CHAPTER 4

Enriched operads and dendroidal sets

This chapter explores how dendroidal sets and their theory fit in with the theory of $\mathcal{E}$-enriched operads. Enriched operads are used to study deformations of algebraic structures. The idea is that if certain algebraic structures are algebras for a given enriched operad $\mathcal{P}$, then by resolving that operad one obtains a new enriched operad whose algebras are $\mathcal{P}$-algebras *up-to-homotopy*. The cradle of such constructions is in topology, notably the W-construction of Boardman and Vogt **[7]** which is recalled below, May's work on loop spaces **[36]**, and Stasheff's work on H-spaces **[43]** (where, even though operads were not yet invented, the author essentially provided an example of an operad enriched in Top). The Boardman-Vogt W-construction can be generalized to monoidal model categories more general than the category of topological spaces, as long as these categories have a suitable notion of an interval. Berger and Moerdijk in a series of papers **[4, 5, 6]** establish a Quillen model structure on enriched operads with a fixed set of objects and give a detailed construction of the Boardman-Vogt W-construction in this more general setting, as well as proving that this Boardman-Vogt construction, when applied to an enriched operad $\mathcal{P}$, yields a cofibrant replacement for $\mathcal{P}$. Specifically for operads enriched in chain complexes, the cobar resolution is a well-known method to resolve operads (e.g.,**[18, 27, 35]**).

Below, the generalized W-construction is used in order to establish the connection between enriched operads and dendroidal sets. As motivation, A_∞-spaces are considered in order to illustrate the general problem of up-to-homotopy algebras of operads. Next, the original Boardman-Vogt W-construction is presented as well as its generalized form. That construction is then utilized to construct the homotopy coherent dendroidal nerve of a given enriched operad thus establishing the relation between enriched operads and dendroidal sets. Following is an extension of the Grothendieck construction from operads to dendroidal sets which is needed in order to apply the general theory to define categories enriched in a dendroidal set. The chapter ends by introducing weak n-categories and some basic properties of them.

4.1. Case study: A_∞-spaces

In this section we look at A_∞-spaces in order to exemplify the kind of problems occurring in the general theory of up-to-homotopy algebras. We do not intend this to be an accurate account of the theory of A_∞-spaces, but rather use it to illustrate a point. For more details and a very precise account of the evolution of ideas and notions, see the introduction in **[35]**. We thus allow ourselves to be somewhat less precise for the sake of greater clarity of the general presentation of the ideas involved.

Consider the following situation. Let X be a topological monoid, i.e., X is a topological space together with a continuous binary operation $\cdot : X \times X \to X$ which makes the set underlying X into a monoid (X, e). If we are now given a homeomorphism $f : X \to Y$ then one can (obviously) transfer the monoid structure from X to Y, in the sense that the function

$$Y \times Y \xrightarrow{f^{-1} \times f^{-1}} X \times X \xrightarrow{\cdot} X \xrightarrow{f} Y$$

is (of course) continuous and makes $(Y, f^{-1}(e))$ into a monoid. Assume now that we are given a weak equivalence $g : X \to Y$, can we still transfer the monoid structure from X to Y?

We can attempt to proceed as follows. Assume g has a homotopy inverse $h : Y \to X$ and choose homotopies

$$H_1 : hg \to id_X$$

and

$$H_2 : gh \to id_Y.$$

We can now define a binary operation on Y, namely

$$Y \times Y \xrightarrow{h \times h} X \times X \xrightarrow{\cdot} X \xrightarrow{g} Y.$$

Let us check whether this operation on Y is associative. Let then $a, b, c \in Y$. On the one hand

$$(ab)c = g(h(a)h(b))c = g(h(g(h(a) \cdot h(b))) \cdot h(c))$$

and on the other hand

$$a(bc) = ag(h(b)h(c)) = g(h(a) \cdot h(g(h(b) \cdot h(c))))$$

so that if $gh \neq id$ or $hg \neq id$ then, in general, equality will not hold. However, the homotopy $H_2 : gh \to id$ specifies for each $y \in Y$ a path $\gamma : [0,1] \to Y$ from $gh(y)$ to y. For $y = h(a)h(b)$ we thus obtain the path $\gamma' : [0,1] \to Y$ defined by

$$\gamma'(t) = g(\gamma(t) \cdot h(c))$$

which is thus a path from $(ab)c$ to $g(h(a) \cdot h(b) \cdot h(c))$. Similarly we can obtain another path from $a(bc)$ to $g(h(a) \cdot h(b) \cdot h(c))$ which together with the first path implies the existence of a path from $a(bc)$ to $(ab)c$. We see thus that the operation need not be associative but it is associative up to homotopy, in the sense that the paths just constructed fit together to form a homotopy between the two functions from $Y \times Y \times Y$ to Y obtained from the binary operation. This observation begins to unfold the kind of structure that can be induced on Y, given a monoid structure on X *and* a chosen homotopy inverse h of g *and* the chosen homotopies realising that homotopy inverse. To fully describe this structure on Y one needs to also consider the various ways to use the binary operation to form functions from $Y \times Y \times Y \times Y$ into Y. These functions can be related to each-other using the given homotopies and the newly created associativity homotopies, and furthermore there is then a natural choice of a homotopy between these homotopies (a so called higher associativity condition). In general, one must consider all possible maps $Y^n \to Y$ for all $k \geq 0$ obtained by the binary operation, and at each stage some new higher associativity relation will emerge.

Evidently, the resulting structure is quite complicated and some way to manage that complexity is needed. In **[43]** Stasheff gives a description of the structure on

Y by means of certain spaces which parametrize all of the various n-ary operations (obtained by repeatedly using the binary operation) as well as homotopies between such operations *and* homotopies between such homotopies and so on. These parametrizing spaces, $\{K_n\}_{n=0}^{\infty}$, were later named associahedra (see **[35]** for a short history of these spaces and their name) and were redefined such that each K_n, with $n \geq 2$, is a convex set in $\mathbb{R}^{n-2}$.

We describe now the first few associahedra. The first space, K_0, is just a point. Since K_0 parametrizes the 0-ary operations, i.e., constants, it implies that Y has one single constant. The next space is K_1 which is also a point, implying there is just one 1-ary operation on Y, namely the identity. The space K_2 is still just a point, which corresponds to the binary operation present on Y. Things get more complicated in the next stage since this is where homotopies start playing a role. The space K_3 is the space $[-1, 1]$. The two endpoints represent the two ternary operations obtained from the binary operation. The entire space K_3 thus parametrizes a homotopy between these two ternary operations. The next space, K_4, is a pentagon. Each of its five vertices corresponds to each of the five 4-ary operations $Y^4 \to T$ obtainable from the binary operation, and each of the sides corresponds to a homotopy between the two operations at the endpoints. The whole pentagon thus corresponds to a higher homotopy relation between the homotopies on its boundary. Things get more and more complicated as we move up in dimensions, yet a concrete definition of all the spaces K_n is possible.

4.2. The Boardman-Vogt W-construction

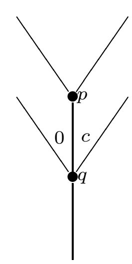
We now address a much more general question, motivated by the above discussion. Let $\mathcal{P}$ be a topological operad. Given a functor $F : \mathcal{P} \to Top$, which we think of as an algebraic structure, and for each $p \in ob(\mathcal{P})$ a homotopy equivalence $Fp \to Gp$ to some space Gp, what is the algebraic structure present on the spaces $\{Gp\}_{p \in ob(\mathcal{P})}$?

We will answer that question in the form of the Boardman-Vogt W-construction. A detailed account (albeit in a slightly different language than that of operads) of that construction can be found in **[7]** so we allow ourselves a more expository presentation aiming at explaining the ideas important for us. The W construction is a functor $W : Operad(Top) \to Operad(Top)$ equipped with a natural transformation (an augmentation) $W \to id$. So, with each topological operad $\mathcal{P}$ there is associated a topological operad $W\mathcal{P}$ and a map of operads $W\mathcal{P} \to \mathcal{P}$. Functors $W\mathcal{P} \to Top$ are then regarded as up-to-homotopy functors from $\mathcal{P}$ to $\mathcal{E}$ and are said to describe up-to-homotopy $\mathcal{P}$-algebras (or simply weak algebras). In that context, a functor $\mathcal{P} \to Top$ is referred to as a strict functor and an ordinary $\mathcal{P}$-algebra as a strict one. The augmentation implies the existence of a functor

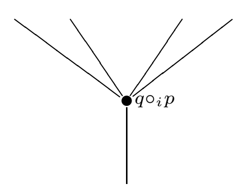
$$\underline{Operad(Top)}(\mathcal{P}, Top) \to \underline{Operad(Top)}(W\mathcal{P}, Top)$$

which views any strict algebra $\mathcal{P} \to Top$ as a weak one.

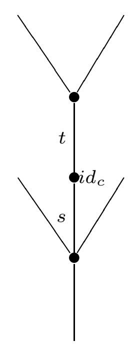
For a topological operad $\mathcal{P}$ we now describe the operad $W\mathcal{P}$. This is essentially the construction presented in **[7]**. For simplicity let us assume that the operads are planar, that is we describe a functor taking a planar operad enriched in Top to another such planar operad. The objects of $W\mathcal{P}$ are the same as those of $\mathcal{P}$. To describe the arrows spaces we consider standard planar trees whose edges are labelled by objects of $\mathcal{P}$ and whose vertices are labelled by operations of $\mathcal{P}$ according to the rule that the objects labelling the input edges of a vertex are equal (in their natural order) to the input of the operation labelling that vertex and similarly the object labelling the output of the vertex is the output object of the operation at the vertex. Moreover, each inner edge in such a tree is given a length $0 \leq t \leq 1$. For objects $p_0, \cdots, p_n \in ob(W\mathcal{P})$ let $A(p_1, \cdots, p_n; p_0)$ be the topological space whose underlying set is the set of all such standard planar labelled trees $\bar{T}$ for which the leaves of $\bar{T}$ are labelled by $p_1, \cdots, p_n$ (in that order) and the root of $\bar{T}$ is labelled by p_0. The topology on $A(p_1, \cdots, p_n; p_0)$ is the evident one induced by the topology of the arrows spaces in $\mathcal{P}$ and the standard topology on the unit interval $[0, 1]$. The space $W\mathcal{P}(p_1, \cdots, p_n; p_0)$ is the quotient of $A(p_1, \cdots, p_n; p_0)$ obtained by the following identifications. If $\bar{T} \in A(p_1, \cdots, p_n; p_0)$ has an inner edge e whose length is 0 then we identify it with the tree $\bar{T}/e$ obtained from $\bar{T}$ by contracting the edge e and labelling the newly formed vertex by the corresponding $\circ_i$-composition of the operations labelling the vertices at the two sides of e (the other labels are as in $\bar{T}$). Thus pictorially we have that locally in the tree a configuration:



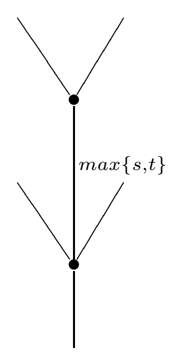
is identified with the configuration:



where the labels of the edges were neglected. Another identification is for a tree $\bar{S}$ with a unary vertex v labelled by an identity. We identify such a tree with the tree $\bar{R}$ obtained by removing the vertex v and identifying its input edge x with its output edge y. The length assigned to the new edge is determined as follows. If it is an outer edge then it has no length. If it is an inner edge then it is assigned the maximum of the lengths of x and y (where if either x or y does not have a length, i.e., it is an outer edge, then its length is considered to be 0). The labelling is as in $\bar{S}$ (notice that the label of the newly formed edge is unique since v was labelled by an identity which means that its input and output were labelled by the same object). Pictorially, this identification identifies the labelled tree

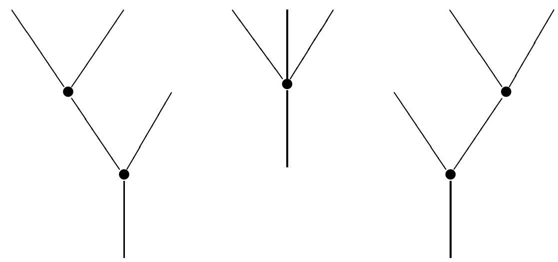


with the tree:



where we neglected the labels of the edges. The composition in $W\mathcal{P}$ is given by grafting such labelled trees, giving the newly formed inner edge length 1.

EXAMPLE 4.2.1. Let $\mathcal{P}$ be the planar operad with a single object and a single n-ary operation in each arity. We consider $\mathcal{P}$ to be a discrete operad in Top. It is easily seen that a functor $\mathcal{P} \to Top$ corresponds to a topological monoid. Let us now calculate the first few arrows spaces in $W\mathcal{P}$. Firstly, $W\mathcal{P}$ has too just one object. We thus use the notation of classical operads, namely $W\mathcal{P}(n)$ for the space of operations of arity n. Clearly $W\mathcal{P}(0)$ is just a one-point space. The space $W\mathcal{P}(1)$ consists of labelled trees with one input. Since in such a tree the only possible label at a vertex is the identify, the identification regarding identities implies that $W\mathcal{P}(1)$ is again just a one-point space. In general, since every unary vertex in a labelled tree in $W\mathcal{P}(n)$ can only be labelled by the identity, and those are then identified with trees not containing unary vertices, it suffices to only consider trees with no unary vertices at all. We call such trees regular. To calculate $W\mathcal{P}(2)$ we need to consider all regular trees with two inputs, but there is just one such tree, the 2-corolla, and it has no inner edges, thus $W\mathcal{P}(2)$ is also a one-point space. Things become more interesting when we calculate $W\mathcal{P}(3)$. We need to consider regular trees with three inputs. There are three such trees, namely



The middle tree contributes a point to the space $W\mathcal{P}(3)$. Each of the other trees has one inner edge and thus contributes the interval $[0,1]$ to the space. The only identification to be made is when the length of one of those inner edges is 0, in which case it is identified with the point corresponding to the middle tree. The space $W\mathcal{P}(3)$ is thus the gluing of two copies of the interval $[0,1]$ where we identify both ends named 0 to a single point. The result is then just a closed interval, $[-1,1]$. However, it is convenient to keep in mind the trees corresponding to each point of this interval. Namely, the tree corresponding to the middle point, 0, is the middle tree. With a point $0 < t \leq 1$ corresponds the tree on the right where the length of the inner edge is t, and with a point $-1 \leq -t < 0$ corresponds the tree on the left where its inner edge is given the length t. In this way one can calculate the entire operad $W\mathcal{P}$. It can then be shown that the spaces $\{W\mathcal{P}(n)\}_{n=0}^{\infty}$ are all homeomorphic to the Stasheff associahedra. An A_∞-space is then an algebra of $W\mathcal{P}$.

An important observation in the W construction given above is the following. In order to construct the space $W\mathcal{P}(p_1, \cdots, p_n; p_0)$ one can proceed as follows. For each labelled standard planar tree $\bar{T}$ as above let $H^{\bar{T}}$ be $H^{\otimes_k}$ where k is the number of inner edges in $\bar{T}$ and $H = [0,1]$, the unit interval. The space $A(p_1, \cdots, p_n; p_0)$ is homeomorphic to

$$\coprod_{\bar{T}} H^{\bar{T}}$$

where $\bar{T}$ varies over all labelled standard planar trees $\bar{T}$ whose leaves are labelled by $p_1, \cdots, p_n$ and whose root is labelled by p_0. The identifications that are then made to construct the space $W\mathcal{P}(p_1, \cdots, p_n; p_0)$ are completely determined by the combinatorics of the various trees $\bar{T}$. This observation is the key to generalizing the W-construction to operads in more general monoidal categories $\mathcal{E}$. What we need is a suitable replacement for the unit interval $[0,1]$ used above to give lengths to the inner edges of the trees. This is done in [**4, 5, 6**] with the notion of an *interval* in a monoidal model category $\mathcal{E}$. In more detail, assume $\mathcal{E}$ is a monoidal model category with a cofibrant unit I. An *interval* in $\mathcal{E}$ is then an object H together with maps

$$I \underset{1}{\overset{0}{\rightrightarrows}} H \xrightarrow{\epsilon} I$$

and

$$H \otimes H \xrightarrow{\vee} H$$

satisfying certain conditions (see [**5**]). In particular, H is an interval in Quillen's sense (see [**40**]), so 0 and 1 together define a cofibration $I \coprod I \to H$, and ϵ is a weak equivalence. In such a setting the W-construction above can be mimicked by gluing together objects of the from $H^{\otimes_k}$ rather then cubes $[0,1]^k$. We thus obtain a functor $W_H : Operad(\mathcal{E}) \to Operad(\mathcal{E})$, which specializes to the topological case in the sense that $W_{[0,1]} = W$. Usually we will just write W instead of W_H. We illustrate this more general construction by an example, referring the reader to **[5]** for more details.

EXAMPLE 4.2.2. Consider the category Cat with the folk model structure. In this monoidal model category we can choose the category H with $ob(H) = \{0,1\}$ and a single isomorphism $0 \to 1$ to be an interval, with the obvious structure maps. Let us consider again (compare Example 4.2.1) the operad $\mathcal{P}$ having one object and one n-ary operation for each $n \geq 0$, this time as a discrete operad in Cat. To calculate $W\mathcal{P}(n)$ we should again consider labelled standard planar trees with lengths. The same argument as above implies that we should only consider regular trees, and a similar calculation shows that $W\mathcal{P}(n)$ is a contractible one-point category for $n = 0,1,2$. Now, to calculate $W\mathcal{P}(3)$ we again consider the three trees as given above. This time the middle tree contributes the category $H^0 = I$. Each of the other trees contributes the category H. The identifications identify the object named 0 in each copy of H to the unique object of I. The result is a contractible category with three objects. In general, the category $W\mathcal{P}(n)$ is a contractible category with $tr(n)$ objects, where $tr(n)$ denotes the number of regular standard planar trees with n leaves. The composition in $W\mathcal{P}$ is given by grafting of such trees.

The generalized Boardman-Vogt W-construction thus provides a definition of up-to-homotopy $\mathcal{P}$-algebras for a wide variety of operads $\mathcal{P}$ enriched in a suitable monoidal model category with a chosen interval. A natural question that arises is, of course, what is the appropriate notion of maps between such weak $\mathcal{P}$-algebras. Luckily, no extra work is needed in order to produce such a notion since we can again use the W-construction to come up with one. The idea is very simple. If, given an operad $\mathcal{P}$, we can find an operad $\mathcal{P}^1$ such that a $\mathcal{P}^1$-algebra consists of two $\mathcal{P}$-algebras and a map of $\mathcal{P}$ algebras between them then it is sensible to define an up-to-homotopy map between weak $\mathcal{P}$-algebras to be a $W(\mathcal{P}^1)$-algebra.

If $\mathcal{P}$ is an operad in Set then we can take $\mathcal{P}^1$ to be $\mathcal{P} \otimes_{BV} \Omega[L_1]$ where L_1 is a linear tree with one vertex. We thus make the following definition:

DEFINITION 4.2.3. Let $\mathcal{P}$ be an operad in Set and $\mathcal{E}$ a symmetric monoidal model category with an interval. An *up-to-homotopy $\mathcal{P}$-algebra* in $\mathcal{E}$ is an algebra for the operad $W(\mathcal{P})$. An *up-to-homotopy map* between up-to-homotopy $\mathcal{P}$-algebras is an algebra for the operad $W(\mathcal{P} \otimes_{BV} \Omega[L_1])$. We will sometimes use the term weak instead of up-to-homotopy.

Recall that L_1 has two edges named 0 and 1. We obtain thus two induced maps $\mathcal{P} \cong \mathcal{P} \otimes_{BV} \Omega[\eta_i] \to \mathcal{P} \otimes_{BV} \Omega[L_1]$, for $i = 0,1$. Given an algebra $W(\mathcal{P} \otimes_{BV} \Omega[L_1]) \to \mathcal{E}$, i.e., a weak map between weak algebras, there are associated two maps $d_i : W(\mathcal{P}) \to \mathcal{E}$, which by definition are weak $\mathcal{P}$-algebras. We agree by convention that d_0 is the domain and d_1 is the codomain of the given map.

An obvious question now is whether the collection of all weak $\mathcal{P}$-algebras and all weak $\mathcal{P}$-maps form a category. The answer is that they usually do not. A simple example is provided by A_∞-spaces where it is known that weak A_∞-maps do not compose associatively. The theory so far already suggests a solution to that problem. For an operad $\mathcal{P}$ in Set consider the operad $\mathcal{P} \otimes_{BV} \Omega[L_n]$. An algebra for such an operad is easily seen to be a sequence of $n+1$ $\mathcal{P}$-algebras $X_0, \cdots, X_n$ together with $\mathcal{P}$-algebra maps:

$$X_0 \to X_1 \to \cdots \to X_n.$$

We proceed as follows:

PROPOSITION 4.2.4. *Let $\mathcal{P}$ be an operad in Set. For each $n \geq 0$ let X_n be the set*

$$Alg(W(\mathcal{P} \otimes_{BV} \Omega[L_n]), \mathcal{E}).$$

The collection $X = \{X_n\}_{n=0}^{\infty}$ is a simplicial set.

PROOF. The proof follows easily by noting that the sequence $\{\mathcal{P} \otimes_{BV} \Omega[L_n]\}_{n=0}^{\infty}$ forms a cosimplicial object in *Operad*. See **[6]** for more details. □

REMARK 4.2.5. In **[7]** the authors make essentially the same definition for the case where $\mathcal{E} = Top$ with the usual interval. They subsequently prove that the resulting simplicial set is a quasi-category.

DEFINITION 4.2.6. We refer to the simplicial set constructed above as the *simplicial set of up-to-homotopy $\mathcal{P}$-algebras.*

4.3. The homotopy coherent nerve

We now use the generalized W-construction in order to define, for an operad $\mathcal{P}$ enriched in a suitable symmetric monoidal model category $\mathcal{E}$, its homotopy coherent dendroidal nerve. This is a dendroidal set which is like the dendroidal nerve construction with homotopies built into it. The main result proved in this section is one identifying a condition on an operad $\mathcal{P}$ in $\mathcal{E}$ that ensures that its homotopy coherent dendroidal nerve is an inner Kan complex. This provides a large family of inner Kan complexes that are rarely ever strict.

DEFINITION 4.3.1. Let $\mathcal{E}$ be a symmetric monoidal model category with an interval H. For each tree $T \in ob(\Omega)$ we may consider the operad $\Omega(T)$ as a discrete operad in $Operad(\mathcal{E})$. Doing so we obtain the probe $\Omega \to Operad(\mathcal{E})$ that sends T to $W(\Omega(T))$. Let

$$dSet \underset{hcN_d}{\overset{hc\tau_d}{\rightleftarrows}} Operad(\mathcal{E})$$

be the associated adjunction. The functor hcN_d is called the *homotopy coherent dendroidal nerve* functor.

Explicitly, given an operad $\mathcal{P}$ in $\mathcal{E}$, its homotopy coherent dendroidal nerve is the dendroidal set given by

$$hcN_d(\mathcal{P})_T = Operad(\mathcal{E})(W(\Omega(T)), \mathcal{P})$$

To better understand this construction let us look more closely at the operads $W(\Omega(T))$. It is convenient to use the functor

$$Symm : Operad(\mathcal{E})_\pi \to Operad(\mathcal{E})$$

which is left adjoint to the obvious forgetful functor from symmetric operads in $\mathcal{E}$ to planar ones. Recall that if T is an object in Ω and $\bar{T}$ is a chosen planar representative of T, then $\Omega(T) = Symm(\Omega_\pi(\bar{T}))$. Since the W-construction commutes with symmetrization (as one easily verifies), it follows that

$$W(T) = Symm(W\Omega_\pi(\bar{T})).$$

REMARK 4.3.2. The operad $W\Omega_\pi(\bar{T})$ is easily described explicitly. The objects of $W(\Omega_\pi(\bar{T}))$ are the objects of $\Omega_\pi(\bar{T})$, i.e., the edges of T. Recall that by a *signature* in $W(\Omega_\pi(\bar{T}))$, we mean a sequence $e_1, \cdots, e_n; e_0$ of objects, i.e., edges of $\bar{T}$. Given a signature $\sigma = (e_1, \cdots, e_n; e_0)$, we have that $W(\Omega_\pi(\bar{T}))(\sigma) = \phi$ (the initial object in $\mathcal{E}$) whenever $\Omega_\pi(\bar{T})(\sigma)$ is empty. If $\Omega_\pi(\bar{T})(\sigma) \neq \phi$ then there is a sub-tree T_σ of T (and a corresponding planar sub-tree $\bar{T}_\sigma$ of $\bar{T}$) whose leaves are $e_1, \cdots, e_n$, and whose root is e_0. We then have that

$$W\Omega_\pi(\bar{T})(e_1, \cdots, e_n; e_0) = \bigotimes_{f \in i(\sigma)} H,$$

where $i(\sigma)$ is the set of *inner* edges of T_σ (or of $\bar{T}_\sigma$). (This last tensor product is to be thought of as the "space" of assignments of lengths to inner edges in $\bar{T}_\sigma$; it is the unit if $i(\sigma)$ is empty.)

The composition operations in the operad $W\Omega_\pi(\bar{T})$ are given in terms of the $\circ_i$-operations as follows. For signatures $\sigma = (e_1, \cdots, e_n; e_0)$ and $\rho = (f_1, \cdots, f_m; e_i)$, the composition map

$$\begin{array}{c} W\Omega_\pi(\bar{T})(e_1, \cdots, e_n; e_0) \otimes W\Omega_\pi(\bar{T})(f_1, \cdots, f_m; e_i) \\ \Big\downarrow \circ_i \\ W\Omega_\pi(\bar{T})(e_1, \cdots, e_{i-1}, f_1, \cdots, f_m, e_{i+1}, \cdots, e_n; e_0) \end{array}$$

is the following one. The trees $\bar{T}_\sigma$ and $\bar{T}_\rho$ can be grafted along e_i to form $\bar{T}_\sigma \circ_{e_i} \bar{T}_\rho$, again a planar sub-tree of $\bar{T}$. In fact

$$\bar{T}_\sigma \circ_{e_i} \bar{T}_\rho = \bar{T}_{\sigma \circ_i \rho}$$

where $\sigma \circ_i \rho$ is the signature $(e_1, \cdots, e_{i-1}, f_1, \cdots, f_m, e_{i+1}, \cdots, e_n; e_0)$, and for the sets of inner edges we have

$$i(\sigma \circ_i \rho) = i(\sigma) \cup i(\rho) \cup \{e_i\}.$$

The required composition is then

$$\begin{array}{ccc} H^{\otimes i(\sigma)} \otimes H^{\otimes i(\rho)} & \dashrightarrow & H^{\otimes i(\sigma \circ_i \rho)} \\ \Big\downarrow \cong & & \Big\downarrow \cong \\ H^{\otimes i(\sigma) \cup i(\rho)} \otimes I & \xrightarrow{id \otimes 1} & H^{\otimes i(\sigma) \cup i(\rho)} \otimes H \end{array}$$

where $1 : I \to H$ is one of the "endpoints" of the interval H, as above.

This description of the operad $W\Omega_\pi(\bar{T})$ is functorial in the planar tree T. In particular, we note that for an inner edge e of T, the tree T/e inherits a planar structure $\bar{T/e}$ from $\bar{T}$, and $W\Omega_\pi(\bar{T/e}) \to W\Omega_\pi(\bar{T})$ is the natural map assigning length 0 to the edge e whenever it occurs (in a sub-tree given by a signature).

The following will result will be useful.

LEMMA 4.3.3. *Let $\mathcal{P}$ be an operad in Set. For any tree $T \in ob(\Omega)$ the equality*

$$N_d\mathcal{P} \otimes \Omega[T] \cong N_d(\mathcal{P} \otimes_{BV} \Omega(T))$$

holds.

PROOF. The dendroidal set $N_d\mathcal{P} \otimes \Omega[T]$ is the following colimit:

$$\varinjlim_{\Omega(S)\to\mathcal{P}} N_d(\Omega(S) \otimes_{BV} \Omega(T)) = \varinjlim_{\Omega(S)\to\mathcal{P}} \Omega[S] \otimes \Omega[T].$$

We show that $N_d(\mathcal{P} \otimes \Omega(T))$ has the required universal property with respect to that diagram. To obtain a cone from the diagram we need an arrow $N_d(\Omega(S) \otimes_{BV} \Omega(T)) \to N_d(\mathcal{P} \otimes_{BV} \Omega(T))$ for each arrow $\Omega(S) \to \mathcal{P}$. Since $\tau_d N_d = id$, such an arrow is the same as an arrow $\Omega(S) \otimes_{BV} \Omega(T) \to \mathcal{P} \otimes_{BV} \Omega(T)$ and the choice for such an arrow is obvious. Assume now that we are given a cone to some other dendroidal set X. Let $t \in N_d(\mathcal{P} \otimes_{BV} \Omega(T))_T$ be a non-degenerate dendrex. Since $\Omega(T)$ is the nerve of the dendroidally ordered set $[T]$ (thus there is at most one arrow for each signature and the domain of each arrow does not contain repeated objects) it follows that there is a unique (up-to-isomorphism) maximal dendrex $s \in N_d(\mathcal{P})_S$ such that t is a dendrex in the dendroidal sub-set $\Omega[S] \otimes \Omega[T] \to N_d(\mathcal{P} \otimes_{BV} \Omega(T))$ given by the map s. This dendrex s corresponds to a map $\Omega(S) \to \mathcal{P}$. If $f : \Omega[S] \otimes \Omega[T] \to X$ is a map in the given cone then $g(t) = f(t)$ defines a unique map $N_d(\mathcal{P} \otimes_{BV} \Omega(T)) \to X$ such that the cones commute, as required for the universal property. □

THEOREM 4.3.4. *Let $\mathcal{E}$ be a symmetric monoidal model category with an interval and $\mathcal{P}$ an operad in Set. We have the following isomorphism of operads*

$$hc\tau_d(N_d\mathcal{P}) \cong W(disc(\mathcal{P})).$$

PROOF. The operad $hc\tau_d(N_d\mathcal{P})$ is given as the colimit

$$\varinjlim_{\Omega(T)\to\mathcal{P}} W\Omega(T).$$

A straightforward inspection of that colimit and the generalized W-construction presented in **[5]** for the special case of a discrete operad show that both colimits are the same. □

An immediate result is that, in the same setting as above, the set

$$\underline{dSet}(N_d\mathcal{P}, hcN_d(\mathcal{E}))_\eta$$

is isomorphic to the set $Operad(\mathcal{E})(W(disc(\mathcal{P})), \mathcal{E})$ of up-to-homotopy $\mathcal{P}$-algebras. More generally we have the following:

PROPOSITION 4.3.5. *Let $\mathcal{E}$ be a symmetric monoidal model category with an interval and $\mathcal{P}$ a discrete operad in $\mathcal{E}$. We then have the following isomorphism:*

$$\underline{dSet}(N_d\mathcal{P}, hcN_d(\mathcal{E}))_T \cong Operad(\mathcal{E})(W(\mathcal{P} \otimes_{BV} \Omega[T]), \mathcal{E}).$$

COROLLARY 4.3.6. *With $\mathcal{E}$ as above we obtain that the simplicial set*

$$i^*(\underline{dSet}(N_d(\mathcal{P}), hcN_d(\mathcal{E})))$$

is isomorphic to the simplicial set of weak $\mathcal{P}$-algebras from Definition 4.2.6 above.

Our aim for the rest of this section is to present a sufficient condition on an operad $\mathcal{P}$ in $\mathcal{E}$ that guarantees that the homotopy coherent dendroidal nerve of $\mathcal{P}$ is an inner Kan complex.

DEFINITION 4.3.7. An operad $\mathcal{P}$ in a symmetric monoidal model category $\mathcal{E}$ is called *locally fibrant* if for each sequence $c_1, \cdots, c_n, c$ of objects of $\mathcal{P}$, the object $\mathcal{P}(c_1, \cdots, c_n; c)$ in $\mathcal{E}$ is fibrant.

THEOREM 4.3.8. *Let $\mathcal{P}$ be a locally fibrant operad in $\mathcal{E}$, where $\mathcal{E}$ is a symmetric monoidal model category with an interval. Then $hcN_d(\mathcal{P})$ is an inner Kan complex.*

PROOF. Consider a tree T and an inner edge e in T. We want to solve the extension problem

$$\begin{array}{ccc} \Lambda^e[T] & \xrightarrow{\varphi} & hcN_d(\mathcal{P}) \\ \downarrow & \nearrow & \\ \Omega[T] & & \end{array}$$

Fix a planar representative $\bar{T}$ of T. Then the derived map $\psi : \Omega[T] \to hcN_d(\mathcal{P})$ corresponds to a map of planar operads

$$\hat{\psi} : W\Omega_\pi(\bar{T}) \to \mathcal{P}.$$

Each face S of T inherits a planar structure $\bar{S}$ from $\bar{T}$, and the given map $\varphi : \Lambda^e[T] \to hcN_d(\mathcal{P})$ corresponds to a map of operads in $\mathcal{E}$,

$$\hat{\varphi} : W(\Lambda^e[T]) \to \mathcal{P},$$

where $W(\Lambda^e[T])$ denotes the colimit of operads in $\mathcal{E}$,

$$W(\Lambda^e[T]) = \varinjlim W(\Omega_\pi(\bar{S})) \tag{1}$$

over all but one of the faces of T. In other words, φ corresponds to a compatible family of maps

$$\hat{\varphi}_S : W(\Omega_\pi(\bar{S})) \to \mathcal{P}.$$

Let us now show the existence of an operad map $\hat{\psi}$ extending the $\hat{\varphi}_S$ for all faces $S \neq T/e$. First, the objects of $\Omega_\pi(\bar{T})$ are the same as those of the colimit in (1), so we already have a map $\psi_0 = \varphi_0$ on objects:

$$\psi_0 : E(T) \to ob(\mathcal{P}).$$

Next, if $\sigma = (e_1, \cdots, e_n; e_0)$ is a signature of T for which $W(\Omega_\pi(\bar{T})) \neq \phi$, if $T_\sigma \subseteq T$ is not all of T, then T_σ is contained in an outer face S of T. So $W(\Omega_\pi(\bar{T}))(\sigma) = W(\Omega_\pi(\bar{T}_\sigma))(\sigma) = W(\Omega_\pi(\bar{S}))(\sigma)$, and we already have a map

$$\hat{\varphi}_\rho(\sigma) : W(\Omega_\pi(\bar{T}))(\sigma) \to \mathcal{P}(\sigma),$$

given by $\hat{\varphi}_S : W(\Omega_\pi(\bar{S})) \to \mathcal{P}$. Thus, the only part of the operad map $\hat{\psi} : W(\Omega_\pi(\bar{T})) \to \mathcal{P}$ not determined by φ is the one for the signature τ where $T_\tau = T$; i.e., $\tau = (e_1, \cdots, e_n; e_0)$ where $e_1, \cdots, e_n$ are all the input edges of $\bar{T}$ (in the planar order) and e_0 is the output edge. For this signature, $\hat{\psi}(\tau)$ is to be a map

$$\hat{\psi} : W(\Omega_\pi(\bar{T})(\tau) = H^{\otimes i(\tau)} \to \mathcal{P}(\tau)$$

which (i) is compatible with the $\hat{\psi}(\sigma) = \hat{\varphi}_S(\sigma)$ for other signatures σ; and (ii) together with these $\hat{\psi}(\sigma)$ respects operad composition. The first condition determines $\hat{\psi}(\tau)$ on the sub-object of $H^{\otimes i(\tau)}$ which is given by a value 1 on one of the factors. Thus, if we write 1 for the map $I \xrightarrow{1} H$ and $\partial H \rightarrowtail H$

for the map $I \coprod I \to H$, and define $\partial H^{\otimes k} \rightarrowtail H^{\otimes k}$ by the Leibniz rule (i.e., $\partial(A \otimes B) = \partial(A) \otimes B \cup A \otimes \partial(b))$, then the problem of finding $\hat{\psi}(\tau)$ comes down to an extension problem of the form

$$\begin{array}{ccc} \partial(H^{\otimes(i(\tau)-\{e\})} \otimes H \cup H^{\otimes(i(\tau)-\{e\})} \otimes I & \longrightarrow & \mathcal{P}(\tau) \\ \downarrow & & \uparrow {\scriptstyle \hat{\psi}(\sigma)} \\ H^{\otimes i(\tau)-\{e\}} \otimes H & \xrightarrow{\cong} & H^{\otimes i(\tau)} \end{array}$$

This extension problem has a solution, because $\mathcal{P}(\tau)$ is fibrant by assumption, and because the left hand map is a trivial cofibration (by repeated use of the push-out product axiom for monoidal model categories). This concludes the proof of the theorem. □

Remark 4.3.9. This result generalizes a result of Cordier and Porter [**10**], namely that the homotopy coherent nerve of a simplicially enriched category with fibrant Hom objects is an inner Kan complex. Indeed, taking $\mathcal{E}$ to be the category of simplicial sets with its usual monoidal model category and the obvious interval we know that any locally fibrant operad in $sSet$ admits a homotopy coherent dendroidal nerve which is an inner Kan complex. Viewing a simplicial category as an operad in $sSet$ in the obvious way gives the desires result.

Recall (Example 1.8.4) that given a set M of object in a symmetric monoidal category $\mathcal{E}$ we can construct the $\mathcal{E}$-enriched operad $\mathcal{P}_M$.

Lemma 4.3.10. *Let $\mathcal{E}$ be a symmetric monoidal model category with an interval. If $M \subseteq ob(\mathcal{E})$ consists of fibrant-cofibrant objects then $hcN_d(\mathcal{P}_M)$ is an inner Kan complex.*

Proof. It is sufficient to show that $\mathcal{P}_M$ is locally fibrant. In a monoidal model category the tensor product of cofibrant objects is again cofibrant and $\mathcal{E}(X, Y)$ is fibrant whenever X is cofibrant and Y is fibrant. It now follows that each Hom object in $\mathcal{P}_M$ is fibrant, as needed. □

Remark 4.3.11. Given a symmetric monoidal model category $\mathcal{E}$ with an interval, let $\mathcal{E}_{cf}$ be the full sub-category of $\mathcal{E}$ spanned by the fibrant-cofibrant objects. A fundamental construction in the theory of model categories is the homotopy category $Ho(\mathcal{E})$, which is again a monoidal category and we may thus consider it as an operad. Recall the theory of homotopy within an inner Kan complex from Section 3.5 and in particular the construction of the homotopy operad $Ho(X)$ of an inner Kan complex X. It is rather simple to verify that the operad $Ho(hcN_d(\mathcal{E}_{cf}))$ is equivalent to $Ho(\mathcal{E})$ and is actually equal to $Ho(\mathcal{E}_{cf})$. In that sense, the theory of homotopy inside a weak Kan complex extends the homotopy theory inside a symmetric monoidal model category with an interval. Notice that the dendroidal nerve $hcN_d(\mathcal{E}_{cf})$ stores much more information than the homotopy category, namely all of the higher homotopies.

4.4. Algebras and the Grothendieck construction

Recall that an operad $\mathcal{P}$ can be used to define an algebraic structure on objects of another operad $\mathcal{E}$. In this section we extend the notion of algebras to dendroidal sets and present a Grothendieck construction for diagrams of dendroidal sets which extends the Grothendieck construction for diagrams of operads (Section 1.7).

DEFINITION 4.4.1. Let E and X be dendroidal sets. The dendroidal set

$$\underline{dSet}(X, E)$$

is called the dendroidal set of X-*algebras* in E and is denoted by $Alg(X, E)$. An element in $Alg(X, E)_\eta$ is called an X-*algebra* in E. An element of $Alg(X, E)_{L_1}$ is called a *map of* X-*algebras in* E. We will also refer to an X-algebra in E as an (X, E)-algebra.

Let us first show that this definition extends the notion of $\mathcal{P}$-algebras in $\mathcal{E}$ for operads.

PROPOSITION 4.4.2. *For operads* $\mathcal{P}$ *and* $\mathcal{E}$

$$Alg(N_d\mathcal{P}, N_d\mathcal{E}) \cong N_d(Alg(\mathcal{P}, \mathcal{E})).$$

PROOF. This is just the statement that N_d commutes with internal Homs. □

Suppose that $Alg(X, E)$ is an inner Kan complex. The existence of composition of dendrices then provides for a notion of composition of maps of (X, E)-algebras, which extends the composition of maps of $(\mathcal{P}, \mathcal{E})$-algebras. Furthermore, the theory of homotopy in a dendroidal set automatically provides a notion of homotopy for (X, E)-algebras, which is particularly well behaved when $Alg(X, E)$ is an inner Kan complex. The exponential property of dendroidal sets can be restated by saying that if E is an inner Kan complex and X is normal then $Alg(X, E)$ is an inner Kan complex and thus enjoys a built-in theory of composition and homotopy. We see thus that by replacing an operad by an inner Kan complex, a much greater generality is obtained (to define algebraic structures in, not necessarily, operads) and still one retains a suitable notion of composition of maps of such algebras and gains immediately a pleasant theory of homotopy.

Later on we will use the theory to obtain a definition for weak n-categories. To that end we will have to use a suitable Grothendieck construction for diagrams of dendroidal sets, which we now turn to.

Let $\mathbb{S}$ be a cartesian category. A functor $X : \mathbb{S}^{op} \to dSet$ is called a *diagram* of dendroidal sets. Our aim is to define a dendroidal set

$$\int_{\mathbb{S}} X$$

obtained by suitably 'gluing' the dendroidal sets $X(S)$ for the various $S \in ob(\mathbb{S})$. It will be convenient to introduce the notion of dendroidal collections. A *dendroidal collection* is a collection of sets $X = \{X_T\}_{T \in ob(\Omega)}$. Each dendroidal set has an obvious underlying dendroidal collection. A map of dendroidal collections $X \to Y$ is a collection of functions $\{X_T \to Y_T\}_{T \in ob(\Omega)}$. There is a natural way of associating an object of $\mathbb{S}$ with each dendrex of $N_d(\mathbb{S})$. For a tree T in Ω, let $leaves(T)$ be the set of leaves of T, and for a leaf l, write $l : \eta \to T$ also for the map sending the unique edge in η to l in T. Then, since $\mathbb{S}$ is assumed to have finite products, each dendrex $t \in N_d(\mathbb{S})_T$ defines an object

$$in(t) = \prod_{l \in leaves(T)} l^*(t)$$

in $\mathbb{S}$. Notice that if $\alpha : S \to T$ is a face map, then by using the canonical symmetries and the projections in $\mathbb{S}$, there is a canonical arrow $in(\alpha) : in(t) \to in(\alpha^* t)$ for any $t \in X_T$. Similarly, if α is a degeneracy map or an isomorphism then one obtains a map $in(\alpha)$, and in fact for any $\alpha : S \to T$ one naturally obtains a map $in(\alpha) : in(t) \to in(\alpha^* t)$ for each $t \in X_T$.

Definition 4.4.3. Let $X : \mathbb{S}^{op} \to dSet$ be a diagram of dendroidal sets where $\mathbb{S}$ is a cartesian category which is thus also an operad. The dendroidal set $\int_{\mathbb{S}} X$ is defined as follows. A dendrex $\Omega[T] \to \int_{\mathbb{S}} X$ is a pair (t, x) such that $t \in N_d(\mathbb{S})_T$ and x is a map of dendroidal collections

$$x : \Omega[T] \to \coprod_{S \in ob(\mathbb{S})} X(S)$$

satisfying the following conditions. For each $r \in \Omega[T]_R$ (that is an arrow $r : R \to T$), we demand that $x(r) \in X(in(r^* t))$. Furthermore we demand the following compatibility condition to hold. For any $r \in \Omega[T]_R$ and any map $\alpha : U \rightarrowtail R$ in Ω

$$\alpha^*(x(r)) = X(in(\alpha)) x(\alpha^*(r)).$$

Remark 4.4.4. A straightforward verification shows that the Grothendieck construction for diagrams of dendroidal sets extends the one for operads (given in Section 1.7) in the following sense. If we have a diagram of operads $X : \mathbb{S}^{op} \to Operad$ and if we write $N_d(X)$ for the diagram of dendroidal sets $\mathbb{S}^{op} \to Operad \to dSet$ obtained by composition with $N_d : Operad \to dSet$, then we have:

$$N_d(\int_{\mathbb{S}} X) \cong \int_{\mathbb{S}} N_d(X).$$

Theorem 4.4.5. *Let $X : \mathbb{S}^{op} \to dSet$ be a diagram of dendroidal sets. If for every $S \in ob(\mathbb{S})$ every $X(S)$ is an inner Kan complex then so is $\int_{\mathbb{S}} X$.*

Proof. Let T be a tree and e an inner edge. We consider the extension problem

$$\begin{array}{ccc} \Lambda^e[T] & \longrightarrow & \int_{\mathbb{S}} X \\ \downarrow & \nearrow & \\ \Omega[T] & & \end{array}$$

The horn $\Lambda^e[T] \to \int_{\mathbb{S}} X$ is given by a compatible collection $\{(r, x_R) : \Omega[R] \to \int_{\mathbb{S}} X\}_{R \neq T/e}$. We wish to construct a dendrex $(t, x_T) : \Omega[T] \to \int_{\mathbb{S}} X$ extending this family. First notice that the collection $\{r\}_{R \neq T/e}$ is an inner horn $\Lambda^e[T] \to hcN_d(\mathbb{S})$ (actually this horn is obtained by composition with the obvious projection $\int_{\mathbb{S}} X \to hcN_d(\mathbb{S})$ sending a dendrex (t, x) to t). We are given that $hcN_d(\mathbb{S})$ is an inner Kan complex and thus there is a filler $t \in hcN_d(\mathbb{S})_T$ for the horn $\{r\}_{R \neq T/e}$. We now wish to define a map of dendroidal collections $x_T : \Omega[T] \to \coprod_{S \in ob(\mathbb{S})} X(S)$ that will extend the given maps x_R for $R \neq T/e$. This condition already determines the value of x_T for any dendrex $r : U \to T$ other then $id : T \to T$ and $\partial_e : T/e \to T$, since for each such r, the tree U factors through one of the faces $R \neq T/e$. To determine $x_T(id_T)$ and $x_T(\partial_e)$ consider the family $\{y_R = x_R(id : R \to R)\}_{R \neq T/e}$. By definition we have that $y_R \in X(in(r))_R$. For each such R let $\alpha_R : R \to T$ be the corresponding face map in Ω. Since $\partial_e^*(t) = r$ we obtain the map $in(\alpha_R) : in(r) \to in(t)$. We can now pull back the collection $\{y_R\}_{R \neq T/e}$ using $X(in(\alpha_R))$ to obtain a collection $\{z_R = X(in(\alpha_R))(y_R)\}_{R \neq T/e}$. This collection is now a horn $\Lambda^e[T] \to X(in(T))$ (this follows from the compatibility conditions in the definition of $\int_{\mathbb{S}} X$). Since $X(in(t))$ is an inner Kan complex there is a filler $u \in X(in(t))_T$ for that horn. We now define $x_T(id : T \to T) = u$ and $x_T(\partial_e : T/e \to T) = \partial_e^*(u)$. Notice that since e is inner we have that $in(t) = in(\partial_e)$ and thus the images of these dendrices are in the correct dendroidal set, namely $X(in(t))$. It follows from our construction that this makes (t, x_T) a dendrex $\Omega[T] \to \int_{\mathbb{S}} X$ which extends the given horn. This concludes the proof. □

4.5. Categories enriched in a dendroidal set

Recall that in Section 1.7 we defined for each set A an operad $\mathcal{C}_A$ such that a $\mathcal{C}_A$-algebra in the operad associated to a symmetric monoidal category $\mathcal{E}$ is the same thing as a category enriched in $\mathcal{E}$, having A as set of objects, that is

$$Alg(\mathcal{C}_A, \mathcal{E}) = Cat(\mathcal{E})_A.$$

Using the dendroidal nerve functor we obtain that

$$N_d(Cat(\mathcal{E})_A) = N_d(Alg(\mathcal{C}_A, \mathcal{E})) \cong Alg(N_d(\mathcal{C}_A), N_d(\mathcal{E})).$$

Based on this, we make the following definition.

Definition 4.5.1. Let X be an arbitrary dendroidal set and $\mathcal{C}_A$ as above. The dendroidal set $Alg(N_d(\mathcal{C}_A), X)$ is called the dendroidal set of *categories enriched in* X having A as set objects and is denoted by $Cat(X)_A$.

We now use the Grothendieck construction in order to obtain the dendroidal set of all categories enriched in X. We already have the obvious functor $Set^{op} \to dSet$ that sends a set A to $Cat(X)_A$.

Definition 4.5.2. Let X be a dendroidal set and let $\mathcal{C}_A$, for each set A, be the operad discussed above. Let $Cat(X)_- : Set^{op} \to dSet$ be the functor that sends a set A to the dendroidal set $Cat(X)_A = Alg(N_d(\mathcal{C}_A), X)$. The dendroidal set of *categories enriched in the dendroidal set* X is

$$Cat(X) = \int_{Set} Cat(X)_-.$$

This construction can be repeated as follows.

Definition 4.5.3. Let X be a dendroidal set. Let $Cat(X)^0 = X$ and define recursively

$$Cat(X)^{n+1} = Cat(Cat(X)^n)$$

for each $n \geq 1$. We call $Cat(X)^n$ the dendroidal set of *n-categories enriched in X*.

Theorem 4.5.4. *If X is an inner Kan complex then for each $n \geq 0$ the dendroidal set $Cat(X)^n$ is an inner Kan complex.*

Proof. For any planar operad $\mathcal{P}$ in Set the dendroidal nerve $N_d(Symm(\mathcal{P}))$ is clearly normal (see Definition 2.6.6) and thus $N_d(\mathcal{C}_A)$ is normal for each set A. From the fact that the the inner Kan complexes form an exponential ideal in $dSet$ with respect to the normal dendroidal sets (Theorem 3.6.3) it follows that each dendroidal set $Cat(X)_A$ is an inner Kan complex and Theorem 4.4.5 then proves that so is $Cat(X)$. □

Thus, for an inner Kan complex X, our definition of the dendroidal set of n-categories enriched in X provides us with a definition of what an n-category enriched in X is, what are functors for such n-categories, a notion of homotopy for such functors together with a composition rule for such functors which is associative up to homotopy.

EXAMPLE 4.5.5. It is rather straightforward to verify that

$$Cat(N_d(Set))^n \cong N_d(Cat^n)$$

for each $n \geq 0$ where Cat^n is the category of strict n-categories with the tensor product of n-categories, viewed as an operad (by Cat^0 we mean just Set). Unfolding this definition one sees that this is just the common definition of strict $(n+1)$-categories as categories enriched in the category of n-categories. More generally, we have that

$$Cat(N_d(Cat^m))^n \cong N_d(Cat^{n+m}).$$

Of course we also have that for a symmetric monoidal category $\mathcal{E}$

$$Cat(N_d(\mathcal{E})) \cong N_d(Cat(\mathcal{E})).$$

We see thus that our notion of a category enriched in a dendroidal set X extends the usual definition of categories enriched in a symmetric monoidal category. We see also that our notion of n-category in X, for $X = N_d(Set)$, captures the notion of strict n-categories.

REMARK 4.5.6. Consider the category Top of compactly generated spaces. This category is a closed monoidal category with weak equivalences the weak homotopy equivalences and fibrations the Serre fibrations ([**40**]). The unit interval $[0,1]$ with the minimum operation acts as an interval for this category in the above sense, so that the homotopy coherent nerve of Top is well defined. We now have the dendroidal set $Cat(hcN_d(Top))$. The dendrices of shape η are then categories weakly enriched in topological spaces in the sense that the composition maps are associative up to specified higher homotopies. In particular, the dendroidal set $A^1 = \underline{dSet}(N_d(\mathcal{C}_*), hcN_d(Top))$ is the dendroidal set of A_∞-spaces ($\mathcal{C}_*$ being $\mathcal{C}_B$ where B is a one-point set). It is now natural to consider the sequence $\{A^n\}_{n=1}^{\infty}$ where $A^{n+1} = \underline{dSet}(N_d(\mathcal{C}_*), A^n)$. It should be interesting to study the relation between n-fold loop spaces and A^n and compare it to the work of Dunn [**12**] and the recent approach of Batanin [**2**] via n-operads.

Given a commutative ring R, the category $Ch(R)$ of graded chain complexes of R-modules is a monoidal model category where the equivalences are the quasi-equivalences and the fibrations are the epimorphisms ([**21**]). An interval in this category is given by $N_*^R(\Delta[1])$ where N_*^R is the normalized chain complex functor. We now have the dendroidal set $Cat(hcN_d(Ch(R)))$ whose dendrices of shape η are essentially A_∞-categories (see [**25**] for a definition and [**6**] for a related discussion).

4.6. Weak n-categories

The definition of strict $(n+1)$-categories as categories enriched in strict n-categories is very appealing and a suitable analogous definition for weak n-categories is desirable. A naive approach to an analogous definition of weak n-categories would proceed along these lines: A weak $(n+1)$-category should be a category weakly enriched in the category of weak n-categories. There are two problems with such a definition. The first is that it is not clear how one should weakly enrich a category and the second, and far more devastating for such an approach, is that while in order to enrich in a category $\mathcal{E}$, that category has to have certain extra structure (mostly that of a symmetric monoidal category, but braided monoidal categories or fc-multicategories are also adequate structures in which to enrich categories [**30**]). The problem becomes apparent when one realizes that for $n > 2$ the collection of weak n-categories and their weak functors should not be expected to even form a category but rather a weak n-category. To proceed we must then consider the extra structure needed to be present on a weak n-category in order to weakly enrich in it and then say what is meant by weakly enriching in it. Hope for a uniform recursive definition of such notions seems remote and the naive approach would appear to fail.

There are of course other approaches to be taken which resulted in a plethora of definitions of weak n-categories (See [**29**] for a survey of ten such definitions). Some of the approaches to a definition can be said to improve on the naive approach dictated above. Using the general theory of dendroidal sets we obtain another such definition, as we now show.

Definition 4.6.1. Let Cat be the category of categories with the folk model structure (see Theorem 1.6.1 for the definition) and the interval H the free-living isomorphism (a two object category with a single isomorphism between them). We define the dendroidal set of weak n-categories $wCat^n$ for $0 \leq n < \infty$ as follows. $wCat^0 = N_d(Set)$ and for $n > 0$:

$$wCat^n = Cat^{n-1}(hcN_d(Cat)).$$

Since every object in Cat is fibrant and cofibrant it follows that $hcN_d(Cat)$ is an inner Kan complex (Lemma 4.3.10) and thus that our definition provides notions of weak n-categories, their maps, homotopy, and compositions.

Let us look more closely at weak n-categories for small n. For $n = 1$ we have $wCat^1 = hcN_d(Cat)$. Recall that a dendrex of shape T in $wCat^1$ is a functor of Cat-enriched operads $W(\Omega(T)) \to Cat$. It is easily seen that $wCat^1_\eta$ is the set of all small categories. A dendrex $F \in wCat^1_{L_1}$ is then just a functor between two categories, while a dendrex in $wCat^1_{L_2}$ corresponds to a choice of three functors $F_1 : A \to B$, $F_2 : B \to C$, and $F_3 : A \to C$ together with a natural isomorphism $\alpha : F_2F_1 \to F_3$. It thus follows that two dendrices $F, F' \in wCat^1_{L_1}$ are homotopic if, and only if, they are naturally isomorphic. We now show that a dendrex $t \in wCat^1_T$, with $|T| \geq 3$ is completely determined by its boundary. We use the following notation. Given a dendrex $t : \Omega[T] \to X$ in a dendroidal set X we denote the map $Sk_k(\Omega[T]) \to \Omega[T] \to X$ by $Sk_k(t)$.

Proposition 4.6.2. *Let T be a tree with $|T| \geq 3$ and t and s two dendrices in $wCat^1_T$. If $Sk_2(t) = Sk_2(s)$ then $t = s$.*

Proof. Consider a functor $F : H^m \to \mathcal{C}$ for some $m \geq 0$. The category H^m is a contractible category with $ob(H^m) = \{0, 1\}^m$ and thus the functor F is completely determined by its value on each of the arrows from the object $(0, \cdots, 0)$ to all other objects. Now, a dendrex $x \in X_T$ is a functor $W(\Omega(T)) \to Cat$ which can be given as a sequence of compatible functors $\{H^{m_i} \to \mathcal{C}_i\}_{i=1}^n$ (see the proof of Theorem 4.3.8 above). Each such functor is determined thus by its image on the special arrows just mentioned. Since these arrows are clearly contained in the image of $Sk_2(\Omega[T]) \longrightarrow \Omega[T] \xrightarrow{x} X$, the result follows. □

We can now deduce the following:

LEMMA 4.6.3. *The dendroidal set $wCat^1 = hcN_d(Cat)$ is 3-coskeletal.*

PROOF. If we can show that $wCat^1$ satisfies the strict inner Kan condition for all trees T with $|T| \geq 3$ then it would follow from Proposition 3.2.5 that $wCat^1$ is 3-coskeletal. Let thus T be such a tree. Since we already know that $wCat^1$ is an inner Kan complex we know that every inner horn $\Lambda^e[T] \to wCat^1$ has a filler t. Suppose that s is also a filler for the same horn. Since $|T| \geq 3$ it follows that $Sk_2(\Omega[T])$ factors through $\Lambda^e[T]$ and thus that $Sk_2(t) = Sk_2(s)$. The above proposition then implies that $t = s$, as needed. □

It is now a straightforward (and somewhat tedious) matter to unpack the definition of a weak 2-category. To identify the relevant dendrices in $wCat^1$ it will be convenient to use the following notation. Given categories $X_1, \cdots, X_n$ and two integers $1 \leq i \leq j \leq n$ we denote by $(X)_i^j$ the category $X_i \times \cdots \times X_j$. A dendrex of $wCat^1$ of shape η is just a category. A dendrex of the shape of a corolla C_n is the same as a choice of $n+1$ categories $X_0, \cdots, X_n$ and a functor $F : (X)_1^n \to X$. Any dendrex of degree 2 is of shape $C_n \circ_i C_m$ and such a dendrex is equivalent to the following data:

(1) A choice of $n+1$ categories $X_0, \cdots, X_n$ and a functor $F : (X)_1^n \to X_0$.
(2) A choice of m categories $Y_0, \cdots, Y_m$ and a functor $G : (Y)_1^m \to X_i$.
(3) A functor $H : (X)_1^{i-1} \times (Y)_1^m \times (X)_{i+1}^n \to X_0$.
(4) A natural isomorphism α between H and $F \circ_i G$.

with $F \circ_i G$ being the obvious functor. A dendrex of degree three can have one of two shapes. Either it is of the shape $C_n \circ_{i,j} (C_m, C_k)$ or of the shape $C_n \circ_i (C_m \circ_j C_k)$. A dendrex of the first shape consists of the following data:

(1) A choice of $n+1$ categories $X_0, \cdots, X_n$ and a functor $F_1 : (X)_1^n \to X_0$.
(2) A choice of m categories $Y_1, \cdots, Y_m$ and a functor $F_2 : (Y)_1^m \to X_i$.
(3) A choice of k categories $Z_1, \cdots, Z_k$ and a functor $F_3 : (Z)_1^k \to X_j$.
(4) A functor $G_1 : (X)_1^{i-1} \times (Y)_1^m \times (X)_{i+1}^n \to X_0$ and a natural isomorphism α_1 between G_1 and the obvious functor $F_1 \circ_i F_2$.
(5) A functor $G_2 : (X)_1^{j-1} \times (Z)_1^k \times (X)_{j+1}^n \to X_0$ and a natural isomorphism α_2 between G_2 and the obvious functor $F_1 \circ_j F_3$.
(6) A functor $H : (X)_1^{i-1} \times (Y)_1^m \times (X)_{i+1}^{j-1} \times (Z)_1^k \times (X)_{j+1}^n \to X_0$ and a natural isomorphism β between H and the obvious functor $F_1 \circ_{i,j} (F_2, F_3)$.

Similarly, a dendrex of shape $C_n \circ_i (C_m \circ_j C_k)$ consists of the following data:

(1) A choice of $n+1$ categories $X_0, \cdots, X_n$ and a functor $F_1 : (X)_1^n \to X_0$.
(2) A choice of m categories $Y_1, \cdots, Y_n$ and a functor $F_2 : (Y)_1^m \to X_i$.
(3) A choice of k categories $Z_1, \cdots, Z_k$ and a functor $F_3 : (Z)_1^k \to Y_j$.
(4) A functor $G_1 : (X)_1^{i-1} \times (Y)_1^m \times (X)_{i+1}^n \to X_0$ and a natural isomorphism α_1 between G_1 and the obvious functor $F_1 \circ_i (F_2)$.
(5) A functor $G_2 : (Y)_1^{j-1} \times (Z)_1^k \times (Y)_{j+1}^m \to Y_0$ and a natural isomorphism α_2 between G_2 and the obvious functor $F_2 \circ_j F_3$.
(6) A functor $H : (X)_1^{i-1} \times (Y)_1^{j-1} \times (Z)_1^k \times (Y)_{j+1}^m \times (X)_{i+1}^n \to X_0$ and a natural isomorphism β between H and the obvious functor $F_1 \circ_i (F_2 \circ_j F_3)$.

Let us now examine what is a weak 2-category $\mathcal{B}$ that has just one object (this is expected to be some kind of a monoidal category). By definition $\mathcal{B}$ is a map $\mathcal{B} : N_d(\mathcal{C}_B) \to wCat^1$ where B is a one-point category. The map $Sk_0(\mathcal{B})$ is just the choice of a category for every dendrex in $N_d(\mathcal{C}_B)_\eta$ (for a fixed η), i.e., it is simply a category M. The map $Sk_1(\mathcal{B})$ amounts to a choice of a functor

$$\gamma : M^n \to M$$

for all $n \geq 0$. We call this functor the *unbiased tensor product* of n-elements and write $(a_1 \otimes \cdots \otimes a_n)$ instead of $\gamma(a_1, \cdots, a_n)$. As a special case we include $n = 0$ which amounts to a map $I \to M$, that is the same as choosing an object in M which is called the *unit*. The map $Sk_2(\mathcal{B})$ amounts to specifying certain isomorphisms as follows. Given objects $a_1, \cdots, a_n$, an integer $0 \leq i \leq n$, and an integer $i \leq j \leq n$ there is an isomorphism

$$\begin{array}{c} (a_1 \otimes \cdots \otimes a_{i-1} \otimes (a_i \otimes \cdots \otimes a_j) \otimes a_{j+1} \otimes \cdots \otimes a_n) \\ \downarrow \\ (a_1 \otimes \cdots \otimes a_n) \end{array}$$

which is natural in each a_k (Recall that we interpret the tensor product of 0 elements to be the chosen unit and so these diagrams include unit laws). The map $Sk_3(\mathcal{B})$ provides two types of coherence constraints for these isomorphisms. To state these constraints we use a similar convention as above; for objects $a_i, \cdots, a_j$ we denote the formal sequence $a_i \otimes \cdots \otimes a_j$ by a_i^j. The first coherence constraint states the commutativity of diagrams of the sort

$$\begin{array}{ccc} (a_1^i \otimes (a_{i+1}^j) \otimes a_{j+1}^k \otimes (a_{k+1}^m) \otimes a_{m+1}^t) & \longrightarrow & (a_1^i \otimes (a_{i+1}^j) \otimes a_{j+1}^t) \\ \downarrow & & \downarrow \\ (a_1^k \otimes (a_{k+1}^m) \otimes a_{m+1}^t) & \longrightarrow & (a_1^t) \end{array}$$

where the arrows are obtained from the given unbiased compositions. The second type of coherence constraints state the commutativity of diagrams of the sort

$$\begin{array}{ccc} (a_1^i \otimes (a_{i+1}^j \otimes (a_{j+1}^k) \otimes a_{k+1}^m) \otimes a_{m+1}^t) & \longrightarrow & (a_1^i \otimes (a_{i+1}^m) \otimes a_{m+1}^t) \\ \downarrow & & \downarrow \\ (a_1^j \otimes (a_{j+1}^k) \otimes a_{k+1}^t) & \longrightarrow & (a_1^t) \end{array}$$

where the arrows are again given by the unbiased tensor products. Lemma 4.6.3 shows that this is precisely the information present in the weak 2-category $\mathcal{B}$.

REMARK 4.6.4. The terminology 'unbiased' is taken from [**31**]. Leinster introduces there the notion of an unbiased monoidal category which is almost identical (and is equivalent) to the notion we arrived at here. The term unbiased refers to the explicitly given tensor products of n objects for all $n \geq 0$ rather than the more usual bias towards a 0-ary tensor product (i.e. a unit) and a binary tensor product. We note also that if one unpacks the notion of a map between weak 2-categories with one object one obtains essentially the same definition as that of a weak monoidal functor between unbiased monoidal categories [**31**].

One can similarly unpack the definition of an arbitrary weak 2-category. Of course the resulting notion will not be identical with that of bicategories (see [**3**] for a definition) but would rather be an unbiased version similar to the situation above. However, we still expect that our notion of weak 2-categories is essentially the same as bicategories in some sense to be made precise. Speculating about possible comparisons between our notion of weak n-categories and other such definitions is very difficult at best and we do not attempt one here.

We end this section by conjecturing about the Baez-Dolan stabilization hypothesis for the notion of weak n-categories just introduced. The Baez-Dolan stabilization hypothesis (see [**1**]) is a general conjecture about weak n-categories that before it can be proved for a specific definition of weak n-categories must first be interpreted and made precise for that definition. For example, Simpson [**42**] states and proves the stabilization hypothesis for Tamsamani's definition of weak n-categories. Let us first explain what the stabilization hypothesis is. Assume for the discussion that some notion of weak n-categories is fixed. A weak n-category typically consists of j dimensional cells for all $0 \leq j \leq n$, such that the 0-cells are the objects, the 1-cells are the arrows, the 2-cells are arrows between arrows and so on. For an integer $k \geq 0$, a *k-monoidal* n-category is a weak $(n+k)$-category that has just one j-cell for each $0 \leq j < k$. For low dimensions we have the following table of k-monoidal n-categories:

	n=0	n=1	n=2
k=0	sets	categories	2-categories
k=1	monoids	monoidal categories	monoidal 2-categories
k=2	commutative monoids	braided monoidal monoidal	braided monoidal 2-categories
k=3	"	symmetric monoidal categories	weakly involutory 2-categories
k=4	"	"	strongly involutory 2-categories
k=5	"	"	"

and it would appear that each column becomes more and more commutative as k increases and stabilizes at $k = n + 2$. The Baez-Dolan hypothesis is that indeed for any reasonable definition of weak n-categories, each column in the table of k-monoidal n-categories stabilizes at $k = n + 2$.

As for our definition of weak n-categories, we now give an interpretation of the stabilization hypothesis and take a small step towards proving it. For each set B we have the operad $\mathcal{C}_B$ such that the dendroidal set $\underline{dSet}(N_d(\mathcal{C}_B), wCat^n)$ is, by definition, the dendroidal set of weak $n+1$ categories whose set of objects is equal to the set B. Let us denote $A = N_d(\mathcal{C}_*)$, where $\mathcal{C}_*$ is $\mathcal{C}_B$ and B is a one-point set. That means that a 1-monoidal n-category is a dendrex of shape η in $\underline{dSet}(A, wCat^n)$. Now, a 2-monoidal n-category should be an $(n+2)$-category with just one object and one arrow, that means that it is a category enriched in 1-monoidal n-categories that has itself just one object. In other words a 2-monoidal n-category is a dendrex of shape η in $\underline{dSet}(A, \underline{dSet}(A, wCat^n))$. Motivated by this, it makes sense to define a $(k+1)$-monoidal n-category to be a category with one object enriched in k-monoidal n-categories. We make this precise in the following definition:

DEFINITION 4.6.5. Let $n \geq 0$ be fixed. For $k \geq 0$ We define recursively the dendroidal set $wCat^n_k$ of *k-monoidal n-categories* as follows. For $k = 0$ we set

$$wCat^n_0 = wCat^n$$

and for $k > 0$

$$wCat^n_k = \underline{dSet}(A, wCat^n_{k-1}).$$

A dendrex of shape η in $wCat^n_k$ is then called a *k-monoidal n-category*.

REMARK 4.6.6. Notice that

$$wCat^n_k = \underline{dSet}(A^{\otimes_k}, wCat^n).$$

CONJECTURE 4.6.7. *(The Baez-Dolan stabilization hypothesis for our notion of n-categories) For a fixed $n \geq 0$, we have the equality*

$$wCat^n_k \cong wCat^n_{n+2}$$

for any $k \geq n + 2$.

As a step towards a proof we make the following conjecture:

CONJECTURE 4.6.8. *For any $n \geq 0$ the dendroidal set $wCat^n_n$ is a strict inner Kan complex.*

REMARK 4.6.9. Note that we already know from the general theory that $wCat^n_n$ is an inner Kan complex.

PROPOSITION 4.6.10. *The conjecture just stated implies the Baez-Dolan stabilization conjecture.*

PROOF. Fix $j > 2$. Using Remark 4.6.6 we proceed as follows. We have to prove that $\underline{dSet}(A^{\otimes_{n+j}}, wCat^n) = \underline{dSet}(A^{\otimes_{n+2}}, wCat^n)$ where we assume that $\underline{dSet}(A^{\otimes_n}, wCat^n)$ is a strict inner Kan complex. By Theorem 3.5.12 there is thus an operad $\mathcal{P}$ such that $\underline{dSet}(A^{\otimes_n}, wCat^n) = N_d(\mathcal{P})$. We now have:

$$\underline{dSet}(A^{\otimes_{n+j}}, wCat^n) = \underline{dSet}(A^{\otimes_j}, \underline{dSet}(A^{\otimes_n}, wCat^n)) = \underline{dSet}(A^{\otimes_j}, N_d(\mathcal{P}))$$

which by adjunction is

$$\underline{Operad}(\tau_d(A^{\otimes_j}), \mathcal{P}).$$

However, A is actually the dendroidal nerve of the symmetric operad As describing associative monoids. By Lemma 3.1.2 we have:

$$\tau_d(A^{\otimes_j}) = \tau_d(N_d(As)^{\otimes_j}) \cong As \otimes_{BV} As \otimes_{BV} \cdots \otimes_{BV} As = As^{\otimes_j}.$$

It is known ([**8**]) that for $j \geq 2$

$$As^{\otimes_j} = Comm,$$

the operad describing commutative monoids and the result now follows. □

4.7. Quillen model structure on $dSet$

We end this chapter and the thesis by giving a conjecture about the existence of a Quillen model structure on $dSet$. We show that if this conjecture is true then X algebras in E for a normal X and an inner Kan complex E have a nice homotopy invariance property. We expect this model structure to be very important in the general future theory of dendroidal sets as well as for the theory of A_∞-spaces, A_∞-categories, weak n-categories, and general up-to-homotopy structures.

To formulate the conjecture about the existence of such a Quillen model structure let us first recall the Joyal model structure on the category of simplicial sets.

First, recall the nerve-realisation adjunction $sSet \underset{N}{\overset{\tau}{\rightleftarrows}} Cat$. For a given simplicial set X, the category $\tau(X)$ is commonly called the *fundamental category* of X and is denoted by $\tau_1(X)$ (terminology taken from [**23**]). We can now define the functor $\tau_0 : sSet \to Set$ where $\tau_0(X)$ is the set of isomorphism classes of objects of the category $\tau_1(X)$. Given two simplicial sets X and Y we write

$$\tau_0(X, Y) = \tau_0(\underline{sSet}(X, Y)).$$

DEFINITION 4.7.1. A *weak categorical equivalence* is a map $f : X \to Y$ of simplicial sets with the property that for any inner Kan simplicial set K (a quasi-category in the terminology of [**24**]) the induced map

$$\tau_0(Y, K) \to \tau_0(X, K)$$

is an isomorphism of sets.

THEOREM 4.7.2. *The category sSet of simplicial sets admits a cartesian Quillen model structure where the weak equivalences are the weak categorical equivalences and the cofibrations are the monomorphisms. Under this model structure the fibrant objects are precisely the inner Kan simplicial sets.*

The proof, and much more theory related to this model structure, which we call the Joyal model structure on simplicial sets, will appear in Joyal's book [**24**].

Mimicking the definition of weak categorical equivalences we proceed as follows. The functor $\tau_d : dSet \to Operad$ gives rise to a functor $\tau_{0_d} : dSet \to Set$ defined for a dendroidal set X by $\tau_{0_d}(X) = \tau_0(i^*(X))$. For dendroidal sets X and Y we define

$$\tau_{0_d}(X, Y) = \tau_{0_d}(\underline{dSet}(X, Y)).$$

DEFINITION 4.7.3. Given two dendroidal sets X and Y, we call a map $f : X \to Y$ a *weak operadic equivalence* if for any inner Kan dendroidal set K the induced map

$$\tau_0(Y, K) \to \tau_0(X, K)$$

is an isomorphism of sets.

Following [**9**] (page 320) we make the following definition:

DEFINITION 4.7.4. Let $f : X \to Y$ be a monomorphism between dendroidal sets. We call f *normal* if for every dendrex $t \in Y_T$ that does not factor through f the only isomorphism of T that fixes t is the identity.

This definition extends Definition 2.6.6 in the sense that a dendroidal set is normal if, and only if, the inclusion $\phi \to X$ is normal.

Remark 4.7.5. Clearly any map between normal dendroidal sets is normal. In particular, if X is normal then the inclusion $Sk_k(X) \to X$ is normal for all $k \geq 0$.

Conjecture 4.7.6. *The category dSet of dendroidal sets admits a Quillen model structure where the weak equivalences are the weak operadic equivalences and the cofibrations are the normal monomorphisms. Furthermore, with the tensor product of dendroidal sets, this model structure is a monoidal model category. Under this model structure the fibrant objects are the inner Kan complexes and the cofibrant objects are the normal dendroidal sets. Moreover, in the diagram*

$$\begin{array}{ccc} Cat & \underset{j^*}{\overset{j_!}{\rightleftarrows}} & Operad \\ {\scriptstyle\tau}\uparrow\downarrow{\scriptstyle N} & & {\scriptstyle\tau_d}\uparrow\downarrow{\scriptstyle N_d} \\ sSet & \underset{i^*}{\overset{i_!}{\rightleftarrows}} & dSet \end{array}$$

from Theorem 3.1.4 we expect all of the adjunctions to be Quillen adjunctions.

Remark 4.7.7. We expect the fact that the inner Kan complexes form an exponential ideal in *dSet* (Corollary 3.6.4) to play a fundamental role in proving the conjecture. We also expect that the functor $K : dSet \to dSet$ given in Lemma 3.7.4, is actually a fibrant replacement functor in this conjectured model structure. Note that we have already shown that $j_!$ and j^* form a Quillen adjunction (Lemma 1.6.5. In **[24]** it is shown that τ and N also form a Quillen adjunction. The fact that $i_!$ preserves cofibrations is obvious and in fact it is quite simple to show that it preserves weak equivalences too, so that $i_!$ and i^* also form a Quillen adjunction. The difficult part is thus proving that τ_d and N_d form a Quillen adjunction.

Assuming the conjecture above holds, we can prove a homotopy invariance property for algebras in an inner Kan complex. Let us first clarify what we mean by such an invariance property. First, recall some terminology from **[23]**. If S is a quasi-category (i.e., an inner Kan simplicial set) a 1-simplex s is called a *weak equivalence* if its image under $\tau : sSet \to Cat$ is invertible. Now, given any 1-simplex $s : \Delta[1] \to S$, we have that s is a weak equivalence if, and only if, it can be extended to a map from S^∞:

$$\begin{array}{ccc} \Delta[1] & \xrightarrow{s} & S \\ \downarrow & \nearrow & \\ S^\infty & & \end{array}$$

where S^∞ is the infinite dimensional sphere, i.e., the nerve of the category H which is the interval in the folk model structure on *Cat*. The vertical map $\Delta[1] \to S^\infty$ is a trivial cofibration in the Joyal model structure. Extending this to dendroidal sets we have:

Definition 4.7.8. Let X be an inner Kan complex. A dendrex $x \in X_{L_1}$ is called a *weak equivalence* if $i^*(x)$ is a weak equivalence in the inner Kan simplicial set $i^*(X)$.

Let E and X be two dendroidal sets and consider the dendroidal set $Alg(X, E)$ of X-algebras in E. Such an algebra is thus a map $A : X \to E$. The map $Sk_0(A) : Sk_0(X) \to Sk_0(E)$ consists of a choice of elements in E_η and we think of A as defining an algebraic structure on the element $Sk_0(A)$. Suppose now that E is an inner Kan complex and A is a fixed X-algebra in E. We say that the algebraic structure on $Sk_0(A)$ given by A has the *homotopy invariance property* if given another choice of elements in X_η, given as a map $A' : Sk_0(X) \to Sk_0(E)$, and for each $w \in E_\eta$ a weak equivalence

$$f_w : A(w) \to A'(w)$$

there is an E-algebra structure on A' and a map of E-algebras $f : A \to A'$ that extends the given f_w.

THEOREM 4.7.9. *Let X be a normal dendroidal set and E an inner Kan complex. If conjecture 4.7.6 holds then all E-algebras in X have the homotopy invariance property.*

PROOF. Assume that we have an algebra $A : X \to E$ and a choice of weak equivalences $f_w : A(w) \to A'(w)$ as above. Since a weak equivalence in X is the same as a map $i_!(S^\infty) \to X$ it follows that the choice of the maps f_w produces a map $f : i_!(S^\infty) \to \underline{dSet}(Sk_0(X), E))$. We now have the following commutative diagram

$$\begin{array}{ccc} \Omega[\eta] & \xrightarrow{A} & \underline{dSet}(X, E) \\ \downarrow & & \downarrow \\ i_!(S^\infty) & \xrightarrow{f} & \underline{dSet}(Sk_0(X), E) \end{array}$$

where the vertical arrow on the right is induced by the cofibration $Sk_0(X) \to X$, (see Remark 4.7.5 for why this is a cofibration) and is thus a fibration (by the push-out product axiom). Notice that a diagonal filler for this diagram corresponds to an element $\bar{f} \in \underline{dSet}(X, E)_{L_1}$, and thus to a map of algebras $\bar{f} : A \to A'$ which extends f - which is what we would like to show. It therefore suffices to show that this diagram has a diagonal filler, which will follow if the left vertical map is a trivial cofibration. Notice that this map is actually the image under $i_! : sSet \to dSet$ of the map $\Delta[1] \to S^\infty$, which is a trivial cofibration in the Joyal model structure on simplicial sets. It thus follows from the assumption that $i_!$ is a left Quillen functor that $\Omega[\eta] \to i_!(S^\infty)$ is indeed a trivial cofibration, which thus completes the proof. □

REMARK 4.7.10. This homotopy invariance property is related to a similar property (Theorem 3.5 in [**4**]) for algebras over cofibrant operads.