CHAPTER 3

Operads and dendroidal sets

This chapter is concerned with the relation between operads, dendroidal sets, and simplicial sets. The relation is established by means of the dendroidal nerve functor which associates with every operad a dendroidal set - its nerve. The notion of an inner Kan dendroidal set is then introduced. This notion is a generalization of a notion given by Boardman and Vogt for simplicial sets in [**7**]. The technique of anodyne extensions is then imported from the theory of simplicial sets and is demonstrated by a simple example. Grafting in dendroidal sets is then discussed as well as homotopy in a dendroidal set. It is shown that with each inner Kan complex one can associate a homotopy operad, which is then used to deduce a characterization of nerves of operads as dendroidal sets satisfying certain strict filling conditions. Following is a proof that the inner Kan complexes form an exponential ideal in the category of dendroidal sets, a result which generalizes a recent result of Joyal [**24**] for simplicial sets. The chapter ends by introducing a process that turns an arbitrary dendroidal set into an inner Kan complex.

3.1. Nerves of operads

The functor relating operads to dendroidal sets is the operadic nerve functor. It is the aim of this section to introduce this functor and study it, and other related functors, in detail.

DEFINITION 3.1.1. Consider the probe $F : \Omega \to \mathit{Operad}$ which sends a tree T to the operad $\Omega(T)$, and the induced adjunction $dSet \underset{N_F}{\overset{|\cdot|_F}{\rightleftarrows}} \mathit{Operad}$. The functor N_F is called the *operadic nerve* functor and will be denoted by N_d. The functor $|\cdot|_F$ is called the *operadic realization* functor and will be denoted by τ_d.

Explicitly, for an operad P, its nerve is the dendroidal set given by

$$N_d(\mathcal{P})_T = \mathit{Operad}(\Omega(T), \mathcal{P}).$$

It is practically a tautology that for any tree $T \in ob(\Omega)$

$$N_d(\Omega(T)) = \Omega[T].$$

Slightly less trivial is the fact that for any operad $\mathcal{P}$

$$\tau_d(N_d(\mathcal{P})) \cong \mathcal{P},$$

a property that will be used on several occasions below.

The categories Cat and $sSet$ are both cartesian closed categories and, with respect to these monoidal structures, both of the functors τ and N are strong monoidal. As we have seen, the categories Operad and $dSet$ also carry a closed monoidal structure, and we turn now to investigate the properties of the functors τ_d and N_d with respect to these monoidal structures.

We would first like to relate the tensor product of dendroidal sets with the Boardman-Vogt tensor product of operads. Recall that the tensor product of dendroidal sets is defined by cocontinuously extending the formula

$$\Omega[T] \otimes \Omega[S] = N([T] \otimes [S])$$

where $N : BrdPoset \to dSet$ is the nerve functor defined in Section 2.5. Since we now also have the nerve functor of operads, we can define a tensor product on $dSet$ by cocontinuously extending the formula

$$\Omega[T] \otimes \Omega[S] = N_d(\Omega(T) \otimes_{BV} \Omega(S)).$$

However, both approaches yield essentially the same monoidal structure. This follows from the easily established equality

$$N([T] \otimes [S]) \cong N_d(\Omega(T) \otimes_{BV} \Omega(S))$$

which holds for any two trees T and S.

To compare the Boardman-Vogt tensor product with the tensor product of dendroidal sets it is convenient to notice first that any operad $\mathcal{P}$ can be written canonically as a colimit of operads of the form $\Omega(T)$, namely

$$\mathcal{P} \cong \varinjlim_{\Omega(T) \to \mathcal{P}} \Omega(T),$$

and since the Boardman-Vogt tensor product of operads is closed, we obtain the formula

$$\mathcal{P} \otimes_{BV} \mathcal{Q} = \varinjlim(\Omega(T) \otimes_{BV} \Omega[S])$$

with the colimit taken over the obvious diagram.

Lemma 3.1.2. *For any two operads $\mathcal{P}$ and $\mathcal{Q}$*

$$\tau_d(N_d(\mathcal{P}) \otimes N_d(\mathcal{Q})) \cong \mathcal{P} \otimes_{BV} \mathcal{Q}.$$

Proof. By definition:

$$N_d(\mathcal{P}) \otimes N_d(\mathcal{Q}) = \varinjlim(\Omega[T] \otimes \Omega[S]).$$

Since

$$N([T] \otimes [S]) \cong N_d(\Omega(T) \otimes_{BV} \Omega(S))$$

we obtain that

$$\tau_d(\Omega[T] \otimes \Omega[S]) = \tau_d(N([T] \otimes [S])) \cong \tau_d(N_d(\Omega(T) \otimes_{BV} \Omega(S)) \cong \Omega(T) \otimes_{BV} \Omega(S).$$

Since τ_d, as a left adjoint, commutes with colimits we obtain that

$$\tau_d(N_d(\mathcal{P}) \otimes N_d(\mathcal{Q})) \cong \varinjlim(\Omega(T) \otimes_{BV} \Omega(S)) \cong \mathcal{P} \otimes_{BV} \mathcal{Q}$$

as claimed. □

Remark 3.1.3. Notice that this lemma implies that the Boardman-Vogt tensor product of operads is completely determined by the tensor product of broad posets. While this fact might not be very important in the general theory, it is remarkable that the quite involved Boardman-Vogt tensor product is already contained within a much simpler notion.

We summarise the relation between categories, operads, simplicial sets, and dendroidal sets in the following theorem.

THEOREM 3.1.4. *In the diagram*

$$\begin{array}{ccc} Cat & \underset{j^*}{\overset{j_!}{\rightleftarrows}} & Operad \\ \tau \uparrow \downarrow N & & \tau_d \uparrow \downarrow N_d \\ sSet & \underset{i^*}{\overset{i_!}{\rightleftarrows}} & dSet \end{array}$$

all pairs of functors are adjunctions with the left adjoint on top or to the left. Furthermore, the following canonical commutativity relations hold

$$\begin{aligned} \tau N &\cong id \\ \tau_d N_d &\cong id \\ i^* i_! &\cong id \\ j^* j_! &\cong id \\ j_! \tau &\cong \tau_d i_! \\ N j^* &\cong i^* N_d \\ i_! N &\cong N_d j_!. \end{aligned}$$

If we consider the cartesian structures on Cat and $sSet$, the Boardman-Vogt tensor product on $Operad$, and the tensor product of dendroidal sets then the four categories are symmetric closed monoidal categories and the functors $i^*, i_!, N, \tau, j^*, j_!$ and τ_d are strong monoidal.

PROOF. The commutativity relations are easily seen to hold. The fact that N is strong monoidal is well known (and easily proved). Proving that $j_!$ is strong monoidal is also easy. To show that $i_!$ is strong monoidal we need to prove that for two simplicial sets X and Y

$$i_!(X \times Y) \cong i_!(X) \otimes i_!(Y).$$

Since $i_!$ is a left adjoint it commutes with colimits, and it therefore follows that it is enough to show that the formula holds for representable simplicial sets, which we now do. Recall that we denote by L_k the linear tree with k vertices. We now have:

$$\begin{aligned} i_!(\Delta[n] \times \Delta[m]) &\cong i_!(N([n]) \times N([m])) \\ &\cong i_!(N([n] \times [m])) \\ &\cong N_d j_!([n] \times [m]) \\ &\cong N_d(j_![n] \otimes_{BV} j_![m]) \\ &\cong N_d(\Omega(L_n) \otimes_{BV} \Omega(L_m)) \\ &\cong \Omega[L_n] \otimes \Omega[L_m] \\ &\cong i_!(\Delta[n]) \otimes i_!(\Delta[m]). \end{aligned}$$

as claimed.

To prove that τ_d is strong monoidal we need to show that

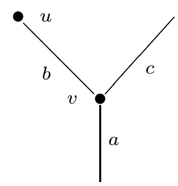
$$\tau_d(X \otimes Y) \cong \tau_d(X) \otimes_{BV} \tau_d(Y)$$

holds for any two dendroidal sets X and Y. Once again it is enough to establish the result for representables, and indeed we have

$$\begin{aligned} \tau_d(\Omega[T] \otimes \Omega[S]) &= \tau_d(N_d(\Omega(T)) \otimes N_d(\Omega(S))) \cong \\ \Omega(T) \otimes_{BV} \Omega(S) &= \tau_d(N_d(\Omega(T))) \otimes_{BV} \tau_d(N_d(\Omega(S))) = \\ \tau_d(\Omega[T]) \otimes_{BV} \tau_d(\Omega[S]) \end{aligned}$$

as required. The rest of the proof follows along similar lines and is omitted. □

REMARK 3.1.5. In general, the canonical map $\tau i^*(X) \to j^*\tau_d(X)$ is not an isomorphism. Consider for example the tree T given by



For the dendroidal set $\Omega[T]$ we have that $i^*\Omega[T]$ is a disjoint union of three copies of $\Omega[\eta]$ and thus $\tau i^*\Omega[T]$ is simply a category with three different objects and non-identity arrows. On the other hand, the operad $\tau_d\Omega[T]$ contains the unary operation $v \circ_1 u : c \to a$ which is thus also present in $j^*\tau_d\Omega[T]$ and so we have that $j^*\tau_d\Omega[T] \ncong \tau i^*\Omega[T]$.

The nerve functor $N : Cat \to sSet$ can easily be shown to commute with internal Homs in the sense that for any two categories $\mathcal{C}$ and $\mathcal{D}$, the equation

$$N(\underline{Cat}(\mathcal{C}, \mathcal{D})) \cong \underline{sSet}(N(\mathcal{C}), N(\mathcal{D}))$$

holds. Lemma 3.1.2 allows us to prove a similar result for the dendroidal nerve functor.

LEMMA 3.1.6. *The dendroidal nerve functor commutes with internal Homs in the sense that for any two operads $\mathcal{P}$ and $\mathcal{Q}$ we have*

$$N_d(\underline{Operad}(\mathcal{P}, \mathcal{Q})) \cong \underline{dSet}(N_d(\mathcal{P}), N_d(\mathcal{Q})).$$

PROOF. For a tree $T \in ob(\Omega)$ we have the equations:

$$\begin{aligned} \underline{dSet}(N_d(\mathcal{P}), N_d(\mathcal{Q}))_T &= dSet(N_d(\mathcal{P}) \otimes \Omega[T], N_d(\mathcal{Q})) = \\ dSet(N_d(\mathcal{P}) \otimes N_d(\Omega(T)), N_d(\mathcal{Q})) &\cong Operad(\tau_d(N_d(\mathcal{P}) \otimes N_d(\Omega(T))), \mathcal{Q}) \cong \\ Operad(\mathcal{P} \otimes_{BV} \Omega(T), \mathcal{Q}) &\cong Operad(\Omega(T), \underline{Operad}(\mathcal{P}, \mathcal{Q})) = \\ N_d(\underline{Operad}(\mathcal{P}, \mathcal{Q}))_T \end{aligned}$$

which prove the claim. □

Another functor that commutes with internal Homs is $i_!$. To show that, we use the useful, and easily verified property, that the the functor $i_! : sSet \to dSet$ embeds simplicial sets in dendroidal sets as a sieve, i.e., that given a simplicial set Y and an arbitrary dendroidal set X, if there is a map $i_!(X) \to Y$ in Ω then $Y = i_!(Y')$ for some simplicial set Y'.

LEMMA 3.1.7. *For simplicial sets X and Y we have:*

$$\underline{dSet}(i_!(X), i_!(Y)) \cong i_!(\underline{sSet}(X, Y)).$$

PROOF. Notice that if T is a tree that is not of the form L_k then

$$\underline{dSet}(i_!(X), i_!(Y))_T = dSet(i_!(X) \otimes \Omega[T], i_!(Y))$$

would be empty if $X \neq \phi$. Now, for a linear tree L_k we have

$$\begin{aligned} \underline{dSet}(i_!(X), i_!(Y))_{L_k} &= dSet(i_!(X) \otimes \Omega[L_k], i_!(Y)) = \\ dSet(i_!(X) \otimes i_!(\Delta[k]), i_!(Y)) &= dSet(i_!(X \times \Delta[k]), i_!(Y)) = \\ sSet(X \times \Delta[k], i^* i_!(Y)) &= sSet(X \times \Delta[k], Y) = \\ \underline{sSet}(X, Y)_k &= i_!(\underline{sSet}(X, Y))_{L_k} \end{aligned}$$

as claimed. □

3.2. Inner Kan complexes

In this section we introduce the notion of inner Kan complexes in the category of dendroidal sets. We start off by motivating the definition, relating it to inner Kan complexes in the category of simplicial sets (also studied under the name ”quasi-categories” by Joyal in **[23, 24]**). Once the definition is given, we provide a class of examples and examine the relation between coskeletality and strict inner Kan complexes, as a first step to characterizing the latter.

Recall that a horn $\Lambda^k[n]$ in the simplicial sense is said to be inner if $0 < k < n$. In **[7]** (page 102) the authors make the following definition:

DEFINITION 3.2.1. A simplicial set X is said to satisfy the restricted Kan condition if every inner horn $\Lambda^k[n] \to X$ can be filled.

We will call such a simplicial set an *inner Kan* simplicial set. The need for such a definition stems from the fact that weak algebraic structures (for example, A_∞-spaces) and their weak maps usually do not form a category. The problem is that, generally, the composition of such maps (if it is at all defined) is not associative. That the notion of an inner Kan simplicial set is at least a plausible replacement of a category is seen by the fact that the nerve of a category always satisfies the restricted Kan condition (we prove a stronger result below). However, there are many simplicial sets that do satisfy this condition without them being nerves of categories, among which lie the simplicial set of A_∞-spaces (as is shown in **[7]**).

In the more recent work **[24]** Joyal is extensively studying inner Kan simplicial sets (which he calls quasi-categories) as an extension of the theory of categories. As Joyal put it himself: ”You find yourself in the situation where most of the results of category theory can be extended to quasi-categories. It’s just that the proof is anything between 10 to 100 times more difficult”. The extra labour needed to prove those theorems arises from the fact that quasi-categories can be thought of as special weak ω-categories, and as such carry with them the complexity of maps between maps between maps between maps..... However, the resulting theory is applicable in many situations where ordinary category theory is too strict.

Continuing with the main theme of this work, that operads are generalized categories, it is very natural to extend the inner Kan condition from simplicial sets to dendroidal sets. This is done by means of the following definition.

DEFINITION 3.2.2. Let X be a dendroidal set. X is said to satisfy the *inner Kan condition* with respect to the tree T if for any inner horn $h : \Lambda^e\Omega[T] \to X$, there is a dendrex $t : \Omega[T] \to X$ such that the diagram

$$\begin{array}{ccc} \Lambda^e\Omega[T] & \xrightarrow{h} & X \\ \downarrow & \nearrow_{t} & \\ \Omega[T] & & \end{array}$$

commutes, where the vertical arrow is the inclusion. If X satisfies the inner Kan condition with respect to all trees T then X is called an *inner Kan complex.* When the filler for the horn is unique we will say that X satisfies the *strict* inner Kan condition and that X is a *strict* inner Kan complex.

The proof of the following proposition, relating the inner Kan condition for simplicial sets and dendroidal sets, is trivial.

PROPOSITION 3.2.3. *Let S be a simplicial set and D a dendroidal set.*

(1) $i_!(S)$ is an inner Kan complex if, and only if, S is.
(2) If D is an inner Kan complex then so is $i^*(D)$.

The following lemma provides a whole class of examples of strict inner Kan complexes.

LEMMA 3.2.4. *Let $\mathcal{P}$ be an operad. The dendroidal set $X = N_d(\mathcal{P})$ is a strict inner Kan complex.*

PROOF. A dendrex $x \in X_T$ is a map $x : \Omega[T] \to N_d(\mathcal{P})$ which, by adjunction, is the same as a map of operads $\Omega_\pi(\bar{T}) \to P$, where $\bar{T}$ is an arbitrary (but fixed) planar representative of T. Since $\Omega_\pi(\bar{T})$ is a free planar operad generated by operations corresponding to the vertices of the tree $\bar{T}$, it follows that x is equivalent to a labelling of $\bar{T}$ as follows. The edges are labelled by objects of $\mathcal{P}$ and the vertices are labelled by operations in $\mathcal{P}$ where the input of such an operation is the tuple of labels of the incoming edges to the vertex and the output is the label of the outgoing edge from the vertex. Any inner horn $\Lambda^e[T] \to N_d(\mathcal{P})$ is easily seen to be equivalent to such a labelling of the tree T and thus determines a unique filler. □

The strict inner Kan condition is very strong and in fact we will show below that the strict inner Kan complexes are precisely those dendroidal sets that are nerves of operads. One can easily turn a strict inner Kan complex into a non-strict one, simply by adding new dendrices that fill already existing horns. More natural examples of inner Kan complexes that are usually not strict will be seen to arise as suitable nerves of operads in a symmetric monoidal model category $\mathcal{E}$, when homotopy is built into the nerve construction. For now, we exhibit the relation between certain strict filling conditions and coskeletality.

PROPOSITION 3.2.5. *Let X be a dendroidal set and $m \geq 2$ an integer. If X satisfies the strict inner Kan condition for all trees T of degree at least m, then X is m-coskeletal.*

PROOF. Let Y be an arbitrary dendroidal set and assume that a map $f : Sk_m(Y) \to Sk_m(X)$ is given. We have to show that f extends uniquely to a map $\hat{f} : Y \to X$. Suppose f were extended to a map $f_k : Sk_k(Y) \to Sk_k(X)$ for $k \geq m$. Let $y \in Sk_{k+1}(Y)$ be a non-degenerate dendrex and assume $y \notin Sk_k(Y)$. So $y \in Y_T$ and T has exactly $k+1$ vertices. Choose an inner horn $\Lambda^e[T]$ (such an inner horn exists since $k \geq 2$). The collection $\{\beta^* y\}_{\beta \neq \partial_e}$ where $\beta : S \to T$ runs over all faces of T, defines a horn $\Lambda^e[T] \to Y$. Since this horn factors through the k-skeleton of Y, we obtain by applying f_k, a horn $\Lambda^e[T] \to X$ in X given by $\{f_k(\beta^* y)\}_{\beta \neq \partial_e}$. Let $f_{k+1}(y) \in X_T$ be the unique filler of that horn. By construction we have that for each $\beta \neq \partial_e$

$$\beta^* f_{k+1}(y) = f_k(\beta^* y).$$

It thus remains to show the same for ∂_e. The dendrices $f_k(\partial_e^* y)$ and $\partial_e^* f_{k+1}(y)$ both have the same boundary and they are both of shape S where S has k vertices. Since $k \geq 2$, S has an inner face, but then it follows that both $f(\partial_e^* y)$ and $\partial_e^* f_{k+1}(y)$ are fillers for the same inner horn in X and they are thus equal. By repeating the process for all non-degenerate dendrices in $Sk_{k+1}(Y)$ it follows that f_k can be extended to $f_{k+1} : Sk_{k+1}(Y) \to Sk_{k+1}(X)$. This holds for all $k \geq m$ which implies that f can be extended to $\hat{f} : Y \to X$.

To show the uniqueness of $\hat{f}$ assume that g is another extension of f. Suppose it has been shown that $\hat{f}$ and g agree on all dendrices of shape T where T has at most $k \geq m$ vertices, and let $y \in X_S$ be a dendrex of shape S where S has $k+1$ vertices. But then $\hat{f}(y)$ and $g(y)$ are both dendrices in X that have the same boundary. Since $k \geq 2$ it follows that these dendrices are both fillers for the same inner horn and so are equal. This proves that $\hat{f} = g$. □

COROLLARY 3.2.6. *Let $\mathcal{P}$ be an operad. Since the dendroidal set $N_d(\mathcal{P})$ is a strict inner Kan complex it follows that it is 2-coskeletal.*

PROPOSITION 3.2.7. *Let X be a dendroidal set and $k \geq 0$ an integer. If X is k-coskeletal then X satisfies the strict inner Kan condition for all trees T with $|T| \geq k+2$.*

PROOF. Let T be a tree with $|T| \geq k+2$ and e an inner face of T. Consider the inner horn extension problem

$$\begin{array}{ccc} \Lambda^e[T] & \longrightarrow & X \\ \downarrow & \nearrow & \\ \Omega[T] & & \end{array}$$

in X. Since X is k-coskeletal, this problem is equivalent to finding the dotted arrow in the diagram

$$\begin{array}{ccc} Sk_k(\Lambda^e[T]) & \longrightarrow & Sk_k(X) \\ \downarrow & \nearrow & \\ Sk_k(\Omega[T]) & & \end{array}$$

The result will follow if we can show that the inclusion $Sk_k(\Lambda^e[T]) \to Sk_k(\Omega[T])$ is an isomorphism. Let $s \in Sk_k(\Omega[T])_S$ be a non-degenerate dendrex of shape S, if we can show that $s \in Sk_k(\Lambda^e[T])$ then we are done. We have that s is a map $s : \Omega[S] \to \Omega[T]$ and S has at most k vertices. Since T has at least $k+2$ vertices it follows that s factors through a sub-face of $\Omega[T]$ of codimension 2, say $\Omega[R] \to \Omega[R'] \to \Omega[T]$ and $\Omega[R'] \to \Omega[T]$ can be chosen to be different from ∂_e (Proposition 2.4.8). Thus s factors through the face $\Omega[R'] \to \Omega[T]$ and thus also through $\Lambda^e[T]$, which means that $s \in \Lambda^e[T]$, as needed. □

3.3. Anodyne extensions

In the theory of simplicial sets **[16]**, anodyne extensions are a technical tool that simplifies proofs significantly. We now develop the equivalent notion for dendroidal sets and provide a simple example that shows how anodyne extensions are typically used.

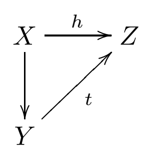
Definition 3.3.1. A class M of monomorphisms in *dSet* is called *saturated* if the following conditions are satisfied:

(1) All isomorphisms are in M.
(2) M is closed under pushouts.
(3) M is closed under retracts.
(4) M is closed under arbitrary sums.
(5) M is closed under countable unions.

See **[19]** for a similar definition for simplicial sets, and a detailed explanation of the closedness properties. Given an arbitrary class of monomorphism B, the saturated class generated by B is simply the intersection of all saturated classes containing B.

Definition 3.3.2. Let B be the class of all inner horn inclusions in *dSet*. The class of *anodyne extensions* is the saturated class generated by B.

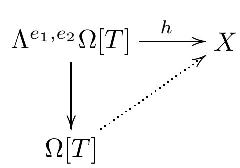
It is easy to show that given any anodyne extension $X \to Y$ and a map $X \to Z$, where Z is an inner Kan complex, there exists an extension $Y \to Z$:



It is precisely this property that makes anodyne extensions useful. Consider the following situation for example. Let e_1 and e_2 be two inner edges in a tree T and let

$$\Lambda^{e_1,e_2}\Omega[T] = \bigcup_{\partial_{e_1},\partial_{e_2}\neq\alpha\in\Phi_1(T)} \partial_\alpha\Omega[T]$$

be the dendroidal sub-set of $\Omega[T]$ which is the union of all of the faces of $\Omega[T]$ except the two inner ones corresponding to e_1 and e_2. Assume one is given the following extension problem:



with the vertical map being the inclusion. If X is an inner Kan complex it is expected that the dotted arrow would exist. This would follow if we can show that the inclusion $\Lambda^{e_1,e_2}\Omega[T] \to \Omega[T]$ is anodyne, as is indeed the case. More generally, for a sub-set $A \subseteq E(T)$ of inner edges let $\Lambda^A[T]$ be the union of all faces of $\Omega[T]$ except those arising by contracting an edge from A, which we denote by

$$\Lambda^A\Omega[T] = \bigcup_{\alpha\in\Phi(T)\setminus A} \partial_\alpha\Omega[T]$$

where we (somewhat loosely) write A also for the set $\{\partial_e \mid e \in A\}$.

Proposition 3.3.3. *For any non-empty $A \subseteq E(T)$ of inner edges in a tree T, the inclusion $\Lambda^A[T] \to \Omega[T]$ is anodyne.*

Proof. By induction on $k = |A|$. If $k = 1$ then the inclusion $\Lambda^A[T] \to \Omega[T]$ is just an inner horn inclusion, thus anodyne. Assume the proposition holds for $1 \leq n < k$ and suppose $|A| = k$. Choose an arbitrary $e \in A$ and put $B = A \backslash \{e\}$. The map $\Lambda^A[T] \to \Omega[T]$ factors as

$$\begin{array}{ccc} \Lambda^A[T] & \longrightarrow & \Lambda^B[T] \\ & \searrow & \downarrow \\ & & \Omega[T] \end{array}$$

The vertical map is anodyne by the induction hypothesis and it therefor suffices to prove that $\Lambda^A[T] \to \Lambda^B[T]$ is anodyne. The following diagram expresses that map as a pushout

$$\begin{array}{ccc} \Lambda^B[T/e] & \longrightarrow & \Lambda^A[T] \\ \downarrow & & \downarrow \\ \Omega[T/e] & \longrightarrow & \Lambda^B[T] \end{array}$$

and since by the induction hypothesis, the map $\Lambda^B[T/e] \to \Omega[T/e]$ is anodyne, the proof is complete. □

3.4. Grafting in an inner Kan complex

We now consider how dendrices in an inner Kan complex can be grafted. Recall that for two trees T and S with $E(T) \cap E(S) = \{l\}$, where l is a leaf of T which is also the root of S, we have the tree $T \circ_l S$ obtained by grafting S onto T along l. Both S and T embed naturally as sub-faces in $T \circ_l S$, which we denote by $S : S \to T \circ_l S$ and $T : T \to T \circ_l S$. These then induce the obvious inclusions $\Omega[S] \to \Omega[T \circ_l S]$ and $\Omega[T] \to \Omega[T \circ_l S]$ and the union of their images in $\Omega[T \circ_l S]$ we denote by $\Omega[T] \cup_l \Omega[S]$.

Lemma 3.4.1. *For any two trees T and S and any leaf l as above, the inclusion $\Omega[T] \cup_l \Omega[S] \to \Omega[T \circ_l S]$ is anodyne.*

Proof. Let us write $R = T \circ_l S$. The case where $T = \eta$ or $S = \eta$ is trivial, we therefore assume that this is not the case. We proceed by induction on $n = |T| + |S|$, the sum of the degrees of T and S. The cases $n = 0$ or $n = 1$ are taken care of by our assumption that $T \neq \eta \neq S$. For the case $n = 2$ the same assumption implies that the inclusion $\Omega[T] \cup_l \Omega[S] \to \Omega[R]$ is an inner horn inclusion and is thus anodyne. Assume then that the result holds for $2 \leq n < k$ and suppose $|T| + |S| = k$.

Let I be the set of all inner edges of R and $\Lambda^I[R]$ the union of all the outer faces of $\Omega[R]$. First notice that $\Omega[T] \cup_l \Omega[S] \to \Omega[R]$ factors as

$$\begin{array}{ccc} \Omega[T] \cup_l \Omega[S] & \longrightarrow & \Lambda^I[R] \\ & \searrow & \downarrow \\ & & \Omega[R] \end{array}$$

and the vertical arrow is anodyne by Proposition 3.3.3. If we can now show that the map

$$\Omega[T] \cup_l \Omega[S] \to \Lambda^I[R]$$

is anodyne then we are done. We do this by exhibiting it as a pushout of an anodyne extension. Recall (Section 2.2.1) that an outer cluster is a vertex v with the property that one of the edges adjacent to it is inner while all the other edges adjacent to it are outer. Let $Cl(T)$ (respectively $Cl(S)$) be the set of all outer clusters in T (respectively S) which do not contain l (respectively the root of S). For each $C \in Cl(T)$ the face of $\Omega[R]$ corresponding to C is isomorphic to $\Omega[(T/C) \circ_l S]$ and the map $\Omega[T/C] \cup_l \Omega[S] \to \Omega[(T/C) \circ_l S]$ is anodyne by the induction hypothesis. Similarly for every $C \in Cl(S)$ the face of $\Omega[R]$ that corresponds to C is isomorphic to $\Omega[T \circ_l (S/C)]$ and the map $\Omega[T] \cup_l \Omega[S/C] \to \Omega[T \circ_l (S/C)]$ is anodyne by the induction hypothesis. The following diagram is a pushout

$$\begin{array}{ccc} \coprod_{C\in Cl(T)}(\Omega[T/C] \cup_l \Omega[S]) \amalg \coprod_{C\in Cl(S)}(\Omega[T] \cup_l \Omega[S/C]) & \longrightarrow & \Omega[T] \cup_l \Omega[S] \\ \downarrow & & \downarrow \\ \coprod_{C\in Cl(T)}(\Omega[(T/C) \circ_l S]) \amalg \coprod_{C\in Cl(S)}(\Omega[T \circ_l (S/C)]) & \longrightarrow & \Lambda^I[R] \end{array}$$

where the map on the left is the coproduct of all of the anodyne extensions just mentioned. Since anodyne extensions are closed under coproducts, it follows that the map on the left of the pushout is anodyne and thus also the one on the right, which is what we set out to prove. This concludes the proof. □

Corollary 3.4.2. *Let X be an inner Kan complex, S and T two trees, and l a leaf of T which is also the root of S such that $T \circ_l S$ is defined. Suppose that $s \in X_S$ and $t \in X_t$ are two dendrices such that $l^*(t) = l^*(s)$ where l denotes both of the obvious maps $\eta \to T$ and $\eta \to S$. It then follows that there is a dendrex $r \in X_{T\circ_l S}$ with the property that $S^*(r) = s$ and $T^*(r) = t$.*

Proof. The two dendrices s and t induce a map $\Omega[T] \cup_l \Omega[S] \to X$. Since $\Omega[T] \cup_l \Omega[S] \to \Omega[T \circ_l S]$ is anodyne it follows that there is an extension $\Omega[T \circ_l S] \to X$. This extension is precisely the required dendrex. □

Consider the special case where both T and S are corollas. The corollary can then be interpreted as saying that suitable dendrices $t \in X_T$ and $s \in X_S$ in an inner Kan complex can be 'composed' along an input. We make this precise in the following definition.

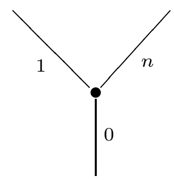
Definition 3.4.3. Let X be a dendroidal set, T and S two corollas (not necessarily with the same number of leaves), and x a leaf of T which is also the root of S such that $T \circ_x S$ is defined. Given two dendrices $t \in X_T$ and $s \in X_S$ we say that they *match along x if*

$$x^*t = x^*s$$

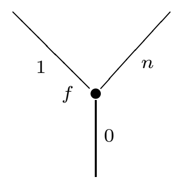
(where x denotes both induced maps $\eta \to S$ and $\eta \to T$). Any dendrex $r \in X_{T\circ_x S}$ with the property that $T^*r = t$ and $S^*r = s$ is called a *composition* of the dendrex s on t along x. We denote this situation by $r \sim t \circ_i s$.

Remark 3.4.4. Notice that usually there need not be a unique dendrex r for which $r \sim t \circ_x s$ and that consequently we cannot talk about *the* composition of two matching dendrices but only about *a* composition of such dendrices.

It is convenient to introduce the following conventions. For each $n \geq 0$ let C_n be the n-corolla:

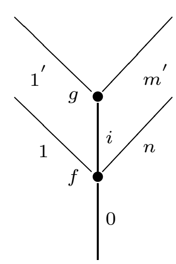


and for each $0 \leq i \leq n$ recall that $i : \eta \to C_n$ denotes the obvious (outer face) map in Ω that sends the unique edge of η to the edge i in C_n. We include here the case C_0, a tree with no leaves and just one vertex. An element $f \in X_{C_n}$ will be denoted by

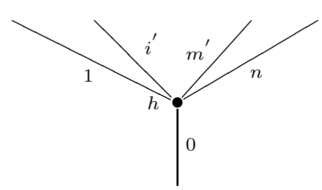


If $C_n^{'}$ is another n-corolla together with an isomorphism $\alpha : C_n^{'} \to C_n$ then we will usually write f again instead of $\alpha^*(f)$. We will use this convention quite often in the sequel, where in each case there will be an obvious choice for the isomorphism α given by the planar representation of the trees at question, which will usually be taken for granted. Given dendrices $f \in X_{C_n}$ and $g \in X_{C_m}$, the definition above does not permit us to consider composing one with the other. To remedy this we proceed as follows.

Definition 3.4.5. Let X be a dendroidal set and let $f \in X_{C_n}$ and $g \in X_{C_m}$ be two dendrices in X. We will say that a dendrex $h \in X_{C_{n+m-1}}$ is a $\circ_i$-composition of f and g if there is a dendrex γ in X as follows (we use the convention just mentioned):



with inner face



We will denote this situation by $h \sim f \circ_i g$ and call γ a *witness* for the composition.

Remark 3.4.6. The notion of composition of dendrices in a dendroidal set is now somewhat ambiguous. However, context will always make it clear which one is meant. Notice that if $h \sim f \circ_i g$ in the second definition then the dendrex i^*f is not equal to i^*g, but it does follow that i^*f is isomorphic to i^*g. We refer to this situation also by saying that f and g match along i, relying again on context to prevent confusion.

Proposition 3.4.7. *In an inner Kan complex every two matching dendrices have at least one composition (using any of the two definitions).*

Proof. This is a special case of Corollary 3.4.2. □

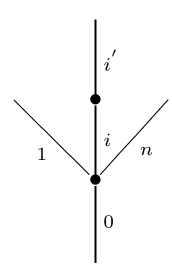
It is straightforward to check that given an operad $\mathcal{P}$, the notion of composition of dendrices in $N_d(\mathcal{P})$ corresponds exactly to the $\circ_i$-composition of arrows in $\mathcal{P}$. So that we see that composition in a dendroidal set is a generalization of composition in an operad.

3.5. Homotopy in an inner Kan complex

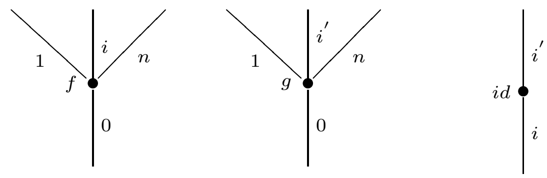
In this section we study a notion of homotopy inherent to a dendroidal set. Basically two dendrices are said to be homotopic if one is a composition of the other with a degenerate dendrex. This notion makes the most sense for dendrices shaped like corollas and indeed we study the homotopy of just such dendrices. We show that this homotopy theory within a dendroidal set is particularly well behaved if the dendroidal set is an inner Kan complex. In that case we show that the obtained homotopy relation is an equivalence relation and we show that it is a congruence for the composition of dendrices as defined in the previous section. From this it follows that with each inner Kan complex one can associate an operad which we call the homotopy operad associated with the inner Kan complex. Using this and other results obtained earlier we prove that a dendroidal set is a strict inner Kan complex if, and only if, it is the nerve of an operad. The ideas presented here generalize similar ideas presented in [**7**].

Using the convention from the end of the previous section we embark with the definition of homotopy.

Definition 3.5.1. Let X be a dendroidal set and let $f, g \in X_{C_n}$. For $1 \leq i \leq n$ we say that *f is homotopic to g along the edge i*, and write $f \sim_i g$, if $g \sim f \circ_i id$ where by id we mean a degeneracy. In more detail, $f \sim_i g$ if there is a dendrex H of shape

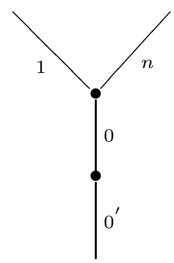


whose three faces are:

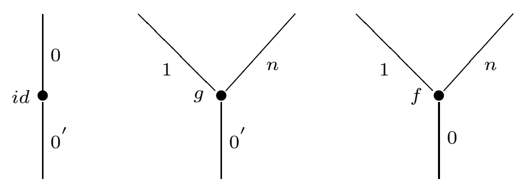


where id in the last tree is a degeneracy of i.

Similarly we will say that f is homotopic to g along the edge 0 and write $f \sim_0 g$ if $g \sim id \circ_0 f$, that is if there is a dendrex of shape



whose three faces are:

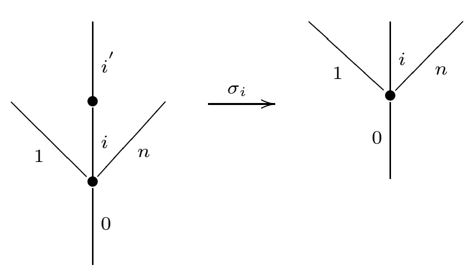


When $f \sim_i g$ for some $0 \leq i \leq n$ we will refer to the corresponding H as a *homotopy* from f to g along i and will sometimes write $H : f \sim_i g$.

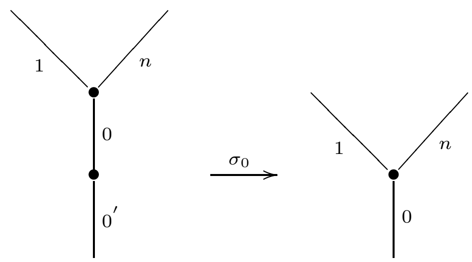
Remark 3.5.2. Notice that in a strict inner Kan complex X the homotopy relation just defined is the identity relation.

Proposition 3.5.3. *Let X be an inner Kan complex. For each $0 \leq i \leq n$ the relation $\sim_i$ on the set X_{C_n} is an equivalence relation.*

Proof. First we prove reflexivity. For $1 \leq i \leq n$ let

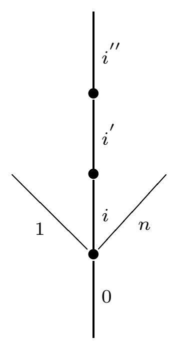


and for $i = 0$ let

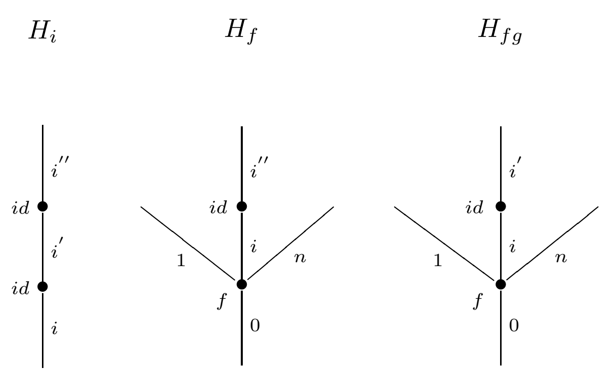


be the obvious degeneracies. It then follows that for any $f \in X_{C_n}$ the dendrex $\sigma_i^*(f)$ is a homotopy from f to f, thus $f \sim_i f$.

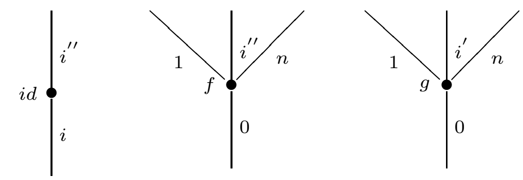
To prove symmetry assume $f \sim_i g$ for some $1 \leq i \leq n$ and let H_{fg} be a homotopy from f to g along i. Consider the tree T:



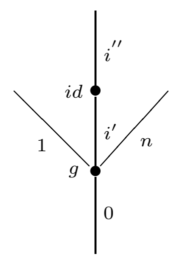
We now describe an inner horn $\Lambda^i[T] \to X$. Such a map is given by specifying three dendrices in X of certain shapes such that their faces match in a suitable way. We describe this map by explicitly writing the mentioned dendrices and their faces:



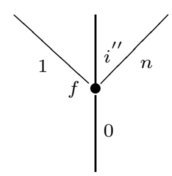
with inner faces of these dendrices:



where H_i is a double degeneracy of i, H_f is a homotopy from f to f (along the branch i) and H_{fg} is the given homotopy from f to g. It is easily checked that the faces indeed match so that we have a horn in X. Let x be a filler for that horn and consider $H_{gf} = \partial_i^*(x)$. This dendrex can be pictured as

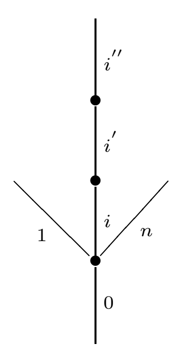


with inner face:

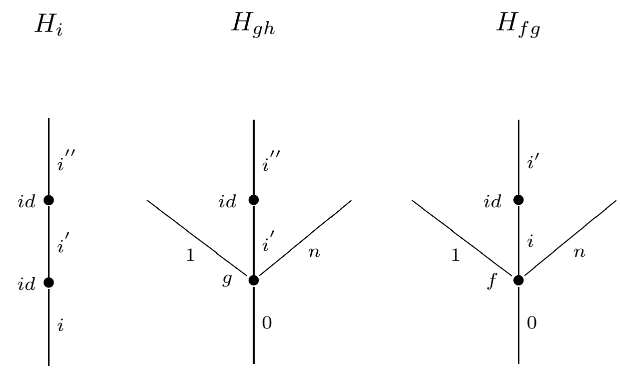


and is thus a homotopy from g to f along i, so that $g \sim_i f$. For $i = 0$ a similar proof works.

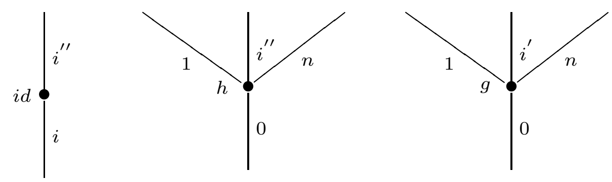
To prove transitivity let $f \sim_i g$ and $g \sim_i h$ for $1 \leq i \leq n$. Let H_{fg} be a homotopy from f to g and let H_{gh} be a homotopy from g to h. We again consider the tree T:



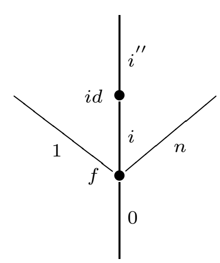
The following is a horn $\Lambda^{i'}[T] \to X$ in X:



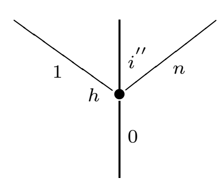
with inner faces being:



Let x be a filler for that horn and let $H_{fh} = \partial_{i'}^*(x)$, this dendrex can be pictured as follows:



with inner face:

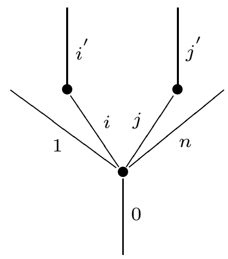


and is thus a homotopy from f to h so that $f \sim_i h$. The proof for $i = 0$ is similar. $\square$

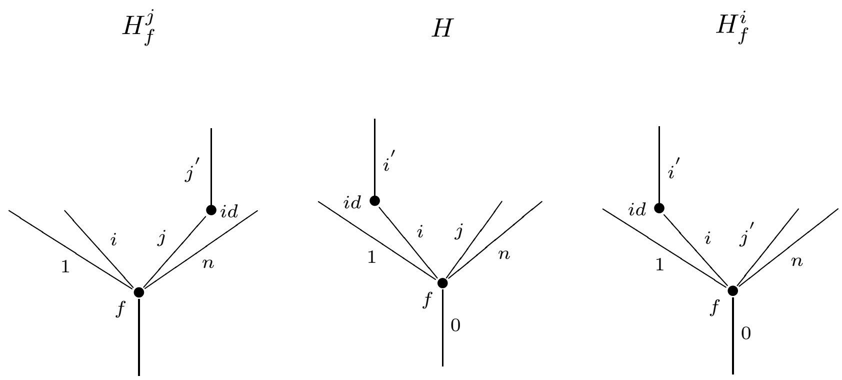
Lemma 3.5.4. *Let X be an inner Kan complex. The relations $\sim_0, \cdots, \sim_n$ on X_{C_n} are all equal.*

Remark 3.5.5. On the basis of this lemma, we will later just write $f \sim g$ instead of $f \sim_i g$.

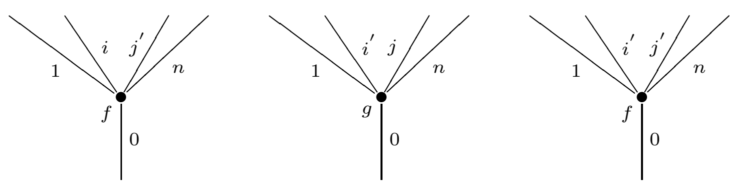
PROOF. Suppose $H : f \sim_i g$ for $1 \leq i \leq n$ and let $1 \leq i < j \leq n$. We consider the tree T:



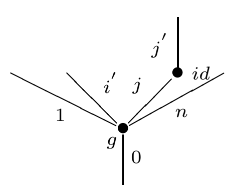
and the following inner horn $\Lambda^i[T] \to X$:



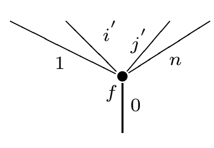
where $H_f^j : f \sim_j f$ and $H_f^i : f \sim_i f$. The inner faces of the three dendrices are



Let x be a filler for this horn, then $\partial_i^*(x)$ is the following dendrex



with inner face:

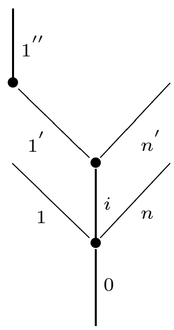


and is thus a homotopy from g to f along the j-th branch. Thus $g \sim_j f$ and so $f \sim_j g$ as well. The other cases to be considered follow in a similar way. $\square$

We now turn to prove that the homotopy equivalence relation behaves well with respect to the composition of dendrices.

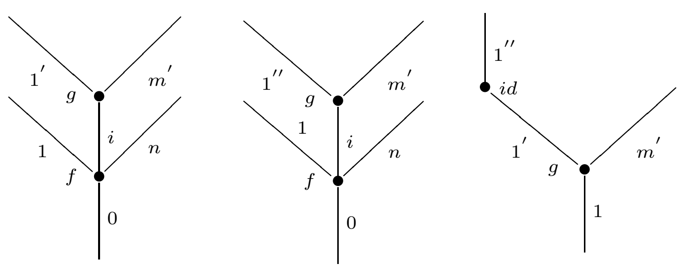
LEMMA 3.5.6. *In an inner Kan complex X, if $h \sim f \circ_i g$ and $h' \sim f \circ_i g$ then $h \sim h'$.*

PROOF. Let γ be a witness for the composition $h \sim f \circ_i g$ and γ' one for the composition $h' \sim f \circ_i g$. We consider the tree T:

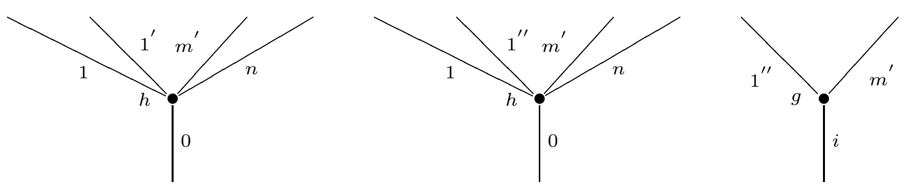


Let $H_g : g \sim_i g$ and consider the following horn $\Lambda^i[T] \to X$

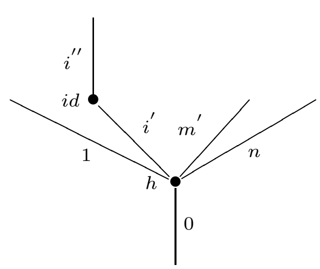
γ γ' H_g



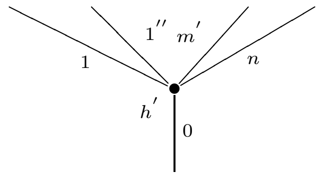
with inner faces



Let x be a filler for this horn. The face $\partial_i^*(x)$ is then the dendrex



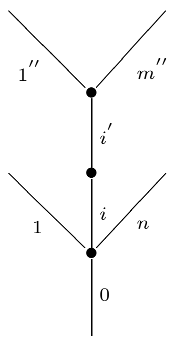
whose inner face is



which proves that $h \sim h'$. □

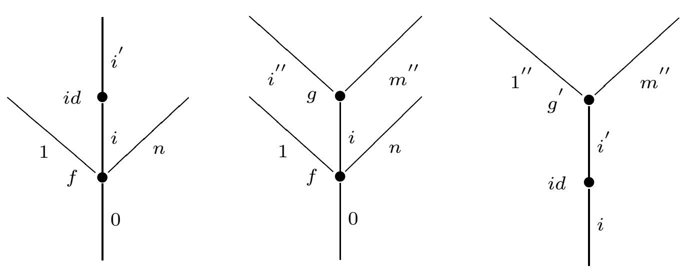
LEMMA 3.5.7. *In an inner Kan complex X, let $f \sim f'$ and $g \sim g'$. If $h \sim f \circ_i g$ and $h' \sim f' \circ_i g'$ then $h \sim h'$.*

PROOF. Let H be a homotopy from f to f' along the edge i, H' a homotopy from from g' to g along the root, and γ a witness for the composition $h \sim f \circ_i g$. We now consider the tree T:

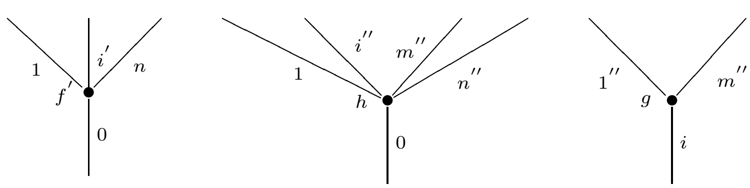


and the inner horn $\Lambda^i[T] \to X$ in X:

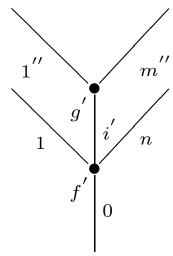
H γ H'



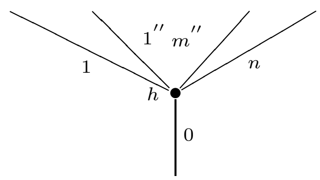
with inner faces:



The missing face of a filler for this horn is then:



with inner face



which proves that $h \sim f' \circ_i g'$, and thus by the previous result also that $h \sim h'$. □

We are now in a position to define the homotopy operad associated with an inner Kan complex. Given an inner Kan complex X and $x_0, \cdots, x_n \in X_\eta$ let $X(x_1, \cdots, x_n; x_0)$ be the set of all dendrices $f \in X_{C_n}$ such that $i^* f = x_i$ for $0 \leq i \leq n$. We now define a collection $Ho(X)$ as follows. The set of objects of $Ho(X)$ is the set X_η. Given objects $x_0, \cdots, x_n$ we put

$$Ho(X)(x_1, \cdots, x_n; x_0) = X(x_1, \cdots, x_n; x_0)/\sim$$

where $\sim$ is the homotopy relation defined above.

THEOREM 3.5.8. *Let X be an inner Kan complex. The composition of dendrices makes the collection $Ho(X)$ into an operad.*

PROOF. Lemma 3.5.7 implies that for $[f] \in Ho(X)(x_1, \cdots, x_n; x)$ and $[g] \in Ho(X)(y_1, \cdots, y_m; x_i)$ the assignment

$$[f] \circ_i [g] = [f \circ_i g]$$

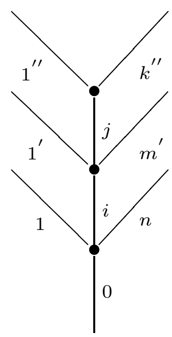
is well defined. This provides the $\circ_i$-compositions of the operad $Ho(X)$. The Σ_n-actions are defined as follows. Given a permutation $\sigma \in \Sigma_n$ let $\sigma : C_n \to C_n$ be the obvious induced map in Ω. The map $\sigma^* : X_{C_n} \to X_{C_n}$ restricts to a function

$$\sigma^* : X(x_1, \cdots, x_n; x) \to X(x_{\sigma(1)}, \cdots, x_{\sigma(n)}; x)$$

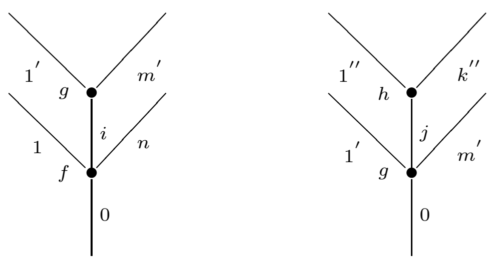
and it is trivial to verify that this map respects the homotopy relation. We thus obtain a map

$$\sigma^* : Ho(X)(x_1, \cdots, x_n; x) \to Ho(X)(x_{\sigma(1)}, \cdots, x_{\sigma(n)}; x).$$

We now need to show that these structure maps make the collection $Ho(X)$ into an operad. The verification is simple and we exemplify it by proving the associativity of the $\circ_i$-compositions. Let $[f] \in Ho(X)(x_1, \cdots, x_n; x)$, $[g] \in Ho(X)(y_1, \cdots, y_m; x_i)$ and $[h] \in Ho(X)(z_1, \cdots, z_k; y_m)$. We need to prove that $[f] \circ ([g] \circ [h]) = ([f] \circ [g]) \circ [h]$ (for simplicity we are neglecting to mention the input along which the compositions are being performed) which is the same as showing that $f \circ (g \circ h) \sim (f \circ g) \circ h$ for any choice of compositions $\psi \sim g \circ h$ and $\varphi \sim f \circ g$. Consider the tree T given by



and consider the anodyne extension $\Lambda^{\{i,j\}}[T] \to \Omega[T]$ (see Proposition 3.3.3). The two given compositions $\psi \sim g \circ h$ and $\varphi \sim f \circ g$ define a map $\Lambda^{\{i,j\}}[T] \to X$ depicted by



whose inner faces are respectively ψ and φ. Let $x \in X_T$ be a dendrex extending this map and $c : C_m \to T$ be the map obtained by contracting both i and j, and put $\rho = c^*x$. It now follows that $\partial_i^*(x)$ is a witness for the the composition $\rho \sim \psi \circ h$ and $\partial_j^*(x)$ is a witness for the composition $\rho \sim f \circ \varphi$, which proves the needed associativity. The other axioms for an operad follow in a similar manner. □

Definition 3.5.9. Given an inner Kan complex X the operad $Ho(X)$ as above is called the *homotopy operad* associated with X.

Remark 3.5.10. In **[7]** the authors construct a homotopy category $Ho(X)$ from an inner Kan simplicial set X. Our construction is a generalization of that one in the sense that for an inner Kan simplicial set X

$$Ho(i_!(X)) \cong j_! Ho(X).$$

The proof is trivial by inspection of these constructions.

We can now relate the homotopy operad of a dendroidal set with its operadic realization.

Proposition 3.5.11. *For any inner Kan complex X, $Ho(X)$ is isomorphic to $\tau_d(X)$.*

Proof. We prove that $Ho(-)$ has the required universal property, that is that for an inner Kan complex X and an operad $\mathcal{P}$ there is a natural bijection between operad maps $Ho(X) \to \mathcal{P}$ and dendroidal maps $X \to N_d(\mathcal{P})$. Let $F : Ho(X) \to \mathcal{P}$ be a map of operads. Since $N_d(\mathcal{P})$ is 2-coskeletal we only need to construct a map

$Sk_2(X) \to N_d(\mathcal{P})$. Since $ob(Ho(X)) = X_\eta$ and $ob(\mathcal{P}) \cong N_d(\mathcal{P})_\eta$, the map F clearly induces a function (the object part function of F) $G_0 : Sk_0(X) \to N_d(\mathcal{P})$. To extend this to a map $G_1 : Sk_1(X) \to N_d(\mathcal{P})$ let $x \in X_{C_n}$. The equivalence class $[x]$ is an operation in $Ho(X)$, and thus $F([x])$ is an operation in $\mathcal{P}$ which clearly defines a dendrex in $N_d(\mathcal{P})$ which we denote by $G_1(x)$. This assignment clearly extends G_0 so that we obtain a map $Sk_1(X) \to N_d(\mathcal{P})$. We now extend this map to a map $G_2 : Sk_2(X) \to N_d(\mathcal{P})$. Let $\gamma \in X_T$ where T is of degree 2. The dendrex γ is a witness for a composition in X of two dendrices, say $h \sim f \circ_i g$, so that in $Ho(X)$ we have that $[h] = [f] \circ_i [g]$. Since F is a map of operads the composition is respected so that γ defines a unique dendrex in $N_d(\mathcal{P})_T$ which we denote by $G_2(\gamma)$. Again, it is easily seen that G_2 extends G_1 so that we obtain a map $G_2 : Sk_2(X) \to N_d(\mathcal{P})$.

Consider now a given map $G : X \to N_d(\mathcal{P})$, that is a map $G_2 : sk_2(X) \to N_d(\mathcal{P})$. We now construct a map $Ho(X) \to \mathcal{P}$. Again we clearly have an obvious function $ob(Ho(X)) \to ob(\mathcal{P})$. Let f now be an operation in $Ho(X)$, that is $f = [f']$ for some dendrex $f' \in X_{C_n}$. Since $N_d(\mathcal{P})_{C_n}$ consists precisely of the operations in $\mathcal{P}$ of arity n, we have that $G(f')$ is such an operation. Since any dendroidal map preserves homotopic dendrices and since the homotopy relation in $N_d(\mathcal{P})$ is the identity we obtain that if $f'' \sim f'$ then $G(f'') = G(f')$. We can thus define $F(f) = G([f])$. It is easy to verify that F is actually a map of operads. Furthermore the two constructions just described are natural and are each other inverses which establishes the required bijection and thus finishes the proof. □

We can now prove the characterization of inner Kan complexes as those dendroidal sets that arise as nerves of operads.

Theorem 3.5.12. *Let X be a dendroidal set. X is a strict inner Kan complex if, and only if, X is the dendroidal nerve of an operad.*

Proof. One direction was proved in Lemma 3.2.4. Assume then that X is an inner Kan complex. We shall prove that $X \cong N_d(Ho(X))$ by showing that the canonical map $X \to N_d(Ho(X))$ is an isomorphism. Since we already know $N_d(Ho(X))$ to be 2-coskeletal we can easily describe the map $X \to N_d(Ho(X))$ simply by stating its value for dendrices shaped like trees with 2 or fewer vertices. The objects of $Ho(X)$ are X_η and $f : X_\eta \to N_d(Ho(X))_\eta$ is the identity. When X has unique fillers the homotopy relation is the identity and thus for any corolla we have $N_d(Ho(X))_{C_n} = X_{C_n}$ and again $f : X_{C_n} \to N_d(Ho(X))_{C_n}$ is the identity. Notice that any tree T with two vertices defines a composition of two operations in $Ho(X)$, which implies that $f : X_T \to N_d(Ho(X))_T$ is again the identity. Coskeletality implies now that f is the identity. □

Remark 3.5.13. This theorem specializes to provide a proof that the strict inner Kan simplicial complexes are precisely the nerves of categories. This result is stated, for example, in **[23]** without proof.

3.6. The exponential property

In this section we are going to prove that the sub-category of $dSet$ spanned by the inner Kan complexes is an exponential ideal with respect to normal dendroidal sets. That means that given an inner Kan complex K and a normal dendroidal set X, the dendroidal set $\underline{dSet}(X, K)$ is an inner Kan complex.

We first reduce the problem to proving that a certain map is an anodyne extension. Given trees S and T, and an inner horn $\Lambda^e[S] \to \Omega[S]$, we may consider the dendroidal sets $\Lambda^e[S]\otimes\Omega[T]$ and $\Omega[S]\otimes\partial\Omega[T]$ as dendroidal sub-sets of $\Omega[S]\otimes\Omega[T]$. As such, their union is a well defined dendroidal sub-set:

$$\Lambda^e[S] \otimes \Omega[T] \cup \Omega[S] \otimes \partial\Omega[T] \rightarrowtail \Omega[S] \otimes \Omega[T].$$

LEMMA 3.6.1. *Assume that for any trees S and T and any inner horn $\Lambda^e[S] \to \Omega[S]$, the inclusion*

$$\Lambda^e[S] \otimes \Omega[T] \cup \Omega[S] \otimes \partial\Omega[T] \rightarrowtail \Omega[S] \otimes \Omega[T]$$

is anodyne. It then follows that the inner Kan complexes form an exponential ideal in dSet with respect to normal dendroidal sets.

PROOF. We have to show that any map of dendroidal sets

$$\varphi : \Lambda^e[S] \otimes X \to K$$

extends to some map

$$\psi : \Omega[S] \otimes X \to K.$$

By writing X as the union of its skeleta,

$$X = \varinjlim Sk_n(X)$$

as in Section 2.6 and using the fact that X admits a normal skeletal filtration, we can build this extension ψ by induction on n. For $n = 0$, $Sk_0(X)$ is a sum of copies of $\Omega[\eta]$, the unit for the tensor product, so obviously the restriction $\varphi_0 : \Lambda^e[S] \otimes Sk_0(X) \to K$ extends to a map

$$\psi_0 : \Omega[S] \otimes Sk_0(X) \to K.$$

Suppose now that we have found an extension $\psi_n : \Omega[S] \otimes Sk_n(X) \to K$ of the restriction $\varphi_n : \Lambda^e[S] \otimes Sk_n(X) \to K$. Consider the following diagram:

$$\begin{array}{ccccccc}
\coprod \Lambda^e[S]\otimes\partial\Omega[T] & & \longrightarrow & & \coprod \Lambda^e[S]\otimes\Omega[T] & & \\
\downarrow & \searrow & & & \downarrow & \searrow & \\
 & & \Lambda^e[S]\otimes Sk_n(X) & \longrightarrow & & & \Lambda^e[S]\otimes Sk_{n+1}(X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\coprod \Omega[S]\otimes\partial\Omega[T] & & \longrightarrow & & \coprod \Omega[S]\otimes\Omega[T] & & \\
 & \searrow & \downarrow & & & \searrow & \downarrow \\
 & & \Omega[S]\otimes Sk_n(X) & \longrightarrow & & & \Omega[S]\otimes Sk_{n+1}(X)
\end{array}$$

In this diagram, the top and bottom faces are pushouts given by the normal skeletal filtration of X. Now inscribe the pushouts U and V in the back and front face, fitting into a square

$$\begin{array}{ccc} U & \longrightarrow & \coprod \Omega[S] \otimes \Omega[T] \\ \downarrow & & \downarrow \\ V & \longrightarrow & \Omega[S] \otimes Sk_{n+1}(X) \end{array}$$

The maps $\psi_n : \Omega[S] \otimes Sk_n(X) \to K$ and $\varphi_{n+1} : \Lambda^e[S] \otimes Sk_{n+1}(X) \to K$ together define a map $V \to K$. So, to find ψ_{n+1}, it suffices to prove that

$$V \rightarrowtail \Omega[S] \otimes Sk_{n+1}(X)$$

is anodyne. But, by a diagram chase argument, the square above is a pushout, so in fact, it suffices to prove that $U \rightarrowtail \coprod \Omega[S] \otimes \Omega[T]$ is anodyne. The latter map is a sum of copies of anodyne extensions as assumed. □

Proposition 3.6.2. *The map*

$$\Lambda^e[S] \otimes \Omega[T] \cup \Omega[S] \otimes \partial\Omega[T] \rightarrowtail \Omega[S] \otimes \Omega[T]$$

is anodyne for any trees S and T and an inner edge e in S.

Proof. The quite technical proof is given in **[38]** Proposition 9.2. □

These two results constitute thus the proof of the following theorem.

Theorem 3.6.3. *The inner Kan dendroidal sets form an exponential ideal with respect to the normal dendroidal sets.*

Corollary 3.6.4. *The inner Kan simplicial sets form an exponential ideal in the category of simplicial sets.*

Proof. Let K be an inner Kan simplicial set and X an arbitrary simplicial set. Clearly the dendroidal set $i_!(X)$ is normal. By Proposition 3.2.3 the dendroidal set $i_!(K)$ is an inner Kan complex, and thus we have that

$$\underline{dSet}(i_!(X), i_!(K))$$

is an inner Kan complex. By Lemma 3.1.7 this dendroidal set is isomorphic to $i_!(\underline{sSet}(X, K))$ and by Proposition 3.2.3 again it follows that $\underline{sSet}(X, K)$ is an inner Kan complex. □

Remark 3.6.5. Since an inner Kan simplicial set is the same as a quasi-category, the above corollary states that quasi-categories form an exponential ideal in simplicial sets. This result was proved by Joyal in **[24]**, though the proof is quite different. The restriction of our proof to simplicial sets resembles more the one given in **[39]**.

3.7. Inner Kan complex generated by a dendroidal set

We end this chapter by introducing a straightforward way of turning an arbitrary dendroidal set into an inner Kan complex. This construction provides thus many examples of inner Kan complexes that may not be strict.

For a dendroidal set X let $Horn_n(X)$ be the set of all inner horns $\Lambda^e[T] \to X$ where $|T| = n$, that do not have a filler in X. So X is an inner Kan complex if, and only if, $Horn_n(X) = \phi$ for all $n \geq 2$.

Proposition 3.7.1. *Let X be a dendroidal set and $n \geq 2$. Consider the dendroidal sets*

$$H_n = \coprod_{h:\Lambda^e[T]\to X} \Lambda^e[T]$$

and

$$F_n = \coprod_{h:\Lambda^e[T]\to X} \Omega[T]$$

where h runs over the set $Horn_n(X)$. Let $H_n \to F_n$ be the obvious inclusion and let $H_n \to X$ be the obvious induced map. Denote by $J_n(X)$ the pushout

$$\begin{array}{ccc} H_n & \longrightarrow & X \\ \downarrow & & \downarrow \\ F_n & \longrightarrow & J_n(X) \end{array}$$

then $X \to J_n(X)$ is anodyne and for every horn $h \in Horn_n(X)$ the horn $\Lambda^e[T] \to X \to J_n(X)$ has a filler.

Proof. Since $H_n \to F_n$ is an anodyne extension (it is a coproduct of inner horn inclusions) so is $X \to J_n(X)$ an anodyne extension (as a pushout of one). It is now immediate that for a horn $h : \Lambda^e[T] \to X$ in $Horn_n(X)$, the dendrex $\Omega[T] \to F_n \to J_n(X)$, where $\Omega[T] \to F_n$ is the summand corresponding to h, is a filler for h. □

Notice that it is not necessarily true that $Horn_n(J_n(X)) = \phi$ since by filling the horns in $Horn_n(X)$ many new horns might have been created. This can easily be remedied as we now show.

Definition 3.7.2. Let X be a dendroidal set and $n \geq 2$. Define the sequence of dendroidal sets $\{X_k\}_{k=0}^{\infty}$ by $X_0 = X$ and $X_{k+1} = J_n(X_k)$. We thus have a sequence of anodyne extensions

$$X \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_k \longrightarrow \cdots$$

We denote the colimit (countable composition) of this sequence by $X \to K_n(X)$.

Proposition 3.7.3. *The map $X \to K_n(X)$ is anodyne and $Horn_n(K_n(X)) = \phi$.*

Proof. The map $X \to K_n(X)$ is an anodyne extension since it is a countable composition of anodyne extensions. To prove that $Horn_n(K_n(X)) = \phi$ let $h : \Lambda^e[T] \to K_n(X)$ with $|T| = n$. We need to show that this horn has a filler in $K_n(X)$. Since such a horn is given by a finite sequence of dendrices in $K_n(X)$ and

$K_n(X)$ is the union of an increasing sequence of dendroidal sets, it follows that the horn factors as

$$\begin{array}{ccc} \Lambda^e[T] & \xrightarrow{h} & X \\ \downarrow{\scriptstyle h'} & \nearrow & \\ X_k & & \end{array}$$

for some X_k as in the definition above. Now, the horn h' has a filler in X_{k+1} which then extends to a filler of h in $K_n(X)$. □

LEMMA 3.7.4. *Let X be a dendroidal set. There exists an inner Kan complex $K(X)$ together with an anodyne extension $X \to K(X)$. Furthermore, this construction is functorial.*

PROOF. Define the sequence $\{X_n\}_{n=1}^{\infty}$ by $X_1 = X$ and $X_{n+1} = K_{n+1}(X_n)$. We let $K(X)$ be the colimit of the induced sequence

$$X_1 \to X_2 \to \cdots$$

A similar argument to the one given in the previous proposition now shows that $Horn_n(K(X)) = \phi$ for all $n \geq 1$ and thus that $K(X)$ is an inner Kan complex. The obvious map $X \to K(X)$ is an anodyne extension as a countable composition of such. The functoriality can easily be established. □

REMARK 3.7.5. Of course one can also use a Quillen small object argument to obtain an inner Kan complex from an arbitrary dendroidal set.