THU-97/16, UVA-ITFA/97/20
hep-th/9706130

Monads, Strings, and M Theory

Christiaan Hofman[†]

Institute for Theoretical Physics
University of Utrecht
Princetonplein 5, 3508 TA Utrecht

Jae-Suk Park[††]

Institute for Theoretical Physics
University of Amsterdam
Valckenierstraat 65, 1018 XE Amsterdam

The recent developments in string theory suggest that the space-time coordinates should be generalized to non-commuting matrices. Postulating this suggestion as the fundamental geometrical principle, we formulate a candidate for covariant second quantized RNS superstrings as a topological field theory in two dimensions. Our construction is a natural non-Abelian extension of the RNS string. It also naturally leads to a model with manifest 11-dimensional covariance, which we conjecture to be a formulation of M theory. The non-commuting space-time coordinates of the strings are further generalized to non-commuting anti-symmetric tensors. The usual space-time picture and the free superstrings appear only in certain special phases of the model. We derive a simple set of algebraic equations, which determine the moduli space of our model. We test some aspects of our conjectual M theory for the case of compactification on T^2.

June, 1997

[†] e-mail: c.m.hofman@fys.ruu.nl
[††] e-mail: park@phys.uva.nl

1. Introduction

It is of no doubt that superstring theory has an underlying higher symmetry unifying those of general relativity and Yang-Mills theory [1]. Furthermore, the current understanding of a web of dualities [2][3] imply that all known superstring theories including 11-dimensional supergravity should be different weak-coupling limits of one underlying quantum theory of some Mystery kind [3][4]. Unfortunately, the precise nature of such an underlying theory still remains obscure. A strong hint pointing towards an underlying geometrical principle of superstrings has emerged from the dramatic revival of D-branes by Polchinski [5]. The description of D-branes, as originally pointed out by Witten, suggests that *the spacetime coordinates of strings should be treated as non-commuting matrices* [6]. This consideration eventually led to the program of matrix theory originated from the proposal of matrix M theory by Banks-Fischler-Shenker-Susskind [7]. More recently Motl [8], Banks-Seiberg [9], and Dijkraaf-Verlinde-Verlinde [10] developed matrix string theory by compactifying M(atrix) theory on a circle [7][11]. Those formulations may be viewed as non-perturbative second quantized Green-Schwarz strings [12] in the light-cone gauge.

The purpose of this paper is to initiate a program toward M-theory closely related with the manifestly covariant Ramond-Neveu-Schwarz (RNS) formulation of string [13]. We believe that the non-commutative nature of spacetime coordinates of strings is clearly directing us to formulate superstring theory in *a phase in which general covariance, as well as other higher symmetry, is unbroken.* The latter proposal was made by Witten almost a decade ago after introducing a new type of generally covariant quantum field theory called topological field theory (TFT) [14]. In fact, there are many similarities between the RNS string and TFT. From the spacetime viewpoint, the world-sheet super-charges transform as scalars, which property is a hall-mark of TFT. It is one of string magics that the RNS formulation of string leads to space-time supersymmetry after the GSO projection [15]. It is very natural to relate the non-commutative nature of "space-time" coordinates with the strings in the unbroken phase of higher symmetry. The purpose of this paper is to demonstrate that superstring theory can indeed be formulated starting from the above two suggestions. Furthermore, our construction will naturally lead us to an underlying model with manifest 11-dimensional covariance. Here the non-commutative "space-time coordinates" of strings will be further generalized to non-commutative anti-symmetric tensors. The usual space-time picture and the free superstrings appear in the various limits of the model after compactifications.

In Sect. 4, we start from a system of ten $N \times N$ matrix functions $X^\mu(\sigma^+, \sigma^-)$ which are functions of two parameters $\sigma^\pm$ parameterizing a cylinder $\Sigma = \mathbb{S}^1 \times \mathbb{R}$ with trivial canonical line bundle. They carry global $SO(9,1)$ vector indices $\mu = 0, \ldots, 9$. We endow our system with the natural metric

$$|\delta X|^2 = \int d\sigma^+ d\sigma^- g_{\mu\nu} \mathrm{Tr}\,(\delta X^\mu \delta X^\nu)\,,$$

where $g_{\mu\nu}$ is the Minkowski metric with signature $(9,1)$. Following the general idea of topological field theory (TFT) [14], we will construct an almost unique theory by gauge fixing the "world-sheet" and "spacetime" Poincaré symmetries. In particular, the obvious symmetry for arbitrary shifts $X^\mu \to X^\mu + \delta X^\mu$ in the "spacetime" viewpoint implies that we are dealing with a topological field theory on the "world-sheet". Now the most natural object to study in our system is the equivariant cohomology. It turns out that the most suitable tool is *the balanced equivariant cohomology* formalized by Dijkraaf and Moore [16]. This is an extremely powerful and simple tool which leads to an almost unique construction of corresponding TFT called Balanced TFT (BTFT). A typical example of a BTFT is the twisted $\mathcal{N} = 4$ super-Yang-Mills theory studied by Vafa and Witten [17]. Our equivariant cohomology can be summarized by a transformation law $Q_\pm X^\mu = i\psi^\mu_\pm$ and the following commutation relations between the two generators $Q_\pm$

$$Q_+^2 = -i\frac{\partial}{\partial\sigma^+} - i\delta_{\phi_{++}}, \qquad \{Q_+, Q_-\} = -2i\delta_{\phi_{+-}}, \qquad Q_-^2 = -i\frac{\partial}{\partial\sigma^-} - i\delta_{\phi_{--}}\,,$$

where δ_ϕ denote the $U(N)$ transformation generated by ϕ. One can regard $Q_\pm$ as the BRST-like charges for the symmetry of the arbitrary shift of X^μ which are nilpotent modulo a $U(N)$ gauge transformation and translations along $\sigma^\pm$. The "world-sheet" Lorentz invariance will be realized by global ghost number symmetry, which should be anomaly-free. We have a unique realization of the algebra and the action functional satisfying our criterion. We will claim that the resulting theory describes a covariant second quantized Ramond-Neveu-Schwarz (RNS) string in the unbroken phase.

Our model has a free string limit where the original RNS string is recovered. The equivariant cohomology generators $Q_\pm$ will be the left and right world-sheet super charges. The ghost fields $\psi^\mu_\pm$ for the shift $X^\mu \to X^\mu + \delta X^\mu$ will be the left and right moving world-sheet fermions. The direct relation of the RNS formalism rather than the space-time supersymmetric Green-Schwarz (GS) formulation is not surprising. In fact, our formulation is a natural and presumably unique generalization of the RNS superstring to incorporate the non-commutative "spacetime" coordinates of strings. We will argue that the transition between the unbroken and broken phases of general covariance should be explained by some of the standard quantum properties of RNS superstring.

Our construction will inevitably lead us to introduce an anti-symmetric tensor of rank 2. We will argue that it is required and compatible with the existence of off-diagonal parts of "space-time coordinates". Our construction will naturally lead to an underlying theory with *manifest eleven-dimensional covariance*, discussed in Sect. 5. The theory is again a stringy BTFT but with anti-symmetric tensors as "space-time coordinates" of strings. We will show that the free RNS string appear in a limit after compactifying the model on a circle. By compactifying further on a circle, we will show the emergency to two types of string limits. The S-duality of type IIB string will be manifest in our formulations. The anti-symmetric tensor "coordinates of strings" $B^{IJ}(\sigma,\tau)$ is somewhat analogous to the membrane in M theory. This motivates us to introduce a new rank 5 anti-symmetric tensor $J^{IJKLM}(\sigma,\tau)$ as the five brane in M theory. We again define a unique extension with the new degree of freedom. This will lead us to find the most important equations in our paper,

$$\begin{aligned} [B_{IK}, B_J{}^K] + \beta[J_{IKLMN}, J_J{}^{KLMN}] &= 0, \\ [B^{IJ}, J_J{}^{KLMN}] &= 0. \end{aligned}$$

Our conjecture will be that the moduli space of M theory is described by the above equations.

In Sect. 2 we discuss the case of constant matrices as the warm-up example, which has some interests in its own right. We will review some relevant properties of the balanced equivariant cohomology and construct, presumably, the simplest balanced topological field theory. In Sect. 3, we will also consider four-dimensional settings of our constructions. We will discuss some close relations with the balanced topological Yang-Mills theory, BTYMT in short, (the Vafa-Witten model of twisted $\mathcal{N}=4$ SYM theory) in four-dimensions. We will argue that BTYMT describes a certain sub-sector of four-dimensional strings in the unbroken phase of general covariance. Here the anti-symmetric tensor fields will play an important role when relating with the monads (the ADHM) construction of instantons. We will use some crucial results of DVV [10] for interpreting our model as a second quantized superstring theory. In our viewpoint, they also demonstrated how some of the known properties of strings can be seen to arise in the unbroken phase.

2. Almost Universal Monads

Throughout this paper we will consider a system (or space) $\overline{W}$ of ten matrices X^μ where $\mu,\nu = 0,\ldots,9$, in the adjoint of an $U(N)$ group [1]. There is a natural $U(N)$ symmetry on acting in this space

$$X^\mu \to gX^\mu g^{-1}, \qquad g \in U(N). \tag{2.1}$$

[1] In general we will allow a matrix X^μ to *degenerate*. This is analogous to the extension of vector bundles to sheaves.

We postulate a $SO(9,1)$ global symmetry acting on the index $\mu, \nu = 0, \ldots, 9$. Under $SO(9,1)$ the X^μ transform as components of a vector. On $\overline{W}$ there is a natural metric which is invariant under $U(N) \times SO(9,1)$

$$|\delta X|^2 = \mathrm{Tr}\,(\eta_{\mu\nu}\delta X^\mu \delta X^\nu)\,, \tag{2.2}$$

where $\eta_{\mu\nu}$ denotes the usual Minkowski metric with signature $(9,1)$.

We want to construct a theory with "spacetime" Poincaré invariance as well as $U(N)$ symmetry. For the $U(N)$ symmetry, we demand the system $\{X^\mu\}$ to be equivalent to the system $\{X'^\mu\}$ if they are related by $X'^\mu = gX^\mu g^{-1}$, for $g \in U(N)$. In general, we can always associate a center of mass coordinate to the X^μ in $\mathbb{R}^{9,1}$ by $x^\mu = N^{-1}\mathrm{Tr}\,X^\mu$. The translations of the base spacetime $\mathbb{R}^{9,1}$ act on the matrices X^μ by $X^\mu \to X^\mu + w^\mu \mathbb{I}_N$. Together with the global $SO(9,1)$ symmetry, we interpret the above as the "spacetime" Poincaré symmetry. The actual spacetime picture emerges when all of the X^μ commute with each other, hence can be simultaneously diagonalized as $X^\mu = diag(x^\mu_\ell)$. By regarding the eigenvalues x^μ_ℓ as coordinates of points (instantons) x_ℓ in $\mathbb{R}^{9,1}$ we get indistinguishable N-tuple of points in $\mathbb{R}^{9,1}$. In this limit, the $U(N)$ symmetry is generically broken down to $U(1)^N$ with the Weyl group acting on the eigenvalues. We will refer to this limit as the broken phase. We should note that all we said above are exactly the properties of the ADHM description [18] of Yang-Mills instantons.

2.1. *Equivariant Cohomology*

In the space of matrices $\overline{W}$ the most natural object is the $U(N)$ equivariant cohomology. We introduce a generator Q_+ of the $U(N)$ equivariant cohomology on $\overline{W}$ satisfying

$$Q_+^2 = -i\delta_{\phi_{++}}, \tag{2.3}$$

where $\delta_{\phi_{++}}$ denote $U(N)$ transformation generated by ϕ_{++}, which is a $N \times N$ matrix in the adjoint representation of $U(N)$. We have the basic action of the algebra

$$Q_+X^\mu = i\psi^\mu_+, \qquad Q_+\psi^\mu_+ = -[\phi_{++}, X^\mu], \qquad Q_+\phi_{++} = 0, \tag{2.4}$$

where ψ^μ_+ is a $N \times N$ matrix with anti-commuting matrix elements. We define an additive quantum number U and assign $U = 1$ to Q_+. We restrict to the $U(N)$-invariant subspace by setting $Q^2_{++} = 0$, which reduces to ordinary cohomology provided that $U(N)$ acts freely. More physically we can interpret the transformation law $Q_+X^\mu = i\psi^\mu_+$ as the BRST-like symmetry for the invariance under the arbitrary shift $X^\mu \to X^\mu + \delta X^\mu$. Thus ψ^μ_+ is nothing but a ghost. The second transformation law in (2.4) involves the redundancy of our description. The general idea of TFT is to study a certain moduli problem using the action functional constructed by gauge fixing the symmetry denoted by Q_+. The moduli space is defined by the solution space, modulo gauge symmetry, of certain field equations (matrices in our case). Then, ψ_+^μ is required to satisfy certain linearized equation as well as to be orthogonal to the direction of the $U(N)$ rotation.

We can extend our equivariant cohomology to its balanced version [16]. In the balanced equivariant cohomology one introduces another fermionic charge Q_- carrying $U = -1$ and the corresponding copy of (2.4),

$$Q_- X^\mu = i\psi_-^\mu, \qquad Q_-\psi_-^\mu = -[\phi_{--}, X^\mu], \qquad Q_-\phi_{--} = 0, \tag{2.5}$$

satisfying

$$Q_-^2 = -i\delta_{\phi_{--}}. \tag{2.6}$$

To make the algebra of our system complete we have to decide about the mutual commutation relation between the two generators $Q_\pm$. The simplest possibility might be $\{Q_+, Q_-\} = 0$. This choice however is inconsistent. Thus we are led to introduce another generator of the $U(N)$ symmetry which has $U = 0$. We have to introduce a new matrix ϕ_{+-} and postulate

$$\{Q_+, Q_-\} = -i2\delta_{\phi_{+-}}. \tag{2.7}$$

The commutation relations (2.3) and (2.6) together with (2.7) determine the superalgebra in a *unique* way. Note that the three separate $U(N)$ symmetry generators $(\phi_{++}, \phi_{+-}, \phi_{--})$ carry $U = (2, 0, -2)$. We will usually denote $\phi_{++} = \phi$, $\phi_{+-} = C$ and $\phi_{--} = \bar{\phi}$. To complete the action of the generators $Q_\pm$ we further have to introduce auxiliary matrices H^μ. They are introduced in the algebra as

$$\begin{aligned} Q_+\psi_-^\mu &= +H^\mu - [C, X^\mu], \\ Q_-\psi_+^\mu &= -H^\mu - [C, X^\mu], \end{aligned} \tag{2.8}$$

which agrees with (2.7), i.e., $\{Q_+, Q_-\}X^\mu = -2i[C, X^\mu]$. To make the algebra closed, we need to impose the following consistent conditions

$$\begin{aligned} &Q_+\bar{\phi} + 2Q_-C = 0, \qquad Q_+^2\bar{\phi} = -i[\phi, \bar{\phi}], \\ &Q_-\phi + 2Q_+C = 0, \qquad Q_-^2\phi = -i[\bar{\phi}, \phi], \end{aligned} \qquad \{Q_+, Q_-\}C = 0. \tag{2.9}$$

These may be seen as the Jacobi identities of the algebra. The solution is

$$\begin{aligned} &Q_+C = i\xi_+, \qquad Q_+\bar{\phi} = -2i\xi_-, \qquad Q_+\xi_- = +\frac{1}{2}[\phi, \bar{\phi}], \qquad Q_+\xi_+ = -[\phi, C], \\ &Q_-C = i\xi_-, \qquad Q_-\phi = -2i\xi_+, \qquad Q_-\xi_+ = -\frac{1}{2}[\phi, \bar{\phi}], \qquad Q_-\xi_- = -[\bar{\phi}, C], \end{aligned} \tag{2.10}$$

Finally consistency with the algebra leads to a transformation of the auxiliary fields H^μ given by

$$\begin{aligned} Q_+ H^\mu &= -i[\phi, \psi_-^\mu] + i[C, \psi_+^\mu] + i[\xi_+, X^\mu], \\ Q_- H^\mu &= +i[\bar{\phi}, \psi_+^\mu] - i[C, \psi_-^\mu] - i[\xi_-, X^\nu], \end{aligned} \tag{2.11}$$

One can check that the algebra is closed.

Before proceeding we summarize the contents of our matrices. We have ten commuting matrices X^μ and their fermionic partners ψ_+^μ and ψ_-^μ, with $U = 1$ and $U = -1$ respectively They carry an $SO(9,1)$ vector index $\mu = 0, \ldots, 9$. We have 10 bosonic auxiliary matrices H^μ carrying $U = 0$ and an $SO(9,1)$ vector index. We also have three bosonic matrices ϕ, C and $\bar{\phi}$ carrying $U = 2$, $U = 0$ and $U = -2$ respectively. Those matrices have superpartners ξ_+ and ξ_- with $U = 1$ and $U = -1$, respectively. Note that our algebra has an internal sl_2 structure [17][16]. The matrices X^μ form ten copies of an sl_2 singlet, (ψ_+^μ, ψ_-^μ) form ten copies of an sl_2 doublet, $(\phi, C, \bar{\phi})$ form an sl_2 triplet and (ξ_+, ξ_-) form an sl_2 doublet. All this can be nicely summarized by the following diagram [16]

$$\begin{array}{ccccccccccc}
U=+2 & & & & & & & & \phi_{++} & & \\
 & & & & & & & & & \searrow & \\
U=+1 & & & & \psi_+^\mu & & & & & & \xi_+ \\
 & & & \nearrow & & \searrow & & & & \nearrow & \\
U=0 & \text{fields} & X^\mu & & & & H^\mu\,, & \text{consistency} & \phi_{+-} & & \\
 & & & \searrow & & \nearrow & & & & \searrow & \\
U=-1 & & & & \psi_-^\mu & & & & & & \xi_- \\
 & & & & & & & & & \nearrow & \\
U=-2 & & & & & & & & \phi_{--} & &
\end{array} \tag{2.12}$$

The sl_2 symmetry of our algebra is referred to as the balanced structure. The symmetry under filliping the signs of the U-number implies that the net U-number of fermionic zero-modes is always zero. We will refer the first multiplet to a vector multiplet.

2.2. Action Functional

Now we have enough machinery to define the action functional, which should have $SO(9,1) \times U(N)$ symmetry and is invariant under the $\mathcal{N} = 2$ symmetry generated by $Q_\pm$. As a BTFT we also require the action functional to be invariant under the sl_2 symmetry. In particular, the action functional should have $U = 0$. The desired action functional turns out to be almost *uniquely* determined.[2] To begin with we define

$$S_1 = Q_+ Q_- \mathcal{F}_1, \tag{2.13}$$

[2] This is a general property of BTFT [16].

derived of a supersymmetry transformation of the action potential

$$\mathcal{F}_1 = -\mathrm{Tr}\,\left(2\psi_+^\mu \psi_{\mu-} + \xi_-\xi_+\right). \tag{2.14}$$

Here $\mathcal{F}_1$ is uniquely determined by the global $SO(9,1)$ and sl_2 symmetries. We find

$$\begin{aligned} S_1 =\mathrm{Tr}\,\Bigg(& 2[\phi, X^\mu][\bar{\phi}, X^\nu] + 2i\psi_-^\mu[\phi, \psi_{\mu-}] + 2i\psi_+^\mu[\bar{\phi}, \psi_{\mu+}] + 4i[C, \psi_+^\mu]\psi_{\mu-} \\ & + 4i[X^\mu, \psi_{\mu+}]\xi_- + 4i[X^\mu, \psi_{\mu-}]\xi_+ - 2[C, X^\mu][C, X_\mu] + 2H^\mu H_\mu \\ & - [\phi, C][\bar{\phi}, C] - i\xi_-[\phi, \xi_-] - i\xi_+[\bar{\phi}, \xi_+] + 2i\xi_+[C, \xi_-] - \frac{1}{4}[\phi, \bar{\phi}]^2 \Bigg). \end{aligned} \tag{2.15}$$

For our purpose the above action functional is not good enough. We need to generate a potential term $V = [X^\mu, X^\nu]^2$ for the X^μ such that these matrices commute in the flat direction. To get this term we need a cubic action potential term $\mathcal{F}_0$. However there are no sl_2 and $SO(9,1)$ invariant combinations of the existing matrices X^μ such that $Q_+Q_-\mathcal{F}_0$ generates this potential. Consequently we have to introduce one more matrix multiplet. We introduce a new adjoint matrix $B^{\mu\nu}$ carrying $U = 0$ which is anti-symmetric in the $SO(9,1)$ indices. We have a corresponding algebra

$$\begin{aligned} & & Q_+\chi_+^{\mu\nu} &= -[\phi, B^{\mu\nu}], \\ Q_+B^{\mu\nu} &= i\chi_+^{\mu\nu}, & Q_+\chi_-^{\mu\nu} &= +H^{\mu\nu} - [C, B^{\mu\nu}], \\ Q_-B^{\mu\nu} &= i\chi_-^{\mu\nu}, & Q_-\chi_+^{\mu\nu} &= -H^{\mu\nu} - [C, B^{\mu\nu}], \\ & & Q_-\chi_-^{\mu\nu} &= -[\bar{\phi}, B^{\mu\nu}], \end{aligned} \tag{2.16}$$

We will refer to the above multiplet as the anti-symmetric tensor multiplet. We define

$$S_0 + S_2 = Q_+Q_-\Big(\mathcal{F}_0 + \mathcal{F}_2\Big), \tag{2.17}$$

with

$$\mathcal{F}_0 = -\mathrm{Tr}\,\left(iB^{\mu\nu}\left([X_\mu, X_\nu] + \frac{1}{3}[B_{\mu\rho}, B_\nu{}^\rho]\right)\right), \qquad \mathcal{F}_2 = -\mathrm{Tr}\,\left(\chi_+^{\mu\nu}\chi_{\mu\nu-}\right), \tag{2.18}$$

which are again the only two sl_2 and $SO(9,1)$ invariants, which do not introduce bare mass.[3]

[3] There are other combinations, which are redundant. We will consider the massive deformations in a later section.

Working through the algebra, we obtain the complete action

$$
\begin{aligned}
S_0 + S_2 =\mathrm{Tr}\Big(& [\phi, B^{\mu\nu}][\bar{\phi}, B^{\mu\nu}] + i\chi_-^{\mu\nu}[\phi, \chi_{\mu\nu-}] + i\chi_+^{\mu\nu}[\bar{\phi}, \chi_{\mu\nu+}] + 2i[C, \chi_+^{\mu\nu}]\chi_{\mu\nu-} \\
& + 2i[B_{\mu\nu}, \chi_+^{\mu\nu}]\xi_- + 2i[B_{\mu\nu}, \chi_-^{\mu\nu}]\xi_+ - [C, B^{\mu\nu}][C, B_{\mu\nu}] + H^{\mu\nu}H_{\mu\nu} \\
& - H^{\mu\nu}\big([X_\mu, X_\nu] + [B_{\mu\rho}, B_\nu{}^\rho]\big) + 2H^\mu[B_{\mu\nu}, X^\nu] - 2iB_{\mu\nu}[\psi_+^\mu, \psi_-^\nu] \\
& - 2iB^{\mu\nu}[\chi_{\mu\rho+}, \chi_\nu\rho] + 2i\chi_-^{\mu\nu}[X_\mu, \psi_{\nu+}] - 2i\chi_+^{\mu\nu}[X_\mu, \psi_{\nu-}]\Big).
\end{aligned} \tag{2.19}
$$

Now we define the total action S by

$$
S = S_0 + S_1 + S_2. \tag{2.20}
$$

We can integrate out the auxiliary matrices H_μ and $H_{\mu\nu}$ by setting

$$
H_\mu = -\frac{1}{2}[B_{\mu\nu}, X^\nu], \qquad H_{\mu\nu} = \frac{1}{2}[X_\mu, X_\nu] + \frac{1}{2}[B_{\mu\rho}, B_\nu{}^\rho] \tag{2.21}
$$

to get

$$
\begin{aligned}
S = \mathrm{Tr}\Big(& 2[\phi, X^\mu][\bar{\phi}, X^\nu] + 2i\psi_-^\mu[\phi, \psi_{\mu-}] + 2i\psi_+^\mu[\bar{\phi}, \psi_{\mu+}] + 4i[C, \psi_+^\mu]\psi_{\mu-} \\
& + 4i[X^\mu, \psi_{\mu+}]\xi_- + 4i[X^\mu, \psi_{\mu-}]\xi_+ - 2[C, X^\mu][C, X_\mu] - [\phi, C][\bar{\phi}, C] \\
& + 2i\xi_+[C, \xi_-] - i\xi_-[\phi, \xi_-] - i\xi_+[\bar{\phi}, \xi_+] - \frac{1}{4}[\phi, \bar{\phi}]^2 + [\phi, B^{\mu\nu}][\bar{\phi}, B_{\mu\nu}] \\
& + 2i[C, \chi_+^{\mu\nu}]\chi_{\mu\nu-} + i\chi_-^{\mu\nu}[\phi, \chi_{\mu\nu-}] + i\chi_+^{\mu\nu}[\bar{\phi}, \chi_{\mu\nu+}] + 2i[B_{\mu\nu}, \chi_+^{\mu\nu}]\xi_- \\
& + 2i[B_{\mu\nu}, \chi_-^{\mu\nu}]\xi_+ - 2iB_{\mu\nu}[\psi_+^\mu, \psi_-^\nu] - 2iB^{\mu\nu}[\chi_{\mu\rho+}, \chi_\nu{}^\rho{}_-] \\
& + 2i\chi_-^{\mu\nu}[X_\mu, \psi_{\nu+}] - 2i\chi_+^{\mu\nu}[X_\mu, \psi_{\nu-}] - [C, B^{\mu\nu}][C, B_{\mu\nu}] \\
& - \frac{1}{4}\left([X_\mu, X_\nu] + \frac{1}{4}[B_{\mu\rho}, B_\nu{}^\rho]\right)^2 - \frac{1}{2}[B^{\mu\rho}, X_\nu]^2\Big).
\end{aligned} \tag{2.22}
$$

This action is invariant under the $Q_\pm$ symmetries after replacing H^μ in (2.8) with the expression in (2.21). As a TFT, we study the fixed points of $Q_\pm$ symmetry. First of all, from (2.8). (2.16) and (2.21), the fixed point equations $Q_\pm\psi_\mp^\mu = 0$ and $Q_\pm\chi_\mp^{\mu\nu} = 0$ imply $H_\mu = H_{\mu\nu} = 0$, which is equivalent to

$$
[X_\mu, X_\nu] + \frac{1}{4}[B_{\mu\rho}, B_\nu{}^\rho] = 0, \qquad [B^{\mu\nu}, X_\nu] = 0, \tag{2.23}
$$

and

$$[C, X_\mu] = [C, B_{\mu\nu}] = 0. \tag{2.24}$$

We also have other fixed point equations

$$[\phi_{AB}, \phi_{A'B'}] = 0, \qquad [\phi_{AB}, X^\mu] = [\phi_{AB}, B^{\mu\nu}] = 0. \tag{2.25}$$

These are the equations for the localization which determine the moduli space we want to study. The $\chi_\mp^{\mu\nu}$, $\psi_\mp^\mu$ and $\xi_\mp$ equations of motion, modulo the $U(N)$ symmetry generated by $\phi, C, \bar{\phi}$ are

$$[X_\mu, \psi_{\pm\nu}] + [B_{\mu\rho}, \chi_{\nu\pm}{}^\rho] = 0, \quad [\chi_\pm^{\mu\nu}, X_\nu] + [B^{\mu\nu}, \psi_{\pm\nu}] = 0, \quad \eta_{\mu\nu}[X^\mu, \psi_\pm^\nu] + [B^{\mu\nu}, \chi_{\mu\nu\pm}] = 0. \tag{2.26}$$

The first two equations can be interpreted as the linearization of (2.23) and the last equation can be interpreted as a kind of Coulomb gauge fixing condition.

We define the partition function Z_0 by

$$Z_0 = \frac{1}{Vol(G)} \int_{\overline{\mathcal{W}} \otimes (Lie(\mathcal{G}) \otimes Lie(\mathcal{G})^*)} \mathcal{D}X\mathcal{D}\psi_+\mathcal{D}\psi_-\mathcal{D}B\mathcal{D}\chi_+\mathcal{D}\chi_-\mathcal{D}\phi\mathcal{D}\bar{\phi}\mathcal{D}C\mathcal{D}\xi_+\mathcal{D}\xi_- \times e^{-S_0}. \tag{2.27}$$

Note that we are dealing with a topological theory so that the stationary phase evaluation is exact. In other words the path integral is localized to the space of supersymmetric minima of the action given by (2.23). At this point we like to emphasis the distinction between X^μ and ϕ_{AB}. Note that we introduced ϕ_{AB} as the generators of $U(N)$ symmetry of the matrices X^μ. In other words the matrices ϕ_{AB} are responsible for *pure gauge degrees of freedom*. So the equations $[\phi_{AB}, \phi_{A'B'}] = 0$ define the *flat* directions. We can diagonalize ϕ_{AB} simultaneously. Then the flat direction can be identified with $Sym^N(\mathbb{R}^3)$. Now we see that, from (2.24), the supersymmetric minimum configuration is a *multiply stratified* space parameterized by a point in $Sym^N(\mathbb{R}^3)$. Note that the supersymmetric minimum depends only on a particular stratum of $Sym^N(\mathbb{R}^3)$ determined by the symmetry breaking pattern of $U(N)$. At generic points in $Sym^N(\mathbb{R}^3)$ the $U(N)$ symmetry is broken down to $U(1)^N$. In a diagonal some non-Abelian symmetry is restored.

The simplest solutions to (2.23) are given by the case where the ten matrices X^μ are mutually commuting and $B^{\mu\nu} = 0$. So $\{X^\mu\}$ can be simultaneously diagonalized. Such a diagonalization depends on a point in $Sym^N(\mathbb{R}^3)$. We can interpret the eigenvalues x_ℓ^μ, $\ell = 1, \ldots, N$, as the positions of N unordered points in a *space-time* $\mathbb{R}^{9,1}$. In other words, we are describing a system of N point-like instantons in ten-dimensional Minkowski space-time as the supersymmetric minimum. Now our abstract global symmetry group $SO(9,1)$ can be interpreted as the Lorentz symmetry of $\mathbb{R}^{9,1}$.

How about more general solutions of (2.23) ? For example, we can imagine solutions with non-vanishing $B^{\mu\nu}$, either commuting or non-commuting one. For non-commuting $B^{\mu\nu}$, leading to non-commuting X_μ, we may use some analogy with the monads (ADHM) construction of Yang-Mills instantons. Those degrees of freedom may be attributed to the size and relative degrees among instantons. For commuting and non-trivial solutions of $B_{\mu\nu}$ we certainly have problems in the space-time interpretations. Furthermore, we can allow more general solutions which break our $SO(9,1)$ symmetry. Then some components of $B_{\mu\nu}$ can be interpreted as positions of instantons living in the lower dimensional space. Such new matrices transform as vectors under the smaller Lorentz group defining another "non-commuting" space-time coordinates of instantons. We will refer to all those solutions as *almost universal instantons*. The systems we are describing can be interpreted as *monads* of such instantons which we will refer to as *almost universal monads*.[4]

There are many other issues concerning the model constructed in the section. Since we will have to repeat those in our description of monadic string, we will not discuss them here. But we like to clarify the role of the anti-symmetric tensor $B^{\mu\nu}$. It was not entirely clear, in the treatment of this section, how we can interpret the eigenvalues of $B^{\mu\nu}$. However, we had to introduce $B^{\mu\nu}$ to define a meaningful theory. Note that $B^{\mu\nu}$ was introduced because of the non-commutative nature of "spacetime" coordinates of instantons and the requirement of*covariance*. Thus we can naturally expect that the existence of "spacetime coordinates" as antisymmetric tensors may be just the direct requirement for the covariant description of the existence of off-diagonal parts of "spacetime coordinates". In the next section, we will discuss these issues for the similar description of instantons in four-dimensions. In later sections, we will return to those points again.

3. Extended Monads and $\mathcal{N}=4$ SYM Theory in Four-Dimensions

In this section we will consider a system of four matrices X^i, $i=0,1,2,3$ rather than ten matrices. To relate with Yang-Mills instantons we assume the X^i to transform as the components of a vector for $SO(4)$. We will repeat the construction of the previous section in the new setting. We will find relations with the monads (the ADHM) description of Yang-Mills instantons. We will discuss the interpretation of "space-time coordinates" which transform as tensors or scalars. We also discuss close relations with the Vafa-Witten model of twisted $\mathcal{N}=4$ SYM theory (or BTYMT) on a four-manifold [17][16]. Using the structure of BTYMT, we will recall Witten's arguments on the unbroken phase of quantum gravity.

[4] Note that above spacetime interpretation are motivated from the ADHM description of Yang-Mills instantons as well as Witten's description of D-instantons (D-branes in general). Witten also mentioned the intriguing similarity between the two cases. This observation is, actually, the starting point of our investigation.

3.1. A Description of Instantons in four-dimensions

This sub-section can be viewed as a continuation of the paper [19], where the monads (the ADHM equation) construction of Yang-Mills instanton was extended in a way motivated by the Vafa-Witten equation of $\mathcal{N}=4$ SYM theory and its relation with the Seiberg-Witten equation. The equations of extended monads are simply the reduction of the Vafa-Witten equations to zero-dimensions. We can repeat the same constructions as in the two previous subsections.

It is possible to break half the supersymmetry maintaining only the symmetry generated by Q_+. An important perturbation satisfying this constraint is given by adding bare mass terms with non-zero U-number to the action. Since the theory in the bulk is U-number anomaly free, such a perturbation does not change the theory unless we take a very special limit. We may view such perturbations as looking to the system through a magnifying glass. Essentially the same perturbation is discussed in [17] and [20]. The resulting theory will be localized to the fixed point locus of this Q_+ symmetry given by

$$\begin{aligned}\frac{1}{2}[X^i,X^j]+\frac{1}{2}[B^{i\ell},B^j{}_\ell]-[C,B^{ij}]=0,\\ [X_i,B^{ij}]+[X^j,C]=0.\end{aligned} \tag{3.1}$$

supplemented by the equations

$$[\phi,\bar\phi]=0, \qquad [\phi,T_i]=0. \tag{3.2}$$

We can decompose the anti-symmetric tensor B_{ij} under $SO(4)$ into its self-dual and the anti-self-dual parts

$$B_{ij}=B^+_{ij}+B^-_{ij}, \tag{3.3}$$

and the two components are orthogonal to each others. Now we can consider the self-dual part of the the equations (3.1)

$$\begin{aligned}\frac{1}{2}[X^i,X^j]^+ +\frac{1}{2}[B^{+i\ell},B^{+j}{}_\ell]-[C,B^{+ij}]=0,\\ [X_i,B^{+ij}]+[X^j,C]=0.\end{aligned} \tag{3.4}$$

The above self-dual truncation is nothing but the Vafa-Witten equations reduced to zero-dimensions [17]. In the paper [19], we referred to them as the equations for extended monads. By using complex $SO(4)$ indices we can rewrite the equations (3.4) as equations for 4 complex $N\times N$ matrices T_a, $a=1,\ldots,4$;

$$\begin{aligned}&[T_1,T_2]+[T_3,T_4]=0,\\ &[T_1,T_1^*]+[T_2,T_2^*]+[T_3,T_4]=0,\end{aligned} \tag{3.5}$$

where T_1 and T_2 are build out of the X^i and T_3 and T_4 come from (C, B^+_{ij}). We only wrote down the first of the equations (3.4). We certainly have solutions of (3.5) with all of the T_a simultaneously diagonalized, $UT_aU^{-1} = t_a$, where the t_a are diagonal matrices. We might interpret the eigenvalues t_i^ℓ of t_i as the positions of N points in $\mathbb{R}^8$. There is however an obvious problem to such an interpretation since T_3 and T_4 do not transform as the components of a vector for $SO(4)$. Note that we can still interpret the eigenvalues of T_1 and T_2 as the positions of points in $\mathbb{R}^4$. In fact, if we set T_3 and T_4 to zero, the equation (3.5) is nothing but the ADHM equations of N point-like (Yang-Mills) instantons in $\mathbb{R}^4$.

A solution to the problem above was presented in the paper [19]. In [19] and [20], the breaking of $Q_\pm$ to Q_+ was realized by extending the Dolbeault version of the balanced equivariant cohomology to incorporate the obvious global symmetry

$$(T_3, T_4) \to (e^{-im\theta}T_3, e^{im\theta}T_4). \tag{3.6}$$

As a result, the fixed point equations (3.2) should be modified to

$$[\phi, \bar{\phi}] = 0, \qquad [\phi, T_1] = [\phi, T_2] = 0, \qquad \begin{aligned} [\phi, T_3] &= +mT_3, \\ [\phi, T_4] &= -mT_4, \end{aligned} \tag{3.7}$$

where m is the bare mass. Now it is obvious that there are no non-trivial diagonal solutions for T_3 and T_4 for $m \neq 0$. Their solutions are always off-diagonal so that we will never be able to interpret them as positions or coordinates in space-time! The situation was described in detail in the paper [19]. We will recall two typical solutions. The solutions of (3.5) and (3.7) are determined by the symmetry breaking pattern of $U(N)$ (via the eigenvalues of ϕ). If the $U(N)$ symmetry is unbroken $T_3 = T_4 = 0$ and T_1 and T_2 should be simultaneously diagonalized. Then we get the ADHM description of N point-like instantons in $\mathbb{R}^4$. If the $U(N)$ symmetry is broken down to $U(N-k) \times U(k)$ we find

$$T_1 = \begin{pmatrix} t_1 & 0 \\ 0 & t_1' \end{pmatrix}, \qquad T_2 = \begin{pmatrix} t_2 & 0 \\ 0 & t_2' \end{pmatrix}, \qquad T_3 = \begin{pmatrix} 0 & \sigma \\ 0 & 0 \end{pmatrix}, \qquad T_4 = \begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix}, \tag{3.8}$$

where σ is $k \times (N-k)$ and π is $(N-k) \times k$ matrices. We have

$$\begin{cases} [t_1, t_2] + \sigma\pi = 0, \\ [t_1, t_1^*] + [t_2, t_2^*] + \sigma\sigma^* - \pi^*\pi = 0, \end{cases} \qquad \begin{cases} [t_1', t_2'] - \pi\sigma = 0, \\ [t_1', t_1'^*] + [t_2', t_2'^*] + \pi\pi^* - \sigma^*\sigma = 0. \end{cases} \tag{3.9}$$

Note that the first and the second set of equations describe $SU(N-k)$ and $SU(k)$ Yang-Mills instantons with instanton numbers k and $(N-k)$ respectively. Now the role of T_3 and T_4 is clear. They carry information about the gauge group and the size of Yang-Mills instantons in $\mathbb{R}^4$.

In the above discussions we restrict our attention to the self-dual part B_{ij}^+ of B_{ij}. This restriction can easily be justified. Recall that B_{ij} is introduced to get the crucial potential term $\mathrm{Tr}\,[X_i, X_j][X^i, X^j]$. We can decompose $[X_i, X_j]$ into self-dual and anti-self-dual parts and show that

$$\mathrm{Tr}\,[X_i, X_j][X^i, X^j] = 2\mathrm{Tr}\,[X_i, X_j]^+[X^i, X^j]^+. \tag{3.10}$$

Thus the anti-self-dual part B_{ij}^- of B_{ij} is redundant. This implies that we are describing essentially the same system with the self-dual anti-symmetric tensor multiplet only.

3.2. The Global $\mathcal{N} = 4$ Super-Yang-Mills Theory

Now we consider $\mathcal{N} = 4$ super-Yang-Mills theory [17][16]. Let M be an arbitrary four-manifold where our $SO(4)$ symmetry is acting. Let E be a $U(N)$ bundle over M and let X^i be the components of a connection. The BTYM theory is defined exactly as in Sect. 2.1 with the same commutation relations (2.3). The only change is that the $U(N)$ gauge transformation acts on X^i by

$$X_i \to gX_ig^{-1} + g\partial_i g^{-1}, \tag{3.11}$$

where $g : M \to U(N)$. In the space of all connections $\mathcal{X}$ we have a natural metric

$$|\delta X|^2 = \int_M d\mu \mathrm{Tr}\,\left(\delta X^i \delta X_i\right), \tag{3.12}$$

where $d\mu$ denotes the measure on M. Every other field transforms in the adjoint representation. The algebra is given by

$$Q_\pm X^i = i\psi_\pm^i, \qquad Q_\pm \psi_\pm^i = D_i\phi_{\pm\pm}, \qquad Q_\pm \psi_\mp^i = \pm H^i + D^i C, \tag{3.13}$$

where D_i is the gauge covariant derivative. The remaining algebra is left unchanged.

The global $\mathcal{N} = 4$ (space-time) supersymmetry requires that the anti-symmetric tensor multiplet is self-dual. Apart from the underlying space-time supersymmetry, as in the previous subsection, the restriction to a self-dual anti-symmetric tensor multiplet is a very natural requirement. Now the potential term becomes the usual kinetic term $\mathrm{Tr}\, F \wedge *F$ of Yang-Mills theory. The well-known fact that

$$\int_M \mathrm{Tr}\, F \wedge *F = 2\int_M \mathrm{Tr}\,\left(F^+ \wedge *F^+\right) + 8\pi^2 k, \tag{3.14}$$

where k denotes the instanton number

$$k = \frac{1}{8\pi^2}\int_M \mathrm{Tr}\,(F \wedge F), \tag{3.15}$$

implies that it is sufficient to introduce the self-dual part of the anti-symmetric tensor multiplet. We can freely add the topological term (3.15) to our action without spoiling anything. The action functional is defined by [16]

$$S = \frac{1}{e^2} Q_+ Q_- \int_M d\mu \Big(\mathcal{F}_0^+ + \mathcal{F}_1 + \mathcal{F}_2^+ \Big), \tag{3.16}$$

where the $\mathcal{F}$'s are given as in (2.18) and the superscript $+$ denote that we only use the self-dual part of the anti-symmetric tensors. Clearly $[X_i, X_j]$ in $\mathcal{F}_0$ should be replaced with the field strength F_{ij}. Here e^2 denotes the Yang-Mills coupling constant which are dimensionless.

4. Monad String Theory

In the previous section we extended the monad (ADHM) description of N point-like instantons in $\mathbb{R}^4$ to $\mathbb{R}^{9,1}$ and construct, presumably the most natural, supersymmetric theory out of it. In this section we will apply the same ideas to describe second quantized superstring theory in $\mathbb{R}^{9,1}$. Throughout this section we will restrict our attentions to classical aspects of the model.

4.1. The Algebra and Action Functional

To begin with we assume our ten matrices X^μ to be matrix functions $X^\mu(\sigma^+, \sigma^-)$ of two world-sheet coordinates. Let $\overline{W}(\sigma^+, \sigma^-)$ be the space of $N \times N$ Hermitian matrix functions. We endow the space $\overline{W}(\sigma^+, \sigma^-)$ with the natural metric

$$|\delta X|^2 = \int d\sigma^+ d\sigma^- \mathrm{Tr}\, (\eta_{\mu\nu} \delta X^\mu \delta X^\nu)\,. \tag{4.1}$$

This metric is invariant under local $U(N)$ symmetry $X^\mu \to g X^\mu g^{-1}$ for $g \in \mathcal{G}$ such that $g : \Sigma \to U(N)$, where Σ denotes the "world-sheet" which is the space of parameters $\sigma^\pm$. As mentioned in the introduction we want to construct a theory by gauge fixing the "spacetime" and "world-sheet" Poincaré symmetry as well as the local $U(N)$ gauge symmetry. By the "spacetime" Poincaré symmetry, we mean the invariance under the global $SO(9,1)$ symmetry acting on the "spacetime" vector index μ and the invariance under arbitrary shift $X^\mu \to X^\mu + \delta X^\mu$. Clearly they are symmetries of our metric (4.1). From the viewpoint of two-dimensional $U(N)$ gauge theory, the "spacetime" Lorentz covariance is just a global symmetry among the fields X^μ. The "spacetime" translation invariance implies that the two-dimensional gauge theory is a TFT.

To get define a system with these properties we can simply extend our balanced $\mathcal{G}$-equivariant cohomology to include translations along the internal directions. We define the commutation relations

$$
\begin{aligned}
Q_+^2 &= -i\left(\frac{\partial}{\partial\sigma^+} + \delta_{\phi_{++}}\right), \\
Q_-^2 &= -i\left(\frac{\partial}{\partial\sigma^-} + \delta_{\phi_{--}}\right).
\end{aligned}
\qquad \{Q_+, Q_-\} = -i\delta_{\phi_{+-}}. \tag{4.2}
$$

This immediately leads to the following basic algebra

$$
\begin{aligned}
Q_+X^\mu &= i\psi_+^\mu, & Q_+\psi_+^\mu &= -D_+X^\mu, \\
Q_-X^\mu &= i\psi_-^\mu, & Q_-\psi_-^\mu &= -D_-X^\mu.
\end{aligned} \tag{4.3}
$$

The above extension is indeed a very natural step. For $Q_\pm X^\mu = i\psi_\pm^\mu$, we can interpret $\psi_\pm^\mu$ as the ghosts for the topological symmetry of the arbitrary shift $X^\mu \to X^\mu + \delta X^\mu$. This description clearly has a redundancy which is the $U(N)$ symmetry and the shift of parameters $\sigma^\pm$ as indicated in (4.2). We may interpret $Q_\pm$ as the balanced equivariant cohomology generators of $U(N) \times P_{\sigma^\pm}$. We will see shortly that the explicit realization of (4.2) requires that the canonical line bundle of Σ is trivial. Naturally, we will consider Σ to be a two-dimensional cylinder. This fits nicely with the description of closed string. We can identify $\sigma^\pm$ with "world-sheet" light-cone coordinates, i.e., $\sigma^\pm = \frac{1}{2}(\sigma \pm \tau)$. For consistency we see from (4.2) that $\phi_{\pm\pm}$ should transform as the components of an $U(N)$ connection. So we can identify ϕ_{++} and ϕ_{--} with the left and right components of the $U(N)$ connection. The global sl_2 symmetry of the balanced equivariant cohomology can be identified with the "world-sheet" Lorentz symmetry. To put it differently, we are just extending the world-sheet supersymmetry to include the $U(N)$ symmetry - the $\mathcal{G}$-equivariant extension of world-sheet supersymmetry.

¿From the construction in the previous section, it is straightforward to get the modified algebra. The algebra (4.3) is supplemented by

$$
\begin{aligned}
Q_+\psi_-^\mu &= +H^\mu - [C, X^\mu], \\
Q_-\psi_+^\mu &= -H^\mu - [C, X^\mu],
\end{aligned} \tag{4.4}
$$

and the algebra of consistency (2.10) should be modified to

$$
\begin{aligned}
&Q_+C = i\xi_+, \qquad
&&\begin{aligned} Q_+\phi_{++} &= 0, \\ Q_-\phi_{++} &= -2i\xi_+, \\ Q_+\phi_{--} &= -2i\xi_-, \\ Q_-\phi_{--} &= 0. \end{aligned}
&&\begin{aligned} Q_+\xi_+ &= -D_+C, \\ Q_+\xi_- &= +\frac{1}{2}F_{+-}, \\ Q_-\xi_+ &= -\frac{1}{2}F_{+-}, \\ Q_-\xi_- &= -D_-C, \end{aligned} \\
&Q_-C = i\xi_-. && &&
\end{aligned} \tag{4.5}
$$

where F_{+-} is the Yang-Mills curvature of the $U(N)$ connection $\phi_{\pm\pm}$.[5] Note also that the U numbers of the covariant derivatives $D_\pm$ are ± 2. The triviality of the canonical line bundle is required since we relate the "world-sheet" vector $\phi_{\pm\pm}$ with a "world-sheet" scalar C via (4.5). The transformation laws for the auxiliary fields H^μ are

$$
\begin{aligned}
Q_+ H^\mu &= -iD_+\psi^\mu_- + i[C,\psi^\mu_+] + i[\xi_+, X^\mu],\\
Q_- H^\mu &= +iD_-\psi^\mu_+ - i[C,\psi^\mu_-] - i[\xi_-, X^\mu].
\end{aligned}
\tag{4.6}
$$

A difference with the previous section is that (ϕ_{++},ϕ_{--}) become an sl_2 doublet and C becomes an sl_2 singlet.

For reasons explained in the previous section we introduce a multiplet $(B^{\mu\nu},\chi^{\mu\nu}_\pm,H^{\mu\nu})$ with the transformation laws

$$
\begin{aligned}
& & Q_+\chi^{\mu\nu}_+ &= -D_+B^{\mu\nu},\\
Q_+B^{\mu\nu} &= i\chi^{\mu\nu}_+, & Q_+\chi^{\mu\nu}_- &= +H^{\mu\nu} - [C,B^{\mu\nu}],\\
Q_-B^{\mu\nu} &= i\chi^{\mu\nu}_-, & Q_-\chi^{\mu\nu}_+ &= -H^{\mu\nu} - [C,B^{\mu\nu}],\\
& & Q_-\chi^{\mu\nu}_- &= -D_-B^{\mu\nu},
\end{aligned}
\tag{4.7}
$$

and

$$
\begin{aligned}
Q_+ H^{\mu\nu} &= -iD_+\chi^{\mu\nu}_- + i[C,\chi^{\mu\nu}_+] + i[\xi_+, B^{\mu\nu}],\\
Q_- H^{\mu\nu} &= +iD_-\chi^{\mu\nu}_+ - i[C,\chi^{\mu\nu}_-] - i[\xi_-, B^{\mu\nu}].
\end{aligned}
\tag{4.8}
$$

The scaling dimensions of $(B^{\mu\nu},\chi^{\mu\nu}_\pm,H^{\mu\nu})$ are $(0,\frac{1}{2},1)$.

The action functional can be defined through a procedure similar to Sect. 2. Only now we have to replace the $U(N)$ trace Tr with $\int d\sigma^+ d\sigma^- \mathrm{Tr}$. The action functional can hence be written in the form

$$
S = Q_+Q_-\left(\int d\sigma^+ d\sigma^-\Big(\mathcal{F}_0+\mathcal{F}_1+\mathcal{F}_2\Big)\right),
\tag{4.9}
$$

where the action potential terms are given by

$$
\begin{aligned}
\mathcal{F}_0 &= -\mathrm{Tr}\left(iB^{\mu\nu}\left([X_\mu,X_\nu]+\frac{R^2}{3}[B_{\mu\rho},B_\nu{}^\sigma]\right)\right),\\
\mathcal{F}_1 &= -\mathrm{Tr}\left(2\psi^\mu_+\psi_{\mu-} + R^2\xi_-\xi_+\right),\\
\mathcal{F}_2 &= -\mathrm{Tr}\left(R^2\chi^{\mu\nu}_+\chi_{\mu\nu-}\right).
\end{aligned}
\tag{4.10}
$$

[5] Under the local gauge transformation the connection A transform as $\delta_\varepsilon A = d_A\varepsilon$.

We made a slightly more general choice for our action potential than in the previous section by introducing a free parameter R. Since our construction is the unique non-Abelian generalization of the RNS string theory, we certainly expect to get the free RNS string in a suitable limit, where the $U(N)$ symmetry breaks down to $U(1)^N$ and all fields can be simultaneously diagonalized. Without introducing the free-parameter R, we do not get the free RNS string. Instead we get a superconformal theory consisting of both the X^μ and $B^{\mu\nu}$ multiplets. Only after introducing R and in the limit $R = 0$, we get the free RNS string.

Although this looks arbitrary, our choice is very natural since the RNS string action entirely comes from the term $\mathrm{Tr}\,(\psi_+^\mu \psi_{\mu-})$ in the action potential $\mathcal{F}_1$. We will see in a later section that the above form of the action-potential originates from eleven-dimensional covariance. It will become clear that our construction is directing us to an underlying theory with eleven dimensional covariance.

The explicit form of the action functional is

$$
\begin{aligned}
S = \int d\sigma^+ d\sigma^- \mathrm{Tr}\Big(& 2D_+X^\mu D_-X_\mu + 2i\psi_-^\mu D_+\psi_{\mu-} + 2i\psi_+^\mu D_-\psi_{\mu+} + 4i[C,\psi_+^\mu]\psi_{\mu-} \\
& + 4i[X^\mu,\psi_{\mu+}]\xi_- + 4i[X^\mu,\psi_{\mu-}]\xi_+ - 2[C,X^\mu][C,X_\mu] + R^2 H^{\mu\nu}H_{\mu\nu} + 2H^\mu H_\mu \\
& - R^2 D_+CD_-C - iR^2\xi_-D_+\xi_- - iR^2\xi_+D_-\xi_+ + 2iR^2\xi_+[C,\xi_-] - \frac{R^2}{4}F_{+-}^2 \\
& - H^{\mu\nu}\left([X_\mu,X_\nu] + R^2[B_{\mu\rho},B_\nu{}^\rho]\right) + 2H_\mu[B^{\mu\nu},X_\nu] + R^2 D_+B^{\mu\nu}D_-B_{\mu\nu} \\
& + iR^2\chi_-^{\mu\nu}D_+\chi_{\mu\nu-} + iR^2\chi_+^{\mu\nu}D_-\chi_{\mu\nu+} + 2iR^2[B_{\mu\nu},\chi_+^{\mu\nu}]\xi_- + 2iR^2[B_{\mu\nu},\chi_-^{\mu\nu}]\xi_+ \\
& + 2iR^2[C,\chi_+^{\mu\nu}]\chi_{\mu\nu-} - R^2[C,B^{\mu\nu}][C,B_{\mu\nu}] - 2iR^2B^{\mu\nu}[\chi_{\mu\rho+},\chi_{\nu-}{}^\rho] \\
& - 2iB_{\mu\nu}[\psi_+^\mu,\psi_-^\nu] + 2i\chi_-^{\mu\nu}[X_\mu,\psi_{\nu+}] - 2i\chi_+^{\mu\nu}[X_\mu,\psi_{\nu-}]\Big).
\end{aligned}
\tag{4.11}
$$

We can integrate out the auxiliary fields $H^{\mu\nu}$ and H^μ by setting

$$
H_{\mu\nu} = \frac{1}{2R^2}\left([X_\mu,X_\nu] + R^2[B_{\mu\rho},B_\nu{}^\rho]\right) \qquad H_\mu = -[B_{\mu\nu},X^\nu]. \tag{4.12}
$$

After this replacement we get

$$
\begin{aligned}
S' = \int d\sigma^+ d\sigma^- \mathrm{Tr} \Big(& 2D_+X^\mu D_-X_\mu + 2i\psi_-^\mu D_+\psi_{\mu-} + 2i\psi_+^\mu D_-\psi_{\mu+} + 4i[C,\psi_+^\mu]\psi_{\mu-} \\
& + 4i[X^\mu,\psi_{\mu+}]\xi_- + 4i[X^\mu,\psi_{\mu-}]\xi_+ - 2[C,X^\mu][C,X_\mu] \\
& - R^2 D_+CD_-C - iR^2\xi_-D_+\xi_- - iR^2\xi_+D_-\xi_+ + 2iR^2\xi_+[C,\xi_-] - \frac{R^2}{4}F_{+-}^2 \\
& - \frac{1}{4R^2}\left([X_\mu,X_\nu] + R^2[B_{\mu\rho},B_\nu{}^\rho]\right)^2 - \frac{1}{2}[B_{\mu\nu},X^\nu]^2 + R^2D_+B^{\mu\nu}D_-B_{\mu\nu} \\
& + iR^2\chi_-^{\mu\nu}D_+\chi_{\mu\nu-} + iR^2\chi_+^{\mu\nu}D_-\chi_{\mu\nu+} + 2iR^2[B_{\mu\nu},\chi_+^{\mu\nu}]\xi_- + 2iR^2[B_{\mu\nu},\chi_-^{\mu\nu}]\xi_+ \\
& + 2iR^2[C,\chi_+^{\mu\nu}]\chi_{\mu\nu-} - R^2[C,B^{\mu\nu}][C,B_{\mu\nu}] - 2iR^2B^{\mu\nu}[\chi_{\mu\rho+},\chi_{\nu-}{}^\rho] \\
& - 2iB_{\mu\nu}[\psi_+^\mu,\psi_-^\nu] + 2i\chi_-^{\mu\nu}[X_\mu,\psi_{\nu+}] - 2i\chi_+^{\mu\nu}[X_\mu,\psi_{\nu-}]\Big).
\end{aligned}
\tag{4.13}
$$

The resulting action is $Q_\pm$ invariant after modifying (4.4) and (4.7) using the replacement (4.12).

4.2. The Free RNS string Limit

As a TFT, the path integral is localized to the fixed point locus of global supersymmetry generated by $Q_\pm$. From (4.5), we can read off one important equations, the fixed point equation $Q_\pm\xi_\mp = 0$;

$$F_{+-} = 0. \tag{4.14}$$

Thus the path integral is always localized to the moduli space of flat $U(N)$ connections. This will significantly simplify our analysis since the connection can be gauged away. Consider a Wilson line for the flat connection

$$U_\gamma = P\exp\int_{\sigma_0}^{\sigma_0+2\pi} A_\sigma d\sigma, \tag{4.15}$$

which can be non-trivial. Associating a Wilson line $\gamma \to U_\gamma$ to a non-contractable loop γ defines a homomorphism $\pi_1(S^1) \to U(N)$, since the parallel transformation along γ depends only on the homotopy class of ρ. Conversely a homomorphism (or representation) $\rho : \pi_1(S^1) \to U(N)$ determines a rank N flat vector bundle E. Thus the moduli space of flat connections can be identified with the representation variety modulo isomorphisms. Of course $\pi_1(S^1) = \mathbb{Z}$, as paths are classified by their winding number. A representation (a $U(N)$ connection) can be either irreducible or reducible. In the latter case the vector bundle decomposes into irreducible factors,

$$E = E_1 \oplus \cdots \oplus E_k. \tag{4.16}$$

Of course such a decomposition is parameterized by the partition $N = \sum \nu_k$ of the rank of the gauge group. Equivalently, non-trivial Wilson lines break the $U(N)$ symmetry ($U(N)$ is broken down to the subgroup that commutes with U_γ).

The other important fixed point equations, $Q_\pm \chi_\mp^{\mu\nu} = 0$, lead to the flat directions

$$[X_\mu, X_\nu] + R^2[B_{\mu\rho}, B_\nu{}^\rho] = 0, \qquad [B_{\mu\nu}, X^\nu] = 0, \qquad [C, X^\mu] = [C, B^{\mu\nu}] = 0. \tag{4.17}$$

We can also examine the equations for fermionic zero-modes from the action functional. By standard arguments in TFT, we see that those equations are nothing but the linearization of the fixed point equations and the Coulomb gauge conditions. Since our model is a BTFT, we do not have net U-number violation in the path integral measure. In the present context the U-number symmetry is just a part of "world-sheet" Lorentz invariance.

In the limit $R^2 = 0$ we get the desired equations $[X_\mu, X_\nu] = 0$. This corresponds to the free RNS string limit. All the R^2 dependent terms can be thrown away and the theory is localized to configurations of commuting matrices. Our balanced equivariant cohomology generators $Q_\pm$ can be identified with the left and right "world-sheet" supersymmetry.

We can rewrite the action functional S defined in (4.9) as a one-parameter family of BTFT's

$$S(R) = -Q_+Q_- \left(\int d\sigma^+ d\sigma^- \left(iB^{\mu\nu}[X_\mu, X_\nu] + 2\psi_+^\mu \psi_{\mu-} \right) \right) + R^2 Q_+Q_- \left(\int d\sigma^+ d\sigma^- \mathcal{V} \right), \tag{4.18}$$

where

$$\mathcal{V} = -\mathrm{Tr} \left(iB^{\mu\nu}[B_{\mu\rho}, B_\nu{}^\sigma] + \chi_+^{\mu\nu} \chi_{\mu\nu-} + \xi_- \xi_+ \right). \tag{4.19}$$

We can regard $S(0)$ as the action functional of N copies of the free RNS string, given by

$$\begin{aligned} S(0) = \int d\sigma^+ d\sigma^- \mathrm{Tr} \Big(& 2D_+X^\mu D_-X_\mu + 2i\psi_-^\mu D_+\psi_{\mu-} + 2i\psi_+^\mu D_-\psi_{\mu+} + 4i[C, \psi_+^\mu]\psi_{\mu-} \\ & + 4i[X^\mu, \psi_{\mu+}]\xi_- + 4i[X^\mu, \psi_{\mu-}]\xi_+ - 2[C, X^\mu][C, X_\mu] + 2H^\mu H_\mu + 2H_\mu[B^{\mu\nu}, X_\nu] \\ & - H^{\mu\nu}[X_\mu, X_\nu] - 2iB_{\mu\nu}[\psi_+^\mu, \psi_-^\nu] + 2i\chi_-^{\mu\nu}[X_\mu, \psi_{\nu+}] - 2i\chi_+^{\mu\nu}[X_\mu, \psi_{\nu-}] \Big). \end{aligned} \tag{4.20}$$

Here the anti-symmetric tensor multiplets are treated as purely auxiliary fields. The integration over $H_{\mu\nu}$ gives the delta-function like constraints

$$[X_\mu, X_\nu] = 0, \tag{4.21}$$

so that our string coordinates $\{X^\mu\}$ commute. The $\chi_\pm^{\mu\nu}$ integration give further delta function constraints

$$[X_\mu, \psi_{\pm\nu}] = 0, \tag{4.22}$$

which are the linearizations of (4.21) We can also treat $(C, \xi_\pm)$ in a similar way. The $\xi_\pm$ integrations give another delta-function gauge constraint

$$[X^\mu, \psi_{\mu\pm}] = 0. \tag{4.23}$$

Finally the $B_{\mu\nu}$ together with C integrations give the constraints

$$[\psi_+^\mu, \psi_-^\nu] = 0, \qquad [H^\mu, X^\nu] = 0. \tag{4.24}$$

The constraints (4.22) and (4.23) are the linearization of (4.21) and the Coulomb gauge conditions respectively. The last condition (4.24) is just the consistency condition.

¿From (4.21), we see that $U(N)$ symmetry is generically broken down to $U(1)^N$. Furthermore, the fixed point equations $Q_\pm \psi_\pm^\mu = -iD_\pm X^\mu = 0$ imply that the $U(N)$ connections should be reducible to have non-trivial solutions.[6] We can conclude that the action $S(0)$ is the straightforward formulation of a gas of free RNS string. In this formulation the off-diagonal part of X^μ plays almost no role, except giving rise to a one-loop determinant from to the localization.

We can gauge away our connection provided that we allow modified periodicity conditions

$$X^\mu(\tau, \sigma_0 + 2\pi) = U_\gamma X^\mu(\tau, \sigma_0) U_\gamma^{-1}. \tag{4.25}$$

We can diagonalize $X^\mu = U x^\mu U^{-1}$. Then the above action of the Wilson line can be identified with conjugation $h x^\mu h^{-1}$ of the Weyl group h on the eigenvalues x^μ of X^μ. Equivalently twisted sectors are parameterized by the moduli space of flat connections.

Now we can follow the general arguments of DVV to interprete our model as second-quantized free-string theory [10]. As far as the bosonic fields are concerned, their arguments essentialy apply also to our model. The fermionic fields are much more difficult to treat. Especially the GSO projection we now need to impose gives some difficulties.

[6] Note that the X^μ are adjoint scalars.

4.3. Monad String as a Deformation of the RNS String Gas

Now we can regard $S(R)$ as a deformation of $S(0)$ parameterized by R. Naively, such a deformation does not change the theory since it is a pure $Q_\pm$ commutator. However, the theory with $S(R)$ is only independent of R, as clarified by Witten [21], if (i) $S(R)$ has a non-degenerate kinetic energy for all R; (ii) if there are no new fixed point to flow in from infinity. Our choice of $\mathcal{V}$ clearly does not satisfy the above criterium. Turning on R introduce the kinetic terms for the anti-symmetric tensor multiplets (via $\mathrm{Tr}\,(\chi_+^{\mu\nu}\chi^{\mu\nu})$) and the gauge multiplets (via $\mathrm{Tr}\,(\xi_+\xi_-)$), as well as extra potential terms and Yukawa couplings for $B_{\mu\nu}$ (via $\mathrm{Tr}\,(B^{\mu\nu}[B_{\mu\rho}, B_\nu{}^\rho])$). Furthermore it changes the fixed points (4.21) via the cubic term in $B^{\mu\nu}$. Then the off-diagonal parts of X^μ will start to play an important role due to the cubic term.

The above discussions also indicate that our construction of the monad string is very natural, once we choose to generalize the string coordinates to matrices. We will see that it also directing us a more general theory with 11-dimensional covariance. It is also more natural to regard the theory with $S(0)$ as a special limit of more fundamental as general theory with $S(R)$. Thus we can interpret the (ten-dimensional) monad string theory as an one-parameter family of theories, which reduces to the RNS string in a special limit.

Remark that the terms in the action arising from $\mathcal{V}$ in (4.19) lead to a well defined theory already by themselves, but only for the fields from the tensor and gauge multiplets. Really a similar action will be the starting point in the next section.

4.4. A Brief Comparison with the Matrix String Theory

At this point, it will be usefull to compare with matrix string theory. For example we can regard $S(0)$ as the covariant RNS version of the free string limit of matrix string theory. In matrix string theory, as beautifully demonstrated by DVV [10], the inverse of the two-dimensional Yang-Mill coupling constant plays the role of type IIA string coupling constant.[7] In the monad string a similar role is given by R^2. However, there are some differences.

i) We will see that turning on R^2 directly leads to a theory with 11-dimensional covariance. In the matrix string theory the relation with 11-dimensional theory is less direct.

ii) Turning on R^2 implies that the free monad strings start to couple with the anti-symmetric tensor multiplets, which are dynamical. In matrix string theory only the off-diagonal parts of X^μ are new contributions.

[7] Note that matrix string theory is two-dimensional $\mathcal{N} = 8$ physical super-Yang-Mills theory. One the other hand, monad string is a TFT in two-dimension and the Yang-Mill coupling play no role. Note also that monad string is not a twisted version of matrix string.

Considering the fact that matrix string theory is defined in the light-cone gauge, it is certainly possible that monad string theory in the light-cone gauge is equivalent the matrix string theory.[8] We should also be very careful about the role of anti-symmetric tensor multiplets, which is absent in matrix string theory. Note that the Yukawa and potential terms are closely related. We introduced the anti-symmetric tensor multiplet to have the necesssary potential term while maintaining 10-dimensional covariance. In the matrix string (and in the light-cone GS formalism) the counterparts of $\psi^{\mu}_{\pm}$ transform as space-time spinors. Thus the covariant (at least in the light-cone gauge) form of Yukawa coupling can be easily written down with the help of the soldering form $\gamma^{i}_{a\dot{a}}$. The appearance of those crucial central charges in the superalgebra is also due to the space-time gamma matrices [22]. Clearly the anti-symmetric tensor multiplet plays a similar role in our model. Thus, it seems to be reasonable to believe that the anti-symmetric tensor multiplet is the cost for a world-sheet supersymmetric formulation and 10-dimensional covariance. On the other hand, we will see that the anti-symmetric tensor multiplet is very important and more fundamental in the 11-dimensional viewpoints.

In the next subsection, we will briefly examine a possible interpretation of the anti-symmetric tensor multiplet from in the ten-dimensional view-point.

4.5. Another Perturbation

As shown earlier, the anti-symmetric tensor multiplet can be regarded as purely auxiliary fields as long as we set $R = 0$. However, we have seemingly mysterious equations

$$[X_{\mu}, B^{\mu\nu}] = 0, \tag{4.26}$$

from (4.20) even in the free string limit. We also note that free monad string theory can not be defined without the $B^{\mu\nu}$-multiplet. However, we can define a free theory of $B^{\mu\nu}$ without the help of the X^{μ}-multiplet. The action functional can be defined as

$$S_B = -Q_+Q_- \int d\sigma^+ d\sigma^- \mathcal{V}, \tag{4.27}$$

where $\mathcal{V}$ is given by (4.19). This can be regarded as a clue that something described by the anti-symmetric tensor multiplet is more fundamental than string itself.

Now we will consider yet another deformation. We consider

$$S(R, m) = S(R) - mQ_+Q_- \int d\sigma^+ d\sigma^- \mathrm{Tr}\left(\frac{i}{2} B^{\mu\nu} B_{\mu\nu}\right), \tag{4.28}$$

[8] This point was suggested to us by H. Verlinde.

where m plays the role of a bare mass for the anti-symmetric tensor multiplet. We have

$$S(R,m) = \int \mathrm{Tr}\, \left(R^2 H^{\mu\nu} H_{\mu\nu} - H^{\mu\nu} \left([X_\mu, X_\nu] + R^2 [B_{\mu\rho}, B_\nu{}^\rho] - m B_{\mu\nu}\right) + im\chi_+^{\mu\nu}\chi_{\mu\nu-}\right) \\ + \ldots . \tag{4.29}$$

We can eliminate $H^{\mu\nu}$ by setting

$$H_{\mu\nu} = \frac{1}{2}\bigg([X_\mu, X_\nu] + R^2[B_{\mu\rho}, B_\nu{}^\rho] - mB_{\mu\nu}\bigg), \tag{4.30}$$

we get

$$\begin{aligned} S'(R,m) =& S'(R) + \int d^2\sigma \mathrm{Tr}\, \left(\frac{m}{2} B^{\mu\nu} \left([X_\mu, X_\nu] + R^2[B_{\mu\rho}, B_\nu{}^\rho]\right) + im\chi_+^{\mu\nu}\chi_{\mu\nu-}\right) \\ & - \frac{m^2}{4}\int d^2\sigma \mathrm{Tr}\, \left(B^{\mu\nu} B_{\mu\nu}\right), \end{aligned} \tag{4.31}$$

where $S'(R)$ is given by (4.13). It is also understood that we integrated out H^μ by setting

$$H_\mu = -[B_{\mu\nu}, X^\nu]. \tag{4.32}$$

This simple looking perturbation is very interesting. The theory is localized to the flat directions given by $H_{\mu\nu} = H^\mu = 0$;

$$\begin{aligned} [X_\mu, X_\nu] + R^2[B_{\mu\rho}, B_\nu{}^\rho] - mB_{\mu\nu} = 0, \\ [X^\mu, B_{\mu\nu}] = 0, \end{aligned} \tag{4.33}$$

Combining these equations, we also have

$$\frac{1}{m}[X^\mu, [X_\mu, X_\nu]] + \frac{R^2}{m}[X^\mu, [B_{\mu\rho}, B_\nu{}^\rho]] = 0. \tag{4.34}$$

Now we will consider two examples.

i) Consider a particular sector of our moduli space such that $\{X^\mu\}$ commutes with each others. We have

$$\begin{aligned} [X_\mu, X_\nu] &= 0, \\ [X^\mu, B_{\mu\nu}] &= 0, \\ [B_{\mu\rho}, B_\nu{}^\rho] &= \frac{m}{R^2} B_{\mu\nu}. \end{aligned} \tag{4.35}$$

ii) For $R = 0$ we can eliminate $B^{\mu\nu}$ from $S(0,m)$ by setting

$$B_{\mu\nu} = \frac{1}{m}[X_\mu, X_\nu]. \tag{4.36}$$

Then the flat direction is given by

$$[X^\mu, [X_\mu, X_\nu]] = 0. \tag{4.37}$$

We can also eliminate $\chi^{\mu\nu}_\pm$ by the simple algebraic equation of motion. Then the action functional $S(0,m)$ can be written as

$$S(0,m) = -\frac{1}{2}(Q_+Q_- - Q_-Q_+)\left(\int d\sigma^+ d\sigma^- \mathrm{Tr}\left(-\frac{1}{2m}[X^\mu, X_\nu][X_\mu, X_\nu] + \psi^\mu_+\psi_{\mu-}\right)\right). \tag{4.38}$$

where the replacement of H^μ in (4.4) with (4.32) is understood. For finite N $S(0,m)$ is equivalent to the unperturbed theory $S(0,0)$. For $N \to \infty$ and if we want to change the commutators to Poisson brackets we will have higher critical points in (4.37).

With an analogy to the matrix theory, the relation (4.36) seems to suggest that $B_{\mu\nu}$ is somehow related to membrane.

5. Eleven Dimensional Covariance

In this section we will provide a more fundamental description. The starting point is an observation that the two multiplets $(X^\mu, \psi^\mu_\pm, H^\mu)$ and $(B^{\mu\nu}, \chi^{\mu\nu}_\pm, H^{\mu\nu})$ can be naturally combined into a single multiplet, which transform as an anti-symmetric second rank tensor under $SO(10,1)$. We will suggest that the resulting theory is a formulation of the sought for M theory.

5.1. The Algebra

Let Σ be a $(1+1)$-dimensional cylinder $S^1 \times R$ with light-cone coordinates $\sigma^\pm = \frac{1}{2}(\sigma \pm \tau)$. Let $I, J, K, L = 0, 1, \ldots, 10$. We introduce an adjoint "world-sheet" scalar field $B^{IJ}(\sigma^+, \sigma^-)$, which transforms as an anti-symmetric second rank tensor under $SO(10,1)$. We denote by g^{IJ} the usual Minkowski metric in $\mathbb{R}^{10,1}$. The $U(N)$ local gauge symmetry acts on B^{IJ} as

$$B^{IJ} \to gB^{IJ}g^{-1}, \qquad g : U(N) \to \Sigma. \tag{5.1}$$

In the space of all fields B^{IJ} we introduce a natural gauge invariant and $SO(10,1)$ invariant metric

$$|\delta B|^2 = \int d\sigma^+ d\sigma^- \mathrm{Tr}\left(B^{IJ}B_{IJ}\right). \tag{5.2}$$

Although the direct geometrical interpretation is obscure, we will still refer to B^{IJ} as the "space-time" coordinates of "strings" in eleven-dimensions. With the above basic setting, we will construct the unique theory in an unbroken phase of the 11-dimensional covariance.

Note that we are not imposing the "space-time" super-Poincaré symmetry. The $\mathcal{G}\times P_{\sigma^\pm}$-equivariant cohomology algebra is given by

$$\begin{array}{ll} & Q_+\chi_+^{IJ}=-D_+B^{IJ},\\ Q_+B^{IJ}=i\chi_+^{IJ}, & Q_+\chi_-^{IJ}=+H^{IJ}-[C,B^{IJ}],\\ Q_-B^{IJ}=i\chi_-^{IJ}, & Q_-\chi_+^{IJ}=-H^{IJ}-[C,B^{IJ}],\\ & Q_-\chi_-^{IJ}=-D_-B^{IJ}, \end{array} \tag{5.3}$$

satisfying the commutation relations

$$Q_\pm^2 B^{IJ}=-i\partial_\pm B^{IJ}-i[\phi_{\pm\pm},B^{IJ}],\qquad \{Q_+,Q_-\}B^{IJ}=-i[C,B^{IJ}]. \tag{5.4}$$

We can interpret $i\chi_\pm^{IJ}$ as ghosts associated with the symmetry under arbitrary shift $B^{IJ}\rightarrow B^{IJ}+\delta B^{IJ}$. As usual $Q_\pm^2=0$ modulo the gauge transformation generated by $\phi_{\pm\pm}$ as well as the "world-sheet" translation along the $\sigma^\pm$ directions, i.e., modulo the redundancy of our system. $\{Q_+,Q_-\}=0$ modulo gauge transformation generated by $\phi_{+-}=C$. As earlier $\phi_\pm$ are the left and right components of an $U(N)$ connection and C is an adjoint scalar on the "world-sheet" with the $Q_\pm$ algebra given by (4.5). The auxiliary fields H^{IJ} transform as

$$\begin{aligned} Q_+H^{IJ}&=-iD_+\chi_-^{IJ}+i[C,\chi_+^{IJ}]+i[\xi_+,B^{IJ}],\\ Q_-H^{IJ}&=+iD_-\chi_+^{IJ}-i[C,\chi_-^{IJ}]-i[\xi_-,B^{IJ}]. \end{aligned} \tag{5.5}$$

We have sl_2 symmetry and an associated additive quantum number U of the above algebra, which can be summarized as usual

$$\begin{array}{c} U=+1\\[2ex] U=0\\[2ex] U=-1 \end{array}\qquad \text{fields}\quad \begin{tikzcd} & \chi_+^{IJ} \arrow[dr] & \\ B^{IJ} \arrow[ur] \arrow[dr] & & H^{IJ}, \\ & \chi_-^{IJ} \arrow[ur] & \end{tikzcd} \tag{5.6}$$

We will call the above multiplet the 11-dimensional anti-symmetric tensor multiplet.

5.2. The Action Functional

Now we define an almost unique $SO(10,1)$ and sl_2 as well as gauge invariant action functional by

$$S_{11}=-Q_+Q_-\int d^+\sigma d\sigma^-\mathrm{Tr}\left(\frac{i}{3}B^{IJ}[B_{IK},B_J{}^K]+\chi_+^{IJ}\chi_{IJ-}+\xi_-\xi_+\right). \tag{5.7}$$

The global $SO(10,1)$ and sl_2 symmetries may be interpreted as the "space-time" and "world-sheet" Lorentz symmetries, respectively. One can regard the above action functional as a BRST quantized version of an underlying theory with unbroken 11-dimensional general covariance. We have

$$\begin{aligned} S_{11} = \int d\sigma^+ d\sigma^- \mathrm{Tr} \Big(& D_+ B^{IJ} D_- B^{IJ} + i\chi_-^{IJ} D_+ \chi_{IJ-} + i\chi_+^{IJ} D_- \chi_{IJ+} + 2i[C, \chi_+^{IJ}]\chi_{IJ-} \\ & + 2i[B_{IJ}, \chi_+^{IJ}]\xi_- + 2i[B_{IJ}, \chi_-^{IJ}]\xi_+ - [C, B^{IJ}][C, B_{IJ}] + H^{IJ} H_{IJ} \\ & - H^{IJ}[B_{IK}, B_J{}^K] - 2iB^{IJ}[\chi_{IK+}, \chi_{J-}^K] - D_+ C D_- C \\ & - i\xi_- D_+ \xi_- - i\xi_+ D_- \xi_+ + 2i\xi_+[C, \xi_-] - \frac{1}{4}F_{+-}^2 \Big). \end{aligned} \tag{5.8}$$

Integrating out H^{IJ} by setting

$$H_{IJ} = \frac{1}{2}[B_{IK}, B_J{}^K], \tag{5.9}$$

we have

$$\begin{aligned} S_{11} = \int d\sigma^+ d\sigma^- \mathrm{Tr} \Big(& D_+ B^{IJ} D_- B^{IJ} + i\chi_-^{IJ} D_+ \chi_{IJ-} + i\chi_+^{IJ} D_- \chi_{IJ+} - D_+ C D_- C \\ & - i\xi_- D_+ \xi_- - i\xi_+ D_- \xi_+ + 2i[B_{IJ}, \chi_+^{IJ}]\xi_- + 2i[B_{IJ}, \chi_-^{IJ}]\xi_+ + 2i[C, \chi_+^{IJ}]\chi_{IJ-} \\ & + 2i\xi_+[C, \xi_-] - 2iB^{IJ}[\chi_{IK+}, \chi_{J-}{}^K] - \frac{1}{4}[B^{IK}, B_{JK}][B_{IL}, B^{JL}] \\ & - [C, B^{IJ}][C, B_{IJ}] - \frac{1}{4}F_{+-}^2 \Big). \end{aligned} \tag{5.10}$$

Now the transformation law (5.3) should be changed to

$$\begin{aligned} Q_+ \chi_-^{IJ} &= +\frac{1}{2}[B^{IK}, B^J{}_K] - [C, B^{IJ}], \\ Q_- \chi_+^{IJ} &= -\frac{1}{2}[B^{IK}, B^J{}_K] - [C, B^{IJ}], \end{aligned} \tag{5.11}$$

The above modification preserves our commutation relations, i.e., $Q_\pm^2 \chi_\mp^{IJ} = -iD_\pm \chi_\mp^{IJ}$, provided that

$$\begin{aligned} iD_- \chi_+^{IJ} - i[B^{IK}, \chi_{K-}^J] - i[C, \chi_-^{IJ}] - i[\xi_-, B^{IJ}] = 0, \\ iD_+ \chi_-^{IJ} + i[B^{IK}, \chi_{K+}^J] - i[C, \chi_+^{IJ}] - i[\xi_+, B^{IJ}] = 0, \end{aligned} \tag{5.12}$$

which are just the equations of motion of $\chi_\pm^{IJ}$. The B^{IJ} equation of motion is

$$\frac{i}{2}D_+ D_- B^{IJ} - \frac{i}{2}[B^{IK}, [B_{KL}, B^{JL}]] + [\chi_+^{IK}, \chi_{J-}^K] + [\xi_+, \chi_-^{IJ}] + [\xi_-, \chi_+^{IJ}] = 0, \tag{5.13}$$

which is a supersymmetry variation of (5.12).

5.3. Back to Ten Dimensions

Now we will break the eleven-dimensional covariance down to the ten-dimensional one. From now on we will label the $SO(10,1)$ vector indeces $I, J, K, L = 1, \ldots, 11$. We fix the 11-dimensional metric $g^{IJ} = \begin{pmatrix} g^{\mu\nu} & 0 \\ 0 & g^{11\,11} \end{pmatrix}$ with $g^{11\,11} = 1/R^2$. Then we define a 10-dimensional (non-commutative) coordinate by $X^\mu = B^{11\mu}$. Similarly, we set $\psi^\mu_\pm = \chi^{11\mu}_\pm$ and $H^\mu = H^{11\mu}$. The supersymmetry algebras (4.3), (4.4), (4.6), (4.7), and (4.8) follow from (5.3) and (5.5). Then the action (5.7) reduces to

$$
\begin{aligned}
S_R = -Q_+Q_- \int d^2\sigma \mathrm{Tr} \Big(iB^{\mu\nu} \Big([X_\mu, X_\nu] + \frac{R^2}{3} [B_{\mu\rho}, B_\nu{}^\rho] \Big) \\
+ R^2 \chi^{\mu\nu}_+ \chi_{\mu\nu -} + 2\psi^\mu_+ \psi_{\mu -} + R^2 \xi_- \xi_+ \Big),
\end{aligned}
\tag{5.14}
$$

where we scaled the action by an overall factor R^2. The above action is exactly the same as (4.9), for which the explicit form is given by (4.11) and (4.13). As discussed in Sect. 4.3 and 4.4, perturbation away from free strings directly lead us the eleven dimensional picture.

At first sight the free string limit looks counter intuitive. It corresponds to the infinite radius $1/R$ of 11th direction of the background 11-dimensional space. Furthermore, it is natural to identify the string coupling constant with R^2. Since however we do not have 11-dimensional string coordinates, the above problem should be reexamined.

First we need to clarify our usage of "compactification" to a circle. For practical purpose the equation (5.14) is the definition of compactification on a circle of our eleven dimensional theory (5.7). In any "world-sheet" formulation of superstring theory, the space-time Lorentz invariance is detected by global symmetry. In terms of $SO(9,1)$ acting on indices $\mu, \nu = 1, \ldots, 10$, $B^{11\mu}$ transform as a vector and $B^{\mu\nu}$ transforms as an anti-symmetric tensor. Since we do not have the 11-th component of the vector (or string coordinates), we can not impose any other conditions apart from the form of the background metric. As for 10-dimensional vectors $B^{11\mu}(\sigma, \tau)$ we may use those as certain "string coordinates" in "space-time". From our viewpoint, any space-time interpretation is just an effective description. The most reasonable description of the model defined by (5.14) is to regard it as a family of theories parameterized by R^2.

Note that we have manifest 11-dimensional covariance. However, we do not have the usual coordinate interpretation of 11-dimensional space-time. Only after the reduction to 10 dimensions we get (non-commutative) coordinates of strings. Now the most difficult question is if our model has a particular limit where an 11-dimensional space-time picture appears. Provided that our model describes M theory, we should certainly expect this [3]. Finding the free string theory as an effective description is very easy in our approach.

However, the appearance of 11-dimensional supergravity can be a very difficult quantum mechanical problem. At this point, we will just leave the difficult problems for the future.

Our approach has another difficult problem. Up to now we did not worry about the GSO projection. It is of no doubt that we need the GSO projection in the free string limit. We expect that the quantum mechanical consistency of our model may determine a particular projection for a particular free string limit. We do not know any direct justification for the above wishful thinking. In the next subsection, we will study our model after compactifying further down to a circle. We will show that our model has two types of string limits, which behave as type IIA and type IIB strings, as well as the predictions based on M theory viewpoints [23]. We may use the examples as the evidence for that our model after proper quantization automatically decide a particular GSO projection at a particular limits.

5.4. A Further Compactification

We can compactify our model further. We will now study the model when compactified on a T^2 in the $10-11$ direction. The background metric is given by

$$g^{IJ} = \begin{pmatrix} g^{ij} & 0 & 0 \\ 0 & \frac{1}{R_{10}^2} & 0 \\ 0 & 0 & \frac{1}{R_{11}^2} \end{pmatrix} \tag{5.15}$$

The index i will refer to the first 9 uncompactified directions. Then we have 2 "space-time coordinates" of strings instead of the one X^i from the last section. These we denote

$$X^i_{(11)} = B^{11i}, \qquad X^i_{(10)} = B^{10i}. \tag{5.16}$$

They will have superpartners $\psi^i_{(1)\pm}$ and $\psi^i_{(2)\pm}$ respectively. Furthermore there is a 9 dimensional scalar $\phi = B^{10\,11}$ with superpartners $\theta_\pm$. We can summarize the supersymmetry by the following diagram

$$\begin{array}{ccccccccccccccc}
 & & \psi^i_{(a)+} & & & & & & \chi^{ij}_+ & & & & & \theta_+ & & \\
 & \nearrow & & \searrow & & & & \nearrow & & \searrow & & & \nearrow & & \searrow & \\
X^i_{(a)} & & & & H^i_{(a)}\,, & & B^{ij} & & & & H^{ij}\,, & \phi & & & & H\,. \\
 & \searrow & & \nearrow & & & & \searrow & & \nearrow & & & \searrow & & \nearrow & \\
 & & \psi^i_{(a)-} & & & & & & \chi^{ij}_- & & & & & \theta_- & &
\end{array} \tag{5.17}$$

Note that we can combine the ϕ-multiplet either with the $X^i_{(11)}$ or with $X^i_{(10)}$ multiplet to get ten-dimensional multiplets $X^\mu_{(11)}$ and $X^\mu_{(10)}$, respectively, at the decompactification limit of one of the directions.

The action will then depend on the parameters R_{10} and R_{11} as

$$S_9 = -Q_+Q_-\int d^2\sigma \mathrm{Tr}\left(2R_{10}^2\psi^i_{(11)+}\psi^{(11)}_{i-} + 2R_{11}^2\psi^i_{(10)+}\psi^{(10)}_{i-} + (R_{10}R_{11})^2\chi^{ij}_+\chi_{ij-}\right.$$
$$+ iB_{ij}\left(R_{10}^2[X^i_{(11)}, X^j_{(11)}] + R_{11}^2[X^i_{(10)}, X^j_{(10)}] + \frac{(R_{10}R_{11})^2}{3}[B_{ik}, B_j{}^k]\right) \tag{5.18}$$
$$\left. + 4i\phi[X^i_{(11)}, X^{(10)}_i] + \theta_+\theta_- + (R_{10}R_{11})^2\xi_-\xi_+\right).$$

This action has one obvious symmetry by exchanging the first and the second 9-dimensional vector multiplets accompanied with $R_{10} \leftrightarrow R_{11}$. This symmetry came from the underlying eleven-dimensional covariance of our model.

We can regard the action $S(R_{10}, R_{11})$ as describing the two-parameter family of theories. To explore the moduli space we consider the potential term

$$V_{11} = \frac{1}{4}\mathrm{Tr}\left([B^{IK}, B_{JK}][B_{IL}, B^{JL}]\right), \tag{5.19}$$

and the flat directions $V_{11} = 0$. In nine dimensions we have

$$\begin{aligned} V_9 =& \mathrm{Tr}\left(\frac{1}{4}\left(\frac{R_{11}}{R_{10}}[X^{(10)}_i, X^{(10)}_j] + \frac{R_{10}}{R_{11}}[X^{(11)}_i, X^{(11)}_j] + R_{10}R_{11}[B^{ik}, B_{jk}]\right)^2\right. \\ &+ \frac{1}{2}\left(\frac{1}{R_{11}}[\phi, X^{(11)}_i] - R_{11}[B_{ik}, X^k_{(10)}]\right)^2 + \frac{1}{2}\left(\frac{1}{R_{10}}[\phi, X^{(10)}_i] + R_{10}[B_{ik}, X^k_{(11)}]\right)^2 \\ &\left. + \frac{1}{2}[X^i_{(11)}, X^{(10)}_i]^2\right), \end{aligned} \tag{5.20}$$

leading to the following flat directions

$$\begin{aligned} \frac{R_{11}}{R_{10}}[X^{(10)}_i, X^{(10)}_j] + \frac{R_{10}}{R_{11}}[X^{(11)}_i, X^{(11)}_j] + R_{10}R_{11}[B^{ik}, B_{jk}] &= 0, \\ \frac{1}{R_{11}}[\phi, X^{(11)}_i] - R_{11}[B_{ik}, X^k_{(10)}] &= 0, \\ \frac{1}{R_{10}}[\phi, X^{(10)}_i] + R_{10}[B_{ik}, X^k_{(11)}] &= 0, \\ [X^i_{(11)}, X^{(10)}_i] &= 0. \end{aligned} \tag{5.21}$$

Now we examine special points in our moduli space where the usual string pictures appears.

1) We consider the limit $R_{11} = 0$ and $R_{10} \to \infty$. This reduces to the free string limit discussed in the previous subsection. From the second equation in (5.21), we see that ϕ

commutes with $X^i_{(11)}$ to form the string coordinates represented by $X^\mu_{(11)}$. Similarly in the limit $R_{11}=\infty$ and $R_{10}=0$, we get another ten-dimensional strings with coordinates $X^\mu_{(10)}$. In both cases the B_{ij}-multiplets are completely decoupled. Those should correspond to the limits for two equivalent IIA strings.

2) Now we consider limit that $R_{10}, R_{11}\to 0$ while taking R_{10}/R_{11} arbitrary. Then our crucial equations (5.21) reduce to

$$
\begin{aligned}
\frac{R_{11}}{R_{10}}[X_i^{(10)},X_j^{(10)}]+\frac{R_{10}}{R_{11}}[X_i^{(11)},X_j^{(11)}]&=0,\\
[\phi,X_i^{(11)}]&=0,\\
[\phi,X_i^{(10)}]&=0,\\
[X^i_{(11)},X_i^{(10)}]&=0.
\end{aligned}
\tag{5.22}
$$

Now ϕ commutes with both $X_i^{(11)}$ and $X_i^{(10)}$. But it is in either the $R_{10}/R_{11}\to 0$ or the $R_{10}/R_{11}\to 0$ limit that one of these coordinates describes free strings. The action functional is effectively given by

$$
\begin{aligned}
S_9=&-Q_+Q_-\int d^2\sigma \mathrm{Tr}\left(2R_{11}^2\psi^i_{(10)+}\psi_{i-}^{(10)}+2R_{10}^2\psi^i_{(11)+}\psi_{i-}^{(11)}+\theta_+\theta_-\right)\\
&+\frac{1}{4}\int d^2\sigma \mathrm{Tr}\left(\frac{R_{10}}{R_{11}}[X_i^{(10)},X_j^{(10)}]+\frac{R_{11}}{R_{10}}[X_i^{(11)},X_j^{(11)}]\right)^2\\
&+\frac{1}{2}\int d^2\sigma \mathrm{Tr}\left(\frac{1}{R_{11}^2}[\phi,X_i^{(11)}]^2+\frac{1}{R_{10}^2}[\phi,X_i^{(10)}]^2\right)\\
&+Yukawa
\end{aligned}
\tag{5.23}
$$

where the Yukawa term has a pattern similar to the potential term. It is also clear that one system is strongly coupled if the other is weakly coupled. Because of the obvious symmetry $X_i^{(10)}\leftrightarrow X_i^{(11)}$ and $R_{10}\leftrightarrow R_{11}$, we have manifestly self-dual system. Whatever system we are describing we find one with manifest and non-perturbative S-duality. These limits of our model should correspond to the type IIB strings. Perturbatively we will only see the usual string action, arising from only one of the sets of coordinates. But in general we find contributions from both of them, and the usual space-time interpretation breaks down.

It will be interesting to see how our approach can be generalized so to give rise to the heterotic and type I strings [24]. This may be done using procedures similar to those in matrix string theory [25].

5.5. A Further Generalization

As argued earlier, it seems that $B^{IJ}(\sigma,\tau)$ is related to the membrane of M theory. Here we merely refer to the membrane M theory as certain degrees of freedom which are required to produce the string theoretic degrees of freedom in lower dimensions after double compactification. ¿From the viewpoint of an observer living in the lower dimensions, certain components of B^{IJ} behave as "space-times coordinates" of strings.

We also expect to have the five-branes of M theory in 11-dimensions. Following the previous discussions, we mean by a five-brane in eleven dimensions an object which transforms as an anti-symmetric 5 rank tensor under $SO(10,1)$. After breaking the 11-dimensional covariance down to the 7-dimensional one, for example, it can be identified with the "space-time" coordinates of strings. We introduce a rank 5 anti-symmetric tensor $J^{IJKLM}(\sigma,\tau)$ which are "world-sheet" adjoint scalars. We have the usual supermultiplet

$$
\begin{array}{ccccc}
 & & \psi_+^{IJKLM} & & \\
 & \nearrow & & \searrow & \\
J^{IJKLM} & & & & H^{IJKLM}\,, \\
 & \searrow & & \nearrow & \\
 & & \psi_-^{IJKLM} & &
\end{array}
\tag{5.24}
$$

and the corresponding super-algebra.

Now we are looking for a cubic action potential term to write down the potential term for J^{IJKLM}. There is no $SO(10,1)$ invariant cubic terms for J^{IJKLM}. The only possibility is to couple with the cubic action potential of B^{IJ}. So we have a more or less unique choice as usual, given by

$$
\begin{aligned}
S_{11}(\beta) = -Q_+Q_- \int d\sigma^+ d\sigma^- \mathrm{Tr}\,\Bigg(&\frac{i}{3}B^{IJ}\Big([B_{IK},{B_J}^K] + 3\beta[J_{IKLMN},{J_J}^{KLMN}]\Big) \\
&+ \chi_+^{IJ}\chi_{IJ-} + \beta\psi_+^{IJKLM}\psi_{IJKLM-} + \xi_-\xi_+\Bigg),
\end{aligned}
\tag{5.25}
$$

where β is a new coupling constant.

If we write down the action explicitely, we have

$$
\begin{aligned}
S_{11}(\beta) = \int d\sigma^+ d\sigma^- \mathrm{Tr} \Big(& D_+ B^{IJ} D_- B_{IJ} + i\chi_-^{IJ} D_+ \chi_{IJ-} + i\chi_+^{IJ} D_- \chi_{IJ+} + 2i[C, \chi_+^{IJ}]\chi_{IJ-} \\
& + 2i[B_{IJ}, \chi_+^{IJ}]\xi_- + 2i[B_{IJ}, \chi_-^{IJ}]\xi_+ - [C, B^{IJ}]^2 + \beta D_+ J^{IJKLM} D_- J_{IJKLM} \\
& + i\beta \boldsymbol{\psi}_-^{IJKLM} D_+ \boldsymbol{\psi}_{IJKLM-} + i\beta \boldsymbol{\psi}_+^{IJ} D_- \boldsymbol{\psi}_{IJKLM+} + 2i\beta [C, \boldsymbol{\psi}_+^{IJKLM}] \boldsymbol{\psi}_{IJKLM-} \\
& + 2i\beta [J_{IJKLM}, \boldsymbol{\psi}_+^{IJKLM}]\xi_- + 2i\beta [J_{IJKLM}, \boldsymbol{\psi}_-^{IJKLM}]\xi_+ - \beta [C, J^{IJKLM}]^2 \\
& + H^{IJ} H_{IJ} - H^{IJ} \left([B_{IK}, B_J{}^K] + \beta [J_{IKLMN}, J_J{}^{KLMN}] \right) + \beta H^2_{IJKLM} \\
& + 2\beta H_{IKLMN} [B^{IJ}, J_J{}^{KLMN}] - 2i\beta B^{IJ} [\boldsymbol{\psi}_{IKLMN+}, J_{J-}{}^{KLMN}] \\
& + 2i\beta \chi_-^{IJ} [\boldsymbol{\psi}_{IKLMN+}, J_J{}^{KLMN}] - 2i\beta \chi_+^{IJ} [\boldsymbol{\psi}_{IKLMN-}, J_J{}^{KLMN}] \\
& - 2i B^{IJ} [\chi_{IK+}, \chi_{J-}{}^K] - D_+ C D_- C - i\xi_- D_+ \xi_- - i\xi_+ D_- \xi_+ \\
& + 2i\xi_+ [C, \xi_-] - \frac{1}{4} F_{+-}^2 \Big).
\end{aligned}
\tag{5.26}
$$

We integrate out H^{IJ} and H^{IJKLM} by setting

$$
\begin{aligned}
H_{IJ} &= \frac{1}{2}[B_{IK}, B_J{}^K] + \frac{\beta}{2}[J_{IKLMN}, J_J{}^{KLMN}], \\
H^{IKLMN} &= -[B^{IJ}, J_J{}^{KLMN}]
\end{aligned}
\tag{5.27}
$$

5.6. Universal Monads and M Theory

Now the two equations

$$
\begin{aligned}
[B_{IK}, B_J{}^K] + \beta [J_{IKLMN}, J_J{}^{KLMN}] &= 0, \\
[B^{IJ}, J_J{}^{KLMN}] &= 0,
\end{aligned}
\tag{5.28}
$$

which define the flat directions, are the most important equations we have. We will call a set of matrices (B, J) satifying (5.28) a *universal monad.* For a constant monad, we may associate a *universal instantons.* The equation (5.28) is the end point of our generalization of the simple matrix equations $[X^\mu, X^\nu] = 0$ describing point-like instantons in 10-dimensions.

Our model is classified by the space of solutions, modulo gauge equivalence, of (5.28). Our conjecture that we are describing M-theory means that the moduli space is identical to that of M theory. According to our conjecture all the information of strings and other extended objects should be encoded in (5.28). Furthermore, as the moduli space of theories, we should be able to find special points where the known string theories are the effective descriptions. We also expect the web of string dualities to be manifest as the symmetry in the bulk. After compactification to lower dimensions we will get a much richer structure of the moduli space. By examining the corresponding reduction of the equation (5.28) we should be able to find numerous theories and mutual relations with eachother.

The detailed examination of the entire moduli space defined by (5.28) is beyond the scope of this paper. We merely want to point out that the theory compactified on T^4 should be very interesting. It is the first dimension where some components of J^{IJKLM} transform as $SO(6,1)$ vectors which give rise to new a set the "space-time coordinates" of strings. Compactifying further down to T^5, we have 5-sets of string coordinates from B^{IJ} and another 5-sets of string coordinates from J^{IJKLM}. By examining the corresponding reduction of (5.28), which describe a 5-dimensional space of theories, we will be able to discover the various different theories. The mutual relations between those theories should follow from a very easy analysis. This may be related with new phenomena in M theory compactified on T^4 and T^5 [26].

6. Further Points to Examine

There are several important issues we ignored in our analysis. First of all, what is the underlying geometrical interpretation of $B^{IJ}(\sigma,\tau)$ and $J^{IJKLM}(\sigma,\tau)$? We already mentioned a possible connection with the membrane and fivebrane of M theory. How this relation comes about we do not know at the moment.

We restricted our attention to classical considerations. The quantization surely will introduce some delicate issues:

1) The spacetime supersymmetry and GSO projection; For the free string limit we certainly have spacetime supersymmetry. However it is not obvious that the spacetime supersymmetry is a generic property of our model in any situation. Our construction indicates that the spacetime interpretation itself is an effective description. Even in the free string limit we need to impose a GSO projection to obtain spacetime supersymmerty. Since the free strings are embbedded into a bigger picture in our model there should be a generalized notion of GSO projection. We speculated that a proper GSO projection could arise via certain quantum consistencies at a particular point in the moduli space. If our speculation is correct, the spacetime supersymmetry itself can be viewed as an effective description. These issues are closely related with the notions of the unbroken and broken phases of general covariance. According to a purely classical argument, our model should

not contain gravitons. However, we found special points corresponding to free strings where gravitons certainly exist. All these issues seem to be subtle and difficult quantum mechnical properties.

2) The critical dimension; It is only the 11-dimensional model, as constructed in this paper, that gives rise to crtitical strings. Since the string appears as an effective description, the usual notion of critical dimension can be meaningless. At least classically, our model can be formulated in arbitrary dimensions with arbitray field contents. Hopefully, the conditions for a consistent quantum theory lead us to the correct dimension for our model.

Acknowledgments

It is our pleasure to thank Erik and Herman Verlinde for illuminating discussions and comments as well as for encouragements. This work is supported by FOM and a pioneer fund of NWO.

References

[1] M. Green, J.H. Schwarz and E. Witten, *Superstring theory*, vol I,II, Cambridge Univ. Press 1987.

[2] C. Hull and P. K. Townsend, *Unity of superstring dualities*, Nucl. Phys. **B 438** (1995) 109, hep-th/9410167

[3] E. Witten, *String theory dynamics in various dimensions*, Nucl. Phys. **B 443** (1995) 85, hep-th/9503124.

[4] J.H. Schwarz, *The power of M theory*, Phys. Lett. **B 367** (1996) 97, hep-th/9510086

[5] J. Polchinski, *Dirichlet-branes and Ramond-Ramond charges*, hep-th/9510017

[6] E. Witten, *Bound states of strings and p-branes*, Nucl. Phys. **B 460** (1996) 335, hep-th/9510135.

[7] T. Banks, W. Fischler, S.H. Shenker, and L. Susskind, *M theory as a matrix model: a conjecture*, Phys. Rev. **D 55** (1997) 5112, hep-th/9610043.

[8] L. Motl, *Proposals on nonperturbative superstring interactions*, hep-th/9701025.

[9] T. Banks and N. Seiberg, *Strings from matrices*, hep-th/9702187.

[10] R. Dijkraaf, E. Verlinde, and H. Verlinde, *Matrix string theory*, hep-th/9703030; R. Dijkraaf, E. Verlinde, and H. Verlinde, *5D black holes and matrix strings*, hep-th/9704018.

[11] W. Taylor, *D-brane field theory on compact spaces*, Phys. Lett. **B 394** (1997) 283, hep-th/9611042

[12] M. Green and J.H. Schwarz, *Supersymmetric dual string theory*, Nucl. Phys. **B 181** (1981) 502; *Supersymmetrical string theories*, Phys. Lett. **B 109** (1982) 444.

[13] P. Ramond, *Dual theory for free fermions*, Phys. Rev. **D 3** (1971) 2415; A. Neveu and J.H. Schwarz, *Factorizable dual model of pions*, Nucl. Phys. **B 31** (1971) 86.

[14] E. Witten, *Topological quantum field theory*, Comm. Math. Phys. **117** (1988) 353; *Introduction to cohomological field theories*, Int. J. Mod. Phys. **A6** (1991) 2273; *Topological Sigma Models*, Comm. Math. Phys. **118** (1988) 411.

[15] F. Gliozzi, J. Sherk and D. Olive, *Supersymmetry, supergravity theories and the dual spinor model*, Nucl. Phys. **B 112** (1977) 253

[16] R. Dijkgraaf, G. Moore, *Balanced topological field theories*, hep-th/9608169

[17] C. Vafa and E. Witten, *A strong coupling test of S-duality*, Nucl. Phys. **B 431** (1994) 3, hep-th/9408074

[18] G. Horrocks, *Vector bundles on the punctured spectrum of a local ring*, Proc. London Math. Soc. **14** (1964) 684 ;
W. Barth, *Some properties of stable rank-2 vector bundles on P_n*, Invent. Math. **42** (1977) 63 ;
M. Atiyah, V. Drinfeld, N. Hitchin and Yu. Manin, *Construction of instantons*, Phys. Lett. **65A** (1978) 185 ;
S.K. Donaldson, *Instantons and geometrical invariant theory*, Comm. Math. Phys. **93** (453) 1984

[19] J.-S. Park, *Monads and D-instantons*, Nucl. Phys. **B 494** (1997) 198 hep-th/9612096.

[20] R. Dijkgraaf, J.-S. Park and B.J. Schroers, *Balanced topological Yang-Mills theory*, to appear.

[21] E. Witten, *Two dimensional gauge theories revisited*, J. Geom. Phys. **9** (1992) 303, hep-th/9204083.

[22] T. Banks and N. Seiberg, *Branes from matrices*, Nucl. Phys. **B 490** (1997) 91, hep-th/9612157

[23] P.S. Aspinwall, *Some relationships between dualities in string theory*, Nucl. Phys. Proc. Suppl. **46** (1996) 30, hep-th/9508154 ;
J.H. Schwarz *An $SL(2,Z)$ multiplet of type IIB superstrings*, hep-th/9508143.

[24] P. Horava and E. Witten, *Heterotic and type I string dynamics from eleven dimensions*, Nucl. Phys. **B 460** (1996) 506, hep-th/9510209.

[25] T. Banks and L. Motl, *Heterotic strings from matrices*, hep-th/9703218 ;
D.A. Lowe, *Heterotic matrix string theory*, hep-th/9704041 ;
S.-J. Rey, *Heterotic matrix strings and their interactions*, hep-th/9704158 ;
P. Horava, *Matrix theory and heterotic strings on tori*, hep-th/9705055 ;
N. Kim and S.-J. Rey, *M(atrix) theory on T^5/Z_2 orbifold and five-branes*, hep-th/9705132

[26] M. Rozali, *Matrix theory and U-duality in seven dimensions*, hep-th/9702136 ;
M. Berkooz, M. Rozali, and N. Seiberg, *Matrix description of M-theory on T^4 and T^5*, hep-th/9704089 ;
N. Seiberg, *Matrix description of M-theory on T^5 and T^5/Z_2*, hep-th/9705221.