3)) \to \mathcal{E}(\mathcal{B}(a_1,a_2) \otimes \mathcal{B}(a_2,a_3), \mathcal{B}(a_1,a_3))$$

that is, a choice of an arrow

$$m : \mathcal{B}(a_1,a_2) \otimes \mathcal{B}(a_2,a_3) \to \mathcal{B}(a_1,a_3)$$

in $\mathcal{E}$. It can now be easily verified that the functoriality condition implies that the various $\mathcal{B}(a,a')$ are the Hom-objects of a category enriched in $\mathcal{E}$ whose set of objects is A, with m the composition arrow.

Consider now the operad $\underline{Operad}(\mathcal{C}_A, \mathcal{E})$. From what we just showed, the objects of this operad are $\mathcal{E}$-enriched categories whose set of objects is the set A.

PROPOSITION 1.7.1. *Let $\mathcal{C}_A$ be as above and $\mathcal{E}$ a symmetric monoidal category. Let $Cat(\mathcal{E})_A$ be the category of all $\mathcal{E}$-enriched categories whose set of objects is A, and arrows those $\mathcal{E}$-enriched functors that are the identity on objects. There is a symmetric monoidal structure on $Cat(\mathcal{E})_A$ and, when we view $Cat(\mathcal{E})_A$ as an operad, we have:*

$$\underline{Operad}(\mathcal{C}_A, \mathcal{E}) \cong Cat(\mathcal{E})_A.$$

PROOF. We first describe the monoidal structure on $Cat(\mathcal{E})_A$. Let $\mathcal{A}$ and $\mathcal{A}'$ be two $\mathcal{E}$-enriched categories with $ob(\mathcal{A}) = ob(\mathcal{A}') = A$. Let $\mathcal{A} \otimes \mathcal{A}'$ be the $\mathcal{E}$-enriched category with set of objects equal to A whose arrow objects for $a_1, a_2 \in A$ is

$$\mathcal{A} \otimes \mathcal{A}'(a_1,a_2) = \mathcal{A}(a_1,a_2) \otimes \mathcal{A}'(a_1,a_2).$$

Composition in this category is defined 'component-wise' in the obvious way. It is routine to verify that this makes $Cat(\mathcal{E})_A$ into a symmetric monoidal category.

We have already established that the objects of $\underline{Operad}(\mathcal{C}_A, \mathcal{E})$ are the objects of $Cat(\mathcal{E})_A$. Given objects $\mathcal{A}_1, \cdots, \mathcal{A}_n, \mathcal{A} \in \underline{Operad}(\mathcal{C}_A, \mathcal{E})$, if we now unfold the definition of a natural transformation $\alpha : \mathcal{A}_1, \cdots, \mathcal{A}_n \to \mathcal{A}$, we readily discover that it corresponds precisely to a functor in $Cat(\mathcal{E})_A$ from $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \to \mathcal{A}$ with the tensor product as just defined, and thus establishes the isomorphism. $\square$

Having an operad $\mathcal{C}_A$ such that $\underline{Operad}(\mathcal{C}_A, \mathcal{E})$ is the operad of $\mathcal{E}$-enriched categories with a fixed set of objects (and special functors), it is natural to look for an operad $\mathcal{C}$ such that $\underline{Operad}(\mathcal{C}, \mathcal{E})$ will be isomorphic to the category $Cat(\mathcal{E})$ of all $\mathcal{E}$-enriched categories. However, such an operad does not exist (here is a sketch of a proof due to Tom Leinster: It suffices to prove that Cat is not monadic over Set^A for any set A. To do that one can show that the regular epimorphisms in a category monadic over Set^A are the coordinate-wise surjections, and are thus closed under composition. However, in Cat the regular epimorphisms are not closed under composition). We are thus led to look for a construction that will assemble the various operads $\underline{Operad}(\mathcal{C}_A, \mathcal{E})$ into one operad that (hopefully) will be isomorphic to $Cat(\mathcal{E})$.

Definition 1.7.2. A *diagram of operads* is a functor $F : \mathbb{B}^{op} \to Operad$ where $\mathbb{B}$ is a cartesian category called the *indexing* category. For an arrow $f : B \to B'$ in $\mathbb{B}$ we will denote the functor $Ff : FB' \to FB$ by f^*.

Example 1.7.3. Let $\mathbb{B} = Set$ with the usual cartesian product of sets. For each set $B \in ob(\mathbb{B})$ consider the operad $\mathcal{C}_{\pi B}$ described above. Any function $f : B \to B'$ induces a functor $F_\pi : \mathcal{C}_{\pi B} \to \mathcal{C}_{\pi B'}$ as follows. On the level of the objects we define

$$F_\pi : ob(\mathcal{C}_{\pi B}) \to ob(\mathcal{C}_{\pi B'})$$

to be the function

$$f \times f : B \times B \to B' \times B'.$$

On the level of operations, the functor F_π is then simply the identity

$$\begin{array}{c} \mathcal{C}_{\pi B}((b_1, b_2), \cdots, (b_{n-1}, b_n); (b_1, b_n)) \\ \downarrow F_\pi \\ \mathcal{C}_{\pi B'}((fb_1, fb_2), \cdots, f(b_{n-1}, fb_n); (fb_1, fb_n)). \end{array}$$

We now define $F = Symm(F_\pi) : \mathcal{C}_B \to \mathcal{C}_{B'}$ and we obtain a functor $\mathbb{B} \to Operad$. The assignment $B \mapsto \underline{Operad}(\mathcal{C}_B, Set)$ is thus contravariant in B and therefore defines a diagram of operads $F : \mathbb{B}^{op} \to Operad$.

Definition 1.7.4. (The Grothendieck construction) Let $F : \mathbb{B}^{op} \to Operad$ be a diagram of operads. We define the operad

$$\int_{\mathbb{B}} F$$

as follows. The objects of $\int_{\mathbb{B}} F$ are pairs (B, p) where $B \in ob(\mathbb{B})$ and $p \in ob(FB)$. An arrow in

$$\int_{\mathbb{B}} F((B_1, p_1), \cdots, (B_n, p_n); (B, p))$$

is a pair (f, ψ) where $f : B_1 \times \cdots \times B_n \to B$ is an arrow in $\mathbb{B}$ and ψ is an operation in $F(B_1 \times \cdots \times B_n)(\pi_1^* p_1, \cdots, \pi_n^* p_n; f^* B)$, where π_i is the canonical projection $B_1 \times \cdots \times B_n \to B_i$.

The composition in $\int_{\mathbb{B}} F$ is given as follows. If (f, ψ) is an operation from $((B_1, p_1), \cdots, (B_n, p_n))$ to (B, p) and for each $1 \leq i \leq n$ we have an operation (f_i, ψ_i) from $((B_1^i, p_1^i), \cdots, (B_{m_i}^i, p_{m_i}^i))$ to (B_i, p_i) then the composition

$$(f, \psi) \circ ((f_1, \psi_1), \cdots, (f_n, \psi_n))$$

is the pair (g, φ) given by the following compositions:

$$g = f \circ (f_1, \cdots, f_n)$$

where we consider $\mathbb{B}$ as an operad via the cartesian structure. To define φ let us denote by $in(h)$ also the object $X_1 \times \cdots \times X_n \in ob(\mathbb{B})$. Since $\mathbb{B}$ is cartesian there are canonical projections $\pi_{(i)} : in(g) \to in(f_i)$, and φ is then the composition

$$(f_1 \times \cdots \times f_n)^*(\psi)(\pi^*_{(1)}\psi_1, \cdots, \pi^*_{(n)}\psi_n)$$

in $F(in(g))$. Given an operation (f, ψ) of arity n and $\sigma \in \Sigma_n$ we define

$$\sigma^*(f, \psi) = (\sigma^* f, \sigma^* \psi)$$

where $\sigma^* f$ is interpreted in (the operad) $\mathbb{B}$. The units are the evident ones, and the fact that the axioms for an operad are satisfied is easily established.

EXAMPLE 1.7.5. For the diagram $F : Set^{op} \to Operad$ given in Example 1.7.3 we obtain that $\int_{Set} F$ is isomorphic to the operad Cat with the usual cartesian structure, as was hoped for.

1.8. Enriched operads

Just as categories can be enriched in a symmetric monoidal category $\mathcal{E}$ by demanding that for any two objects A, B in the category one has an object $\mathcal{C}(A, B) \in ob(\mathcal{E})$ such that the composition and identity operations are now arrows in $\mathcal{E}$ making suitable diagrams commute (see **[26]**), so can operads be enriched in the same manner. It is possible to extend most of what was mentioned above to enriched operads, however we will only introduce here that part of the theory that is relevant for rest of this work.

DEFINITION 1.8.1. An *$\mathcal{E}$-enriched planar operad* $\mathcal{P}$ consists of a set $ob(\mathcal{P})$, whose elements are called the *objects* of the operad, and for every signature

$$p_1, \cdots, p_n, p \in ob(\mathcal{P})$$

an object of $\mathcal{E}$

$$\mathcal{P}(p_1, \cdots, p_n; p) \in ob(\mathcal{E})$$

called the object of *arrows* from the input $(p_1, \cdots, p_n)$ to the output p. Furthermore, there are composition arrows in $\mathcal{E}$

$$\begin{array}{c} \mathcal{P}(p_1, \cdots, p_n; p) \otimes \mathcal{P}(p^1_1, \cdots, p^1_{m_1}; p_1) \otimes \cdots \otimes \mathcal{P}(p^n_1, \cdots, p^n_{m_n}; p_n) \\ \Big\downarrow \gamma \\ \mathcal{P}(p^1_1, \cdots, p^1_{m_1}, \cdots, p^n_1, \cdots, p^n_{m_n}; p) \end{array}$$

for all possible signatures as indicated. For each object $p \in ob(\mathcal{E})$ there is also an arrow $id_p : I \to \mathcal{P}(p, p)$, where I is the monoidal unit in $\mathcal{E}$. These arrows should satisfy certain commutativity axioms that express associativity of the composition and unit laws.

An *$\mathcal{E}$-enriched (symmetric) operad* is the same data as above together with actions of the symmetric groups, which again satisfy certain diagrams expressing the equivariance of the composition with respect to these actions.

REMARK 1.8.2. In **[35]** these diagrams are explicitly given for the special case where the operad contains just one object. In **[26]** these diagrams are given in the case where all object arrows with $\mathcal{P}(p_1, \cdots, p_n; p)$ for $n \neq 1$ are empty (i.e., the initial object). The needed diagrams for our definition are then a merging of these two kinds of diagrams. For more details see **[6, 15]**.

The category $Operad(\mathcal{E})$ is the category of $\mathcal{E}$-enriched operads with the evident notion of $\mathcal{E}$-enriched functors between $\mathcal{E}$-enriched operads.

REMARK 1.8.3. It is trivial to check that for $\mathcal{E} = Set$ with the cartesian structure

$$Operad(\mathcal{E}) = Operad$$

EXAMPLE 1.8.4. Let $\mathcal{E}$ be a symmetric closed monoidal category and $M \subseteq ob(\mathcal{E})$. We then have the $\mathcal{E}$-enriched operad $\mathcal{P}_M$ given by:

$$ob(\mathcal{P}_M) = M$$

and for objects $p_1, \cdots, p_n, p \in M$

$$\mathcal{P}_M(p_1, \cdots, p_n; p) = \underline{\mathcal{E}}(p_1 \otimes \cdots \otimes p_n; p)$$

with the obvious operadic structure (compare with Example 1.1.4). When $M = ob(\mathcal{E})$ we will simply write $\hat{\mathcal{E}}$ or even just $\mathcal{E}$ for the enriched operad $\mathcal{P}_{ob(\mathcal{E})}$.

Every enriched operad $\mathcal{P}$ in $Operad(\mathcal{E})$ has an underlying operad $\mathcal{P}_0$ defined as follows. The objects of $\mathcal{P}_0$ are those of $\mathcal{P}$ and for objects $p_1, \cdots, p_n, p \in ob(\mathcal{P})$ we have

$$\mathcal{P}_0(p_1, \cdots, p_n; p) = \mathcal{E}(I, \mathcal{P}(p_1, \cdots, p_n; p)),$$

that is the set of arrow in $\mathcal{E}$ from the unit I to $\mathcal{P}(p_1, \cdots, p_n; p)$. The operad structure is the evident one. This actually defines a functor $(-)_0 : Operad(\mathcal{E}) \to Operad$ which has a left adjoint which we now describe (for the case where $\mathcal{E}$ has colimits). For a set A let $I[A]$ be the coproduct of A copies of the unit I. The functor $disc : Operad \to Operad(\mathcal{E})$ sends an operad $\mathcal{P} \in Operad$ to the enriched operad $disc(\mathcal{P})$ that has the same objects as $\mathcal{P}$ and, for objects $p_1, \cdots, p_n; p \in ob(disc(\mathcal{P}))$ has the object of operations

$$disc(\mathcal{P})(p_1, \cdots, p_n; p) = I[\mathcal{P}(p_1, \cdots, p_n; p)].$$

These constructions are direct generalizations of the corresponding construction for enriched categories (see **[26]**). The proof that $Operad \underset{(-)_0}{\overset{disc}{\rightleftarrows}} Operad_{\mathcal{E}}$ is an adjunction follows in just the same way as the analogous result for categories. An operad which is in the image of *disc* will be called a *discrete* operad.

1.9. Comparison with the usual terminology

In this section we compare our definitions with the classical notions related to operads. This is just meant to justify our definitions by showing that they agree with the classical ones. The proofs to all the claims we make are trivial and thus omitted.

DEFINITION 1.9.1. Let $\mathcal{E}$ be a symmetric closed monoidal category. A *classical operad* in $\mathcal{E}$ is an $\mathcal{E}$-enriched operad $\mathcal{P}$ such that $ob(\mathcal{P})$ is a one-point set.

Let $\mathcal{P}$ be a classical operad. Since $ob(\mathcal{P})$ is just a one point set, say $\{\star\}$, the operad is given by specifying for each $n \geq 0$ an object $\mathcal{P}(\star, \cdots, \star; \star)$ of $\mathcal{E}$ where $\star$ appears $n+1$ times. We can thus denote it simply by $\mathcal{P}(n)$. If we now rewrite the axioms for an operad in terms of $\mathcal{P}(n)$ we obtain a description of a classical operad which is identical to the definition in the literature (see e.g., [**18, 35, 36**]). More explicitly, Let $\mathcal{P}$ be a classical operad in $\mathcal{E}$. $\mathcal{P}$ is then given by a sequence $\{\mathcal{P}(n)\}_{n=0}^{\infty}$ of objects of $\mathcal{E}$ together with an arrow $I \to \mathcal{P}(1)$ (the unit of the unique object) and composition functions

$$\mathcal{P}(n) \otimes \mathcal{P}(m_1) \otimes \cdots \otimes \mathcal{P}(m_n) \to \mathcal{P}(m_1 + \cdots + m_n)$$

for all sequences of natural numbers $n, m_1, \cdots, m_n$, satisfying the appropriate unit and associativity constraints.

The notion of a map of classical operads is then defined in the obvious way and it is easily seen that it agrees with our definition. Given a classical operad $\mathcal{P}$, a $\mathcal{P}$-algebra is, by definition, a map of operads $\mathcal{P} \to End_X$ from $\mathcal{P}$ to the so called endomorphism (classical) operad. This operad is defined by

$$End_X(n) = \underline{\mathcal{E}}(X^{\otimes n}, X)$$

where the composition is given by substitution and the symmetric groups act by permuting the variables. This is actually just a special case of Example 1.8.4, namely

$$End_X = \mathcal{P}_{\{X\}}.$$

Lemma 1.9.2. *Let $\mathcal{P}$ be a classical operad in $\mathcal{E}$. Then a $\mathcal{P}$-algebra $A : \mathcal{P} \to End_X$ corresponds to a map of enriched operads $B : \mathcal{P} \to \mathcal{E}$ such that $B(\star) = X$.*

Notice that if $\mathcal{P} = disc(\mathcal{P}')$ is a discrete operad then a $\mathcal{P}$-algebra $\mathcal{P} \to \mathcal{E}$ is the same as a $\mathcal{P}'$-algebra $\mathcal{P}' \to \mathcal{E}$. We can thus form the operad $\underline{Operad}(\mathcal{P}', \mathcal{E})$ and obtain the operad of $\mathcal{P}$-algebras. This is a slightly richer structure on the collection of $\mathcal{P}$-algebras than the usual category of $\mathcal{P}$-algebras presented in the literature, namely it forms an operad and not just a category.

CHAPTER 2

Dendroidal sets

In this chapter the category of dendroidal sets is introduced and some of its basic properties are studied. Starting the chapter is a motivating problem arising from the nerve construction of categories and from our approach that operads are a generalization of categories. Then the simplicial category is briefly recalled together with some adjunctions related to it. The simplicial category is then extended in two different ways (which are proven to be equivalent) to what we call the dendroidal category, which is then used to define the category of dendroidal sets. The chapter ends by studying a certain closed monoidal structure on the category of dendroidal sets and a generalization of the skeletal filtration of simplicial sets to dendroidal sets.

2.1. Motivation - simplicial sets and nerves of categories

The simplicial category Δ can be defined in several different ways. Each such definition gives a different point of view on the category and is useful in different situations. We present here three definitions of the category Δ. The fact that these definitions produce isomorphic categories is well known and can easily be proven.

Definition 2.1.1. (Algebraic definition of Δ) Consider for each $n \geq 0$ the linearly ordered set $[n] = \{0 < 1 < \cdots < n\}$. The category Δ_A is the full sub-category of *PoSet* (the category of partially ordered sets) spanned by the objects $\{[n] \mid n \geq 0\}$.

This definition can be rephrased by saying that Δ_A is a skeleton of the category of non-empty, finite linearly ordered sets. This is the most common definition of the simplicial category Δ.

Definition 2.1.2. (Categorical definition of Δ) Consider for each $n \geq 0$ the category $[n]$ whose objects are $\{0, 1, \cdots, n\}$ and such that for $0 \leq i, j \leq n$ there is exactly one arrow $i \to j$ whenever $i \leq j$. The category Δ_C is the full sub-category of *Cat* spanned by the objects $\{[n] \mid n \geq 0\}$.

The equivalence between these two definitions is just the observation that any poset is precisely a small category with at most one arrow between any two of its objects.

Definition 2.1.3. (Geometric definition of Δ) Consider for each $n \geq 0$ the space $\Delta^n = \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1}\}$ with the sub-space topology. The objects of the category Δ_G are $\{\Delta^n | n \geq 0\}$ and the maps are generated by face inclusions and degeneracies (see **[19]**, page 3).

Usually we will just write Δ for the simplicial category whose objects are $[n]$ for $n \geq 0$, and will use whichever definition is most convenient. The geometric definition implies that there is a functor $\Delta \to Top$, i.e., the inclusion $\Delta_G \to Top$. Taking this functor as a probe (see the preliminaries) yields an adjunction

$$sSet \underset{N}{\overset{|-|}{\rightleftarrows}} Top$$

where N is usually called the singular complex functor and $|-|$ is called the geometric realization functor (see [**19**]). On the other hand the categorical definition implies the existence of a functor $\Delta \to Cat$, which is just the inclusion functor $\Delta_C \to Cat$. When taken as a probe, this functor yields the adjunction

$$sSet \underset{N}{\overset{|-|}{\rightleftarrows}} Cat$$

where this time N is the nerve functor and $|-|$ is usually denoted by τ (see [**23**]). Lastly, the algebraic definition also implies the existence of a functor $\Delta \to PoSet$ (again the inclusion functor $\Delta_A \to PoSet$) and thus an adjunction

$$sSet \underset{N}{\overset{|-|}{\rightleftarrows}} PoSet.$$

However, this adjunction is not particularly useful.

Staying true to the main principal of the previous chapter, namely that operads are a natural extension of categories, the adjunction

$$sSet \underset{N}{\overset{|-|}{\rightleftarrows}} Cat$$

should catch our attention. We thus ask whether it is possible to define the nerve of an operad. More concretely we can ask whether in the diagram

$$\begin{array}{ccc} sSet & \xleftarrow{\ N\ } & Cat \\ {\scriptstyle i_!}\big\downarrow & & \big\downarrow{\scriptstyle j_!} \\ ? & \xleftarrow{\ N_d\ } & Operad \end{array}$$

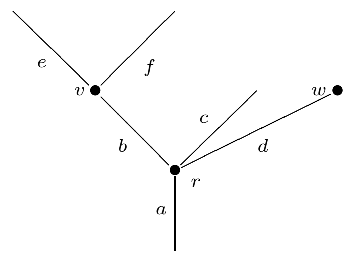
the question mark can be replaced by a category and the dotted arrows be filled in a natural way by functors such that N_d would send an operad to its (not yet defined) nerve. Our approach to answering this question will be to extend the category Δ to a bigger category Ω and then use an appropriate probe $\Omega \to Operad$ that will produce an adjunction of which the right adjoint will be N_d. Each point of view of the category Δ suggests a different way to extend it. We will consider below the categorical definition and the algebraic one, and extend both (to isomorphic categories). Each approach will have its merits, as we shall see.

2.2. An operadic definition of the dendroidal category

We consider here the categorical definition (Definition 2.1.2) of Δ and extend it to a bigger category by means of special operads induced by trees.

DEFINITION 2.2.1. Let T be a planar tree. The planar operad *generated* by T, denoted $\Omega_\pi(T)$, is the following free planar operad. We define the collection C on the set $E(T)$ of edges of T as follows. For each vertex v with $in(v) = (e_1, \cdots, e_n)$ and $out(v) = e_0$ we set $C(e_1, \cdots, e_n; e_0)$ to be a one-point set. These are the only non-empty sets in the collection C. We now define $\Omega_\pi(T)$ to be $\mathcal{F}_\pi(C)$, the free planar operad on the collection C (see Section 1.2).

EXAMPLE 2.2.2. For the tree T given by



$\Omega_\pi(T)$ has six objects, $a, b, \cdots, f$ and the following generating operations:

$$r \in \Omega_\pi(T)(b, c, d; a),$$

$$w \in \Omega_\pi(T)(-; d)$$

and

$$v \in \Omega_\pi(T)(e, f; b).$$

The other operations are units (such as $1_b \in \Omega_\pi(T)(b; b)$) and formal compositions of the generating operations (such as $r \circ_1 v \in \Omega_\pi(T)(e, f, c, d; a)$).

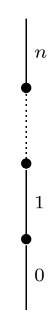
DEFINITION 2.2.3. Let T be a non-planar tree. The operad *generated* by T, denoted by $\Omega(T)$, is defined as follows. Let $\bar{T}$ be a planar representative of T, then

$$\Omega(T) = Symm(\Omega_\pi(\bar{T})).$$

It is clear that the definition does not depend on the chosen planar representative $\bar{T}$. In fact a different choice amounts to choosing a different set of generating operations for $\Omega(T)$.

DEFINITION 2.2.4. (Operadic definition of Ω) The *dendroidal* category Ω is the full sub-category of *Operad* whose objects are the operads of the form $\Omega(T)$ where T is a non-planar rooted tree.

To see how Δ embeds in Ω consider for each $n \geq 0$ the linear tree L_n



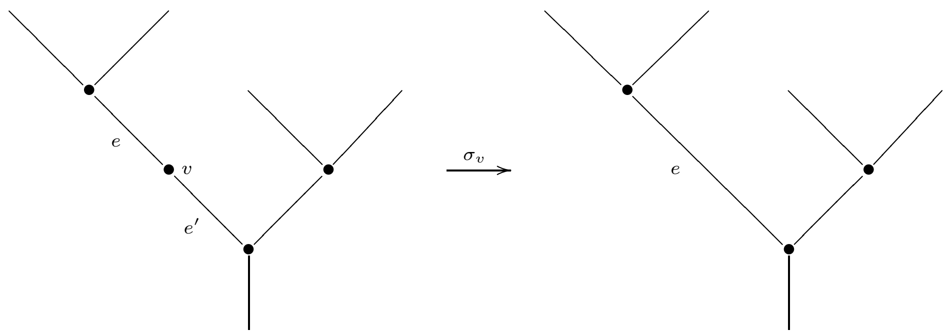
It is trivial to verify that

$$\Omega(L_n) = j_!([n])$$

and that the functor $i : \Delta \to \Omega$ sending $[n]$ to $\Omega(L_n)$ is an embedding of categories. This constitutes our operadic extension of Δ.

2.2.1. Faces and degeneracy maps. Exactly as for Δ, the maps in Ω are generated by special kinds of maps which we now describe.

Let T be a tree and $v \in T$ a vertex of valence 1 with $in(v) = e$ and $out(v) = e'$. Consider the tree T/v, obtained from T by deleting the vertex v and the edge e'. There is an operad map, i.e, an arrow $\sigma_v : T \to T/v$ in Ω, which sends the operation in $\Omega(T)$ generated by v to the unit 1_e in $\Omega(T/v)$. For example:

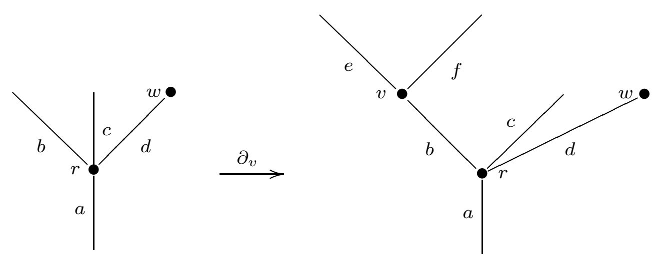


An arrow in Ω of this kind will be called a *degeneracy (map).*

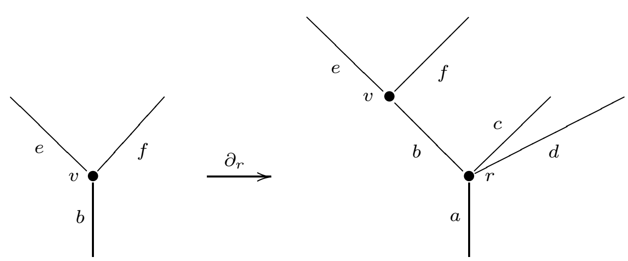
Consider now a tree T and a vertex v in T with exactly one inner edge attached to it (such a vertex will be called an *outer cluster*), one can obtain a new tree T/v by deleting v and all the outer edges attached to it. The operad $\Omega(T/v)$ associated to T/v is simply a sub-operad of the one associated to T, and this inclusion of operads defines an arrow in Ω denoted

$$\partial_v : T/v \to T.$$

An arrow in Ω of this kind is called an *outer face (map).* For example



and (to emphasize that it is sometimes possible to remove the root of the tree T)



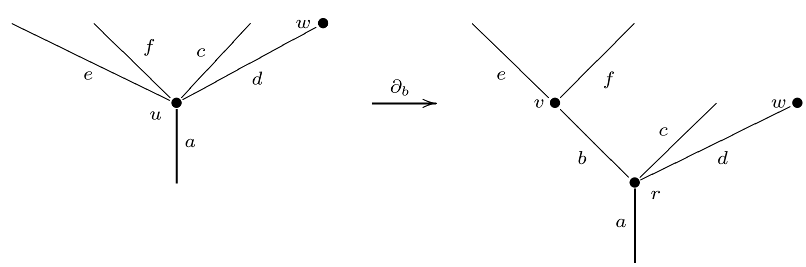
are both outer faces.

Moreover, for a corolla C_n and an edge e of C_n (necessarily outer) there is an associated outer face map

$$e : \eta_e \to T$$

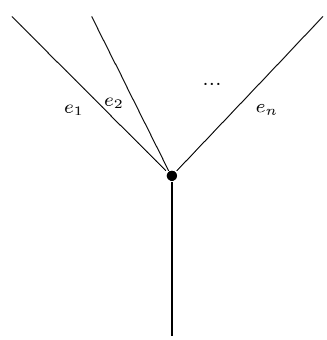
sending the unique edge e of η_e to e in C_n.

Given a tree T and an inner edge e in T, one can obtain a new tree T/e by contracting the edge e. There is a canonical map of operads $\partial_e : \Omega(T/e) \to \Omega(T)$ which sends the new vertex in T/e (obtained by merging the two vertices attached to e) into the appropriate composition of these two vertices in $\Omega(T)$. The corresponding arrow $\partial_e : T/e \to T$ in Ω is called an *inner face (map).* For example



where $u = r \circ_1 v$.

Lastly, we mention the isomorphisms in Ω. Of course there may be non-trivial isomorphisms from a tree to itself, for example, for the corolla C_n whose input edges are $e_1, \cdots, e_n$:



any permutation $\varphi \in \Sigma_n$ defines an automorphism of C_n in Ω.

DEFINITION 2.2.5. The *degree* of a tree T, denoted by $|T|$, is the number of vertices in T.

It is easily seen that degeneracy maps decrease degree by 1, face maps (outer or inner) increase degree by 1, and isomorphisms preserve degree.

THEOREM 2.2.6. *Any map* $T \xrightarrow{f} T'$ *in* Ω *can be written uniquely as* $f = \varphi\pi\delta$, *where* δ *is a composition of degeneracy maps,* π *is an isomorphism, and* φ *is a composition of (inner and outer) face maps.*

The proof will be given below once we develop the algebraic definition of Ω.

2.3. An algebraic definition of the dendroidal category

Our aim now is to extend the algebraic definition of Δ (Definition 2.1.1). This approach is technically more involved than the previous one since we first have to enlarge the category of posets (to what we call broad posets) and then find a suitable algebraic characterization for the analogue of linear orders. Our plan is thus as follows. We start by developing the notion of a broad poset. The basic principal is that a broad poset stands in the same relation to a poset as does an operad to a category. Once the category of broad posets is defined we notice that it carries a natural symmetric closed monoidal structure. We then turn to the analogue of a linear order for broad posets, which we call dendroidally ordered sets. The algebraic definition of the dendroidal category is given, and we then prove that this definition is equivalent to the operadic one.

2.3.1. Broad posets. For a set A we denote by A^* the free monoid on A. That is

$$A^* = \bigcup_{n=0}^{\infty} A^n$$

with concatenation of tuples as the monoid operation. The set A^0 is a singleton set which consists of the unique tuple of length 0, denoted by ϵ, which is the unit of the monoid. We denote elements of A^* by $\overrightarrow{a}$ and identify an element a with the 1-tuple (a). We use the notation $a \in \overrightarrow{a}$ to indicate that a occurs in the tuple $\overrightarrow{a}$. If $\overrightarrow{a} \in A^*$ is of length n and $\sigma \in \Sigma_n$ then by permuting the components (that is $(a_1, \cdots, a_n)\sigma = (a_{\sigma(1)}, \cdots, a_{\sigma(n)})$), we obtain a right action of Σ_n on the set A^n.

A *broad relation* is a pair (A, R) where A is a set and R is a sub-set of $A \times A^*$. As is common with ordinary relations, we use the notation $aR\overrightarrow{a}$ instead of $(a, \overrightarrow{a}) \in R$.

Definition 2.3.1. A *broad poset* is a broad relation (A, R) satisfying:

(1) Reflexivity: aRa holds for any $a \in A$.
(2) Transitivity: If $aR(a_1, \cdots, a_n)$ and $a_iR\overrightarrow{b_i}$ hold for $1 \le i \le n$ then $aR\overrightarrow{b_1} \cdot \cdots \cdot \overrightarrow{b_n}$.
(3) Anti-symmetry: If $aR\overrightarrow{b}$ and $bR\overrightarrow{a}$ hold while $a \in \overrightarrow{a}$ and $b \in \overrightarrow{b}$ then $a = b$.
(4) Permutability: If $a \le \overrightarrow{a}$ and $\overrightarrow{a}$ has length n, then $a \le \overrightarrow{a}\sigma$ holds for any $\sigma \in \Sigma_n$.

When (A, R) is a broad poset we denote R by $\le$. The meaning of $<$ is then defined in the usual way.

Remark 2.3.2. In the definition above one can obviously drop condition four and retain a sensible definition of what we call a *non-symmetric (*or a *planar)* broad poset. In that context we will sometime refer to a broad poset as a *symmetric* broad poset. We will see below that there is a close connection between symmetric operads and broad posets. There is a similar connection between planar operads and planar broad posets.

Example 2.3.3. The site http://genealogy.math.ndsu.nodak.edu of the math genealogy project lists mathematicians and their students. This allows us to define the following broad poset. The set A is the set of mathematicians. We say that $a \le (a_1, \cdots, a_n)$ if mathematicians $a_1, \cdots, a_n$ (in no particular order) are students of mathematician a. We assume every student has exactly one well-defined adviser.

By agreeing to the convention that $a \leq a$ for every mathematician a, and closing under transitivity we obtain a broad relation which is clearly a broad poset.

A *map* of broad posets $f : A \to B$ is a set function preserving the broad poset structure, that is if $a \leq \overrightarrow{a}$ then $f(a) \leq f(\overrightarrow{a})$ where $f(\overrightarrow{a})$ is defined component-wise (namely, $f(a_1, \cdots, a_n) = (f(a_1), \cdots, f(a_n))$).

DEFINITION 2.3.4. We denote by *BrdPoset* the category of all broad posets and their maps.

Recall that a poset A can be considered as a category $\mathcal{C}$ whose objects are the elements of A and such that there is precisely one arrow $a \to a'$ in $\mathcal{C}$ whenever $a \leq a'$. One obtains thus a functor $Poset \to Cat$. Similarly, given a symmetric (respectively planar) broad poset B one can define a symmetric (respectively planar) operad $\mathcal{P}$ whose objects are the elements of B and such that there is exactly one operation in $\mathcal{P}(b_1, \cdots, b_n; b)$ whenever $b \leq (b_1, \cdots, b_n)$. In that way one obtains a functor $BrdPoset \to Operad$.

It is obvious how an ordinary poset can be viewed as a broad poset and that this defines an embedding

$$k_! : Poset \to BrdPoset.$$

This functor has a right adjoint $k^* : BrdPoset \to Poset$, that sends a broad poset A to the poset $k^*(A) = A$ where $a \leq b$ holds in $k^*(A)$ exactly when $a \leq b$ holds in A. Consider the endofunctor $R : Poset \to Poset$ that sends a poset A to the same set with the reversed partial order. One may now easily establish that in the following diagram

$$\begin{array}{ccc} Poset & \underset{Rk^*}{\overset{k_! R}{\rightleftarrows}} & BrdPoset \\ \downarrow & & \downarrow \\ Cat & \underset{j^*}{\overset{j_!}{\rightleftarrows}} & Operad \end{array}$$

both squares commute. This constitutes our extension of *Poset* to the category *BrdPoset* and establishes the relation to categories and operads. Notice that the use of the endofunctor R is needed because of the convention that in a broad poset A a relation $a \leq a_1, \cdots, a_n$ is translated in its corresponding operad to an arrow from $a_1, \cdots, a_n$ to a (arrows go from big to small), while in a poset a relation $a \leq b$ is translated in its corresponding category to an arrow from a to b (arrows go from small to big).

2.3.2. Closed monoidal structure on *BrdPoset*. The category *Poset* is cartesian closed with the obvious product of posets. The category *BrdPoset* is also cartesian closed in such a way as to make $k_!$ a strong monoidal functor. However, there is another closed monoidal structure that also extends the one on *Poset*, which we now describe.

DEFINITION 2.3.5. Let A and B be two broad posets. Their *tensor product* $A \otimes B$ is the set $A \times B$ with the minimal broad poset structure in which

1. For every $a \in A$ if $b \leq (b_1, \cdots, b_n)$ then $(a, b) \leq ((a, b_1), \cdots, (a, b_n))$.
2. For every $b \in B$ if $a \leq (a_1, \cdots, a_m)$ then $(a, b) \leq ((a_1, b), \cdots, (a_m, b))$.

THEOREM 2.3.6. *The category BrdPoset with the tensor product of broad posets is a symmetric closed monoidal category, and $k_!$: Poset $\to$ BrdPoset is strong monoidal.*

PROOF. A singleton set with the trivial broad poset structure is clearly the unit for the tensor product. It is easily verified that $\otimes$ makes *BrdPoset* into a symmetric monoidal category, so all that is left to do is to describe the internal Hom. Given two broad posets A and B, the set $\underline{BrdPoset}(A, B)$ of all broad poset maps from A to B is made into a broad poset by defining

$$f \leq (f_1, \cdots, f_n)$$

to hold if for every $a \in A$

$$f(a) \leq (f_1(a), \cdots, f_n(a))$$

holds in B. It is an easy matter to verify that this broad poset is the internal Hom with respect to the tensor product of broad posets. The fact that $k_! : Poset \to BrdPoset$ is strong monoidal is trivial. $\square$

2.3.3. Dendroidally ordered sets. Above we extended the category *Poset* to the category *BrdPoset* of broad posets. The algebraic definition of Δ identifies it as a certain full sub-category of *Poset* by considering linear orders. Our aim now is to identify those objects of *BrdPoset* that generalize linear orders in a suitable way.

A broad poset $(A, \leq)$ induces a partial order relation on A as follows. For $a, b \in A$ we say that a is *dominated* by b and write $a \leq_d b$ if there is a $\overrightarrow{b} \in A^*$ such that $a \leq \overrightarrow{b}$ and $b \in \overrightarrow{b}$. It is immediately seen that $\leq_d$ is indeed a partial order. The broad poset $\leq$ also induces a partial order relation on the set A^* as follows. For $\overrightarrow{a} = (a_1, \cdots, a_n)$ and $\overrightarrow{b}$ in A^* we say that $\overrightarrow{a} \leq \overrightarrow{b}$ if there are $\overrightarrow{b_1}, \cdots, \overrightarrow{b_n}$ such that $\overrightarrow{b} = \overrightarrow{b_1} \cdot \cdots \cdot \overrightarrow{b_n}$ and such that $a_i \leq \overrightarrow{b_i}$ for each $1 \leq i \leq n$. Notice that this does not conflict with our abuse of notation which identifies the 1-tuple (a) with a.

DEFINITION 2.3.7. Let A be a broad poset. An element $r \in A$ such that for all $a \in A$

$$r \leq_d a$$

is called the *root* of A. Clearly, if a root exists then it is unique.

For $a \in A$ let $\hat{a}$ be the set $\{\overrightarrow{a} \in A^* | a < \overrightarrow{a}\}$.

DEFINITION 2.3.8. Let A be a broad poset and $a \in A$. Assume that the set $\hat{a}$, as a sub-set of A^*, has a smallest element which is unique up to symmetry. We will call such a smallest element a *representative* of the *successors* of a and will denote it (somewhat ambiguously) by $s(a)$. An element $a \in A$ for which $\hat{a}$ is empty is called a *leaf*.

By "a smallest element which is unique up to symmetry" we mean the following. An element $\overrightarrow{a}$ which is a smallest element with respect to the poset $\leq$ on A^*, such that if $\overrightarrow{b}$ is another smallest element then they are both of the same length n, and there is a $\sigma \in \Sigma_n$ such that $\overrightarrow{a} = \overrightarrow{b}\sigma$. Note that if $\overrightarrow{a}$ is a representative of $s(a)$ then $\overrightarrow{a}\sigma$ is again a representative of $s(a)$ for any $\sigma \in \Sigma_n$ with n the length of $\overrightarrow{a}$.

EXAMPLE 2.3.9. In the math genealogy example, a leaf is a mathematician with no students. For a mathematician a, the tuple $s(a)$ is a list of the students of a in an arbitrary order. In this example there is no root.

DEFINITION 2.3.10. A broad poset $(A, \leq)$ is called *finite* if the set $\leq$ is finite. A is called *minimal* if whenever

$$a \leq (a_1, \cdots, a_n)$$

$a_i \neq a_j$ for $i \neq j$.

Notice that the finiteness of A as a broad poset implies that of A as a set, but not vice-versa.

DEFINITION 2.3.11. Let A be a finite broad poset. A is called *dendroidally* ordered if

(1) A has a root.
(2) For every $a \in A$ either a is a leaf or a has successors.
(3) A is minimal.

REMARK 2.3.12. If A is a dendroidally ordered set, minimality implies that for each $a \in A$ the tuple $s(a)$ does not contain the same element twice. We can therefore consider $s(a)$ unambiguously as a set. We shall do this from now on.

It is obvious that if $A \neq \emptyset$ is a finite poset which is linearly ordered, then the broad poset $k_!(A)$ is dendroidally ordered. This is thus our extension of the notion of a linear order from the category of posets to the category of broad posets.

DEFINITION 2.3.13. (Algebraic definition of Ω) The *dendroidal* category Ω is the full sub-category of $BrdPoset$ spanned by the dendroidally ordered sets.

The embedding of Δ in Ω using the algebraic definition is obvious, we simply send the linearly ordered set $[n] \in ob(\Delta)$ to the dendroidally ordered set $k_!([n])$. This concludes our algebraic extension of the simplicial category to the dendroidal category.

2.3.4. Grafting in *DenOrd*. We discuss now how dendroidally ordered sets can be grafted. We obtain a decomposition of a dendroidally ordered set as the grafting of certain dendroidally ordered sub-sets of it, much like the fundamental decomposition of trees (Proposition 0.2.6). It is precisely this property that will imply the equivalence of the two definitions of the dendroidal category given above.

DEFINITION 2.3.14. Let A and B be two dendroidally ordered sets with $A \cap B = \{y\}$, where y is a leaf of A and the root of B. The *grafting* of B on A, denoted by $A \circ B$, is the set $A \cup B$ together with the broad relation in which $x \leq (x_1, \cdots, x_n)$ holds if one of the following conditions is satisfied:

(1) $x \leq (x_1, \cdots, x_n)$ holds in A.
(2) $x \leq (x_1, \cdots, x_n)$ holds in B.
(3) $x \in A$ and there are $\overrightarrow{a_1}, \overrightarrow{a_2} \in A^*$ and $\overrightarrow{b} \in B^*$ such that

$$(x_1, \cdots, x_n) = \overrightarrow{a_1} \cdot \overrightarrow{b} \cdot \overrightarrow{a_2}$$

and

$$x \leq \overrightarrow{a_1} \cdot y \cdot \overrightarrow{a_2}$$

holds in A and

$$y \leq \overrightarrow{b}$$

holds in B.

It is easily seen that the grafting of two dendroidally ordered sets is again a dendroidally ordered set. By repeated grafting one can define a full grafting operation

$$A \circ (B_1, \cdots, B_n)$$

which is unambiguously defined whenever the sets B_i are pairwise disjoint, the set $\{r_{B_i}\}_{i=1}^{n}$, consisting of the roots of the dendroidally ordered sets B_i, is equal to the set of leaves of A, and each B_i meets A at one edge.

Maps of dendroidally ordered sets can also be grafted as explained in the following proposition whose proof is trivial.

Proposition 2.3.15. *Let $f : A \to A'$ and $g : B \to B'$ be two maps of dendroidally ordered sets. Suppose $A \cap B = \{a\}$ and $A' \cap B' = \{f(a)\}$ where a and $f(a)$ are leaves in, respectively, A and A'. Assume further that the root of B is a and that the root of B' is $f(a)$ and that $g(a) = f(a)$. Then the function $f \circ g : A \circ B \to A' \circ B'$ given by*

$$f \circ g(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

is well defined and is a map of dendroidally ordered sets.

Remark 2.3.16. When f is the identity we will denote $f \circ g$ by $A \circ g$ and when g is the identity we will denote $f \circ g$ by $f \circ B$. We use the notation $f \circ (g_1, \cdots, g_n)$ for repeated grafting of maps (under the obvious compatibility conditions on the given maps).

For a dendroidally ordered set A and $a \in A$ let

$$A_a = \{a' \in A | a \leq_d a'\}$$

with the induced broad relation from A. It is immediate that A_a is again dendroidally ordered. For a dendroidally ordered set A with root r and $s(r) = \{a_1, \cdots, a_n\}$ let

$$A_{root} = \{r, a_1, \cdots, a_n\}$$

with the induced broad order from A (which is obviously a dendroidal order).

Proposition 2.3.17. *(Fundamental decomposition of dendroidally ordered sets) Let A be a dendroidally ordered set with root r and $s(r) = \{a_1, \cdots, a_n\}$. Then $A = A_{root} \circ (A_{a_1}, \cdots, A_{a_n})$.*

Proof. First notice that $A_{a_i} \cap A_{root} = \{a_i\}$. It is generally true for any $a \in A$ that $a \notin s(a)$, so that if $r \in A_{a_i}$ then, since r is the smallest element in $(A, \leq_d)$ it follows that $r = a_i$, but then $r \in s(r)$, a contradiction. If $a_j \in A_{a_i}$ and $a_j \neq a_i$ then $a_i \leq_d a_j$ which means that there is $\overrightarrow{a} \in A^*$ with $a_j \in \overrightarrow{a}$ and $a_i \leq \overrightarrow{a}$. But transitivity and

$$r \leq (a_1, \cdots, a_n)$$

imply

$$r \leq (a_1, \cdots, a_{i-1}) \cdot \overrightarrow{a} \cdot (a_{i+1}, \cdots, a_n)$$

which contradicts the minimality of A (since the latter tuple on the right contains a_j twice). Thus the only element of A which can be in $A_{root} \cap A_{a_i}$ is a_i which is clearly there. It follows that $A_{root} \cap A_{a_i} = \{a_i\}$ and thus that the grafting $A_{root} \circ (A_{a_1}, \cdots, A_{a_n})$ is well defined.

Next we notice that $A_{root} \cup A_{a_1} \cup \cdots \cup A_{a_n} = A$, since for any $a \in A$ one has $r \leq_d a$, which means that there is an $\overrightarrow{a}$ such that $a \in \overrightarrow{a}$ and $r \leq \overrightarrow{a}$. If $\overrightarrow{a} = r$ then $a = r$ and we are done, otherwise $\overrightarrow{a} \in \hat{r}$ and by definition we then have that $s(r) \leq \overrightarrow{a}$. This means that $\overrightarrow{a}$ can be written as $\overrightarrow{a_1} \cdot \cdots \cdot \overrightarrow{a_n}$ in such a way that $a_i \leq \overrightarrow{a_i}$. Since $a \in \overrightarrow{a}$ it follows that there is $1 \leq j \leq n$ for which $a \in \overrightarrow{a_j}$, which implies that $a_j \leq_d a$, and so $a \in A_{a_j}$. We see thus that the underlying set of $A_{root} \circ (A_{a_1}, \cdots, A_{a_n})$ is the same as that of A and it is now easy to see that the broad order defined by the grafting operation is the original one on A. □

2.3.5. Classification of dendroidally ordered sets. We now establish the connection between dendroidally ordered sets and trees - thus justifying the use of the term 'dendroidal'.

DEFINITION 2.3.18. Let T be a tree. We define a dendroidally ordered set, $[T]$, whose underlying set is $E(T)$ (the set of edges of T), by induction on the number k of vertices in the tree T. If $T = \eta$ (the tree with one edge and no leaves) then the broad poset structure on $[\eta]$ is just $e \leq e$ for the unique edge e in η. If T is an n-corolla C_n with root r and leaves $\{a_1, \cdots, a_n\}$ then the broad poset structure on $[C_n]$ is the one in which $r \leq \overrightarrow{a}$ where $\overrightarrow{a}$ is any permutation of $(a_1, \cdots, a_n)$. Obviously these two broad posets make $[\eta]$ and $[C_n]$ into dendroidally ordered sets and thus the cases $k = 0, 1$ are covered. Suppose now that T has more then 1 vertex and write $T = T_{root} \circ (T_{e_1}, \cdots, T_{e_n})$ (as in 0.2.6). We then define $[T] = [T_{root}] \circ ([T_{e_1}], \cdots, [T_{e_n}])$, where the grafting is that of dendroidally ordered sets.

We now wish to associate with any dendroidally ordered set A a tree T such that $A = [T]$. To do that we introduce the notion of the degree of A, which allows for induction on dendroidally ordered sets.

A pair $(a, \overrightarrow{a})$ is called a *link* in a broad poset A if $a < \overrightarrow{a}$ and if $a < \overrightarrow{b} < \overrightarrow{a}$ does not hold for any choice of $\overrightarrow{b}$. We say that two links $(a, \overrightarrow{a})$ and $(a, \overrightarrow{a'})$ are equivalent if there is a permutation σ such that $\overrightarrow{a} \cdot \sigma = \overrightarrow{a'}$. The number of equivalence classes of links in a broad poset A is the *degree* of A and is denoted by $|A|$. It can easily be shown that for dendroidally ordered sets A and B the equality:

$$|A \circ B| = |A| + |B|$$

holds, whenever $A \circ B$ is defined.

LEMMA 2.3.19. *Let A be a dendroidally ordered set. There is a tree $Tr(A)$ for which $A = [Tr(A)]$.*

PROOF. By induction on n, the degree of A. If $|A| = 0$ or $|A| = 1$ then the claim is obvious. Assume the statement holds for dendroidally ordered sets of degree smaller then n and let A be of degree n. Write $A = A_{root} \circ (A_{a_1}, \cdots, A_{a_n})$, and let $Tr(A) = Tr(A_{root}) \circ (Tr(A_{a_1}), \cdots, Tr(A_{a_n}))$. By the definition of $[-]$ and the induction hypothesis it follows that $[Tr(A)] = A$. □

We summarize the properties of the two constructions relating trees and dendroidally ordered sets in the following theorem.

THEOREM 2.3.20. *(Classification of dendroidally ordered sets) The above constructions, associating with any tree T a dendroidally ordered set $[T]$, and with a dendroidally ordered set A a tree $Tr(A)$ have the following properties:*

1) $[Tr(A)] = A$.

2) $Tr([T]) = T$.

3) Whenever one of the sides of the equation $[T \circ S] = [T] \circ [S]$ is defined so is the other, and in that case the equation holds.

4) Whenever one of the sides of the equation $Tr(A \circ B) = Tr(A) \circ Tr(B)$ is defined so is the other, and in that case the equation holds.

5) The two constructions $Tr(-)$ and $[-]$ are unique with respect to properties 1-4.

PROOF. The proofs of the parts that were not already given follow by an easy induction and are therefore omitted. □

Under this correspondence each concept of trees can be translated to a concept of dendroidally ordered sets and vice-versa. for instance, if T is a tree and $[T]$ is its corresponding dendroidally ordered set then the root of T is the root of $[T]$, a vertex in T is a link in $[T]$, and so on. Notice also that $|T| = |[T]|$ and $|Tr(A)| = |A|$.

2.3.6. The equivalence of the two definitions of the dendroidal category. We now prove that the algebraic and operadic definitions of the category Ω are equivalent, and we recast the notation and definitions of the operadic definition in the algebraic one. We also provide a proof of Theorem 2.2.6.

THEOREM 2.3.21. *The algebraic and operadic definitions of Ω are equivalent.*

PROOF. Let Ω_O be the dendroidal category as given in Definition 2.2.4 (operadic definition) and let Ω_A be the dendroidal category as given in Definition 2.3.13 (algebraic definition). The precise meaning of the statement is that these two categories are isomorphic. Given a dendroidally ordered set $A \in ob(\Omega_A)$ we have the tree $Tr(A)$ associated with it from Lemma 2.3.19. It is easily seen that the assignment $A \mapsto Tr(A)$ extends to a functor $Tr : \Omega_A \to \Omega_O$. Similarly, the assignment $T \mapsto [T]$ extends to a functor $[-] : \Omega_O \to \Omega_A$, which is the inverse of Tr. □

REMARK 2.3.22. From now on we will denote the dendroidal category by Ω. We consider the objects of Ω to be non-planar rooted trees and we regard an arrow $T \to S$ in Ω between two such trees either as a map of dendroidally ordered sets $[T] \to [S]$ or as a map of operads $\Omega(T) \to \Omega(S)$, depending on which point of view is more convenient at the time.

Given a dendroidally ordered set A of degree n we wish now to identify its degree $n-1$ dendroidally ordered sub-sets. An element $a \in A$ which is not a leaf and not the root will be called an *inner* element, otherwise it is an *outer* element. Given a link $(a, (a_1, \cdots, a_n))$ and a set C consisting of any n of the elements $a, a_1, \cdots, a_n$, if the elements of C are all outer then C is called an *outer cluster* of A.

If A is a dendroidally ordered set and $B \subseteq A$, we will denote by A/B the subset $A \backslash B$ with the induced broad poset structure. For an element a we write A/a as shorthand for $A/\{a\}$.

PROPOSITION 2.3.23. *(Characterization of maximal dendroidally ordered subsets) Let A be a dendroidally ordered set of degree n. If a is an inner element of*

A then A/a is dendroidally ordered and $|A/a| = n-1$. If C is an outer cluster of A then A/C is dendroidally ordered and $|A/C| = n-1$. If $B \subseteq A$ and B with the broad order induced by A is dendroidally ordered of degree $n-1$ then $B = A/a$ for a unique inner element $a \in A$ or $B = A/C$ for a unique outer cluster C (the meaning of 'or' should be taken in the exclusive sense).

PROOF. The proof is straightforward and thus omitted. □

Consider a dendroidally ordered set A and its corresponding tree $Tr(A)$. One can easily verify the following assertions. An inner element in A corresponds to an inner edge in $Tr(A)$ while an outer element corresponds either to a leaf or to the root of $Tr(A)$. An outer cluster C in A corresponds to a vertex v of valence n together with a choice of n of the edges adjacent to v that are all outer edges. Furthermore the tree $Tr(A/a)$ for an inner element a is equal to the tree $Tr(A)/e$ obtained from the tree $Tr(A)$ by contracting the inner edge e corresponding to a. Similarly the tree $Tr(A/C)$ for an outer cluster C is equal to the tree $Tr(A)/C$ obtained from $Tr(A)$ by removing the outer edges (which may or may not contain the root) corresponding to the outer elements in C.

DEFINITION 2.3.24. Let A be a dendroidally ordered set of degree n. Any inclusion $B \to A$ of a dendroidally ordered sub-set B of degree $n-1$ in A is called a *face map*. If $B = A/a$ then this inclusion is denoted by ∂_a and is called the *inner face map* associated to a. if $B = A/C$ then the inclusion is denoted by ∂_C and is called the *outer face map* associated to C.

Again one can easily see that an inner face map $A/a \to A$ corresponds exactly to an inner face map $Tr(A/e) \to Tr(A)$ and similarly for an outer face map.

DEFINITION 2.3.25. Let A be a dendroidally ordered set and $l = (a_1, a_2)$ a unary link in A. The map $\sigma_l : A \to A/a_2$ defined by

$$\sigma_l(x) = \begin{cases} x & x \neq a_2 \\ a_1 & x = a_2 \end{cases}$$

is a map of dendroidally ordered sets and is called the *degeneracy map* associated with the unary link l.

Comparing this definition of a degeneracy map with the operadic one we easily see that (under the identification of Ω_A with Ω_O) both definitions agree.

We now turn to the isomorphisms in Ω. Let A and B be two dendroidally ordered sets and $f : A \to B$ a function. It is easily verified that f is an isomorphism of dendroidally ordered sets if, and only if, f sends the root of A to the root of B and if for each $a \in A$

$$f(s(a)) = s(f(a)).$$

We are now going to prove Theorem 2.2.6. By the discussion so far it is obvious that we can state and prove the theorem in the setting of dendroidally ordered sets instead of that of operads. This turns out to be a slightly more convenient framework for a precise proof. We prepare for this proof with the following simple proposition whose proof we omit.

PROPOSITION 2.3.26. *If the map $\alpha : B \to B'$ of dendroidally ordered sets is an inner face (respectively outer face, degeneracy, isomorphism) then for any dendroidally ordered set A, the map $A \circ \alpha : A \circ B \to A \circ B'$ is an inner face (respectively outer face, degeneracy, isomorphism) whenever the grafting is defined.*

THEOREM 2.3.27. *(Restatement of Theorem 2.2.6) Any arrow $f : A \to B$ in Ω decomposes uniquely as*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\varphi} \\ A' & \xrightarrow{\pi} & B' \end{array}$$

where $\delta : A \to A'$ is a composition of degeneracy maps, $\pi : A' \to B'$ is an isomorphism, and $\varphi : B' \to B$ is a composition of face maps.

PROOF. We prove this by induction on $n = |A| + |B|$. If $n = 0$ or $n = 1$ the proof is trivial. Assume the assertion holds for $1 \leq n < m$ and let $f : A \to B$ be a map such that $|A| + |B| = m$. We consider four cases. First assume that $f(r_A) = b \neq r_B$ where r_A (respectively r_B) is the root of A (respectively B). In that case f factors through B_b:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow^{f'} & \uparrow{\scriptstyle\phi'} \\ & & B_b \end{array}$$

where ϕ' is the obvious inclusion of B_b into B which is clearly a composition of face maps (recall that $B_b = \{b' \in B \mid b \leq_d b'\}$). Since $|B_b| < |B|$ the induction hypothesis implies that f' can be factored as:

$$\begin{array}{ccc} A & \xrightarrow{f'} & B_b \\ \downarrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\phi} \\ A' & \xrightarrow{\pi} & B' \end{array}$$

and adjoining the map ϕ' to this decomposition yields the desired factorization of f.

We now consider the case where $f(r_A) = r_B$ and $f(s(r_A)) = s(r_B)$. Let $s(r_A) = \{a_1, \cdots, a_k\}$ and $s(r_B) = \{b_1, \cdots, b_k\}$ with $f(a_i) = b_i$. In that case, by restricting f to A_{a_i}, one obtains a map $f_i : A_{a_i} \to B_{b_i}$. Let $A_{root} = \{r_A, a_1, \cdots, a_k\}$ with the broad order induced by A and define B_{root} similarly. Let $f_{root} : A_{root} \to B_{root}$ be the restriction of f to A_{root}. The map f can be written as $f_{root} \circ (f_{a_1}, \cdots, f_{a_k})$. By the induction hypothesis each f_i decomposes as

$$\begin{array}{ccc} A_{a_i} & \xrightarrow{f_i} & B_{b_i} \\ \downarrow{\scriptstyle\delta_i} & & \uparrow{\scriptstyle\phi_i} \\ A'_{a_i} & \xrightarrow{\pi_i} & B'_{b_i} \end{array}$$

Let $A' = A_{root} \circ (A'_{a_1}, \cdots, A'_{a_k})$. The maps δ_i can then be grafted to produce the map $A_{root} \circ (\delta_1, \cdots, \delta_k) : A_{root} \circ (A_1, \cdots, A_k) \to A_{root} \circ (A'_{a_1}, \cdots, A'_{a_k})$ and since $A = A_{root} \circ (A_1, \cdots, A_k)$ we obtain a map $\delta : A \to A'$. It follows from the preceding proposition that δ is a composition of degeneracies. Similarly define $B' = B_{root} \circ (B'_{b_1}, \cdots, B'_{b_k})$ and then the ϕ_i together form a map $\phi : B' \to B$ which is a composition of face maps. Lastly the π_i also assemble themselves to give a map

$\pi : A' \to B'$ which is an isomorphism. These maps form the needed factorization of f.

The third case is when $f(r_A) = r_B$ but $f(s(r_A)) \neq s(r_B)$ and $f(x) \neq r_B$ for any $x \neq r_A$. Let $s(r_A) = \{a_1, \cdots, a_k\}$ and $f(a_i) = b_i$. Notice that an element $x \in B$ such that $r_B <_d x <_d b_i$ for some i is, of course, inner. Let $\hat{B}$ be the dendroidally ordered sub-set of B obtained by removing all of those inner elements. The inclusion $\hat{\phi} : \hat{B} \to B$ is then obviously a composition of (inner) face maps, and the map f factors through it as

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow^{\hat{f}} & \uparrow{\scriptstyle \hat{\phi}} \\ & & \hat{B} \end{array}$$

Since $f(s(r_A)) \neq s(r_B)$ the induction hypothesis applies to $\hat{f}$ which then factors as

$$\begin{array}{ccc} A & \xrightarrow{\hat{f}} & \hat{B} \\ \downarrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \phi} \\ A' & \xrightarrow{\pi} & \hat{B}' \end{array}$$

and adjoining $\hat{\phi}$ to this decomposition yields the desired factorization of f.

The last case to be considered is when $f(r_A) = r_B$ and $f(s(r_A)) \neq f(s(r_B))$ while for at least one $x \neq r_A$ we have $f(x) = r_B$. This implies that $s(r_A)$ consists of just one element a and $f(a) = r_B$. Thus (r_A, a) is a link. Let $\sigma : A \to A'$ be the degeneracy associated with it. Since $f(r_A) = f(a) = r_B$ it follows that f factors through σ as

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & A' \\ & \searrow^{f} & \downarrow{\scriptstyle f'} \\ & & B \end{array}$$

The induction hypothesis for f' together with σ now provide the required decomposition. The uniqueness of the decomposition follows by a rather straightforward induction and is omitted. □

2.4. Dendroidal sets - basic definitions

We now introduce the category of dendroidal sets, its basic terminology and properties, and establish its relation to simplicial sets.

Definition 2.4.1. The category *dSet* of *dendroidal sets* is the presheaf category on the dendroidal category Ω, that is it is the category of functors $\Omega^{op} \to Set$ and natural transformations.

A *dendroidal set* is thus a functor $X : \Omega^{op} \to Set$. Given a tree T in Ω we denote the set $X(T)$ by X_T and for each $\alpha : T \to S$ in Ω we denote the function $X(\alpha) : X_S \to X_T$ by α^*. An element $x \in X_T$ is called a *dendrex* of *shape* T, or a T-dendrex. A map of dendroidal sets $f : X \to Y$ is a natural transformation between the given functors. Such a natural transformation consists of functions (all denoted) $f : X_T \to Y_T$ for each tree T, such that for any $\alpha : T \to S$ in Ω and $x \in X_S$

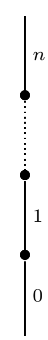
$$\alpha^* f(x) = f(\alpha^* x).$$

DEFINITION 2.4.2. Let T be a tree. The representable presheaf $\Omega(-, T) : \Omega^{op} \to$ *Set* is called the *standard T-dendrex* and is denoted by $\Omega[T]$. Explicitly we have for each tree S

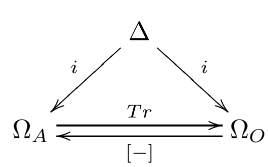
$$\Omega[T]_S = \Omega(S, T).$$

From the Yoneda lemma it follows that given any dendroidal set X, a T-dendrex $x \in X_T$ corresponds bijectively to a map of dendroidal sets $\iota_x : \Omega[T] \to X$. We will usually simply write x instead of ι_x, conveniently identifying a dendrex with its associated map. Notice that $\Omega[T]$ is functorial in T in the sense that given an arrow $\alpha : S \to T$ in Ω, there is the obvious induced dendroidal map $\Omega[\alpha] : \Omega[S] \to \Omega[T]$. A *dendroidal sub-set* Y of a dendroidal set X consists of, for each tree T, a sub-set $Y_T \subseteq X_T$ such that Y, endowed with the obvious structure from X, is a dendroidal set. Given a dendroidal set X and for each T a subset $Y_T \subseteq X_T$, we call the smallest dendroidal sub-set $\bar{Y}$ of X that contains Y (i.e., $Y_T \subseteq \bar{Y}_T$ for each T) the dendroidal set *generated* by Y.

We now define the basic functors that relate dendroidal sets to simplicial sets. These functors will be used often to relate definitions and results regarding dendroidal sets to simplicial sets and vice-versa. Since we have two (equivalent) definitions for Ω we need to be a bit more precise. We thus, very temporarily, use the notation Ω_A and Ω_O for (respectively) the algebraic and operadic definitions of Ω. Recall the functor $i : \Delta \to \Omega_O$ that sends $[n] \in ob(\Delta)$ to the operad $\Omega(L_n)$ where L_n is the tree depicted by



and the functor $i : \Delta \to \Omega_A$ that sends the poset $[n]$ to itself viewed as a dendroidally ordered set. It is clear that under the isomorphism $\Omega_A \underset{[-]}{\overset{Tr}{\rightleftarrows}} \Omega_O$ given in Theorem 2.3.21, the diagram



commutes. That means that we have one well-defined embedding $i : \Delta \to \Omega$ which from now on is fixed. This functor induces a restriction functor $i^* : dSet \to sSet$

which sends a dendroidal set X to the simplicial set

$$i^*(X)_n = X_{i(n)}.$$

This functor has both a left and a right adjoint (by Kan extension, see **[34]**) of which the left adjoint is of significance. The left adjoint $i_! : sSet \to dSet$ sends a simplicial set X to the dendroidal set given by

$$i_!(X)_T = \begin{cases} X_n, & \text{if } T \cong i([n]). \\ \phi, & \text{otherwise.} \end{cases}$$

This functor is full and faithful and thus embeds $sSet$ in $dSet$.

DEFINITION 2.4.3. Let T be a tree and $\alpha : S \to T$ a face map in Ω. The α-*face* of $\Omega[T]$, denoted by $\partial_\alpha \Omega[T]$, is the dendroidal sub-set of $\Omega[T]$ which is the image of the map $\Omega[\alpha] : \Omega[S] \to \Omega[T]$.

Thus we have that

$$\partial_\alpha \Omega[T]_R = \{ R \longrightarrow S \xrightarrow{\alpha} T \mid R \to S \in \Omega[S]_R \}$$

When α is obtained by contracting an inner edge e in T we denote ∂_α by ∂_e.

DEFINITION 2.4.4. Let T be a tree. The *boundary* of $\Omega[T]$ is the dendroidal sub-set $\partial\Omega[T]$ of $\Omega[T]$ obtained as the union of all the faces of $\Omega[T]$. That is

$$\partial\Omega[T] = \bigcup_{\alpha \in \Phi_1(T)} \partial_\alpha \Omega[T]$$

where $\Phi_1(T)$, is the set of all faces of T.

DEFINITION 2.4.5. Let T be a tree and $\alpha \in \Phi_1(T)$ a face of T. The α-*horn* in $\Omega[T]$ is the dendroidal sub-set $\Lambda^\alpha[T]$ of $\Omega[T]$ which is the union of all the faces of T except $\partial_\alpha \Omega[T]$, that is

$$\Lambda^\alpha[T] = \bigcup_{\beta \neq \alpha \in \Phi_1(T)} \partial_\beta \Omega[T].$$

The horn is called an *inner horn* if α is an inner face, otherwise it is called an *outer horn*. We will denote an inner horn $\Lambda^\alpha[T]$ by $\Lambda^e[T]$, where e is the contracted inner edge in T that defines the inner face $\alpha = \partial_e : T/e \to T$. A horn in a dendroidal set X is a map of dendroidal sets $\Lambda^\alpha[T] \to X$. It is inner (respectively outer) if the horn $\Lambda^\alpha[T]$ is inner (respectively outer).

REMARK 2.4.6. It is trivial to verify that these notions for dendroidal sets extend the common ones for simplicial sets in the sense, for example, that for the simplicial horn $\Lambda^k[n] \subseteq \Delta[n]$, the dendroidal set

$$i_!(\Lambda^k[n]) \subseteq i_!(\Delta[n]) = \Omega[L_n]$$

(where L_n is the linear tree with n vertices as described above) is a horn in the dendroidal sense. Furthermore, the horn $\Lambda^k[n]$ is inner (i.e., $0 < k < n$) if, and only if, the horn $i_!(\Lambda^k[n])$ is inner. A similar remark holds for the rest of the notions just introduced.

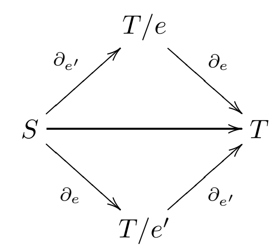
Both the boundary $\partial\Omega[T]$ and the horns $\Lambda^\alpha[T]$ in $\Omega[T]$ can be described as colimits as follows.

DEFINITION 2.4.7. Let $T_1 \to T_2 \to \cdots \to T_n$ be a sequence of n face maps in Ω. We call the composition of these maps a *sub-face* of T_n of *codimension* n.

Notice then that a face $S \to T$ is a sub-face of T of codimension 1.

PROPOSITION 2.4.8. *Let $S \to T$ be a sub-face of T of codimension 2. The map $S \to T$ decomposes in precisely two different ways as a composition of faces.*

PROOF. Consider $[S]$ and $[T]$, the dendroidally ordered sets associated with the trees S and T. We have to consider several cases in which the map $S \to T$ can be obtained. Assume thus that $S \to T$ is the composition of two inner face maps. That means that $[S] = [T]/\{e, e'\}$ where e and e' are both inner elements of $[T]$. It is obvious then that $S \to T$ decomposes only as



The other cases involve removing outer clusters (see 2.2.1) as well and are proved similarly. □

Let $\Phi_2(T)$ be the set of all sub-faces of T of codimension 2. The proposition implies that for each $\beta : S \to T \in \Phi_2(T)$ there are two maps $\beta_1 : S \to T_1$ and $\beta_2 : S \to T_2$ through which β factors. Using these maps we can form two maps γ_1 and γ_2

$$\coprod_{S \to T \in \Phi_2(T)} \Omega[S] \rightrightarrows \coprod_{R \to T \in \Phi_1(T)} \Omega[R]$$

where γ_i $(i = 1, 2)$ has component $\Omega[S] \xrightarrow{\Omega[\beta_i]} \Omega[T_i] \longrightarrow \coprod \Omega[R]$ for each $\beta : S \to T \in \Phi_2(T)$.

LEMMA 2.4.9. *Let T be a tree in Ω. With notation as above we have that the boundary $\partial\Omega[T]$ is the coequalizer*

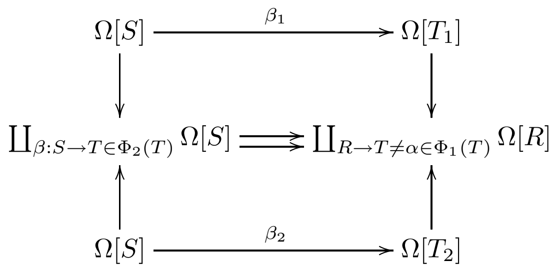
$$\coprod_{S \to T \in \Phi_2(T)} \Omega[S] \rightrightarrows \coprod_{R \to T \in \Phi_1(T)} \Omega[R] \to \partial\Omega[T]$$

for the two maps γ_1, γ_2 constructed above.

PROOF. The required universal property of $\partial\Omega[T]$ is easily established. □

COROLLARY 2.4.10. *A map of dendroidal sets $\partial\Omega[T] \to X$ corresponds exactly to a sequence $\{x_R\}_{R \to T \in \Phi_1(T)}$ of dendrices whose faces match, in the sense that for each sub-face $\beta : S \to T$ of codimension 2 we have $\beta_1^*(x_{T_1}) = \beta_2^*(x_{T_2})$.*

We have a similar presentation for horns. For a fixed face $\alpha : S \to T \in \Phi_1(T)$ consider the parallel arrows defined by making the following diagram commute



where the vertical arrows are the canonical injections into the coproduct and where we use the same notation as above.

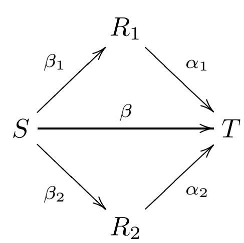
LEMMA 2.4.11. *Let T be a tree in Ω and α a face of T. In the diagram*

$$\coprod_{S\to T\in\Phi_2(T)} \Omega[S] \rightrightarrows \coprod_{R\to T\neq\alpha\in\Phi_1(T)} \Omega[R] \to \Lambda^\alpha[T]$$

the dendroidal set $\Lambda^\alpha[T]$ is the coequalizers of the two maps constructed above.

PROOF. Again, the verification of the universal property is simple and thus omitted. □

COROLLARY 2.4.12. *A horn $\Lambda^\alpha[T] \to X$ in X corresponds exactly to a sequence $\{x_R\}_{R\to T\neq\alpha\in\Phi_1(T)}$ of dendrices that agree on common faces in the sense that if $\beta : S \to T$ is a sub-face of codimension 2 which factors as*



where $\alpha_i \neq \alpha$ $(i = 1, 2)$ then

$$\beta_1^*(x_{R_1}) = \beta_2^*(x_{R_2}).$$

REMARK 2.4.13. In the special case where the tree T is linear we obtain the equivalent result for simplicial sets. Namely, the presentation of the boundary $\partial\Delta[n]$ and of the horn $\Lambda^k[n]$ as colimits of standard simplices, and the description of a horn $\Lambda^k[n] \to X$ in a simplicial set X (see, respectively, **[19]** page 8, page 9, and Corollary 3.2).

We end this section by introducing the terminology of faces, sub-faces, and so on for dendrices in a dendroidal set.

DEFINITION 2.4.14. Let $\alpha : S \to T$ be a map in Ω and X a dendroidal set. Given a dendrex $t \in X_T$ we refer to the dendrex $\alpha^* t \in X_S$ as

(1) a *face* (respectively *inner face, outer face*) of t if α is a face (respectively inner face, outer face) of T.
(2) a *sub-face* of t if α is a sub-face of T.

(3) *isomorphic* to t if α is an isomorphism.
(4) a *degeneracy* of t if α is a composition of degeneracies.

2.5. Closed monoidal structure on the category of dendroidal sets

Just like any presheaf category, $dSet$ is a cartesian closed category ([**33**]). This cartesian product extends the cartesian product of simplicial sets in the sense that for two simplicial sets X and Y we have

$$i_!(X \times Y) \cong i_!(X) \times i_!(Y).$$

However, there is another closed monoidal structure on $dSet$ with a very strong connection to the Boardman-Vogt tensor product of operads, as we shall see below. In this section we introduce this monoidal structure and study it in detail.

In a presheaf category a closed monoidal structure is completely determined (up to isomorphism) by the tensor product of representables. This follows easily since the tensor product, being closed, preserves colimits and since every object in a presheaf category is a colimit of representables (see the preliminaries). In more detail, suppose $\otimes$ is a closed monoidal structure on $dSet$ and let X and Y be two dendroidal sets. Write

$$X = \varinjlim \Omega[T]$$

and

$$Y = \varinjlim \Omega[S]$$

canonically as in Section 0.1.1. Then we obtain that

$$X \otimes Y = \varinjlim \Omega[T] \otimes \varinjlim \Omega[S] \cong \varinjlim (\Omega[T] \otimes \Omega[S])$$

and we see that $X \otimes Y$ can be expressed in terms of $\Omega[T] \otimes \Omega[S]$. We thus need to define $\otimes$ only for representable dendroidal sets. To that end consider the functor $\Omega \to BrdPoset$ which sends a tree T to the dendroidally ordered set $[T]$. Taken as a probe (Section 0.1.1) it induces a nerve-realisation adjunction

$$dSet \underset{N}{\rightleftarrows} BrdPoset$$

of which the left adjoint shall remain nameless. We now define, for two trees T and S in Ω, the tensor product of the associated representable dendroidal sets by

$$\Omega[T] \otimes \Omega[S] = N([T] \otimes [S])$$

where $[T] \otimes [S]$ is the tensor product in the category $BrdPoset$ (Definition 2.3.5). We can now define the tensor product in $dSet$.

DEFINITION 2.5.1. Let X and Y be two dendroidal sets. Their tensor product is given as follows. Write X and Y canonically as colimits as above. Then

$$X \otimes Y = \varinjlim \Omega[T] \otimes \varinjlim \Omega[S] = \varinjlim (\Omega[T] \otimes \Omega[S]).$$

THEOREM 2.5.2. *The above defined tensor product of dendroidal sets turns* $dSet$ *into a symmetric closed monoidal category.*

PROOF. This follows from general theorems of category theory as presented in [**11, 26**]. ☐

The internal Hom is explicitly given as follows. Let X and Y be two dendroidal sets. Their internal Hom is the dendroidal set $\underline{dSet}(X,Y)$ whose set of T-dendrices is given by

$$\underline{dSet}(X,Y)_T = dSet(X \otimes \Omega[T], Y)$$

and the dendroidal structure is given in the obvious way.

The monoidal structure on $dSet$ extends the cartesian product in $sSet$ in the following sense.

Lemma 2.5.3. *For any two simplicial sets X and Y*

$$i_!(X) \otimes i_!(Y) \cong i_!(X \times Y).$$

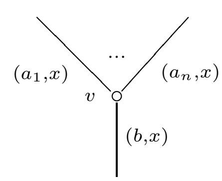
Proof. The statement will follow if we can show that it holds for representable simplicial sets. This will be proved in Theorem 3.1.4 after we have developed some more of the theory of dendroidal sets. □

Let us study the simplest (yet important) case of the tensor product of two representable dendroidal sets. To that end let us fix two trees S and T. Our aim is to exhibit $\Omega[S] \otimes \Omega[T]$ as a union of some of its dendrices, which carry a natural partial order.

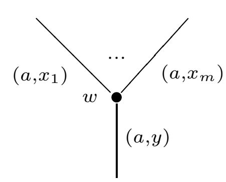
By definition, $\Omega[S] \otimes \Omega[T]$ is the dendroidal set $N([S] \otimes [T])$, where $[S] \otimes [T]$ is the tensor product of broad posets and $N(-)$ is the functor $N : BrdPoset \to dSet$ obtained from the probe $\Omega \to BrdPoset$ sending T to $[T]$. An R-dendrex in $\Omega[S] \otimes \Omega[T]$ is thus a map

$$[R] \to [S] \otimes [T]$$

of broad posets. It is easily seen that this dendrex is non-degenerate if, and only if, its underlying set function is injective. For the purpose of characterizing these maps, let us think of the vertices of S as being *white* (drawn $\circ$) and those of T as being *black* (drawn $\bullet$). Consider a tree R whose set of edges is the set $E(S) \times E(T)$. A vertex in such a tree that looks like this:



where v is a vertex in S with input edges $a_1, \cdots, a_n$ and output edge b, while x is an edge of T, will be called a *white* vertex. Similarly a vertex in R that looks like this:



where w is a vertex in T with input edges $x_1, \cdots, x_m$ and output edge y, while a is an edge in S, will be called a *black* vertex.

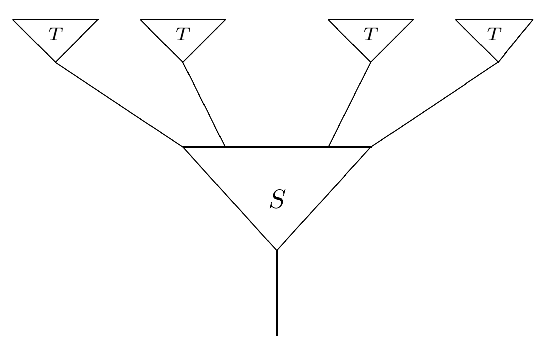
Definition 2.5.4. An (S,T)-tree is a tree R whose set of edges is the set $E(T) \times E(S)$ in which every vertex is either white or black.

Obviously, any (S,T)-tree R gives rise to an injective map $f : [R] \to [S] \otimes [T]$ (just the inclusion), and thus corresponds to a non-degenerate dendrex in $\Omega[S] \otimes \Omega[T]$. Moreover, every such non-degenerate dendrex is isomorphic (see Definition 2.4.14) to a dendrex obtained this way. Among all (S,T)-trees there are certain trees that are maximal in the following sense:

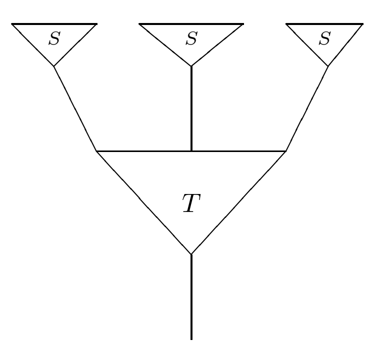
DEFINITION 2.5.5. An (S,T)-tree R is called a *percolation* tree if the root of R is (r_S, r_T) where r_S (respectively r_T) is the root of S (respectively T) and if each leaf of R is of the form (l_S, l_T) where l_S (respectively l_T) is a leaf of S (respectively T).

It is easily verified that every (S,T)-tree can be extended (not necessarily uniquely) to a percolation tree. In that sense the percolation trees are maximal and it follows that every non-degenerate vertex in $\Omega[S] \otimes \Omega[T]$ is isomorphic to a sub-face (see Definition 2.4.14) of a dendrex given by a percolation tree.

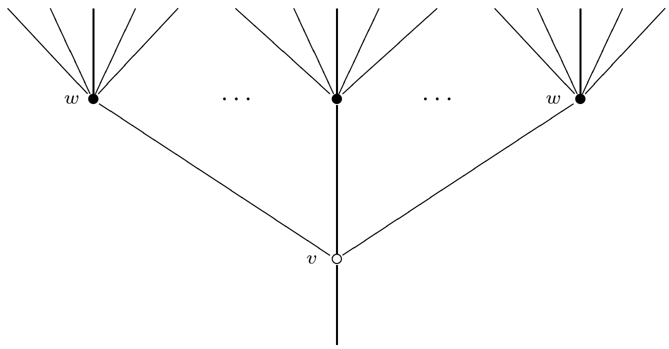
All the possible percolation trees R_i come in a natural partial order. The minimal tree R_1 in the poset is the one obtained by stacking a copy of the black tree T on top of each of the input edges of the white tree S. Or, more precisely, on the bottom of T_1 there is a copy $S \otimes r_T$ of the tree S all whose edges are (a, r_T) where r_T the root of T. For each input edge b of S, a copy of T is grafted on the edge (b, r) of $S \otimes r$, whose edges are (b, x). The maximal tree R_N in the poset is the similar tree with copies of the white tree S grafted on each of the input edges of the black tree. Pictorially R_1 looks like



and R_N looks like

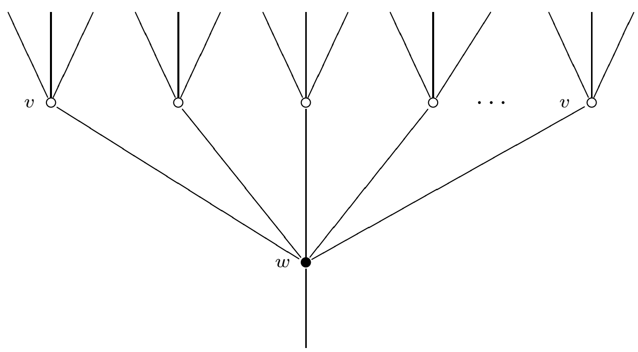


The intermediate trees R_k $(1 < k < N)$ are obtained by letting the black vertices in R_1 slowly percolate in all possible ways towards the root of the tree. Each R_k is obtained from an earlier R_l by replacing a configuration



(1)

in R_l by the configuration



(2)

in R_k. More explicitly, let P be the portion of the tree R_l shaped like (1). If we denote by R' the part of R_l below the vertex v and by $R'_1, \cdots, R'_m$ the parts of R_l above the edges depicted as leaves in (1) then we can write

$$R_l = R' \circ P \circ (R'_1, \cdots, R'_m).$$

The tree R_k is now given as

$$R_k = R' \circ P' \circ (R'_1, \cdots, R'_m)$$

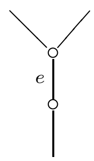
where P' is a tree that looks like (2). Notice that the grafting is well defined since the output edges of P' are precisely the roots of the various R_i, where only the order changed. Since we are dealing with non-planar trees we don't have to rearrange the trees R_i. When this is the case we say that R_k is obtained by a *single percolation step* from R_l and denote this by $R_l \leq R_k$. This defines a partial order on the set of all percolation trees.

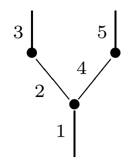
As mentioned, each percolation tree R_k corresponds to a dendrex in $\Omega[S] \otimes \Omega[T]$ of shape R_k and thus to a map $m : \Omega[R_k] \to \Omega[S] \otimes \Omega[T]$. We denote by $m(R_k)$ the image of $\Omega[R_k]$ under this map. We summarize the above discussion in the following lemma.

LEMMA 2.5.6. *(Shuffle presentation of $\Omega[S] \otimes \Omega[T]$) With notation as above we have the equality*

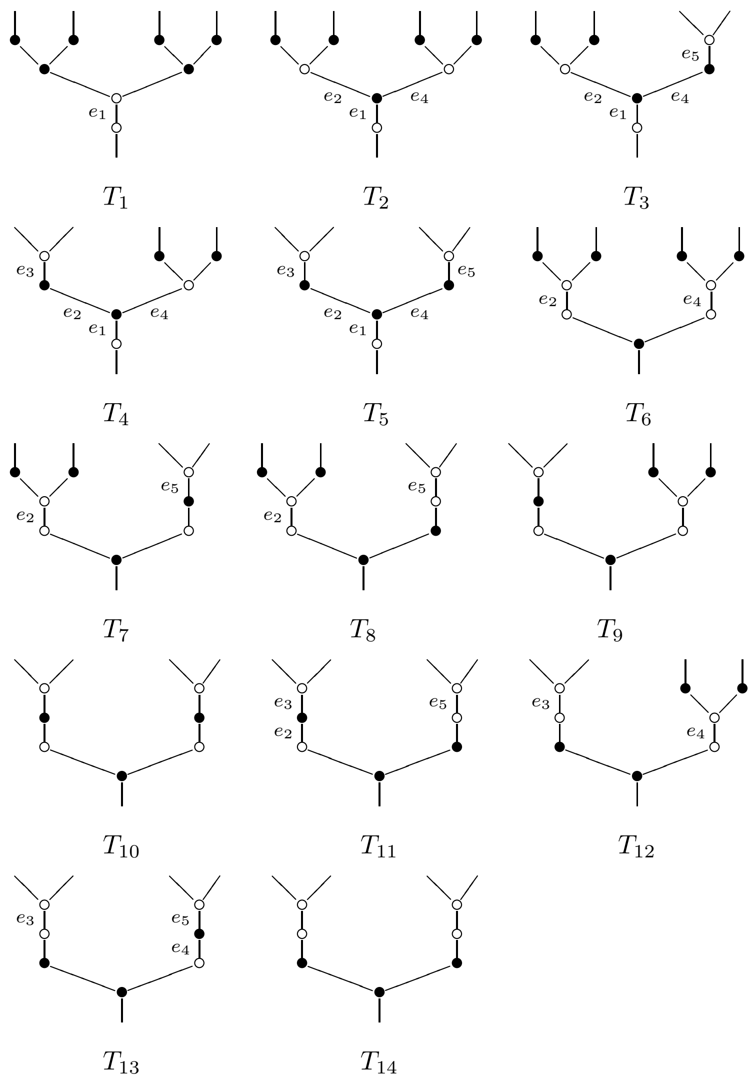
$$\Omega[S] \otimes \Omega[T] = \bigcup_{k=1}^{N} m(R_k).$$

EXAMPLE 2.5.7. We illustrate this shuffle presentation with the following two trees S and T; here, we have singled out one particular edge e in S, numbered the edges of T as $1, \cdots, 5$, and denoted the edge (e, i) in R_k by e_i.

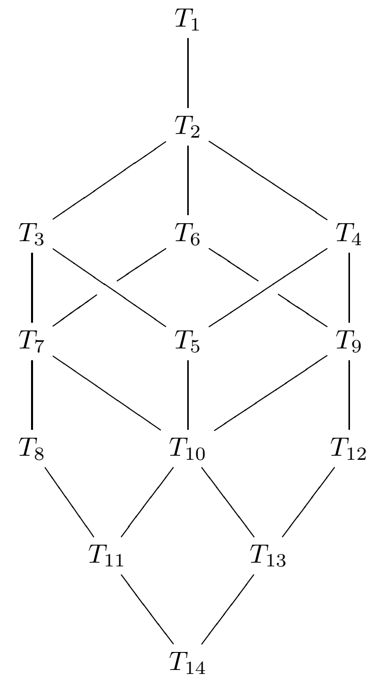
$S =$ 

$T =$ 

There are 14 percolation schemes $T_1, \cdots, T_{14}$ in this case:



The poset structure on the percolation trees above is:



2.6. Skeletal filtration

We present now a useful filtration of a dendroidal set based on non-degenerate elements.

Definition 2.6.1. Let X be a dendroidal set. A dendrex $t \in X_T$ is called *degenerate* if it is a degeneracy of some dendrex s (see Definition 2.4.14). Otherwise t is called *non-degenerate*. The *degree* of the dendrex t is equal to the degree of the tree T, i.e., the number of vertices of T.

For example, any dendrex $t \in X_T$ where T has no unary vertices is non-degenerate. One can also easily see that every degenerate dendrex $t \in X_T$ is a degeneracy of a non-degenerate dendrex. This dendrex is unique up to an isomorphism.

Definition 2.6.2. Let X be a dendroidal set and $n \geq 0$ a natural number. The *n-skeleton* of X is the dendroidal set $Sk_n(X) \subseteq X$ generated by the dendrices of X of degree less then or equal to n. There is an obvious inclusion $Sk_n(X) \subseteq Sk_{n+1}(X)$.

Clearly we have that

$$X = \bigcup_{n=0}^{\infty} Sk_n(X)$$

and we refer to this presentation of X as the skeletal filtration of X.

Remark 2.6.3. This filtration of a dendroidal set relates to the standard filtration of a simplicial set as follows. If X is a simplicial set with skeletal filtration $X_0 \subseteq X_1 \subseteq \cdots$ then $i_!(X_0) \subseteq i_!(X_1) \subseteq \cdots$ is isomorphic to the skeletal filtration of the dendroidal set $i_!(X)$.

Definition 2.6.4. A dendroidal set X is called *n-skeletal* if, given a dendroidal set Y, every map $Sk_n(X) \to Sk_n(Y)$ extends uniquely, along the inclusion $Sk_n(Y) \to Y$, to a map $X \to Y$. Similarly, X is called *n-coskeletal* if given an arbitrary dendroidal set Z, every map $Sk_n(Z) \to Sk_n(X)$ extends uniquely , along the inclusion $Sk_n(Z) \to Z$, to a map $Y \to X$.

Once again, for a simplicial set X, the dendroidal set $i_!(X)$ is n-skeletal (respectively n-coskeletal) if, and only if, X is n-skeletal (respectively n-coskeletal).

Notice that

$$Sk_0(X) = \coprod_{x \in X_\eta} \Omega[\eta]$$

where η is some fixed unit tree. For $n > 0$ consider now the following diagram:

$$\begin{array}{ccc} \coprod_{(t,T)} \partial\Omega[T] & \longrightarrow & Sk_{n-1}(X) \\ \downarrow & & \downarrow \\ \coprod_{(t,T)} \Omega[T] & \longrightarrow & Sk_n(X) \end{array}$$

where the coproduct ranges over all isomorphism classes of pairs (t, T) in the category of elements of X, where $|T| = n$ and $t \in X_T$ is non-degenerate. In more detail, two pairs (t, T) and (s, S) where $|T| = |S| = n$ and both $t \in X_T$ and $s \in X_S$ are non-degenerate are isomorphic if there is an isomorphism $\alpha : S \to T$ such that $\alpha^* t = s$. In the coproduct above we choose one representative of each such isomorphism class of pairs.

Definition 2.6.5. Let X be a dendroidal set. The skeletal filtration of X is said to be *normal* if for each $n > 0$, the square above is a pushout. We then say that X admits a *normal skeletal filtration*.

Following Cisinski **[9]** we make the following definition:

Definition 2.6.6. A dendroidal set X is called *normal* if for each non-degenerate dendrex $t \in X_T$, the only isomorphism $\alpha : T \to T$ that fixes t (i.e., $\alpha^*(t) = t$) is the identity.

Cisinski develops a very rich theory of certain presheaf categories and a special case of his theory is the following theorem:

Lemma 2.6.7. *A dendroidal set X is normal if, and only if, it admits a normal skeletal filtration.*

Proof. The proof is a special case of Lemma 8.1.34 in **[9]** after noting that Ω is a skeletal category (Definition 8.1.1 in **[9]**). □

Clearly, all representable dendroidal sets are normal. Below we will see examples of other normal dendroidal sets. To get a better intuition for normal dendroidal sets let us give an example of a dendroidal set which is not normal. Consider the dendroidal set $X = \Omega[C_2]$ where C_2 is a 2-corolla. For this dendroidal set, X_η consists of three dendrices and X_{C_2} consists of two dendrices with an evident action of $\mathbb{Z}_2$. Consider now the dendroidal set $Y = X/\mathbb{Z}_2$ obtained from X by identifying the two dendrices in X_{C_2}. That means that Y_η consists of two elements and Y_{C_2} of one element. It is then clear that Y doesn't satisfy the condition for normality (the unique dendrex in Y_{C_2} is fixed by a non-trivial isomorphism) and it can also be seen directly that the skeletal filtration of Y is not normal.

CHAPTER 3

Operads and dendroidal sets

This chapter is concerned with the relation between operads, dendroidal sets, and simplicial sets. The relation is established by means of the dendroidal nerve functor which associates with every operad a dendroidal set - its nerve. The notion of an inner Kan dendroidal set is then introduced. This notion is a generalization of a notion given by Boardman and Vogt for simplicial sets in [**7**]. The technique of anodyne extensions is then imported from the theory of simplicial sets and is demonstrated by a simple example. Grafting in dendroidal sets is then discussed as well as homotopy in a dendroidal set. It is shown that with each inner Kan complex one can associate a homotopy operad, which is then used to deduce a characterization of nerves of operads as dendroidal sets satisfying certain strict filling conditions. Following is a proof that the inner Kan complexes form an exponential ideal in the category of dendroidal sets, a result which generalizes a recent result of Joyal [**24**] for simplicial sets. The chapter ends by introducing a process that turns an arbitrary dendroidal set into an inner Kan complex.

3.1. Nerves of operads

The functor relating operads to dendroidal sets is the operadic nerve functor. It is the aim of this section to introduce this functor and study it, and other related functors, in detail.

Definition 3.1.1. Consider the probe $F : \Omega \to \mathit{Operad}$ which sends a tree T to the operad $\Omega(T)$, and the induced adjunction $dSet \underset{N_F}{\overset{|\cdot|_F}{\rightleftarrows}} \mathit{Operad}$. The functor N_F is called the *operadic nerve* functor and will be denoted by N_d. The functor $|\cdot|_F$ is called the *operadic realization* functor and will be denoted by τ_d.

Explicitly, for an operad P, its nerve is the dendroidal set given by

$$N_d(\mathcal{P})_T = \mathit{Operad}(\Omega(T), \mathcal{P}).$$

It is practically a tautology that for any tree $T \in ob(\Omega)$

$$N_d(\Omega(T)) = \Omega[T].$$

Slightly less trivial is the fact that for any operad $\mathcal{P}$

$$\tau_d(N_d(\mathcal{P})) \cong \mathcal{P},$$

a property that will be used on several occasions below.

The categories Cat and $sSet$ are both cartesian closed categories and, with respect to these monoidal structures, both of the functors τ and N are strong monoidal. As we have seen, the categories Operad and $dSet$ also carry a closed monoidal structure, and we turn now to investigate the properties of the functors τ_d and N_d with respect to these monoidal structures.

We would first like to relate the tensor product of dendroidal sets with the Boardman-Vogt tensor product of operads. Recall that the tensor product of dendroidal sets is defined by cocontinuously extending the formula

$$\Omega[T] \otimes \Omega[S] = N([T] \otimes [S])$$

where $N : BrdPoset \to dSet$ is the nerve functor defined in Section 2.5. Since we now also have the nerve functor of operads, we can define a tensor product on $dSet$ by cocontinuously extending the formula

$$\Omega[T] \otimes \Omega[S] = N_d(\Omega(T) \otimes_{BV} \Omega(S)).$$

However, both approaches yield essentially the same monoidal structure. This follows from the easily established equality

$$N([T] \otimes [S]) \cong N_d(\Omega(T) \otimes_{BV} \Omega(S))$$

which holds for any two trees T and S.

To compare the Boardman-Vogt tensor product with the tensor product of dendroidal sets it is convenient to notice first that any operad $\mathcal{P}$ can be written canonically as a colimit of operads of the form $\Omega(T)$, namely

$$\mathcal{P} \cong \varinjlim_{\Omega(T) \to \mathcal{P}} \Omega(T),$$

and since the Boardman-Vogt tensor product of operads is closed, we obtain the formula

$$\mathcal{P} \otimes_{BV} \mathcal{Q} = \varinjlim(\Omega(T) \otimes_{BV} \Omega[S])$$

with the colimit taken over the obvious diagram.

Lemma 3.1.2. *For any two operads $\mathcal{P}$ and $\mathcal{Q}$*

$$\tau_d(N_d(\mathcal{P}) \otimes N_d(\mathcal{Q})) \cong \mathcal{P} \otimes_{BV} \mathcal{Q}.$$

Proof. By definition:

$$N_d(\mathcal{P}) \otimes N_d(\mathcal{Q}) = \varinjlim(\Omega[T] \otimes \Omega[S]).$$

Since

$$N([T] \otimes [S]) \cong N_d(\Omega(T) \otimes_{BV} \Omega(S))$$

we obtain that

$$\tau_d(\Omega[T] \otimes \Omega[S]) = \tau_d(N([T] \otimes [S])) \cong \tau_d(N_d(\Omega(T) \otimes_{BV} \Omega(S)) \cong \Omega(T) \otimes_{BV} \Omega(S).$$

Since τ_d, as a left adjoint, commutes with colimits we obtain that

$$\tau_d(N_d(\mathcal{P}) \otimes N_d(\mathcal{Q})) \cong \varinjlim(\Omega(T) \otimes_{BV} \Omega(S)) \cong \mathcal{P} \otimes_{BV} \mathcal{Q}$$

as claimed. □

Remark 3.1.3. Notice that this lemma implies that the Boardman-Vogt tensor product of operads is completely determined by the tensor product of broad posets. While this fact might not be very important in the general theory, it is remarkable that the quite involved Boardman-Vogt tensor product is already contained within a much simpler notion.

We summarise the relation between categories, operads, simplicial sets, and dendroidal sets in the following theorem.

Theorem 3.1.4. *In the diagram*

$$\begin{array}{ccc} Cat & \underset{j^*}{\overset{j_!}{\rightleftarrows}} & Operad \\ \tau \uparrow \downarrow N & & \tau_d \uparrow \downarrow N_d \\ sSet & \underset{i^*}{\overset{i_!}{\rightleftarrows}} & dSet \end{array}$$

all pairs of functors are adjunctions with the left adjoint on top or to the left. Furthermore, the following canonical commutativity relations hold

$$\begin{aligned} &\tau N \cong id \\ &\tau_d N_d \cong id \\ &i^* i_! \cong id \\ &j^* j_! \cong id \\ &j_! \tau \cong \tau_d i_! \\ &N j^* \cong i^* N_d \\ &i_! N \cong N_d j_!. \end{aligned}$$

If we consider the cartesian structures on Cat and $sSet$, the Boardman-Vogt tensor product on $Operad$, and the tensor product of dendroidal sets then the four categories are symmetric closed monoidal categories and the functors $i^*, i_!, N, \tau, j^*, j_!$ and τ_d are strong monoidal.

Proof. The commutativity relations are easily seen to hold. The fact that N is strong monoidal is well known (and easily proved). Proving that $j_!$ is strong monoidal is also easy. To show that $i_!$ is strong monoidal we need to prove that for two simplicial sets X and Y

$$i_!(X \times Y) \cong i_!(X) \otimes i_!(Y).$$

Since $i_!$ is a left adjoint it commutes with colimits, and it therefore follows that it is enough to show that the formula holds for representable simplicial sets, which we now do. Recall that we denote by L_k the linear tree with k vertices. We now have:

$$\begin{aligned} i_!(\Delta[n] \times \Delta[m]) &\cong i_!(N([n]) \times N([m])) \\ &\cong i_!(N([n] \times [m])) \\ &\cong N_d j_!([n] \times [m]) \\ &\cong N_d(j_![n] \otimes_{BV} j_![m]) \\ &\cong N_d(\Omega(L_n) \otimes_{BV} \Omega(L_m)) \\ &\cong \Omega[L_n] \otimes \Omega[L_m] \\ &\cong i_!(\Delta[n]) \otimes i_!(\Delta[m]). \end{aligned}$$

as claimed.

To prove that τ_d is strong monoidal we need to show that

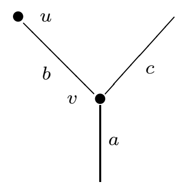
$$\tau_d(X \otimes Y) \cong \tau_d(X) \otimes_{BV} \tau_d(Y)$$

holds for any two dendroidal sets X and Y. Once again it is enough to establish the result for representables, and indeed we have

$$\begin{array}{rcl} \tau_d(\Omega[T]\otimes\Omega[S]) & = & \tau_d(N_d(\Omega(T))\otimes N_d(\Omega(S)))\cong \\ \Omega(T)\otimes_{BV}\Omega(S) & = & \tau_d(N_d(\Omega(T)))\otimes_{BV}\tau_d(N_d(\Omega(S))) = \\ \tau_d(\Omega[T])\otimes_{BV}\tau_d(\Omega[S]) & & \end{array}$$

as required. The rest of the proof follows along similar lines and is omitted. □

REMARK 3.1.5. In general, the canonical map $\tau i^*(X)\to j^*\tau_d(X)$ is not an isomorphism. Consider for example the tree T given by



For the dendroidal set $\Omega[T]$ we have that $i^*\Omega[T]$ is a disjoint union of three copies of $\Omega[\eta]$ and thus $\tau i^*\Omega[T]$ is simply a category with three different objects and non-identity arrows. On the other hand, the operad $\tau_d\Omega[T]$ contains the unary operation $v\circ_1 u : c\to a$ which is thus also present in $j^*\tau_d\Omega[T]$ and so we have that $j^*\tau_d\Omega[T]\ncong\tau i^*\Omega[T]$.

The nerve functor $N : Cat\to sSet$ can easily be shown to commute with internal Homs in the sense that for any two categories $\mathcal{C}$ and $\mathcal{D}$, the equation

$$N(\underline{Cat}(\mathcal{C},\mathcal{D}))\cong\underline{sSet}(N(\mathcal{C}),N(\mathcal{D}))$$

holds. Lemma 3.1.2 allows us to prove a similar result for the dendroidal nerve functor.

LEMMA 3.1.6. *The dendroidal nerve functor commutes with internal Homs in the sense that for any two operads $\mathcal{P}$ and $\mathcal{Q}$ we have*

$$N_d(\underline{Operad}(\mathcal{P},\mathcal{Q}))\cong\underline{dSet}(N_d(\mathcal{P}),N_d(\mathcal{Q})).$$

PROOF. For a tree $T\in ob(\Omega)$ we have the equations:

$$\begin{array}{rcl} \underline{dSet}(N_d(\mathcal{P}),N_d(\mathcal{Q}))_T & = & dSet(N_d(\mathcal{P})\otimes\Omega[T],N_d(\mathcal{Q})) = \\ dSet(N_d(\mathcal{P})\otimes N_d(\Omega(T)),N_d(\mathcal{Q})) & \cong & Operad(\tau_d(N_d(\mathcal{P})\otimes N_d(\Omega(T))),\mathcal{Q})\cong \\ Operad(\mathcal{P}\otimes_{BV}\Omega(T),\mathcal{Q}) & \cong & Operad(\Omega(T),\underline{Operad}(\mathcal{P},\mathcal{Q})) = \\ N_d(\underline{Operad}(\mathcal{P},\mathcal{Q}))_T & & \end{array}$$

which prove the claim. □

Another functor that commutes with internal Homs is $i_!$. To show that, we use the useful, and easily verified property, that the the functor $i_! : sSet\to dSet$ embeds simplicial sets in dendroidal sets as a sieve, i.e., that given a simplicial set Y and an arbitrary dendroidal set X, if there is a map $i_!(X)\to Y$ in Ω then $Y=i_!(Y')$ for some simplicial set Y'.

LEMMA 3.1.7. *For simplicial sets X and Y we have:*

$$\underline{dSet}(i_!(X),i_!(Y))\cong i_!(\underline{sSet}(X,Y)).$$

PROOF. Notice that if T is a tree that is not of the form L_k then

$$\underline{dSet}(i_!(X), i_!(Y))_T = dSet(i_!(X) \otimes \Omega[T], i_!(Y))$$

would be empty if $X \neq \phi$. Now, for a linear tree L_k we have

$$\begin{aligned} \underline{dSet}(i_!(X), i_!(Y))_{L_k} &= dSet(i_!(X) \otimes \Omega[L_k], i_!(Y)) = \\ dSet(i_!(X) \otimes i_!(\Delta[k]), i_!(Y)) &= dSet(i_!(X \times \Delta[k]), i_!(Y)) = \\ sSet(X \times \Delta[k], i^* i_!(Y)) &= sSet(X \times \Delta[k], Y) = \\ \underline{sSet}(X, Y)_k &= i_!(\underline{sSet}(X, Y))_{L_k} \end{aligned}$$

as claimed. □

3.2. Inner Kan complexes

In this section we introduce the notion of inner Kan complexes in the category of dendroidal sets. We start off by motivating the definition, relating it to inner Kan complexes in the category of simplicial sets (also studied under the name "quasi-categories" by Joyal in **[23, 24]**). Once the definition is given, we provide a class of examples and examine the relation between coskeletality and strict inner Kan complexes, as a first step to characterizing the latter.

Recall that a horn $\Lambda^k[n]$ in the simplicial sense is said to be inner if $0 < k < n$. In **[7]** (page 102) the authors make the following definition:

DEFINITION 3.2.1. A simplicial set X is said to satisfy the restricted Kan condition if every inner horn $\Lambda^k[n] \to X$ can be filled.

We will call such a simplicial set an *inner Kan* simplicial set. The need for such a definition stems from the fact that weak algebraic structures (for example, A_∞-spaces) and their weak maps usually do not form a category. The problem is that, generally, the composition of such maps (if it is at all defined) is not associative. That the notion of an inner Kan simplicial set is at least a plausible replacement of a category is seen by the fact that the nerve of a category always satisfies the restricted Kan condition (we prove a stronger result below). However, there are many simplicial sets that do satisfy this condition without them being nerves of categories, among which lie the simplicial set of A_∞-spaces (as is shown in **[7]**).

In the more recent work **[24]** Joyal is extensively studying inner Kan simplicial sets (which he calls quasi-categories) as an extension of the theory of categories. As Joyal put it himself: "You find yourself in the situation where most of the results of category theory can be extended to quasi-categories. It's just that the proof is anything between 10 to 100 times more difficult". The extra labour needed to prove those theorems arises from the fact that quasi-categories can be thought of as special weak ω-categories, and as such carry with them the complexity of maps between maps between maps between maps..... However, the resulting theory is applicable in many situations where ordinary category theory is too strict.

Continuing with the main theme of this work, that operads are generalized categories, it is very natural to extend the inner Kan condition from simplicial sets to dendroidal sets. This is done by means of the following definition.

DEFINITION 3.2.2. Let X be a dendroidal set. X is said to satisfy the *inner Kan condition* with respect to the tree T if for any inner horn $h : \Lambda^e\Omega[T] \to X$, there is a dendrex $t : \Omega[T] \to X$ such that the diagram

$$\begin{array}{ccc} \Lambda^e\Omega[T] & \xrightarrow{h} & X \\ \downarrow & \nearrow_{t} & \\ \Omega[T] & & \end{array}$$

commutes, where the vertical arrow is the inclusion. If X satisfies the inner Kan condition with respect to all trees T then X is called an *inner Kan complex.* When the filler for the horn is unique we will say that X satisfies the *strict* inner Kan condition and that X is a *strict* inner Kan complex.

The proof of the following proposition, relating the inner Kan condition for simplicial sets and dendroidal sets, is trivial.

PROPOSITION 3.2.3. *Let S be a simplicial set and D a dendroidal set.*

(1) $i_!(S)$ is an inner Kan complex if, and only if, S is.
(2) If D is an inner Kan complex then so is $i^*(D)$.

The following lemma provides a whole class of examples of strict inner Kan complexes.

LEMMA 3.2.4. *Let $\mathcal{P}$ be an operad. The dendroidal set $X = N_d(\mathcal{P})$ is a strict inner Kan complex.*

PROOF. A dendrex $x \in X_T$ is a map $x : \Omega[T] \to N_d(\mathcal{P})$ which, by adjunction, is the same as a map of operads $\Omega_\pi(\bar{T}) \to P$, where $\bar{T}$ is an arbitrary (but fixed) planar representative of T. Since $\Omega_\pi(\bar{T})$ is a free planar operad generated by operations corresponding to the vertices of the tree $\bar{T}$, it follows that x is equivalent to a labelling of $\bar{T}$ as follows. The edges are labelled by objects of $\mathcal{P}$ and the vertices are labelled by operations in $\mathcal{P}$ where the input of such an operation is the tuple of labels of the incoming edges to the vertex and the output is the label of the outgoing edge from the vertex. Any inner horn $\Lambda^e[T] \to N_d(\mathcal{P})$ is easily seen to be equivalent to such a labelling of the tree T and thus determines a unique filler. □

The strict inner Kan condition is very strong and in fact we will show below that the strict inner Kan complexes are precisely those dendroidal sets that are nerves of operads. One can easily turn a strict inner Kan complex into a non-strict one, simply by adding new dendrices that fill already existing horns. More natural examples of inner Kan complexes that are usually not strict will be seen to arise as suitable nerves of operads in a symmetric monoidal model category $\mathcal{E}$, when homotopy is built into the nerve construction. For now, we exhibit the relation between certain strict filling conditions and coskeletality.

PROPOSITION 3.2.5. *Let X be a dendroidal set and $m \geq 2$ an integer. If X satisfies the strict inner Kan condition for all trees T of degree at least m, then X is m-coskeletal.*

PROOF. Let Y be an arbitrary dendroidal set and assume that a map $f : Sk_m(Y) \to Sk_m(X)$ is given. We have to show that f extends uniquely to a map $\hat{f} : Y \to X$. Suppose f were extended to a map $f_k : Sk_k(Y) \to Sk_k(X)$ for $k \geq m$. Let $y \in Sk_{k+1}(Y)$ be a non-degenerate dendrex and assume $y \notin Sk_k(Y)$. So $y \in Y_T$ and T has exactly $k+1$ vertices. Choose an inner horn $\Lambda^e[T]$ (such an inner horn exists since $k \geq 2$). The collection $\{\beta^* y\}_{\beta \neq \partial_e}$ where $\beta : S \to T$ runs over all faces of T, defines a horn $\Lambda^e[T] \to Y$. Since this horn factors through the k-skeleton of Y, we obtain by applying f_k, a horn $\Lambda^e[T] \to X$ in X given by $\{f_k(\beta^* y)\}_{\beta \neq \partial_e}$. Let $f_{k+1}(y) \in X_T$ be the unique filler of that horn. By construction we have that for each $\beta \neq \partial_e$

$$\beta^* f_{k+1}(y) = f_k(\beta^* y).$$

It thus remains to show the same for ∂_e. The dendrices $f_k(\partial_e^* y)$ and $\partial_e^* f_{k+1}(y)$ both have the same boundary and they are both of shape S where S has k vertices. Since $k \geq 2$, S has an inner face, but then it follows that both $f(\partial_e^* y)$ and $\partial_e^* f_{k+1}(y)$ are fillers for the same inner horn in X and they are thus equal. By repeating the process for all non-degenerate dendrices in $Sk_{k+1}(Y)$ it follows that f_k can be extended to $f_{k+1} : Sk_{k+1}(Y) \to Sk_{k+1}(X)$. This holds for all $k \geq m$ which implies that f can be extended to $\hat{f} : Y \to X$.

To show the uniqueness of $\hat{f}$ assume that g is another extension of f. Suppose it has been shown that $\hat{f}$ and g agree on all dendrices of shape T where T has at most $k \geq m$ vertices, and let $y \in X_S$ be a dendrex of shape S where S has $k+1$ vertices. But then $\hat{f}(y)$ and $g(y)$ are both dendrices in X that have the same boundary. Since $k \geq 2$ it follows that these dendrices are both fillers for the same inner horn and so are equal. This proves that $\hat{f} = g$. □

Corollary 3.2.6. *Let $\mathcal{P}$ be an operad. Since the dendroidal set $N_d(\mathcal{P})$ is a strict inner Kan complex it follows that it is 2-coskeletal.*

Proposition 3.2.7. *Let X be a dendroidal set and $k \geq 0$ an integer. If X is k-coskeletal then X satisfies the strict inner Kan condition for all trees T with $|T| \geq k+2$.*

Proof. Let T be a tree with $|T| \geq k+2$ and e an inner face of T. Consider the inner horn extension problem

$$\begin{array}{ccc} \Lambda^e[T] & \longrightarrow & X \\ \downarrow & \nearrow & \\ \Omega[T] & & \end{array}$$

in X. Since X is k-coskeletal, this problem is equivalent to finding the dotted arrow in the diagram

$$\begin{array}{ccc} Sk_k(\Lambda^e[T]) & \longrightarrow & Sk_k(X) \\ \downarrow & \nearrow & \\ Sk_k(\Omega[T]) & & \end{array}$$

The result will follow if we can show that the inclusion $Sk_k(\Lambda^e[T]) \to Sk_k(\Omega[T])$ is an isomorphism. Let $s \in Sk_k(\Omega[T])_S$ be a non-degenerate dendrex of shape S, if we can show that $s \in Sk_k(\Lambda^e[T])$ then we are done. We have that s is a map $s : \Omega[S] \to \Omega[T]$ and S has at most k vertices. Since T has at least $k+2$ vertices it follows that s factors through a sub-face of $\Omega[T]$ of codimension 2, say $\Omega[R] \to \Omega[R'] \to \Omega[T]$ and $\Omega[R'] \to \Omega[T]$ can be chosen to be different from ∂_e (Proposition 2.4.8). Thus s factors through the face $\Omega[R'] \to \Omega[T]$ and thus also through $\Lambda^e[T]$, which means that $s \in \Lambda^e[T]$, as needed. □

3.3. Anodyne extensions

In the theory of simplicial sets [**16**], anodyne extensions are a technical tool that simplifies proofs significantly. We now develop the equivalent notion for dendroidal sets and provide a simple example that shows how anodyne extensions are typically used.

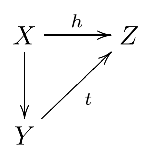
Definition 3.3.1. A class M of monomorphisms in *dSet* is called *saturated* if the following conditions are satisfied:

(1) All isomorphisms are in M.
(2) M is closed under pushouts.
(3) M is closed under retracts.
(4) M is closed under arbitrary sums.
(5) M is closed under countable unions.

See [**19**] for a similar definition for simplicial sets, and a detailed explanation of the closedness properties. Given an arbitrary class of monomorphism B, the saturated class generated by B is simply the intersection of all saturated classes containing B.

Definition 3.3.2. Let B be the class of all inner horn inclusions in *dSet*. The class of *anodyne extensions* is the saturated class generated by B.

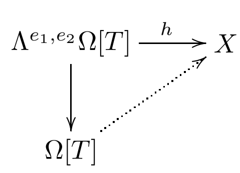
It is easy to show that given any anodyne extension $X \to Y$ and a map $X \to Z$, where Z is an inner Kan complex, there exists an extension $Y \to Z$:



It is precisely this property that makes anodyne extensions useful. Consider the following situation for example. Let e_1 and e_2 be two inner edges in a tree T and let

$$\Lambda^{e_1,e_2}\Omega[T] = \bigcup_{\partial_{e_1},\partial_{e_2}\neq\alpha\in\Phi_1(T)} \partial_\alpha\Omega[T]$$

be the dendroidal sub-set of $\Omega[T]$ which is the union of all of the faces of $\Omega[T]$ except the two inner ones corresponding to e_1 and e_2. Assume one is given the following extension problem:



with the vertical map being the inclusion. If X is an inner Kan complex it is expected that the dotted arrow would exist. This would follow if we can show that the inclusion $\Lambda^{e_1,e_2}\Omega[T] \to \Omega[T]$ is anodyne, as is indeed the case. More generally, for a sub-set $A \subseteq E(T)$ of inner edges let $\Lambda^A[T]$ be the union of all faces of $\Omega[T]$ except those arising by contracting an edge from A, which we denote by

$$\Lambda^A\Omega[T] = \bigcup_{\alpha\in\Phi(T)\setminus A} \partial_\alpha\Omega[T]$$

where we (somewhat loosely) write A also for the set $\{\partial_e \mid e \in A\}$.

Proposition 3.3.3. *For any non-empty $A \subseteq E(T)$ of inner edges in a tree T, the inclusion $\Lambda^A[T] \to \Omega[T]$ is anodyne.*

Proof. By induction on $k = |A|$. If $k = 1$ then the inclusion $\Lambda^A[T] \to \Omega[T]$ is just an inner horn inclusion, thus anodyne. Assume the proposition holds for $1 \leq n < k$ and suppose $|A| = k$. Choose an arbitrary $e \in A$ and put $B = A\backslash\{e\}$. The map $\Lambda^A[T] \to \Omega[T]$ factors as

$$\begin{array}{ccc} \Lambda^A[T] & \longrightarrow & \Lambda^B[T] \\ & \searrow & \downarrow \\ & & \Omega[T] \end{array}$$

The vertical map is anodyne by the induction hypothesis and it therefor suffices to prove that $\Lambda^A[T] \to \Lambda^B[T]$ is anodyne. The following diagram expresses that map as a pushout

$$\begin{array}{ccc} \Lambda^B[T/e] & \longrightarrow & \Lambda^A[T] \\ \downarrow & & \downarrow \\ \Omega[T/e] & \longrightarrow & \Lambda^B[T] \end{array}$$

and since by the induction hypothesis, the map $\Lambda^B[T/e] \to \Omega[T/e]$ is anodyne, the proof is complete. $\square$

3.4. Grafting in an inner Kan complex

We now consider how dendrices in an inner Kan complex can be grafted. Recall that for two trees T and S with $E(T) \cap E(S) = \{l\}$, where l is a leaf of T which is also the root of S, we have the tree $T \circ_l S$ obtained by grafting S onto T along l. Both S and T embed naturally as sub-faces in $T \circ_l S$, which we denote by $S : S \to T \circ_l S$ and $T : T \to T \circ_l S$. These then induce the obvious inclusions $\Omega[S] \to \Omega[T \circ_l S]$ and $\Omega[T] \to \Omega[T \circ_l S]$ and the union of their images in $\Omega[T \circ_l S]$ we denote by $\Omega[T] \cup_l \Omega[S]$.

Lemma 3.4.1. *For any two trees T and S and any leaf l as above, the inclusion $\Omega[T] \cup_l \Omega[S] \to \Omega[T \circ_l S]$ is anodyne.*

Proof. Let us write $R = T \circ_l S$. The case where $T = \eta$ or $S = \eta$ is trivial, we therefore assume that this is not the case. We proceed by induction on $n = |T|+|S|$, the sum of the degrees of T and S. The cases $n = 0$ or $n = 1$ are taken care of by our assumption that $T \neq \eta \neq S$. For the case $n = 2$ the same assumption implies that the inclusion $\Omega[T] \cup_l \Omega[S] \to \Omega[R]$ is an inner horn inclusion and is thus anodyne. Assume then that the result holds for $2 \leq n < k$ and suppose $|T| + |S| = k$.

Let I be the set of all inner edges of R and $\Lambda^I[R]$ the union of all the outer faces of $\Omega[R]$. First notice that $\Omega[T] \cup_l \Omega[S] \to \Omega[R]$ factors as

$$\begin{array}{ccc} \Omega[T] \cup_l \Omega[S] & \longrightarrow & \Lambda^I[R] \\ & \searrow & \downarrow \\ & & \Omega[R] \end{array}$$

and the vertical arrow is anodyne by Proposition 3.3.3. If we can now show that the map

$$\Omega[T] \cup_l \Omega[S] \to \Lambda^I[R]$$

is anodyne then we are done. We do this by exhibiting it as a pushout of an anodyne extension. Recall (Section 2.2.1) that an outer cluster is a vertex v with the property that one of the edges adjacent to it is inner while all the other edges adjacent to it are outer. Let $Cl(T)$ (respectively $Cl(S)$) be the set of all outer clusters in T (respectively S) which do not contain l (respectively the root of S). For each $C \in Cl(T)$ the face of $\Omega[R]$ corresponding to C is isomorphic to $\Omega[(T/C) \circ_l S]$ and the map $\Omega[T/C] \cup_l \Omega[S] \to \Omega[(T/C) \circ_l S]$ is anodyne by the induction hypothesis. Similarly for every $C \in Cl(S)$ the face of $\Omega[R]$ that corresponds to C is isomorphic to $\Omega[T \circ_l (S/C)]$ and the map $\Omega[T] \cup_l \Omega[S/C] \to \Omega[T \circ_l (S/C)]$ is anodyne by the induction hypothesis. The following diagram is a pushout

$$\begin{array}{ccc} \coprod_{C\in Cl(T)}(\Omega[T/C] \cup_l \Omega[S]) \amalg \coprod_{C\in Cl(S)}(\Omega[T] \cup_l \Omega[S/C]) & \longrightarrow & \Omega[T] \cup_l \Omega[S] \\ \downarrow & & \downarrow \\ \coprod_{C\in Cl(T)}(\Omega[(T/C) \circ_l S]) \amalg \coprod_{C\in Cl(S)}(\Omega[T \circ_l (S/C)]) & \longrightarrow & \Lambda^I[R] \end{array}$$

where the map on the left is the coproduct of all of the anodyne extensions just mentioned. Since anodyne extensions are closed under coproducts, it follows that the map on the left of the pushout is anodyne and thus also the one on the right, which is what we set out to prove. This concludes the proof. □

COROLLARY 3.4.2. *Let X be an inner Kan complex, S and T two trees, and l a leaf of T which is also the root of S such that $T \circ_l S$ is defined. Suppose that $s \in X_S$ and $t \in X_t$ are two dendrices such that $l^*(t) = l^*(s)$ where l denotes both of the obvious maps $\eta \to T$ and $\eta \to S$. It then follows that there is a dendrex $r \in X_{T\circ_l S}$ with the property that $S^*(r) = s$ and $T^*(r) = t$.*

PROOF. The two dendrices s and t induce a map $\Omega[T] \cup_l \Omega[S] \to X$. Since $\Omega[T] \cup_l \Omega[S] \to \Omega[T \circ_l S]$ is anodyne it follows that there is an extension $\Omega[T \circ_l S] \to X$. This extension is precisely the required dendrex. □

Consider the special case where both T and S are corollas. The corollary can then be interpreted as saying that suitable dendrices $t \in X_T$ and $s \in X_S$ in an inner Kan complex can be 'composed' along an input. We make this precise in the following definition.

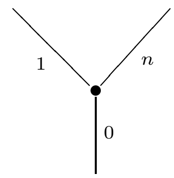
DEFINITION 3.4.3. Let X be a dendroidal set, T and S two corollas (not necessarily with the same number of leaves), and x a leaf of T which is also the root of S such that $T \circ_x S$ is defined. Given two dendrices $t \in X_T$ and $s \in X_S$ we say that they *match along x if*

$$x^*t = x^*s$$

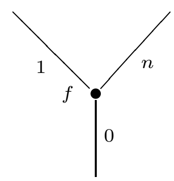
(where x denotes both induced maps $\eta \to S$ and $\eta \to T$). Any dendrex $r \in X_{T\circ_x S}$ with the property that $T^*r = t$ and $S^*r = s$ is called a *composition* of the dendrex s on t along x. We denote this situation by $r \sim t \circ_i s$.

REMARK 3.4.4. Notice that usually there need not be a unique dendrex r for which $r \sim t \circ_x s$ and that consequently we cannot talk about *the* composition of two matching dendrices but only about *a* composition of such dendrices.

It is convenient to introduce the following conventions. For each $n \geq 0$ let C_n be the n-corolla:

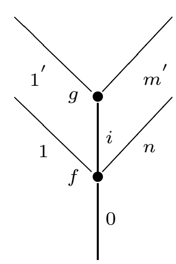


and for each $0 \leq i \leq n$ recall that $i : \eta \to C_n$ denotes the obvious (outer face) map in Ω that sends the unique edge of η to the edge i in C_n. We include here the case C_0, a tree with no leaves and just one vertex. An element $f \in X_{C_n}$ will be denoted by

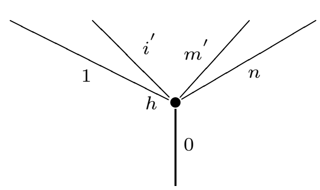


If $C_n^{'}$ is another n-corolla together with an isomorphism $\alpha : C_n^{'} \to C_n$ then we will usually write f again instead of $\alpha^*(f)$. We will use this convention quite often in the sequel, where in each case there will be an obvious choice for the isomorphism α given by the planar representation of the trees at question, which will usually be taken for granted. Given dendrices $f \in X_{C_n}$ and $g \in X_{C_m}$, the definition above does not permit us to consider composing one with the other. To remedy this we proceed as follows.

DEFINITION 3.4.5. Let X be a dendroidal set and let $f \in X_{C_n}$ and $g \in X_{C_m}$ be two dendrices in X. We will say that a dendrex $h \in X_{C_{n+m-1}}$ is a $\circ_i$-composition of f and g if there is a dendrex γ in X as follows (we use the convention just mentioned):



with inner face



We will denote this situation by $h \sim f \circ_i g$ and call γ a *witness* for the composition.

REMARK 3.4.6. The notion of composition of dendrices in a dendroidal set is now somewhat ambiguous. However, context will always make it clear which one is meant. Notice that if $h \sim f \circ_i g$ in the second definition then the dendrex i^*f is not equal to i^*g, but it does follow that i^*f is isomorphic to i^*g. We refer to this situation also by saying that f and g match along i, relying again on context to prevent confusion.

PROPOSITION 3.4.7. *In an inner Kan complex every two matching dendrices have at least one composition (using any of the two definitions).*

PROOF. This is a special case of Corollary 3.4.2. □

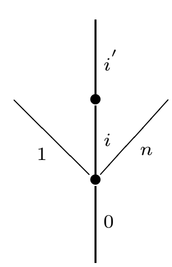
It is straightforward to check that given an operad $\mathcal{P}$, the notion of composition of dendrices in $N_d(\mathcal{P})$ corresponds exactly to the $\circ_i$-composition of arrows in $\mathcal{P}$. So that we see that composition in a dendroidal set is a generalization of composition in an operad.

3.5. Homotopy in an inner Kan complex

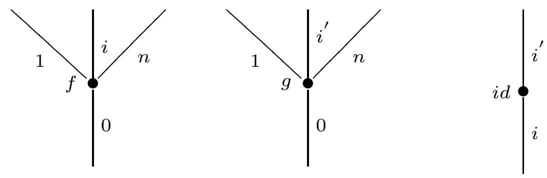
In this section we study a notion of homotopy inherent to a dendroidal set. Basically two dendrices are said to be homotopic if one is a composition of the other with a degenerate dendrex. This notion makes the most sense for dendrices shaped like corollas and indeed we study the homotopy of just such dendrices. We show that this homotopy theory within a dendroidal set is particularly well behaved if the dendroidal set is an inner Kan complex. In that case we show that the obtained homotopy relation is an equivalence relation and we show that it is a congruence for the composition of dendrices as defined in the previous section. From this it follows that with each inner Kan complex one can associate an operad which we call the homotopy operad associated with the inner Kan complex. Using this and other results obtained earlier we prove that a dendroidal set is a strict inner Kan complex if, and only if, it is the nerve of an operad. The ideas presented here generalize similar ideas presented in [**7**].

Using the convention from the end of the previous section we embark with the definition of homotopy.

DEFINITION 3.5.1. Let X be a dendroidal set and let $f, g \in X_{C_n}$. For $1 \leq i \leq n$ we say that *f is homotopic to g along the edge i,* and write $f \sim_i g$, if $g \sim f \circ_i id$ where by id we mean a degeneracy. In more detail, $f \sim_i g$ if there is a dendrex H of shape

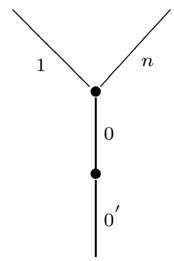


whose three faces are:

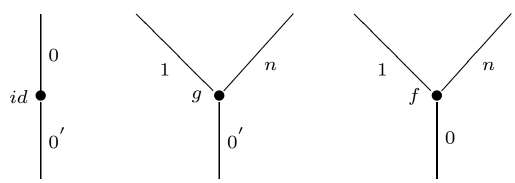


where id in the last tree is a degeneracy of i.

Similarly we will say that f is homotopic to g along the edge 0 and write $f \sim_0 g$ if $g \sim id \circ_0 f$, that is if there is a dendrex of shape



whose three faces are:

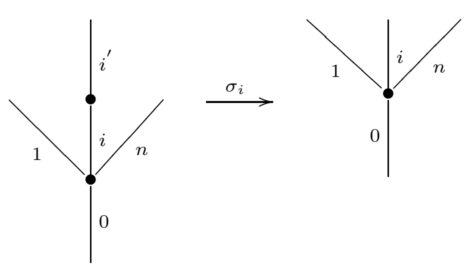


When $f \sim_i g$ for some $0 \leq i \leq n$ we will refer to the corresponding H as a *homotopy* from f to g along i and will sometimes write $H : f \sim_i g$.

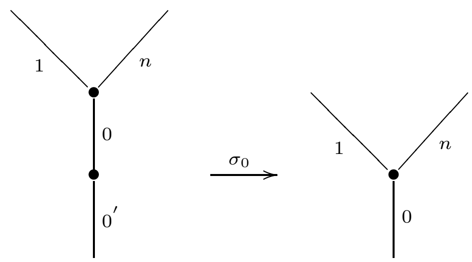
Remark 3.5.2. Notice that in a strict inner Kan complex X the homotopy relation just defined is the identity relation.

Proposition 3.5.3. *Let X be an inner Kan complex. For each $0 \leq i \leq n$ the relation $\sim_i$ on the set X_{C_n} is an equivalence relation.*

Proof. First we prove reflexivity. For $1 \leq i \leq n$ let

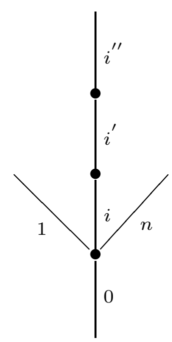


and for $i = 0$ let

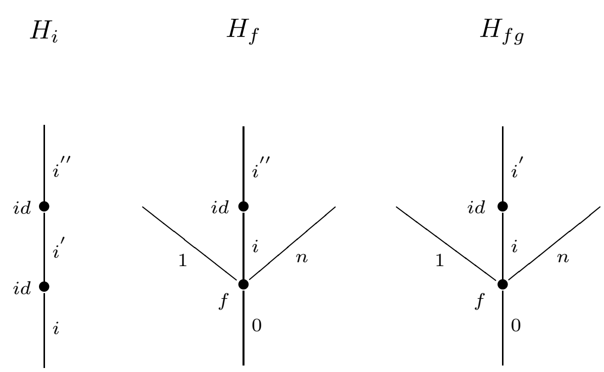


be the obvious degeneracies. It then follows that for any $f \in X_{C_n}$ the dendrex $\sigma_i^*(f)$ is a homotopy from f to f, thus $f \sim_i f$.

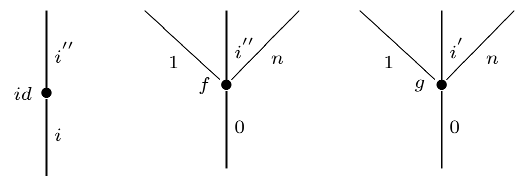
To prove symmetry assume $f \sim_i g$ for some $1 \leq i \leq n$ and let H_{fg} be a homotopy from f to g along i. Consider the tree T:



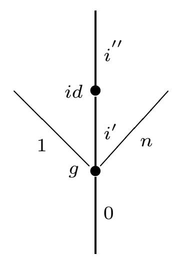
We now describe an inner horn $\Lambda^i[T] \to X$. Such a map is given by specifying three dendrices in X of certain shapes such that their faces match in a suitable way. We describe this map by explicitly writing the mentioned dendrices and their faces:



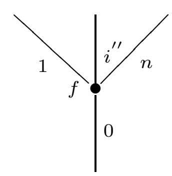
with inner faces of these dendrices:



where H_i is a double degeneracy of i, H_f is a homotopy from f to f (along the branch i) and H_{fg} is the given homotopy from f to g. It is easily checked that the faces indeed match so that we have a horn in X. Let x be a filler for that horn and consider $H_{gf} = \partial_i^*(x)$. This dendrex can be pictured as

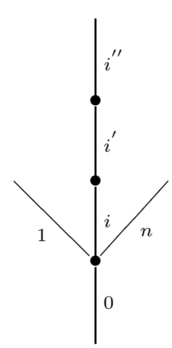


with inner face:



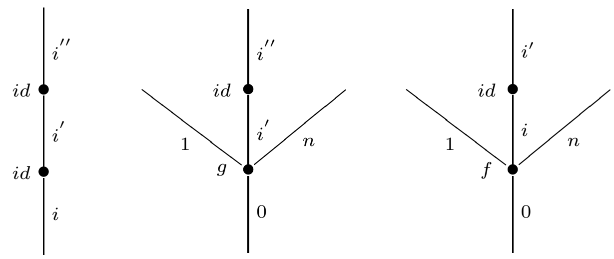
and is thus a homotopy from g to f along i, so that $g \sim_i f$. For $i = 0$ a similar proof works.

To prove transitivity let $f \sim_i g$ and $g \sim_i h$ for $1 \leq i \leq n$. Let H_{fg} be a homotopy from f to g and let H_{gh} be a homotopy from g to h. We again consider the tree T:

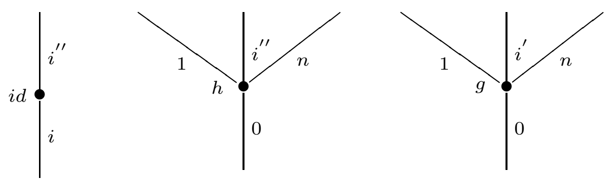


The following is a horn $\Lambda^{i'}[T] \to X$ in X:

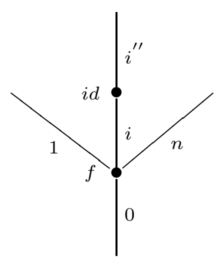
H_i $\quad$ H_{gh} $\quad$ H_{fg}



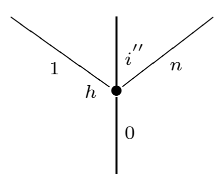
with inner faces being:



Let x be a filler for that horn and let $H_{fh} = \partial^*_{i'}(x)$, this dendrex can be pictured as follows:



with inner face:

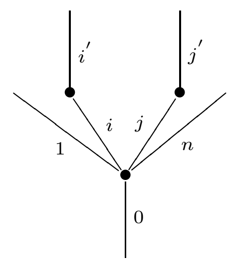


and is thus a homotopy from f to h so that $f \sim_i h$. The proof for $i = 0$ is similar. $\square$

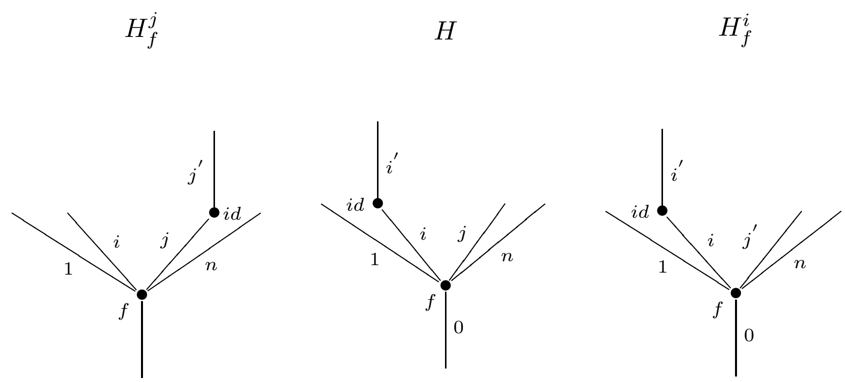
Lemma 3.5.4. *Let X be an inner Kan complex. The relations $\sim_0, \cdots, \sim_n$ on X_{C_n} are all equal.*

Remark 3.5.5. On the basis of this lemma, we will later just write $f \sim g$ instead of $f \sim_i g$.

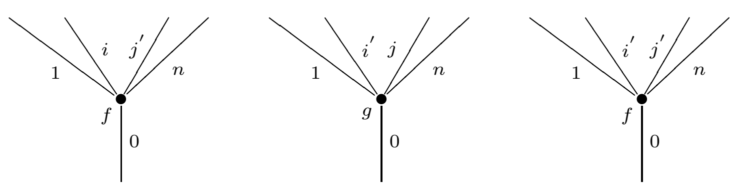
Proof. Suppose $H : f \sim_i g$ for $1 \leq i \leq n$ and let $1 \leq i < j \leq n$. We consider the tree T:



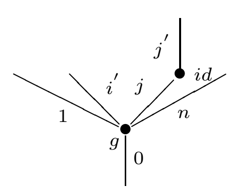
and the following inner horn $\Lambda^i[T] \to X$:



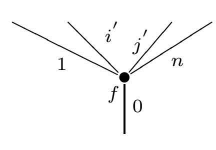
where $H_f^j : f \sim_j f$ and $H_f^i : f \sim_i f$. The inner faces of the three dendrices are



Let x be a filler for this horn, then $\partial_i^*(x)$ is the following dendrex



with inner face:

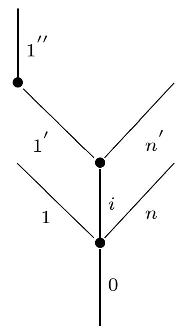


and is thus a homotopy from g to f along the j-th branch. Thus $g \sim_j f$ and so $f \sim_j g$ as well. The other cases to be considered follow in a similar way. □

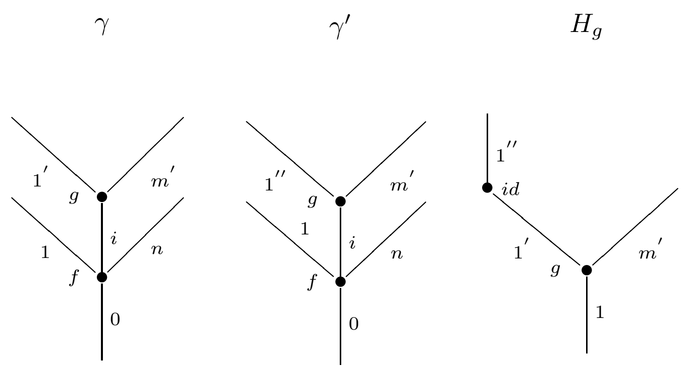
We now turn to prove that the homotopy equivalence relation behaves well with respect to the composition of dendrices.

LEMMA 3.5.6. *In an inner Kan complex X, if $h \sim f \circ_i g$ and $h' \sim f \circ_i g$ then $h \sim h'$.*

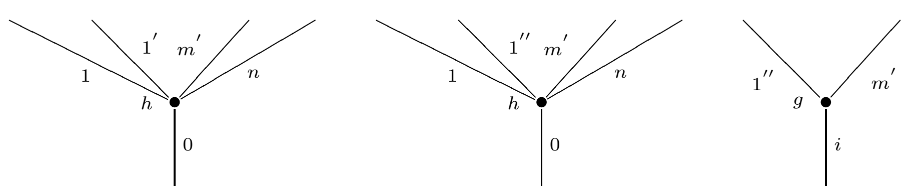
PROOF. Let γ be a witness for the composition $h \sim f \circ_i g$ and γ' one for the composition $h' \sim f \circ_i g$. We consider the tree T:



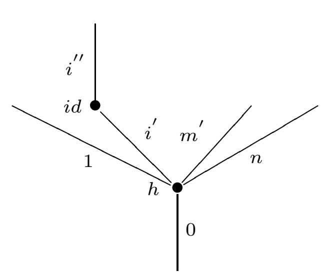
Let $H_g : g \sim_i g$ and consider the following horn $\Lambda^i[T] \to X$



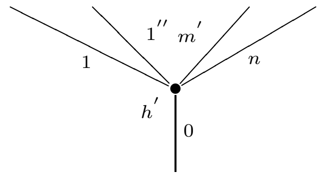
with inner faces



Let x be a filler for this horn. The face $\partial_i^*(x)$ is then the dendrex



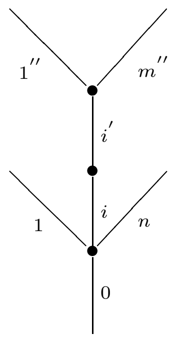
whose inner face is



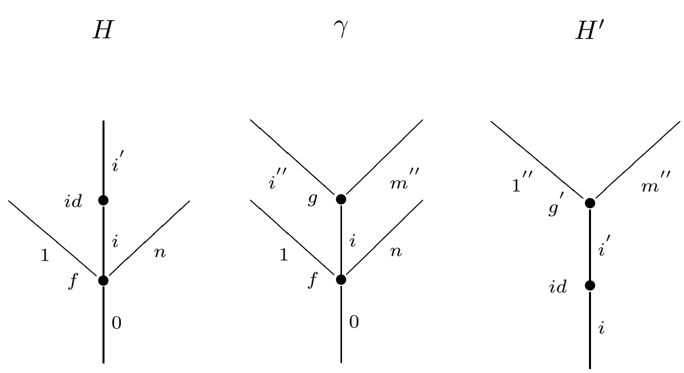
which proves that $h \sim h'$. □

LEMMA 3.5.7. *In an inner Kan complex X, let $f \sim f'$ and $g \sim g'$. If $h \sim f \circ_i g$ and $h' \sim f' \circ_i g'$ then $h \sim h'$.*

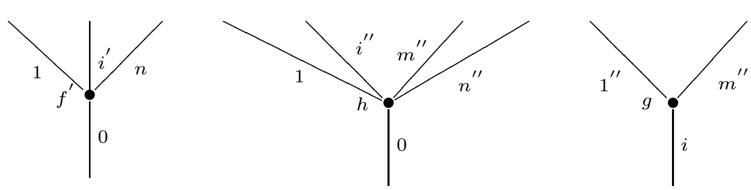
PROOF. Let H be a homotopy from f to f' along the edge i, H' a homotopy from from g' to g along the root, and γ a witness for the composition $h \sim f \circ_i g$. We now consider the tree T:



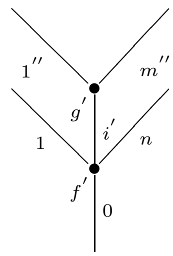
and the inner horn $\Lambda^i[T] \to X$ in X:



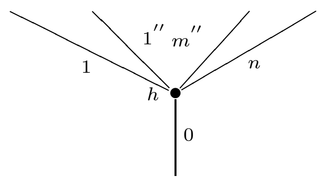
with inner faces:



The missing face of a filler for this horn is then:



with inner face



which proves that $h \sim f' \circ_i g'$, and thus by the previous result also that $h \sim h'$. □

We are now in a position to define the homotopy operad associated with an inner Kan complex. Given an inner Kan complex X and $x_0, \cdots, x_n \in X_\eta$ let $X(x_1, \cdots, x_n; x_0)$ be the set of all dendrices $f \in X_{C_n}$ such that $i^* f = x_i$ for $0 \leq i \leq n$. We now define a collection $Ho(X)$ as follows. The set of objects of $Ho(X)$ is the set X_η. Given objects $x_0, \cdots, x_n$ we put

$$Ho(X)(x_1, \cdots, x_n; x_0) = X(x_1, \cdots, x_n; x_0)/\sim$$

where $\sim$ is the homotopy relation defined above.

THEOREM 3.5.8. *Let X be an inner Kan complex. The composition of dendrices makes the collection $Ho(X)$ into an operad.*

PROOF. Lemma 3.5.7 implies that for $[f] \in Ho(X)(x_1, \cdots, x_n; x)$ and $[g] \in Ho(X)(y_1, \cdots, y_m; x_i)$ the assignment

$$[f] \circ_i [g] = [f \circ_i g]$$

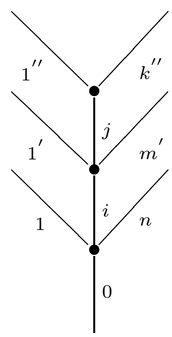
is well defined. This provides the $\circ_i$-compositions of the operad $Ho(X)$. The Σ_n-actions are defined as follows. Given a permutation $\sigma \in \Sigma_n$ let $\sigma : C_n \to C_n$ be the obvious induced map in Ω. The map $\sigma^* : X_{C_n} \to X_{C_n}$ restricts to a function

$$\sigma^* : X(x_1, \cdots, x_n; x) \to X(x_{\sigma(1)}, \cdots, x_{\sigma(n)}; x)$$

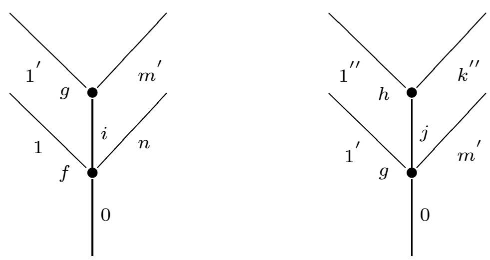
and it is trivial to verify that this map respects the homotopy relation. We thus obtain a map

$$\sigma^* : Ho(X)(x_1, \cdots, x_n; x) \to Ho(X)(x_{\sigma(1)}, \cdots, x_{\sigma(n)}; x).$$

We now need to show that these structure maps make the collection $Ho(X)$ into an operad. The verification is simple and we exemplify it by proving the associativity of the $\circ_i$-compositions. Let $[f] \in Ho(X)(x_1, \cdots, x_n; x)$, $[g] \in Ho(X)(y_1, \cdots, y_m; x_i)$ and $[h] \in Ho(X)(z_1, \cdots, z_k; y_m)$. We need to prove that $[f]\circ([g]\circ[h]) = ([f]\circ[g])\circ[h]$ (for simplicity we are neglecting to mention the input along which the compositions are being performed) which is the same as showing that $f \circ (g \circ h) \sim (f \circ g) \circ h$ for any choice of compositions $\psi \sim g \circ h$ and $\varphi \sim f \circ g$. Consider the tree T given by



and consider the anodyne extension $\Lambda^{\{i,j\}}[T] \to \Omega[T]$ (see Proposition 3.3.3). The two given compositions $\psi \sim g \circ h$ and $\varphi \sim f \circ g$ define a map $\Lambda^{\{i,j\}}[T] \to X$ depicted by



whose inner faces are respectively ψ and φ. Let $x \in X_T$ be a dendrex extending this map and $c : C_m \to T$ be the map obtained by contracting both i and j, and put $\rho = c^*x$. It now follows that $\partial_i^*(x)$ is a witness for the the composition $\rho \sim \psi \circ h$ and $\partial_j^*(x)$ is a witness for the composition $\rho \sim f \circ \varphi$, which proves the needed associativity. The other axioms for an operad follow in a similar manner. □

Definition 3.5.9. Given an inner Kan complex X the operad $Ho(X)$ as above is called the *homotopy operad* associated with X.

Remark 3.5.10. In **[7]** the authors construct a homotopy category $Ho(X)$ from an inner Kan simplicial set X. Our construction is a generalization of that one in the sense that for an inner Kan simplicial set X

$$Ho(i_!(X)) \cong j_! Ho(X).$$

The proof is trivial by inspection of these constructions.

We can now relate the homotopy operad of a dendroidal set with its operadic realization.

Proposition 3.5.11. *For any inner Kan complex X, $Ho(X)$ is isomorphic to $\tau_d(X)$.*

Proof. We prove that $Ho(-)$ has the required universal property, that is that for an inner Kan complex X and an operad $\mathcal{P}$ there is a natural bijection between operad maps $Ho(X) \to \mathcal{P}$ and dendroidal maps $X \to N_d(\mathcal{P})$. Let $F : Ho(X) \to \mathcal{P}$ be a map of operads. Since $N_d(\mathcal{P})$ is 2-coskeletal we only need to construct a map

$Sk_2(X) \to N_d(\mathcal{P})$. Since $ob(Ho(X)) = X_\eta$ and $ob(\mathcal{P}) \cong N_d(\mathcal{P})_\eta$, the map F clearly induces a function (the object part function of F) $G_0 : Sk_0(X) \to N_d(\mathcal{P})$. To extend this to a map $G_1 : Sk_1(X) \to N_d(\mathcal{P})$ let $x \in X_{C_n}$. The equivalence class $[x]$ is an operation in $Ho(X)$, and thus $F([x])$ is an operation in $\mathcal{P}$ which clearly defines a dendrex in $N_d(\mathcal{P})$ which we denote by $G_1(x)$. This assignment clearly extends G_0 so that we obtain a map $Sk_1(X) \to N_d(\mathcal{P})$. We now extend this map to a map $G_2 : Sk_2(X) \to N_d(\mathcal{P})$. Let $\gamma \in X_T$ where T is of degree 2. The dendrex γ is a witness for a composition in X of two dendrices, say $h \sim f \circ_i g$, so that in $Ho(X)$ we have that $[h] = [f] \circ_i [g]$. Since F is a map of operads the composition is respected so that γ defines a unique dendrex in $N_d(\mathcal{P})_T$ which we denote by $G_2(\gamma)$. Again, it is easily seen that G_2 extends G_1 so that we obtain a map $G_2 : Sk_2(X) \to N_d(\mathcal{P})$.

Consider now a given map $G : X \to N_d(\mathcal{P})$, that is a map $G_2 : sk_2(X) \to N_d(\mathcal{P})$. We now construct a map $Ho(X) \to \mathcal{P}$. Again we clearly have an obvious function $ob(Ho(X)) \to ob(\mathcal{P})$. Let f now be an operation in $Ho(X)$, that is $f = [f']$ for some dendrex $f' \in X_{C_n}$. Since $N_d(\mathcal{P})_{C_n}$ consists precisely of the operations in $\mathcal{P}$ of arity n, we have that $G(f')$ is such an operation. Since any dendroidal map preserves homotopic dendrices and since the homotopy relation in $N_d(\mathcal{P})$ is the identity we obtain that if $f'' \sim f'$ then $G(f'') = G(f')$. We can thus define $F(f) = G([f])$. It is easy to verify that F is actually a map of operads. Furthermore the two constructions just described are natural and are each other inverses which establishes the required bijection and thus finishes the proof. □

We can now prove the characterization of inner Kan complexes as those dendroidal sets that arise as nerves of operads.

THEOREM 3.5.12. *Let X be a dendroidal set. X is a strict inner Kan complex if, and only if, X is the dendroidal nerve of an operad.*

PROOF. One direction was proved in Lemma 3.2.4. Assume then that X is an inner Kan complex. We shall prove that $X \cong N_d(Ho(X))$ by showing that the canonical map $X \to N_d(Ho(X))$ is an isomorphism. Since we already know $N_d(Ho(X))$ to be 2-coskeletal we can easily describe the map $X \to N_d(Ho(X))$ simply by stating its value for dendrices shaped like trees with 2 or fewer vertices. The objects of $Ho(X)$ are X_η and $f : X_\eta \to N_d(Ho(X))_\eta$ is the identity. When X has unique fillers the homotopy relation is the identity and thus for any corolla we have $N_d(Ho(X))_{C_n} = X_{C_n}$ and again $f : X_{C_n} \to N_d(Ho(X))_{C_n}$ is the identity. Notice that any tree T with two vertices defines a composition of two operations in $Ho(X)$, which implies that $f : X_T \to N_d(Ho(X))_T$ is again the identity. Coskeletality implies now that f is the identity. □

REMARK 3.5.13. This theorem specializes to provide a proof that the strict inner Kan simplicial complexes are precisely the nerves of categories. This result is stated, for example, in **[23]** without proof.

3.6. The exponential property

In this section we are going to prove that the sub-category of $dSet$ spanned by the inner Kan complexes is an exponential ideal with respect to normal dendroidal sets. That means that given an inner Kan complex K and a normal dendroidal set X, the dendroidal set $\underline{dSet}(X, K)$ is an inner Kan complex.

We first reduce the problem to proving that a certain map is an anodyne extension. Given trees S and T, and an inner horn $\Lambda^e[S] \to \Omega[S]$, we may consider the dendroidal sets $\Lambda^e[S] \otimes \Omega[T]$ and $\Omega[S] \otimes \partial\Omega[T]$ as dendroidal sub-sets of $\Omega[S] \otimes \Omega[T]$. As such, their union is a well defined dendroidal sub-set:

$$\Lambda^e[S] \otimes \Omega[T] \cup \Omega[S] \otimes \partial\Omega[T] \rightarrowtail \Omega[S] \otimes \Omega[T].$$

LEMMA 3.6.1. *Assume that for any trees S and T and any inner horn $\Lambda^e[S] \to \Omega[S]$, the inclusion*

$$\Lambda^e[S] \otimes \Omega[T] \cup \Omega[S] \otimes \partial\Omega[T] \rightarrowtail \Omega[S] \otimes \Omega[T]$$

is anodyne. It then follows that the inner Kan complexes form an exponential ideal in dSet with respect to normal dendroidal sets.

PROOF. We have to show that any map of dendroidal sets

$$\varphi : \Lambda^e[S] \otimes X \to K$$

extends to some map

$$\psi : \Omega[S] \otimes X \to K.$$

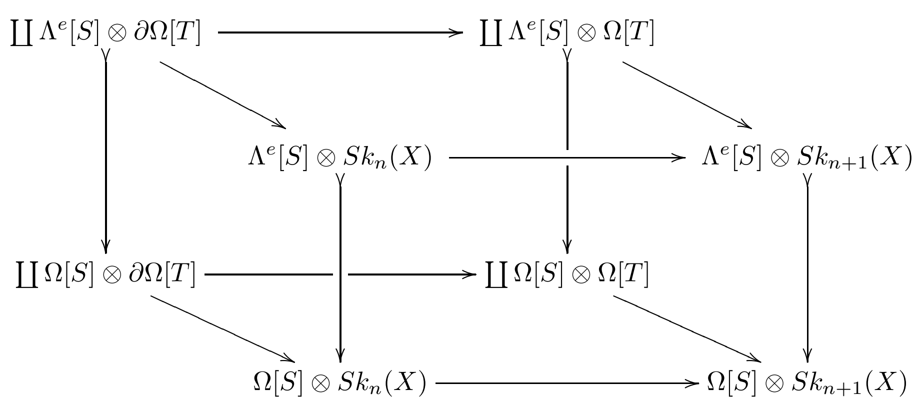
By writing X as the union of its skeleta,

$$X = \varinjlim Sk_n(X)$$

as in Section 2.6 and using the fact that X admits a normal skeletal filtration, we can build this extension ψ by induction on n. For $n = 0$, $Sk_0(X)$ is a sum of copies of $\Omega[\eta]$, the unit for the tensor product, so obviously the restriction $\varphi_0 : \Lambda^e[S] \otimes Sk_0(X) \to K$ extends to a map

$$\psi_0 : \Omega[S] \otimes Sk_0(X) \to K.$$

Suppose now that we have found an extension $\psi_n : \Omega[S] \otimes Sk_n(X) \to K$ of the restriction $\varphi_n : \Lambda^e[S] \otimes Sk_n(X) \to K$. Consider the following diagram:



In this diagram, the top and bottom faces are pushouts given by the normal skeletal filtration of X. Now inscribe the pushouts U and V in the back and front face, fitting into a square

$$\begin{array}{ccc} U & \longrightarrow & \coprod \Omega[S] \otimes \Omega[T] \\ \downarrow & & \downarrow \\ V & \longrightarrow & \Omega[S] \otimes Sk_{n+1}(X) \end{array}$$

The maps $\psi_n : \Omega[S] \otimes Sk_n(X) \to K$ and $\varphi_{n+1} : \Lambda^e[S] \otimes Sk_{n+1}(X) \to K$ together define a map $V \to K$. So, to find ψ_{n+1}, it suffices to prove that

$$V \rightarrowtail \Omega[S] \otimes Sk_{n+1}(X)$$

is anodyne. But, by a diagram chase argument, the square above is a pushout, so in fact, it suffices to prove that $U \rightarrowtail \coprod \Omega[S] \otimes \Omega[T]$ is anodyne. The latter map is a sum of copies of anodyne extensions as assumed. □

PROPOSITION 3.6.2. *The map*

$$\Lambda^e[S] \otimes \Omega[T] \cup \Omega[S] \otimes \partial\Omega[T] \rightarrowtail \Omega[S] \otimes \Omega[T]$$

is anodyne for any trees S and T and an inner edge e in S.

PROOF. The quite technical proof is given in **[38]** Proposition 9.2. □

These two results constitute thus the proof of the following theorem.

THEOREM 3.6.3. *The inner Kan dendroidal sets form an exponential ideal with respect to the normal dendroidal sets.*

COROLLARY 3.6.4. *The inner Kan simplicial sets form an exponential ideal in the category of simplicial sets.*

PROOF. Let K be an inner Kan simplicial set and X an arbitrary simplicial set. Clearly the dendroidal set $i_!(X)$ is normal. By Proposition 3.2.3 the dendroidal set $i_!(K)$ is an inner Kan complex, and thus we have that

$$\underline{dSet}(i_!(X), i_!(K))$$

is an inner Kan complex. By Lemma 3.1.7 this dendroidal set is isomorphic to $i_!(\underline{sSet}(X, K))$ and by Proposition 3.2.3 again it follows that $\underline{sSet}(X, K)$ is an inner Kan complex. □

REMARK 3.6.5. Since an inner Kan simplicial set is the same as a quasi-category, the above corollary states that quasi-categories form an exponential ideal in simplicial sets. This result was proved by Joyal in **[24]**, though the proof is quite different. The restriction of our proof to simplicial sets resembles more the one given in **[39]**.

3.7. Inner Kan complex generated by a dendroidal set

We end this chapter by introducing a straightforward way of turning an arbitrary dendroidal set into an inner Kan complex. This construction provides thus many examples of inner Kan complexes that may not be strict.

For a dendroidal set X let $Horn_n(X)$ be the set of all inner horns $\Lambda^e[T] \to X$ where $|T| = n$, that do not have a filler in X. So X is an inner Kan complex if, and only if, $Horn_n(X) = \phi$ for all $n \geq 2$.

Proposition 3.7.1. *Let X be a dendroidal set and $n \geq 2$. Consider the dendroidal sets*

$$H_n = \coprod_{h:\Lambda^e[T]\to X} \Lambda^e[T]$$

and

$$F_n = \coprod_{h:\Lambda^e[T]\to X} \Omega[T]$$

where h runs over the set $Horn_n(X)$. Let $H_n \to F_n$ be the obvious inclusion and let $H_n \to X$ be the obvious induced map. Denote by $J_n(X)$ the pushout

$$\begin{array}{ccc} H_n & \longrightarrow & X \\ \downarrow & & \downarrow \\ F_n & \longrightarrow & J_n(X) \end{array}$$

then $X \to J_n(X)$ is anodyne and for every horn $h \in Horn_n(X)$ the horn $\Lambda^e[T] \to X \to J_n(X)$ has a filler.

Proof. Since $H_n \to F_n$ is an anodyne extension (it is a coproduct of inner horn inclusions) so is $X \to J_n(X)$ an anodyne extension (as a pushout of one). It is now immediate that for a horn $h : \Lambda^e[T] \to X$ in $Horn_n(X)$, the dendrex $\Omega[T] \to F_n \to J_n(X)$, where $\Omega[T] \to F_n$ is the summand corresponding to h, is a filler for h. □

Notice that it is not necessarily true that $Horn_n(J_n(X)) = \phi$ since by filling the horns in $Horn_n(X)$ many new horns might have been created. This can easily be remedied as we now show.

Definition 3.7.2. Let X be a dendroidal set and $n \geq 2$. Define the sequence of dendroidal sets $\{X_k\}_{k=0}^{\infty}$ by $X_0 = X$ and $X_{k+1} = J_n(X_k)$. We thus have a sequence of anodyne extensions

$$X \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_k \longrightarrow \cdots$$

We denote the colimit (countable composition) of this sequence by $X \to K_n(X)$.

Proposition 3.7.3. *The map $X \to K_n(X)$ is anodyne and $Horn_n(K_n(X)) = \phi$.*

Proof. The map $X \to K_n(X)$ is an anodyne extension since it is a countable composition of anodyne extensions. To prove that $Horn_n(K_n(X)) = \phi$ let $h : \Lambda^e[T] \to K_n(X)$ with $|T| = n$. We need to show that this horn has a filler in $K_n(X)$. Since such a horn is given by a finite sequence of dendrices in $K_n(X)$ and

$K_n(X)$ is the union of an increasing sequence of dendroidal sets, it follows that the horn factors as

$$\begin{array}{ccc} \Lambda^e[T] & \xrightarrow{h} & X \\ \downarrow{\scriptstyle h'} & \nearrow & \\ X_k & & \end{array}$$

for some X_k as in the definition above. Now, the horn h' has a filler in X_{k+1} which then extends to a filler of h in $K_n(X)$. □

LEMMA 3.7.4. *Let X be a dendroidal set. There exists an inner Kan complex $K(X)$ together with an anodyne extension $X \to K(X)$. Furthermore, this construction is functorial.*

PROOF. Define the sequence $\{X_n\}_{n=1}^{\infty}$ by $X_1 = X$ and $X_{n+1} = K_{n+1}(X_n)$. We let $K(X)$ be the colimit of the induced sequence

$$X_1 \to X_2 \to \cdots$$

A similar argument to the one given in the previous proposition now shows that $Horn_n(K(X)) = \phi$ for all $n \geq 1$ and thus that $K(X)$ is an inner Kan complex. The obvious map $X \to K(X)$ is an anodyne extension as a countable composition of such. The functoriality can easily be established. □

REMARK 3.7.5. Of course one can also use a Quillen small object argument to obtain an inner Kan complex from an arbitrary dendroidal set.

CHAPTER 4

Enriched operads and dendroidal sets

This chapter explores how dendroidal sets and their theory fit in with the theory of $\mathcal{E}$-enriched operads. Enriched operads are used to study deformations of algebraic structures. The idea is that if certain algebraic structures are algebras for a given enriched operad $\mathcal{P}$, then by resolving that operad one obtains a new enriched operad whose algebras are $\mathcal{P}$-algebras *up-to-homotopy*. The cradle of such constructions is in topology, notably the W-construction of Boardman and Vogt **[7]** which is recalled below, May's work on loop spaces **[36]**, and Stasheff's work on H-spaces **[43]** (where, even though operads were not yet invented, the author essentially provided an example of an operad enriched in Top). The Boardman-Vogt W-construction can be generalized to monoidal model categories more general than the category of topological spaces, as long as these categories have a suitable notion of an interval. Berger and Moerdijk in a series of papers **[4, 5, 6]** establish a Quillen model structure on enriched operads with a fixed set of objects and give a detailed construction of the Boardman-Vogt W-construction in this more general setting, as well as proving that this Boardman-Vogt construction, when applied to an enriched operad $\mathcal{P}$, yields a cofibrant replacement for $\mathcal{P}$. Specifically for operads enriched in chain complexes, the cobar resolution is a well-known method to resolve operads (e.g.,**[18, 27, 35]**).

Below, the generalized W-construction is used in order to establish the connection between enriched operads and dendroidal sets. As motivation, A_∞-spaces are considered in order to illustrate the general problem of up-to-homotopy algebras of operads. Next, the original Boardman-Vogt W-construction is presented as well as its generalized form. That construction is then utilized to construct the homotopy coherent dendroidal nerve of a given enriched operad thus establishing the relation between enriched operads and dendroidal sets. Following is an extension of the Grothendieck construction from operads to dendroidal sets which is needed in order to apply the general theory to define categories enriched in a dendroidal set. The chapter ends by introducing weak n-categories and some basic properties of them.

4.1. Case study: A_∞-spaces

In this section we look at A_∞-spaces in order to exemplify the kind of problems occurring in the general theory of up-to-homotopy algebras. We do not intend this to be an accurate account of the theory of A_∞-spaces, but rather use it to illustrate a point. For more details and a very precise account of the evolution of ideas and notions, see the introduction in **[35]**. We thus allow ourselves to be somewhat less precise for the sake of greater clarity of the general presentation of the ideas involved.

Consider the following situation. Let X be a topological monoid, i.e., X is a topological space together with a continuous binary operation $\cdot : X \times X \to X$ which makes the set underlying X into a monoid (X, e). If we are now given a homeomorphism $f : X \to Y$ then one can (obviously) transfer the monoid structure from X to Y, in the sense that the function

$$Y \times Y \xrightarrow{f^{-1}\times f^{-1}} X \times X \xrightarrow{\cdot} X \xrightarrow{f} Y$$

is (of course) continuous and makes $(Y, f^{-1}(e))$ into a monoid. Assume now that we are given a weak equivalence $g : X \to Y$, can we still transfer the monoid structure from X to Y?

We can attempt to proceed as follows. Assume g has a homotopy inverse $h : Y \to X$ and choose homotopies

$$H_1 : hg \to id_X$$

and

$$H_2 : gh \to id_Y.$$

We can now define a binary operation on Y, namely

$$Y \times Y \xrightarrow{h\times h} X \times X \xrightarrow{\cdot} X \xrightarrow{g} Y.$$

Let us check whether this operation on Y is associative. Let then $a, b, c \in Y$. On the one hand

$$(ab)c = g(h(a)h(b))c = g(h(g(h(a) \cdot h(b))) \cdot h(c))$$

and on the other hand

$$a(bc) = ag(h(b)h(c)) = g(h(a) \cdot h(g(h(b) \cdot h(c))))$$

so that if $gh \neq id$ or $hg \neq id$ then, in general, equality will not hold. However, the homotopy $H_2 : gh \to id$ specifies for each $y \in Y$ a path $\gamma : [0,1] \to Y$ from $gh(y)$ to y. For $y = h(a)h(b)$ we thus obtain the path $\gamma' : [0,1] \to Y$ defined by

$$\gamma'(t) = g(\gamma(t) \cdot h(c))$$

which is thus a path from $(ab)c$ to $g(h(a) \cdot h(b) \cdot h(c))$. Similarly we can obtain another path from $a(bc)$ to $g(h(a) \cdot h(b) \cdot h(c))$ which together with the first path implies the existence of a path from $a(bc)$ to $(ab)c$. We see thus that the operation need not be associative but it is associative up to homotopy, in the sense that the paths just constructed fit together to form a homotopy between the two functions from $Y \times Y \times Y$ to Y obtained from the binary operation. This observation begins to unfold the kind of structure that can be induced on Y, given a monoid structure on X *and* a chosen homotopy inverse h of g *and* the chosen homotopies realising that homotopy inverse. To fully describe this structure on Y one needs to also consider the various ways to use the binary operation to form functions from $Y \times Y \times Y \times Y$ into Y. These functions can be related to each-other using the given homotopies and the newly created associativity homotopies, and furthermore there is then a natural choice of a homotopy between these homotopies (a so called higher associativity condition). In general, one must consider all possible maps $Y^n \to Y$ for all $k \geq 0$ obtained by the binary operation, and at each stage some new higher associativity relation will emerge.

Evidently, the resulting structure is quite complicated and some way to manage that complexity is needed. In **[43]** Stasheff gives a description of the structure on

Y by means of certain spaces which parametrize all of the various n-ary operations (obtained by repeatedly using the binary operation) as well as homotopies between such operations *and* homotopies between such homotopies and so on. These parametrizing spaces, $\{K_n\}_{n=0}^{\infty}$, were later named associahedra (see **[35]** for a short history of these spaces and their name) and were redefined such that each K_n, with $n \geq 2$, is a convex set in $\mathbb{R}^{n-2}$.

We describe now the first few associahedra. The first space, K_0, is just a point. Since K_0 parametrizes the 0-ary operations, i.e., constants, it implies that Y has one single constant. The next space is K_1 which is also a point, implying there is just one 1-ary operation on Y, namely the identity. The space K_2 is still just a point, which corresponds to the binary operation present on Y. Things get more complicated in the next stage since this is where homotopies start playing a role. The space K_3 is the space $[-1, 1]$. The two endpoints represent the two ternary operations obtained from the binary operation. The entire space K_3 thus parametrizes a homotopy between these two ternary operations. The next space, K_4, is a pentagon. Each of its five vertices corresponds to each of the five 4-ary operations $Y^4 \to T$ obtainable from the binary operation, and each of the sides corresponds to a homotopy between the two operations at the endpoints. The whole pentagon thus corresponds to a higher homotopy relation between the homotopies on its boundary. Things get more and more complicated as we move up in dimensions, yet a concrete definition of all the spaces K_n is possible.

4.2. The Boardman-Vogt W-construction

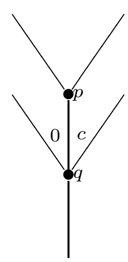
We now address a much more general question, motivated by the above discussion. Let $\mathcal{P}$ be a topological operad. Given a functor $F : \mathcal{P} \to Top$, which we think of as an algebraic structure, and for each $p \in ob(\mathcal{P})$ a homotopy equivalence $Fp \to Gp$ to some space Gp, what is the algebraic structure present on the spaces $\{Gp\}_{p \in ob(\mathcal{P})}$?

We will answer that question in the form of the Boardman-Vogt W-construction. A detailed account (albeit in a slightly different language than that of operads) of that construction can be found in **[7]** so we allow ourselves a more expository presentation aiming at explaining the ideas important for us. The W construction is a functor $W : Operad(Top) \to Operad(Top)$ equipped with a natural transformation (an augmentation) $W \to id$. So, with each topological operad $\mathcal{P}$ there is associated a topological operad $W\mathcal{P}$ and a map of operads $W\mathcal{P} \to \mathcal{P}$. Functors $W\mathcal{P} \to Top$ are then regarded as up-to-homotopy functors from $\mathcal{P}$ to $\mathcal{E}$ and are said to describe up-to-homotopy $\mathcal{P}$-algebras (or simply weak algebras). In that context, a functor $\mathcal{P} \to Top$ is referred to as a strict functor and an ordinary $\mathcal{P}$-algebra as a strict one. The augmentation implies the existence of a functor

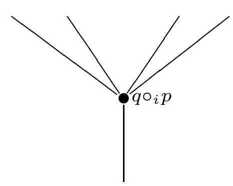
$$\underline{Operad(Top)}(\mathcal{P}, Top) \to \underline{Operad(Top)}(W\mathcal{P}, Top)$$

which views any strict algebra $\mathcal{P} \to Top$ as a weak one.

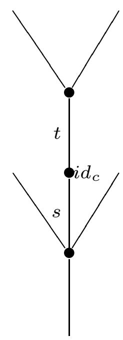
For a topological operad $\mathcal{P}$ we now describe the operad $W\mathcal{P}$. This is essentially the construction presented in **[7]**. For simplicity let us assume that the operads are planar, that is we describe a functor taking a planar operad enriched in Top to another such planar operad. The objects of $W\mathcal{P}$ are the same as those of $\mathcal{P}$. To describe the arrows spaces we consider standard planar trees whose edges are labelled by objects of $\mathcal{P}$ and whose vertices are labelled by operations of $\mathcal{P}$ according to the rule that the objects labelling the input edges of a vertex are equal (in their natural order) to the input of the operation labelling that vertex and similarly the object labelling the output of the vertex is the output object of the operation at the vertex. Moreover, each inner edge in such a tree is given a length $0 \leq t \leq 1$. For objects $p_0, \cdots, p_n \in ob(W\mathcal{P})$ let $A(p_1, \cdots, p_n; p_0)$ be the topological space whose underlying set is the set of all such standard planar labelled trees $\bar{T}$ for which the leaves of $\bar{T}$ are labelled by $p_1, \cdots, p_n$ (in that order) and the root of $\bar{T}$ is labelled by p_0. The topology on $A(p_1, \cdots, p_n; p_0)$ is the evident one induced by the topology of the arrows spaces in $\mathcal{P}$ and the standard topology on the unit interval $[0, 1]$. The space $W\mathcal{P}(p_1, \cdots, p_n; p_0)$ is the quotient of $A(p_1, \cdots, p_n; p_0)$ obtained by the following identifications. If $\bar{T} \in A(p_1, \cdots, p_n; p_0)$ has an inner edge e whose length is 0 then we identify it with the tree $\bar{T}/e$ obtained from $\bar{T}$ by contracting the edge e and labelling the newly formed vertex by the corresponding $\circ_i$-composition of the operations labelling the vertices at the two sides of e (the other labels are as in $\bar{T}$). Thus pictorially we have that locally in the tree a configuration:



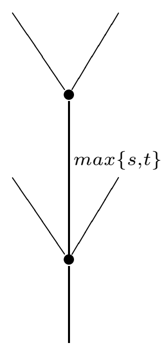
is identified with the configuration:



where the labels of the edges were neglected. Another identification is for a tree $\bar{S}$ with a unary vertex v labelled by an identity. We identify such a tree with the tree $\bar{R}$ obtained by removing the vertex v and identifying its input edge x with its output edge y. The length assigned to the new edge is determined as follows. If it is an outer edge then it has no length. If it is an inner edge then it is assigned the maximum of the lengths of x and y (where if either x or y does not have a length, i.e., it is an outer edge, then its length is considered to be 0). The labelling is as in $\bar{S}$ (notice that the label of the newly formed edge is unique since v was labelled by an identity which means that its input and output were labelled by the same object). Pictorially, this identification identifies the labelled tree

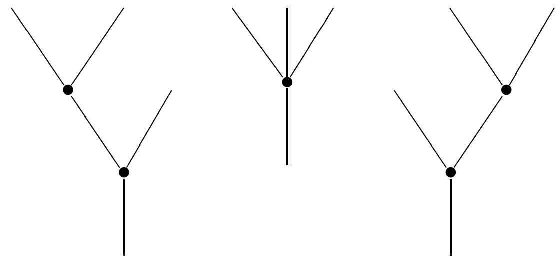


with the tree:



where we neglected the labels of the edges. The composition in $W\mathcal{P}$ is given by grafting such labelled trees, giving the newly formed inner edge length 1.

EXAMPLE 4.2.1. Let $\mathcal{P}$ be the planar operad with a single object and a single n-ary operation in each arity. We consider $\mathcal{P}$ to be a discrete operad in Top. It is easily seen that a functor $\mathcal{P} \to Top$ corresponds to a topological monoid. Let us now calculate the first few arrows spaces in $W\mathcal{P}$. Firstly, $W\mathcal{P}$ has too just one object. We thus use the notation of classical operads, namely $W\mathcal{P}(n)$ for the space of operations of arity n. Clearly $W\mathcal{P}(0)$ is just a one-point space. The space $W\mathcal{P}(1)$ consists of labelled trees with one input. Since in such a tree the only possible label at a vertex is the identify, the identification regarding identities implies that $W\mathcal{P}(1)$ is again just a one-point space. In general, since every unary vertex in a labelled tree in $W\mathcal{P}(n)$ can only be labelled by the identity, and those are then identified with trees not containing unary vertices, it suffices to only consider trees with no unary vertices at all. We call such trees regular. To calculate $W\mathcal{P}(2)$ we need to consider all regular trees with two inputs, but there is just one such tree, the 2-corolla, and it has no inner edges, thus $W\mathcal{P}(2)$ is also a one-point space. Things become more interesting when we calculate $W\mathcal{P}(3)$. We need to consider regular trees with three inputs. There are three such trees, namely



The middle tree contributes a point to the space $W\mathcal{P}(3)$. Each of the other trees has one inner edge and thus contributes the interval $[0,1]$ to the space. The only identification to be made is when the length of one of those inner edges is 0, in which case it is identified with the point corresponding to the middle tree. The space $W\mathcal{P}(3)$ is thus the gluing of two copies of the interval $[0,1]$ where we identify both ends named 0 to a single point. The result is then just a closed interval, $[-1,1]$. However, it is convenient to keep in mind the trees corresponding to each point of this interval. Namely, the tree corresponding to the middle point, 0, is the middle tree. With a point $0 < t \leq 1$ corresponds the tree on the right where the length of the inner edge is t, and with a point $-1 \leq -t < 0$ corresponds the tree on the left where its inner edge is given the length t. In this way one can calculate the entire operad $W\mathcal{P}$. It can then be shown that the spaces $\{W\mathcal{P}(n)\}_{n=0}^{\infty}$ are all homeomorphic to the Stasheff associahedra. An A_∞-space is then an algebra of $W\mathcal{P}$.

An important observation in the W construction given above is the following. In order to construct the space $W\mathcal{P}(p_1,\cdots,p_n;p_0)$ one can proceed as follows. For each labelled standard planar tree $\bar{T}$ as above let $H^{\bar{T}}$ be $H^{\otimes_k}$ where k is the number of inner edges in $\bar{T}$ and $H=[0,1]$, the unit interval. The space $A(p_1,\cdots,p_n;p_0)$ is homeomorphic to

$$\coprod_{\bar{T}} H^{\bar{T}}$$

where $\bar{T}$ varies over all labelled standard planar trees $\bar{T}$ whose leaves are labelled by $p_1,\cdots,p_n$ and whose root is labelled by p_0. The identifications that are then made to construct the space $W\mathcal{P}(p_1,\cdots,p_n;p_0)$ are completely determined by the combinatorics of the various trees $\bar{T}$. This observation is the key to generalizing the W-construction to operads in more general monoidal categories $\mathcal{E}$. What we need is a suitable replacement for the unit interval $[0,1]$ used above to give lengths to the inner edges of the trees. This is done in [**4, 5, 6**] with the notion of an *interval* in a monoidal model category $\mathcal{E}$. In more detail, assume $\mathcal{E}$ is a monoidal model category with a cofibrant unit I. An *interval* in $\mathcal{E}$ is then an object H together with maps

$$I \underset{1}{\overset{0}{\rightrightarrows}} H \xrightarrow{\epsilon} I$$

and

$$H \otimes H \xrightarrow{\vee} H$$

satisfying certain conditions (see [**5**]). In particular, H is an interval in Quillen's sense (see [**40**]), so 0 and 1 together define a cofibration $I \coprod I \to H$, and ϵ is a weak equivalence. In such a setting the W-construction above can be mimicked by gluing together objects of the from $H^{\otimes_k}$ rather then cubes $[0,1]^k$. We thus obtain a functor $W_H : Operad(\mathcal{E}) \to Operad(\mathcal{E})$, which specializes to the topological case in the sense that $W_{[0,1]} = W$. Usually we will just write W instead of W_H. We illustrate this more general construction by an example, referring the reader to **[5]** for more details.

Example 4.2.2. Consider the category Cat with the folk model structure. In this monoidal model category we can choose the category H with $ob(H) = \{0, 1\}$ and a single isomorphism $0 \to 1$ to be an interval, with the obvious structure maps. Let us consider again (compare Example 4.2.1) the operad $\mathcal{P}$ having one object and one n-ary operation for each $n \geq 0$, this time as a discrete operad in Cat. To calculate $W\mathcal{P}(n)$ we should again consider labelled standard planar trees with lengths. The same argument as above implies that we should only consider regular trees, and a similar calculation shows that $W\mathcal{P}(n)$ is a contractible one-point category for $n = 0, 1, 2$. Now, to calculate $W\mathcal{P}(3)$ we again consider the three trees as given above. This time the middle tree contributes the category $H^0 = I$. Each of the other trees contributes the category H. The identifications identify the object named 0 in each copy of H to the unique object of I. The result is a contractible category with three objects. In general, the category $W\mathcal{P}(n)$ is a contractible category with $tr(n)$ objects, where $tr(n)$ denotes the number of regular standard planar trees with n leaves. The composition in $W\mathcal{P}$ is given by grafting of such trees.

The generalized Boardman-Vogt W-construction thus provides a definition of up-to-homotopy $\mathcal{P}$-algebras for a wide variety of operads $\mathcal{P}$ enriched in a suitable monoidal model category with a chosen interval. A natural question that arises is, of course, what is the appropriate notion of maps between such weak $\mathcal{P}$-algebras. Luckily, no extra work is needed in order to produce such a notion since we can again use the W-construction to come up with one. The idea is very simple. If, given an operad $\mathcal{P}$, we can find an operad $\mathcal{P}^1$ such that a $\mathcal{P}^1$-algebra consists of two $\mathcal{P}$-algebras and a map of $\mathcal{P}$ algebras between them then it is sensible to define an up-to-homotopy map between weak $\mathcal{P}$-algebras to be a $W(\mathcal{P}^1)$-algebra.

If $\mathcal{P}$ is an operad in Set then we can take $\mathcal{P}^1$ to be $\mathcal{P} \otimes_{BV} \Omega[L_1]$ where L_1 is a linear tree with one vertex. We thus make the following definition:

Definition 4.2.3. Let $\mathcal{P}$ be an operad in Set and $\mathcal{E}$ a symmetric monoidal model category with an interval. An *up-to-homotopy $\mathcal{P}$-algebra* in $\mathcal{E}$ is an algebra for the operad $W(\mathcal{P})$. An *up-to-homotopy map* between up-to-homotopy $\mathcal{P}$-algebras is an algebra for the operad $W(\mathcal{P} \otimes_{BV} \Omega[L_1])$. We will sometimes use the term weak instead of up-to-homotopy.

Recall that L_1 has two edges named 0 and 1. We obtain thus two induced maps $\mathcal{P} \cong \mathcal{P} \otimes_{BV} \Omega[\eta_i] \to \mathcal{P} \otimes_{BV} \Omega[L_1]$, for $i = 0, 1$. Given an algebra $W(\mathcal{P} \otimes_{BV} \Omega[L_1]) \to \mathcal{E}$, i.e., a weak map between weak algebras, there are associated two maps $d_i : W(\mathcal{P}) \to \mathcal{E}$, which by definition are weak $\mathcal{P}$-algebras. We agree by convention that d_0 is the domain and d_1 is the codomain of the given map.

An obvious question now is whether the collection of all weak $\mathcal{P}$-algebras and all weak $\mathcal{P}$-maps form a category. The answer is that they usually do not. A simple example is provided by A_∞-spaces where it is known that weak A_∞-maps do not compose associatively. The theory so far already suggests a solution to that problem. For an operad $\mathcal{P}$ in Set consider the operad $\mathcal{P} \otimes_{BV} \Omega[L_n]$. An algebra for such an operad is easily seen to be a sequence of $n+1$ $\mathcal{P}$-algebras $X_0, \cdots, X_n$ together with $\mathcal{P}$-algebra maps:

$$X_0 \to X_1 \to \cdots \to X_n.$$

We proceed as follows:

PROPOSITION 4.2.4. *Let $\mathcal{P}$ be an operad in Set. For each $n \geq 0$ let X_n be the set*

$$Alg(W(\mathcal{P} \otimes_{BV} \Omega[L_n]), \mathcal{E}).$$

The collection $X = \{X_n\}_{n=0}^{\infty}$ is a simplicial set.

PROOF. The proof follows easily by noting that the sequence $\{\mathcal{P} \otimes_{BV} \Omega[L_n]\}_{n=0}^{\infty}$ forms a cosimplicial object in *Operad*. See **[6]** for more details. □

REMARK 4.2.5. In **[7]** the authors make essentially the same definition for the case where $\mathcal{E} = Top$ with the usual interval. They subsequently prove that the resulting simplicial set is a quasi-category.

DEFINITION 4.2.6. We refer to the simplicial set constructed above as the *simplicial set of up-to-homotopy $\mathcal{P}$-algebras.*

4.3. The homotopy coherent nerve

We now use the generalized W-construction in order to define, for an operad $\mathcal{P}$ enriched in a suitable symmetric monoidal model category $\mathcal{E}$, its homotopy coherent dendroidal nerve. This is a dendroidal set which is like the dendroidal nerve construction with homotopies built into it. The main result proved in this section is one identifying a condition on an operad $\mathcal{P}$ in $\mathcal{E}$ that ensures that its homotopy coherent dendroidal nerve is an inner Kan complex. This provides a large family of inner Kan complexes that are rarely ever strict.

DEFINITION 4.3.1. Let $\mathcal{E}$ be a symmetric monoidal model category with an interval H. For each tree $T \in ob(\Omega)$ we may consider the operad $\Omega(T)$ as a discrete operad in $Operad(\mathcal{E})$. Doing so we obtain the probe $\Omega \to Operad(\mathcal{E})$ that sends T to $W(\Omega(T))$. Let

$$dSet \underset{hcN_d}{\overset{hc\tau_d}{\rightleftarrows}} Operad(\mathcal{E})$$

be the associated adjunction. The functor hcN_d is called the *homotopy coherent dendroidal nerve* functor.

Explicitly, given an operad $\mathcal{P}$ in $\mathcal{E}$, its homotopy coherent dendroidal nerve is the dendroidal set given by

$$hcN_d(\mathcal{P})_T = Operad(\mathcal{E})(W(\Omega(T)), \mathcal{P})$$

To better understand this construction let us look more closely at the operads $W(\Omega(T))$. It is convenient to use the functor

$$Symm : Operad(\mathcal{E})_\pi \to Operad(\mathcal{E})$$

which is left adjoint to the obvious forgetful functor from symmetric operads in $\mathcal{E}$ to planar ones. Recall that if T is an object in Ω and $\bar{T}$ is a chosen planar representative of T, then $\Omega(T) = Symm(\Omega_\pi(\bar{T}))$. Since the W-construction commutes with symmetrization (as one easily verifies), it follows that

$$W(T) = Symm(W\Omega_\pi(\bar{T})).$$

REMARK 4.3.2. The operad $W\Omega_\pi(\bar{T})$ is easily described explicitly. The objects of $W(\Omega_\pi(\bar{T}))$ are the objects of $\Omega_\pi(\bar{T})$, i.e., the edges of T. Recall that by a *signature* in $W(\Omega_\pi(\bar{T}))$, we mean a sequence $e_1, \cdots, e_n; e_0$ of objects, i.e., edges of $\bar{T}$. Given a signature $\sigma = (e_1, \cdots, e_n; e_0)$, we have that $W(\Omega_\pi(\bar{T}))(\sigma) = \phi$ (the initial object in $\mathcal{E}$) whenever $\Omega_\pi(\bar{T})(\sigma)$ is empty. If $\Omega_\pi(\bar{T})(\sigma) \neq \phi$ then there is a sub-tree T_σ of T (and a corresponding planar sub-tree $\bar{T}_\sigma$ of $\bar{T}$) whose leaves are $e_1, \cdots, e_n$, and whose root is e_0. We then have that

$$W\Omega_\pi(\bar{T})(e_1, \cdots, e_n; e_0) = \bigotimes_{f \in i(\sigma)} H,$$

where $i(\sigma)$ is the set of *inner* edges of T_σ (or of $\bar{T}_\sigma$). (This last tensor product is to be thought of as the "space" of assignments of lengths to inner edges in $\bar{T}_\sigma$; it is the unit if $i(\sigma)$ is empty.)

The composition operations in the operad $W\Omega_\pi(\bar{T})$ are given in terms of the $\circ_i$-operations as follows. For signatures $\sigma = (e_1, \cdots, e_n; e_0)$ and $\rho = (f_1, \cdots, f_m; e_i)$, the composition map

$$\begin{array}{c} W\Omega_\pi(\bar{T})(e_1, \cdots, e_n; e_0) \otimes W\Omega_\pi(\bar{T})(f_1, \cdots, f_m; e_i) \\ \big\downarrow \circ_i \\ W\Omega_\pi(\bar{T})(e_1, \cdots, e_{i-1}, f_1, \cdots, f_m, e_{i+1}, \cdots, e_n; e_0) \end{array}$$

is the following one. The trees $\bar{T}_\sigma$ and $\bar{T}_\rho$ can be grafted along e_i to form $\bar{T}_\sigma \circ_{e_i} \bar{T}_\rho$, again a planar sub-tree of $\bar{T}$. In fact

$$\bar{T}_\sigma \circ_{e_i} \bar{T}_\rho = \bar{T}_{\sigma \circ_i \rho}$$

where $\sigma \circ_i \rho$ is the signature $(e_1, \cdots, e_{i-1}, f_1, \cdots, f_m, e_{i+1}, \cdots, e_n; e_0)$, and for the sets of inner edges we have

$$i(\sigma \circ_i \rho) = i(\sigma) \cup i(\rho) \cup \{e_i\}.$$

The required composition is then

$$\begin{array}{ccc} H^{\otimes i(\sigma)} \otimes H^{\otimes i(\rho)} & \dashrightarrow & H^{\otimes i(\sigma \circ_i \rho)} \\ \big\downarrow \cong & & \big\downarrow \cong \\ H^{\otimes i(\sigma) \cup i(\rho)} \otimes I & \xrightarrow{id \otimes 1} & H^{\otimes i(\sigma) \cup i(\rho)} \otimes H \end{array}$$

where $1 : I \to H$ is one of the "endpoints" of the interval H, as above.

This description of the operad $W\Omega_\pi(\bar{T})$ is functorial in the planar tree T. In particular, we note that for an inner edge e of T, the tree T/e inherits a planar structure $\bar{T/e}$ from $\bar{T}$, and $W\Omega_\pi(\bar{T/e}) \to W\Omega_\pi(\bar{T})$ is the natural map assigning length 0 to the edge e whenever it occurs (in a sub-tree given by a signature).

The following will result will be useful.

LEMMA 4.3.3. *Let $\mathcal{P}$ be an operad in Set. For any tree $T \in ob(\Omega)$ the equality*

$$N_d\mathcal{P} \otimes \Omega[T] \cong N_d(\mathcal{P} \otimes_{BV} \Omega(T))$$

holds.

PROOF. The dendroidal set $N_d\mathcal{P} \otimes \Omega[T]$ is the following colimit:

$$\varinjlim_{\Omega(S)\to\mathcal{P}} N_d(\Omega(S) \otimes_{BV} \Omega(T)) = \varinjlim_{\Omega(S)\to\mathcal{P}} \Omega[S] \otimes \Omega[T].$$

We show that $N_d(\mathcal{P} \otimes \Omega(T))$ has the required universal property with respect to that diagram. To obtain a cone from the diagram we need an arrow $N_d(\Omega(S) \otimes_{BV} \Omega(T)) \to N_d(\mathcal{P} \otimes_{BV} \Omega(T))$ for each arrow $\Omega(S) \to \mathcal{P}$. Since $\tau_d N_d = id$, such an arrow is the same as an arrow $\Omega(S) \otimes_{BV} \Omega(T) \to \mathcal{P} \otimes_{BV} \Omega(T)$ and the choice for such an arrow is obvious. Assume now that we are given a cone to some other dendroidal set X. Let $t \in N_d(\mathcal{P} \otimes_{BV} \Omega(T))_T$ be a non-degenerate dendrex. Since $\Omega(T)$ is the nerve of the dendroidally ordered set $[T]$ (thus there is at most one arrow for each signature and the domain of each arrow does not contain repeated objects) it follows that there is a unique (up-to-isomorphism) maximal dendrex $s \in N_d(\mathcal{P})_S$ such that t is a dendrex in the dendroidal sub-set $\Omega[S] \otimes \Omega[T] \to N_d(\mathcal{P} \otimes_{BV} \Omega(T))$ given by the map s. This dendrex s corresponds to a map $\Omega(S) \to \mathcal{P}$. If $f : \Omega[S] \otimes \Omega[T] \to X$ is a map in the given cone then $g(t) = f(t)$ defines a unique map $N_d(\mathcal{P} \otimes_{BV} \Omega(T)) \to X$ such that the cones commute, as required for the universal property. □

THEOREM 4.3.4. *Let $\mathcal{E}$ be a symmetric monoidal model category with an interval and $\mathcal{P}$ an operad in Set. We have the following isomorphism of operads*

$$hc\tau_d(N_d\mathcal{P}) \cong W(disc(\mathcal{P})).$$

PROOF. The operad $hc\tau_d(N_d\mathcal{P})$ is given as the colimit

$$\varinjlim_{\Omega(T)\to\mathcal{P}} W\Omega(T).$$

A straightforward inspection of that colimit and the generalized W-construction presented in **[5]** for the special case of a discrete operad show that both colimits are the same. □

An immediate result is that, in the same setting as above, the set

$$\underline{dSet}(N_d\mathcal{P}, hcN_d(\mathcal{E}))_\eta$$

is isomorphic to the set $Operad(\mathcal{E})(W(disc(\mathcal{P})), \mathcal{E})$ of up-to-homotopy $\mathcal{P}$-algebras. More generally we have the following:

PROPOSITION 4.3.5. *Let $\mathcal{E}$ be a symmetric monoidal model category with an interval and $\mathcal{P}$ a discrete operad in $\mathcal{E}$. We then have the following isomorphism:*

$$\underline{dSet}(N_d\mathcal{P}, hcN_d(\mathcal{E}))_T \cong Operad(\mathcal{E})(W(\mathcal{P} \otimes_{BV} \Omega[T]), \mathcal{E}).$$

COROLLARY 4.3.6. *With $\mathcal{E}$ as above we obtain that the simplicial set*

$$i^*(\underline{dSet}(N_d(\mathcal{P}), hcN_d(\mathcal{E})))$$

is isomorphic to the simplicial set of weak $\mathcal{P}$-algebras from Definition 4.2.6 above.

Our aim for the rest of this section is to present a sufficient condition on an operad $\mathcal{P}$ in $\mathcal{E}$ that guarantees that the homotopy coherent dendroidal nerve of $\mathcal{P}$ is an inner Kan complex.

DEFINITION 4.3.7. An operad $\mathcal{P}$ in a symmetric monoidal model category $\mathcal{E}$ is called *locally fibrant* if for each sequence $c_1, \cdots, c_n, c$ of objects of $\mathcal{P}$, the object $\mathcal{P}(c_1, \cdots, c_n; c)$ in $\mathcal{E}$ is fibrant.

THEOREM 4.3.8. *Let $\mathcal{P}$ be a locally fibrant operad in $\mathcal{E}$, where $\mathcal{E}$ is a symmetric monoidal model category with an interval. Then $hcN_d(\mathcal{P})$ is an inner Kan complex.*

PROOF. Consider a tree T and an inner edge e in T. We want to solve the extension problem

$$\begin{array}{ccc} \Lambda^e[T] & \xrightarrow{\varphi} & hcN_d(\mathcal{P}) \\ \big\downarrow & \nearrow & \\ \Omega[T] & & \end{array}$$

Fix a planar representative $\bar{T}$ of T. Then the derived map $\psi : \Omega[T] \to hcN_d(\mathcal{P})$ corresponds to a map of planar operads

$$\hat{\psi} : W\Omega_\pi(\bar{T}) \to \mathcal{P}.$$

Each face S of T inherits a planar structure $\bar{S}$ from $\bar{T}$, and the given map $\varphi : \Lambda^e[T] \to hcN_d(\mathcal{P})$ corresponds to a map of operads in $\mathcal{E}$,

$$\hat{\varphi} : W(\Lambda^e[T]) \to \mathcal{P},$$

where $W(\Lambda^e[T])$ denotes the colimit of operads in $\mathcal{E}$,

$$W(\Lambda^e[T]) = \varinjlim W(\Omega_\pi(\bar{S})) \qquad (1)$$

over all but one of the faces of T. In other words, φ corresponds to a compatible family of maps

$$\hat{\varphi}_S : W(\Omega_\pi(\bar{S})) \to \mathcal{P}.$$

Let us now show the existence of an operad map $\hat{\psi}$ extending the $\hat{\varphi}_S$ for all faces $S \neq T/e$. First, the objects of $\Omega_\pi(\bar{T})$ are the same as those of the colimit in (1), so we already have a map $\psi_0 = \varphi_0$ on objects:

$$\psi_0 : E(T) \to ob(\mathcal{P}).$$

Next, if $\sigma = (e_1, \cdots, e_n; e_0)$ is a signature of T for which $W(\Omega_\pi(\bar{T})) \neq \phi$, if $T_\sigma \subseteq T$ is not all of T, then T_σ is contained in an outer face S of T. So $W(\Omega_\pi(\bar{T}))(\sigma) = W(\Omega_\pi(\bar{T}_\sigma))(\sigma) = W(\Omega_\pi(\bar{S}))(\sigma)$, and we already have a map

$$\hat{\varphi}_\rho(\sigma) : W(\Omega_\pi(\bar{T}))(\sigma) \to \mathcal{P}(\sigma),$$

given by $\hat{\varphi}_S : W(\Omega_\pi(\bar{S})) \to \mathcal{P}$. Thus, the only part of the operad map $\hat{\psi} : W(\Omega_\pi(\bar{T})) \to \mathcal{P}$ not determined by φ is the one for the signature τ where $T_\tau = T$; i.e., $\tau = (e_1, \cdots, e_n; e_0)$ where $e_1, \cdots, e_n$ are all the input edges of $\bar{T}$ (in the planar order) and e_0 is the output edge. For this signature, $\hat{\psi}(\tau)$ is to be a map

$$\hat{\psi} : W(\Omega_\pi(\bar{T})(\tau) = H^{\otimes i(\tau)} \to \mathcal{P}(\tau)$$

which (i) is compatible with the $\hat{\psi}(\sigma) = \hat{\varphi}_S(\sigma)$ for other signatures σ; and (ii) together with these $\hat{\psi}(\sigma)$ respects operad composition. The first condition determines $\hat{\psi}(\tau)$ on the sub-object of $H^{\otimes i(\tau)}$ which is given by a value 1 on one of the factors. Thus, if we write 1 for the map $I \overset{1}{\rightarrowtail} H$ and $\partial H \rightarrowtail H$

for the map $I \coprod I \to H$, and define $\partial H^{\otimes k} \rightarrowtail H^{\otimes k}$ by the Leibniz rule (i.e., $\partial(A \otimes B) = \partial(A) \otimes B \cup A \otimes \partial(b)$), then the problem of finding $\hat{\psi}(\tau)$ comes down to an extension problem of the form

$$\begin{array}{ccc}
\partial(H^{\otimes(i(\tau)-\{e\})} \otimes H \cup H^{\otimes(i(\tau)-\{e\})} \otimes I & \longrightarrow & \mathcal{P}(\tau) \\
\big\downarrow & & \uparrow{\scriptstyle \hat{\psi}(\sigma)} \\
H^{\otimes i(\tau)-\{e\}} \otimes H & \xrightarrow{\cong} & H^{\otimes i(\tau)}
\end{array}$$

This extension problem has a solution, because $\mathcal{P}(\tau)$ is fibrant by assumption, and because the left hand map is a trivial cofibration (by repeated use of the push-out product axiom for monoidal model categories). This concludes the proof of the theorem. □

Remark 4.3.9. This result generalizes a result of Cordier and Porter [**10**], namely that the homotopy coherent nerve of a simplicially enriched category with fibrant Hom objects is an inner Kan complex. Indeed, taking $\mathcal{E}$ to be the category of simplicial sets with its usual monoidal model category and the obvious interval we know that any locally fibrant operad in $sSet$ admits a homotopy coherent dendroidal nerve which is an inner Kan complex. Viewing a simplicial category as an operad in $sSet$ in the obvious way gives the desires result.

Recall (Example 1.8.4) that given a set M of object in a symmetric monoidal category $\mathcal{E}$ we can construct the $\mathcal{E}$-enriched operad $\mathcal{P}_M$.

Lemma 4.3.10. *Let $\mathcal{E}$ be a symmetric monoidal model category with an interval. If $M \subseteq ob(\mathcal{E})$ consists of fibrant-cofibrant objects then $hcN_d(\mathcal{P}_M)$ is an inner Kan complex.*

Proof. It is sufficient to show that $\mathcal{P}_M$ is locally fibrant. In a monoidal model category the tensor product of cofibrant objects is again cofibrant and $\mathcal{E}(X, Y)$ is fibrant whenever X is cofibrant and Y is fibrant. It now follows that each Hom object in $\mathcal{P}_M$ is fibrant, as needed. □

Remark 4.3.11. Given a symmetric monoidal model category $\mathcal{E}$ with an interval, let $\mathcal{E}_{cf}$ be the full sub-category of $\mathcal{E}$ spanned by the fibrant-cofibrant objects. A fundamental construction in the theory of model categories is the homotopy category $Ho(\mathcal{E})$, which is again a monoidal category and we may thus consider it as an operad. Recall the theory of homotopy within an inner Kan complex from Section 3.5 and in particular the construction of the homotopy operad $Ho(X)$ of an inner Kan complex X. It is rather simple to verify that the operad $Ho(hcN_d(\mathcal{E}_{cf}))$ is equivalent to $Ho(\mathcal{E})$ and is actually equal to $Ho(\mathcal{E}_{cf})$. In that sense, the theory of homotopy inside a weak Kan complex extends the homotopy theory inside a symmetric monoidal model category with an interval. Notice that the dendroidal nerve $hcN_d(\mathcal{E}_{cf})$ stores much more information than the homotopy category, namely all of the higher homotopies.

4.4. Algebras and the Grothendieck construction

Recall that an operad $\mathcal{P}$ can be used to define an algebraic structure on objects of another operad $\mathcal{E}$. In this section we extend the notion of algebras to dendroidal sets and present a Grothendieck construction for diagrams of dendroidal sets which extends the Grothendieck construction for diagrams of operads (Section 1.7).

DEFINITION 4.4.1. Let E and X be dendroidal sets. The dendroidal set

$$\underline{dSet}(X, E)$$

is called the dendroidal set of X-*algebras* in E and is denoted by $Alg(X, E)$. An element in $Alg(X, E)_\eta$ is called an X-*algebra* in E. An element of $Alg(X, E)_{L_1}$ is called a *map of* X-*algebras in* E. We will also refer to an X-algebra in E as an (X, E)-algebra.

Let us first show that this definition extends the notion of $\mathcal{P}$-algebras in $\mathcal{E}$ for operads.

PROPOSITION 4.4.2. *For operads* $\mathcal{P}$ *and* $\mathcal{E}$

$$Alg(N_d\mathcal{P}, N_d\mathcal{E}) \cong N_d(Alg(\mathcal{P}, \mathcal{E})).$$

PROOF. This is just the statement that N_d commutes with internal Homs. □

Suppose that $Alg(X, E)$ is an inner Kan complex. The existence of composition of dendrices then provides for a notion of composition of maps of (X, E)-algebras, which extends the composition of maps of $(\mathcal{P}, \mathcal{E})$-algebras. Furthermore, the theory of homotopy in a dendroidal set automatically provides a notion of homotopy for (X, E)-algebras, which is particularly well behaved when $Alg(X, E)$ is an inner Kan complex. The exponential property of dendroidal sets can be restated by saying that if E is an inner Kan complex and X is normal then $Alg(X, E)$ is an inner Kan complex and thus enjoys a built-in theory of composition and homotopy. We see thus that by replacing an operad by an inner Kan complex, a much greater generality is obtained (to define algebraic structures in, not necessarily, operads) and still one retains a suitable notion of composition of maps of such algebras and gains immediately a pleasant theory of homotopy.

Later on we will use the theory to obtain a definition for weak n-categories. To that end we will have to use a suitable Grothendieck construction for diagrams of dendroidal sets, which we now turn to.

Let $\mathbb{S}$ be a cartesian category. A functor $X : \mathbb{S}^{op} \to dSet$ is called a *diagram* of dendroidal sets. Our aim is to define a dendroidal set

$$\int_{\mathbb{S}} X$$

obtained by suitably 'gluing' the dendroidal sets $X(S)$ for the various $S \in ob(\mathbb{S})$. It will be convenient to introduce the notion of dendroidal collections. A *dendroidal collection* is a collection of sets $X = \{X_T\}_{T \in ob(\Omega)}$. Each dendroidal set has an obvious underlying dendroidal collection. A map of dendroidal collections $X \to Y$ is a collection of functions $\{X_T \to Y_T\}_{T \in ob(\Omega)}$. There is a natural way of associating an object of $\mathbb{S}$ with each dendrex of $N_d(\mathbb{S})$. For a tree T in Ω, let $leaves(T)$ be the set of leaves of T, and for a leaf l, write $l : \eta \to T$ also for the map sending the unique edge in η to l in T. Then, since $\mathbb{S}$ is assumed to have finite products, each dendrex $t \in N_d(\mathbb{S})_T$ defines an object

$$in(t) = \prod_{l \in leaves(T)} l^*(t)$$

in $\mathbb{S}$. Notice that if $\alpha : S \to T$ is a face map, then by using the canonical symmetries and the projections in $\mathbb{S}$, there is a canonical arrow $in(\alpha) : in(t) \to in(\alpha^* t)$ for any $t \in X_T$. Similarly, if α is a degeneracy map or an isomorphism then one obtains a map $in(\alpha)$, and in fact for any $\alpha : S \to T$ one naturally obtains a map $in(\alpha) : in(t) \to in(\alpha^* t)$ for each $t \in X_T$.

Definition 4.4.3. Let $X : \mathbb{S}^{op} \to dSet$ be a diagram of dendroidal sets where $\mathbb{S}$ is a cartesian category which is thus also an operad. The dendroidal set $\int_{\mathbb{S}} X$ is defined as follows. A dendrex $\Omega[T] \to \int_{\mathbb{S}} X$ is a pair (t, x) such that $t \in N_d(\mathbb{S})_T$ and x is a map of dendroidal collections

$$x : \Omega[T] \to \coprod_{S \in ob(\mathbb{S})} X(S)$$

satisfying the following conditions. For each $r \in \Omega[T]_R$ (that is an arrow $r : R \to T$), we demand that $x(r) \in X(in(r^* t))$. Furthermore we demand the following compatibility condition to hold. For any $r \in \Omega[T]_R$ and any map $\alpha : U \rightarrowtail R$ in Ω

$$\alpha^*(x(r)) = X(in(\alpha))x(\alpha^*(r)).$$

Remark 4.4.4. A straightforward verification shows that the Grothendieck construction for diagrams of dendroidal sets extends the one for operads (given in Section 1.7) in the following sense. If we have a diagram of operads $X : \mathbb{S}^{op} \to Operad$ and if we write $N_d(X)$ for the diagram of dendroidal sets $\mathbb{S}^{op} \to Operad \to dSet$ obtained by composition with $N_d : Operad \to dSet$, then we have:

$$N_d(\int_{\mathbb{S}} X) \cong \int_{\mathbb{S}} N_d(X).$$

Theorem 4.4.5. *Let $X : \mathbb{S}^{op} \to dSet$ be a diagram of dendroidal sets. If for every $S \in ob(\mathbb{S})$ every $X(S)$ is an inner Kan complex then so is $\int_{\mathbb{S}} X$.*

Proof. Let T be a tree and e an inner edge. We consider the extension problem

$$\begin{array}{ccc} \Lambda^e[T] & \longrightarrow & \int_{\mathbb{S}} X \\ \downarrow & \nearrow & \\ \Omega[T] & & \end{array}$$

The horn $\Lambda^e[T] \to \int_{\mathbb{S}} X$ is given by a compatible collection $\{(r, x_R) : \Omega[R] \to \int_{\mathbb{S}} X\}_{R \neq T/e}$. We wish to construct a dendrex $(t, x_T) : \Omega[T] \to \int_{\mathbb{S}} X$ extending this family. First notice that the collection $\{r\}_{R \neq T/e}$ is an inner horn $\Lambda^e[T] \to hcN_d(\mathbb{S})$ (actually this horn is obtained by composition with the obvious projection $\int_{\mathbb{S}} X \to hcN_d(\mathbb{S})$ sending a dendrex (t, x) to t). We are given that $hcN_d(\mathbb{S})$ is an inner Kan complex and thus there is a filler $t \in hcN_d(\mathbb{S})_T$ for the horn $\{r\}_{R \neq T/e}$. We now wish to define a map of dendroidal collections $x_T : \Omega[T] \to \coprod_{S \in ob(\mathbb{S})} X(S)$ that will extend the given maps x_R for $R \neq T/e$. This condition already determines the value of x_T for any dendrex $r : U \to T$ other then $id : T \to T$ and $\partial_e : T/e \to T$, since for each such r, the tree U factors through one of the faces $R \neq T/e$. To determine $x_T(id_T)$ and $x_T(\partial_e)$ consider the family $\{y_R = x_R(id : R \to R)\}_{R \neq T/e}$. By definition we have that $y_R \in X(in(r))_R$. For each such R let $\alpha_R : R \to T$ be the corresponding face map in Ω. Since $\partial_e^*(t) = r$ we obtain the map $in(\alpha_R) : in(r) \to in(t)$. We can now pull back the collection $\{y_R\}_{R \neq T/e}$ using $X(in(\alpha_R))$ to obtain a collection $\{z_R = X(in(\alpha_R))(y_R)\}_{R \neq T/e}$. This collection is now a horn $\Lambda^e[T] \to X(in(T))$ (this follows from the compatibility conditions in the definition of $\int_{\mathbb{S}} X$). Since $X(in(t))$ is an inner Kan complex there is a filler $u \in X(in(t))_T$ for that horn. We now define $x_T(id : T \to T) = u$ and $x_T(\partial_e : T/e \to T) = \partial_e^*(u)$. Notice that since e is inner we have that $in(t) = in(\partial_e)$ and thus the images of these dendrices are in the correct dendroidal set, namely $X(in(t))$. It follows from our construction that this makes (t, x_T) a dendrex $\Omega[T] \to \int_{\mathbb{S}} X$ which extends the given horn. This concludes the proof. □

4.5. Categories enriched in a dendroidal set

Recall that in Section 1.7 we defined for each set A an operad $\mathcal{C}_A$ such that a $\mathcal{C}_A$-algebra in the operad associated to a symmetric monoidal category $\mathcal{E}$ is the same thing as a category enriched in $\mathcal{E}$, having A as set of objects, that is

$$Alg(\mathcal{C}_A, \mathcal{E}) = Cat(\mathcal{E})_A.$$

Using the dendroidal nerve functor we obtain that

$$N_d(Cat(\mathcal{E})_A) = N_d(Alg(\mathcal{C}_A, \mathcal{E})) \cong Alg(N_d(\mathcal{C}_A), N_d(\mathcal{E})).$$

Based on this, we make the following definition.

Definition 4.5.1. Let X be an arbitrary dendroidal set and $\mathcal{C}_A$ as above. The dendroidal set $Alg(N_d(\mathcal{C}_A), X)$ is called the dendroidal set of *categories enriched in* X having A as set objects and is denoted by $Cat(X)_A$.

We now use the Grothendieck construction in order to obtain the dendroidal set of all categories enriched in X. We already have the obvious functor $Set^{op} \to dSet$ that sends a set A to $Cat(X)_A$.

Definition 4.5.2. Let X be a dendroidal set and let $\mathcal{C}_A$, for each set A, be the operad discussed above. Let $Cat(X)_- : Set^{op} \to dSet$ be the functor that sends a set A to the dendroidal set $Cat(X)_A = Alg(N_d(\mathcal{C}_A), X)$. The dendroidal set of *categories enriched in the dendroidal set* X is

$$Cat(X) = \int_{Set} Cat(X)_-.$$

This construction can be repeated as follows.

Definition 4.5.3. Let X be a dendroidal set. Let $Cat(X)^0 = X$ and define recursively

$$Cat(X)^{n+1} = Cat(Cat(X)^n)$$

for each $n \geq 1$. We call $Cat(X)^n$ the dendroidal set of *n-categories enriched in X*.

Theorem 4.5.4. *If X is an inner Kan complex then for each $n \geq 0$ the dendroidal set $Cat(X)^n$ is an inner Kan complex.*

Proof. For any planar operad $\mathcal{P}$ in Set the dendroidal nerve $N_d(Symm(\mathcal{P}))$ is clearly normal (see Definition 2.6.6) and thus $N_d(\mathcal{C}_A)$ is normal for each set A. From the fact that the the inner Kan complexes form an exponential ideal in $dSet$ with respect to the normal dendroidal sets (Theorem 3.6.3) it follows that each dendroidal set $Cat(X)_A$ is an inner Kan complex and Theorem 4.4.5 then proves that so is $Cat(X)$. □

Thus, for an inner Kan complex X, our definition of the dendroidal set of n-categories enriched in X provides us with a definition of what an n-category enriched in X is, what are functors for such n-categories, a notion of homotopy for such functors together with a composition rule for such functors which is associative up to homotopy.

EXAMPLE 4.5.5. It is rather straightforward to verify that

$$Cat(N_d(Set))^n \cong N_d(Cat^n)$$

for each $n \geq 0$ where Cat^n is the category of strict n-categories with the tensor product of n-categories, viewed as an operad (by Cat^0 we mean just Set). Unfolding this definition one sees that this is just the common definition of strict $(n+1)$-categories as categories enriched in the category of n-categories. More generally, we have that

$$Cat(N_d(Cat^m))^n \cong N_d(Cat^{n+m}).$$

Of course we also have that for a symmetric monoidal category $\mathcal{E}$

$$Cat(N_d(\mathcal{E})) \cong N_d(Cat(\mathcal{E})).$$

We see thus that our notion of a category enriched in a dendroidal set X extends the usual definition of categories enriched in a symmetric monoidal category. We see also that our notion of n-category in X, for $X = N_d(Set)$, captures the notion of strict n-categories.

REMARK 4.5.6. Consider the category Top of compactly generated spaces. This category is a closed monoidal category with weak equivalences the weak homotopy equivalences and fibrations the Serre fibrations ([**40**]). The unit interval $[0,1]$ with the minimum operation acts as an interval for this category in the above sense, so that the homotopy coherent nerve of Top is well defined. We now have the dendroidal set $Cat(hcN_d(Top))$. The dendrices of shape η are then categories weakly enriched in topological spaces in the sense that the composition maps are associative up to specified higher homotopies. In particular, the dendroidal set $A^1 = \underline{dSet}(N_d(\mathcal{C}_*), hcN_d(Top))$ is the dendroidal set of A_∞-spaces ($\mathcal{C}_*$ being $\mathcal{C}_B$ where B is a one-point set). It is now natural to consider the sequence $\{A^n\}_{n=1}^\infty$ where $A^{n+1} = \underline{dSet}(N_d(\mathcal{C}_*), A^n)$. It should be interesting to study the relation between n-fold loop spaces and A^n and compare it to the work of Dunn [**12**] and the recent approach of Batanin [**2**] via n-operads.

Given a commutative ring R, the category $Ch(R)$ of graded chain complexes of R-modules is a monoidal model category where the equivalences are the quasi-equivalences and the fibrations are the epimorphisms ([**21**]). An interval in this category is given by $N_*^R(\Delta[1])$ where N_*^R is the normalized chain complex functor. We now have the dendroidal set $Cat(hcN_d(Ch(R)))$ whose dendrices of shape η are essentially A_∞-categories (see [**25**] for a definition and [**6**] for a related discussion).

4.6. Weak n-categories

The definition of strict $(n+1)$-categories as categories enriched in strict n-categories is very appealing and a suitable analogous definition for weak n-categories is desirable. A naive approach to an analogous definition of weak n-categories would proceed along these lines: A weak $(n+1)$-category should be a category weakly enriched in the category of weak n-categories. There are two problems with such a definition. The first is that it is not clear how one should weakly enrich a category and the second, and far more devastating for such an approach, is that while in order to enrich in a category $\mathcal{E}$, that category has to have certain extra structure (mostly that of a symmetric monoidal category, but braided monoidal categories or fc-multicategories are also adequate structures in which to enrich categories [**30**]). The problem becomes apparent when one realizes that for $n > 2$ the collection of weak n-categories and their weak functors should not be expected to even form a category but rather a weak n-category. To proceed we must then consider the extra structure needed to be present on a weak n-category in order to weakly enrich in it and then say what is meant by weakly enriching in it. Hope for a uniform recursive definition of such notions seems remote and the naive approach would appear to fail.

There are of course other approaches to be taken which resulted in a plethora of definitions of weak n-categories (See [**29**] for a survey of ten such definitions). Some of the approaches to a definition can be said to improve on the naive approach dictated above. Using the general theory of dendroidal sets we obtain another such definition, as we now show.

Definition 4.6.1. Let Cat be the category of categories with the folk model structure (see Theorem 1.6.1 for the definition) and the interval H the free-living isomorphism (a two object category with a single isomorphism between them). We define the dendroidal set of weak n-categories $wCat^n$ for $0 \leq n < \infty$ as follows. $wCat^0 = N_d(Set)$ and for $n > 0$:

$$wCat^n = Cat^{n-1}(hcN_d(Cat)).$$

Since every object in Cat is fibrant and cofibrant it follows that $hcN_d(Cat)$ is an inner Kan complex (Lemma 4.3.10) and thus that our definition provides notions of weak n-categories, their maps, homotopy, and compositions.

Let us look more closely at weak n-categories for small n. For $n = 1$ we have $wCat^1 = hcN_d(Cat)$. Recall that a dendrex of shape T in $wCat^1$ is a functor of Cat-enriched operads $W(\Omega(T)) \to Cat$. It is easily seen that $wCat^1_\eta$ is the set of all small categories. A dendrex $F \in wCat^1_{L_1}$ is then just a functor between two categories, while a dendrex in $wCat^1_{L_2}$ corresponds to a choice of three functors $F_1 : A \to B$, $F_2 : B \to C$, and $F_3 : A \to C$ together with a natural isomorphism $\alpha : F_2F_1 \to F_3$. It thus follows that two dendrices $F, F' \in wCat^1_{L_1}$ are homotopic if, and only if, they are naturally isomorphic. We now show that a dendrex $t \in wCat^1_T$, with $|T| \geq 3$ is completely determined by its boundary. We use the following notation. Given a dendrex $t : \Omega[T] \to X$ in a dendroidal set X we denote the map $Sk_k(\Omega[T]) \to \Omega[T] \to X$ by $Sk_k(t)$.

Proposition 4.6.2. *Let T be a tree with $|T| \geq 3$ and t and s two dendrices in $wCat^1_T$. If $Sk_2(t) = Sk_2(s)$ then $t = s$.*

Proof. Consider a functor $F : H^m \to \mathcal{C}$ for some $m \geq 0$. The category H^m is a contractible category with $ob(H^m) = \{0, 1\}^m$ and thus the functor F is completely determined by its value on each of the arrows from the object $(0, \cdots, 0)$ to all other objects. Now, a dendrex $x \in X_T$ is a functor $W(\Omega(T)) \to Cat$ which can be given as a sequence of compatible functors $\{H^{m_i} \to \mathcal{C}_i\}_{i=1}^n$ (see the proof of Theorem 4.3.8 above). Each such functor is determined thus by its image on the special arrows just mentioned. Since these arrows are clearly contained in the image of $Sk_2(\Omega[T]) \longrightarrow \Omega[T] \xrightarrow{x} X$, the result follows. $\square$

We can now deduce the following:

Lemma 4.6.3. *The dendroidal set $wCat^1 = hcN_d(Cat)$ is 3-coskeletal.*

Proof. If we can show that $wCat^1$ satisfies the strict inner Kan condition for all trees T with $|T| \geq 3$ then it would follow from Proposition 3.2.5 that $wCat^1$ is 3-coskeletal. Let thus T be such a tree. Since we already know that $wCat^1$ is an inner Kan complex we know that every inner horn $\Lambda^e[T] \to wCat^1$ has a filler t. Suppose that s is also a filler for the same horn. Since $|T| \geq 3$ it follows that $Sk_2(\Omega[T])$ factors through $\Lambda^e[T]$ and thus that $Sk_2(t) = Sk_2(s)$. The above proposition then implies that $t = s$, as needed. $\square$

It is now a straightforward (and somewhat tedious) matter to unpack the definition of a weak 2-category. To identify the relevant dendrices in $wCat^1$ it will be convenient to use the following notation. Given categories $X_1, \cdots, X_n$ and two integers $1 \leq i \leq j \leq n$ we denote by $(X)_i^j$ the category $X_i \times \cdots \times X_j$. A dendrex of $wCat^1$ of shape η is just a category. A dendrex of the shape of a corolla C_n is the same as a choice of $n+1$ categories $X_0, \cdots, X_n$ and a functor $F : (X)_1^n \to X$. Any dendrex of degree 2 is of shape $C_n \circ_i C_m$ and such a dendrex is equivalent to the following data:

1. A choice of $n+1$ categories $X_0, \cdots, X_n$ and a functor $F : (X)_1^n \to X_0$.
2. A choice of m categories $Y_0, \cdots, Y_m$ and a functor $G : (Y)_1^m \to X_i$.
3. A functor $H : (X)_1^{i-1} \times (Y)_1^m \times (X)_{i+1}^n \to X_0$.
4. A natural isomorphism α between H and $F \circ_i G$.

with $F \circ_i G$ being the obvious functor. A dendrex of degree three can have one of two shapes. Either it is of the shape $C_n \circ_{i,j} (C_m, C_k)$ or of the shape $C_n \circ_i (C_m \circ_j C_k)$. A dendrex of the first shape consists of the following data:

1. A choice of $n+1$ categories $X_0, \cdots, X_n$ and a functor $F_1 : (X)_1^n \to X_0$.
2. A choice of m categories $Y_1, \cdots, Y_m$ and a functor $F_2 : (Y)_1^m \to X_i$.
3. A choice of k categories $Z_1, \cdots, Z_k$ and a functor $F_3 : (Z)_1^k \to X_j$.
4. A functor $G_1 : (X)_1^{i-1} \times (Y)_1^m \times (X)_{i+1}^n \to X_0$ and a natural isomorphism α_1 between G_1 and the obvious functor $F_1 \circ_i F_2$.
5. A functor $G_2 : (X)_1^{j-1} \times (Z)_1^k \times (X)_{j+1}^n \to X_0$ and a natural isomorphism α_2 between G_2 and the obvious functor $F_1 \circ_j F_3$.
6. A functor $H : (X)_1^{i-1} \times (Y)_1^m \times (X)_{i+1}^{j-1} \times (Z)_1^k \times (X)_{j+1}^n \to X_0$ and a natural isomorphism β between H and the obvious functor $F_1 \circ_{i,j} (F_2, F_3)$.

Similarly, a dendrex of shape $C_n \circ_i (C_m \circ_j C_k)$ consists of the following data:

1. A choice of $n+1$ categories $X_0, \cdots, X_n$ and a functor $F_1 : (X)_1^n \to X_0$.
2. A choice of m categories $Y_1, \cdots, Y_n$ and a functor $F_2 : (Y)_1^m \to X_i$.
3. A choice of k categories $Z_1, \cdots, Z_k$ and a functor $F_3 : (Z)_1^k \to Y_j$.
4. A functor $G_1 : (X)_1^{i-1} \times (Y)_1^m \times (X)_{i+1}^n \to X_0$ and a natural isomorphism α_1 between G_1 and the obvious functor $F_1 \circ_i (F_2)$.
5. A functor $G_2 : (Y)_1^{j-1} \times (Z)_1^k \times (Y)_{j+1}^m \to Y_0$ and a natural isomorphism α_2 between G_2 and the obvious functor $F_2 \circ_j F_3$.
6. A functor $H : (X)_1^{i-1} \times (Y)_1^{j-1} \times (Z)_1^k \times (Y)_{j+1}^m \times (X)_{i+1}^n \to X_0$ and a natural isomorphism β between H and the obvious functor $F_1 \circ_i (F_2 \circ_j F_3)$.

Let us now examine what is a weak 2-category $\mathcal{B}$ that has just one object (this is expected to be some kind of a monoidal category). By definition $\mathcal{B}$ is a map $\mathcal{B} : N_d(\mathcal{C}_B) \to wCat^1$ where B is a one-point category. The map $Sk_0(\mathcal{B})$ is just the choice of a category for every dendrex in $N_d(\mathcal{C}_B)_\eta$ (for a fixed η), i.e., it is simply a category M. The map $Sk_1(\mathcal{B})$ amounts to a choice of a functor

$$\gamma : M^n \to M$$

for all $n \geq 0$. We call this functor the *unbiased tensor product* of n-elements and write $(a_1 \otimes \cdots \otimes a_n)$ instead of $\gamma(a_1, \cdots, a_n)$. As a special case we include $n = 0$ which amounts to a map $I \to M$, that is the same as choosing an object in M which is called the *unit*. The map $Sk_2(\mathcal{B})$ amounts to specifying certain isomorphisms as follows. Given objects $a_1, \cdots, a_n$, an integer $0 \leq i \leq n$, and an integer $i \leq j \leq n$ there is an isomorphism

$$\begin{array}{c} (a_1 \otimes \cdots \otimes a_{i-1} \otimes (a_i \otimes \cdots \otimes a_j) \otimes a_{j+1} \otimes \cdots \otimes a_n) \\ \downarrow \\ (a_1 \otimes \cdots \otimes a_n) \end{array}$$

which is natural in each a_k (Recall that we interpret the tensor product of 0 elements to be the chosen unit and so these diagrams include unit laws). The map $Sk_3(\mathcal{B})$ provides two types of coherence constraints for these isomorphisms. To state these constraints we use a similar convention as above; for objects $a_i, \cdots, a_j$ we denote the formal sequence $a_i \otimes \cdots \otimes a_j$ by a_i^j. The first coherence constraint states the commutativity of diagrams of the sort

$$\begin{array}{ccc} (a_1^i \otimes (a_{i+1}^j) \otimes a_{j+1}^k \otimes (a_{k+1}^m) \otimes a_{m+1}^t) & \longrightarrow & (a_1^i \otimes (a_{i+1}^j) \otimes a_{j+1}^t) \\ \downarrow & & \downarrow \\ (a_1^k \otimes (a_{k+1}^m) \otimes a_{m+1}^t) & \longrightarrow & (a_1^t) \end{array}$$

where the arrows are obtained from the given unbiased compositions. The second type of coherence constraints state the commutativity of diagrams of the sort

$$\begin{array}{ccc} (a_1^i \otimes (a_{i+1}^j \otimes (a_{j+1}^k) \otimes a_{k+1}^m) \otimes a_{m+1}^t) & \longrightarrow & (a_1^i \otimes (a_{i+1}^m) \otimes a_{m+1}^t) \\ \downarrow & & \downarrow \\ (a_1^j \otimes (a_{j+1}^k) \otimes a_{k+1}^t) & \longrightarrow & (a_1^t) \end{array}$$

where the arrows are again given by the unbiased tensor products. Lemma 4.6.3 shows that this is precisely the information present in the weak 2-category $\mathcal{B}$.

Remark 4.6.4. The terminology 'unbiased' is taken from [**31**]. Leinster introduces there the notion of an unbiased monoidal category which is almost identical (and is equivalent) to the notion we arrived at here. The term unbiased refers to the explicitly given tensor products of n objects for all $n \geq 0$ rather than the more usual bias towards a 0-ary tensor product (i.e. a unit) and a binary tensor product. We note also that if one unpacks the notion of a map between weak 2-categories with one object one obtains essentially the same definition as that of a weak monoidal functor between unbiased monoidal categories [**31**].

One can similarly unpack the definition of an arbitrary weak 2-category. Of course the resulting notion will not be identical with that of bicategories (see [**3**] for a definition) but would rather be an unbiased version similar to the situation above. However, we still expect that our notion of weak 2-categories is essentially the same as bicategories in some sense to be made precise. Speculating about possible comparisons between our notion of weak n-categories and other such definitions is very difficult at best and we do not attempt one here.

We end this section by conjecturing about the Baez-Dolan stabilization hypothesis for the notion of weak n-categories just introduced. The Baez-Dolan stabilization hypothesis (see [**1**]) is a general conjecture about weak n-categories that before it can be proved for a specific definition of weak n-categories must first be interpreted and made precise for that definition. For example, Simpson [**42**] states and proves the stabilization hypothesis for Tamsamani's definition of weak n-categories. Let us first explain what the stabilization hypothesis is. Assume for the discussion that some notion of weak n-categories is fixed. A weak n-category typically consists of j dimensional cells for all $0 \leq j \leq n$, such that the 0-cells are the objects, the 1-cells are the arrows, the 2-cells are arrows between arrows and so on. For an integer $k \geq 0$, a *k-monoidal* n-category is a weak $(n+k)$-category that has just one j-cell for each $0 \leq j < k$. For low dimensions we have the following table of k-monoidal n-categories:

	n=0	n=1	n=2
k=0	sets	categories	2-categories
k=1	monoids	monoidal categories	monoidal 2-categories
k=2	commutative monoids	braided monoidal monoidal	braided monoidal 2-categories
k=3	”	symmetric monoidal categories	weakly involutory 2-categories
k=4	”	”	strongly involutory 2-categories
k=5	”	”	”

and it would appear that each column becomes more and more commutative as k increases and stabilizes at $k = n+2$. The Baez-Dolan hypothesis is that indeed for any reasonable definition of weak n-categories, each column in the table of k-monoidal n-categories stabilizes at $k = n+2$.

As for our definition of weak n-categories, we now give an interpretation of the stabilization hypothesis and take a small step towards proving it. For each set B we have the operad $\mathcal{C}_B$ such that the dendroidal set $\underline{dSet}(N_d(\mathcal{C}_B), wCat^n)$ is, by definition, the dendroidal set of weak $n+1$ categories whose set of objects is equal to the set B. Let us denote $A = N_d(\mathcal{C}_*)$, where $\mathcal{C}_*$ is $\mathcal{C}_B$ and B is a one-point set. That means that a 1-monoidal n-category is a dendrex of shape η in $\underline{dSet}(A, wCat^n)$. Now, a 2-monoidal n-category should be an $(n+2)$-category with just one object and one arrow, that means that it is a category enriched in 1-monoidal n-categories that has itself just one object. In other words a 2-monoidal n-category is a dendrex of shape η in $\underline{dSet}(A, \underline{dSet}(A, wCat^n))$. Motivated by this, it makes sense to define a $(k+1)$-monoidal n-category to be a category with one object enriched in k-monoidal n-categories. We make this precise in the following definition:

DEFINITION 4.6.5. Let $n \geq 0$ be fixed. For $k \geq 0$ We define recursively the dendroidal set $wCat^n_k$ of *k-monoidal n-categories* as follows. For $k = 0$ we set

$$wCat^n_0 = wCat^n$$

and for $k > 0$

$$wCat^n_k = \underline{dSet}(A, wCat^n_{k-1}).$$

A dendrex of shape η in $wCat^n_k$ is then called a *k-monoidal n-category.*

REMARK 4.6.6. Notice that

$$wCat^n_k = \underline{dSet}(A^{\otimes_k}, wCat^n).$$

CONJECTURE 4.6.7. *(The Baez-Dolan stabilization hypothesis for our notion of n-categories) For a fixed $n \geq 0$, we have the equality*

$$wCat^n_k \cong wCat^n_{n+2}$$

for any $k \geq n + 2$.

As a step towards a proof we make the following conjecture:

CONJECTURE 4.6.8. *For any $n \geq 0$ the dendroidal set $wCat^n_n$ is a strict inner Kan complex.*

REMARK 4.6.9. Note that we already know from the general theory that $wCat^n_n$ is an inner Kan complex.

PROPOSITION 4.6.10. *The conjecture just stated implies the Baez-Dolan stabilization conjecture.*

PROOF. Fix $j > 2$. Using Remark 4.6.6 we proceed as follows. We have to prove that $\underline{dSet}(A^{\otimes_{n+j}}, wCat^n) = \underline{dSet}(A^{\otimes_{n+2}}, wCat^n)$ where we assume that $\underline{dSet}(A^{\otimes_n}, wCat^n)$ is a strict inner Kan complex. By Theorem 3.5.12 there is thus an operad $\mathcal{P}$ such that $\underline{dSet}(A^{\otimes_n}, wCat^n) = N_d(\mathcal{P})$. We now have:

$$\underline{dSet}(A^{\otimes_{n+j}}, wCat^n) = \underline{dSet}(A^{\otimes_j}, \underline{dSet}(A^{\otimes_n}, wCat^n)) = \underline{dSet}(A^{\otimes_j}, N_d(\mathcal{P}))$$

which by adjunction is

$$\underline{Operad}(\tau_d(A^{\otimes_j}), \mathcal{P}).$$

However, A is actually the dendroidal nerve of the symmetric operad As describing associative monoids. By Lemma 3.1.2 we have:

$$\tau_d(A^{\otimes_j}) = \tau_d(N_d(As)^{\otimes_j}) \cong As \otimes_{BV} As \otimes_{BV} \cdots \otimes_{BV} As = As^{\otimes_j}.$$

It is known ([**8**]) that for $j \geq 2$

$$As^{\otimes_j} = Comm,$$

the operad describing commutative monoids and the result now follows. $\square$

4.7. Quillen model structure on *dSet*

We end this chapter and the thesis by giving a conjecture about the existence of a Quillen model structure on $dSet$. We show that if this conjecture is true then X algebras in E for a normal X and an inner Kan complex E have a nice homotopy invariance property. We expect this model structure to be very important in the general future theory of dendroidal sets as well as for the theory of A_∞-spaces, A_∞-categories, weak n-categories, and general up-to-homotopy structures.

To formulate the conjecture about the existence of such a Quillen model structure let us first recall the Joyal model structure on the category of simplicial sets.

First, recall the nerve-realisation adjunction $sSet \underset{N}{\overset{\tau}{\rightleftarrows}} Cat$. For a given simplicial set X, the category $\tau(X)$ is commonly called the *fundamental category* of X and is denoted by $\tau_1(X)$ (terminology taken from [**23**]). We can now define the functor $\tau_0 : sSet \to Set$ where $\tau_0(X)$ is the set of isomorphism classes of objects of the category $\tau_1(X)$. Given two simplicial sets X and Y we write

$$\tau_0(X, Y) = \tau_0(\underline{sSet}(X, Y)).$$

DEFINITION 4.7.1. A *weak categorical equivalence* is a map $f : X \to Y$ of simplicial sets with the property that for any inner Kan simplicial set K (a quasi-category in the terminology of [**24**]) the induced map

$$\tau_0(Y, K) \to \tau_0(X, K)$$

is an isomorphism of sets.

THEOREM 4.7.2. *The category sSet of simplicial sets admits a cartesian Quillen model structure where the weak equivalences are the weak categorical equivalences and the cofibrations are the monomorphisms. Under this model structure the fibrant objects are precisely the inner Kan simplicial sets.*

The proof, and much more theory related to this model structure, which we call the Joyal model structure on simplicial sets, will appear in Joyal's book [**24**].

Mimicking the definition of weak categorical equivalences we proceed as follows. The functor $\tau_d : dSet \to Operad$ gives rise to a functor $\tau_{0_d} : dSet \to Set$ defined for a dendroidal set X by $\tau_{0_d}(X) = \tau_0(i^*(X))$. For dendroidal sets X and Y we define

$$\tau_{0_d}(X, Y) = \tau_{0_d}(\underline{dSet}(X, Y)).$$

DEFINITION 4.7.3. Given two dendroidal sets X and Y, we call a map $f : X \to Y$ a *weak operadic equivalence* if for any inner Kan dendroidal set K the induced map

$$\tau_0(Y, K) \to \tau_0(X, K)$$

is an isomorphism of sets.

Following [**9**] (page 320) we make the following definition:

DEFINITION 4.7.4. Let $f : X \to Y$ be a monomorphism between dendroidal sets. We call f *normal* if for every dendrex $t \in Y_T$ that does not factor through f the only isomorphism of T that fixes t is the identity.

This definition extends Definition 2.6.6 in the sense that a dendroidal set is normal if, and only if, the inclusion $\phi \to X$ is normal.

Remark 4.7.5. Clearly any map between normal dendroidal sets is normal. In particular, if X is normal then the inclusion $Sk_k(X) \to X$ is normal for all $k \geq 0$.

Conjecture 4.7.6. *The category dSet of dendroidal sets admits a Quillen model structure where the weak equivalences are the weak operadic equivalences and the cofibrations are the normal monomorphisms. Furthermore, with the tensor product of dendroidal sets, this model structure is a monoidal model category. Under this model structure the fibrant objects are the inner Kan complexes and the cofibrant objects are the normal dendroidal sets. Moreover, in the diagram*

$$\begin{array}{ccc} Cat & \underset{j^*}{\overset{j_!}{\rightleftarrows}} & Operad \\ {\scriptstyle\tau}\uparrow\downarrow{\scriptstyle N} & & {\scriptstyle\tau_d}\uparrow\downarrow{\scriptstyle N_d} \\ sSet & \underset{i^*}{\overset{i_!}{\rightleftarrows}} & dSet \end{array}$$

from Theorem 3.1.4 we expect all of the adjunctions to be Quillen adjunctions.

Remark 4.7.7. We expect the fact that the inner Kan complexes form an exponential ideal in $dSet$ (Corollary 3.6.4) to play a fundamental role in proving the conjecture. We also expect that the functor $K : dSet \to dSet$ given in Lemma 3.7.4, is actually a fibrant replacement functor in this conjectured model structure. Note that we have already shown that $j_!$ and j^* form a Quillen adjunction (Lemma 1.6.5. In **[24]** it is shown that τ and N also form a Quillen adjunction. The fact that $i_!$ preserves cofibrations is obvious and in fact it is quite simple to show that it preserves weak equivalences too, so that $i_!$ and i^* also form a Quillen adjunction. The difficult part is thus proving that τ_d and N_d form a Quillen adjunction.

Assuming the conjecture above holds, we can prove a homotopy invariance property for algebras in an inner Kan complex. Let us first clarify what we mean by such an invariance property. First, recall some terminology from **[23]**. If S is a quasi-category (i.e., an inner Kan simplicial set) a 1-simplex s is called a *weak equivalence* if its image under $\tau : sSet \to Cat$ is invertible. Now, given any 1-simplex $s : \Delta[1] \to S$, we have that s is a weak equivalence if, and only if, it can be extended to a map from S^∞:

$$\begin{array}{ccc} \Delta[1] & \xrightarrow{s} & S \\ \downarrow & \nearrow & \\ S^\infty & & \end{array}$$

where S^∞ is the infinite dimensional sphere, i.e., the nerve of the category H which is the interval in the folk model structure on Cat. The vertical map $\Delta[1] \to S^\infty$ is a trivial cofibration in the Joyal model structure. Extending this to dendroidal sets we have:

Definition 4.7.8. Let X be an inner Kan complex. A dendrex $x \in X_{L_1}$ is called a *weak equivalence* if $i^*(x)$ is a weak equivalence in the inner Kan simplicial set $i^*(X)$.

Let E and X be two dendroidal sets and consider the dendroidal set $Alg(X, E)$ of X-algebras in E. Such an algebra is thus a map $A : X \to E$. The map $Sk_0(A) : Sk_0(X) \to Sk_0(E)$ consists of a choice of elements in E_η and we think of A as defining an algebraic structure on the element $Sk_0(A)$. Suppose now that E is an inner Kan complex and A is a fixed X-algebra in E. We say that the algebraic structure on $Sk_0(A)$ given by A has the *homotopy invariance property* if given another choice of elements in X_η, given as a map $A' : Sk_0(X) \to Sk_0(E)$, and for each $w \in E_\eta$ a weak equivalence

$$f_w : A(w) \to A'(w)$$

there is an E-algebra structure on A' and a map of E-algebras $f : A \to A'$ that extends the given f_w.

THEOREM 4.7.9. *Let X be a normal dendroidal set and E an inner Kan complex. If conjecture 4.7.6 holds then all E-algebras in X have the homotopy invariance property.*

PROOF. Assume that we have an algebra $A : X \to E$ and a choice of weak equivalences $f_w : A(w) \to A'(w)$ as above. Since a weak equivalence in X is the same as a map $i_!(S^\infty) \to X$ it follows that the choice of the maps f_w produces a map $f : i_!(S^\infty) \to \underline{dSet}(Sk_0(X), E))$. We now have the following commutative diagram

$$\begin{array}{ccc} \Omega[\eta] & \xrightarrow{A} & \underline{dSet}(X, E) \\ \downarrow & & \downarrow \\ i_!(S^\infty) & \xrightarrow{f} & \underline{dSet}(Sk_0(X), E) \end{array}$$

where the vertical arrow on the right is induced by the cofibration $Sk_0(X) \to X$, (see Remark 4.7.5 for why this is a cofibration) and is thus a fibration (by the push-out product axiom). Notice that a diagonal filler for this diagram corresponds to an element $\bar{f} \in \underline{dSet}(X, E)_{L_1}$, and thus to a map of algebras $\bar{f} : A \to A'$ which extends f - which is what we would like to show. It therefore suffices to show that this diagram has a diagonal filler, which will follow if the left vertical map is a trivial cofibration. Notice that this map is actually the image under $i_! : sSet \to dSet$ of the map $\Delta[1] \to S^\infty$, which is a trivial cofibration in the Joyal model structure on simplicial sets. It thus follows from the assumption that $i_!$ is a left Quillen functor that $\Omega[\eta] \to i_!(S^\infty)$ is indeed a trivial cofibration, which thus completes the proof. □

REMARK 4.7.10. This homotopy invariance property is related to a similar property (Theorem 3.5 in [**4**]) for algebras over cofibrant operads.

Bibliography

[1] J. Baez, J. Dolan, Categorification, Contemp. Math. AMS, 230 (1998), 1-36.

[2] M.A. Batanin, The combinatorics of iterated loop spaces, preprint, math.CT/0301221.

[3] J. Bénabou, Introduction to bicategories, in J. Bénabou et al. (eds.), Reports of the Midwest Category Seminar, Lecture Notes in Mathematics 47, Springer (1967).

[4] C. Berger, I. Moerdijk, Axiomatic homotopy theory for operads, Comment. Math. Helv. 78 (2003), 805-831.

[5] C. Berger, I. Moerdijk, The Boardman-Vogt resolution of operads in monoidal model categories, Topology 45, (2006), 807-849.

[6] C. Berger, I. Moerdijk, Resolution of coloured operads and rectification of homotopy algebras, preprint, math.AT/0512576.

[7] J.M. Boardman, R.M. Vogt, Homotopy Invariant Algebraic Structures on Topological Spaces, Lecture Notes in Mathematics, Vol. 347, Springer, Berlin, (1973).

[8] J.M. Boardman, R.M. Vogt, Tensor products of theories and applications to infinite loop spaces. J. Pure Appl. Algebra 14 (1979), no. 2, 117-129.

[9] C.D. Cisinski, Les préfaisceaux comme modèles des types d'homotopie, Astérisque 308, Société mathématique de France, (2006).

[10] J.M. Cordier, T. Porter, Vogt's theorem on categories of homotopy coherent diagrams, Math. Proc. Cambridge Philos. Soc. 100, (1986), 65-90.

[11] B.J. Day, On Closed Categories of Functors, Report of the Midwest Category Seminar IV, Lecture Notes in Mathematics, Vol. 137, Springer, Berlin, (1970).

[12] G. Dunn, Tensor product of operads and iterated loop spaces, J. Pure and Applied Algebra 50 (1988) 237-258.

[13] W.G. Dwyer, J. Spalinsky, Homotopy theories and model categories, Handbook of algebraic topology, 73-126, North-Holland, Amsterdam (1995).

[14] S. Eilenberg, S. Mac lane, General theory of natural equivalences, Trans. Amer. Math. Soc. 58, (1945) 231-294.

[15] A.D. Elmendorf, M.A. Mandell, Rings, modules, and algebras in infinite loop spaces, Adv. Math. 205 (2006), no. 1, 163-228.

[16] P. Gabriel, M. Zisman, Calculus of Fractions and Homotopy Theory, Ergebnisse der Mathematik, Vol. 35, Springer-Verlag, (1967).

[17] E, Getzler, J.D.S. Jones, Operads, homotopy algebra and iterated integrals for double loop spaces, preprint, hep-th/9403055.

[18] V. Ginzburg, M.M. Kapranov, Koszul duality for operads, Duke Math. J. 76 (1994), 203-272.

[19] P.G. Goerss, J.F. Jardine, Simplicial Homotopy Theory, Progress in Mathematics Vol. 174, (1999), Birkhäuser.

[20] P.S. Hirschhorn, Model Categories and Their Localizations, Math. Surveys and Monographs, Vol. 99, Amer. Math. Soc., (2003).

[21] M. Hovey, Model Categories, AMS Mathematical Survery and Monographs, Vol. 63, American Mathematical Society, (1999).

[22] A. Joyal, M. Tierney, Strong stacks and classifying spaces, Category Theory (Como, 1990), 213-236.

[23] A. Joyal, Quasi-categories and Kan complexes, Journal of Pure and Applied Algebra Vol. 175, (2002).

[24] A. Joyal, Theory of Quasi-Categories, in preparation.

[25] B. Keller, A_∞-algebras, modules and functor categories, Trends in representation theory of algebras and related topics, 67-93 Contemp. Math., 406, Amer. Math. Soc., Providence, RI, (2006).

[26] G.M. Kelly, Basic Concepts of Enriched Category Theory, London Mathematical Society Lecture Notes Series, Vol. 64, Cambridge University Press, Cambridge, (1982).

[27] M. Kontsevich, Y. Soibelman, Deformations of algebras over operads and Deligne's conjecture, Math. Phys, Stud. 21 (2000), 255-307.

[28] S. Lack, Homotopy theoretic aspects of 2-monads, preprint, math.CT/0607646.

[29] T. Leinster, A survey of definitions of n-category, Theory and Applications of Category Theory 10, (2002), 1-70.

[30] T. Leinster, Generalized enrichment of categories, Journal of Pure and Applied Algebra 168, (2002), no. 2-3, 391-406.

[31] T. Leinster, Higher Operads, Higher Categories, London Mathematical Society Lecture Note Series 298, Cambridge University Press, (2004).

[32] J.L. Loday, J. Stasheff, A. Voronov, (eds.) (1997), Operads: Proceedings of Renaissance Conference, AMS.

[33] S. Mac Lane, I. Moerdijk, Sheaves in Geometry and Logic, Universitext, Springer-Verlag, (1994).

[34] S. Mac Lane, Categories for the Working Mathematician, Graduate Texts in Mathematics, Vol. 5, Springer-Verlag, (1998), second edition.

[35] M. Markl, S. Shnider, J. Stasheff, Operads in Algebra, Topology, and Physics, Mathematical Surveys and Monographs 96, (2002), AMS.

[36] J.P May, The Geometry of Iterated Loop spaces, Lectures Notes in Mathematics 271, Springer, Berlin, (1972).

[37] I. Moerdijk, I. Weiss, Dendroidal sets, accepted for publication in Algebraic and Geometric Topology.

[38] I. Moerdijk, I. Weiss, On weak Kan complexes in the category of dendroidal sets, preprint, math.AT/0701295.

[39] J. Nichols-Barrer, On Quasi-Categories as a Foundations for Higher Algebraic Stacks, MIT PhD thesis, (2007).

[40] D. Quillen, Homotopical Algebra, Lecture Notes in Mathematics, Vol. 43, Springer, Berlin, (1967).

[41] C. Rezk, A model category for categories, unpublished, available: http://www.math.uiuc.edu/~rezk/papers.html

[42] C. Simpson, On the Breen-Baez-Dolan stabilization hypothesis for Tamsamani's weak n-categories, math.CT/9810058.

[43] J. Stasheff, Homotopy associativity of H-spaces I, Trans. Amer. Math. Soc. 108 (1963), p. 275-292.

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Samenvatting

In dit proefschrift wordt het begrip boomachtige verzameling geïntroduceerd. Eenvoudig gezegd, kan dat begrip als een veralgemenisering van het concept simpliciale verzameling gezien worden. Maar terwijl simpliciale verzamelingen vooral zeer geschikt zijn voor het bestuderen van topologische ruimtes, zijn boomachtige verzamelingen ergens anders nuttig voor.

Simpliciale verzamelingen hebben een sterk verband met categorieën. De wiskundige Grothendieck heeft aan elke kleine categorie een simpliciale verzameling gerelateerd die de nerf van die categorie heet. Deze constructie levert een functor van de categorie van alle kleine categorieën naar de categorie van simpliciale verzamelingen. Een simpliciale verzameling die de nerf van een categorie is kan gekarakteriseerd worden met behulp van een zekere opvul-conditie. Namelijk, een simpliciale verzameling is de nerf van een categorie dan en slechts dan als elke inwendige hoorn in de simpliciale verzameling op precies één manier opgevuld kan worden. Als we de eenduidigheid van de opvulling laten vallen dan komen we aan het begrip quasi-categorie dat recentelijk door Joyal is bestudeerd.

Een ander concept van veel belang in dit proefschrift is dat van een operade. Operaden werden in de jaren '70 geïntroduceerd in het gebied van de algebraïsche topologie en zijn sindsdien nogal populair geworden, voornamelijk in het onderzoeken van verzwakte algebraïsche structuren. Ofschoon de fundamentelen van de operadentheorie algemeen bekend zijn, geven wij in dit proefschrift een andere benadering tot operaden. Wij beschouwen operaden als een veralgemenisering van het begrip categorie op de volgende manier. In een categorie heeft elke pijl een domein en een codomein die allebei een object van de categorie moeten zijn. Als we in plaats van een object een rij van objecten toestaan als het domein van een pijl dan komen we aan het concept van operade.

De nerf-constructie van categorieën en het gezichtspunt van operaden als een uitbereiding van categorieën leidt tot de vraag of men de nerf van een operade kan definiëren. Om deze vraag te beantwoorden definiëren we de categorie Ω. De objecten van deze categorie zijn zekere niet-planaire gewortelde bomen en de pijlen tussen twee bomen zijn de pijlen tussen zekere operaden die met de bomen geassocieerd zijn. Deze categorie bevat de simpliciale categorie Δ als een volledige subcategorie. De categorie van boomachtige verzamelingen is dan de categorie van preschoven op Ω en de nerf-functor van categorieën kan uitgebreid worden tot een boomachtige nerf-functor van operaden. Het karakteriseren van simpliciale verzamelingen die de nerf van een categorie zijn kan veralgemeniseerd worden om de nerven van operaden te karakteriseren. Preciezer gezegd, het begrip van een hoorn blijft zinvol ook in de context van boomachtige verzamelingen en dus kan over opvulcondities in een boomachtige verzameling gesproken worden. Een boomachtige verzameling is dan de nerf van een categorie dan en slechts dan als elke inwendige hoorn op precies één manier opgevuld kan worden. Als we de eenduidigheid laten vallen dan komen we aan het begrip inwendig Kan complex. In het bijzonder, is elke quasi-categorie ook een inwendig Kan complex.

Quasi-categorieën worden soms beschouwd als een speciale vorm van zwakke ω-categorieën. Op dezelfde manier kunnen inwendige Kan complexen als zwakke ω-operaden gezien worden. Inwendige Kan complexen genieten een rijke structuur. Er is een interne notie van homotopie en van samenstelling en deze samenstelling is associatief op homotopie na. Bovendien is de categorie van boomachtige verzamelingen een monoïdaal gesloten categorie en de inwendige Kan complexen vormen een exponentieel ideaal met betrekking tot normale boomachtige verzamelingen. De algemene theorie van boomachtige verzamelingen is zeer geschikt om operaden in de context van homotopietheorie te onderzoeken. We definiëren een verfijning van de boomachtige nerf functor voor het geval dat de operade verrijkt is in een monoïdaal model-categorie. Deze functor heet de homotopie-coherente nerf functor. We bewijzen dat onder bepaalde voorwaarden de homotopie-coherente nerf van een operade een inwendig Kan complex is.

Een toepassing van de theorie is het definiëren van hoger dimensionele categoriën. We geven een algemene definitie van een categorie verrijkt in een boomachtige verzameling die geïtereerd kan worden om een definitie van verrijkt n-dimensionele categorieën te krijgen. Als we dan dit proces uitvoeren met de homotopie-coherente nerf van de categorie van kleine categorieën uitgerust met de folkloristische modelstructuur, krijgen we een definitie van zwakke n-categorieën. Uit de algemene theorie volgt dan meteen een definitie van functoren tussen deze categorieën, een notie van homotopie tussen zulke functoren en een samenstelling van functoren.

Bepaalde resultaten in dit werk zijn veralgemeniseringen van bekende resultaten. Bijvoorbeeld: Het bewijs dat de inwendige Kan complexen een exponentieel ideaal zijn, levert als speciaal geval een bewijs dat quasi-categorieën een exponentieel ideaal in de categorie van simpliciale verzamelingen zijn. Dit resultaat werd recentelijk door Joyal bewezen maar ons bewijs is anders. Ons bewijs dat onder bepaalde voorwaarden, de homotopie-coherente boomachtige nerf van een operade een inwendig Kan complex is, levert, samen met andere stellingen, een bewijs van een resultaat van Cordier en Porter.

Het proefschrift eindigt met een bespreking van een mogelijke modelstructuur op de categorie van boomachtige verzamelingen. We bewijzen dat als dit inderdaad een modelstructuur is, er dan een zekere homotopie-invariantie eigenschap geldt voor algebraïsche structuren op elementen van een inwendig kan complex. Deze eigenschap is vergelijkbaar met een homotopie invariantie eigenschap in modelcategoriën van Berger en Moerdijk. We vermoeden dat de theorie van boomachtige verzamelingen veel toepassingen zal vinden in onder andere, algebraïsche (en abstracte) homotopietheorie, hoger-dimensionele categorieën en A_∞-ruimtes, en een nieuwe benadering tot het bestuderen van operaden oplevert.

Acknowledgements

Numerous people contributed, directly and indirectly, to the completion and development of this thesis, some of whom I would particularly like to mention.

My supervisor, Ieke Moerdijk, who always seemed to know the answers to questions I didn't even ask yet. The collaboration with him that led to the formation of this thesis was an extraordinary experience for which I am very grateful. His guidance and the countless hours we spent discussing ideas as well as his always excellent remarks shaped to a large extent both the content and form of this thesis.

Emanuel Farjoun, who guided me in my very first steps into mathematical research and with whom I had countless enthusiastic excursions to categorical realms. Steve Shnider, with whom I had many useful and enjoyable conversations that can be said to have spawned the ideas that led to this thesis. Paul-André Melliès, with whom I had a wonderful time in Paris and who was one of the first people to hear about my ideas in pre-baked form. His keen questions aided immensely in fine-tuning the theory. Immeasurable gratitude goes to Michael Shulman who did a wonderful job reading early versions both of the thesis and the articles on which it is based, picking out mistakes and suggesting many useful corrections.

My research colleagues at the university of Utrecht, Alex, Andor, Benno, Camilo, Claire, David, Erik, Federico, Javier, Georgio, Jaap, and Marius for providing a wonderful work environment. Between mathematical discussions and games of table-tennis and chess, they each contributed in her or his unique way to the wonderful atmosphere that was always present in our work group. I particularly wish to thank Benno with whom I had the pleasure of sharing an office and to whom I owe much of my proficiency in Dutch and Andor with whom I had many discussions that led to important insights.

Other colleagues at the university, Jean, Martijn, Patty, Paul, and Thea, who were always very glad to help with any kind of problem. These people are also partly responsible for improving my Dutch skills and I thank them for their encouragement.

My students, both in The Netherlands and in Israel, for their attendance and desire to learn which forced me to occasionally take some time off from research. Especially to my students in The Netherlands, thank you very much for your patience with my struggles in Dutch.

Very special and warm thanks go to my mother who by cutting ties with her own past provided me with a better future. For giving me the freedom and guidance to make my own decisions and for always supporting me with my choices.

Rahel, whose endless support and trust in me were invaluable for, among many things, the completion of this thesis. But far more importantly thank you for making everyday life so much more enjoyable!

Curriculum vitae

The author of this thesis was born on January 24th 1977 in Jerusalem, Israel. In 1993, instead of going to high school, he started following computer science courses at the Open University in Israel. After a period of contemplation between computer science and mathematics he decided, in 1998, to enroll for the B.Sc. programme of mathematics at the Hebrew University in Jerusalem. During his bachelors studies he worked as a private tutor for mathematics students and as a corrector in various courses, and obtained his B.Sc. degree cum laude in 2001. He then continued for the M.Sc. programme at the Hebrew University under the supervision of Prof. Farjoun, carrying out research in the field of operad theory. During his masters he worked as a teaching assistant in the course "Infinitesimal Calculus", was part of a team that wrote a textbook for the same course, and was active in an Arab-Israeli peace movement and in an organization for the demilitarization of the Israeli society. He won two years in a row the Klein Scholarship for excellence in teaching and obtained his M.Sc. degree cum laude in 2003. He then started his position as AiO at Utrecht Universiteit under the supervision of Prof. Moerdijk. He worked as a teaching assistant in various courses and was a teacher of the introductory course "Wat is Wiskunde?" during the last two years of his employment at the university.