CHAPTER 2

Dendroidal sets

In this chapter the category of dendroidal sets is introduced and some of its basic properties are studied. Starting the chapter is a motivating problem arising from the nerve construction of categories and from our approach that operads are a generalization of categories. Then the simplicial category is briefly recalled together with some adjunctions related to it. The simplicial category is then extended in two different ways (which are proven to be equivalent) to what we call the dendroidal category, which is then used to define the category of dendroidal sets. The chapter ends by studying a certain closed monoidal structure on the category of dendroidal sets and a generalization of the skeletal filtration of simplicial sets to dendroidal sets.

2.1. Motivation - simplicial sets and nerves of categories

The simplicial category Δ can be defined in several different ways. Each such definition gives a different point of view on the category and is useful in different situations. We present here three definitions of the category Δ. The fact that these definitions produce isomorphic categories is well known and can easily be proven.

Definition 2.1.1. (Algebraic definition of Δ) Consider for each $n \geq 0$ the linearly ordered set $[n] = \{0 < 1 < \cdots < n\}$. The category Δ_A is the full sub-category of *PoSet* (the category of partially ordered sets) spanned by the objects $\{[n] \mid n \geq 0\}$.

This definition can be rephrased by saying that Δ_A is a skeleton of the category of non-empty, finite linearly ordered sets. This is the most common definition of the simplicial category Δ.

Definition 2.1.2. (Categorical definition of Δ) Consider for each $n \geq 0$ the category $[n]$ whose objects are $\{0, 1, \cdots, n\}$ and such that for $0 \leq i, j \leq n$ there is exactly one arrow $i \to j$ whenever $i \leq j$. The category Δ_C is the full sub-category of *Cat* spanned by the objects $\{[n] \mid n \geq 0\}$.

The equivalence between these two definitions is just the observation that any poset is precisely a small category with at most one arrow between any two of its objects.

Definition 2.1.3. (Geometric definition of Δ) Consider for each $n \geq 0$ the space $\Delta^n = \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1}\}$ with the sub-space topology. The objects of the category Δ_G are $\{\Delta^n | n \geq 0\}$ and the maps are generated by face inclusions and degeneracies (see **[19]**, page 3).

Usually we will just write Δ for the simplicial category whose objects are $[n]$ for $n \geq 0$, and will use whichever definition is most convenient. The geometric definition implies that there is a functor $\Delta \to Top$, i.e., the inclusion $\Delta_G \to Top$. Taking this functor as a probe (see the preliminaries) yields an adjunction

$$\begin{tikzcd} sSet \arrow[r, shift left, "|-|"] & Top \arrow[l, shift left, "N"] \end{tikzcd}$$

where N is usually called the singular complex functor and $|-|$ is called the geometric realization functor (see **[19]**). On the other hand the categorical definition implies the existence of a functor $\Delta \to Cat$, which is just the inclusion functor $\Delta_C \to Cat$. When taken as a probe, this functor yields the adjunction

$$\begin{tikzcd} sSet \arrow[r, shift left, "|-|"] & Cat \arrow[l, shift left, "N"] \end{tikzcd}$$

where this time N is the nerve functor and $|-|$ is usually denoted by τ (see **[23]**). Lastly, the algebraic definition also implies the existence of a functor $\Delta \to PoSet$ (again the inclusion functor $\Delta_A \to PoSet$) and thus an adjunction

$$\begin{tikzcd} sSet \arrow[r, shift left, "|-|"] & PoSet. \arrow[l, shift left, "N"] \end{tikzcd}$$

However, this adjunction is not particularly useful.

Staying true to the main principal of the previous chapter, namely that operads are a natural extension of categories, the adjunction

$$\begin{tikzcd} sSet \arrow[r, shift left, "|-|"] & Cat \arrow[l, shift left, "N"] \end{tikzcd}$$

should catch our attention. We thus ask whether it is possible to define the nerve of an operad. More concretely we can ask whether in the diagram

$$\begin{tikzcd} sSet \arrow[d, dotted, "i_!"] & Cat \arrow[l, "N"'] \arrow[d, "j_!"] \\ ? & Operad \arrow[l, dotted, "N_d"'] \end{tikzcd}$$

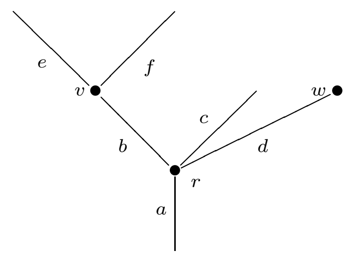
the question mark can be replaced by a category and the dotted arrows be filled in a natural way by functors such that N_d would send an operad to its (not yet defined) nerve. Our approach to answering this question will be to extend the category Δ to a bigger category Ω and then use an appropriate probe $\Omega \to Operad$ that will produce an adjunction of which the right adjoint will be N_d. Each point of view of the category Δ suggests a different way to extend it. We will consider below the categorical definition and the algebraic one, and extend both (to isomorphic categories). Each approach will have its merits, as we shall see.

2.2. An operadic definition of the dendroidal category

We consider here the categorical definition (Definition 2.1.2) of Δ and extend it to a bigger category by means of special operads induced by trees.

DEFINITION 2.2.1. Let T be a planar tree. The planar operad *generated* by T, denoted $\Omega_\pi(T)$, is the following free planar operad. We define the collection C on the set $E(T)$ of edges of T as follows. For each vertex v with $in(v) = (e_1, \cdots, e_n)$ and $out(v) = e_0$ we set $C(e_1, \cdots, e_n; e_0)$ to be a one-point set. These are the only non-empty sets in the collection C. We now define $\Omega_\pi(T)$ to be $\mathcal{F}_\pi(C)$, the free planar operad on the collection C (see Section 1.2).

EXAMPLE 2.2.2. For the tree T given by



$\Omega_\pi(T)$ has six objects, $a, b, \cdots, f$ and the following generating operations:

$$r \in \Omega_\pi(T)(b, c, d; a),$$

$$w \in \Omega_\pi(T)(-; d)$$

and

$$v \in \Omega_\pi(T)(e, f; b).$$

The other operations are units (such as $1_b \in \Omega_\pi(T)(b; b)$) and formal compositions of the generating operations (such as $r \circ_1 v \in \Omega_\pi(T)(e, f, c, d; a)$).

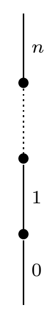
DEFINITION 2.2.3. Let T be a non-planar tree. The operad *generated* by T, denoted by $\Omega(T)$, is defined as follows. Let $\bar{T}$ be a planar representative of T, then

$$\Omega(T) = Symm(\Omega_\pi(\bar{T})).$$

It is clear that the definition does not depend on the chosen planar representative $\bar{T}$. In fact a different choice amounts to choosing a different set of generating operations for $\Omega(T)$.

DEFINITION 2.2.4. (Operadic definition of Ω) The *dendroidal* category Ω is the full sub-category of *Operad* whose objects are the operads of the form $\Omega(T)$ where T is a non-planar rooted tree.

To see how Δ embeds in Ω consider for each $n \geq 0$ the linear tree L_n



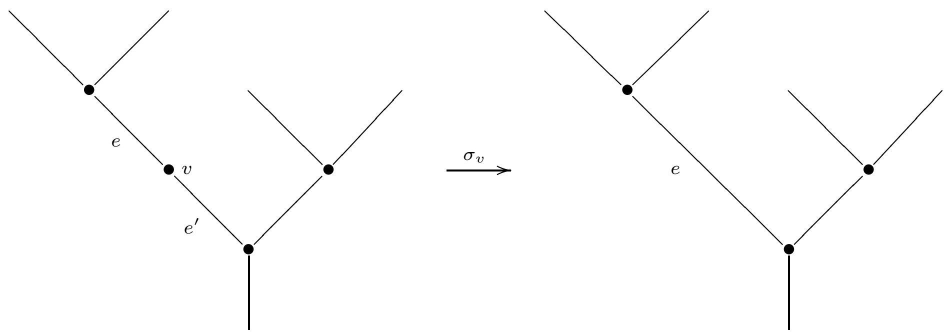
It is trivial to verify that

$$\Omega(L_n) = j_!([n])$$

and that the functor $i : \Delta \to \Omega$ sending $[n]$ to $\Omega(L_n)$ is an embedding of categories. This constitutes our operadic extension of Δ.

2.2.1. Faces and degeneracy maps. Exactly as for Δ, the maps in Ω are generated by special kinds of maps which we now describe.

Let T be a tree and $v \in T$ a vertex of valence 1 with $in(v) = e$ and $out(v) = e'$. Consider the tree T/v, obtained from T by deleting the vertex v and the edge e'. There is an operad map, i.e, an arrow $\sigma_v : T \to T/v$ in Ω, which sends the operation in $\Omega(T)$ generated by v to the unit 1_e in $\Omega(T/v)$. For example:

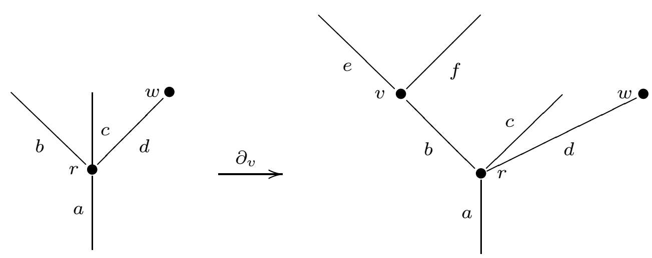


An arrow in Ω of this kind will be called a *degeneracy (map).*

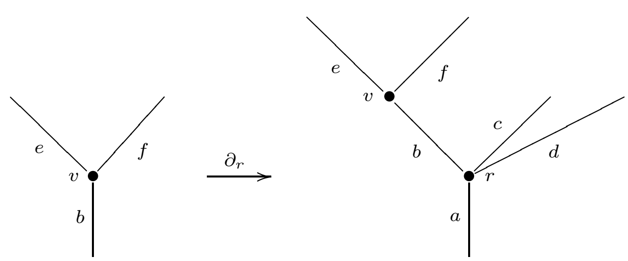
Consider now a tree T and a vertex v in T with exactly one inner edge attached to it (such a vertex will be called an *outer cluster*), one can obtain a new tree T/v by deleting v and all the outer edges attached to it. The operad $\Omega(T/v)$ associated to T/v is simply a sub-operad of the one associated to T, and this inclusion of operads defines an arrow in Ω denoted

$$\partial_v : T/v \to T.$$

An arrow in Ω of this kind is called an *outer face (map).* For example



and (to emphasize that it is sometimes possible to remove the root of the tree T)



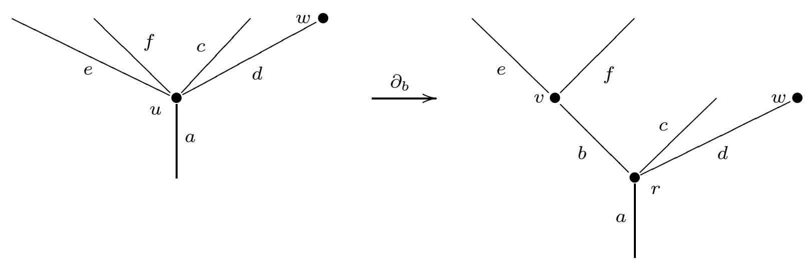
are both outer faces.

Moreover, for a corolla C_n and an edge e of C_n (necessarily outer) there is an associated outer face map

$$e : \eta_e \to T$$

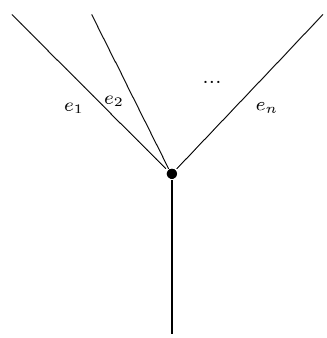
sending the unique edge e of η_e to e in C_n.

Given a tree T and an inner edge e in T, one can obtain a new tree T/e by contracting the edge e. There is a canonical map of operads $\partial_e : \Omega(T/e) \to \Omega(T)$ which sends the new vertex in T/e (obtained by merging the two vertices attached to e) into the appropriate composition of these two vertices in $\Omega(T)$. The corresponding arrow $\partial_e : T/e \to T$ in Ω is called an *inner face (map)*. For example



where $u = r \circ_1 v$.

Lastly, we mention the isomorphisms in Ω. Of course there may be non-trivial isomorphisms from a tree to itself, for example, for the corolla C_n whose input edges are $e_1, \cdots, e_n$:



any permutation $\varphi \in \Sigma_n$ defines an automorphism of C_n in Ω.

DEFINITION 2.2.5. The *degree* of a tree T, denoted by $|T|$, is the number of vertices in T.

It is easily seen that degeneracy maps decrease degree by 1, face maps (outer or inner) increase degree by 1, and isomorphisms preserve degree.

THEOREM 2.2.6. *Any map* $T \xrightarrow{f} T'$ *in* Ω *can be written uniquely as* $f = \varphi\pi\delta$*, where* δ *is a composition of degeneracy maps,* π *is an isomorphism, and* φ *is a composition of (inner and outer) face maps.*

The proof will be given below once we develop the algebraic definition of Ω.

2.3. An algebraic definition of the dendroidal category

Our aim now is to extend the algebraic definition of Δ (Definition 2.1.1). This approach is technically more involved than the previous one since we first have to enlarge the category of posets (to what we call broad posets) and then find a suitable algebraic characterization for the analogue of linear orders. Our plan is thus as follows. We start by developing the notion of a broad poset. The basic principal is that a broad poset stands in the same relation to a poset as does an operad to a category. Once the category of broad posets is defined we notice that it carries a natural symmetric closed monoidal structure. We then turn to the analogue of a linear order for broad posets, which we call dendroidally ordered sets. The algebraic definition of the dendroidal category is given, and we then prove that this definition is equivalent to the operadic one.

2.3.1. Broad posets. For a set A we denote by A^* the free monoid on A. That is

$$A^* = \bigcup_{n=0}^{\infty} A^n$$

with concatenation of tuples as the monoid operation. The set A^0 is a singleton set which consists of the unique tuple of length 0, denoted by ϵ, which is the unit of the monoid. We denote elements of A^* by $\overrightarrow{a}$ and identify an element a with the 1-tuple (a). We use the notation $a \in \overrightarrow{a}$ to indicate that a occurs in the tuple $\overrightarrow{a}$. If $\overrightarrow{a} \in A^*$ is of length n and $\sigma \in \Sigma_n$ then by permuting the components (that is $(a_1, \cdots, a_n)\sigma = (a_{\sigma(1)}, \cdots, a_{\sigma(n)})$), we obtain a right action of Σ_n on the set A^n.

A *broad relation* is a pair (A, R) where A is a set and R is a sub-set of $A \times A^*$. As is common with ordinary relations, we use the notation $aR\overrightarrow{a}$ instead of $(a, \overrightarrow{a}) \in R$.

Definition 2.3.1. A *broad poset* is a broad relation (A, R) satisfying:

(1) Reflexivity: aRa holds for any $a \in A$.
(2) Transitivity: If $aR(a_1, \cdots, a_n)$ and $a_iR\overrightarrow{b_i}$ hold for $1 \le i \le n$ then $aR\overrightarrow{b_1} \cdot \cdots \cdot \overrightarrow{b_n}$.
(3) Anti-symmetry: If $aR\overrightarrow{b}$ and $bR\overrightarrow{a}$ hold while $a \in \overrightarrow{a}$ and $b \in \overrightarrow{b}$ then $a = b$.
(4) Permutability: If $a \le \overrightarrow{a}$ and $\overrightarrow{a}$ has length n, then $a \le \overrightarrow{a}\sigma$ holds for any $\sigma \in \Sigma_n$.

When (A, R) is a broad poset we denote R by $\le$. The meaning of $<$ is then defined in the usual way.

Remark 2.3.2. In the definition above one can obviously drop condition four and retain a sensible definition of what we call a *non-symmetric (* or a *planar)* broad poset. In that context we will sometime refer to a broad poset as a *symmetric* broad poset. We will see below that there is a close connection between symmetric operads and broad posets. There is a similar connection between planar operads and planar broad posets.

Example 2.3.3. The site http://genealogy.math.ndsu.nodak.edu of the math genealogy project lists mathematicians and their students. This allows us to define the following broad poset. The set A is the set of mathematicians. We say that $a \le (a_1, \cdots, a_n)$ if mathematicians $a_1, \cdots, a_n$ (in no particular order) are students of mathematician a. We assume every student has exactly one well-defined adviser.

By agreeing to the convention that $a \leq a$ for every mathematician a, and closing under transitivity we obtain a broad relation which is clearly a broad poset.

A *map* of broad posets $f : A \to B$ is a set function preserving the broad poset structure, that is if $a \leq \overrightarrow{a}$ then $f(a) \leq f(\overrightarrow{a})$ where $f(\overrightarrow{a})$ is defined component-wise (namely, $f(a_1, \cdots, a_n) = (f(a_1), \cdots, f(a_n))$).

DEFINITION 2.3.4. We denote by $BrdPoset$ the category of all broad posets and their maps.

Recall that a poset A can be considered as a category $\mathcal{C}$ whose objects are the elements of A and such that there is precisely one arrow $a \to a'$ in $\mathcal{C}$ whenever $a \leq a'$. One obtains thus a functor $Poset \to Cat$. Similarly, given a symmetric (respectively planar) broad poset B one can define a symmetric (respectively planar) operad $\mathcal{P}$ whose objects are the elements of B and such that there is exactly one operation in $\mathcal{P}(b_1, \cdots, b_n; b)$ whenever $b \leq (b_1, \cdots, b_n)$. In that way one obtains a functor $BrdPoset \to Operad$.

It is obvious how an ordinary poset can be viewed as a broad poset and that this defines an embedding

$$k_! : Poset \to BrdPoset.$$

This functor has a right adjoint $k^* : BrdPoset \to Poset$, that sends a broad poset A to the poset $k^*(A) = A$ where $a \leq b$ holds in $k^*(A)$ exactly when $a \leq b$ holds in A. Consider the endofunctor $R : Poset \to Poset$ that sends a poset A to the same set with the reversed partial order. One may now easily establish that in the following diagram

$$\begin{tikzcd} Poset \arrow[r, shift left, "k_! R"] \arrow[d] & BrdPoset \arrow[l, shift left, "Rk^*"] \arrow[d] \\ Cat \arrow[r, shift left, "j_!"] & Operad \arrow[l, shift left, "j^*"] \end{tikzcd}$$

both squares commute. This constitutes our extension of $Poset$ to the category $BrdPoset$ and establishes the relation to categories and operads. Notice that the use of the endofunctor R is needed because of the convention that in a broad poset A a relation $a \leq a_1, \cdots, a_n$ is translated in its corresponding operad to an arrow from $a_1, \cdots, a_n$ to a (arrows go from big to small), while in a poset a relation $a \leq b$ is translated in its corresponding category to an arrow from a to b (arrows go from small to big).

2.3.2. Closed monoidal structure on *BrdPoset*. The category $Poset$ is cartesian closed with the obvious product of posets. The category $BrdPoset$ is also cartesian closed in such a way as to make $k_!$ a strong monoidal functor. However, there is another closed monoidal structure that also extends the one on $Poset$, which we now describe.

DEFINITION 2.3.5. Let A and B be two broad posets. Their *tensor product* $A \otimes B$ is the set $A \times B$ with the minimal broad poset structure in which

1. For every $a \in A$ if $b \leq (b_1, \cdots, b_n)$ then $(a, b) \leq ((a, b_1), \cdots, (a, b_n))$.
2. For every $b \in B$ if $a \leq (a_1, \cdots, a_m)$ then $(a, b) \leq ((a_1, b), \cdots, (a_m, b))$.

THEOREM 2.3.6. *The category BrdPoset with the tensor product of broad posets is a symmetric closed monoidal category, and $k_!$: Poset $\rightarrow$ BrdPoset is strong monoidal.*

PROOF. A singleton set with the trivial broad poset structure is clearly the unit for the tensor product. It is easily verified that $\otimes$ makes *BrdPoset* into a symmetric monoidal category, so all that is left to do is to describe the internal Hom. Given two broad posets A and B, the set $\underline{BrdPoset}(A, B)$ of all broad poset maps from A to B is made into a broad poset by defining

$$f \leq (f_1, \cdots, f_n)$$

to hold if for every $a \in A$

$$f(a) \leq (f_1(a), \cdots, f_n(a))$$

holds in B. It is an easy matter to verify that this broad poset is the internal Hom with respect to the tensor product of broad posets. The fact that $k_! : Poset \rightarrow BrdPoset$ is strong monoidal is trivial. $\square$

2.3.3. Dendroidally ordered sets. Above we extended the category *Poset* to the category *BrdPoset* of broad posets. The algebraic definition of Δ identifies it as a certain full sub-category of *Poset* by considering linear orders. Our aim now is to identify those objects of *BrdPoset* that generalize linear orders in a suitable way.

A broad poset $(A, \leq)$ induces a partial order relation on A as follows. For $a, b \in A$ we say that a is *dominated* by b and write $a \leq_d b$ if there is a $\overrightarrow{b} \in A^*$ such that $a \leq \overrightarrow{b}$ and $b \in \overrightarrow{b}$. It is immediately seen that $\leq_d$ is indeed a partial order. The broad poset $\leq$ also induces a partial order relation on the set A^* as follows. For $\overrightarrow{a} = (a_1, \cdots, a_n)$ and $\overrightarrow{b}$ in A^* we say that $\overrightarrow{a} \leq \overrightarrow{b}$ if there are $\overrightarrow{b_1}, \cdots, \overrightarrow{b_n}$ such that $\overrightarrow{b} = \overrightarrow{b_1} \cdot \cdots \cdot \overrightarrow{b_n}$ and such that $a_i \leq \overrightarrow{b_i}$ for each $1 \leq i \leq n$. Notice that this does not conflict with our abuse of notation which identifies the 1-tuple (a) with a.

DEFINITION 2.3.7. Let A be a broad poset. An element $r \in A$ such that for all $a \in A$

$$r \leq_d a$$

is called the *root* of A. Clearly, if a root exists then it is unique.

For $a \in A$ let $\hat{a}$ be the set $\{\overrightarrow{a} \in A^* | a < \overrightarrow{a}\}$.

DEFINITION 2.3.8. Let A be a broad poset and $a \in A$. Assume that the set $\hat{a}$, as a sub-set of A^*, has a smallest element which is unique up to symmetry. We will call such a smallest element a *representative* of the *successors* of a and will denote it (somewhat ambiguously) by $s(a)$. An element $a \in A$ for which $\hat{a}$ is empty is called a *leaf*.

By "a smallest element which is unique up to symmetry" we mean the following. An element $\overrightarrow{a}$ which is a smallest element with respect to the poset $\leq$ on A^*, such that if $\overrightarrow{b}$ is another smallest element then they are both of the same length n, and there is a $\sigma \in \Sigma_n$ such that $\overrightarrow{a} = \overrightarrow{b}\sigma$. Note that if $\overrightarrow{a}$ is a representative of $s(a)$ then $\overrightarrow{a}\sigma$ is again a representative of $s(a)$ for any $\sigma \in \Sigma_n$ with n the length of $\overrightarrow{a}$.

EXAMPLE 2.3.9. In the math genealogy example, a leaf is a mathematician with no students. For a mathematician a, the tuple $s(a)$ is a list of the students of a in an arbitrary order. In this example there is no root.

DEFINITION 2.3.10. A broad poset $(A, \leq)$ is called *finite* if the set $\leq$ is finite. A is called *minimal* if whenever

$$a \leq (a_1, \cdots, a_n)$$

$a_i \neq a_j$ for $i \neq j$.

Notice that the finiteness of A as a broad poset implies that of A as a set, but not vice-versa.

DEFINITION 2.3.11. Let A be a finite broad poset. A is called *dendroidally* ordered if

(1) A has a root.
(2) For every $a \in A$ either a is a leaf or a has successors.
(3) A is minimal.

REMARK 2.3.12. If A is a dendroidally ordered set, minimality implies that for each $a \in A$ the tuple $s(a)$ does not contain the same element twice. We can therefore consider $s(a)$ unambiguously as a set. We shall do this from now on.

It is obvious that if $A \neq \emptyset$ is a finite poset which is linearly ordered, then the broad poset $k_!(A)$ is dendroidally ordered. This is thus our extension of the notion of a linear order from the category of posets to the category of broad posets.

DEFINITION 2.3.13. (Algebraic definition of Ω) The *dendroidal* category Ω is the full sub-category of $BrdPoset$ spanned by the dendroidally ordered sets.

The embedding of Δ in Ω using the algebraic definition is obvious, we simply send the linearly ordered set $[n] \in ob(\Delta)$ to the dendroidally ordered set $k_!([n])$. This concludes our algebraic extension of the simplicial category to the dendroidal category.

2.3.4. Grafting in *DenOrd*. We discuss now how dendroidally ordered sets can be grafted. We obtain a decomposition of a dendroidally ordered set as the grafting of certain dendroidally ordered sub-sets of it, much like the fundamental decomposition of trees (Proposition 0.2.6). It is precisely this property that will imply the equivalence of the two definitions of the dendroidal category given above.

DEFINITION 2.3.14. Let A and B be two dendroidally ordered sets with $A \cap B = \{y\}$, where y is a leaf of A and the root of B. The *grafting* of B on A, denoted by $A \circ B$, is the set $A \cup B$ together with the broad relation in which $x \leq (x_1, \cdots, x_n)$ holds if one of the following conditions is satisfied:

(1) $x \leq (x_1, \cdots, x_n)$ holds in A.
(2) $x \leq (x_1, \cdots, x_n)$ holds in B.
(3) $x \in A$ and there are $\overrightarrow{a_1}, \overrightarrow{a_2} \in A^*$ and $\overrightarrow{b} \in B^*$ such that

$$(x_1, \cdots, x_n) = \overrightarrow{a_1} \cdot \overrightarrow{b} \cdot \overrightarrow{a_2}$$

and

$$x \leq \overrightarrow{a_1} \cdot y \cdot \overrightarrow{a_2}$$

holds in A and
$$y \leq \overrightarrow{b}$$
holds in B.

It is easily seen that the grafting of two dendroidally ordered sets is again a dendroidally ordered set. By repeated grafting one can define a full grafting operation
$$A \circ (B_1, \cdots, B_n)$$
which is unambiguously defined whenever the sets B_i are pairwise disjoint, the set $\{r_{B_i}\}_{i=1}^{n}$, consisting of the roots of the dendroidally ordered sets B_i, is equal to the set of leaves of A, and each B_i meets A at one edge.

Maps of dendroidally ordered sets can also be grafted as explained in the following proposition whose proof is trivial.

PROPOSITION 2.3.15. *Let $f : A \to A'$ and $g : B \to B'$ be two maps of dendroidally ordered sets. Suppose $A \cap B = \{a\}$ and $A' \cap B' = \{f(a)\}$ where a and $f(a)$ are leaves in, respectively, A and A'. Assume further that the root of B is a and that the root of B' is $f(a)$ and that $g(a) = f(a)$. Then the function $f \circ g : A \circ B \to A' \circ B'$ given by*
$$f \circ g(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$
is well defined and is a map of dendroidally ordered sets.

REMARK 2.3.16. When f is the identity we will denote $f \circ g$ by $A \circ g$ and when g is the identity we will denote $f \circ g$ by $f \circ B$. We use the notation $f \circ (g_1, \cdots, g_n)$ for repeated grafting of maps (under the obvious compatibility conditions on the given maps).

For a dendroidally ordered set A and $a \in A$ let
$$A_a = \{a' \in A | a \leq_d a'\}$$
with the induced broad relation from A. It is immediate that A_a is again dendroidally ordered. For a dendroidally ordered set A with root r and $s(r) = \{a_1, \cdots, a_n\}$ let
$$A_{root} = \{r, a_1, \cdots, a_n\}$$
with the induced broad order from A (which is obviously a dendroidal order).

PROPOSITION 2.3.17. *(Fundamental decomposition of dendroidally ordered sets) Let A be a dendroidally ordered set with root r and $s(r) = \{a_1, \cdots, a_n\}$. Then $A = A_{root} \circ (A_{a_1}, \cdots, A_{a_n})$.*

PROOF. First notice that $A_{a_i} \cap A_{root} = \{a_i\}$. It is generally true for any $a \in A$ that $a \notin s(a)$, so that if $r \in A_{a_i}$ then, since r is the smallest element in $(A, \leq_d)$ it follows that $r = a_i$, but then $r \in s(r)$, a contradiction. If $a_j \in A_{a_i}$ and $a_j \neq a_i$ then $a_i \leq_d a_j$ which means that there is $\overrightarrow{a} \in A^*$ with $a_j \in \overrightarrow{a}$ and $a_i \leq \overrightarrow{a}$. But transitivity and
$$r \leq (a_1, \cdots, a_n)$$
imply
$$r \leq (a_1, \cdots, a_{i-1}) \cdot \overrightarrow{a} \cdot (a_{i+1}, \cdots, a_n)$$
which contradicts the minimality of A (since the latter tuple on the right contains a_j twice). Thus the only element of A which can be in $A_{root} \cap A_{a_i}$ is a_i which

is clearly there. It follows that $A_{root} \cap A_{a_i} = \{a_i\}$ and thus that the grafting $A_{root} \circ (A_{a_1}, \cdots, A_{a_n})$ is well defined.

Next we notice that $A_{root} \cup A_{a_1} \cup \cdots \cup A_{a_n} = A$, since for any $a \in A$ one has $r \leq_d a$, which means that there is an $\overrightarrow{a}$ such that $a \in \overrightarrow{a}$ and $r \leq \overrightarrow{a}$. If $\overrightarrow{a} = r$ then $a = r$ and we are done, otherwise $\overrightarrow{a} \in \hat{r}$ and by definition we then have that $s(r) \leq \overrightarrow{a}$. This means that $\overrightarrow{a}$ can be written as $\overrightarrow{a_1} \cdot \cdots \cdot \overrightarrow{a_n}$ in such a way that $a_i \leq \overrightarrow{a_i}$. Since $a \in \overrightarrow{a}$ it follows that there is $1 \leq j \leq n$ for which $a \in \overrightarrow{a_j}$, which implies that $a_j \leq_d a$, and so $a \in A_{a_j}$. We see thus that the underlying set of $A_{root} \circ (A_{a_1}, \cdots, A_{a_n})$ is the same as that of A and it is now easy to see that the broad order defined by the grafting operation is the original one on A. □

2.3.5. Classification of dendroidally ordered sets. We now establish the connection between dendroidally ordered sets and trees - thus justifying the use of the term 'dendroidal'.

Definition 2.3.18. Let T be a tree. We define a dendroidally ordered set, $[T]$, whose underlying set is $E(T)$ (the set of edges of T), by induction on the number k of vertices in the tree T. If $T = \eta$ (the tree with one edge and no leaves) then the broad poset structure on $[\eta]$ is just $e \leq e$ for the unique edge e in η. If T is an n-corolla C_n with root r and leaves $\{a_1, \cdots, a_n\}$ then the broad poset structure on $[C_n]$ is the one in which $r \leq \overrightarrow{a}$ where $\overrightarrow{a}$ is any permutation of $(a_1, \cdots, a_n)$. Obviously these two broad posets make $[\eta]$ and $[C_n]$ into dendroidally ordered sets and thus the cases $k = 0, 1$ are covered. Suppose now that T has more then 1 vertex and write $T = T_{root} \circ (T_{e_1}, \cdots, T_{e_n})$ (as in 0.2.6). We then define $[T] = [T_{root}] \circ ([T_{e_1}], \cdots, [T_{e_n}])$, where the grafting is that of dendroidally ordered sets.

We now wish to associate with any dendroidally ordered set A a tree T such that $A = [T]$. To do that we introduce the notion of the degree of A, which allows for induction on dendroidally ordered sets.

A pair $(a, \overrightarrow{a})$ is called a *link* in a broad poset A if $a < \overrightarrow{a}$ and if $a < \overrightarrow{b} < \overrightarrow{a}$ does not hold for any choice of $\overrightarrow{b}$. We say that two links $(a, \overrightarrow{a})$ and $(a, \overrightarrow{a'})$ are equivalent if there is a permutation σ such that $\overrightarrow{a} \cdot \sigma = \overrightarrow{a'}$. The number of equivalence classes of links in a broad poset A is the *degree* of A and is denoted by $|A|$. It can easily be shown that for dendroidally ordered sets A and B the equality:

$$|A \circ B| = |A| + |B|$$

holds, whenever $A \circ B$ is defined.

Lemma 2.3.19. *Let A be a dendroidally ordered set. There is a tree $Tr(A)$ for which $A = [Tr(A)]$.*

Proof. By induction on n, the degree of A. If $|A| = 0$ or $|A| = 1$ then the claim is obvious. Assume the statement holds for dendroidally ordered sets of degree smaller then n and let A be of degree n. Write $A = A_{root} \circ (A_{a_1}, \cdots, A_{a_n})$, and let $Tr(A) = Tr(A_{root}) \circ (Tr(A_{a_1}), \cdots, Tr(A_{a_n}))$. By the definition of $[-]$ and the induction hypothesis it follows that $[Tr(A)] = A$. □

We summarize the properties of the two constructions relating trees and dendroidally ordered sets in the following theorem.

THEOREM 2.3.20. *(Classification of dendroidally ordered sets) The above constructions, associating with any tree T a dendroidally ordered set $[T]$, and with a dendroidally ordered set A a tree $Tr(A)$ have the following properties:*
1) $[Tr(A)] = A$.
2) $Tr([T]) = T$.
3) Whenever one of the sides of the equation $[T \circ S] = [T] \circ [S]$ is defined so is the other, and in that case the equation holds.
4) Whenever one of the sides of the equation $Tr(A \circ B) = Tr(A) \circ Tr(B)$ is defined so is the other, and in that case the equation holds.
5) The two constructions $Tr(-)$ and $[-]$ are unique with respect to properties 1-4.

PROOF. The proofs of the parts that were not already given follow by an easy induction and are therefore omitted. □

Under this correspondence each concept of trees can be translated to a concept of dendroidally ordered sets and vice-versa. for instance, if T is a tree and $[T]$ is its corresponding dendroidally ordered set then the root of T is the root of $[T]$, a vertex in T is a link in $[T]$, and so on. Notice also that $|T| = |[T]|$ and $|Tr(A)| = |A|$.

2.3.6. The equivalence of the two definitions of the dendroidal category. We now prove that the algebraic and operadic definitions of the category Ω are equivalent, and we recast the notation and definitions of the operadic definition in the algebraic one. We also provide a proof of Theorem 2.2.6.

THEOREM 2.3.21. *The algebraic and operadic definitions of Ω are equivalent.*

PROOF. Let Ω_O be the dendroidal category as given in Definition 2.2.4 (operadic definition) and let Ω_A be the dendroidal category as given in Definition 2.3.13 (algebraic definition). The precise meaning of the statement is that these two categories are isomorphic. Given a dendroidally ordered set $A \in ob(\Omega_A)$ we have the tree $Tr(A)$ associated with it from Lemma 2.3.19. It is easily seen that the assignment $A \mapsto Tr(A)$ extends to a functor $Tr : \Omega_A \to \Omega_O$. Similarly, the assignment $T \mapsto [T]$ extends to a functor $[-] : \Omega_O \to \Omega_A$, which is the inverse of Tr. □

REMARK 2.3.22. From now on we will denote the dendroidal category by Ω. We consider the objects of Ω to be non-planar rooted trees and we regard an arrow $T \to S$ in Ω between two such trees either as a map of dendroidally ordered sets $[T] \to [S]$ or as a map of operads $\Omega(T) \to \Omega(S)$, depending on which point of view is more convenient at the time.

Given a dendroidally ordered set A of degree n we wish now to identify its degree $n-1$ dendroidally ordered sub-sets. An element $a \in A$ which is not a leaf and not the root will be called an *inner* element, otherwise it is an *outer* element. Given a link $(a, (a_1, \cdots, a_n))$ and a set C consisting of any n of the elements $a, a_1, \cdots, a_n$, if the elements of C are all outer then C is called an *outer cluster* of A.

If A is a dendroidally ordered set and $B \subseteq A$, we will denote by A/B the subset $A \backslash B$ with the induced broad poset structure. For an element a we write A/a as shorthand for $A/\{a\}$.

PROPOSITION 2.3.23. *(Characterization of maximal dendroidally ordered subsets) Let A be a dendroidally ordered set of degree n. If a is an inner element of*

A then A/a is dendroidally ordered and $|A/a| = n-1$. If C is an outer cluster of A then A/C is dendroidally ordered and $|A/C| = n-1$. If $B \subseteq A$ and B with the broad order induced by A is dendroidally ordered of degree $n-1$ then $B = A/a$ for a unique inner element $a \in A$ or $B = A/C$ for a unique outer cluster C (the meaning of 'or' should be taken in the exclusive sense).

PROOF. The proof is straightforward and thus omitted. $\square$

Consider a dendroidally ordered set A and its corresponding tree $Tr(A)$. One can easily verify the following assertions. An inner element in A corresponds to an inner edge in $Tr(A)$ while an outer element corresponds either to a leaf or to the root of $Tr(A)$. An outer cluster C in A corresponds to a vertex v of valence n together with a choice of n of the edges adjacent to v that are all outer edges. Furthermore the tree $Tr(A/a)$ for an inner element a is equal to the tree $Tr(A)/e$ obtained from the tree $Tr(A)$ by contracting the inner edge e corresponding to a. Similarly the tree $Tr(A/C)$ for an outer cluster C is equal to the tree $Tr(A)/C$ obtained from $Tr(A)$ by removing the outer edges (which may or may not contain the root) corresponding to the outer elements in C.

DEFINITION 2.3.24. Let A be a dendroidally ordered set of degree n. Any inclusion $B \to A$ of a dendroidally ordered sub-set B of degree $n-1$ in A is called a *face map*. If $B = A/a$ then this inclusion is denoted by ∂_a and is called the *inner face map* associated to a. if $B = A/C$ then the inclusion is denoted by ∂_C and is called the *outer face map* associated to C.

Again one can easily see that an inner face map $A/a \to A$ corresponds exactly to an inner face map $Tr(A/e) \to Tr(A)$ and similarly for an outer face map.

DEFINITION 2.3.25. Let A be a dendroidally ordered set and $l = (a_1, a_2)$ a unary link in A. The map $\sigma_l : A \to A/a_2$ defined by

$$\sigma_l(x) = \begin{cases} x & x \neq a_2 \\ a_1 & x = a_2 \end{cases}$$

is a map of dendroidally ordered sets and is called the *degeneracy map* associated with the unary link l.

Comparing this definition of a degeneracy map with the operadic one we easily see that (under the identification of Ω_A with Ω_O) both definitions agree.

We now turn to the isomorphisms in Ω. Let A and B be two dendroidally ordered sets and $f : A \to B$ a function. It is easily verified that f is an isomorphism of dendroidally ordered sets if, and only if, f sends the root of A to the root of B and if for each $a \in A$

$$f(s(a)) = s(f(a)).$$

We are now going to prove Theorem 2.2.6. By the discussion so far it is obvious that we can state and prove the theorem in the setting of dendroidally ordered sets instead of that of operads. This turns out to be a slightly more convenient framework for a precise proof. We prepare for this proof with the following simple proposition whose proof we omit.

PROPOSITION 2.3.26. *If the map $\alpha : B \to B'$ of dendroidally ordered sets is an inner face (respectively outer face, degeneracy, isomorphism) then for any dendroidally ordered set A, the map $A \circ \alpha : A \circ B \to A \circ B'$ is an inner face (respectively outer face, degeneracy, isomorphism) whenever the grafting is defined.*

THEOREM 2.3.27. *(Restatement of Theorem 2.2.6) Any arrow $f : A \to B$ in Ω decomposes uniquely as*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\delta"] & B \\ A' \arrow[r, "\pi"] & B' \arrow[u, "\varphi"'] \end{tikzcd}$$

where $\delta : A \to A'$ is a composition of degeneracy maps, $\pi : A' \to B'$ is an isomorphism, and $\varphi : B' \to B$ is a composition of face maps.

PROOF. We prove this by induction on $n = |A| + |B|$. If $n = 0$ or $n = 1$ the proof is trivial. Assume the assertion holds for $1 \leq n < m$ and let $f : A \to B$ be a map such that $|A| + |B| = m$. We consider four cases. First assume that $f(r_A) = b \neq r_B$ where r_A (respectively r_B) is the root of A (respectively B). In that case f factors through B_b:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f'"'] & B \\ & B_b \arrow[u, "\phi'"'] \end{tikzcd}$$

where ϕ' is the obvious inclusion of B_b into B which is clearly a composition of face maps (recall that $B_b = \{b' \in B \mid b \leq_d b'\}$). Since $|B_b| < |B|$ the induction hypothesis implies that f' can be factored as:

$$\begin{tikzcd} A \arrow[r, "f'"] \arrow[d, "\delta"] & B_b \\ A' \arrow[r, "\pi"] & B' \arrow[u, "\phi"'] \end{tikzcd}$$

and adjoining the map ϕ' to this decomposition yields the desired factorization of f.

We now consider the case where $f(r_A) = r_B$ and $f(s(r_A)) = s(r_B)$. Let $s(r_A) = \{a_1, \cdots, a_k\}$ and $s(r_B) = \{b_1, \cdots, b_k\}$ with $f(a_i) = b_i$. In that case, by restricting f to A_{a_i}, one obtains a map $f_i : A_{a_i} \to B_{b_i}$. Let $A_{root} = \{r_A, a_1, \cdots, a_k\}$ with the broad order induced by A and define B_{root} similarly. Let $f_{root} : A_{root} \to B_{root}$ be the restriction of f to A_{root}. The map f can be written as $f_{root} \circ (f_{a_1}, \cdots, f_{a_k})$. By the induction hypothesis each f_i decomposes as

$$\begin{tikzcd} A_{a_i} \arrow[r, "f_i"] \arrow[d, "\delta_i"] & B_{b_i} \\ A'_{a_i} \arrow[r, "\pi_i"] & B'_{b_i} \arrow[u, "\phi_i"'] \end{tikzcd}$$

Let $A' = A_{root} \circ (A'_{a_1}, \cdots, A'_{a_k})$. The maps δ_i can then be grafted to produce the map $A_{root} \circ (\delta_1, \cdots, \delta_k) : A_{root} \circ (A_1, \cdots, A_k) \to A_{root} \circ (A'_{a_1}, \cdots, A'_{a_k})$ and since $A = A_{root} \circ (A_1, \cdots, A_k)$ we obtain a map $\delta : A \to A'$. It follows from the preceding proposition that δ is a composition of degeneracies. Similarly define $B' = B_{root} \circ (B'_{b_1}, \cdots, B'_{b_k})$ and then the ϕ_i together form a map $\phi : B' \to B$ which is a composition of face maps. Lastly the π_i also assemble themselves to give a map

$\pi : A' \to B'$ which is an isomorphism. These maps form the needed factorization of f.

The third case is when $f(r_A) = r_B$ but $f(s(r_A)) \neq s(r_B)$ and $f(x) \neq r_B$ for any $x \neq r_A$. Let $s(r_A) = \{a_1, \cdots, a_k\}$ and $f(a_i) = b_i$. Notice that an element $x \in B$ such that $r_B <_d x <_d b_i$ for some i is, of course, inner. Let $\hat{B}$ be the dendroidally ordered sub-set of B obtained by removing all of those inner elements. The inclusion $\hat{\phi} : \hat{B} \to B$ is then obviously a composition of (inner) face maps, and the map f factors through it as

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "\hat{f}"'] & B \\ & \hat{B} \arrow[u, "\hat{\phi}"'] \end{tikzcd}$$

Since $f(s(r_A)) \neq s(r_B)$ the induction hypothesis applies to $\hat{f}$ which then factors as

$$\begin{tikzcd} A \arrow[r, "\hat{f}"] \arrow[d, "\delta"] & \hat{B} \\ A' \arrow[r, "\pi"] & \hat{B}' \arrow[u, "\phi"'] \end{tikzcd}$$

and adjoining $\hat{\phi}$ to this decomposition yields the desired factorization of f.

The last case to be considered is when $f(r_A) = r_B$ and $f(s(r_A)) \neq f(s(r_B))$ while for at least one $x \neq r_A$ we have $f(x) = r_B$. This implies that $s(r_A)$ consists of just one element a and $f(a) = r_B$. Thus (r_A, a) is a link. Let $\sigma : A \to A'$ be the degeneracy associated with it. Since $f(r_A) = f(a) = r_B$ it follows that f factors through σ as

$$\begin{tikzcd} A \arrow[r, "\sigma"] \arrow[dr, "f"'] & A' \arrow[d, "f'"] \\ & B \end{tikzcd}$$

The induction hypothesis for f' together with σ now provide the required decomposition. The uniqueness of the decomposition follows by a rather straightforward induction and is omitted. □

2.4. Dendroidal sets - basic definitions

We now introduce the category of dendroidal sets, its basic terminology and properties, and establish its relation to simplicial sets.

DEFINITION 2.4.1. The category *dSet* of *dendroidal sets* is the presheaf category on the dendroidal category Ω, that is it is the category of functors $\Omega^{op} \to Set$ and natural transformations.

A *dendroidal set* is thus a functor $X : \Omega^{op} \to Set$. Given a tree T in Ω we denote the set $X(T)$ by X_T and for each $\alpha : T \to S$ in Ω we denote the function $X(\alpha) : X_S \to X_T$ by α^*. An element $x \in X_T$ is called a *dendrex* of *shape* T, or a T-dendrex. A map of dendroidal sets $f : X \to Y$ is a natural transformation between the given functors. Such a natural transformation consists of functions (all denoted) $f : X_T \to Y_T$ for each tree T, such that for any $\alpha : T \to S$ in Ω and $x \in X_S$

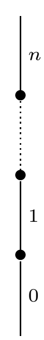
$$\alpha^* f(x) = f(\alpha^* x).$$

DEFINITION 2.4.2. Let T be a tree. The representable presheaf $\Omega(-, T) : \Omega^{op} \to Set$ is called the *standard T-dendrex* and is denoted by $\Omega[T]$. Explicitly we have for each tree S

$$\Omega[T]_S = \Omega(S, T).$$

From the Yoneda lemma it follows that given any dendroidal set X, a T-dendrex $x \in X_T$ corresponds bijectively to a map of dendroidal sets $\iota_x : \Omega[T] \to X$. We will usually simply write x instead of ι_x, conveniently identifying a dendrex with its associated map. Notice that $\Omega[T]$ is functorial in T in the sense that given an arrow $\alpha : S \to T$ in Ω, there is the obvious induced dendroidal map $\Omega[\alpha] : \Omega[S] \to \Omega[T]$. A *dendroidal sub-set* Y of a dendroidal set X consists of, for each tree T, a sub-set $Y_T \subseteq X_T$ such that Y, endowed with the obvious structure from X, is a dendroidal set. Given a dendroidal set X and for each T a subset $Y_T \subseteq X_T$, we call the smallest dendroidal sub-set $\bar{Y}$ of X that contains Y (i.e., $Y_T \subseteq \bar{Y}_T$ for each T) the dendroidal set *generated* by Y.

We now define the basic functors that relate dendroidal sets to simplicial sets. These functors will be used often to relate definitions and results regarding dendroidal sets to simplicial sets and vice-versa. Since we have two (equivalent) definitions for Ω we need to be a bit more precise. We thus, very temporarily, use the notation Ω_A and Ω_O for (respectively) the algebraic and operadic definitions of Ω. Recall the functor $i : \Delta \to \Omega_O$ that sends $[n] \in ob(\Delta)$ to the operad $\Omega(L_n)$ where L_n is the tree depicted by



and the functor $i : \Delta \to \Omega_A$ that sends the poset $[n]$ to itself viewed as a dendroidally ordered set. It is clear that under the isomorphism $\Omega_A \underset{[-]}{\overset{Tr}{\rightleftarrows}} \Omega_O$ given in Theorem 2.3.21, the diagram

$$\begin{tikzcd} & \Delta \arrow[dl, "i"'] \arrow[dr, "i"] & \\ \Omega_A \arrow[rr, shift left, "Tr"] & & \Omega_O \arrow[ll, shift left, "{[-]}"] \end{tikzcd}$$

commutes. That means that we have one well-defined embedding $i : \Delta \to \Omega$ which from now on is fixed. This functor induces a restriction functor $i^* : dSet \to sSet$

which sends a dendroidal set X to the simplicial set

$$i^*(X)_n = X_{i(n)}.$$

This functor has both a left and a right adjoint (by Kan extension, see [**34**]) of which the left adjoint is of significance. The left adjoint $i_! : sSet \to dSet$ sends a simplicial set X to the dendroidal set given by

$$i_!(X)_T = \begin{cases} X_n, & \text{if } T \cong i([n]). \\ \phi, & \text{otherwise.} \end{cases}$$

This functor is full and faithful and thus embeds $sSet$ in $dSet$.

DEFINITION 2.4.3. Let T be a tree and $\alpha : S \to T$ a face map in Ω. The *α-face* of $\Omega[T]$, denoted by $\partial_\alpha \Omega[T]$, is the dendroidal sub-set of $\Omega[T]$ which is the image of the map $\Omega[\alpha] : \Omega[S] \to \Omega[T]$.

Thus we have that

$$\partial_\alpha \Omega[T]_R = \{ \; R \longrightarrow S \xrightarrow{\alpha} T \;\mid R \to S \in \Omega[S]_R \}$$

When α is obtained by contracting an inner edge e in T we denote ∂_α by ∂_e.

DEFINITION 2.4.4. Let T be a tree. The *boundary* of $\Omega[T]$ is the dendroidal sub-set $\partial\Omega[T]$ of $\Omega[T]$ obtained as the union of all the faces of $\Omega[T]$. That is

$$\partial\Omega[T] = \bigcup_{\alpha \in \Phi_1(T)} \partial_\alpha \Omega[T]$$

where $\Phi_1(T)$, is the set of all faces of T.

DEFINITION 2.4.5. Let T be a tree and $\alpha \in \Phi_1(T)$ a face of T. The *α-horn* in $\Omega[T]$ is the dendroidal sub-set $\Lambda^\alpha[T]$ of $\Omega[T]$ which is the union of all the faces of T except $\partial_\alpha \Omega[T]$, that is

$$\Lambda^\alpha[T] = \bigcup_{\beta \neq \alpha \in \Phi_1(T)} \partial_\beta \Omega[T].$$

The horn is called an *inner horn* if α is an inner face, otherwise it is called an *outer horn*. We will denote an inner horn $\Lambda^\alpha[T]$ by $\Lambda^e[T]$, where e is the contracted inner edge in T that defines the inner face $\alpha = \partial_e : T/e \to T$. A horn in a dendroidal set X is a map of dendroidal sets $\Lambda^\alpha[T] \to X$. It is inner (respectively outer) if the horn $\Lambda^\alpha[T]$ is inner (respectively outer).

REMARK 2.4.6. It is trivial to verify that these notions for dendroidal sets extend the common ones for simplicial sets in the sense, for example, that for the simplicial horn $\Lambda^k[n] \subseteq \Delta[n]$, the dendroidal set

$$i_!(\Lambda^k[n]) \subseteq i_!(\Delta[n]) = \Omega[L_n]$$

(where L_n is the linear tree with n vertices as described above) is a horn in the dendroidal sense. Furthermore, the horn $\Lambda^k[n]$ is inner (i.e., $0 < k < n$) if, and only if, the horn $i_!(\Lambda^k[n])$ is inner. A similar remark holds for the rest of the notions just introduced.

Both the boundary $\partial\Omega[T]$ and the horns $\Lambda^\alpha[T]$ in $\Omega[T]$ can be described as colimits as follows.

Definition 2.4.7. Let $T_1 \to T_2 \to \cdots \to T_n$ be a sequence of n face maps in Ω. We call the composition of these maps a *sub-face* of T_n of *codimension* n.

Notice then that a face $S \to T$ is a sub-face of T of codimension 1.

Proposition 2.4.8. *Let $S \to T$ be a sub-face of T of codimension 2. The map $S \to T$ decomposes in precisely two different ways as a composition of faces.*

Proof. Consider $[S]$ and $[T]$, the dendroidally ordered sets associated with the trees S and T. We have to consider several cases in which the map $S \to T$ can be obtained. Assume thus that $S \to T$ is the composition of two inner face maps. That means that $[S] = [T]/\{e, e'\}$ where e and e' are both inner elements of $[T]$. It is obvious then that $S \to T$ decomposes only as

$$\begin{tikzcd}
& T/e \arrow[dr, "\partial_e"] & \\
S \arrow[ur, "\partial_{e'}"] \arrow[rr] \arrow[dr, "\partial_e"'] & & T \\
& T/e' \arrow[ur, "\partial_{e'}"'] &
\end{tikzcd}$$

The other cases involve removing outer clusters (see 2.2.1) as well and are proved similarly. $\square$

Let $\Phi_2(T)$ be the set of all sub-faces of T of codimension 2. The proposition implies that for each $\beta : S \to T \in \Phi_2(T)$ there are two maps $\beta_1 : S \to T_1$ and $\beta_2 : S \to T_2$ through which β factors. Using these maps we can form two maps γ_1 and γ_2

$$\coprod_{S \to T \in \Phi_2(T)} \Omega[S] \rightrightarrows \coprod_{R \to T \in \Phi_1(T)} \Omega[R]$$

where γ_i $(i = 1, 2)$ has component $\Omega[S] \xrightarrow{\Omega[\beta_i]} \Omega[T_i] \longrightarrow \coprod \Omega[R]$ for each $\beta : S \to T \in \Phi_2(T)$.

Lemma 2.4.9. *Let T be a tree in Ω. With notation as above we have that the boundary $\partial\Omega[T]$ is the coequalizer*

$$\coprod_{S \to T \in \Phi_2(T)} \Omega[S] \rightrightarrows \coprod_{R \to T \in \Phi_1(T)} \Omega[R] \to \partial\Omega[T]$$

for the two maps γ_1, γ_2 constructed above.

Proof. The required universal property of $\partial\Omega[T]$ is easily established. $\square$

Corollary 2.4.10. *A map of dendroidal sets $\partial\Omega[T] \to X$ corresponds exactly to a sequence $\{x_R\}_{R \to T \in \Phi_1(T)}$ of dendrices whose faces match, in the sense that for each sub-face $\beta : S \to T$ of codimension 2 we have $\beta_1^*(x_{T_1}) = \beta_2^*(x_{T_2})$.*

We have a similar presentation for horns. For a fixed face $\alpha : S \to T \in \Phi_1(T)$ consider the parallel arrows defined by making the following diagram commute

$$\begin{tikzcd}
\Omega[S] \arrow[r, "\beta_1"] \arrow[d] & \Omega[T_1] \arrow[d] \\
\coprod_{\beta:S\to T\in\Phi_2(T)} \Omega[S] \arrow[r, shift left] \arrow[r, shift right] & \coprod_{R\to T\neq\alpha\in\Phi_1(T)} \Omega[R] \\
\Omega[S] \arrow[r, "\beta_2"] \arrow[u] & \Omega[T_2] \arrow[u]
\end{tikzcd}$$

where the vertical arrows are the canonical injections into the coproduct and where we use the same notation as above.

Lemma 2.4.11. *Let T be a tree in Ω and α a face of T. In the diagram*

$$\coprod_{S\to T\in\Phi_2(T)} \Omega[S] \rightrightarrows \coprod_{R\to T\neq\alpha\in\Phi_1(T)} \Omega[R] \to \Lambda^\alpha[T]$$

the dendroidal set $\Lambda^\alpha[T]$ is the coequalizers of the two maps constructed above.

Proof. Again, the verification of the universal property is simple and thus omitted. $\square$

Corollary 2.4.12. *A horn $\Lambda^\alpha[T] \to X$ in X corresponds exactly to a sequence $\{x_R\}_{R\to T\neq\alpha\in\Phi_1(T)}$ of dendrices that agree on common faces in the sense that if $\beta : S \to T$ is a sub-face of codimension 2 which factors as*

$$\begin{tikzcd}
& R_1 \arrow[dr, "\alpha_1"] & \\
S \arrow[ur, "\beta_1"] \arrow[rr, "\beta"] \arrow[dr, "\beta_2"'] & & T \\
& R_2 \arrow[ur, "\alpha_2"'] &
\end{tikzcd}$$

where $\alpha_i \neq \alpha$ ($i = 1, 2$) then

$$\beta_1^*(x_{R_1}) = \beta_2^*(x_{R_2}).$$

Remark 2.4.13. In the special case where the tree T is linear we obtain the equivalent result for simplicial sets. Namely, the presentation of the boundary $\partial\Delta[n]$ and of the horn $\Lambda^k[n]$ as colimits of standard simplices, and the description of a horn $\Lambda^k[n] \to X$ in a simplicial set X (see, respectively, **[19]** page 8, page 9, and Corollary 3.2).

We end this section by introducing the terminology of faces, sub-faces, and so on for dendrices in a dendroidal set.

Definition 2.4.14. Let $\alpha : S \to T$ be a map in Ω and X a dendroidal set. Given a dendrex $t \in X_T$ we refer to the dendrex $\alpha^* t \in X_S$ as

(1) a *face* (respectively *inner face, outer face*) of t if α is a face (respectively inner face, outer face) of T.
(2) a *sub-face* of t if α is a sub-face of T.

(3) *isomorphic* to t if α is an isomorphism.
(4) a *degeneracy* of t if α is a composition of degeneracies.

2.5. Closed monoidal structure on the category of dendroidal sets

Just like any presheaf category, $dSet$ is a cartesian closed category ([**33**]). This cartesian product extends the cartesian product of simplicial sets in the sense that for two simplicial sets X and Y we have

$$i_!(X \times Y) \cong i_!(X) \times i_!(Y).$$

However, there is another closed monoidal structure on $dSet$ with a very strong connection to the Boardman-Vogt tensor product of operads, as we shall see below. In this section we introduce this monoidal structure and study it in detail.

In a presheaf category a closed monoidal structure is completely determined (up to isomorphism) by the tensor product of representables. This follows easily since the tensor product, being closed, preserves colimits and since every object in a presheaf category is a colimit of representables (see the preliminaries). In more detail, suppose $\otimes$ is a closed monoidal structure on $dSet$ and let X and Y be two dendroidal sets. Write

$$X = \varinjlim \Omega[T]$$

and

$$Y = \varinjlim \Omega[S]$$

canonically as in Section 0.1.1. Then we obtain that

$$X \otimes Y = \varinjlim \Omega[T] \otimes \varinjlim \Omega[S] \cong \varinjlim (\Omega[T] \otimes \Omega[S])$$

and we see that $X \otimes Y$ can be expressed in terms of $\Omega[T] \otimes \Omega[S]$. We thus need to define $\otimes$ only for representable dendroidal sets. To that end consider the functor $\Omega \to BrdPoset$ which sends a tree T to the dendroidally ordered set $[T]$. Taken as a probe (Section 0.1.1) it induces a nerve-realisation adjunction

$$dSet \underset{N}{\overset{\longrightarrow}{\longleftarrow}} BrdPoset$$

of which the left adjoint shall remain nameless. We now define, for two trees T and S in Ω, the tensor product of the associated representable dendroidal sets by

$$\Omega[T] \otimes \Omega[S] = N([T] \otimes [S])$$

where $[T] \otimes [S]$ is the tensor product in the category $BrdPoset$ (Definition 2.3.5). We can now define the tensor product in $dSet$.

Definition 2.5.1. Let X and Y be two dendroidal sets. Their tensor product is given as follows. Write X and Y canonically as colimits as above. Then

$$X \otimes Y = \varinjlim \Omega[T] \otimes \varinjlim \Omega[S] = \varinjlim (\Omega[T] \otimes \Omega[S]).$$

Theorem 2.5.2. *The above defined tensor product of dendroidal sets turns dSet into a symmetric closed monoidal category.*

Proof. This follows from general theorems of category theory as presented in [**11, 26**]. □

The internal Hom is explicitly given as follows. Let X and Y be two dendroidal sets. Their internal Hom is the dendroidal set $\underline{dSet}(X,Y)$ whose set of T-dendrices is given by

$$\underline{dSet}(X,Y)_T = dSet(X \otimes \Omega[T], Y)$$

and the dendroidal structure is given in the obvious way.

The monoidal structure on $dSet$ extends the cartesian product in $sSet$ in the following sense.

Lemma 2.5.3. *For any two simplicial sets X and Y*

$$i_!(X) \otimes i_!(Y) \cong i_!(X \times Y).$$

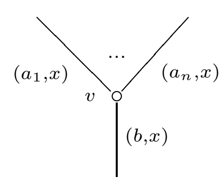
Proof. The statement will follow if we can show that it holds for representable simplicial sets. This will be proved in Theorem 3.1.4 after we have developed some more of the theory of dendroidal sets. □

Let us study the simplest (yet important) case of the tensor product of two representable dendroidal sets. To that end let us fix two trees S and T. Our aim is to exhibit $\Omega[S] \otimes \Omega[T]$ as a union of some of its dendrices, which carry a natural partial order.

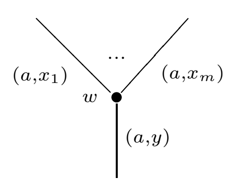
By definition, $\Omega[S] \otimes \Omega[T]$ is the dendroidal set $N([S] \otimes [T])$, where $[S] \otimes [T]$ is the tensor product of broad posets and $N(-)$ is the functor $N : BrdPoset \to dSet$ obtained from the probe $\Omega \to BrdPoset$ sending T to $[T]$. An R-dendrex in $\Omega[S] \otimes \Omega[T]$ is thus a map

$$[R] \to [S] \otimes [T]$$

of broad posets. It is easily seen that this dendrex is non-degenerate if, and only if, its underlying set function is injective. For the purpose of characterizing these maps, let us think of the vertices of S as being *white* (drawn $\circ$) and those of T as being *black* (drawn $\bullet$). Consider a tree R whose set of edges is the set $E(S) \times E(T)$. A vertex in such a tree that looks like this:



where v is a vertex in S with input edges $a_1, \cdots, a_n$ and output edge b, while x is an edge of T, will be called a *white* vertex. Similarly a vertex in R that looks like this:



where w is a vertex in T with input edges $x_1, \cdots, x_m$ and output edge y, while a is an edge in S, will be called a *black* vertex.

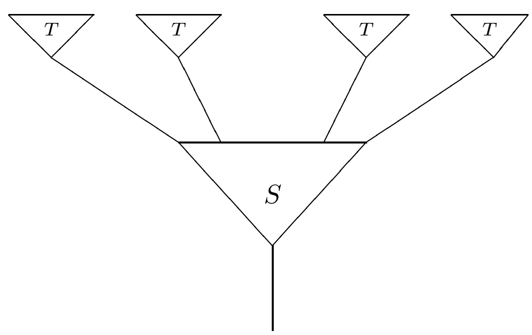
Definition 2.5.4. An (S,T)-tree is a tree R whose set of edges is the set $E(T) \times E(S)$ in which every vertex is either white or black.

Obviously, any (S,T)-tree R gives rise to an injective map $f : [R] \to [S] \otimes [T]$ (just the inclusion), and thus corresponds to a non-degenerate dendrex in $\Omega[S] \otimes \Omega[T]$. Moreover, every such non-degenerate dendrex is isomorphic (see Definition 2.4.14) to a dendrex obtained this way. Among all (S,T)-trees there are certain trees that are maximal in the following sense:

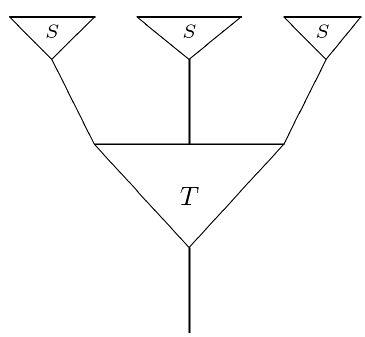
DEFINITION 2.5.5. An (S,T)-tree R is called a *percolation* tree if the root of R is (r_S, r_T) where r_S (respectively r_T) is the root of S (respectively T) and if each leaf of R is of the form (l_S, l_T) where l_S (respectively l_T) is a leaf of S (respectively T).

It is easily verified that every (S,T)-tree can be extended (not necessarily uniquely) to a percolation tree. In that sense the percolation trees are maximal and it follows that every non-degenerate vertex in $\Omega[S] \otimes \Omega[T]$ is isomorphic to a sub-face (see Definition 2.4.14) of a dendrex given by a percolation tree.

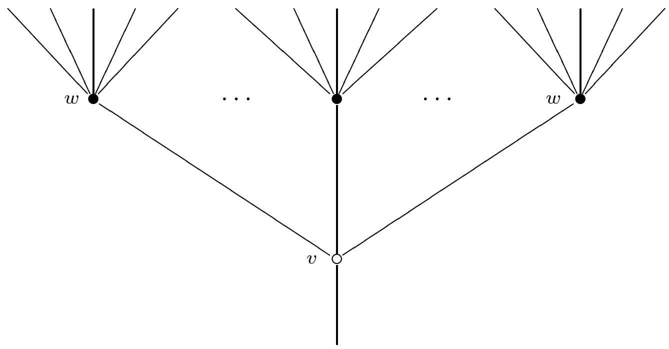
All the possible percolation trees R_i come in a natural partial order. The minimal tree R_1 in the poset is the one obtained by stacking a copy of the black tree T on top of each of the input edges of the white tree S. Or, more precisely, on the bottom of T_1 there is a copy $S \otimes r_T$ of the tree S all whose edges are (a, r_T) where r_T the root of T. For each input edge b of S, a copy of T is grafted on the edge (b,r) of $S \otimes r$, whose edges are (b,x). The maximal tree R_N in the poset is the similar tree with copies of the white tree S grafted on each of the input edges of the black tree. Pictorially R_1 looks like



and R_N looks like

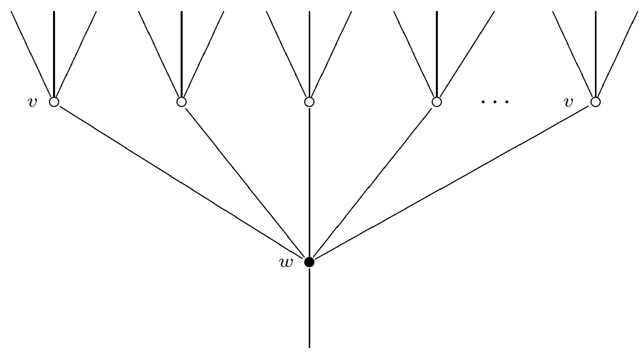


The intermediate trees R_k $(1 < k < N)$ are obtained by letting the black vertices in R_1 slowly percolate in all possible ways towards the root of the tree. Each R_k is obtained from an earlier R_l by replacing a configuration



(1)

in R_l by the configuration



(2)

in R_k. More explicitly, let P be the portion of the tree R_l shaped like (1). If we denote by R' the part of R_l below the vertex v and by $R'_1, \cdots, R'_m$ the parts of R_l above the edges depicted as leaves in (1) then we can write

$$R_l = R' \circ P \circ (R'_1, \cdots, R'_m).$$

The tree R_k is now given as

$$R_k = R' \circ P' \circ (R'_1, \cdots, R'_m)$$

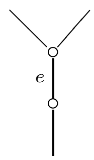
where P' is a tree that looks like (2). Notice that the grafting is well defined since the output edges of P' are precisely the roots of the various R_i, where only the order changed. Since we are dealing with non-planar trees we don't have to rearrange the trees R_i. When this is the case we say that R_k is obtained by a *single percolation step* from R_l and denote this by $R_l \leq R_k$. This defines a partial order on the set of all percolation trees.

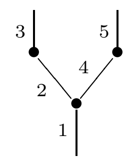
As mentioned, each percolation tree R_k corresponds to a dendrex in $\Omega[S] \otimes \Omega[T]$ of shape R_k and thus to a map $m : \Omega[R_k] \to \Omega[S] \otimes \Omega[T]$. We denote by $m(R_k)$ the image of $\Omega[R_k]$ under this map. We summarize the above discussion in the following lemma.

LEMMA 2.5.6. *(Shuffle presentation of $\Omega[S] \otimes \Omega[T]$) With notation as above we have the equality*

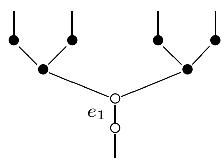
$$\Omega[S] \otimes \Omega[T] = \bigcup_{k=1}^{N} m(R_k).$$

EXAMPLE 2.5.7. We illustrate this shuffle presentation with the following two trees S and T; here, we have singled out one particular edge e in S, numbered the edges of T as $1, \cdots, 5$, and denoted the edge (e, i) in R_k by e_i.

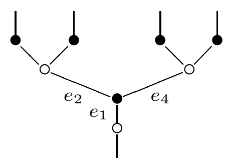
$S =$ 

$T =$ 

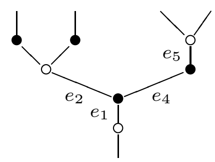
There are 14 percolation schemes $T_1, \cdots, T_{14}$ in this case:



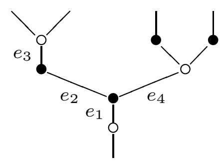
T_1



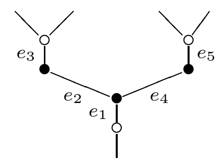
T_2



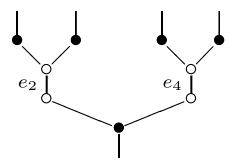
T_3



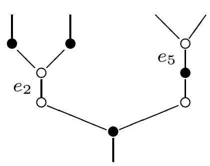
T_4



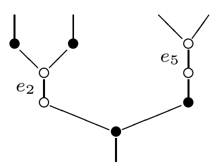
T_5



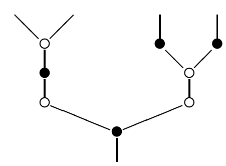
T_6

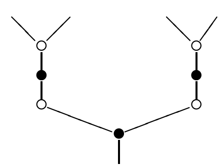


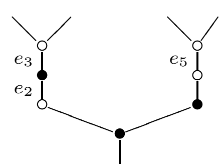
T_7



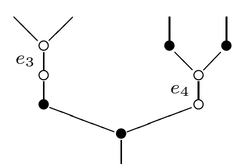
T_8


T_9

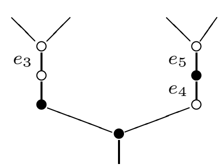

T_{10}



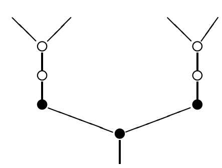
T_{11}



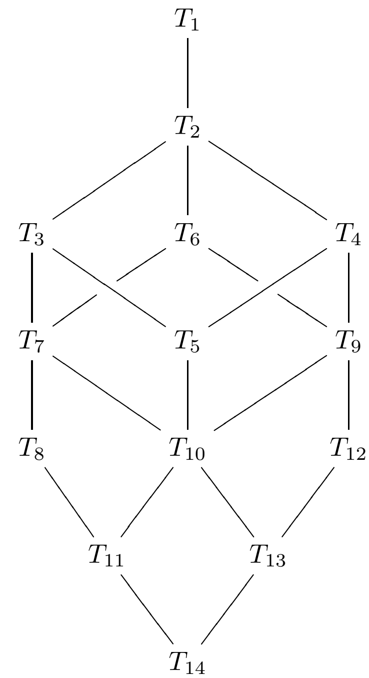
T_{12}



T_{13}


T_{14}

The poset structure on the percolation trees above is:



2.6. Skeletal filtration

We present now a useful filtration of a dendroidal set based on non-degenerate elements.

DEFINITION 2.6.1. Let X be a dendroidal set. A dendrex $t \in X_T$ is called *degenerate* if it is a degeneracy of some dendrex s (see Definition 2.4.14). Otherwise t is called *non-degenerate*. The *degree* of the dendrex t is equal to the degree of the tree T, i.e., the number of vertices of T.

For example, any dendrex $t \in X_T$ where T has no unary vertices is non-degenerate. One can also easily see that every degenerate dendrex $t \in X_T$ is a degeneracy of a non-degenerate dendrex. This dendrex is unique up to an isomorphism.

DEFINITION 2.6.2. Let X be a dendroidal set and $n \geq 0$ a natural number. The *n-skeleton* of X is the dendroidal set $Sk_n(X) \subseteq X$ generated by the dendrices of X of degree less then or equal to n. There is an obvious inclusion $Sk_n(X) \subseteq Sk_{n+1}(X)$.

Clearly we have that

$$X = \bigcup_{n=0}^{\infty} Sk_n(X)$$

and we refer to this presentation of X as the skeletal filtration of X.

REMARK 2.6.3. This filtration of a dendroidal set relates to the standard filtration of a simplicial set as follows. If X is a simplicial set with skeletal filtration $X_0 \subseteq X_1 \subseteq \cdots$ then $i_!(X_0) \subseteq i_!(X_1) \subseteq \cdots$ is isomorphic to the skeletal filtration of the dendroidal set $i_!(X)$.

Definition 2.6.4. A dendroidal set X is called *n-skeletal* if, given a dendroidal set Y, every map $Sk_n(X) \to Sk_n(Y)$ extends uniquely, along the inclusion $Sk_n(Y) \to Y$, to a map $X \to Y$. Similarly, X is called *n-coskeletal* if given an arbitrary dendroidal set Z, every map $Sk_n(Z) \to Sk_n(X)$ extends uniquely , along the inclusion $Sk_n(Z) \to Z$, to a map $Y \to X$.

Once again, for a simplicial set X, the dendroidal set $i_!(X)$ is n-skeletal (respectively n-coskeletal) if, and only if, X is n-skeletal (respectively n-coskeletal).

Notice that

$$Sk_0(X) = \coprod_{x \in X_\eta} \Omega[\eta]$$

where η is some fixed unit tree. For $n > 0$ consider now the following diagram:

$$\begin{tikzcd} \coprod_{(t,T)} \partial\Omega[T] \arrow[r] \arrow[d] & Sk_{n-1}(X) \arrow[d] \\ \coprod_{(t,T)} \Omega[T] \arrow[r] & Sk_n(X) \end{tikzcd}$$

where the coproduct ranges over all isomorphism classes of pairs (t, T) in the category of elements of X, where $|T| = n$ and $t \in X_T$ is non-degenerate. In more detail, two pairs (t, T) and (s, S) where $|T| = |S| = n$ and both $t \in X_T$ and $s \in X_S$ are non-degenerate are isomorphic if there is an isomorphism $\alpha : S \to T$ such that $\alpha^* t = s$. In the coproduct above we choose one representative of each such isomorphism class of pairs.

Definition 2.6.5. Let X be a dendroidal set. The skeletal filtration of X is said to be *normal* if for each $n > 0$, the square above is a pushout. We then say that X admits a *normal skeletal filtration*.

Following Cisinski **[9]** we make the following definition:

Definition 2.6.6. A dendroidal set X is called *normal* if for each non-degenerate dendrex $t \in X_T$, the only isomorphism $\alpha : T \to T$ that fixes t (i.e., $\alpha^*(t) = t$) is the identity.

Cisinski develops a very rich theory of certain presheaf categories and a special case of his theory is the following theorem:

Lemma 2.6.7. *A dendroidal set X is normal if, and only if, it admits a normal skeletal filtration.*

Proof. The proof is a special case of Lemma 8.1.34 in **[9]** after noting that Ω is a skeletal category (Definition 8.1.1 in **[9]**). $\square$

Clearly, all representable dendroidal sets are normal. Below we will see examples of other normal dendroidal sets. To get a better intuition for normal dendroidal sets let us give an example of a dendroidal set which is not normal. Consider the dendroidal set $X = \Omega[C_2]$ where C_2 is a 2-corolla. For this dendroidal set, X_η consists of three dendrices and X_{C_2} consists of two dendrices with an evident action of $\mathbb{Z}_2$. Consider now the dendroidal set $Y = X/\mathbb{Z}_2$ obtained from X by identifying the two dendrices in X_{C_2}. That means that Y_η consists of two elements and Y_{C_2} of one element. It is then clear that Y doesn't satisfy the condition for normality (the unique dendrex in Y_{C_2} is fixed by a non-trivial isomorphism) and it can also be seen directly that the skeletal filtration of Y is not normal.