Gauge Bundles and Born-Infeld on the Noncommutative Torus

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Abstract

In this paper, we describe non-abelian gauge bundles with magnetic and electric fluxes on higher dimensional noncommutative tori. We give an explicit construction of a large class of bundles with nonzero magnetic 't Hooft fluxes. We discuss Morita equivalence between these bundles. The action of the duality is worked out in detail for the four-torus. As an application, we discuss Born-Infeld on this torus, as a description of compactified string theory. We show that the resulting theory, including the fluctuations, is manifestly invariant under the T-duality group SO(4,4;Z). The U-duality invariant BPS mass-formula is discussed shortly. We comment on a discrepancy of this result with that of a recent calculation.

1. INTRODUCTION

In the last year, noncommutative geometry, and especially the noncommutative torus, got exciting new applications in compactifications of M-theory. This started with [1], where compactification of M-theory on the noncommutative two-torus was studied. Many more discussions of M- and string theory compactifcations on these geometries followed, for example [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. The results of the last two papers are slightly different from ours.

The noncommutative torus basically is a flat compact space, where, in contrast to the 'classical' torus, the flat coordinates have nonzero commutation relations. These new compactifications were identified as string compactifications with nonzero B-flux [1, 4]. A nice property of the noncommutative torus, is the appearance of an Morita equivalence, which is a mathematical equivalence between gauge bundles on different tori. It was noted that in the connection to string theory Morita equivalence expresses T-duality [1, 7].

In the papers cited above, mainly the two-torus was considered. In this paper we describe gauge theories on the noncommutative torus in higher dimensions. A different, more abstract approach to this was also taken in [7]. There it was also found that for the d-torus, the Morita equivalences form an $\mathrm{SO}(d, d; \mathbb{Z})$ group, which is precisely the group needed for the identification as T-duality. We also give an explicit construction for a large class of gauge bundles with nonzero magnetic fluxes. Morita equivalence will come out automatically in this construction. We also include the electric variables in the discussion. For the case of the four-torus, we discuss these matters in some more detail.

String theory compactified on a four-torus, can contain several wrapped D-branes. The dynamics of D0-branes on the four-torus is known to be described by a gauge theory on the dual four-torus [16, 17]. The resulting four-dimensional gauge theory can be viewed as the world-volume theory of the D4-brane. As is well-known, the world-volume theory of a D-brane is given by a Born-Infeld type theory [18]. Therefore at low energies, the system of D0-branes should be described by this Born-Infeld theory. The D4-brane system may contain bound states of wrapped D2-branes, D0-branes, and fundamental strings, as well as momentum. These different charges manifest themselves in the gauge theory as the different fluxes. So the different charge sectors of the Type IIA string theory are identified with different topological sectors in the gauge theory. The charges and the corresponding fluxes are presented in table 1.

In [1], it was argued that in the presence of a constant background B-field, the dual torus on which the gauge theory lives should be replaced by a noncommutative torus. The noncommutative parameter θ should be identified with this constant B-field. On the other hand, the moduli on the T-dual D4-brane side, are modified as they should be given by the T-dual expression. Especially, the metric is different from the metric on dual momentum torus, which is just the inverse metric. So it seems that the theory of the D0-branes is different from the effective world-volume theory on the D4-brane. The two descriptions should however be closely related. So we still expect that the system should be described by a gauge theory of Born-Infeld type.

This paper is a generalization to higher dimensional noncommutative tori of an earlier

charge	D0-brane	D4-brane	Yang-Mills
N	D0	D4	rank
M_{ij}	D2	D2	$\int \mathrm{Tr}\, F_{ij}$
k	D4	D0	$\int \mathrm{Tr}\, \frac{1}{2} F \wedge F$
n^i	momentum	winding	$\int \mathrm{Tr}\, E^i$
m_i	winding	momentum	$\int \mathrm{Tr}\, P_i$

TABLE 1: The different charges from the point of view of the D0-branes, the T-dual D4-branes, and the interpretation as fluxes in the gauge theory.

paper [15]. But the main focus of this paper will be the study of general gauge bundles on the noncommutative tori in higher dimensions.

In section 2, we give a detailed description of gauge bundles on the noncommutative torus. We give explicit representations of the gauge field on the noncommutative d-torus for situations with magnetic and electric fluxes. We also discuss flux quantization on the noncommutative torus. We go into some depth concerning the electric flux quantization, because of the confusion in the recent literature. The action of duality transformations on the fluxes is discussed in more detail in four dimensions.

In section 3, the Born-Infeld theory living on the noncommutative four-torus will be introduced. The Hamiltonian of this theory will be fixed by requiring to have the same dependence on the global zero-modes, as this was already correctly described in the naive generalization. We discuss a *manifest* invariance of the Hamiltonian under the full T-duality group $\mathrm{SO}(4,4;\mathbb{Z})$.

Section 4 concludes with some discussion on what we found.

2.   THE NONCOMMUTATIVE TORUS

In this section we describe the noncommutative torus in some detail. We are mainly interested in an explicit description of gauge bundles. As we will have in mind the gauge theory on the noncommutative torus, the description will be in terms of the gauge connection, and not the sections of the bundles. We start with some generalities on noncommutative tori and bundles on them. Then we describe in detail the abelian and non-abelian gauge bundles.

NONCOMMUTATIVE GEOMETRY AND GAUGE BUNDLES

It is quite often convenient to describe a geometry in terms of the (complex) functions on the manifold. As we can multiply and sum up these functions, they form an algebra. In the case of ordinary geometry, this algebra is of course always commutative. Usually one also wants to define an (Hermitian) inner product on the space of functions. The

algebra then becomes what is called a C^*-algebra. It turns out that there is a one-to-one correspondence between commutative C^*-algebras and classical geometries.

In noncommutative geometry, one starts from a C^*-algebra, but the demand that it should be a commutative algebra is dropped. This turns out to give a quite general framework to study geometries which do not have a simple local description, such as fractal geometries.

Really the simplest example of a noncommutative geometry is the noncommutative torus. The classical torus is described in terms of the set of single valued functions. These are of course generated by the Fourier modes $U_i = \mathrm{e}^{2\pi i x_i}$. To define the noncommutative torus in d dimensions, we start from the functions, which again are taken to be generated by Fourier modes. We will also denote these modes by U_i. Obviously, the algebra of functions is generated by these d operators. Although in the classical case these modes obviously commute, for the noncommutative torus we assume that they do not commute, but have commutation relations of the form

$$U_j U_k = \mathrm{e}^{2\pi i \theta^{jk}} U_k U_j, \qquad \text{or} \qquad [x^j, x^k] = \frac{\theta^{jk}}{2\pi i}, \tag{1}$$

where θ_{ij} is a constant anti-symmetric tensor. It is called the noncommutative parameter, as it characterizes the noncommutativeness of the torus. The noncommutative torus defined by this relation is denoted $\mathbb{T}^d_\theta$. Note that the above commutation relation can also be expressed in terms of a commutation relation between the local coordinates x_i as $[x_j, x_k] = 2\pi i \theta^{jk}$, which is essentially the commutation relation on a quantum phase space. Therefore the noncommutative torus is often referred to as the quantum torus. We will not use this name here, as we want to see the noncommutative geometry really as a classical geometry.

Except for the functions on the torus, we will also need a set of derivations. These will effectively replace the ordinary derivative operators on the torus, and can be considered as such. They are defined through the commutation relation with the algebra of functions. It is of obviously enough to give the commutation relations with the coordinates x^k

$$[\partial_j, x^k] = \delta^k_j. \tag{2}$$

Note that this is taken over without change from the classical situation. Also, remark that these derivations commute among themselves. This property will be very convenient, and allows one to consistently define differential forms on the noncommutative torus [19, 7].

Another important ingredient for us is the definition of bundles on the torus. Also these bundles are in noncommutative geometry defined in terms of the sections. This generalizes the trivial (U(1)) bundle, which has the functions as sections, and is therefore already defined above. An important property of these sections is that they can be multiplied (on the right) by a function. This makes them into (right-) modules of the C^*-algebra of functions. This characterization is the defining property of bundles on noncommutative geometry. More important for us is some extra structure on these bundles, namely the gauge connection. A connection on a bundle in noncommutative geometry is defined

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through the same requirement as in ordinary geometry, namely it has to be a derivation. This can be stated as

$$\nabla_j(\psi f) = (\nabla_j \psi)f + \psi \partial_j f, \tag{3}$$

where ψ is a section of the bundle and f is any function on the noncommutative torus.

Like for bundles on ordinary geometry, any connection can be written in terms of the derivations we defined above and a gauge field as $\nabla_j = \partial_j + i\mathcal{A}_j$. Now the derivation condition above states that the gauge field $\mathcal{A}$ should commute with all functions on the torus. Usually, this just says that it is a local function on the torus. In the noncommutative case, this is more subtle as we know that the functions do not commute among themselves. Therefore we first have to find a set of modified coordinates which commute with the coordinates x^i. It is easily seen that they are given by

$$\tilde{x}^i = x^i + \frac{i\theta^{ij}}{2\pi}\partial_j. \tag{4}$$

The derivative corrections are chosen such that the translations which the functions x_i generate are cancelled. Analogously to the function modes U_i, we introduce modes $\widetilde{U}_i = e^{2\pi i \tilde{x}^i}$. The gauge field should thus be functions of the $\tilde{x}^i$. In general, they will take of course values in the Lie-algebra of the gauge group.

Note that the modes $\tilde{x}^i$ satisfy commutation relations that are similar to that of the coordinates x^i, but with opposite noncommutative parameter

$$[\tilde{x}^i, \tilde{x}^j] = -\frac{\theta^{ij}}{2\pi i}. \tag{5}$$

The modes $\widetilde{U}_i$ therefore generate a C^*-algebra with parameter $-\theta^{ij}$.

An essential notion for gauge bundles are local gauge transformations. These are given by gauge group-valued functions which act only fiber-wise. In noncommutative geometry, these should therefore be functions also of the $\tilde{x}^i$. This is also necessary for the gauge field to remain a function of the $\tilde{x}^i$ after a gauge transformation.

To define a definite gauge bundle via its sections over the torus, we have to give periodicity conditions. This means that a translation, generated by $e^{2\pi i \partial_i}$, acting on a section is equal to a gauge transformation $\Omega(\widetilde{U})$. We can restate this by introducing a set of modified translation operators for the sections

$$T_i = e^{\partial_i} \Omega_i(\tilde{x})^{-1}. \tag{6}$$

The allowed sections of the bundle are then simply the invariants of these operators, while the gauge connection should be invariant under conjugation by these operators. It turns out that all modules of the algebra of functions, and hence all bundles on the noncommutative torus, can be characterized in this way. The translation operators T_i should commute among themselves [1]; this is equivalent with the cocycle condition on the Ω_i.

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[1]In general this is only true if there is a fundamental bundle, otherwise they may commute up to elements of the center of the gauge group. In this paper, we assume that a fundamental bundle does exist. This is because we have in mind a relation to D-branes in string theory, and in these systems the endpoints of fundamental strings are fundamentals.

For example, for the trivial abelian gauge bundle, so all magnetic and higher fluxes are zero, the Ω_i are all trivial. It is easily seen that the sections are just the functions, generated by the U_i. A simple connection on this bundle can also be given just by the derivations

$$\nabla_i^0 = \partial_i, \tag{7}$$

which obviously commutes with the translation operators T_j. The gauge connection should be a function of the $\widetilde{U}_j = e^{2\pi i \tilde{x}^i}$, which commutes with the translation operators. It is then easily seen that it can be given as a power series in these modes

$$\mathcal{A}_i = \sum_{p \in \mathbb{Z}^d} \mathcal{A}_{i,p} \, e^{2\pi i p_j \tilde{x}^j} \,. \tag{8}$$

The gauge field should also satisfy a reality condition, which translates because of the choice above to $\mathcal{A}_{j,p}^\dagger = \mathcal{A}_{j,-p}$.

Trivial U(N) bundles are also easily constructed like this, by simply letting $\mathcal{A}_j$ take values in the corresponding Lie-algebra.

NON-ABELIAN GAUGE BUNDLES

We shall now turn to the description of non-abelian gauge bundles on the noncommutative torus. We also describe bundles with nontrivial fluxes. We assume here that the torus has even dimension $d = 2g$. We study gauge bundle with gauge groups U(N) for arbitrary N, and magnetic fluxes M. There will be some restrictions on these fluxes. We do not know if these restrictions we will meet must be imposed in general, or if there are situations possible with more general fluxes. All the higher fluxes, such as the instanton number, will be, in the situations wer are able to describe, completely determined by the data N and M. To be precise, the 'integral Chern characters' [2] $\mathrm{ch}_k(F)$ for $k \geq 1$ are given by

$$\mathrm{ch}_k(F) = \frac{1}{k!} \mathrm{Tr}\, F^k = \frac{M^k}{k! N^{k-1}}. \tag{9}$$

To define a particular U(N) bundle with the above magnetic fluxes M can be defined through the translation operators T_i, defining the periodic boundary conditions in the different directions. Bundles with magnetic fluxes can be obtained by including SU(N) rotations in these boundary conditions. These SU(N) matrices, which we denote V_i, have to satisfy the commutation relations

$$V_i V_j = e^{2\pi i M_{ij}/N} \, V_j V_i, \tag{10}$$

where M_{ij} is the integral matrix of fluxes. These matrices can only be constructed in certain cases, that is when N is big enough to support them. An explicit construction of

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[2]We will see shortly that this is not the true Chern character in the noncommutative case, but it is closely related. In fact, what we mean here is the quantity we refer to as $\mu(F)$.

these matrices was worked out in [22], and is discussed in Appendix A. There also these restrictions are discussed. Using these matrices, appropriate boundary conditions can be defined through the translation operators

$$T_i = e^{\partial_i} \, V_i \, e^{2\pi i \alpha_{ij} \tilde{x}^j}, \tag{11}$$

where α is an appropriate matrix of abelian fluxes. It can be found from the requirement that the T_i must commute. The solution to this constraint is not unique, but different solutions are gauge equivalent.

We now turn to the explicit form the gauge field must take. The connection should satisfy the correct commutation relations with the Fourier modes U_i, as given in (3), and it should commute with the translation operators T_i. Therefore we first construct the algebra of operators which commute with both the T_i and the U_i. These then will be the modes from which the gauge field is constructed. This means that $\mathcal{A}$ will be given by an expansion in these operators. They are the non-abelian generalization of the abelian operators $\widetilde{U}_i$.

To write down the modes of the gauge field in the adjoint, we first introduce an integral matrix κ which is as close the the inverse of M as possible. To define it, we first introduce the matrix $n = \mathrm{diag}(n_1 \mathbb{1}, \cdots, n_g \mathbb{1})$, where $n_i = \gcd(M_i, N)$. Here we block-diagonalized M, $M = \mathrm{diag}(M_1 \epsilon, \cdots, M_g \epsilon)$, and $\mathbb{1}$ is the two-by-two unit matrix. Now κ and another integral matrix λ are chosen such that

$$M\kappa + N\lambda = n. \tag{12}$$

Note that in the basis where M and n are block-diagonal, also these two matrices are, $\kappa = \mathrm{diag}(\kappa_1 \epsilon, \cdots, \kappa_g \epsilon)$ and $\lambda = \mathrm{diag}(\lambda_1 \mathbb{1}, \cdots, \lambda_g \mathbb{1})$. Note that then all four matrices we introduced commute. We define $\nu = Nn^{-1}$ and $\mu = Mn^{-1}$, which both are integral matrices. In terms of these matrices, (12) can be written

$$\mu\kappa + \nu\lambda = \mathbb{1}. \tag{13}$$

With the matrix κ at hand we can write the modes, which we shall denote Z_i, as

$$Z_i = e^{2\pi i \beta_{ij} \tilde{x}^j} \prod_j V_j^{\kappa_{ij}}, \tag{14}$$

where the matrix β can be found from the requirement that Z_i are inert under conjugation by the T_i (this means that they are global sections of the gauge bundle). These modes satisfy commutation relations

$$Z_i Z_j = e^{-2\pi i \hat{\theta}_{ij}} \, Z_j Z_i, \qquad \text{where} \qquad \hat{\theta} = (\kappa + \lambda\theta)(\nu - \mu\theta)^{-1}. \tag{15}$$

The linear connection can easily be found from the representations of T_i and Z_i, when we use the ansatz

$$\exp(\nabla_i^0) = T_i \prod_j Z_j^{\mu_{ji}}. \tag{16}$$

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In this expression, the presence of the T_i automatically guarantees the correct derivation condition (3). The exponents of the Z_i in this expression are precisely such that the SU(N) contribution (the part involving the V_i)). Therefore, we can solve for ∇_i^0 by taking the formal logarithm.

The zero-mode of the field strength is then found from the linear connection

$$F^0 = \frac{1}{2\pi i}[\nabla^0, \nabla^0] = (N - M\theta)^{-1}MI. \tag{17}$$

Note that the precise choice for α drops out, therefore this field strength is gauge invariant, which it should be as it is an abelian connection. The complete connection can now be written as $\nabla_i = \nabla_i^0 + i\mathcal{A}_i(Z)$, where $\mathcal{A}_i$ can has an expansion in the non-abelian modes Z_i, analogous to (8).

CHERN CHARACTER

We now review the known expression of the Chern character on the noncommutative torus. For a more detailed discussion, see the book of Connes [19].

The Chern character of a gauge bundle on the noncommutative torus is the element of the even cohomology

$$\mathrm{ch}(F) = \mathrm{Tr}_\theta\, \mathrm{e}^F, \tag{18}$$

where Tr_θ is the natural trace on the noncommutative torus. The element above should be interpreted as an element of the cohomology.

Any gauge bundle on the noncommutative torus can be identified with an element of the K-group K_0 of the noncommutative torus. We call the corresponding element $\mu(F)$. This can be seen as an element of the integral even cohomology $H^{even}(\mathbb{T}_\theta^d, \mathbb{Z})$. On the classical torus $\mu(F)$ can be identified with the Chern character. On the noncommutative torus this is no longer true. In fact, the Chern character can be expressed in terms of $\mu(F)$ as [20, 21]

$$\mathrm{ch}(F) = \mathrm{e}^{-\theta} \lrcorner \mu(F), \tag{19}$$

where $\lrcorner$ denotes contraction. Note that θ should be considered a bivector, so that it can be contracted with forms in a coordinate invariant way. This element is not integral any more. Note however that the top chern character $\int d^d x\, \mathrm{ch}(F)$ is still integral. This is because it can be related to an index, analogous to the classical torus [19].

In two dimensions, the element $\mu(F)$ can be identified with

$$\mu(F) = N + Mdx^{12}. \tag{20}$$

Here we use a shorthand notation $dx^{12} = dx^1 \wedge dx^2$. The Chern Character can then be written

$$(N + M\theta) + Mdx^{12}. \tag{21}$$

We get immediately the normalization of the trace, $\mathrm{Tr}_\theta\, I = N + M\theta$. The magnetic flux, as it is the top Chern character in two dimensions, is still an integer.

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In four dimensions, the formulas get a bit more involved. The element $\mu(F)$ can be written

$$\mu(F) = N + \frac{1}{2}M_{ij}dx^{ij} + kdx^{1234}.$$ (22)

The Chern character of this bundle takes the form

$$\mathrm{ch}(F) = \left(N + \frac{1}{2}\theta^{ij}M_{ij} + \frac{1}{2}\theta \wedge \theta k\right) + \frac{1}{2}(M_{ij} + {}^*\theta_{ij}k)dx^{ij} + kdx^{1234}.$$ (23)

Again, the top Chern character, which is now the instanton number k, is integral. From this we can read off the zero modes of the operators I, F and $F \wedge F$. Note that for the gauge connection that we explicitly described earlier, the matrix M_{ij} in the formulas above is equals the matrix of fluxes appearing in the field strength.

QUANTIZATION OF THE ELECTRIC FLUX

We now study the periodicity of the gauge field. As the $U(N)$ gauge field takes values in a compact space, there are certain periodicity conditions. Most important is the $U(1)$ factor, which gives a periodicity $\mathcal{A}_i \to \mathcal{A}_i + 2\pi$. This periodicity is generated by the electric field operator $E^i = \delta/\delta\mathcal{A}_i$. On the commutative torus this shift is generated by a gauge transformation, with $\Omega = \exp 2\pi i x^i$. On the noncommutative torus this is not a local gauge transition function, as we saw above. The gauge transformation that is most close to this on the noncommutative torus is $\Omega = \exp(2\pi i x^i + \theta^{ij}\nabla_j)$.[3] Note that this is a function of the $\tilde{x}^i$, if we combine the derivative in ∇ with the coordinates. Also note that this gauge transformation commutes with the T_i. We comment on the necessity for this below. We find

$$\nabla_j \to e^{\theta^{ik}\nabla_k}\nabla_j e^{-\theta^{ik}\nabla_k} -2\pi i\delta_{ij}.$$ (24)

Now the shift part is generated by the electric field zero mode $\int \mathrm{Tr}\, E^i$, while the covariant derivative generates a translation, which can be interpreted as the action of the total momentum operator, $\int \mathrm{Tr}_\theta\, P_i$. The gauge transformation on the gauge field above is therefore generated on the wave function by the operator

$$\exp\left(2\pi i \int \mathrm{Tr}_\theta\left(E^i + \theta^{ik}P_k\right)\right).$$ (25)

As it is a true gauge transformation (Ω is single valued on the noncommutative torus), this operation should act trivial on the wave function. Therefore, the quantization of the electric flux is modified on the noncommutative torus to

$$\int \mathrm{Tr}_\theta\, E^i = n^i - \theta^{ij}m_j, \qquad \text{where} \qquad \int \mathrm{Tr}_\theta\, P_i = m_i.$$ (26)

Here both n^i and m_i are integers. Note that the total momentum is still quantized in the usual way, because they are related to the periodicity of the torus.

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[3]This corrects a similar expression for the gauge transformation in an earlier paper.

The conclusions we derived above for the periodicity of the gauge field is different from the one calculated in [13, 14]. As remarked in the last paper, and as we shall see shortly in more detail, this has an important consequence for the BPS spectrum of gauge theories. It is therefore useful to comment in some more detail on what is going on. First of all, it is essential to identify the global gauge transformations correctly. Technically, they should be global sections of the principal bundle corresponding to the gauge bundle at hand. In the formulation that we have been using, this just means that they should commute with the translation operators T_i. We already argued that any gauge transformation (either global or local) is a function of the $\tilde{x}^i$. Therefore we find that *global* gauge transformations are single-valued functions of the Z_i. For example the gauge transformation we have been using above satisfies this criterium, because both U_i and ∇_j commute with the T_i. The global gauge transformations that were used in the two papers mentioned above, and which let them to conclude that the periodicity was different, where generated by the Z_i themselves. As these are roughly proportional to $\exp 2\pi i x^i/(N + M\theta)$ it seems that the periodicity of the gauge field is proportional to $(N + M\theta)^{-1}$. Now how can we explain this discrepancy? The difference arises not because we use a different gauge transformation, but because the gauge-transformed gauge field should not directly be identified with the shifted gauge field. More careful inspection shows that it must be identified in general with a gauge field that is *both shifted and translated*. The fact that also a translation should be considered can be seen if we write the full gauge connection $\nabla_i^0 + i\mathcal{A}_i(Z)$. Acting with a global gauge transformation, for example Z_i, simply shifts the linear connection by $(N + M\theta)^{-1}$ (it could not do anything else). But this gauge transformation also effects the fluctuation part $\mathcal{A}(Z)$. (This is in fact true also for the commutative torus). Now the commutation relations (15) show that $\log Z_i$ acts as a derivation on functions of Z. Therefore, the gauge transformation Z_i acts as an exponentiated derivation, or a translation. To be precise, it can be shown that for the gauge bundles we have constructed above, the operator Z_i acts on functions of Z as

$$Z_i f(Z) Z_i^{-1} = \mathrm{e}^{(\kappa + \lambda\theta)^{ij}\nabla_j} f(Z) \, \mathrm{e}^{-(\kappa + \lambda\theta)^{ij}\nabla_j} . \tag{27}$$

If we now interpret the shift in the gauge field as a result of the gauge transformation Z_i, including this translational contribution, we find that it is in fact generated by the operator

$$\exp\left(2\pi i \int \mathrm{Tr}_\theta \left(\lambda(E + \theta P) + \kappa P \right) \right). \tag{28}$$

Now because λ and κ are integral matrices by definition, this is in perfect agreement with the quantization for E and P that we derived above. Note that the translation over κ is over an integral times the period of the torus. Therefore, it should automatically act trivially.

MORITA EQUIVALENCE

We saw above that the modes Z_i of the non-abelian gauge field generate an algebra (15) of the same type as the abelian modes $\widetilde{U}_i$. Therefore, we can identify the modes Z_i

with modes of an abelian gauge field. This identifies the non-abelian gauge field $\mathcal{A}$ with an abelian gauge field $\widehat{\mathcal{A}}$. The corresponding abelian gauge bundle lives on a different noncommutative torus. Comparing the commutation relations of the Z_i with those of the $\widetilde{U}_i$, we see that this should be a torus with noncommutative parameter $\hat{\theta}$. So we find that the *twisted non-abelian* gauge field $\mathcal{A}$ on the non-commutative torus $\mathbb{T}^2_\theta$ can be identified with a *trivial* gauge field $\widehat{\mathcal{A}}$ on the dual torus $\widehat{\mathbb{T}}^2_{\hat{\theta}}$. The gauge group is the unbroken part of the gauge group $\mathrm{U}(N_0)$, where N_0 is defined in Appendix A. This relation is a special kind of *Morita equivalence* [19, 7]. Note that $\hat{\theta}$ is related by an integral fractional linear transformation to the old θ. The matrix which induces this transformation is the $\mathrm{SO}(d, d; \mathbb{Z})$ matrix

$$S = \begin{pmatrix} \lambda & \kappa \\ -\mu & \nu \end{pmatrix}. \tag{29}$$

This then should be the $\mathrm{SO}(d, d; \mathbb{Z})$ duality transformation which maps the data (N, M) to $(N_0, 0)$, where $\mathrm{U}(N_0)$ is the unbroken part of the gauge group. Later in this section we shall see how $\mathrm{SO}(d, d; \mathbb{Z})$ acts explicitly on the data of the gauge bundle when $d = 4$.

The relation of the linear connection ∇^0 to the corresponding abelian connection $\widehat{\nabla}^0_i = \widehat{\partial}_i$, which lives on a bundle on the dual torus $\widehat{\mathbb{T}}^d_{\hat{\theta}}$, can easily be found from the transformation of the coordinates, which gives $\widehat{\partial} = (\beta^t)^{-1}\partial$. The result is

$$\widehat{\nabla}^0_i = (\nu - \mu\theta)_{ij}\, \nabla^0_j + 2\pi i (\mu - \nu\alpha)_{ij} x^j. \tag{30}$$

Note that the scaling factor in front of the covariant derivative is gauge invariant, that is independent of the precise choice for α.

The transformation of the total connection should be given by essentially the same formula. Therefore the gauge field $\mathcal{A}$ transforms with the matrix $\nu - \mu\theta$. The transformation of the field strength is now easily derived

$$\widehat{F} = (\nu - \mu\theta)F(\nu - \mu\theta)^t + \mu(\nu - \mu\theta)^t I. \tag{31}$$

Note that this relation is completely gauge invariant. The scaling of the field strength and the gauge field can be undone by a linear coordinate transformation. Also, the shift guarantees that the flux on the dual torus becomes zero, making the dual bundle trivial.

From (31) we can read off that under a general $\mathrm{SO}(d, d; \mathbb{Z})$ duality transformation

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SO}(d, d; \mathbb{Z}), \tag{32}$$

(the fluctuation part of) the gauge field and the field strength transform as

$$\mathcal{A} \to (C\theta + D)\mathcal{A}, \qquad F \to (C\theta + D)F(C\theta + D)^t + C(C\theta + D)^t I. \tag{33}$$

The transformation of the electric field $E^i = \delta\mathcal{L}/\delta\mathcal{A}_i$ can most easily be found from the requirement that $\mathrm{Tr}_\theta\, E^i \dot{\mathcal{A}}_i$ should be invariant, and the fact that the trace transforms according to

$$\mathrm{Tr}_\theta \to \sqrt{\det(C\theta + D)}^{-1} \mathrm{Tr}_\theta. \tag{34}$$

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Therefore the electric field has to transform as

$$E \to \sqrt{\det(C\theta + D)} \, [(C\theta + D)^t]^{-1} E. \tag{35}$$

Note that all these transformations are correct in any dimension.

The transformation rules for the field strength and electric fields can now be used to find the transformations of the integral fluxes. we saw before, that in the calculation of the fluxes, the electric flux mixes with the momentum. Indeed, from the transformations of the electric and magnetic fields, we derive the following transformation rules

$$E + \theta P \to \sqrt{\det(C\theta + D)} \Big(A(E + \theta P) + BP \Big), \qquad P \to \sqrt{\det(C\theta + D)} \Big(C(E + \theta P) + DP \Big). \tag{36}$$

These rules imply that the electric and momentum fluxes n^i and m_i combine into a vector of the Morita group $SO(d, d)$. Also the rank, magnetic flux and fluxes of higher exterior powers of the field strength get mixed as we saw before. The transformation of these fluxes is much more involved, and we will not perform this here. It turns out that they transform in a spinor of $SO(d, d)$. As a hint towards this, note they transform as the even forms, which can be identified with a bispinor, of the $SO(d)$ rotation group. Furthermore, the spinor of $SO(d, d)$ can be identified with the bispinor of the $SO(d)$ subgroup.

A nice property of the above transformation rules of the fields, is that they are independent of the Lagrangian that is used, as long as it is invariant under Morita equivalence. So, as we shall discuss later, this is correct for the Born-Infeld theory on the noncommutative torus. Also, because of the invariance of the action shown in [7], it is also correct for Yang-Mills theory. At least, if we assume the correct transformation rules for the parameters of the theory.

Morita Equivalence in Four Dimensions

In four dimensions, the Morita equivalences form an $SO(4, 4; \mathbb{Z})$ group. We saw above that the noncommutative parameter θ transforms with fractional linear transformations under this group. As noted above, the fluxes of powers of the field strength transform in the spinor $\mathbf{8_s}$ of $SO(4, 4; \mathbb{Z})$. These are the rank N the magnetic fluxes M_{ij} and the instanton number k. It is useful to represent them into a matrix, which transforms in the $\mathbf{8_v} \otimes \mathbf{8_c}$ of $SO(4, 4; \mathbb{Z})$.[4] In Appendix B, we give an explicitly representation of the spinors, which we can use to identify this matrix. The electric flux and the total momentum transforms, as noted above, in the vector $\mathbf{8_v}$ of the duality group. We find for this matrix and vector

$$\mathcal{N} = \begin{pmatrix} N\mathbb{1} & {}^*M \\ M & -k\mathbb{1} \end{pmatrix}, \qquad \mathcal{M} = \begin{pmatrix} n \\ m \end{pmatrix}. \tag{37}$$

The explicit transformation rules are given by

$$\mathcal{N} \to \pm S\mathcal{N}\widetilde{S}^{-1}, \qquad \mathcal{M} \to S\mathcal{M} \tag{38}$$

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[4]Note that this product contains a $\mathbf{8_s}$ factor.

where S and $\widetilde{S}$ are the $\mathrm{SO}(4,4;\mathbb{Z})$ transformations acting on the $\mathbf{8_v}$ and $\mathbf{8_c}$ respectively. The extra plus/minus sign is added to reflect the fact that the quantities are spinors. The sign should be chosen such that the rank N is always positive. There is an invariant

$$Nk - \frac{1}{2} M \wedge M. \tag{39}$$

The invariance follows easily, because it can be related ot the determinant of $\mathcal{N}$.

The group $\mathrm{SO}(4,4;\mathbb{Z})$ is generated by three types of elements. We shortly list these generators below, and discuss their action on the fluxes.

MAPPING CLASS GROUP. These are just the basis change transformations of the four-torus. They are embedded in $\mathrm{SO}(4,4;\mathbb{Z})$ as

$$S_A = \widetilde{S}_A = \begin{pmatrix} (A^t)^{-1} & 0 \\ 0 & A \end{pmatrix}, \tag{40}$$

where $A \in \mathrm{SL}(4,\mathbb{Z})$ is the element of the mapping class group. It is easily seen that they leave N and k invariant, and transform M_{ij} in the proper way as an antisymmetric matrix. Also the parameter θ transforms appropriately as an antisymmetric matrix.

INTEGRAL SHIFT TRANSFORMATIONS. They are generated by matrices of the form

$$S_\Theta = \begin{pmatrix} \mathbb{1} & 0 \\ \Theta & \mathbb{1} \end{pmatrix}, \qquad \widetilde{S}_\Theta = \begin{pmatrix} \mathbb{1} & -{}^*\Theta \\ 0 & \mathbb{1} \end{pmatrix}, \tag{41}$$

where Θ may be any antisymmetric matrix with integral entries. They shift the magnetic fluxes M by an amount $N\Theta$. The transformation of the instanton number k is precisely such that the combination (39) is invariant. The parameter θ of the noncommutative torus transforms according to

$$\theta^{-1} \to \theta^{-1} + \Theta. \tag{42}$$

NAHM TRANSFORMATIONS. These are generated by two-dimensional versions of the Nahm transformation. For example, the Nahm transformation in the 12-direction is generated by the matrices

$$S_{12} = \begin{pmatrix} 0 & 0 & \mathbb{1} & 0 \\ 0 & \mathbb{1} & 0 & 0 \\ \mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix}, \qquad \widetilde{S}_{12} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon \\ 0 & 0 & \epsilon & 0 \\ 0 & \epsilon & 0 & 0 \end{pmatrix}, \tag{43}$$

where $\epsilon = i\sigma^2$ and $\mathbb{1}$ is the 2-by-2 unit matrix. The fluxes transform according to

$$N \to M_{12}, \qquad k \to -M_{34}, \qquad M_{12} \to -N, \qquad M_{34} \to k, \qquad M_{1i} \leftrightarrow \pm M_{2i}, \tag{44}$$

13

where $i = 3, 4$ in the last item. Note that it correctly interchanges the rank and the 12-component of the magnetic flux. The usual Nahm transformation, involving all four directions, is generated by the product $S_{1234} = -\widetilde{S}_{1234} = S_{12}S_{34}$. It interchanges the rank N with the instanton number k, and replaces the magnetic flux M by the negative of its Hodge-dual, $-^*M$.

In the correspondence to string theory, Morita equivalence should be related to T-duality. The Nahm transformations are then related to the pure T-dualities. The careful reader may have noticed that there is no element corresponding to T-duality in a single direction. The reason for this, is that it takes us from the Type IIA strings to the Type IIB strings. Therefore, it is not an equivalence of the corresponding gauge theory, as this is only related to the Type IIA description. In fact, such a transformation is an element of $O(4, 4; \mathbb{Z})$ and not of $SO(4, 4; \mathbb{Z})$. Such a transformation interchanges magnetic fluxes with electric fluxes and momenta. This is the reason why they can not be seen so easily in this formulation.

3. BORN-INFELD ON THE NONCOMMUTATIVE TORUS

In this section, we discuss Born-Infeld theory on the noncommutative four-torus as an application of the Morita equivalence. Our main interest will therefore be the construction of a manifestly invariant Hamiltonian for the Born-Infeld theory on the noncommutative four-torus.

LAGRANGIAN AND HAMILTONIAN ON THE CLASSICAL TORUS

The correct effective description of D-branes, at least for vanishing Kalb-Ramond field, has been known for quite some time [18], and is described by a Born-Infeld action. This action for the abelian gauge theory is given by

$$S_{BI} = \frac{1}{\ell_s} \int d^4x \, dt \, \frac{-1}{\lambda_s} \sqrt{\det(-G - \mathcal{F})} + \frac{1}{2}C_1 \wedge \mathcal{F} \wedge \mathcal{F} + C_3 \wedge \mathcal{F}, \qquad (45)$$

where $\mathcal{F} = F - B$. The coupling constant in front is the ten-dimensional string coupling. Note that in Type IIA theory there are a one-form and a three-form RR field.

We can write the Hamiltonian density in a form which at first sight looks more invariant, by combining the various operators into a matrix and a vector

$$\mathcal{B} = \begin{pmatrix} I & {}^*F^{ij} \\ F_{ij} & -\frac{1}{2}F \wedge F \end{pmatrix}, \qquad \mathcal{E} = \begin{pmatrix} E^i \\ P_i \end{pmatrix}. \qquad (46)$$

These are such that their zero-modes are precisely the matrix and vector of fluxes in (37). The momentum density is given by $P_i = F_{ij}E^j$. Using these combinations, we can express

the Hamiltonian density into a form which looks already quite invariant

$$\mathcal{H} = \frac{1}{\ell_{pl}}\sqrt{\frac{1}{4\lambda_6}\operatorname{tr}\left(\mathbb{G}\mathcal{B}\widetilde{\mathbb{G}}\mathcal{B}^t\right) + \lambda_6(\mathcal{E} + \mathcal{B}\mathbb{C})^t\mathbb{G}(\mathcal{E} + \mathcal{B}\mathbb{C})}. \qquad (47)$$

Here, the moduli are encoded into the following matrices

$$\mathbb{G} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}, \qquad \mathbb{C} = \begin{pmatrix} {}^*C_3 \\ -C_1 - {}^*B\,{}^*C_3 \end{pmatrix}, \qquad (48)$$

and $\widetilde{\mathbb{G}}$ is similar to $\mathbb{G}$, with G replaced by $\widetilde{G}^{-1} = \sqrt{\det G}\,G^{-1}$, and B by ${}^*B$. In order to have that at least the zero-mode contribution is invariant under $\mathrm{SO}(4,4;\mathbb{Z})$, they should transform according to

$$\mathbb{G} \to (S^t)^{-1}\mathbb{G}S^{-1}, \qquad \widetilde{\mathbb{G}} \to \widetilde{S}\widetilde{\mathbb{G}}\widetilde{S}^t, \qquad \mathbb{C} \to \widetilde{S}\mathbb{C}. \qquad (49)$$

These can be seen to give exactly the T-duality transformations for the moduli. The coupling is the effective coupling in the six-dimensional transverse world, given by $\lambda_6 = \lambda_s/(\det G)^{1/4}$.

BPS-Spectrum

The BPS mass levels were calculated in [23], and also turn out to be given by a completely U-duality invariant expression. The result of that paper can be written in our notation in the form

$$\ell_{pl}^2 M_{BPS}^2 = \frac{1}{4\lambda_6}\operatorname{tr}\left(\mathbb{G}\mathcal{N}\widetilde{\mathbb{G}}\mathcal{N}^t\right) + \lambda_6\mathcal{M}^t\mathbb{G}\mathcal{M} \qquad (50)$$

$$+\sqrt{\mathcal{M}^t J\mathcal{N}\widetilde{\mathbb{G}}\mathcal{N}^t J\mathcal{M} + \frac{\lambda_6^2}{4}(\mathcal{M}^t J\mathcal{M})^2 + \frac{1}{\lambda_6^2}\left(\frac{1}{8}\operatorname{tr}(\mathcal{N}^t J\mathcal{N}J)\right)^2}.$$

For notational simplicity we took $\mathbb{C} = 0$. The manifest invarance under the $\mathrm{SO}(4,4;\mathbb{Z})$ T-duality group is obvious from the transformation of tghe various components.

Note that this result is certainly correct for $N = 1$ and $b = 0$, as we know that then the Born-Infeld theory is correct. For different values of N and b, this expression is the unique expression for the BPS-mass which is invariant under the full expected U-duality group $\mathrm{E}_5(\mathbb{Z}) = \mathrm{SO}(5,5;\mathbb{Z})$. Therefore, whatever theory should describe the D0-branes, we know that it must always give rise to these BPS-masses. For a review on U-duality in the context of D-branes and M-theory, see [24].

Also note that the contribution from the electric field is different from [14]. The difference is a consequence of the correct periodicity of the gauge field $\mathcal{A}$, and therefore the quantization of the electric field; as has been argued in the last section.

HAMILTONIAN ON THE NONCOMMUTATIVE TORUS

On the noncommutative torus, the operators $\mathcal{B}$ and $\mathcal{E}$ can not be directly identified with the local fields as in (46). The reason for this is that with this identification, they have different zero-modes. We should therefore replace them with different ones, with the same integral zero-modes. First let us denote the right-hand-sides of (46) on the noncommutative torus by $\mathcal{B}_\theta$ and $\mathcal{E}_\theta$. Denoting the zero modes of $\mathcal{B}_\theta$ and $\mathcal{E}_\theta$ by $\mathcal{N}_\theta$ and $\mathcal{M}_\theta$ respectively, we find that we can write

$$\int \mathrm{Tr}_\theta \, \mathcal{B}_\theta \equiv \mathcal{N}_\theta = S_\theta^t \mathcal{N} (\widetilde{S}_\theta^t)^{-1}, \qquad \int \mathrm{Tr}_\theta \, \mathcal{E}_\theta \equiv \mathcal{M}_\theta = S_\theta^t \mathcal{M}. \qquad (51)$$

where Tr_θ denotes the trace for the bundle on the noncommutative torus, and S_θ and $\widetilde{S}_\theta$ are theta-shift matrices as in (41). The same relation should then apply for the local operators. It was argued in [1] and other papers, that the noncommutative parameter θ should be identified in string theory compactifications with the T-dual B-field. After making this identification, the Hamiltonian can be written

$$\mathcal{H} = \frac{1}{\ell_{pl}} \mathrm{Tr}_\theta \sqrt{\frac{1}{4\lambda_6} \mathrm{tr}\left(\mathcal{G}\mathcal{B}_\theta \widetilde{\mathcal{G}} \mathcal{B}_\theta^t\right) + \lambda_6 (\mathcal{E}_\theta + \mathcal{B}_\theta \mathcal{C})^t \mathcal{G}(\mathcal{E}_\theta + \mathcal{B}_\theta \mathcal{C})}, \qquad (52)$$

where now the moduli are given by

$$\mathcal{G} = \begin{pmatrix} g^{-1} & 0 \\ 0 & g \end{pmatrix}, \qquad \widetilde{\mathcal{G}} = \begin{pmatrix} \tilde{g} & 0 \\ 0 & \tilde{g}^{-1} \end{pmatrix}, \qquad \mathcal{C} = \begin{pmatrix} {}^*c_3 \\ -c_1 \end{pmatrix}. \qquad (53)$$

Here g and c are the metric and RR-fields in the T-dual picture. Note that the T-dual B-field only appears as the noncommutative parameter. Note that the appearance of the metric in the Hamiltonian is such that the metric in the gauge theory is identified with g^{-1}, which is the inverse of the metric the N D0-branes see. This is exactly the metric appropriately for the dual momentum torus.

When taking the trace in the above formula for the Hamiltonian density, one has to be careful about the ordering of the operators. It was argued that for the non-abelian generalization of the Born-Infeld, as it follows from string theory, one should take the symmetric trace [25]. As we are ultimately trying to describe string theory, we shall also assume that the trace here is a symmetric trace. For the paper at hand, this will however not be very important.

MANIFEST SO$(4,4;\mathbb{Z})$ DUALITY

Now that we have constructed the theory on the noncommutative torus, we can start to analyze it. We already saw that we have the correct BPS mass levels, invariant even under the full U-duality group.

As we already noted before, the local operators $\mathcal{E}$ and $\mathcal{B}$ should transform, modulo normalization, as a spinor and vector under this duality group, because the zero-modes of

these operators do so. In order to find the correct transformation of the zero-modes, the local operators should transform according to the rule

$$\mathcal{B}' = dS\mathcal{B}\widetilde{S}^{-1}, \qquad \mathcal{E}' = dS\mathcal{E} \tag{54}$$

where $d = \operatorname{Tr}_\theta I / \operatorname{Tr}_{\theta'} I$ takes care of the difference in normalization. In noncommutative geometry, it has the interpretation of the scaling of an abstract dimension of the fiber of the bundle over the noncommutative torus (which is not necessarily integral on the noncommutative torus). To appreciate this, note that for the classical torus it equals the scaling of the rank N.

Let us see what this means for the fields on the noncommutative torus. Before we noted that problems arose because the unit operator transforms into a combination of the unit operator and the field strength. So let us try to find a transformation such that the transformation of the unit operator does not get additional contributions. It turns out that, with the transformation rule (54) and the relation of $\mathcal{E}$ and $\mathcal{B}$ to the fields on the noncommutative torus, the noncommutative parameter θ must transform with fractional linear transformations

$$\theta' = (A\theta + B)(C\theta + D)^{-1}. \tag{55}$$

Using this, we can calculate the scaling of the trace from the formula for the trace of the unit

$$d = \frac{\operatorname{Tr}_\theta I}{\operatorname{Tr}_{\theta'} I} = \sqrt{\det(C\theta + D)}. \tag{56}$$

Using the various relations, we find that in the end the objects $\mathcal{E}_\theta$ and $\mathcal{B}_\theta$ transform with the matrix

$$S_{\theta'}^t S (S_\theta^t)^{-1} = \begin{pmatrix} ((C\theta + D)^t)^{-1} & 0 \\ C & C\theta + D \end{pmatrix}, \quad \widetilde{S}_{\theta'}^t \widetilde{S} (\widetilde{S}_\theta^t)^{-1} = \begin{pmatrix} (\widetilde{A}\,{}^*\theta - \widetilde{B})^{-1} & 0 \\ \widetilde{A}^t & \widetilde{A}^t\,{}^*\theta - \widetilde{B}^t \end{pmatrix}. \tag{57}$$

Now we can extract the transformation of the field strength F and the electric field E. They are exactly the transformation that is expected from Morita equivalence for bundles on the noncommutative torus. Furthermore, it turns out that in this limit the unit operator I remains unchanged. This is a very important consistency check, because the unit operator has no fluctuations, while all the other local operators indeed have fluctuation contributions. Hence we find that the $SO(4, 4; \mathbb{Z})$ Morita equivalence is a manifest symmetry of the Born-Infeld theory.

4. Discussion and Conclusion

In this paper we gave some explicit constructions of gauge bundles on noncommutative tori. We saw that Morita equivalence is a very natural identification of sections of the adjoint bundles. We were able to construct bundles with magnetic fluxes. But for the bundles we could construct the higher topological characters, such as the instanton number, coming

were completely from the abelian part. Fully non-abelian instantons on the torus, even the classical torus, are very hard to construct. But as we have seen, the construction of fluxes on the noncommutative tori is basically the same as for the classical tori. So this problem is not essentially due to the noncommutativeness of the geometry, but simply of our lack of understanding of these true non-abelian instantons. Related to this is the extra constraint on the magnetic fluxes, as discussed in Appendix A. This is due to the fact that the contribution from the non-abelian part and the abelian part of the gauge field to the instanton number should always add up to an integer, although they are not separately restricted to integer values. However, when the non-abelian contribution vanishes this gives – at least in four dimensions – exactly the restriction on the magnetic fluxes discussed in the appendix [22]. It would inded be very nice to give a construction also for these non-abelian instantons. But as we have seen, the cases we could describe already give enough information on bundles transform under Morita equivalence.

In the last section we saw that it is possible to write down a Born-Infeld action on the noncommutative four-torus which reproduces exactly the correct BPS-spectrum expected for the D4-brane. A subtlety arises in the identification of the noncommutative parameter θ with the dual B-field b, as they transform differently under the $SO(d,d;\mathbb{Z})$ T-duality transformations. Only in the limit of small volume $g \to 0$ do their transformations match, and can we consistently identify these parameters. In [15] we argued that this is precisely the limit where we expect the Yang-Mills description to be correct. Note that although the volume itself should be zero, the scaling of the metric and the shape are still important in proving duality invariance. The identification of T-duality transformations with Morita equivalence between the bundles makes it possible to even leave the BPS-limit.In fact, noncommutative geometry gives a Born-Infeld theory which is fully invariant under these $SO(d,d;\mathbb{Z})$ transformation, precisely because it can be described by Morita equivalences. This is a definite improvement over the description using classical geometry, where this is not the case. Up till now nobody has been able to capture also the rest of the U-duality group $E_{d+1}(\mathbb{Z})$ as a manifest duality. Our BPS-spectrum up till $d = 4$ is however still invariant under this group. IT would be interesting to see if somehow also these other dualities could be captured in terms of a noncommutative description of some kind.

Appendix A.   Twist-Eating Solutions

In the description of the quasi periodicity conditions for the fluxes we needed $SU(N)$ matrices V_i satisfying commutation relations

$$V_i V_j = e^{2\pi i M_{ij}/N} V_j V_i, \qquad i,j = 1,\ldots,d, \tag{58}$$

where M_{ij} is an anti-symmetric integral matrix. We assume that $d = 2g$. The solutions to these equations are called twist-eating solutions. There is a direct relation to these commutation relations and the Heisenberg algebra. This relation is the key to an explicit construction of these solutions, as described in [22]. We will only give the result of this construction here.

First we write the fluxes in a canonical block-diagonal form

$$M = \begin{pmatrix} M_1\epsilon & & \\ & M_2\epsilon & \\ & & \ddots \end{pmatrix}, \tag{59}$$

using the $SL(d,\mathbb{Z})$ symmetry. So there are g fluxes M_i. From now on we let $i, j, \ldots$ run from 1 to g. Introduce $n_i = \gcd(M_i, N)$ and $N_i = N/n_i$. It turns out [22] that twist eating solutions can only exists if $N_1 \cdots N_g | N$. We write $N = N_1 \cdots N_g N_0$. When this restriction is satisfied, an explicit construction of the twist-eating solution can be given by the tensor product of several $SU(N_i)$ factors as

$$V_{2i-1} = \mathbb{1}_{N_1} \otimes \cdots \otimes P_{N_i} \otimes \cdots \otimes \mathbb{1}_{N_g} \otimes \mathbb{1}_{N_0}, \tag{60}$$

$$V_{2i} = \mathbb{1}_{N_1} \otimes \cdots \otimes (Q_{N_i})^{M_i/n_i} \otimes \cdots \otimes \mathbb{1}_{N_g} \otimes \mathbb{1}_{N_0}, \tag{61}$$

where P_N and Q_N are $SU(N)$ clock and shift matrices

$$Q_N = \begin{pmatrix} 1 & & & \\ & e^{\frac{2\pi i}{N}} & & \\ & & e^{\frac{4\pi i}{N}} & \\ & & & \ddots \end{pmatrix}, \qquad P_N = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ 1 & & & 0 \end{pmatrix}. \tag{62}$$

The part of $SU(N)$ commuting with all the matrices V_i are obviously the $SU(N_0)$ matrices of the form $\mathbb{1}_{N_1} \otimes \cdots \otimes \mathbb{1}_{N_g} \otimes V_0$. Hence the part of the gauge group $U(N)$ left unbroken by this configuration of fluxes is $U(N_0)$. (The $U(1)$ part, as it is the center of $U(N)$, also remains unbroken).

Appendix B.   SO(4, 4) Clifford Algebra

The algebra of $SO(4,4)$ has, just as for $SO(8)$, three inequivalent eight-dimensional representations: a vector, a spinor and a conjugate spinor. In the following we will use 8-valued indices $I, J, \ldots$ for the vector, $A, B, \ldots$ for the spinor and $\dot{A}, \dot{B}, \ldots$ for the conjugate spinor.

We shall construct Dirac matrices relative to the spinor representation of $SO(4,4)$. Using triality the Dirac matrices relative to the other two representations can be found by permuting the indices. The index structure is $\Gamma^A_{I\dot{B}}$ and $\Gamma^A_{\dot{A}J}$.

To construct the Dirac matrices Γ^A, we start from Dirac matrices for $SO(3,3)$, which will be embedded into the spinor representation. Hence for this embedding we have the decompositions $\mathbf{8_s} \to \mathbf{6} \oplus \mathbf{1} \oplus \mathbf{1}$ and $\mathbf{8_v}, \mathbf{8_c} \to \mathbf{4_s} \oplus \mathbf{4_c}$. The $SO(3,3)$ Dirac matrices will be denoted γ^a, where $a, b, \ldots = 1, \ldots, 6$ is the vector index for $SO(3,3)$. They can be represented by the following 6 real matrices

$$\gamma^a = \gamma^a_- = \epsilon \otimes \eta^a_-, \qquad \gamma^{a+3} = \gamma^a_+ = \tau_1 \otimes \eta^a_+ \qquad a = 1,2,3. \tag{63}$$

Here the $\eta^a_\pm$ form a basis of (anti-) self-dual antisymmetric 4×4 matrices $*\eta^a_\pm = \pm\eta^a_\pm$. They can be represented by

$$\begin{aligned}
\eta^a_+ : & \qquad \epsilon \otimes \tau_1, & \epsilon \otimes \tau_3, & \qquad 1 \otimes \epsilon, \\
\eta^a_- : & \qquad \tau_1 \otimes \epsilon, & \tau_3 \otimes \epsilon, & \qquad \epsilon \otimes 1.
\end{aligned} \tag{64}$$

In all the formulas above we used the matrices

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{65}$$

The chirality operator is $\gamma^7 = \gamma^1 \cdots \gamma^6 = \tau_3 \otimes \mathbb{1}$.

The $SO(4,4)$ Dirac matrices can now be represented by $\Gamma^A = \tau_1 \otimes \gamma^A$ for $A = 1, \ldots, 7$, and $\Gamma^8 = \epsilon \otimes \mathbb{1}$. Explicitly we have

$$\Gamma^A = \begin{pmatrix} 0 & \gamma^A_{I\dot{B}} \\ \gamma^A_{\dot{A}J} & 0 \end{pmatrix}, \qquad A = 1, \ldots, 7 \quad \text{and} \quad \Gamma^8 = \begin{pmatrix} 0 & \delta_{I\dot{B}} \\ -\delta_{\dot{A}J} & 0 \end{pmatrix}. \tag{66}$$

Let $\psi_A = (\psi_a, \psi_7, \psi_8)$ be a spinor in $\mathbf{8_s}$ (here a runs from 1 to 6). We can use the explicit representation of the gamma-matrices to embed this spinor into $\mathbf{8_v} \otimes \mathbf{8_c}$, via

$$\Psi_{I\dot{B}} = \psi_A \gamma^A_{I\dot{B}} = \begin{pmatrix} \psi_8 + \psi_7 & \psi_a \eta^a \\ *(\psi_a \eta^a) & \psi_8 - \psi_7 \end{pmatrix} \tag{67}$$

The generators of $SO(4,4)$ on $\mathbf{8_v} \oplus \mathbf{8_c}$ are the matrices Γ^{AB}. They can be decomposed into diagonal blocks, acting on the two irreducible factors. We find

$$\Gamma^{AB} = \mathbb{1} \otimes \gamma^{AB}, \qquad \Gamma^{8A} = \tau_3 \otimes \gamma^A, \qquad A, B = 1, \ldots, 7. \tag{68}$$

From this we find the action on the vector representation through matrices

$$\Gamma^{AB}_{IJ} = \gamma^{AB}_{IJ}. \qquad \Gamma^{8A}_{IJ} = \gamma^A_{IJ}, \qquad A, B = 1, \ldots, 7, \tag{69}$$

and on the conjugate spinor by

$$\Gamma^{AB}_{\dot{A}\dot{B}} = \gamma^{AB}_{\dot{A}\dot{B}}. \qquad \Gamma^{8A}_{\dot{A}\dot{B}} = -\gamma^A_{\dot{A}\dot{B}}, \qquad A, B = 1, \ldots, 7. \tag{70}$$

They all leave the quadratic form given by the matrix $J = \tau_1 \otimes 1 \otimes 1$ invariant. This means that $XJ + JX^t = 0$ (both in the vector, spinor and conjugate spinor representation). This implies that the generators are of the form

$$X = \begin{pmatrix} U + u \mathbb{1} & V \\ W & -U^t - u \mathbb{1} \end{pmatrix}, \qquad \widetilde{X} = \begin{pmatrix} U - u \mathbb{1} & -^*W \\ -^*V & -U^t + u \mathbb{1} \end{pmatrix}. \tag{71}$$

where V and W are anti-symmetric and U is traceless. Here X and $\widetilde{X}$ are the generators in the vector and the conjugate spinor representation respectively. The relatiion between the two can be found using the explicit relation beteween the two representations as described above.

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