

Emergent Quantum Mechanics and Emergent Symmetries*

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Abstract

Quantum mechanics is ‘emergent’ if a statistical treatment of large scale phenomena in a locally deterministic theory requires the use of quantum operators. These quantum operators may allow for symmetry transformations that are not present in the underlying deterministic system. Such theories allow for a natural explanation of the existence of gauge equivalence classes (gauge orbits), including the equivalence classes generated by general coordinate transformations. Thus, local gauge symmetries and general coordinate invariance could be emergent symmetries, and this might lead to new alleys towards understanding the flatness problem of the Universe.

July 31, 2007

*Presented at the 13th International Symposium on Particles, Strings and Cosmology, *PASCOS*, Imperial College, London, July 6, 2007

1. Starting points

The 19th century view of the physical world was that all dynamical laws of Nature were *deterministic*, which means that every single event in the Universe was assumed to be an irrevocable consequence of configurations in its past; according to Laplace the fate of every single atom at all times in the future has been sealed right from the beginning, by the boundary conditions of the Universe at $t = 0$. A theory of the world would be most satisfactory if it not only uniquely fixes the local dynamical laws, but also provides for one particular unavoidable choice of the boundary conditions, not only in the spacelike directions, but also at time $t = 0$, the moment that would now be called the “Big Bang”. This would have been the ultimate deterministic scenario. Today it is usually dismissed as being totally at odds with what has been learned from the theory of quantum mechanics. Today, it certainly seems as if the only theories we have are the ones where the outcomes of most experiments can only be predicted in terms of probabilities. The root of this situation is the occurrence of a large number of non-commuting operators, describing the values of measurable quantities.

The observation we wish to investigate further is that non-commuting operators do not necessarily imply a breakdown of determinism. One may perfectly well introduce non-commuting operators if the evolution law itself is still deterministic. For instance, in studying the planetary system, it would be totally legal to introduce an operator that interchanges the positions of Mars and Venus, or an operator that changes the tilt of the rotation axis of the Earth. Such operators would ‘violate Bell’s inequalities’ just like the ones we use in ordinary quantum mechanics[1][2]. Of course, one must ask what the circumstances may be that would necessitate such a step, in a theory that nevertheless is deterministic. If we study the evolution of planetary systems or bouncing billiard balls, non-commuting operators never appear to play any substantial role¹. A completely satisfactory answer to this question has not yet been found, but there do exist systems where one could strongly suspect that exactly this happens; these theories are *complemented* by the addition of non-commuting operators.

We expect the non-commuting operators to begin playing more essential roles in systems that have a *dynamical vacuum solution*. This is one dynamical, chaotic solution that may appear to be completely static at large distance scales, but microscopically it is not static at all. For simplicity, we refer to this microscopical scale as the ‘Planck scale’. According to our scenario, it will be fundamentally impossible to recover deterministic rules of behavior at large scales, just because things are surrounded by a chaotic vacuum already at the Planck scale (or thereabouts). So we should not be surprised at having an *effective* theory at large scales that produces statistical fluctuations in its answers to what might be expected to be observed at large scales. We suspect that the introduction of non-commuting operators may be essential to understand the statistical features at large scales. This is suspected to be what we call quantum mechanics today.

It should be emphasized that, in this view, we are not opting for a quantum mechanics

¹There is one important exception: the infinitesimal displacement operators, which are needed to define, for instance, Lyapunov exponents; see also Section 2

that would be only “approximate”. Non-commuting operators evolve quantum mechanically even in deterministic systems (think of how the Earth-Mars exchange operator would evolve in time). Quantum mechanics as a description of the evolution laws of operators is probably exactly valid in the same sense that the laws of thermodynamics follow exactly from classical (or quantum) mechanics at atomic scales. In this work, we furthermore claim that the mathematical features of quantum mechanics indeed can arise naturally, in particular the need to introduce wave functions, whose amplitudes squared are to be interpreted as probabilities. There are problems, however, and it will be important to confront these directly, rather than try to hide them. Thus, quantum mechanics as it is known today has become the theory enabling us to produce the *best possible predictions* for the future, given as much information as we can give about the system’s past, in any conceivable experimental setup. Quantum mechanics is *not* a description of the actual course of events between past and future.

Quantum mechanics will exactly reproduce the statistical features of Nature at a local scale, in our laboratories. The only effect our present considerations will have on the pursuit of an improved, accurate theory of quantum gravity and cosmology is that the universe in itself is required to be controlled by equations beyond quantum mechanics. The argument for this is simple. Quantum mechanics has been tailored by us to describe the statistical outcomes of experiments when repeated many times, locally in some laboratory. We may well assume this theory to be exact in describing local statistics. The entire Universe, however (in particular when we are talking about a closed universe), is itself an ‘experiment’ carried out only once, and all events in it are unique. The question whether a single event took place or not can only be answered by ‘yes’ or ‘no’, but there is no probabilistic answer. A theory that yields ‘maybe’ as an answer should be recognized as an inaccurate theory. If this is what we should believe, then only deterministic theories describing the entire cosmos should be accepted. There can be no ‘quantum cosmology’.

New light might be shed precisely on those problems that are connected to our attempts to describe Nature at the Planck scale, as well as at the cosmological scale. A notorious difficulty is the mystery of the cosmological constant, which appears to be fantastically small yet not zero.[3] We think that indeed we have something to say about that. This problem cannot go away as long as we hold on to quantum mechanics as a primary foundation of our theories rather than an emergent effect.

If the vacuum state indeed is a complicated dynamical solution of local deterministic equations, this implies among others that two systems can never be completely isolated from one another. They are connected by the dynamical degrees of freedom of the surrounding vacuum. This should explain why isolated systems appear to feature stochastic behavior; they simply aren’t isolated at all.

We shall furthermore argue that *symmetries* such as rotation symmetry, translation, Lorentz invariance, but also local gauge symmetries and coordinate reparametrization invariance, might all be *emergent*, together with quantum mechanics itself. They are exact locally, but they are not properties of the underlying ToE; they certainly are not a property of the boundary conditions of the Universe. We furthermore conjecture that *information* is not conserved in the deterministic description[4]. This assumption appears

to be necessary in order to obtain a chaotic yet relatively ordered vacuum state, and also was found to be desired in order to understand the positivity of the hamiltonian.

The notion of ‘free will’ is given a meaning that deviates from most standard views on the subject;[5][6][7] ours[8] appears to go back as far as Benedict de Spinoza: an individual’s actions are completely dictated by laws of Nature, yet this does not exempt us from our responsibilities for our actions. The ‘free will postulate’ is presently seen as an axiom in the reconstruction of the interpretation of quantum mechanics. This postulate will not be needed in the form usually given. We replace it by the condition that a theory or model of Nature should give appropriate responses for every conceivable initial state. So, in investigating our model, we have the ‘freedom’ to choose our initial state at will; whatever that state is, the model should tell me what will happen next. This is referred to as the ‘unconstrained initial state’ requirement.[9]

2. Mathematics: emergent symmetry on a lattice

Consider a system with deterministic laws[10][11]:

$$\frac{d}{dt}\vec{x}(t) = \vec{f}(\vec{x}) , \quad (2.1)$$

where \vec{x} represents a set of variables such as the positions and the velocities of the planets at a given time, and \vec{f} can be any ordinary vector function. It allows us to introduce quantum operators such as

$$\hat{p} = -i\frac{\partial}{\partial\vec{x}} ; \quad (2.2)$$

$$\hat{H} = \hat{p} \cdot \vec{f}(\vec{x}) + \vec{g}(\vec{x}) . \quad (2.3)$$

The latter operator is not the standard energy observable for things such as planets, but the generator of their time evolution:

$$\frac{d}{dt}\vec{x}(t) = -i[\vec{x}(t), \hat{H}] = \vec{f}(\vec{x}) . \quad (2.4)$$

Thus, we have a description of a deterministic system in a quantum mechanical language. This language allows us to identify emergent symmetries, but not yet many rigorous examples are known or understood. One interesting example of an emergent symmetry, or rather the emergent enhancement of a given symmetry, is the following.[12] Consider a model of spins on a discrete lattice in one space, one time dimension (extension of this model to more dimensions is straightforward). The sites on the lattice are indicated by the integers (x, t) , where t is a discretized time variable, but only the *even* sites are occupied by a single boolean variable $\sigma_{x,t}$. So,

$$\sigma_{x,t} = \pm 1 , \quad x + t = \text{even} . \quad (2.5)$$

The equation of motion is chosen to be

$$\sigma_{x+1,t+1} = \sigma_{x,t} \sigma_{x+2,t} \sigma_{x+1,t-1} . \quad (2.6)$$

This system has an obvious classical translation symmetry:

$$\begin{pmatrix} x \\ t \end{pmatrix} \rightarrow \begin{pmatrix} x + \delta x \\ t + \delta t \end{pmatrix} ; \quad \delta x + \delta t = \text{even} . \quad (2.7)$$

We now claim that the quantum symmetry is larger. To see this, we call $\sigma_{x,t} = \sigma_{x,t}^3$, and we introduce the quantum operators $\sigma_{x,t}^1$ on the odd sites: $x + t = \text{odd}$. On a Cauchy surface, here defined to be the lines $t = t_0$ and $t = t_0 - 1$, the operator σ_{x,t_0}^1 is defined to switch the variables $\sigma_{x,t-1}^3$, $\sigma_{x-1,t}^3$, and $\sigma_{x+1,t}^3$, leaving the others fixed. One quickly establishes how these few switches propagate in time, both to the future and to the past: the operator σ_{x,t_0}^1 switches all variables $\sigma_{y,t}^3$ for which $|y - x| < |t - t_0|$.

By inspection then, one finds that the switches produced by four σ^1 operations that are arranged in an elementary square cancel each other out completely, hence

$$\sigma_{x,t}^1 \sigma_{x-1,t+1}^1 \sigma_{x+1,t-1}^1 \sigma_{x,t+2}^1 = 1 . \quad (2.8)$$

This is exactly the same equation as the equation of motion for the σ^3 operators, eq. (2.6)! Note, that the σ^1 operators are only defined on the odd lattice sites. The transformation

$$\sigma_{x,t}^3 \leftrightarrow \sigma_{x+1,t}^1 , \quad (2.9)$$

is our first example of an emergent symmetry due to the introduction of a non-commutative operator such as σ^1 . It adds the translations over odd distances in space-time to the symmetry translations (2.7) that we already had.

3. Harmonic oscillators

One way to continue our discussion is to imagine elementary building blocks for a realistic model. A realistic model of the universe could consist of a large number of tiny segments that are mutually interacting with their neighbors. These tiny segments could be discrete, like the lattice elements of the previous section, but it is better to re-introduce a time continuum. Even if time were to be discrete in some model, one can still imagine filling up the time segments between the discrete events, even if no events take place in these continua. One now first formulates how the segments would interact if they were not coupled to their neighbors, and then adds the interaction, assuming all of these to be deterministic.

If every segment has only a finite number of different states to choose from, then inevitably a non-interacting segment would evolve in a cyclic motion: it would be periodic. So, prior to the inclusion of the interaction, our non-interacting universe will consist of many segments moving periodically. Let us first describe that motion using our new “quantum” language. Let a segment have a period T , and call the periodic variable φ , taking values in the segment $[0, 2\pi)$. Our quantum hamiltonian is taken to be²

$$H = \omega p ; \quad \omega = \frac{2\pi}{T} ; \quad p = -i \frac{\partial}{\partial \varphi} . \quad (3.1)$$

²the function $g(\varphi)$ in Eq. (2.3) is taken to be zero here.

Because of the periodicity, we must have

$$e^{-iHT} = 1 \quad \rightarrow \quad H = \frac{2\pi n}{T} = \omega n ; \quad n = 0, \pm 1, \pm 2, \dots \quad (3.2)$$

This is exactly the spectrum of a quantum harmonic oscillator, except for the emergence of negative energy states. The states with $n < 0$ are forbidden, for some reason. Of course, we also do not have the ‘vacuum energy’ $\frac{1}{2}\omega$, which would have emerged if we would have required $e^{-iHT} = -1$. This minus sign is as harmless as an overall addition of $\frac{1}{2}\omega$ to the hamiltonian, but perhaps it will mean something in a more complete theory; we will ignore it for the time being.

The disappearance of the negative energy modes is very troublesome, however. In a single oscillator, one might still say that energy is conserved, and once it is chosen to be positive, it will stay positive. However, when two or more of these systems interact, they might exchange energy, and we will have to explain why the real world appears to have interactions only in such a way that only positive energy states are occupied. This is a very difficult problem, and, disguised one way or another, it keeps popping up throughout our investigations. It still has not been solved in a completely satisfactory manner, but we can try to handle this difficulty, and one then reaches a number of quite interesting conclusions. In short, our problem is this: in a deterministic theory, one can reproduce quantum-like mathematics in a multitude of ways, but in many cases one encounters hamiltonian functions that are either periodic (in case time is taken to be discrete), or not bounded from below (when time is continuous). Can the real world nevertheless be approximated by, or rather exactly reproduced in, some deterministic model? What then causes the hamiltonian of the real world to be bounded below, with a very special lowest energy state, the ‘vacuum’, as a result? Without this positivity of H , we would not have thermodynamics. The hamiltonian is conjugated to time. Is there something about time that we are not handling correctly?

To get some ideas, consider two periodic systems, with periods $T_1 = 2\pi/\omega_1$ and $T_2 = 2\pi/\omega_2$. It would be described by the hamiltonian

$$H = \omega_1 n_1 + \omega_2 n_2 . \quad (3.3)$$

As soon as interactions are added (which can easily be imagined to be deterministic as well), transition amplitudes will be proportional to $\langle n_1, n_2 | H^{\text{int}} | m_1, m_2 \rangle$, where $n_{1,2}$ and $m_{1,2}$ take the entire range of integral values between $-\infty$ and ∞ . This would make the state $|0, 0\rangle$ unstable. States with *both* n_1 and n_2 negative would not cause a problem; we could interpret these as the bra states. It is the states $n_1 < 0, n_2 > 0$, or vice versa, which must somehow be excluded.

A very crude, but nevertheless suggestive observation is the following. Let us write the time evolution operator as $e^{-i(H_1 t_1 + H_2 t_2)}$. Since φ is proportional to time, we could actually replace t_i here by φ_i/ω_i , but the present notation is more suggestive. Then, if E_i are the eigenstates of H_i ,

$$E_1 t_1 + E_2 t_2 = \frac{1}{2} \left((E_1 + E_2)(t_1 + t_2) + (E_1 - E_2)(t_1 - t_2) \right) , \quad (3.4)$$

so that we see an ‘uncertainty relation’ not only saying that $E_i \delta t_i \approx \frac{1}{2}$, but also

$$2\delta(t_1 + t_2) \approx 1/(E_1 + E_2) , \quad 2\delta(t_1 - t_2) \approx 1/|E_1 - E_2| . \quad (3.5)$$

Since we demand only to have states with $E_1 E_2 \geq 0$, one easily derives

$$E_1 + E_2 \geq |E_1 - E_2| \quad \rightarrow \quad \delta(t_1 + t_2) \leq \delta(t_1 - t_2) , \quad (3.6)$$

The spread in relative time differences must be larger than the spread in average time! This might indicate that there is information loss: only the information about average time must play a role in the interactions, whereas information about relative time, or relative positions, is gradually lost.

4. Information loss

The prototype model with information loss is a simple automaton having four states $\{i\}$, $i = 1, \dots, 4$, with the following evolution rule (here discrete in time):

$$\{1\} \rightarrow \{2\} \rightarrow \{3\} \rightarrow \{1\} ; \quad \{4\} \rightarrow \{2\} . \quad (4.1)$$

In such a simple model, one could decide once and for all that state $\{4\}$ is superfluous; after one time step it never is reached anymore. In a complicate universe, however, it is impossibly difficult to distinguish the states, like state $\{4\}$, that can never be reached, from the states that continue to reappear in a periodic motion; what is more, the universe is too large anyway to ever reach a perfectly periodic mode. Rather than removing the superfluous states, it is technically more convenient to combine states into equivalence classes. Two states are equivalent iff within a definite amount of time these two states evolve into exactly the same state. Thus, states $\{1\}$ and state $\{4\}$ form one equivalence class, which we denote as the Dirac ket state $|1\rangle$. Thus, our model is unitary only in terms of these equivalence classes: $|1\rangle \rightarrow |2\rangle \rightarrow |3\rangle \rightarrow |1\rangle$.

In our model of two interacting periodic automata, as described in the previous section, we now assume that, *before* the interaction is switched on, information loss is to be added to the system, such that information concerning $\varphi_1 - \varphi_2$ is erased, while information concerning $\varphi_1 + \varphi_2$ is preserved³. Only then interactions can be admitted. After the interaction, one then again obtains a periodic system, so that the procedure can be repeated to describe all interactions.

Some of the details of this process have been further described in Ref[11]. One not only sees the need for information loss to occur in the deterministic system, one also sees that the information loss will be huge. There are two places in the real world where we expect this information loss to play decisive roles. One is the physics of black holes. According to the holographic principle, the amount on information in this universe should grow no faster than one bit of information per Planckian surface element, as in black holes. The surface area of the universe is huge, so there is a gigantic amount of information that is

³the coefficients of proportionality indeed have to be adapted, see ref[11].

not lost, but in Planckian units, the volume of the universe is far bigger, and all bits and bytes in the 3-dimensional bulk apparently do disappear.

The other place where one might suspect information loss to take place is in non-Abelian gauge theories. indeed, the information equivalence classes do resemble the gauge equivalence classes. In practice, they seem to play the same role: information concerning the gauge parameters of a theory does not seem to propagate in time. Rather than saying that the gauge degrees of freedom are “unphysical”, one might conclude that the gauge parameters contain information that, somewhere during the evolution of our fields, got lost: two states in a different gauge may evolve both into the same state in the same gauge. Needless to say that this is only a conjecture, but it seems to make sense.

5. Emergent symmetries

Thus, unitarity of the quantum evolution only holds in terms of the gauge equivalence classes, as is routinely taken for granted in gauge theories. Local gauge invariance thus could be interpreted as an emergent symmetry. What about other symmetries such as translation, rotation, and Lorentz invariance? There is one easy way to elevate discrete symmetries into continuous ones using quantum operators. Consider for example a discrete displacement operator U , defined on wave functions $|\{\psi(x)\}\rangle$ by

$$U|\{\psi(x)\}\rangle = |\{\psi(x-1)\}\rangle , \quad (5.1)$$

or by the commutation rule

$$xU = U(x+1) . \quad (5.2)$$

Its eigenstates are defined by

$$U|p, r\rangle = e^{ip}|p, r\rangle , \quad (5.3)$$

where $-\pi < p \leq \pi$, and r describes other quantum numbers. A fractional displacement operator can then be defined by

$$U(a) = e^{iap} , \quad (5.4)$$

where now a can be any (real or complex) number. Symmetry under discrete displacements then also implies symmetry under the fractional displacements (5.4), which is unitary for real a . The only price paid for this extension is that, for fractional a , the transformation transforms “ontological” states into quantum superpositions, even if the underlying dynamics is deterministic.

Thus, our world could be deterministic and discrete at the Planck scale, but since we only know about continuous symmetries, we are always dealing with quantum superpositions.

If the non-Abelian gauge classes are indeed information equivalence classes, an even more daring conjecture could be that the same could be true for the classes of the coordinate transformations in general relativity. Their being continuous could be an emergent

phenomenon as described above, and their gauge orbits could be the information equivalence classes. This could mean that the “ontological” theory could be defined on an ‘ordinary’ flat space-time. The nice feature of this conjecture would be that it would explain the strange preference of our Universe for flat spacial dimensions: dark matter and dark energy cancel out (in the spacelike direction).

6. A simple model

In this section, we describe a simple model from which we can derive an existence theorem:

For any quantum system there exists at least one deterministic model that reproduces all its dynamics after prequantization.

To keep the argument transparent, we consider only a finite dimensional subspace of Hilbert space. Let Schrödinger’s equation be

$$\frac{d\psi}{dt} = -iH\psi ; \quad H = \begin{pmatrix} H_{11} & \cdots & H_{1N} \\ \vdots & \ddots & \vdots \\ H_{N1} & \cdots & H_{NN} \end{pmatrix} . \quad (6.1)$$

We now claim that this system, with the same Hamiltonian, is reproduced by the following deterministic system. There are two degrees of freedom, ω and φ , of which the latter is periodic: $\varphi \in [0, 2\pi)$, or $\psi(\omega, \varphi) = \psi(\omega, \varphi + 2\pi)$. Now, take as classical equations of motion:

$$\frac{d\varphi(t)}{dt} = \omega(t) \quad ; \quad (6.2)$$

$$\frac{d\omega(t)}{dt} = -\kappa f(\omega) f'(\omega) , \quad f(\omega) = \det(H - \omega) . \quad (6.3)$$

The function $f(\omega)$ has zeros exactly at the eigenvalues of H . Its derivative, the function $f'(\omega)$, has zeros between the zeros of f , see Fig. 1.

This way, we achieve the situation that all eigenvalues ω_i , $i = 1, \dots, N$ are attractive zeros, as can be read off from the Figure. Depending on the initial configuration, one of the eigenvalues of the matrix H is rapidly approached. If $\omega_{i+\frac{1}{2}}$ is defined to be the zero of $f'(\omega)$ between ω_i and ω_{i+1} , then the regions $\omega_{i-\frac{1}{2}} < \omega < \omega_{i+\frac{1}{2}}$ are the equivalence classes associated to the final state $\omega = \omega_i$. Since the convergence towards ω_i is exponential in time, one can say that for all practical purposes the limit value ω_i is soon reached, at which point the system indeed enters into a limit cycle. At this cycle, the variable φ is periodic in time with period $T = 2\pi/\omega$. The pre-quantum states can be Fourier transformed in φ :

$$\psi(\omega, \varphi) = \sum_n e^{in\varphi} \psi_n(\omega) . \quad (6.4)$$

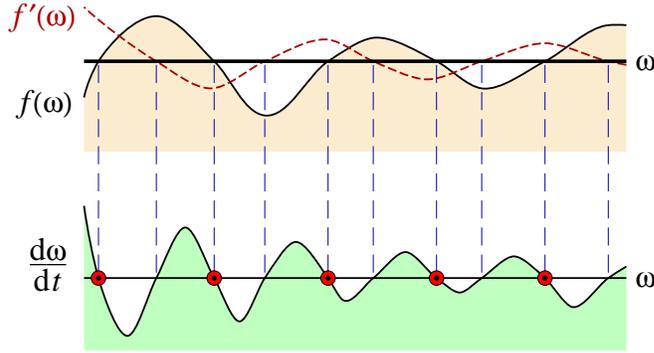


Figure 1: Above: sketches of $f(\omega)$ and of $f'(\omega)$ (dotted). Below: a sketch of the product $-\kappa f(\omega) f'(\omega)$. The red dots indicate its attractive zeros, which now coincide with the zeros of f .

At $t \rightarrow \infty$, we have

$$\psi(\omega, \varphi, t) \rightarrow \sum_n \psi_n(\omega_i, 0) e^{in(\varphi - \omega_i t)}. \quad (6.5)$$

At this point, we can argue that ω_i is the ontological energy. In this particular model, we can be even more explicit. The quantum number n is absolutely conserved, so, we can use the superselection rule that the universe settles for one particular value for n , say, $n = 1$. Then, the states $\psi_1(\omega_i)$ can be exactly identified with the energy eigenstates of the original quantum system. This completes our mathematical mapping. In this model, interference is a formality: we are always free to consider probabilistic superpositions, described by superimposing wave functions ψ_i with different i .

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