CHAPTER 1

Operads

The theory of operads is a rich and well established one as seen through the works of May [**36**], Ginzburg and Kapranov [**18**], Boardman and Vogt [**7**], and Getzler and Jones [**17**], just to name a few. It is the aim of this chapter to introduce operads and their basic theory and is thus expository in nature. However, our approach is vastly different than the classical one, in which operads are introduced as algebraic structures modelled after the endomorphism operad [**36**]. Instead, our approach is categorical in the sense that we view operads as a direct generalization of categories. In fact, what we call operads are usually named symmetric multicategories [**31**] or coloured operads [**6**]. Our approach is very close to Leinster's [**31**]. Our decision to use the term 'operad' throughout is a mix of personal preference and arbitrariness, simply since a choice must be made. We attempt no justification for our choice nor do we claim that it is better than any other terminology. For the sake of clarity then we emphasize again that by an operad we mean a symmetric multicategory or, equivalently, a symmetric coloured operad (in the category of sets). While the treatment of operads presented here is very elementary and contains a lot of known results, it is sprinkled with new simple results that arise naturally by taking the categorical approach. Most notably, Sections 4-7 contain new results all of which relate to either known results in operad theory or in category theory.

The chapter starts by giving the definition of operads, maps of operads (functors), and natural transformations where an attempt to parallel the development of the theory to that of category theory is made. The construction of free operads is then introduced which facilitates the definition of operads using generators and relations, followed by an examination of limits and colimits of operads. The Boardman-Vogt tensor product of operads is presented together with a proof that this makes the category of operads into a symmetric closed monoidal category. Then the 'folk' Quillen model structure on the category of small categories is extended to operads, and it is shown that with the Boardman-Vogt tensor product the category of operads is a symmetric closed monoidal model category. Following is a presentation of a Grothendieck construction for operads. The chapter ends with a consideration of enriched operads and a comparison of our notation with the classical one.

1.1. Operads, functors, and natural transformations

In [**14**] the authors explain that categories are defined in order to be able to define functors, which in turn are defined to facilitate the definition of natural transformations. We develop the basic definitions of operad theory along the same lines.

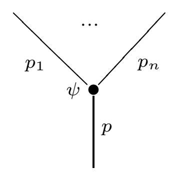
Definition 1.1.1. A *planar operad* $\mathcal{P}$ is given by specifying objects and operations and supplying a composition function on the operations, which satisfies unit and associativity axioms. In detail:

Objects and operations

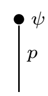
There is a specified set of objects $ob(\mathcal{P})$. For each sequence of objects

$$p_1, \cdots, p_n, p,$$

also called a *signature*, there is a set $\mathcal{P}(p_1, \cdots, p_n; p)$ of *operations* or *arrows.* Such an operation ψ will be depicted as



and will be said to have $(p_1, \cdots, p_n)$ as *input* and p as *output,* and to be of *arity* n. It is assumed that each operation has a well defined input and output, in other words if $\mathcal{P}(p_1, \cdots, p_n; p_0) \cap \mathcal{P}(q_1, \cdots, q_m; q_0) \neq \emptyset$ then $m = n$ and $p_i = q_i$ for $0 \leq i \leq n$. For each object p there is an operation $id_p \in \mathcal{P}(p; p)$ called the *identity* on p. We allow n to be 0, in which case ψ will be denoted by



Composition function

The operations can be composed in the following way. Given a signature $p_1, \cdots, p_n, p$, and for each $1 \leq i \leq n$, another sequence of objects $p^i_1, \cdots p^i_{m_i}$, there is a *composition* function

$$\begin{array}{c} \mathcal{P}(p_1, \cdots, p_n; p) \times \mathcal{P}(p^1_1, \cdots, p^1_{m_1}; p_1) \times \cdots \times \mathcal{P}(p^n_1, \cdots, p^n_{m_n}; p_n) \\ \big\downarrow \\ \mathcal{P}(p^1_1, \cdots, p^1_{m_1}, \cdots, p^n_1, \cdots, p^n_{m_n}; p) \end{array}$$

If $\psi \in \mathcal{P}(p_1, \cdots, p_n; p)$ and $\psi_i \in \mathcal{P}(p^i_1, \cdots, p^i_{m_i}; p_i)$, we denote by $\psi \circ (\psi_1, \cdots, \psi_n)$ (or simply by $\psi(\psi_1, \cdots, \psi_n)$) the image of $(\psi, \psi_1, \cdots, \psi_n)$ under the composition function.

Axioms

The identities are required to satisfy

$$id_p(\psi) = \psi$$

and

$$\varphi(id_{p_1}, \cdots, id_{p_n}) = \varphi$$

whenever the compositions are defined. Furthermore, the composition is required to be *associative* in the sense that given

$$\psi \in \mathcal{P}(p_1, \cdots, p_n; p),$$

for each $1 \leq i \leq n$ an operation

$$\psi_i \in \mathcal{P}(p_1^i, \cdots, p_{m_i}^i; p_i),$$

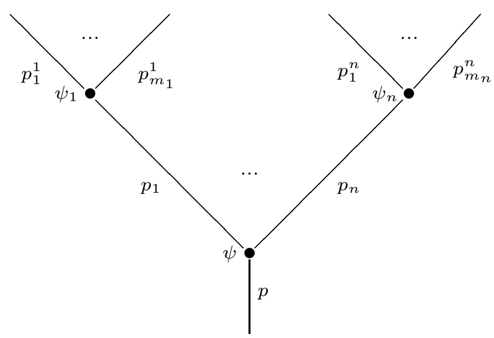
and for each pair (i,j) with $1 \leq i \leq n$ and $1 \leq j \leq m_i$ an operation ψ_j^i with output p_j^i (and an arbitrary input), the composition

$$(\psi(\psi_1, \cdots, \psi_n))(\psi_1^1, \cdots, \psi_{m_1}^1, \psi_1^2, \cdots, \psi_{m_2}^2, \cdots, \psi_1^n, \cdots, \psi_{m_n}^n)$$

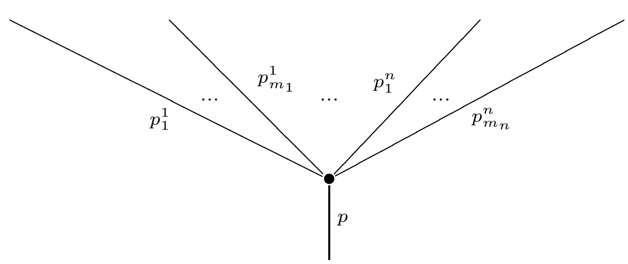
should be equal to the composition

$$\psi(\psi_1(\psi_1^1, \cdots, \psi_{n_1}^1), \cdots, \psi_n(\psi_1^n, \cdots, \psi_{m_n}^n)).$$

Some light can be shed on the definition by considering certain labelled planar trees. In more detail, given an operad $\mathcal{P}$ and a planar tree T one can consider labelling the edges and vertices of the tree T respectively with objects and operations of the operad $\mathcal{P}$. We call T a *labelled* tree if each edge e is labelled by an object $p_e \in ob(\mathcal{P})$ and if each vertex v with $in(v) = (e_1, \cdots, e_n)$ and $out(v) = e$ is labelled by an operation $\psi_v \in \mathcal{P}(p_{e_1}, \cdots, p_{e_n}; p_e)$. Using this language one can interpret the composition function in the operad as follows. Given a labelled tree T



the composition function associates to it a labelled corolla (that is, just an operation) of the shape of the tree obtained from the one above by contracting all of the inner edges, where the labelling is as follows. All of the edges of the corolla are also edges in the original tree (namely the outer ones) and they retain their labels from T. The sole vertex of the corolla is then labelled by $\psi \circ (\psi_1, \cdots, \psi_n)$. Visually, the composition associated to the tree above is depicted by the following labelled corolla:



We can refine the composition a bit by introducing the so called $\circ_i$-compositions. Given $p_1, \cdots, p_n, p$ and $q_1, \cdots, q_m$ objects in $\mathcal{P}$, the $\circ_i$-composition for $1 \leq i \leq n$

is the function

$$\mathcal{P}(p_1, \cdots, p_n; p) \times \mathcal{P}(q_1, \cdots, q_m; p_i) \xrightarrow{\circ_i} \mathcal{P}(p_1, \cdots, p_{i-1}, q_1, \cdots, q_m, p_{i+1}, \cdots, p_n; p)$$

defined by

$$\psi \circ_i \varphi = \psi(id_{p_1}, \cdots, id_{p_{i-1}}, \varphi, id_{p_{i+1}}, \cdots, id_{p_n}).$$

We can again use planar trees to get a geometric picture of the $\circ_i$-compositions. Given any labelled planar tree S and an inner edge e in it, there is a natural labelling of the tree S/e, obtained from S by contracting e. The labelling of S/e is as follows. In S/e there is just one vertex which does not appear in S, all other vertices and edges other then e occur in S as well and retain their labels. Let v be the new vertex in S/e and suppose e leads from the vertex u to w, so $in(w) = (u_1, \cdots, u_k)$ and $u_j = u$ for some $1 \leq j \leq k$. The label of v in S/e is then defined to be $\psi_w \circ_j \psi_u$.

By sequentially contracting all of the inner edges in S we obtain a labelled corolla $c(S)$. It is a direct consequence of the associativity axiom that the label of the only vertex in $c(S)$ is independent of the chosen order in which edges are contracted. We will sometimes refer to a labelled tree S as a *composition scheme* in $\mathcal{P}$ and will then refer to the uniquely labelled corolla $c(S)$ (or rather to the operation labelling its unique vertex) as the *composition* of the composition scheme.

It is obvious that under the suitable associativity conditions of the various $\circ_i$-compositions, an operad can equivalently be given by a set of objects together with $\circ_i$-compositions. See **[35]** for more details on defining operads via $\circ_i$-compositions for the special case where the operad in question has just one object. The extension to the general case is trivial.

Definition 1.1.2. An *operad* (or a *symmetric* operad) is a planar operad together with actions of the symmetric groups as follows. Given a permutation $\sigma \in \Sigma_n$ there is a function $\sigma^* : \mathcal{P}(p_1, \cdots, p_n; p) \to \mathcal{P}(p_{\sigma(1)}, \cdots, p_{\sigma(n)}; p)$. These functions are required to define a right action of Σ_n (that is, $(\sigma\tau)^* = \tau^*\sigma^*$, and for the identity permutation $id \in \Sigma_n$ we have $id^* = id$) and to respect compositions in the following sense. Given operations $\psi_0, \cdots, \psi_n$ for which the composition $\psi_0 \circ (\psi_1, \cdots, \psi_n)$ is defined, and permutations $\sigma_0, \cdots, \sigma_n$ where $\sigma_i \in \Sigma_{k_i}$ (with k_i the arity of ψ_i), the equation

$$\sigma_0^*(\psi_0) \circ (\sigma_{\sigma_0(1)}^*(\psi_{\sigma_0(1)}), \cdots, \sigma_{\sigma_0(n)}^*(\psi_{\sigma_0(n)})) = [\sigma_0 \circ (\sigma_1, \cdots, \sigma_n)]^*(\psi_0 \circ (\psi_1, \cdots, \psi_n))$$

holds. The permutation $\sigma_0 \circ (\sigma_1, \cdots, \sigma_n)$ is the block permutation product of the given permutations, which is the evident one equating inputs on both sides (see **[31]** for more details, page 77 under 'operad of symmetries', and **[35]**).

Remark 1.1.3. It is easily seen that an operad (planar or symmetric) that has only operations of arity 1 is the same thing as a category. More precisely, given such an operad $\mathcal{P}$ we define the category $j^*(\mathcal{P})$ by setting

$$ob(j^*(\mathcal{P})) = ob(\mathcal{P})$$

and for objects $p, p' \in ob(\mathcal{P})$ we set

$$j^*(\mathcal{P})(p, p') = \mathcal{P}(p; p').$$

The units and composition are induced by those in $\mathcal{P}$ in the obvious way. The result is a category by the unit and associativity axioms for operads.

EXAMPLE 1.1.4. There are many examples of operads given in the literature (see e.g., [**18, 35, 36**]), the vast majority of which have just one object. We wish to present here a different family of operads, namely, those obtained from symmetric monoidal categories. Let $\mathcal{E}$ be any symmetric monoidal category and M a subset of $ob(\mathcal{E})$. The operad $\mathcal{P}_M$ is defined as follows. The set of objects of $\mathcal{P}_M$ is M and the set of operations with input $(A_1, \cdots, A_n)$ and output B is the set $\mathcal{E}(A_1 \otimes \cdots \otimes A_n, B)$ where some choice for the repeated tensoring was made. The composition and units in $\mathcal{P}_M$ are the evident ones, and the symmetric groups Σ_n act by permuting the variables. The operad axioms follow immediately from the usual coherence theorems for symmetric monoidal categories. We will usually write $\hat{\mathcal{E}}$ instead of $\mathcal{P}_{ob(\mathcal{E})}$, or just $\mathcal{E}$ where context will prevent confusion.

DEFINITION 1.1.5. Let $\mathcal{P}$ and $\mathcal{Q}$ be two planar operads. A *map of planar operads* $F : \mathcal{P} \to \mathcal{Q}$ is a function $F : ob(\mathcal{P}) \to ob(\mathcal{Q})$ and for any sequence of objects $p_1, \cdots, p_n, p$ in $ob(\mathcal{P})$ a function

$$F : \mathcal{P}(p_1, \cdots, p_n; p) \to \mathcal{Q}(Fp_1, \cdots, Fp_n; Fp)$$

such that

$$F(\psi(\psi_1, \cdots, \psi_n)) = F(\psi)(F\psi_1, \cdots, F\psi_n)$$

holds whenever the compositions make sense. Furthermore, for any $p \in ob(\mathcal{P})$ we demand that $F(id_p) = id_{Fp}$. If $\mathcal{P}$ and $\mathcal{Q}$ are both symmetric then a *map of symmetric operads* $F : \mathcal{P} \to \mathcal{Q}$ is the same as above with the extra condition that for any permutation $\sigma \in \Sigma_n$ one has

$$F(\sigma^*(\psi)) = \sigma^*(F(\psi))$$

for any operation ψ with input of length n.

REMARK 1.1.6. When $\mathcal{P}$ and $\mathcal{Q}$ are both categories (that is, they have only unary operations) it is immediate to see that a map of operads (planar or symmetric) $F : \mathcal{P} \to \mathcal{Q}$ is the same thing as a functor. For this reason we will also use the word *functor* to refer to maps of operads.

The category $Operad_\pi$ is the category of all planar operads and functors between them with the obvious notion of composition of functors and the evident identity functors. Likewise, $Operad$ is the category of all symmetric operads and their maps. The remarks above allude to the fact that the category $Operad$ can be seen as an extension of the category Cat. We now make this relation precise. Given a category $\mathcal{C}$ we can construct a planar operad $j_!\mathcal{C}$ by setting

$$ob(j_!\mathcal{C}) = ob(\mathcal{C})$$

and for objects $c, c' \in ob(j_!\mathcal{C})$ we define

$$j_!\mathcal{C}(c; c') = \mathcal{C}(c, c').$$

Composition and units are induced from $\mathcal{C}$ in the obvious way to make $j_!\mathcal{C}$ into an operad. Notice that since in $j_!\mathcal{C}$ all operations are unary (that is they have just one input), each symmetry group Σ_n acts trivially on the operations of the planar operad $j_!\mathcal{C}$. It follows that $j_!\mathcal{C}$ can also be considered as a symmetric operad. We thus obtain two functors (both named $j_!$):

$$\begin{array}{ccc} & Cat & \\ {}^{j_!}\swarrow & & \searrow^{j_!} \\ Operad_\pi & & Operad \end{array}$$

that view a category as an operad (planar or symmetric) all of which operations are unary. Clearly both of these functors are fully faithful. We will thus consider Cat to be embedded in $Operad$ and in $Operad_\pi$ via $j_!$. These two functors both have right adjoints which send a (planar or symmetric) operad $\mathcal{P}$ to the category $j^*(\mathcal{P})$ whose objects are the objects of $\mathcal{P}$, and whose arrows for any two objects $p, p' \in Ob(j^*\mathcal{P})$ are given by

$$j^*\mathcal{P}(p, p') = \mathcal{P}(p; p').$$

The identities and the compositions are as in $\mathcal{P}$. Somewhat less formally, we see that inside any operad there is a category which is the *linear* part of the operad. We will freely use category theoretic terms and notation when referring to this category. Thus for example, the meaning of "ψ is an isomorphism in the operad $\mathcal{P}$" should be interpreted as "$j^*\psi$ is an isomorphism in the category $j^*\mathcal{P}$". Again we use the same name, j^*, for both functors $Operad_\pi \to Cat$ and $Operad \to Cat$.

There is an obvious forgetful functor $U : Operad \to Operad_\pi$ which simply forgets the symmetric group actions. This functor has a left adjoint

$$Symm : Operad_\pi \to Operad,$$

called the *symmetrization* functor, which we now describe. Let $\mathcal{P}$ be a planar operad. The objects of $Symm(\mathcal{P})$ are the same as the objects of $\mathcal{P}$. To describe the operations in $Symm(\mathcal{P})$ let $p_1, \cdots, p_n, p \in ob(Symm(\mathcal{P}))$. For each $\sigma \in \Sigma_n$ let

$$\mathcal{P}_\sigma(p_1, \cdots, p_n; p) = \{\sigma\} \times \mathcal{P}(p_{\sigma^{-1}(1)}, \cdots, p_{\sigma^{-1}(n)}; p).$$

We now define

$$Symm(\mathcal{P})(p_1, \cdots, p_n; p) = \coprod_{\sigma \in \Sigma_n} \mathcal{P}_\sigma(p_1, \cdots, p_n; p).$$

The unit id_p for $p \in ob(Symm(\mathcal{P}))$ is (id, id_p) and the symmetric groups Σ_n can now freely act on the various operations in $Symm(\mathcal{P})$ as we now describe. Let $\tau \in \Sigma_n$ be a permutation, we define

$$\tau^* : Symm(\mathcal{P})(p_1, \cdots, p_n; p) \to Symm(\mathcal{P})(p_{\tau(1)}, \cdots, p_{\tau(n)}; p)$$

on $(\sigma, \psi) \in \mathcal{P}_\sigma(p_1, \cdots, p_n; p)$ by

$$\tau^*(\sigma, \psi) = (\sigma\tau, \psi) \in \mathcal{P}_{\sigma\tau}(p_{\tau(1)}, \cdots, p_{\tau(n)}; p).$$

This obviously defines a right action of the symmetric groups. To define the composition let $\psi_0 \in Symm(\mathcal{P})(p_1, \cdots, p_n; p)$ and $\psi_i \in Symm(\mathcal{P})(p_1^i, \cdots, p_{m_i}^i; p_i)$ for $1 \le i \le n$ be operations in $Symm(\mathcal{P})$. By definition we have then that $\psi_i = (\tau_i, \varphi_i)$ with

$$\varphi_0 \in \mathcal{P}(p_{\tau_0^{-1}(1)}, \cdots, p_{\tau_0^{-1}(n)}; p)$$

and for $1 \le i \le n$

$$\varphi_i \in \mathcal{P}(p^i_{\tau_i^{-1}(1)}, \cdots, p^i_{\tau_i^{-1}(m_i)}; p_i).$$

We can thus use the composition in $\mathcal{P}$ to obtain the operation

$$\varphi = \varphi_0 \circ (\varphi_{\tau_0^{-1}(1)}, \cdots, \varphi_{\tau_0^{-1}(n)}).$$

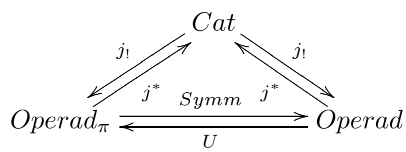
Calculating the domain of φ we see that

$$(\tau \circ (\tau_1, \cdots, \tau_n), \varphi) \in Symm(\mathcal{P})(p_1^1, \cdots, p_{n_1}^1, \cdots, p_1^n, \cdots, p_{m_n}^n; p),$$

(where $\tau \circ (\tau_1, \cdots, \tau_n)$ is again the block permutation product) and we define this operation to be the composition $\psi_0 \circ (\psi_1, \cdots, \psi_n)$ in $Symm(\mathcal{P})$. The verification of the rest of the axioms is straightforward.

We summarize the information given above relating categories, planar operads, and symmetric operads in the following theorem.

THEOREM 1.1.7. *The six functors described above fit into the triangle*

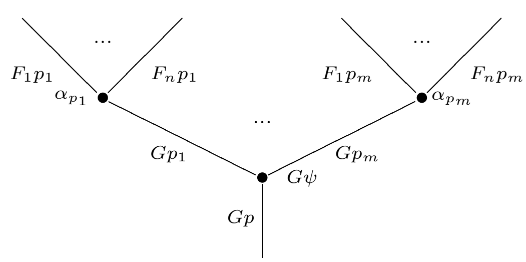


where each pair of functors is an adjunction (with the left adjoint on top), and the following equations hold:

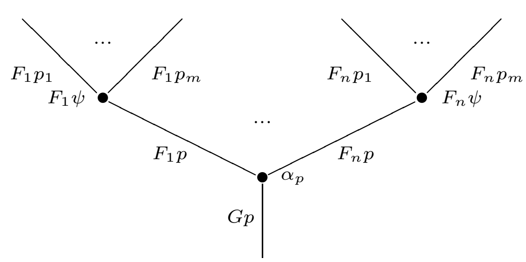
(1) $j^* j_! \cong id$ (these are actually two equalities)
(2) $j^* \circ Symm = j^*$ as functors from $Operad_\pi$ to Cat
(3) $j^* \circ U = j^*$ as functors from $Operad$ to Cat

We now turn to define natural transformations for operads.

DEFINITION 1.1.8. Let $F_i : \mathcal{P} \to \mathcal{Q}$ for $1 \leq i \leq n$ and $G : \mathcal{P} \to \mathcal{Q}$ be $n+1$ functors between symmetric operads. A *natural transformation* α from $(F_1, \cdots, F_n)$ to G is a family $\{\alpha_p\}_{p \in ob(\mathcal{P})}$, where $\alpha_p \in \mathcal{Q}(F_1 p, \cdots, F_n p; Gp)$ and is called the *component* of the natural transformation at p, satisfying the following property. Given any operation $\psi \in \mathcal{P}(p_1, \cdots, p_m; p)$ consider the following composition schemes in $\mathcal{Q}$



and



and let φ_1 and φ_2 be the compositions in $\mathcal{Q}$ of, respectively, the first and second composition schemes. We demand that $\varphi_2 = \sigma^*_{m,n}(\varphi_1)$, where $\sigma_{m,n}$ is the obvious permutation equating the inputs of both operations.

Remark 1.1.9. It is trivial to check that when $\mathcal{P}$ and $\mathcal{Q}$ are categories, if α is a natural transformation from $(F_1, \cdots, F_n)$ to G then $n = 1$ *and* α is exactly the same thing as a natural transformation in the categorical sense. It is also immediate to verify that given a natural transformation $\alpha : F \to G$ in the operadic sense, the family $\{\alpha_p\}_{p \in ob(\mathcal{P})}$ is a natural transformation between the functors $j^*(F)$ and $j^*(G)$ in the categorical sense.

Notice as well that the symmetric actions play a vital role in the definition. One cannot define natural transformations between functors of planar operads unless the domain consists of a single functor. This is a significant difference between the category of planar operads and that of symmetric operads.

Natural transformations can be composed as follows. Fix two operads $\mathcal{P}$ and $\mathcal{Q}$. Suppose $\alpha : (F_1, \cdots, F_n) \to F$ and $\beta^i : (F^i_1, \cdots, F^i_{m_i}) \to F_i$ for $1 \le i \le n$ are natural transformations where all the functors are from $\mathcal{P}$ to $\mathcal{Q}$. The *composition* of these natural transformations is the natural transformation

$$\alpha \circ (\beta^1, \cdots, \beta^n) : (F^1_1, \cdots, F^1_{m_1}, \cdots, F^n_1, \cdots, F^n_{m_n}) \to F$$

that for each object $p \in ob(\mathcal{P})$ has the component

$$[\alpha \circ (\beta^1, \cdots, \beta^n)]_p = \alpha_p \circ (\beta^1_p, \cdots, \beta^n_p).$$

The verification of the naturality is routine.

Proposition 1.1.10. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two operads. We denote the set of all functors $F : \mathcal{P} \to \mathcal{Q}$ by $ob(Func(\mathcal{P}, \mathcal{Q}))$. Given functors $F_1, \cdots, F_n, F \in ob(Func(\mathcal{P}, \mathcal{Q}))$ let $Func(\mathcal{P}, \mathcal{Q})(F_1, \cdots, F_n; F)$ be the set of all natural transformations*

$$\alpha : F_1, \cdots, F_n \to F.$$

The composition of natural transformations defined above makes $Func(\mathcal{P}, \mathcal{Q})$ into a symmetric operad (with the obvious units and Σ_n-actions).

Proof. The proof is completely routine and thus omitted. □

Remark 1.1.11. As noted, the symmetries in the operad play a crucial role in the definition of $Func(\mathcal{P}, \mathcal{Q})$. If $\mathcal{P}$ and $\mathcal{Q}$ were planar operads we would still be able to consider the collection of all functors between them, but in order to obtain some sensible structure on it we would have to restrict ourselves to those natural transformations that have a single functor as domain. This is done very briefly in [**31**] (page 87 under the name 'transformation') and we recount it here. Let $\mathcal{P}$ and $\mathcal{Q}$ be two planar operads. We denote by $ob(Func(\mathcal{P}, \mathcal{Q}))$ the set of all functors $F : \mathcal{P} \to \mathcal{Q}$ as above. For two functors $F, F' : \mathcal{P} \to \mathcal{Q}$ we denote by $Func(\mathcal{P}, \mathcal{Q})(F, F')$ the set of all natural transformations $\alpha : F \to F'$. Composition of such natural transformations still makes sense and makes $Func(\mathcal{P}, \mathcal{Q})$ into a category.

We can summarize the discussion so far by noticing that the above remarks imply that both *Operad* and $Operad_\pi$ are strict 2-categories. For the category $Operad_\pi$ this follows from the fact that for two planar operads we have that $Func(\mathcal{P}, \mathcal{Q})$ is a category. As for the category *Operad*, we showed that for two symmetric operads $\mathcal{P}$ and $\mathcal{Q}$, $Func(\mathcal{P},\mathcal{Q})$ is a symmetric operad. By considering the category $j^*Func(\mathcal{P},\mathcal{Q})$ we see that *Operad*, too, is a strict 2-category. We now have:

THEOREM 1.1.12. *Consider Cat, Operad, and* $Operad_\pi$ *as strict* 2*-categories. The functors* $j^* : Operad \to Cat$ *and* $j^* : Operad_\pi \to Cat$ *extend naturally to strict* 2*-functors.*

PROOF. The proof is trivial. □

EXAMPLE 1.1.13. Consider the symmetric operad *Comm* given as follows. *Comm* has one object $\star$, and for each $n \geq 0$ there is just one operation in

$$Comm(\star,\cdots,\star;\star),$$

with $\star$ in the domain repeated n times, which is denoted by m_n. There is now just one way to define an operad structure. Namely, the unit $id_\star$ is m_1, composition is given by

$$m_n \circ (m_{k_1},,\cdots,m_{k_n}) = m_{k_1+\cdots+k_n},$$

and all Σ_n actions are trivial. All of the axioms for an operad are trivially satisfied. We also consider the category *Set* of small sets, which we consider as a monoidal category via the cartesian product. Recall (see Example 1.1.4) that we then have the operad $\widehat{Set}$ which we denote by *Set* again. Suppose $F : Comm \to Set$ is a functor. Such a functor consists of a function $F : ob(Comm) \to ob(Set)$, which amounts to a choice of a set A. The functor F consists further of a function $Comm(\star,\cdots,\star;\star) \to Set(A^n, A)$, that is simply a choice of a function $F(m_n) : A^n \to A$ for each $n \geq 0$. For $n = 0$ this is a map $F(m_0) : A^0 \to A$, i.e., a map $I \to A$ where I is a one-point set, so it is just a choice of a constant $e \in A$. We have thus a constant in A and for every $n \geq 1$ an n-ary operation $F(m_n) : A^n \to A$.

Let us now examine the consequences of the functoriality of F. First of all, by definition, m_1 is mapped to the identity. Furthermore, in *Comm* we have that

$$m_2 \circ (m_1, m_2) = m_3 = m_2 \circ (m_2, m_1)$$

from which it follows that

$$F(m_2) \circ (id, F(m_2)) = F(m_3) = F(m_2) \circ (F(m_2), id),$$

which implies that $F(m_2)$ is an associative binary operation. In *Comm* we also have the relation

$$m_2 \circ (m_1, m_0) = m_1 = m_2 \circ (m_0, m_1)$$

that is

$$F(m_2) \circ (id, e) = F(m_1) = F(m_2) \circ (e, id)$$

which means that e is a two-sided inverse for the binary operation $F(m_2)$. We thus see that $(A, F(m_2), e)$ is a monoid. Lastly, since F commutes with the Σ_n-actions it follows that

$$F(m_n \cdot \sigma) = F(m_n) \cdot \sigma$$

holds for every $\sigma \in \Sigma_n$. Since σ acts trivially in *Comm* we obtain that

$$F(m_n) = F(m_n) \cdot \sigma$$

holds for each $\sigma \in \Sigma_n$. Specifically for $n = 2$ and for the twist permutation $\sigma \in \Sigma_2$, we obtain (by the fact that in *Set* the symmetric groups act by permuting the variables) that $F(m_2)$ is a commutative operation. A is thus a commutative monoid. All the other relations in the operad $Comm$ impose no new conditions on the monoid, since they all just express general associativity and commutativity for various tuples of elements of A. Conversely it is clear that given a commutative monoid A, one can construct a functor $F : Comm \to Set$, such that $F(\star) = A$, $F(m_0)$ is the unit of the monoid, and $F(m_2)$ is the binary operation.

Let $F_1, \cdots, F_n, F$ be functors from $Comm$ to Set and let $A_1, \cdots, A_n, A$ be their corresponding commutative monoids. We now examine a natural transformation $\alpha : F_1, \cdots, F_n \to F$. To start with, α consists of just one component, namely $\alpha_\star : Set(F_1(\star), \cdots, F_n(\star); F(\star))$, i.e., a function $\alpha_\star : A_1 \times \cdots \times A_n \to A$. Following the definition of a natural transformation one sees that this function $\alpha_\star$ respects the binary composition in the sense that if we endow $A_1 \times \cdots \times A_n$ with the obvious commutative monoid structure then $\alpha_\star$ is a map of commutative monoids. The converse is also true and we actually obtain the following. Let $ComMon$ be the category of commutative monoids. The usual product of monoids makes $ComMon$ into a symmetric monoidal category and we may thus consider it as a symmetric operad. We then have

$$Func(Comm, Set) \cong CommMon$$

as operads.

This example illustrates a more general phenomenon, namely that operads can be used to describe algebraic structures on objects of other operads. Thus given two operads $\mathcal{P}$ and $\mathcal{Q}$ and a functor $F : \mathcal{P} \to \mathcal{Q}$, there are two ways to think about F. One way is to think of $\mathcal{P}$ and $\mathcal{Q}$ as algebraic structures and of F as a mapping preserving this structure. The other is to think of $\mathcal{P}$ as modeling an algebraic structure and of $\mathcal{Q}$ as an operad upon whose objects we wish to define that algebraic structure. The functor $F : \mathcal{P} \to \mathcal{Q}$ can then be thought of as defining an algebraic structure in Q by realizing inside $\mathcal{Q}$ the model encoded in $\mathcal{P}$. It is useful to make a semantic distinction between these two interpretations of a functor. We thus give the second interpretation a different name.

Definition 1.1.14. Let $\mathcal{P}$ and $\mathcal{E}$ be two operads. An *algebra* for $\mathcal{P}$ in $\mathcal{E}$, or a $(\mathcal{P}, \mathcal{E})$-*algebra,* is a functor

$$A : \mathcal{P} \to \mathcal{E}.$$

For such an algebra we say that A defines an algebraic structure on the family of objects given by $F : ob(\mathcal{P}) \to ob(\mathcal{E})$.

Remark 1.1.15. The choice of the letter $\mathcal{E}$ for the codomain operad is meant to make the distinction between the two different roles of the operads clear.

We end this section by generalizing some basic properties of categories, functors, and natural transformations to our setting of operads. These results are chosen since they will be used in the sequel. Of course many other results can be generalized along the same lines.

Given a natural transformation $\alpha : F \to G$, we call α a *natural isomorphism* if each component of α is an isomorphism in $\mathcal{Q}$.

Definition 1.1.16. Let $\mathcal{P}$ and $\mathcal{Q}$ be two operads. We say that they are *equivalent* provided that there are two functors $F : \mathcal{P} \to \mathcal{Q}$ and $G : \mathcal{Q} \to \mathcal{P}$ together with two natural isomorphisms $\alpha : id_{\mathcal{P}} \to GF$ and $\beta : id_{\mathcal{Q}} \to FG$. We then call F an *equivalence* from $\mathcal{P}$ to $\mathcal{Q}$.

REMARK 1.1.17. For two categories $\mathcal{C}$ and $\mathcal{D}$, it is obvious that $\mathcal{C}$ and $\mathcal{D}$ are equivalent if, and only if, $j_!\mathcal{C}$ and $j_!\mathcal{D}$ are equivalent operads. It is also clear that if $\mathcal{P}$ and $\mathcal{Q}$ are equivalent operads then $j^*\mathcal{P}$ and $j^*\mathcal{Q}$ are equivalent categories. The converse implication is (in general) not true, as can easily be seen.

DEFINITION 1.1.18. A functor $F : \mathcal{P} \to \mathcal{Q}$ is *essentially surjective* if j^*F is. F is called *full* if for any signature $p_1, \cdots, p_n, p$ the function

$$F : \mathcal{P}(p_1, \cdots, p_n; p) \to \mathcal{Q}(Fp_1, \cdots, Fp_n; Fp)$$

is surjective. F is called *faithful* if the function above is injective. It is called *fully faithful* if that map is a bijection.

LEMMA 1.1.19. *Let $F : \mathcal{P} \to \mathcal{Q}$ be a functor between two operads. $\mathcal{F}$ is an equivalence of operads if, and only if, $\mathcal{F}$ is fully faithful and essentially surjective.*

PROOF. The proof is just like that of the corresponding result for categories ([**34**] Theorem 1, page 93). □

1.2. Free operads and operads given by generators and relations

We now turn to the construction of free operads (planar and symmetric). This will be done in terms of the standard planar trees defined in the preliminaries. (see [**31**], page 85, for a slightly different approach). We will then use this construction to define operads by generators and relations. The constructions given here are a special case of similar constructions in the theory of enriched operads (see e.g., [**6, 18**]).

DEFINITION 1.2.1. Let A be a set. A *collection* C on the set A is a family of sets $C(a_1, \cdots, a_n; a_0)$ where $a_i \in A$ and $n \geq 0$ varies over all natural numbers. An *arrow* $(A, C) \to (A', C')$ between collections is a function $f : A \to A'$ and a family of functions (all denoted f)

$$f : C(a_1, \cdots, a_n; a) \to C'(fa_1, \cdots, fa_n; fa).$$

We denote by Col the category of all collections and their arrows.

Evidently every planar operad $\mathcal{P}$ has an *underlying* collection C on the set $ob(\mathcal{P})$ given for $p_1, \cdots, p_n, p \in ob(\mathcal{P})$ simply by

$$C(p_1, \cdots, p_n; p) = \mathcal{P}(p_1, \cdots, p_n; p).$$

We thus obtain a functor $C_\pi : Operad_\pi \to Col$.

We shall now construct the left adjoint $\mathcal{F}_\pi : Col \to Operad_\pi$ of $C_\pi : Operad_\pi \to Col$.

Let C be a collection on a set A. We are going to define a planar operad $\mathcal{P}$ with $ob(\mathcal{P}) = A$. To define the arrows we consider standard planar trees labelled by the elements of A and elements from the collection C. Let T be a standard planar tree. A labelling of T is a choice of an element $a_e \in A$ for any edge $e \in E(T)$ and for any vertex v in T with $in(v) = (e_1, \cdots, e_n)$ and $out(v) = e_0$, an element $c_v \in C(a_{e_1}, \cdots, a_{e_n}; a_{e_0})$. Let LT be the set of all labelled standard planar trees. For such trees we will use the notation $in(T)$ to refer to the tuple of leaves $(l_1, \cdots, l_n)$ of T and also (so long that it is clear which one we mean) to the tuple of labels of the leaves $(a_{l_1}, \cdots, a_{l_n})$. Similarly $out(T)$ will refer both to the root of T and to the label of the root.

We can now define the arrows in $\mathcal{P}$. For objects $p_1, \cdots, p_n, p \in ob(\mathcal{P})$, we define

$$\mathcal{P}(p_1, \cdots, p_n; p) = \{T \in LT \mid in(T) = (p_1, \cdots, p_n), out(T) = p\}.$$

Composition is obtained by grafting labelled trees as follows. Given

$$\psi_0 \in \mathcal{P}(p_1, \cdots, p_n; p)$$

and

$$\psi_i \in \mathcal{P}(p_1^i, \cdots, p_{m_i}^i; p_i)$$

let φ be the standard planar tree obtained by grafting the root of each of ψ_i onto the i-th leaf of ψ_0. The labelling of φ is obtained by copying the labelling of each ψ_i onto the obvious sub-tree in φ corresponding to ψ_i. This composition is clearly associative. The unit at an object a is simply the tree η labelled by a itself. This completes the construction of $\mathcal{P}$. It is now trivial to confirm that we obtain a functor $\mathcal{F}_\pi : Col \to Operad_\pi$ which is in fact the left adjoint of $C_\pi : Operad_\pi \to Col$. A planar operad obtained in this way is called a *free* planar operad. Thus, for a collection C on a set A, a map of planar operads $G : \mathcal{F}C \to \mathcal{Q}$ is completely determined by a function $g : A \to ob(\mathcal{Q})$ and a family of functions

$$G : C(a_1, \cdots, a_n; a) \to \mathcal{Q}(fa_1, \cdots, fa_n; fa).$$

Given an arbitrary $\psi \in \mathcal{F}C(p_1, \cdots, p_n; p)$, consider the corresponding composition scheme in $\mathcal{Q}$ obtained by labelling each vertex in the planar standard planar tree representing ψ by its image under G in Q. We call this the composition scheme associated with ψ and denote it by $cs(\psi)$. It follows that $G\psi$ is the composition in $\mathcal{Q}$ of $cs(\psi)$.

We can use this construction to describe operads using generators and relations, much like the description of certain groups by generators and relations. This will become very handy when we discuss the closed monoidal structure on *Operad*. Let $\mathcal{P}$ be the free planar operad on the collection C. We refer to C as *generators*. A set of *relations* in $\mathcal{P}$ is a family of sets $R = \{R_{p_1, \cdots, p_n; p_0}\}_{p_i \in ob(\mathcal{P})}$ where $R_{p_1, \cdots, p_n; p_0}$ is a relation on the set $\mathcal{P}(p_1, \cdots, p_n; p_0)$. For two operations $\psi, \psi' \in \mathcal{P}(p_1, \cdots, p_n; p)$ we write $\psi \sim \psi'$ if $(\psi, \psi') \in R_{p_1, \cdots, p_n, p_0}$. A set of relations R is called *normal* if each $R_{p_1, \cdots, p_n, p_0}$ is an equivalence relation which is a congruence for the composition in $\mathcal{P}$ in the sense that given $\psi_0, \cdots, \psi_n$ and $\psi'_0, \cdots, \psi'_n$ with $\psi_i \sim \psi'_i$ for each $0 \leq i \leq n$ then (whenever the composition is defined)

$$\psi_0 \circ (\psi_1, \cdots, \psi_n) \sim \psi'_0 \circ (\psi'_1, \cdots, \psi'_n)$$

holds. Since the intersection of normal relations is again a normal relation, it follows that given any relation R in $\mathcal{P}$, there is a unique smallest normal relation R' that contains it. We call this R' the *normal* relation *generated* by R.

It is now clear that given a normal relation R' in $\mathcal{P}$, there is an operad $\mathcal{P}/R'$ given by

$$ob(\mathcal{P}/R') = ob(\mathcal{P})$$

and for objects $p_1, \cdots, p_n, p_0 \in ob(\mathcal{P}/R')$ we set

$$(\mathcal{P}/R')(p_1, \cdots, p_n; p_0) = \mathcal{P}(p_1, \cdots, p_n; p_0)/\sim$$

with the obvious operadic structure induced from $\mathcal{P}$.

Definition 1.2.2. Let C be a collection and R a set of relations in the planar operad $\mathcal{F}_\pi C$. The planar operad $\mathcal{F}_\pi C/R'$, where R' is the normal relation generated by R, is called the planar operad *generated* by the *generators* C and the *relations* R.

Obviously the same can be applied to symmetric operads. The only thing we need to do is modify the definition of a normal relation R' in a free symmetric operad $\mathcal{P}$ to involve the symmetric group actions as well. In detail, let C be a collection. A set of relations R in the symmetric operad $\mathcal{P} = Symm(\mathcal{F}_\pi C)$ is a set of relations in the planar operad underlying $\mathcal{P}$. R is called *normal* if it is normal in the planar sense and if given $\psi \sim \psi'$ and $\sigma \in \Sigma_n$ (n being the arity of ψ), then

$$\sigma^*\psi \sim \sigma^*\psi'.$$

Just as before, given any set of relations R, there exists a unique smallest normal set of relations R' containing R. Clearly, for a normal set of relations R, one can define the operad $\mathcal{P}/R$ just as above.

Definition 1.2.3. Let C be a collection and R a set of relations in the symmetric operad $Symm(\mathcal{F}_\pi C)$. The symmetric operad $\mathcal{F}_\pi C/R'$, where R' is the normal set of relations generated by R, is called the symmetric operad *generated* by the *generators* C and the *relations* R.

Remark 1.2.4. In the special case where in the collection C all sets not of the form $C(a;b)$ are empty, one may interpret C as defining a directed graph (in the traditional sense of the word). It is easily verified that in that case the operad $\mathcal{F}_\pi C$ has only unary operations and is thus (essentially) a category. This category is of course isomorphic to the free category on the graph given by C.

When we use this construction to describe operads, we will usually not define the set R of relations in the way given above. Rather, we will just give a list of the equations between various operations that we wish to force.

It is clear from our construction that if the (planar or symmetric) operad $\mathcal{P}$ is generated by C and R, then, given any operad $\mathcal{Q}$, a map of operads $F : \mathcal{P} \to \mathcal{Q}$ corresponds exactly to a function $f : ob(\mathcal{P}) \to ob(\mathcal{Q})$ and a family of functions

$$f : C(p_1, \cdots, p_n; p) \to \mathcal{Q}(fp_1, \cdots, fp_n; fp)$$

mapping the generators of $\mathcal{P}$ to operations in $\mathcal{Q}$ such that if $\psi \sim \psi'$ in $\mathcal{F}C$ then the compositions of the composition schemes $cs(\psi)$ and $cs(\psi')$ in $\mathcal{Q}$, which correspond to ψ and ψ', are equal.

Remark 1.2.5. One can obviously define the notion of a symmetric collection and proceed to construct the free symmetric operad on a symmetric collection. This is the more usual approach in the literature (e.g., [**6**]) yet for our purposes in this work the above (slightly simpler) construction is sufficient.

1.3. Limits and colimits in the category of operads

In this section we prove that the category *Operad* is small complete and small cocomplete. We give explicit constructions for products, coproducts, equalizers, and coequalizers which of course suffice to prove that all small limits and colimits exist (see [**34**] Theorem 2, page 113). We also obtain the easy result that the functor $ob : Operad \to Set$ that sends an operad $\mathcal{P}$ to $ob(\mathcal{P})$ preserves both limits and colimits. We wish to point out that the existence of limits and colimits of operads follow from general category theory (the category of operads is defined by a finite limit theory and thus is locally finitely presentable). Our decision to give an explicit construction is motivated by two considerations. One is to stress the analogy with category theory, since the limits and colimits of operads are constructed in essentially the same way as limits and colimits of categories. The other consideration is to emphasize the difference from the construction of limits and colimits of operads in the classical sense. The common construction of limits and colimits of operads usually consist of a diagram of operads with just one object and then calculate the (co)limit inside the category of operads with just one object, which is of course very different then the (co)limit of the same diagram inside the category of all operads.

THEOREM 1.3.1. *The category Operad is small complete.*

PROOF. It is sufficient to prove that *Operad* has equalizers and small products. Let $\mathcal{P} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{Q}$ be an equalizer diagram. We construct an operad $\mathcal{Q}$ as follows. The set of objects of $\mathcal{R}$ is the equalizer

$$ob(\mathcal{R}) \xrightarrow{e} ob(\mathcal{P}) \underset{G}{\overset{F}{\rightrightarrows}} ob(\mathcal{Q})$$

in *Set*, which we view as a subset of $ob(\mathcal{P})$. Given objects $r_1, \cdots, r_n, r_0 \in ob(\mathcal{R})$ we have that $Fr_i = Gr_i = r_i^{'}$. Let the set of operations from $(r_1, \cdots, r_n)$ to r_0 be the equalizer

$$\mathcal{R}(r_1, \cdots, r_n; r) \xrightarrow{e} \mathcal{P}(r_1, \cdots, r_n; r) \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{Q}(r_1', \cdots, r_n'; r')$$

where again we view $\mathcal{R}(r_1, \cdots, r_n; r_0)$ as a subset of $\mathcal{P}(r_1, \cdots, r_n; r_0)$. The operadic structure on $\mathcal{R}$ is induced from that of $\mathcal{P}$ in the obvious way. This makes $\mathcal{R}$ into an operad in such a way that all of the above given equalizing maps (all called) e, form together a map of operads $e : \mathcal{R} \to \mathcal{P}$. It is easily verified that this makes $\mathcal{R}$ into an equalizer of the diagram $\mathcal{P} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{Q}$. Given a small collection $\{\mathcal{P}_i\}_{i \in I}$ of operads we can similarly construct a product for this family such that

$$ob(\prod_{i \in I} \mathcal{P}_i) = \prod_{i \in I} ob(\mathcal{P}_i).$$

We omit the details. □

THEOREM 1.3.2. *The category Operad is small cocomplete.*

PROOF. Again we just need to show that *Operad* has all coequalizers and all small coproducts. Consider a coequalizer diagram

$$\mathcal{P} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{Q}$$

of operads. Let C be the collection underlying the operad $\mathcal{Q}$. The coequalizer we are looking for is then the operad generated by C and the relations

$$F\psi = G\psi$$

for any operation ψ in $\mathcal{P}$, as well as all relations describing the composition in $\mathcal{Q}$. Coproducts are constructed similarly, and we omit the rest of the details. □

COROLLARY 1.3.3. *The functor $ob : Operad \to Set$ which sends an operad $\mathcal{P}$ to the set $ob(\mathcal{P})$ preserves all small limits and all small colimits.*

PROOF. This follows either by inspection of the constructions of limits and colimits in *Operad* or from the fact that $ob : Operad \to Set$ has both a left and a right adjoint (as can easily be seen). □

1.4. Yoneda's lemma

In this section we briefly study how the Yoneda lemma extends from category theory to operad theory. To that end we introduce representable functors for operads, a construction that, from the point of view of operads as a tool to describe algebraic structures, associates with each operad $\mathcal{P}$ some canonical algebras, namely those functors that are represented by the objects of $\mathcal{P}$.

DEFINITION 1.4.1. Let $\mathcal{P}$ be an operad and $q \in ob(\mathcal{P})$. The *representable functor* $\mathcal{P}(q^*, -) : \mathcal{P} \to Set$ is the functor of operads defined as follows. For an object $p \in ob(\mathcal{P})$ we have

$$\mathcal{P}(q^*, -)(p) = \coprod_{n=0}^{\infty} \mathcal{P}(q^n; p) = \mathcal{P}(q^*, p)$$

where q^n is the tuple $(q, \cdots, q)$ with q occurring n times. Given $\psi \in \mathcal{P}(p_1, \cdots, p_m; p)$ we define the operation $\mathcal{P}(q^*, \psi) \in Set(\mathcal{P}(q^*, p_1), \cdots, \mathcal{P}(q^*, p_m); \mathcal{P}(q^*, p))$, i.e., a function

$$\begin{array}{c} \coprod \mathcal{P}(q^n; p_1) \times \cdots \times \coprod \mathcal{P}(q^n; p_m) \\ \Big\downarrow \\ \coprod \mathcal{P}(q^n; p) \end{array}$$

as follows. For $(\psi_1, \cdots, \psi_m)$ with $\psi_i \in \mathcal{P}(q^{n_i}; p_i)$ we define $\mathcal{P}(q^*, \psi)(\psi_1, \cdots, \psi_m)$ to be $\psi \circ (\psi_1, \cdots, \psi_m)$, which has input $q^{n_1 + \cdots + n_m}$ and output p and is thus an element of $\mathcal{P}(q^*, p)$.

It is trivial to prove that $\mathcal{P}(p^*, -)$ is indeed a functor.

REMARK 1.4.2. This extends the usual definition of representable functors in the theory of categories in the sense that given a category $\mathcal{C}$ and an object $C \in ob(\mathcal{C})$ the representable functor $j_!(\mathcal{C})(C^*, -)$ is naturally isomorphic to $j_!(\mathcal{C}(C, -))$. This follows since for any $D \in ob(\mathcal{C})$, by definition

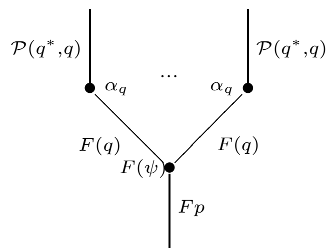
$$j_!(\mathcal{C})(C^*, D) = \coprod_{n=0}^{\infty} j_!\mathcal{C}(C^n; D)$$

however for $n \neq 1$ the set $j_!\mathcal{C}(C^n; D)$ is empty while for $n = 1$ it is exactly $\mathcal{C}(C, D)$.

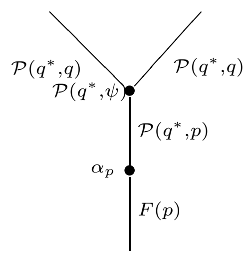
LEMMA 1.4.3. *(Yoneda for operads) Let $\mathcal{P}$ be an operad, $q \in ob(\mathcal{P})$, and $F : \mathcal{P} \to Set$ a functor. There is a natural bijection between the set of natural transformations $\alpha : \mathcal{P}(q^*, -) \to F$ and the set $F(q)$.*

PROOF. First we show that a natural transformation $\alpha : \mathcal{P}(q^*, -) \to F$ is completely determined by $\alpha_q(id_q)$, where $\alpha_q : \mathcal{P}(q^*, q) \to F(q)$ is the component of

α at q. To that end let $\psi \in \mathcal{P}(q^*, p)$, that is $\psi \in \mathcal{P}(q^k; p)$ for some $k \geq 0$. Naturality of α with respect to ψ implies that the composition of the composition scheme



is equal to that of the composition scheme



We now chase the value of $(id_q, \cdots, id_q)$ along both schemes. From the first one we obtain the value $F(\psi)(\alpha_q(id_q), \cdots, \alpha_q(id_q))$, while from the second one we obtain the value $\alpha_p(\mathcal{P}(q^*, \psi)(id_q, \cdots, id_q)) = \alpha_p(\psi \circ (id_q, \cdots, id_q)) = \alpha_p(\psi)$. Since both compositions are equal, we see that

$$\alpha_p(\psi) = F(\psi)(\alpha_q(id_q), \cdots, \alpha_q(id_q))$$

and thus that α is completely determined by $\alpha_q(id_q)$. Furthermore, a straightforward verification shows that for any fixed $a \in F(q)$, the formula

$$\alpha_p(\psi) = F(\psi)(a, \cdots, a)$$

for all p and $\psi \in \mathcal{P}(q^*, p)$, defines a natural transformation $\alpha(a) : \mathcal{P}(q^*, -) \to F$. It now follows that the assignment $\alpha \mapsto \alpha_q(id_q)$ has an inverse function, namely $a \mapsto \alpha(a)$. $\square$

Example 1.4.4. Consider the operad *Comm* from Example 1.1.13 whose algebras are commutative monoids. There is precisely one representable functor $Comm(\star^*, -) : Comm \to Set$ since *Comm* has just one object. It is easy to verify that the commutative monoid corresponding to that representable functor is the free commutative monoid on one object. The correspondence between natural transformations $Comm(\star^*, -) \to F$ and the set $F(\star)$ is precisely the universal property of free commutative monoid on one object. Representable functors for other operads usually yield some 'free' algebras as well.

We end this section by noting that a Yoneda embedding does not exist for operads. The Yoneda embedding for categories states that the assignment $C \mapsto \mathcal{C}(C, -)$ is an embedding

$$\mathcal{C}^{op} \to \underline{Cat}(\mathcal{C}, Set).$$

Such a result is not possible for operads since for an operad $\mathcal{P}$ the opposite 'operad' $\mathcal{P}^{op}$ does not exist. Instead $\mathcal{P}^{op}$ can be defined to have the structure of an 'anti-operad' where each operation has one input and possibly many (or no) outputs. However there is then no natural definition of an arrow from an anti-operad to an operad.

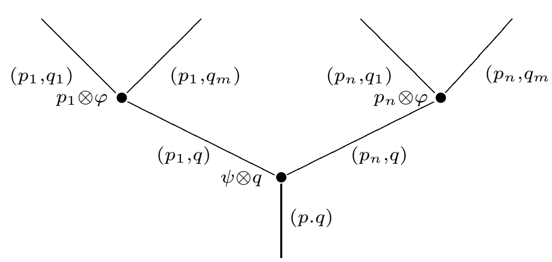
1.5. Closed monoidal structure on the category of operads

The category *Cat* is a cartesian closed category for which the internal Hom $\underline{Cat}(\mathcal{C}, \mathcal{D})$ is formed by taking functors as objects, and natural transformations as arrows. In [**7**], Boardman and Vogt define a tensor product for topological operads. In this section we show that essentially the same tensor product can be defined in the context of our notion of operads and we show that it turns the category *Operad* into a symmetric closed monoidal category in an analogous way.

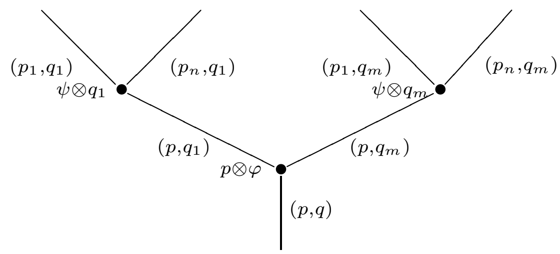
Definition 1.5.1. (The Boardman-Vogt tensor product) Let $\mathcal{P}$ and $\mathcal{Q}$ be two symmetric operads. The *Boardman-Vogt tensor product* of these operads is the symmetric operad $\mathcal{P} \otimes_{BV} \mathcal{Q}$ with $ob(\mathcal{P} \otimes_{BV} \mathcal{Q}) = ob(\mathcal{P}) \times ob(\mathcal{Q})$ given in terms of generators and relations as follows. Let C be the collection on $ob(\mathcal{P}) \times ob(\mathcal{Q})$ which contains the following generators. For each $q \in ob(\mathcal{Q})$ and each operation $\psi \in \mathcal{P}(p_1, \cdots, p_n; p)$ there is a generator $\psi \otimes_{bv} q$ in $C((p_1, q), \cdots, (p_n, q); (p, q))$ and for each $p \in ob(\mathcal{P})$ and an operation $\varphi \in \mathcal{Q}(q_1, \cdots, q_m; q)$ there is a generator $p \otimes_{bv} \varphi$ in $C((p, q_1), \cdots, (p, q_m); (p, q))$. There are five types of relations among the arrows generated by these generators:

1) $(\psi \otimes_{bv} q) \circ ((\psi_1 \otimes_{bv} q), \cdots, (\psi_n \otimes_{bv} q)) = (\psi \circ (\psi_1, \cdots, \psi_n)) \otimes_{bv} q$
2) $\sigma^*(\psi \otimes_{bv} q) = (\sigma^* \psi) \otimes_{bv} q$
3) $(p \otimes_{bv} \varphi) \circ ((p \otimes_{bv} \varphi_1), \cdots, (p \otimes_{bv} \varphi_m)) = p \otimes_{bv} (\varphi \circ (\varphi_1, \cdots, \varphi_m))$
4) $\sigma^*(p \otimes_{bv} \varphi) = p \otimes_{bv} (\sigma^* \varphi)$
5) $(\psi \otimes_{bv} q) \circ ((p_1 \otimes_{bv} \varphi), \cdots, (p_n \otimes_{bv} \varphi)) = \sigma^*_{m,n}((p \otimes_{bv} \varphi) \circ ((\psi, q_1), \cdots, (\psi, q_m)))$

By the relations above we mean every possible choice of operations for which the compositions are defined. The relations of type 1 and 2 ensure that for any $q \in ob(\mathcal{P})$, the map $p \mapsto (p, q)$ naturally extends to a map of operads $\mathcal{P} \to \mathcal{P} \otimes_{BV} \mathcal{Q}$. Similarly, the relations of type 3 and 4 guarantee that for each $p \in ob(\mathcal{P})$, the map $q \mapsto (p, q)$ naturally extends to a map of operads $\mathcal{Q} \to \mathcal{P} \otimes_{BV} \mathcal{Q}$. The relation of type 5 can be pictured as follows. The left hand side is an operation in the free operad, represented by the labelled planar tree



while the right hand side is given by the tree



before applying $\sigma^*_{m,n}$, which is the same permutation that was used in the definition of natural transformation, in order to equate the domain of the second operation to that of the first one. We call this type of relation the *interchange* relation.

THEOREM 1.5.2. *The category Operad with the Boardman-Vogt tensor product is a symmetric closed monoidal category.*

PROOF. The fact that the Boardman-Vogt tensor product makes *Operad* into a symmetric monoidal category is a straightforward verification and is omitted. We now describe the internal Hom. Let $\mathcal{Q}$ and $\mathcal{R}$ be two operads. We are going to prove that

$$\underline{Operad}(\mathcal{Q},\mathcal{R}) = Func(\mathcal{Q},\mathcal{R}),$$

that is $ob(\underline{Operad}(\mathcal{Q},\mathcal{R}))$ are all functors $\mathcal{Q}\to\mathcal{R}$ and for such functors $F_1,\cdots,F_n,G$, the operations with input $F_1,\cdots,F_n$ and output G are the natural transformations from $(F_1,\cdots,F_n)$ to G.

We need to construct a bijection

$$Operad(\mathcal{P}\otimes_{BV}\mathcal{Q},\mathcal{R}) \cong Operad(\mathcal{P},\underline{Operad}(\mathcal{Q},\mathcal{R}))$$

natural in $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$. Let $F:\mathcal{P}\otimes_{BV}\mathcal{Q}\to\mathcal{R}$ be a functor. For each $p\in ob(\mathcal{P})$ we need to construct a functor $F_p:\mathcal{Q}\to\mathcal{R}$. This functor is given on objects $q\in ob(\mathcal{Q})$ and operations φ in $\mathcal{Q}$ by

$$F_p(q) = F(p,q)$$

and

$$F_p(\varphi) = p\otimes_{bv}\varphi$$

which is obviously functorial. Actually, F_p is just the composition

$$\mathcal{Q}\longrightarrow\mathcal{P}\otimes_{BV}\mathcal{Q}\xrightarrow{F}\mathcal{R}$$

where the first functor is the one sending q to (p,q) mentioned above, right after the definition of the Boardman-Vogt tensor product. If we are now given an operation $\psi\in\mathcal{P}(p_1,\cdots,p_n;p)$, we need to construct a natural transformation

$$\alpha(\psi) = \alpha : (F_{p_1},\cdots,F_{p_n})\to F_p.$$

The component of this natural transformation at $q\in ob(\mathcal{Q})$ is the arrow

$$\alpha_q = F(\psi\otimes_{bv} q)\in\mathcal{R}(F(p_1,q),\cdots,F(p_n,q);F(p,q)) = \mathcal{R}(F_{p_1}q,\cdots,F_{p_n}q;F_pq).$$

To verify that $\alpha(p)$ is indeed a natural transformation we need to show that given an operation $\varphi\in\mathcal{Q}(q_1,\cdots,q_n;q)$ the two composition schemes from the definition of a natural transformation yield the same operation. In our case these two composition schemes are the two trees which appear in the interchange relation, with F applied to each edge and vertex. Since F is a functor, the interchange relation guarantees that $\alpha(p)$ is a natural transformation.

To go in the other direction, let $G : \mathcal{P} \to \underline{Operad}(\mathcal{Q}, \mathcal{R})$ be a functor. To construct a functor $H : \mathcal{P} \otimes_{BV} \mathcal{Q} \to \mathcal{R}$ we need to specify it on the objects and on the generators of $\mathcal{P} \otimes_{BV} \mathcal{Q}$ such that the relations are satisfied. For an object $(p, q) \in ob(\mathcal{P} \otimes_{BV} \mathcal{Q})$ let

$$H(p, q) = G(p)(q)$$

and for a generator of the form $p \otimes_{bv} \varphi$ we define

$$H(p \otimes_{bv} \varphi) = G(p)(\varphi).$$

For a generator of the form $\psi \otimes_{bv} q$ where $\psi \in \mathcal{P}(p_1, \cdots, p_n; p)$, we have the natural transformation $G(\psi) : (G(p_1), \cdots, G(p_n) \to G(p)$. We then define

$$H(\psi \otimes_{bv} q) = G(\psi)_q$$

to be the component of $G(\psi)$ at q. We omit the rest of the details. □

Remark 1.5.3. Notice that the symmetric actions in the definition of an operad again proved to be crucial for the definition of the Boardman-Vogt tensor product. This again illustrates the significant differences between the category of planar operads and symmetric operads. One can of course define a Boardman-Vogt style tensor product for planar operads too, simply by leaving out the interchange relation, and obtain a closed monoidal structure. However, the corresponding internal Hom is not well behaved as we show below.

Recall again the operad *Comm* (Example 1.1.13) whose algebras are commutative monoids. We have seen that $Func(Comm, Set)$ is isomorphic to the operad of commutative monoids with the cartesian product. We now make the following definition.

Definition 1.5.4. Let $\mathcal{P}$ and $\mathcal{E}$ be operads. We denote by $Alg(\mathcal{P}, \mathcal{E})$ the operad $\underline{Operad}(\mathcal{P}, \mathcal{E})$ and refer to it as the operad of $\mathcal{P}$ algebras in $\mathcal{E}$, or as the operad of $(\mathcal{P}, \mathcal{E})$-operads.

This is again just a shift in focus regarding the roles that the two operads play (see Remark 1.1.15). Notice that the objects of $Alg(\mathcal{P}, \mathcal{E})$ are precisely the $\mathcal{P}$-algebras in $\mathcal{E}$.

The internal Hom captures thus the notion of $\mathcal{P}$-algebras in $\mathcal{E}$, provides a notion of operations between such algebras (namely, natural transformations), and in such a way that they form themselves an operad.

Remark 1.5.5. Let us return now to discuss the differences between symmetric and non-symmetric operads. Assume that we consider $Operad_\pi$ as a closed monoidal category via the modified Boardman-Vogt tensor product (i.e., without the interchange relations). Let As_π be the planar operad that has just one object and one n-ary operation of each arity. If we now inspect the operad

$$\underline{Operad_\pi}(As_\pi, Set)$$

we easily see that the objects correspond to associative monoids and that unary arrows correspond to maps between the corresponding associative monoids. However, arrows of arity $n > 1$ fail to preserve the monoid structures, precisely because of the lack of symmetries. On the other hand, for the symmetric operad

$As = Symm(As_\pi)$, it is easy to confirm that

$$\underline{Operad}(As, Set) \cong Mon,$$

where we view Mon, the category of associative monoids, as an operad via the usual cartesian product of monoids.

In short, if we define $Alg_\pi(\mathcal{P}, \mathcal{E}) = \underline{Operad_\pi}(\mathcal{P}, \mathcal{E})$ for planar operads $\mathcal{P}$ and $\mathcal{E}$ then we have

$$j^*(Alg_\pi(\mathcal{P}, \mathcal{E})) \cong j^*(Alg(Symm(\mathcal{P}), Symm(\mathcal{E})))$$

but in general

$$Alg_\pi(\mathcal{P}, \mathcal{E}) \ncong Alg(Symm(\mathcal{P}), Symm(\mathcal{E})).$$

Thus planar operads fail to capture the correct notion of multi-maps of algebras.

We end this section by noting that given two operads $\mathcal{P}$ and $\mathcal{Q}$, one has the equality:

$$Alg(\mathcal{P}, Alg(\mathcal{Q}, \mathcal{E})) \cong Alg(\mathcal{P} \otimes_{BV} \mathcal{Q}, \mathcal{E}) \cong Alg(\mathcal{Q}, Alg(\mathcal{P}, \mathcal{E})).$$

This property can loosely be stated by saying that $\mathcal{P} \otimes_{BV} \mathcal{Q}$-algebras in $\mathcal{E}$ are the same as $\mathcal{P}$ algebras in $\mathcal{Q}$-algebras in $\mathcal{E}$ and, at the same time, the same as $\mathcal{Q}$-algebras in $\mathcal{P}$-algebras in $\mathcal{E}$.

1.6. Quillen model structure on the category of operads

In this section we introduce a Quillen model structure which is a direct generalization of the 'folk' Quillen model structure on Cat. Rezk, in an unpublished manuscript [**41**], gave a complete proof of this model structure, which we will now recall. More recently, Joyal and Tierney [**22**] establish the same model structure as a special case in the much more general context of internal categories in a topos. Again, the same model structure is established by Lack [**28**] as a special case in the context of model structures on 2-categories.

Theorem 1.6.1. *The category Cat admits a cartesian closed Quillen model structure where:*

1) The weak equivalences are the categorical equivalences.

2) The cofibrations are those functors $F : \mathcal{C} \to \mathcal{D}$ that are injective on objects.

3) The fibrations are those functors $F : \mathcal{C} \to D$ such that for any $c \in Ob(\mathcal{C})$ and each isomorphism $\psi : Fc \to d$ in $\mathcal{D}$, there exists an isomorphism $\phi : c \to c'$ for which $F\phi = \psi$.

We refer to this model structure as the folk model structure on Cat. The proof itself is not at all difficult and constitutes one of the rare examples of non-trivial, interesting Quillen model structures which are easily proved by elementary means. As stated, this model structure extends, to what we call the folk model structure, to the category $Operad$ as we now prove.

Theorem 1.6.2. *The category Operad admits a Quillen model structure where:*

1) The weak equivalences are the operadic equivalences.

2) The cofibrations are those functors $F : \mathcal{P} \to \mathcal{Q}$ that are injective on objects.

3) The fibrations are those functors $F : \mathcal{P} \to \mathcal{Q}$ such that for any $p \in Ob(\mathcal{P})$ and each isomorphism $\psi : Fp \to q$ in $\mathcal{Q}$, there exists an isomorphism $\phi : p \to p'$ for which $F\phi = \psi$.

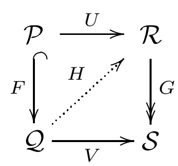
PROOF. Notice that a functor $F : \mathcal{P} \to \mathcal{Q}$ is a fibration (respectively cofibration) if, and only if, j^*F is a fibration (respectively cofibration). Notice as well that a functor $F : \mathcal{P} \to \mathcal{Q}$ is a trivial fibration if, and only if, the function $ob(F) : ob(\mathcal{P}) \to ob(\mathcal{Q})$ is surjective and F is fully faithful. We now set out to prove the Quillen axioms.

M1 (Existence of limits and colimits): As discussed above, *Operad* has all small limits and small colimits (Theorem 1.3.1 and 1.3.2).

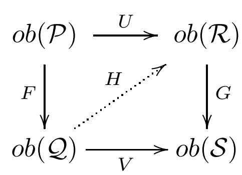
M2 (2 out of 3 property): Obviously holds.

M3 (Closed under retracts): Can easily be established.

M4 (Liftings): Consider the square



where F is a cofibration and G is a fibration. We need to prove the existence of a lift H making the diagram commute, whenever F or G is a weak equivalence. Assume first that G is a weak equivalence. Applying the object functor (that sends an operad $\mathcal{P}$ to the set $ob(\mathcal{P})$) to the lifting diagram we obtain



where F is injective and G is surjective. We can thus find a lift H. Let now $\psi \in \mathcal{Q}(q_1, \cdots, q_n; q)$, and consider $V(\psi) \in \mathcal{S}(Vq_1, \cdots, Vq_n; Vq)$. Since G is fully faithful and $HG = V$ on the level of objects, we obtain that the function

$$G : \mathcal{R}(Hq_1, \cdots, Hq_n; Hq) \to \mathcal{S}(Vq_1, \cdots, Vq_n; Vq)$$

is an isomorphism. We now define $H(\psi) = G^{-1}(V(\psi))$. It is easily checked that this (uniquely) extends H and makes it into the desired lift.

Assume now that F is a trivial cofibration. We can thus construct a functor $F' : \mathcal{Q} \to \mathcal{P}$ such that

$$F' \circ F = id_{\mathcal{P}}$$

together with a natural isomorphism $\alpha : F \circ F' \to id_{\mathcal{Q}}$. We can moreover choose α such that for each $p \in ob(\mathcal{P})$, the component at Fp is given by

$$\alpha_{Fp} = id_{Fp}.$$

To define $H : ob(\mathcal{Q}) \to ob(\mathcal{R})$ let $q \in ob(\mathcal{Q})$ and consider the object $VFF'q \in ob(\mathcal{S})$. Since

$$VFF'q = GUF'q$$

it follows from the definition of fibration that there is an object $H(q)$ and an isomorphism

$$\beta_q : UF'q \to Hq$$

in $\mathcal{R}$ such that

$$GHq = Vq$$

and

$$G\beta_q = V\alpha_q.$$

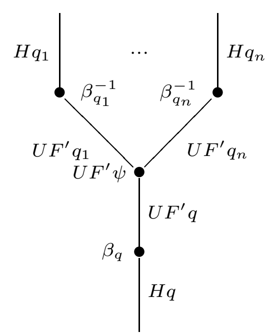
We can also choose β such that for every $p \in ob(\mathcal{P})$

$$HFp = Up$$

and

$$\beta_{Fp} = id_{Up}.$$

Let now $\psi \in \mathcal{Q}(q_1, \cdots, q_n; q)$ and define $H(\psi)$ to be the composition of the following composition scheme in $\mathcal{R}$:



The resulting H is easily seen to be a functor and the desired lift.

M5 (Factorizations): Let $F : \mathcal{P} \to \mathcal{Q}$ be a functor. We first construct a factorization of F into a trivial cofibration followed by a fibration. Construct first the following operad $\mathcal{P}'$ with

$$ob(\mathcal{P}') = \{(p, \varphi, q) \in ob(\mathcal{P}) \times \mathcal{Q}(Fp, q) \times ob(\mathcal{Q}) \mid \varphi \text{ is an isomorphism}\}$$

and, for objects $(p_1, \varphi_1, q_1), \cdots, (p_n, \varphi_n, q_n), (p, \varphi, q)$, the arrows

$$\mathcal{P}'((p_1, \varphi_1, q_1), \cdots, (p_n, \varphi_n, q_n); (p, \varphi, q)) = \mathcal{P}(p_1, \cdots, p_n; p)$$

with the obvious operadic structure. If we now define $G : \mathcal{P} \to \mathcal{P}'$ on objects $p \in ob(\mathcal{P})$ by

$$G(p) = (p, id_{Fp}, Fp)$$

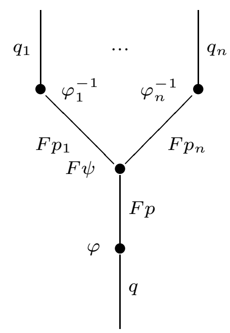
and for and arrow $\psi \in \mathcal{P}(p_1, \cdots, p_n; p)$ by

$$G(\psi) = \psi$$

we evidently get a functor, which is clearly a trivial cofibration. We now define the functor $H : \mathcal{P}' \to Q$ on objects $(p, \varphi, q) \in ob(\mathcal{P}')$ by

$$H(p, \varphi, q) = q$$

and on an arrow $\psi \in \mathcal{P}'((p_1, \varphi_1, q_1), \cdots, (p_n, \varphi_n, q_n); (p, \varphi, q))$ to be the composition of the composition scheme



Clearly, H is a fibration since if $f : H(p, \varphi, q) \to q'$ is an isomorphism in $\mathcal{Q}$ then $(p, f\varphi, q')$ is also an object of $\mathcal{Q}$ and id_p is an isomorphism in $\mathcal{P}'$ from (p, φ, q) to $(p, f\varphi, q')$ which, by definition, maps under H to $f\varphi \circ F(id_p) \circ \varphi^{-1} = f$. Since we obviously have that $F = H \circ G$ we have the desired factorization.

We now proceed to prove that F can be factored as a composition of a cofibration followed by a trivial fibration. Let $\mathcal{Q}'$ be the operad with

$$ob(\mathcal{Q}') = ob(\mathcal{P}) \coprod ob(\mathcal{Q})$$

and with arrows defined as follows. Given an object $x \in ob(\mathcal{Q}')$ let (somewhat ambiguously)

$$Fx = \begin{cases} x, & \text{if } x \in ob(\mathcal{Q}) \\ Fx, & \text{if } x \in ob(\mathcal{P}) \end{cases}$$

Now, for objects $x_1, \cdots, x_n, x \in ob(\mathcal{Q}')$ let

$$\mathcal{Q}'(x_1, \cdots, x_n; x) = \mathcal{Q}(Fx_1, \cdots, Fx_n; Fx).$$

The operad structure is the evident one. If we now define a functor $G : \mathcal{P} \to Q'$ for an object $p \in ob(\mathcal{P})$ and an arrow $\psi \in \mathcal{P}(p_1, \cdots, p_n; p)$ by

$$Gp = p$$

and

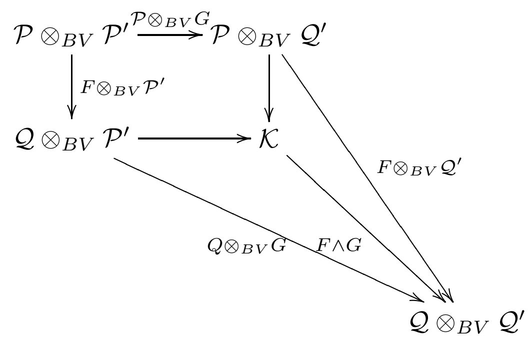
$$G\psi = F\psi$$

then we obviously obtain a cofibration. We now define $H : \mathcal{Q}' \to \mathcal{Q}$ as follows. Given an object $x \in ob(\mathcal{Q}')$, if $x \in ob(\mathcal{P})$ then we set $Hx = Fx$ and if $x \in ob(\mathcal{Q})$ then we set $Hx = x$ (thus in our slightly ambiguous notation we have that $Hx = Fx$). Given an arrow $\psi \in \mathcal{Q}'(x_1, \cdots, x_n; x)$, defining $H\psi = \psi$ makes H into a functor, clearly fully faithful. Moreover H is a fibration as can easily be seen. Since obviously $F = H \circ G$ the proof is complete. □

Note that all operads are both fibrant and cofibrant under this model structure.

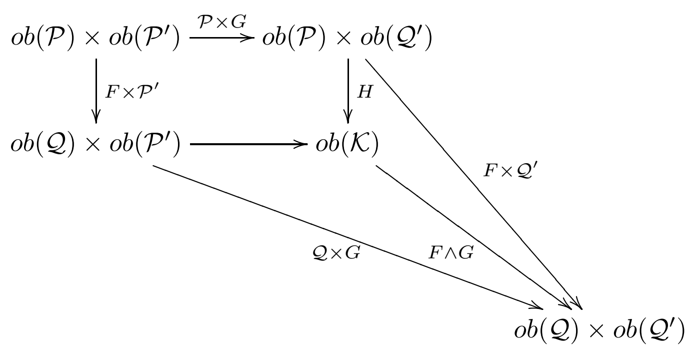
THEOREM 1.6.3. *The category Operad with the Boardman-Vogt tensor product and the model structure defined above is a monoidal model category.*

PROOF. Since all objects are cofibrant we only have to prove that given two cofibrations $F : \mathcal{P} \hookrightarrow \mathcal{Q}$ and $G : \mathcal{P}' \hookrightarrow \mathcal{Q}'$, the push-out corner map $F \wedge G$



is a cofibration which is a trivial cofibration if F is a trivial cofibration.

Since in general $ob(\mathcal{P} \otimes_{BV} \mathcal{Q}) = ob(\mathcal{P}) \times ob(\mathcal{Q})$ and since $ob : Operad \to Set$ commutes with colimits, if we apply the functor ob we obtain the following diagram



which is again a pushout. We are given that F and G are injective from which follows that $F \times \mathcal{P}'$ and $\mathcal{P} \times G$ are also injective. It is now easy to verify that $F \wedge G$ is injective as well which proves that the operad map $F \wedge G : \mathcal{K} \to \mathcal{Q} \otimes_{BV} \mathcal{Q}'$ is a cofibration.

Assume now that F in the first diagram is also a weak equivalence, i.e., an operadic equivalence. It is trivial to verify that $F \otimes_{BV} \mathcal{P}'$ is also an equivalence. Thus $F \times \mathcal{P}'$ is a trivial cofibration. Since trivial cofibrations are closed under cobase change it follows that H is a trivial cofibration. Since $F \times \mathcal{Q}'$ is too an equivalence, the two out of three property implies that $F \wedge G$ is a trivial cofibration. □

REMARK 1.6.4. Considering categories as operads, it is easily seen that in the proofs above every construction applied to categories yields again a category. For this reason these proofs can be restricted to the case of categories to give a proof of the folk model structure on *Cat*. Such a proof is essentially identical to the one given in [**41**].

LEMMA 1.6.5. *The adjunction* $Operad \underset{j_!}{\overset{j^*}{\rightleftarrows}} Cat$ *is a Quillen adjunction.*

PROOF. It is enough to prove that $j_!$ preserves cofibrations and trivial cofibrations. Actually it is trivial to verify the much stronger property that both j^* and $j_!$ preserve fibrations, cofibrations, and weak equivalences. □

We end our treatment of the model structure on *Operad* with the following:

THEOREM 1.6.6. *The operadic model structure on Operad is cofibrantly generated.*

PROOF. Let $*$ be the operad with one object and just one arrow (the identity on $*$) and let H be the free living isomorphism operad, which has two objects and, besides the necessary identities, just one isomorphism between the two objects. It is a triviality to check that a functor $F : \mathcal{P} \to \mathcal{Q}$ is a fibration if, and only if, it has the right lifting property with respect to (any one of the two possible functors) $* \to H$.

To characterize the trivial fibrations by right lifting properties we will need to consider several other operads. First of all, it is clear that if a functor $F : \mathcal{P} \to \mathcal{Q}$ has the right lifting property with respect to $\phi \to *$ then $F : ob(\mathcal{P}) \to ob(\mathcal{Q})$ is surjective (where ϕ is the initial operad with no objects). For each $n \geq 1$ consider the operad Ar_n that has $n+1$ objects $\{0, 1, \cdots, n\}$ and is generated by a single arrow from $(1, \cdots, n)$ to 0. Thus a functor $Ar_n \to \mathcal{P}$ is just a choice of an arrow in $\mathcal{P}$ of arity n. Let ∂Ar_n be the sub-operad of Ar_n that contains all the objects of Ar_n but only the identity arrows. It now easily follows that if a functor $F : \mathcal{P} \to \mathcal{Q}$ has the right lifting property with respect to the inclusion $\partial Ar_n \to Ar_n$ then for any objects $p_1, \cdots, p_n, p \in ob(\mathcal{P})$, the function

$$F : \mathcal{P}(p_1, \cdots, p_n; p) \to \mathcal{Q}(Fp_1, \cdots, Fp_n; Fp)$$

is surjective. Consider now the operad PAr_n with $n+1$ objects $\{0, 1, \cdots, n\}$ generated by two different arrows from $(1, \cdots, n)$ to 0 and the obvious map $PAr_n \to Ar_n$ which identifies those two arrows. If a functor $F : \mathcal{P} \to \mathcal{Q}$ has the right lifting property with respect to $PAr_n \to Ar_n$ then the map

$$F : \mathcal{P}(p_1, \cdots, p_n; p) \to \mathcal{Q}(Fp_1, \cdots, Fp_n; Fp)$$

is injective. Combining these results we see that if a functor $F : \mathcal{P} \to \mathcal{Q}$ has the right lifting property with respect to the set of functors

$$\{\phi \to *\} \cup \{\partial Ar_n \to Ar_n \mid n \geq 0\} \cup \{PAr_n \to Ar_n \mid n \geq 0\}$$

then F is fully faithful and $F : ob(\mathcal{P}) \to ob(\mathcal{Q})$ is surjective, which implies that F is a trivial fibration. Finally, since all the functors just mentioned are cofibrations it follows that all trivial fibrations have the right lifting property with respect to them. This then proves that the trivial fibrations are exactly those functors having the right lifting property with respect to that set. □

1.7. Grothendieck construction for operads

We now turn to the definition of a Grothendieck construction for diagrams of operads. This construction is useful if one wishes to 'glue' a suitably parametrized family of operads into one operad. We start by giving an example where such a gluing procedure is required and then proceed to the construction itself.

For a fixed set A we consider the planar operad $\mathcal{C}_{\pi A}$ whose objects are

$$ob(\mathcal{C}_{\pi A}) = A \times A$$

and for a given signature $(a_1, a_2), (a_2, a_3) \cdots, (a_{n-1}, a_n); (a_1, a_n)$ there is a single operation in

$$\mathcal{C}_{\pi A}((a_1, a_2), (a_2, a_3), \cdots, (a_{n-1}, a_n); (a_1, a_n))$$

and for every $a \in A$ there is one operation in

$$\mathcal{C}_{\pi A}(;(a,a)).$$

There are no other operations except for those just mentioned. The operadic structure is now uniquely determined. We denote

$$\mathcal{C}_A = Symm(\mathcal{C}_{\pi A})$$

the symmetrization of $\mathcal{C}_{\pi A}$.

Let $\mathcal{E}$ be a symmetric monoidal category and let us consider a functor $\mathcal{C}_A \to \mathcal{E}$ where we view $\mathcal{E}$ as a symmetric operad. By adjunction this is the same as a functor $\mathcal{B} : \mathcal{C}_{\pi A} \to \mathcal{E}$ where $\mathcal{E}$ is now considered as a planar operad. Such a functor F consists of a function $\mathcal{B} : ob(\mathcal{C}_{\pi A}) \to ob(\mathcal{E})$, that is a choice of an object $\mathcal{B}(a, a')$ for any two elements $a, a' \in A$. Further, the operations of $\mathcal{C}_{\pi A}$ are to be mapped to operations of $\mathcal{E}$, so for each $a \in A$ we have a map

$$\mathcal{C}_{\pi A}(;(a,a)) \to \mathcal{E}(;\mathcal{B}(a,a))$$

which is just a choice of an arrow

$$id_a : I \to \mathcal{B}(a,a)$$

in $\mathcal{E}$, where I is the monoidal unit. Furthermore, for any two elements $a_1, a_2 \in A$ there is a map

$$\mathcal{C}_{\pi A}((a_1,a_2),(a_2,a_3);(a_1,a_3)) \to \mathcal{E}(\mathcal{B}(a_1,a_2) \otimes \mathcal{B}(a_2,a_3), \mathcal{B}(a_1,a_3))$$

that is, a choice of an arrow

$$m : \mathcal{B}(a_1,a_2) \otimes \mathcal{B}(a_2,a_3) \to \mathcal{B}(a_1,a_3)$$

in $\mathcal{E}$. It can now be easily verified that the functoriality condition implies that the various $\mathcal{B}(a, a')$ are the Hom-objects of a category enriched in $\mathcal{E}$ whose set of objects is A, with m the composition arrow.

Consider now the operad $\underline{Operad}(\mathcal{C}_A, \mathcal{E})$. From what we just showed, the objects of this operad are $\mathcal{E}$-enriched categories whose set of objects is the set A.

PROPOSITION 1.7.1. *Let $\mathcal{C}_A$ be as above and $\mathcal{E}$ a symmetric monoidal category. Let $Cat(\mathcal{E})_A$ be the category of all $\mathcal{E}$-enriched categories whose set of objects is A, and arrows those $\mathcal{E}$-enriched functors that are the identity on objects. There is a symmetric monoidal structure on $Cat(\mathcal{E})_A$ and, when we view $Cat(\mathcal{E})_A$ as an operad, we have:*

$$\underline{Operad}(\mathcal{C}_A, \mathcal{E}) \cong Cat(\mathcal{E})_A.$$

PROOF. We first describe the monoidal structure on $Cat(\mathcal{E})_A$. Let $\mathcal{A}$ and $\mathcal{A}'$ be two $\mathcal{E}$-enriched categories with $ob(\mathcal{A}) = ob(\mathcal{A}') = A$. Let $\mathcal{A} \otimes \mathcal{A}'$ be the $\mathcal{E}$-enriched category with set of objects equal to A whose arrow objects for $a_1, a_2 \in A$ is

$$\mathcal{A} \otimes \mathcal{A}'(a_1,a_2) = \mathcal{A}(a_1,a_2) \otimes \mathcal{A}'(a_1,a_2).$$

Composition in this category is defined 'component-wise' in the obvious way. It is routine to verify that this makes $Cat(\mathcal{E})_A$ into a symmetric monoidal category.

We have already established that the objects of $\underline{Operad}(\mathcal{C}_A, \mathcal{E})$ are the objects of $Cat(\mathcal{E})_A$. Given objects $\mathcal{A}_1, \cdots, \mathcal{A}_n, \mathcal{A} \in \underline{Operad}(\mathcal{C}_A, \mathcal{E})$, if we now unfold the definition of a natural transformation $\alpha : \mathcal{A}_1, \cdots, \mathcal{A}_n \to \mathcal{A}$, we readily discover that it corresponds precisely to a functor in $Cat(\mathcal{E})_A$ from $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \to \mathcal{A}$ with the tensor product as just defined, and thus establishes the isomorphism. □

Having an operad $\mathcal{C}_A$ such that $\underline{Operad}(\mathcal{C}_A, \mathcal{E})$ is the operad of $\mathcal{E}$-enriched categories with a fixed set of objects (and special functors), it is natural to look for an operad $\mathcal{C}$ such that $\underline{Operad}(\mathcal{C}, \mathcal{E})$ will be isomorphic to the category $Cat(\mathcal{E})$ of all $\mathcal{E}$-enriched categories. However, such an operad does not exist (here is a sketch of a proof due to Tom Leinster: It suffices to prove that Cat is not monadic over Set^A for any set A. To do that one can show that the regular epimorphisms in a category monadic over Set^A are the coordinate-wise surjections, and are thus closed under composition. However, in Cat the regular epimorphisms are not closed under composition). We are thus led to look for a construction that will assemble the various operads $\underline{Operad}(\mathcal{C}_A, \mathcal{E})$ into one operad that (hopefully) will be isomorphic to $Cat(\mathcal{E})$.

DEFINITION 1.7.2. A *diagram of operads* is a functor $F : \mathbb{B}^{op} \to Operad$ where $\mathbb{B}$ is a cartesian category called the *indexing* category. For an arrow $f : B \to B'$ in $\mathbb{B}$ we will denote the functor $Ff : FB' \to FB$ by f^*.

EXAMPLE 1.7.3. Let $\mathbb{B} = Set$ with the usual cartesian product of sets. For each set $B \in ob(\mathbb{B})$ consider the operad $\mathcal{C}_{\pi B}$ described above. Any function $f : B \to B'$ induces a functor $F_\pi : \mathcal{C}_{\pi B} \to \mathcal{C}_{\pi B'}$ as follows. On the level of the objects we define

$$F_\pi : ob(\mathcal{C}_{\pi B}) \to ob(\mathcal{C}_{\pi B'})$$

to be the function

$$f \times f : B \times B \to B' \times B'.$$

On the level of operations, the functor F_π is then simply the identity

$$\begin{array}{c} \mathcal{C}_{\pi B}((b_1, b_2), \cdots, (b_{n-1}, b_n); (b_1, b_n)) \\ \Big\downarrow F_\pi \\ \mathcal{C}_{\pi B'}((fb_1, fb_2), \cdots, f(b_{n-1}, fb_n); (fb_1, fb_n)). \end{array}$$

We now define $F = Symm(F_\pi) : \mathcal{C}_B \to \mathcal{C}_{B'}$ and we obtain a functor $\mathbb{B} \to Operad$. The assignment $B \mapsto \underline{Operad}(\mathcal{C}_B, Set)$ is thus contravariant in B and therefore defines a diagram of operads $F : \mathbb{B}^{op} \to Operad$.

DEFINITION 1.7.4. (The Grothendieck construction) Let $F : \mathbb{B}^{op} \to Operad$ be a diagram of operads. We define the operad

$$\int_{\mathbb{B}} F$$

as follows. The objects of $\int_{\mathbb{B}} F$ are pairs (B, p) where $B \in ob(\mathbb{B})$ and $p \in ob(FB)$. An arrow in

$$\int_{\mathbb{B}} F((B_1, p_1), \cdots, (B_n, p_n); (B, p))$$

is a pair (f, ψ) where $f : B_1 \times \cdots \times B_n \to B$ is an arrow in $\mathbb{B}$ and ψ is an operation in $F(B_1 \times \cdots \times B_n)(\pi_1^* p_1, \cdots, \pi_n^* p_n; f^* B)$, where π_i is the canonical projection $B_1 \times \cdots \times B_n \to B_i$.

The composition in $\int_{\mathbb{B}} F$ is given as follows. If (f, ψ) is an operation from $((B_1, p_1), \cdots, (B_n, p_n))$ to (B, p) and for each $1 \leq i \leq n$ we have an operation (f_i, ψ_i) from $((B_1^i, p_1^i), \cdots, (B_{m_i}^i, p_{m_i}^i))$ to (B_i, p_i) then the composition

$$(f, \psi) \circ ((f_1, \psi_1), \cdots, (f_n, \psi_n))$$

is the pair (g, φ) given by the following compositions:

$$g = f \circ (f_1, \cdots, f_n)$$

where we consider $\mathbb{B}$ as an operad via the cartesian structure. To define φ let us denote by $in(h)$ also the object $X_1 \times \cdots \times X_n \in ob(\mathbb{B})$. Since $\mathbb{B}$ is cartesian there are canonical projections $\pi_{(i)} : in(g) \to in(f_i)$, and φ is then the composition

$$(f_1 \times \cdots \times f_n)^*(\psi)(\pi_{(1)}^*\psi_1, \cdots, \pi_{(n)}^*\psi_n)$$

in $F(in(g))$. Given an operation (f, ψ) of arity n and $\sigma \in \Sigma_n$ we define

$$\sigma^*(f, \psi) = (\sigma^* f, \sigma^* \psi)$$

where $\sigma^* f$ is interpreted in (the operad) $\mathbb{B}$. The units are the evident ones, and the fact that the axioms for an operad are satisfied is easily established.

Example 1.7.5. For the diagram $F : Set^{op} \to Operad$ given in Example 1.7.3 we obtain that $\int_{Set} F$ is isomorphic to the operad Cat with the usual cartesian structure, as was hoped for.

1.8. Enriched operads

Just as categories can be enriched in a symmetric monoidal category $\mathcal{E}$ by demanding that for any two objects A, B in the category one has an object $\mathcal{C}(A, B) \in ob(\mathcal{E})$ such that the composition and identity operations are now arrows in $\mathcal{E}$ making suitable diagrams commute (see **[26]**), so can operads be enriched in the same manner. It is possible to extend most of what was mentioned above to enriched operads, however we will only introduce here that part of the theory that is relevant for rest of this work.

Definition 1.8.1. An *$\mathcal{E}$-enriched planar operad* $\mathcal{P}$ consists of a set $ob(\mathcal{P})$, whose elements are called the *objects* of the operad, and for every signature

$$p_1, \cdots, p_n, p \in ob(\mathcal{P})$$

an object of $\mathcal{E}$

$$\mathcal{P}(p_1, \cdots, p_n; p) \in ob(\mathcal{E})$$

called the object of *arrows* from the input $(p_1, \cdots, p_n)$ to the output p. Furthermore, there are composition arrows in $\mathcal{E}$

$$\begin{array}{c} \mathcal{P}(p_1, \cdots, p_n; p) \otimes \mathcal{P}(p_1^1, \cdots, p_{m_1}^1; p_1) \otimes \cdots \otimes \mathcal{P}(p_1^n, \cdots, p_{m_n}^n; p_n) \\ \downarrow \gamma \\ \mathcal{P}(p_1^1, \cdots, p_{m_1}^1, \cdots, p_1^n, \cdots, p_{m_n}^n; p) \end{array}$$

for all possible signatures as indicated. For each object $p \in ob(\mathcal{E})$ there is also an arrow $id_p : I \to \mathcal{P}(p, p)$, where I is the monoidal unit in $\mathcal{E}$. These arrows should satisfy certain commutativity axioms that express associativity of the composition and unit laws.

An *$\mathcal{E}$-enriched (symmetric) operad* is the same data as above together with actions of the symmetric groups, which again satisfy certain diagrams expressing the equivariance of the composition with respect to these actions.

REMARK 1.8.2. In **[35]** these diagrams are explicitly given for the special case where the operad contains just one object. In **[26]** these diagrams are given in the case where all object arrows with $\mathcal{P}(p_1, \cdots, p_n; p)$ for $n \neq 1$ are empty (i.e., the initial object). The needed diagrams for our definition are then a merging of these two kinds of diagrams. For more details see **[6, 15]**.

The category $Operad(\mathcal{E})$ is the category of $\mathcal{E}$-enriched operads with the evident notion of $\mathcal{E}$-enriched functors between $\mathcal{E}$-enriched operads.

REMARK 1.8.3. It is trivial to check that for $\mathcal{E} = Set$ with the cartesian structure

$$Operad(\mathcal{E}) = Operad$$

EXAMPLE 1.8.4. Let $\mathcal{E}$ be a symmetric closed monoidal category and $M \subseteq ob(\mathcal{E})$. We then have the $\mathcal{E}$-enriched operad $\mathcal{P}_M$ given by:

$$ob(\mathcal{P}_M) = M$$

and for objects $p_1, \cdots, p_n, p \in M$

$$\mathcal{P}_M(p_1, \cdots, p_n; p) = \underline{\mathcal{E}}(p_1 \otimes \cdots \otimes p_n; p)$$

with the obvious operadic structure (compare with Example 1.1.4). When $M = ob(\mathcal{E})$ we will simply write $\hat{\mathcal{E}}$ or even just $\mathcal{E}$ for the enriched operad $\mathcal{P}_{ob(\mathcal{E})}$.

Every enriched operad $\mathcal{P}$ in $Operad(\mathcal{E})$ has an underlying operad $\mathcal{P}_0$ defined as follows. The objects of $\mathcal{P}_0$ are those of $\mathcal{P}$ and for objects $p_1, \cdots, p_n, p \in ob(\mathcal{P})$ we have

$$\mathcal{P}_0(p_1, \cdots, p_n; p) = \mathcal{E}(I, \mathcal{P}(p_1, \cdots, p_n; p)),$$

that is the set of arrow in $\mathcal{E}$ from the unit I to $\mathcal{P}(p_1, \cdots, p_n; p)$. The operad structure is the evident one. This actually defines a functor $(-)_0 : Operad(\mathcal{E}) \to Operad$ which has a left adjoint which we now describe (for the case where $\mathcal{E}$ has colimits). For a set A let $I[A]$ be the coproduct of A copies of the unit I. The functor $disc : Operad \to Operad(\mathcal{E})$ sends an operad $\mathcal{P} \in Operad$ to the enriched operad $disc(\mathcal{P})$ that has the same objects as $\mathcal{P}$ and, for objects $p_1, \cdots, p_n; p \in ob(disc(\mathcal{P}))$ has the object of operations

$$disc(\mathcal{P})(p_1, \cdots, p_n; p) = I[\mathcal{P}(p_1, \cdots, p_n; p)].$$

These constructions are direct generalizations of the corresponding construction for enriched categories (see **[26]**). The proof that $Operad \underset{(-)_0}{\overset{disc}{\rightleftarrows}} Operad_{\mathcal{E}}$ is an adjunction follows in just the same way as the analogous result for categories. An operad which is in the image of *disc* will be called a *discrete* operad.

1.9. Comparison with the usual terminology

In this section we compare our definitions with the classical notions related to operads. This is just meant to justify our definitions by showing that they agree with the classical ones. The proofs to all the claims we make are trivial and thus omitted.

DEFINITION 1.9.1. Let $\mathcal{E}$ be a symmetric closed monoidal category. A *classical operad* in $\mathcal{E}$ is an $\mathcal{E}$-enriched operad $\mathcal{P}$ such that $ob(\mathcal{P})$ is a one-point set.

Let $\mathcal{P}$ be a classical operad. Since $ob(\mathcal{P})$ is just a one point set, say $\{\star\}$, the operad is given by specifying for each $n \geq 0$ an object $\mathcal{P}(\star, \cdots, \star; \star)$ of $\mathcal{E}$ where $\star$ appears $n+1$ times. We can thus denote it simply by $\mathcal{P}(n)$. If we now rewrite the axioms for an operad in terms of $\mathcal{P}(n)$ we obtain a description of a classical operad which is identical to the definition in the literature (see e.g., [**18, 35, 36**]). More explicitly, Let $\mathcal{P}$ be a classical operad in $\mathcal{E}$. $\mathcal{P}$ is then given by a sequence $\{\mathcal{P}(n)\}_{n=0}^{\infty}$ of objects of $\mathcal{E}$ together with an arrow $I \to \mathcal{P}(1)$ (the unit of the unique object) and composition functions

$$\mathcal{P}(n) \otimes \mathcal{P}(m_1) \otimes \cdots \otimes \mathcal{P}(m_n) \to \mathcal{P}(m_1 + \cdots + m_n)$$

for all sequences of natural numbers $n, m_1, \cdots, m_n$, satisfying the appropriate unit and associativity constraints.

The notion of a map of classical operads is then defined in the obvious way and it is easily seen that it agrees with our definition. Given a classical operad $\mathcal{P}$, a $\mathcal{P}$-algebra is, by definition, a map of operads $\mathcal{P} \to End_X$ from $\mathcal{P}$ to the so called endomorphism (classical) operad. This operad is defined by

$$End_X(n) = \underline{\mathcal{E}}(X^{\otimes n}, X)$$

where the composition is given by substitution and the symmetric groups act by permuting the variables. This is actually just a special case of Example 1.8.4, namely

$$End_X = \mathcal{P}_{\{X\}}.$$

LEMMA 1.9.2. *Let $\mathcal{P}$ be a classical operad in $\mathcal{E}$. Then a $\mathcal{P}$-algebra $A : \mathcal{P} \to End_X$ corresponds to a map of enriched operads $B : \mathcal{P} \to \mathcal{E}$ such that $B(\star) = X$.*

Notice that if $\mathcal{P} = disc(\mathcal{P}')$ is a discrete operad then a $\mathcal{P}$-algebra $\mathcal{P} \to \mathcal{E}$ is the same as a $\mathcal{P}'$-algebra $\mathcal{P}' \to \mathcal{E}$. We can thus form the operad $\underline{Operad}(\mathcal{P}', \mathcal{E})$ and obtain the operad of $\mathcal{P}$-algebras. This is a slightly richer structure on the collection of $\mathcal{P}$-algebras than the usual category of $\mathcal{P}$-algebras presented in the literature, namely it forms an operad and not just a category.