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Non-Commutative Black Hole Algebra and String Theory from Gravity

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We generalize the action found by 't Hooft, which describes the gravitational interaction between ingoing and outgoing particles in the neighbourhood of a black hole. The effect of this back-reaction is that of a shock wave, and it provides a mechanism for recovering information about the momentum of the incoming particles. The new action also describes particles with transverse momenta and takes into account the transverse curvature of the hole, and has the form of a string theory action. Apart from the Polyakov term found by 't Hooft, we also find an antisymmetric tensor, which is here related to the momentum of the particles. At the quantum level, the identification between position and momentum operators leads to four non-commuting coordinates. A certain relation to M(atrix) theory is proposed.

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Introduction

The S-matrix approach to the quantum black hole is an attempt to solve the information paradox [1]. This arose from Hawking's discovery of black hole radiation and its thermal character. The claim made by several researchers is that the inclusion of the gravitational interaction between the ingoing particles and the Hawking radiation that comes out of the hole will restore predictability; more exactly, that in this way we will obtain a unitary mapping between an initial pure state before the hole was formed and a pure final state after its evaporation. Independently of whether this claim is true or not, it is a fact that Hawking neglected the effect of these particles on the metric, and that is in any case an important calculation to make, if one wants to make accurate statements about black hole microscopy. For one thing, these effects are not negligible if the particles have Planckian energies, as they do when they fall into the hole.

So the first step was to compute the back-reaction of highly energetic particles near the horizon on the metric of the hole. Dray and 't Hooft found [2] that the effect is that of a shift in the position of the horizon, generally known as shockwave, and through this shift one had a mechanism to recover the information about the momentum state of the particles, essentially because the strength of the shift depends on the momentum. 't Hooft was then able to find the S-matrix describing the process [3]. This turned out to be related to string theory, including the Nambu-Goto action of a Euclidean string (or, more properly, of a membrane at one instant in time) that is exchanged between the particles. These ideas were further studied in reference [4], and made contact with the intuitive "membrane paradigm" in [5].

The next step after finding the S-matrix was to see what kind of Hilbert space it worked on. This turned out to be a difficult problem, with several technical complications. One of them was how to obtain a discrete spectrum for this Hilbert space, with the right number of microstates, according to the Bekenstein-Hawking entropy formula. Another problem was the inclusion of the transverse gravitational forces felt by the ingoing and the outgoing particles and their transverse momenta, as all calculations had been done in the Rindler approximation. Some attempts at this were made [6], but not completely successful.

In this paper we will show that the covariant generalization of the action appearing in the S-matrix is not the one previously advocated [3], but has to be slightly modified. The action one obtains is that of bosonic string theory, including the antisymmetric tensor $B_{\mu\nu}$ with a field strength $H = \mathrm{d}B$, which turns out to be the Hodge dual of the momentum distribution. We will show that 't Hooft's equations for the shockwave can be seen as field equations arising from this string action. Our treatment is fully covariant, so that the particles are allowed to have momenta in any direction. When quantizing the theory, the equations of motion give rise to a set of four non-commuting coordinates which nevertheless still depend on the string degrees of freedom. We will argue that this problem could be cured by considering our description to be an effective one, so that the string is actually made up of particles coming in at definite positions, with a limited number, N, of them. At the end a possible connection with M-theory is made.

The requirement of conformal invariance for this string theory will give us an expression for Einstein's equation which presumably contains solutions that describe a black hole interacting with matter. The detailed study of this is left for a future paper [7].

In appendix A this formalism is extended to an arbitrary number of dimensions.

1 A covariant action for the gravitational interaction

One of our aims is to include the transverse gravitational forces and transverse momenta in the S-matrix approach of 't Hooft. One would also like to improve the algebra obtained in [6], which gives an uncertainty relation between the light-cone coordinates when there are particles travelling at very high energies. Therefore we need the covariant generalization of the black hole action

$$S = \int \mathrm{d}\sigma \mathrm{d}\tau \left(T\, U \triangle V - P_{\mathrm{U}} U + P_{\mathrm{V}} V \right), \tag{1}$$

with the definition $\triangle = \partial_\sigma^2 + \partial_\tau^2$. σ and τ are coordinates that parametrize the surface of the horizon. Since this action was derived in the Rindler approximation, they are set equal to the transverse coordinates X and Y (in Schwarzchild spacetime, they correspond to the angles θ and ϕ). U and V are light-cone coordinates, describing the location of the incoming and outgoing particles. The "string tension" T is defined by $T = \frac{1}{8\pi G}$, where G is Newton's constant. Expression (1) was found in reference [8] to be the action appearing in the S-matrix for highly energetic scattering near a black hole. The S-matrix gives the momentum state of the outgoing particles once that of the ingoing particles is known, and *vice versa*. The covariant generalization must account for the relative minus sign between the momenta (the importance of this is shown in appendix B), which comes from complex conjugation in the bra of the S-matrix. We expect these relative sign differences to appear when we include more general states like $|P_{\mathrm{U}}, P_{\mathrm{V}}, P_{\mathrm{X}}, P_{\mathrm{Y}}\rangle$, and they will not be removable by a redefinition of coordinates. A general S-matrix element will be of the form

$${}_{\mathrm{out}}\langle P'_{\mathrm{U}}, P'_{\mathrm{V}}, P'_{\mathrm{X}}, P'_{\mathrm{Y}} | P_{\mathrm{U}}, P_{\mathrm{V}}, P_{\mathrm{X}}, P_{\mathrm{Y}} \rangle_{\mathrm{in}}. \tag{2}$$

However, it is difficult to calculate this matrix element in the way it was done when one had the longitudinal directions only. Instead, we generalize covariantly the expression found by 't Hooft, and in particular the action. Thisis a much easier task to do.

All the ingredients we need for this are present in [8], but we will repeat the line of reasoning below. The arguments to be presented next were used in [8] to improve the commutator between the longitudinal coordinates, from which one could then find an algebra for the surface elements

$$W^{\mu\nu} = \epsilon^{ij} \partial_i X^\mu \partial_j X^\nu. \tag{3}$$

These were combined in a certain way and integrated over some region of the horizon to give the SU(2) algebra. This had the advantage that one got rid of the $\tilde{\sigma}$-dependence, obtaining a discrete spectrum, see [9]. In this paper, however, we will regard the covariant approach of [8] as more fundamental, not only because it was used in the derivation of the algebra of the operators (3), but also because it makes full contact with string theory. As the main tools needed to covariantly generalize the action (1) are the same one uses to derive the algebra of the surface elements (3), the validity of our calculation is the same as that of the algebra.

So the first thing to take into account is the relative minus sign between the terms $P_{\mathrm{U}} U$ and $P_{\mathrm{V}} V$ in the action (1). For that purpose we dispose of the four-dimensional tensor

$\epsilon_{\mu\nu \mathrm{XY}}$. We use the convention $U=1$, $V=2$, $X=3$, $Y=4$ and $\epsilon^{1234}=\epsilon^{12}_{34}=-\epsilon_{1234}=1$. $\epsilon_{\mu\nu\mathrm{XY}}$ would automatically project μ,ν to the values $1,2$, if contracted in the product

$$\epsilon_{\mu\nu 34}\, P^\mu X^\nu, \tag{4}$$

giving us the result (1). But now because of covariance we have to replace the term $\epsilon_{\mu\nu 34}$ by the full tensor $\epsilon_{\mu\nu\alpha\beta}$, so the two lower indices have to be projected onto the indices corresponding to the membrane:

$$\epsilon_{\mu\nu 34} = \frac{1}{2}\,\epsilon_{\mu\nu\alpha\beta}\,\epsilon^{ij}\delta^\alpha_i\delta^\beta_j, \tag{5}$$

where i,j run over the values $3,4$. In addition, we want the action to be invariant under reparametrizations of the membrane coordinates, in order that i,j can also be identified with the $1,2$-directions of spacetime. This will allow us to embed our string in spacetime as we wish. As 't Hooft remarked, this can be done be means of $\partial_i X^\mu = \delta^\mu_i$, which holds in the Rindler approximation. So if this is the correct generalization, we get a factor

$$\frac{1}{2}\,\epsilon_{\mu\nu\alpha\beta}\,\epsilon^{ij}\partial_i X^\alpha \partial_j X^\beta, \tag{6}$$

which in the approximation reduces to $\epsilon_{\mathrm{UV}\sigma\tau}$[2], as seen. So inserting this back in (4), the total action becomes

$$S = -\frac{1}{2}\int \mathrm{d}^2\tilde\sigma\,\left(T\,\sqrt{h}h^{ij}\,g_{\mu\nu}\partial_i X^\mu\partial_j X^\nu + \epsilon_{\mu\nu\alpha\beta}P^\mu X^\nu\,\epsilon^{ij}\partial_i X^\alpha\partial_j X^\beta\right). \tag{7}$$

Notice that the only arguments we used here were covariance and that we must get the appropriate limit (1) when we go back to the Rindler approximation.

Now, the variation $X^\mu \to X^\mu + \delta X^\mu$ of (7) yields the equation of motion

$$\triangle X^\mu = -\frac{1}{T\sqrt{h}}\,\epsilon^{\mu\nu}_{\alpha\beta}\,\epsilon^{ij}\left(\frac{3}{2}\,\partial_i X^\alpha\partial_j X^\beta P_\nu + X^\beta\partial_i X^\alpha\partial_j P_\nu\right), \tag{8}$$

where we have defined a Laplacian $\triangle = \frac{1}{\sqrt{h}}\,\partial_i\sqrt{h}\,h^{ij}\partial_j$. Because our fields are distributions that should be integrated over some region of the horizon, we can use partial integration. The equation of motion becomes

$$\triangle X^\mu = -\frac{1}{2T\sqrt{h}}\,\epsilon^{\mu\nu}_{\alpha\beta}\,\epsilon^{ij}\partial_i X^\alpha\partial_j X^\beta P_\nu. \tag{9}$$

In the Rindler gauge, this equation clearly reduces to equation (71) in appendix B, which was found by 't Hooft. For simplicity, we have taken the metric $g_{\mu\nu}(X)$ to be constant, so our spacetime is Ricci-flat. In a more precise calculation, it of course has to be varied as well, and (9) receives a correction (see (53) in appendix A), but this is a good starting point for the cases we are interested in at the moment (where the metric is flat).

The physical meaning of the factor $\epsilon^{ij}\partial_i X^\alpha\partial_j X^\beta$ in the second term of (7) is still unclear. We would like to be able to interpret it in terms of Fourier transforms of the bras and kets in the S-matrix, but that is not straightforward to do. What we seem to learn from this is that when there are particles travelling in all directions, the solutions

[2] σ,τ here stand for the transverse coordinates $\tilde\sigma$, not to be confused with indices running from 1 to 4.

of the Schrödinger equation are not simply plane waves, but very complicated functions which depend on the geometry induced by these particles on the horizon.

As said, it is difficult to derive our action from first principles, as was done in the longitudinal case. But there is one important physical effect which we do see from (9) and that one also expected: the shift in an arbitrary direction X^μ receives contributions from the particles travelling in *all* perpendicular directions, and not only from one of them. So now the ingoing particles will also affect each other, and the same for the outgoing ones. For example, two ingoing particles with momenta $P_{\rm V}$ and $P'_{\rm V}$, respectively, in the V-direction, and momenta $P_{\rm X}$ and $P'_{\rm Y}$, respectively, in the transverse directions, will interact with each other. This fact could not be found from an action like the one in appendix B (see equation (62)), since the summation over all perpendicular directions in (9) is essential. So the distinction between ingoing and outgoing particles, typical of the Rindler gauge, has lost its fundamental meaning. It will manifest itself only in the sign of the momentum in the longitudinal directions, after we have chosen a gauge. What is now important is to know the total momentum distribution in a certain direction.

Next we try to quantize the model. P_μ was defined as the operator working on the Hilbert space that is canonically conjugated to X^μ, satysfying the following relation:

$$[P_\mu(\tilde\sigma), X^\nu(\tilde\sigma')] = -i\,\delta^\nu_\mu\,\delta(\tilde\sigma-\tilde\sigma'). \tag{10}$$

This corresponds to the momentum of the particles. Commuting both sides of (9) with X^ν and applying (10), we easily get

$$[X^\mu(\tilde\sigma), X^\nu(\tilde\sigma')] = -i\,F^{\mu\nu}(\tilde\sigma-\tilde\sigma'), \tag{11}$$

where

$$F^{\mu\nu}(\tilde\sigma-\tilde\sigma') \equiv \frac{\ell^2_{\rm Pl}}{2}\,\epsilon^{\mu\nu}_{\alpha\beta}\,\epsilon^{ij}\partial_i X^\alpha \partial_j X^\beta f(\tilde\sigma-\tilde\sigma'), \tag{12}$$

$\ell^2_{\rm Pl}$ being Planck's length. f is the Green function defined by

$$\triangle f(\tilde\sigma-\tilde\sigma') = -\frac{1}{8\pi\sqrt{h}}\,\delta(\tilde\sigma-\tilde\sigma'). \tag{13}$$

This is exactly the commutator postulated in [8], which has been derived here by covariant generalization of the action (1). The only requirement was that it must give us the right equations of motion (see (71) in appendix B) in the Rindler limit, because those are the only expressions whose correctness is without doubt. We did neglect a higher-order correction to (11) coming from the fact that $F^{\mu\nu}$ is itself also an operator, and for simplicity considered Ricci-flat metrics only. Notice that in the flat-space limit, there is a well defined Fourier transformation that relates the two. In a curved space, however, equation (10) will receive $\mathcal{O}(\ell^2_{\rm Pl})$ corrections, which, however, affect (11) only by a term of $\mathcal{O}(\ell^4_{\rm Pl})$.

This commutator can be inverted to give a relation between the momenta. After some algebra, we easily find

$$[P_\mu(\tilde\sigma), P_\nu(\tilde\sigma')] = i\ell^{-2}_{\rm Pl}\,\epsilon_{\mu\nu\alpha\beta}\frac{W^{\alpha\beta}}{W^2}\,f^{-1}(\tilde\sigma-\tilde\sigma'), \tag{14}$$

where $W^2 \equiv W_{\alpha\beta}W^{\alpha\beta}$.

There are several subtleties about the commutator (10). One can wonder whether P is really the momentum canonically conjugated to X or not. The main problem is that it is not defined in the usual way in field theory, like

$$P_\mu(\cdot,t) = \frac{\delta L}{\delta(\partial_t X^\mu)}, \tag{15}$$

where t is some time variable and the dot denotes all the fields P may depend on. Rather, it was assumed to be a known function of the world-sheet variables $\tilde{\sigma}$ (this function is explicitly given in equation (23) of section 2), which is allowed if we work in the momentum representation, as we are doing (see (2)). It was then inserted in the action in such a way that it gives the right equations of motion when we vary X (so we did not vary the action with respect to P because P is a known function; it is not integrated over in the S-matrix, see equation (21) in section 2). This problem is closely related to that of time and of finding a Hamiltonian formulation for the problem, and at present we do not know how to solve it. However, for other forces different from gravity are switched off, we do know that (10) must be true because P enters in the Einstein equations as being the momentum of the photons. In a momentum representation, the latter is known and hence this picture is consistent. Therefore, the momentum must be conjugated to the position of the particle. It is probably true that this should be realised as a constraint in the theory[3], but have not yet succeed doing this[4].

Let us write (9) for the transverse fields explicitly. We get

$$\begin{aligned} T\sqrt{h}\,\triangle U &= -W^{XY}\,P_{\rm V} + W^{UY}\,P_{\rm X} - W^{UX}\,P_{\rm Y} \\ T\sqrt{h}\,\triangle V &= +W^{XY}\,P_{\rm U} - W^{VY}\,P_{\rm X} + W^{VX}\,P_{\rm Y}, \end{aligned} \tag{16}$$

with W defined as in (3). We thus indeed explicitly see that momenta in all directions contribute to the shift. These equations of course give us equation (71) in the limit.

The exact commutator is now

$$[U(\tilde{\sigma}), V(\tilde{\sigma})] = -\frac{i\ell_{\rm Pl}^2}{\sqrt{h}}\,\epsilon^{ij}\partial_i X \partial_j Y\, f(\tilde{\sigma}-\tilde{\sigma}'), \tag{17}$$

which in the gauge $\tilde{\sigma} = (X, Y)$ becomes 't Hooft's commutator

$$[U(\tilde{\sigma}), V(\tilde{\sigma})] = -i\ell_{\rm Pl}^2\, f(\tilde{\sigma}-\tilde{\sigma}'). \tag{18}$$

It is important to see that our equations are manifestly covariant. For the transverse fields, we get from (11)

$$[X(\tilde{\sigma}), Y(\tilde{\sigma})] = -\frac{i\ell_{\rm Pl}^2}{\sqrt{h}}\,\epsilon^{ij}\partial_i U \partial_j V\, f(\tilde{\sigma}-\tilde{\sigma}'), \tag{19}$$

so that, if we apply a Lorentz transformation on the Rindler gauge and take the membrane coordinates to be $\tilde{\sigma} = (U, V)$, we get

$$[X(\tilde{\sigma}), Y(\tilde{\sigma})] = -i\ell_{\rm Pl}^2\, f(\tilde{\sigma}-\tilde{\sigma}'). \tag{20}$$

[3] We thank Steven Carlip for drawing this to our attention.

[4] One possibility for going to the Hamiltonian formalism is to Wick rotate one of the two world-sheet coordinates, identifying it with the time of the string; it is, however, not clear whether this is consistent with the shockwave interpretation. The interaction would then be by exchange of a string that is frozen at the horizon, evolving in time along one of the angular coordinates. A less exotic possibility is to add an extra, third dimension that accounts for the time evolution of the membrane in time. We believe that finding a Hamiltonian formalism will solve the problem about momentum.

2 The physical degrees of freedom

One possible source of criticism is that in the Rindler gauge $\tilde{\sigma} = (X, Y)$, the commutator (19) is not well defined because on the right-hand side we have functionals depending on $\tilde{\sigma}$ and $\tilde{\sigma}'$, while the left-hand side would just be $[\sigma, \tau]$. But in this gauge, this commutator is not valid. From (9) we see that, because $\triangle X \sim P_{\rm Y}$ and $\triangle Y \sim P_{\rm X}$, if $(X, Y) = (\sigma, \tau)$ the momentum in the transverse directions is zero, $P_{\rm X} = 0$ and $P_{\rm Y} = 0$. So in that case the transverse coordinates are not physical fields, and their commutator vanishes. This is consistent with 't Hooft's calculation, where the identification of the world-sheet and two of the target space coordinates forced the degrees of freedom to be reduced from four to two. Here we have decoupled these coordinates and allowed X and Y to be any function of σ, τ, so we have a non-linear sigma-model with four bosons living on a two-dimensional space. Therefore, up to a normalization constant, the matrix element (2) is

$$\langle P_{\rm out} | P_{\rm in} \rangle = \mathcal{N} \int \mathcal{D}U(\tilde{\sigma})\, \mathcal{D}V(\tilde{\sigma})\, \mathcal{D}X(\tilde{\sigma})\, \mathcal{D}Y(\tilde{\sigma})\, \mathcal{D}h^{ij}(\tilde{\sigma})\, \exp iS\,[U, V, X, Y, h^{ij}]. \tag{21}$$

Which functions are allowed for $X(\tilde{\sigma})$ and $Y(\tilde{\sigma})$ will follow from the equations of motion, just like for U and V. For example, in the Rindler gauge, one usually takes

$$\begin{aligned} P_{\rm U}(\tilde{\sigma}) &= \sum_{i=1}^{N} p_{\rm U}^i\, \delta(\tilde{\sigma} - \tilde{\sigma}^i) \\ P_{\rm V}(\tilde{\sigma}) &= \sum_{i=1}^{N} p_{\rm V}^i\, \delta(\tilde{\sigma} - \tilde{\sigma}^i), \end{aligned} \tag{22}$$

where $\tilde{\sigma}^i$ is the location of the ith particle. In this picture, $P(\tilde{\sigma})$ is the total momentum distribution, which tells us how many particles are going in and out. It has a finite number, N, of contributions. p^i is the momentum of each particle. Now we can do the same for the X and Y directions, taking generally

$$P_\mu(\tilde{\sigma}) = \sum_{i=1}^{N} p_\mu^i\, \delta(\tilde{\sigma} - \tilde{\sigma}^i). \tag{23}$$

The solution that then follows from (9) is

$$X^\mu(\tilde{\sigma}) = X_0^\mu(\tilde{\sigma}) + \frac{\ell_{\rm Pl}^2}{2} \sum_{i=1}^{N} p_\nu^i\, \epsilon_{\alpha\beta}^{\mu\nu}\, W^{\alpha\beta}(\tilde{\sigma}^i)\, f(\tilde{\sigma} - \tilde{\sigma}^i), \tag{24}$$

$X_0^\mu(\tilde{\sigma})$ being a solution of the equation

$$\triangle X_0^\mu(\tilde{\sigma}) = 0. \tag{25}$$

So it is just the solution of the equations of motion of the free string, and can be expanded in Fourier modes α_n^μ, as usual in string theory[5].

Notice that in (21) both the in and the out states contain particles moving in the U and the V directions. Here a difference from the original S-matrix arises. The labels "in" or "out" no longer have to do with the direction in which the particles are travelling.

[5] Because it is an Euclidean wave equation, the boundary conditions are different from the usual ones. This will be explained elsewhere.

Rather, they represent the initial and final states of the particles, whatever their spatial configuration may be. The hole is the interaction region, an intermediate state that is integrated over in the path integral. This probably means that its mass has a complex component, as has been advocated in [8][4]. That is, we think, the essence of the S-matrix description, but one will nor have full understanding of this question until one has a Hamiltonian formulation, because then one will have a picture with a preferred time variable.

Like for 't Hooft's Lagrangian, one still has to impose a condition on the physical solutions which follows from the equations of motion of the world-sheet metric. The latter has also to be varied, because in (21) we integrate over all possible metrics. So the two-dimensional stress-energy tensor must vanish,

$$T_{ij} = -\frac{2}{T\sqrt{h}}\frac{\delta S}{\delta h^{ij}} = 0. \tag{26}$$

The result is the usual constraint for the metric on the string to be the metric induced by spacetime:

$$h_{ij} = g_{\mu\nu}\,\partial_i X^\mu \partial_j X^\nu. \tag{27}$$

This amounts to the condition that the positive frequency components of the Virasoro generators annihilate physical states. This brings us back to the restriction that there are only $d-2$ physical bosons instead of d (the spacetime dimension), like in 't Hooft's case, but now we can do this in a fully covariant way, imposing this condition at the end. In general, however, due to global effects and the presence of particles, the gauge will be more involved than just the Rindler gauge.

Classically, the stress-energy tensor (26) is automatically traceless, without imposing the equations of motion. This is because, at the classical level, the action (7) has conformal symmetry. Quantum mechanically, however, the trace can be non-vanishing if there are anomalies, which nevertheless can be cancelled by imposing very special conditions on, for example, the dimensionality of the spacetime. Since —as we will next show— our model is just a string theory, the same arguments used in critical string theory to demand this symmetry at the quantum level can be applied to our case. Therefore, in this paper we will not go into the rather involved discussion of whether or not anomalies can occur in more general cases, but will only consider the case that this conformal symmetry is maintained quantum mechanically.

As said, (7) has a common interpretation in string theory, for which the equations that follow from conformal invariance are known. The action can be written in the following way:

$$S = -\frac{1}{2}\int \mathrm{d}^2\tilde{\sigma}\,\left(T\,\sqrt{h}h^{ij}\,g_{\mu\nu}\partial_i X^\mu \partial_j X^\nu + B_{\mu\nu}(X)\,\epsilon^{ij}\partial_i X^\mu \partial_j X^\nu\right), \tag{28}$$

which is exactly the action of a bosonic string propagating on a manifold with a graviton background $g_{\mu\nu}(X)$ and an antisymmetric tensor field $B_{\mu\nu}(X)$. The antisymmetric tensor is, in our case,

$$B_{\mu\nu}(X) = \epsilon_{\mu\nu\alpha\beta}\,P^\alpha X^\beta. \tag{29}$$

Hence the shockwave modifies the background where it propagates by giving it a non-vanishing torsion. The equations of motion, which shift the position of the particles in the

perpendicular directions, are similar to the influence of a magnetic field on an electron in a cyclotron.

3 A conformally invariant horizon

The analogy with string theory discovered in [8] is now found to be more accurate: we have the same field content as in string theory, except for the dilaton field. We could not derive its existence, which is only possible if, from the beginning, one fully takes into account the curvature of the horizon, since the dilaton couples to the two-dimensional Ricci scalar, which was zero in 't Hooft's calculation. Also, the dilaton does not have a clear physical interpretation in the black hole context yet, but we know it must be there, since it is part of the spectrum of the string that is obtained by a Virasoro decomposition of the fields satisfying the wave equation (8). The dilaton term is needed to ensure conformal invariance and finiteness at the quantum level, and so in this section we study the case that this conformal anomaly is indeed cancelled, although it remains a subtle issue how this invariance is realized (see the discussion in appendix A). Therefore we introduce by hand the following (non-Weyl-invariant) dilaton term to the action, in the usual way in string theory:

$$S_\phi = -\frac{1}{4\pi}\int \mathrm{d}^2\tilde{\sigma}\,\sqrt{h}\,\phi(X)\,R^{(2)}, \tag{30}$$

where $R^{(2)}$ is the two-dimensional Ricci scalar. For Ricci-flat metrics like, for example, $h_{ij} = \lambda\,\delta_{ij}$, with constant λ, this term is zero and thus consistent with 't Hooft's result.

The system (28) with the extra term (30) has been well studied [10]. The mentioned scale invariance turns out to require the following β-functions for each of the fields to vanish[6]:

$$\begin{aligned}
\beta^\phi &= \frac{d-26}{48\pi^2} + \frac{G}{4\pi^2}\left[4\,(\nabla\phi)^2 - 4\,\nabla^2\phi - R + \frac{1}{12}H^2\right] + \mathcal{O}\left(G^2\right)\\
\beta^g_{\mu\nu} &= R_{\mu\nu} - \frac{1}{4}\,H^{\alpha\beta}_{\mu}H_{\nu\alpha\beta} + 2\,\nabla_\mu\nabla_\nu\phi + \mathcal{O}\left(G\right)\\
\frac{1}{4G}\,\beta^B_{\mu\nu} &= \nabla_\alpha H^\alpha_{\mu\nu} - 2\,(\nabla_\alpha\phi)\,H^\alpha_{\mu\nu} + \mathcal{O}\left(G\right),
\end{aligned} \tag{31}$$

where Newton's constant now plays the role of the string constant α' and $R_{\mu\nu}$ is the four-dimensional Ricci tensor. In our notation, $H^2 \equiv H_{\mu\nu\alpha}H^{\mu\nu\alpha}$. The antisymmetric tensor field strength, $H_{\mu\nu\alpha}$, is defined by $H = \mathrm{d}B$, and, in this case, equals

$$H_{\mu\nu\alpha} = -3\,\epsilon_{\mu\nu\alpha\beta}\,P^\beta, \tag{32}$$

hence it is the dual of the momentum. By definition, it satisfies the Bianchi identity.

We see from (31) that, in the presence of a dilaton, we are restricted to $d = 26$ if we want to cancel the conformal anomaly, which is the usual result in bosonic string theory.

Notice that both the β-function for the dilaton and that for the momentum torsion are linear in the Newton coupling constant. This is because their coefficient in the action is smaller by a power of G than that of the Polyakov term. For the dilaton we just

[6] We are grateful to Yolanda Lozano for clarifying the analogy with string theory to us and suggesting to calculate the β-functions.

assumed that it had the same power of G as in string theory. But for the momentum torsion tensor, this is a consequence of the equations of motion for the shockwave, since the momentum comes in with the first power of G on the right-hand side of Einstein's equation. Therefore, as remarked in [10], the *classical* contributions of the dilaton and the momentum torsion to the anomaly are of the same order as the one-loop *quantum* contribution of the $g_{\mu\nu}$ coupling.

Equations (31) are the equations of motion of the following effective action:

$$S_{\text{eff}} = \int \mathrm{d}^d x \sqrt{g} e^{-\phi} \left[R - \frac{1}{12} H^2 + 4 \left(\nabla \phi \right)^2 \right], \tag{33}$$

which, as shown in [10], can be obtained from the Chapline-Manton [11] supergravity action after rescaling of the four-dimensional metric.

If we would require the action to be supersymmmetric, which is not hard to do, standard results of string theory would require the number of dimensions to be $d = 10$. In that case, because 't Hooft's calculation is independent of the number of transverse dimensions, and the generalization is straightforward, the shock wave would not be a membrane (with time left out), but a Euclidean 7-brane (see Appendix A). The latter is also needed if we want to apply the results to Schwarzschild spacetime, where (in Rindler gauge) the membrane is identified with the horizon of the black hole[7]. So to be consistent we have to regard our model as an effective description that arises after compacification down to 4 dimensions. Therefore it is important to have the full action, which after compactification reduces to (28). In fact, this action is easy to find:

$$\begin{aligned} S &= -\frac{T_d}{2} \int \mathrm{d}^{d-2}\sigma \left(\sqrt{h}\, h^{ij} g_{\mu\nu}\, \partial_i X^\mu \partial_j X^\nu - (d-4) \sqrt{h} \right) \\ &- \frac{1}{(d-2)!} \int \mathrm{d}^{d-2}\sigma\, B_{\mu_1 \cdots \mu_{d-2}}(X)\, \epsilon^{i_1 \cdots i_{d-2}}\, \partial_{i_1} X^{\mu_1} \cdots \partial_{i_{d-2}} X^{\mu_{d-2}}, \end{aligned} \tag{34}$$

where the antisymmtric tensor is given in appendix A and the other results of section 1 are generalized to arbitrary dimension.

One of course would like to have also some physical motivation for these extra dimensions, but at present we are not in a position to say very much about this. It turns out [9] that electromagnetic interactions can be included quite naturally in this formalism as a fifth Kaluza-Klein dimension, but it is hard to extend the theory to include the other interactions.

Combining the first and the second of equations (31) and applying Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = -8\pi G\, T_{\mu\nu}, \tag{35}$$

gives us a stress-energy tensor equal to[8]

$$T_{\mu\nu} = \frac{9}{2} \left(P_\mu P_\nu - \frac{1}{2} P^2 g_{\mu\nu} \right) - 2\, \nabla_\mu \nabla_\nu \phi + 2\, g_{\mu\nu} \nabla^2 \phi - 2\, g_{\mu\nu} \left(\nabla \phi \right)^2. \tag{36}$$

The first part of this expression is reminding of the energy-momentum tensor of a perfect relativistic fluid, where P_μ plays the role of the fluid velocity u_μ. This analogy will be studied in [7].

[7] We thank Gijsbert Zwart for pointing this out to us.

[8] One must not forget that there are higher-order corrections to these equations.

Finally, we seem to be back to a dynamical Einstein equation (35). This presumably describes black holes with matter falling in and going out and interacting gravitationally at the horizon [7]. This is exactly what we wanted, because the aim of 't Hooft's calculation was to include the back-reaction on the metric. So the next step will be to look for physically sensible solutions of (35) and then calculate Hawking's temperature (for a similar approach, in a somewhat different context, see [12]).

4 A discrete algebra

The analogy with string theory is beautiful, but when considering the physical meaning of the theory one has to recall that equation (28) only has a meaning as an effective action. In partucular, we would like to give the equations of motion and relation (9) a meaning in terms of single particles. The operators $X^\mu(\tilde{\sigma})$, as they are treated here, indeed give the distribution of ingoing and outgoing particles on the surface of the horizon. Hence we have to go back to a discrete representation, where we have N of these particles.

The first thing to remark (see [3]) is that the physical Fock space of this theory is very different from ordinary Fock space, because in (23) the momentum distribution does not distinguish between different particles that are at the same position on the surface of the horizon. Thus, the total number of particles is not well defined in the usual sense or, more accurately, it is defined by the number of "lattice sites"[9]. The discrepancy with usual field theory lies in the fact that in the low-energy limit, one is not interested in what happens when two particles are at exactly the same location, because the cross-sections of scattering processes, dominated by the low-energy interactions, are much larger than that. But when the gravitational force dominates, this question becomes relevant, even when the radial separations are not small.

So we have to apply (23) to the above results. To do this we will have to integrate with test functions that live on the horizon, in the following way:

$$\int \mathrm{d}^2\tilde{\sigma}\, F(\tilde{\sigma})\, I(\tilde{\sigma}) = \sum_{i=1}^{N} F^i\, I(\tilde{\sigma}^i), \tag{37}$$

where F can carry any other spacetime index and I is an arbitrary function. These distributions thus satisfy

$$F(\tilde{\sigma}) = \sum_{i=1}^{N} F^i\, \delta(\tilde{\sigma} - \tilde{\sigma}^i). \tag{38}$$

On the other hand, for large N we have

$$\int \mathrm{d}^2\tilde{\sigma}\, F(\tilde{\sigma}) \simeq \frac{A}{N} \sum_{i=1}^{N} F(\tilde{\sigma}^i), \tag{39}$$

where A is the area of the horizon, $A = 16\pi M^2$. Thus we have

$$F^i = \frac{A}{N}\, F(\tilde{\sigma}^i). \tag{40}$$

[9] We thank Leonard Susskind for the remark that this heuristic terminology should not be taken literally. These "lattice points" of course do not need to be fixed. They must rather be understood as the positions of the 0-particles on the membrane [3][13].

From this and equation (14) one can get an algebra for the momentum of each particle. To do this we again go to the Rindler approximation, where we identify two spacetime directions with σ and τ. So in that approximation,

$$\frac{1}{2}\,\epsilon_{\mu\nu\alpha\beta}\,\epsilon^{ij}\partial_i X^\alpha \partial_j X^\beta = \epsilon_{\mu\nu\sigma\tau} \equiv \epsilon_{\mu\nu}, \tag{41}$$

where the indices μ, ν are now restricted to the longitudinal plane, and $W_{\alpha\beta}W^{\alpha\beta} = 2$.

Writing down equation (14) for the specific case $\tilde{\sigma} = \tilde{\sigma}^i$, $\tilde{\sigma}' = \tilde{\sigma}^j$, we get

$$[P_\mu(\tilde{\sigma}^i), P_\nu(\tilde{\sigma}^j)] = i\ell_{\rm Pl}^{-2}\,\epsilon_{\mu\nu}\,f^{-1}(\tilde{\sigma}^i - \tilde{\sigma}^j); \tag{42}$$

now comparing (23) with (38), and using equation (40), we get

$$[p^i_\mu, p^j_\nu] = i\epsilon_{\mu\nu}\left(\frac{16\pi M^2}{N}\right)^2 f^{-1}(\tilde{\sigma}^i - \tilde{\sigma}^j). \tag{43}$$

For example, for a large black hole ($M \to \infty$) in four dimensions, f is approximately given by

$$-\frac{1}{2T}\log\frac{\|\tilde{\sigma}^i - \tilde{\sigma}^j\|^2}{\ell_{\rm Pl}^2}, \tag{44}$$

for any two particles i and j. Now if we assume all particles to be homogeneously distributed throughout the horizon, the mean distance between two of them will be $\|\tilde{\sigma}^i - \tilde{\sigma}^j\| \simeq \sqrt{A/N} = \sqrt{16\pi M^2/N}$, and so

$$[p^i_\mu, p^j_\nu]_{i\neq j} = -i\,\epsilon_{\mu\nu}\,\frac{32\pi M^4}{GN^2\log\frac{16\pi M^2}{N\ell_{\rm Pl}^4}}. \tag{45}$$

We of course consider this discrete model, especially the last part, where we did a kind of mean-field theory, as a toy model, purely as an indication of the direction in which one has to search for an algebra with a finite number of degrees of freedom. Nor did we derive that the number of particles is finite or that there is something like a minimal length on the horizon. However, in spacetime a minimal length scale *does* explicitly appear in equation (11). The unusual fact about this is that it depends on the location of the particles on the world-sheet by means of the propagator $f(\tilde{\sigma} - \tilde{\sigma}')$.

5 M(atrix) theory and gravity

At first sight, the commutator (11) looks very much like what is found in M(atrix) theory, because it is proportional to the orientation tensor (3) (the "Poisson bracket" of the membrane): it is its Hodge dual. However, one must be very careful. This commutator comes from Dirac quantization, and not from a matrix representation of the fields, like in matrix theory. The striking fact is that the coordinates do not commute, not in the sense of matrix theory, where the matrices are non-commuting but the matrix elements themselves are numbers, but in the sense of quantum mechanics; our prescription is really a Dirac quantization condition. The commutator (11) gives us a rule for making the transition from the classical to the quantum Euclidean string (or membrane at one instant

of time) in the presence of highly energetic particles. So, though at a speculative stage, and although the noncommutativity has a different origin here and in matrix theory, we wish to push the analogy forward to see if we can learn something about matrix theory from this model.

In matrix theory, one has the representation[10]

$$\frac{1}{N}\{\cdot\} \leftrightarrow [\cdot]_{\mathrm{M}}, \tag{46}$$

where N is the cutoff imposed to regularize the membrane. Now in our case, the quantum mechanical commutator is given by calculating the dual of the Poisson bracket,

$$i\ell_{\mathrm{Pl}}^2\, \epsilon_{\alpha\beta}^{\mu\nu}\, \{x^\alpha, x^\beta\}\, f, \tag{47}$$

i.e., quantization is determined by the replacement

$$*\ \{\cdot\} \to \frac{1}{i\ell_{\mathrm{Pl}}^2 f}\,[\cdot], \tag{48}$$

$*$ denoting the Hodge dual[11]. So $\ell_{\mathrm{Pl}}^2 f$ seems to play the role of $\frac{1}{N}$ in matrix theory. It also gives a minimal length scale in spacetime. One is therefore tempted to make the identification[12] $\ell_{\mathrm{Pl}}^2 f \equiv \frac{1}{N}$, or $G \sim \frac{1}{N}$; the Fourier expansion would then be truncated like $N \sim \frac{1}{\ell_{\mathrm{Pl}}^2}$. Physically, this means that the membrane is naturally regularized by Planck's length if one takes the momenta of the particles into account. So we propose the following series:

$$\frac{1}{N} * \{\cdot\} \ \leftrightarrow\ *[\cdot]_{\mathrm{M}} \ \to\ \frac{1}{i\hbar}[\cdot]. \tag{49}$$

The first correspondence is the matrix representation of membrane theory, with the specific choice $N \sim \frac{1}{G}$; the second is Dirac's quantization condition. If our proposal is correct, roughly speaking we have one degree of freedom per Planck area, as 't Hooft has suggested in his brick-wall model. At large distances, one of course recovers the commutative limit $N \to \infty$, meaning that the particles form a continuum and the horizon behaves classically.

Also the Fock space of our theory resembles very much that of matrix theory. We have interpreted the membrane as made up of a finite number, N, of particles. So the membrane description is only an effective one, which appears in the large-distance limit. These ideas have long been advocated by 't Hooft in, for example, [3].

It is clear that very much has to be done to understand and check, if possible, this analogy, in particular the choice $N \sim \frac{1}{\ell_{\mathrm{Pl}}^2}$, as well as to understand the M(atrix) theory proposal itself. In particular, although the factor N explicitly appears in our formulae, suggesting that in the limit $N \to \infty$ both the left- and right-hand sides of (46) are zero, in principle one can get rid of this N-dependence by a redefinition of the fields X. But in that case, the generators of the group diverge and the membrane becomes an infinite

[10] We use the following notation: x^α are the classical functions representing X^α, which can be expanded in a truncated summation over membrane modes; the M in $[\cdot]_{\mathrm{M}}$ stands for "matrix commutator".

[11] We now work in 4 dimensions, see appendix A for a generalization.

[12] Notice that the truncation of the series then depends on $\|\tilde{\sigma} - \tilde{\sigma}'\|$. We are not sure that this is a reasonable assumption, but, as already remarked, the cutoff of spacetime turns out to depend on the propagator on the world-sheet. A more conservative viewpoint is to take simply $N = \frac{1}{\ell_{\mathrm{Pl}}^2}$.

plane. It is unclear whether this picture is physically meaningful[13], and the dependence on the choice of basis of the M(atrix) theory proposal has to be studied as well. Also, the commutator (11) receives a higher-order correction which could modify the resemblance to matrix theory. Note [14], however, that, in the same way, for large but finite N there are higher order corrections to the relation between the algebra of area-preserving diffeomorphisms and the SU(N) algebra, so that only in the limit are both goups the same.

Conclusion

One of the questions one would ultimately like to answer[14] concerns the number of microstates of the hole. Although a trial to get some insight into this has been made in section 4, we are still far from an answer. In particular, one would like to be able to calculate the entropy. Yet something can be learned directly from the action appearing in the S-matrix. The first term is the Polyakov term, which, imposing the constraint $h_{ij} = g_{\mu\nu}\,\partial_i X^\mu \partial_j X^\nu$, becomes the Nambu-Goto term, namely, the horizon area. More generally, it describes the world volume of the p-brane. So this suggests that, in any dimension, 't Hooft's action, which is Euclidean[15], is related to the entropy as far as the first term is concerned. It is possible that the second term gives corrections to the entropy due to the presence of ingoing and outgoing particles (in fact, the equation of motion (9) describes the oscillations of the membrane, whose area increases with the ingoing particles and decreases with the outgoing ones), and that these corrections can be included in the area-term like in a Born-Infeld action, but more work is needed in this direction. Another possible approach is to calculate the entropy from the Einstein equations one obtains in the low-energy limit.

There are other subtle points about 't Hooft's S-matrix, especially when one considers realistic, "astrophysical", black holes [15]. One, however, has to realize that the S-matrix obtained so far cannot be the whole story, but is a crude approximation to a microscopic description of the horizon. Nevertheless we did learn something from this microscopic model. The dynamics of particles at high energies, which carry shockwaves with them, can be described by a string theory action in a background including a torsion, which is related to the momentum of the particles. Furthermore, the coordinates of these particles obey the uncertainty principle, with an uncertainty proportional to Newton's constant.

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[13] This is analogous to rescaling the operators x and p of quantum mechanics, that satisfy $[x,p] = i\hbar$, by a factor of $\sqrt{\hbar}$: $X \equiv \frac{x}{\sqrt{\hbar}}$, $P \equiv \frac{p}{\sqrt{\hbar}}$, so that they obey $[X,P] = i$ even in the limit $\hbar \to 0$; in that limit, however, they diverge, and the physical interpretation as the position and momentum of a particle is in any case different; the noncommutativity in the classical limit is rather an artifact of blowing up a very small region of phase space by an infinite factor. This is similar to what happens in the case of the membrane. Therefore, one has to be careful about these rescalings.

[14] We are grateful to Finn Larsen for this remark.

[15] However, there is a factor of i in front of the action which we do not understand.

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Appendix A. Generalization to d dimensions

In this appendix we generalize some results of sections 1 and 2 to any dimension $d > 4$. This will be useful if one wants to work in the critical dimensions $d = 26$ or $d = 10$. First of all, the antisymmetric tensor appearing in (34) is equal to

$$B_{\mu_1\cdots\mu_{d-2}}(X) = \epsilon_{\mu_1\cdots\mu_{d-2}\alpha\beta}\, P^\alpha X^\beta, \tag{50}$$

so the action is

$$\begin{aligned} S &= -\frac{T_d}{2}\int \mathrm{d}^{d-2}\sigma \left(\sqrt{h}\, h^{ij} g_{\mu\nu}\, \partial_i X^\mu \partial_j X^\nu - (d-4)\sqrt{h}\right) \\ &- \frac{1}{(d-2)!}\int \mathrm{d}^{d-2}\sigma\, \epsilon_{\mu_1\cdots\mu_{d-2}\alpha\beta}\, P^\alpha X^\beta\, \epsilon^{i_1\cdots i_{d-2}} \partial_{i_1} X^{\mu_1} \cdots \partial_{i_{d-2}} X^{\mu_{d-2}}. \end{aligned} \tag{51}$$

The field-strength $H = \mathrm{d}B$ is then given by

$$H_{\mu_1\cdots\mu_{d-1}} = -(-1)^d\,(d-1)\,\epsilon_{\mu_1\cdots\mu_{d-1}\alpha}\, P^\alpha. \tag{52}$$

The equations of motion of (51) are [16]

$$\partial_i\left(\sqrt{h}\, h^{ij}\, g_{\mu\nu}\, \partial_j X^\nu\right) + h^{ij}\, g_{\mu\nu}\, \Gamma^\nu{}_{\alpha\beta}\, \partial_i X^\alpha \partial_j X^\beta = \frac{1}{(d-1)!\, T_d}\, H_{\mu\mu_1\cdots\mu_{d-2}}\, W^{\mu_1\cdots\mu_{d-2}}, \tag{53}$$

where the orientation tensor is now

$$W^{\mu_1\cdots\mu_{d-2}} = \epsilon^{i_1\cdots i_{d-2}}\, \partial_{i_1} X^{\mu_1} \cdots \partial_{i_{d-2}} X^{\mu_{d-2}}. \tag{54}$$

For a flat spacetime metric, this reduces to

$$\triangle X^\mu = \frac{1}{(d-1)!\, T_d\, \sqrt{h}}\, H^\mu{}_{\mu_1\cdots\mu_{d-2}}\, W^{\mu_1\cdots\mu_{d-2}}, \tag{55}$$

which agrees with the supergravity result obtained in the appendix of [17] for the shock wave in $d = 11$. Requirement (10) gives us

$$[H_{\mu_1\cdots\mu_{d-1}}(\tilde\sigma), X^\nu(\tilde\sigma')] = \frac{i(-1)^d\,(d-1)}{\sqrt{h}}\, \epsilon_{\mu_1\cdots\mu_{d-2}}{}^\nu\, \delta(\tilde\sigma - \tilde\sigma') \tag{56}$$

and thus

$$[X^\mu(\tilde\sigma), X^\nu(\tilde\sigma')] = -i\ell_{\mathrm{Pl}}^2\, \frac{1}{(d-2)!}\, \epsilon^{\mu\nu}{}_{\mu_1\cdots\mu_{d-2}}\, W^{\mu_1\cdots\mu_{d-2}}\, f(\tilde\sigma - \tilde\sigma'). \tag{57}$$

Notice that for $d \neq 4$, the propagator is not a logarithmic function, but (for a flat world-sheet metric)

$$f(\tilde\sigma - \tilde\sigma') = -\frac{1}{T_d\,(d-4)\,\Omega_{d-2}}\, \frac{1}{\|\tilde\sigma - \tilde\sigma'\|^{d-4}}, \tag{58}$$

where Ω_{d-2} is the surface area of a unit sphere in $d-1$ dimensions.

A possible problem is that, for our model, conformal invariance only occurs in 4 dimensions, because only then is the horizon a Euclidean string. This is at first sight inconsistent with the fact that conformal invariance requires $d = 26$, or $d = 10$ after adding fermions. However, conformal invariance re-emerges when letting the compactification radius go to zero (see reference [18] for conformal invariance arising from the usual diffeomorphism invariance of the membrane, after double-dimensional reduction). What happens in the presence of Kaluza-Klein modes or winding in the compactified dimensions, and whether conformal invariance arises after compactification as a manifestation of some other symmetry of the higher-dimensional theory, is still unknown. This will be subject of a future investigation.

Appendix B. Uniqueness of the covariant action

In this appendix we argue that the expression which should reduce to equation (1) in the appropriate limit and has until now been used as its covariant generalization should be slightly modified, in the way done in section 1.

The earlier proposed candidate [3] is

$$S \stackrel{?}{=} \int \mathrm{d}\sigma \mathrm{d}\tau \left(-\frac{T}{2} \sqrt{h} h^{ij} g_{\mu\nu}\, \partial_i X^\mu \partial_j X^\nu + P_\mu(\sigma,\tau)\, X^\mu(\sigma,\tau) \right), \tag{59}$$

where h_{ij} is the metric on the world-sheet, $h \equiv \det h_{ij}$, $g_{\mu\nu}$ is the four-dimensional metric in target space, $\tilde{\sigma} = (\sigma, \tau)$ are the coordinates on the world-sheet, and the string tension is $T = \frac{1}{8\pi G}$. Notice that this action is generally covariant in the world-sheet and in target space coordinates, and under reparametrizations of h_{ij}. P_μ now transforms as a density in the world-sheet coordinates (in contrast to our convention of section 1).

The limit without transverse momenta and curvature is then obtained taking a conformally flat metric on the world-sheet,

$$h_{ij} = \lambda(\sigma,\tau)\, \delta_{ij}, \tag{60}$$

for arbitrary λ, and working in the Minkowski approximation to Kruskal coordinates, which holds at points near to the horizon, where $r \simeq 2M$: $X^\mu = (U, V, X, Y)$. The metric is then

$$\mathrm{d}s^2 = 2\,\mathrm{d}U\mathrm{d}V + \mathrm{d}X^2 + \mathrm{d}Y^2. \tag{61}$$

Up to a sign redefinition of one of the coordinates, (59) gives us the same equations of motion as (1).

But we now show that the covariant action (59) cannot be consistently quantized. Variation of the action gives[16]

$$P_\mu \stackrel{?}{=} -T g_{\mu\nu} \triangle X^\nu. \tag{62}$$

Now we would like to promote the p's, which come from Fourier transforming the X^μ-fields in the S-matrix, to operators that satisfy (10). But if we impose this equation on

[16] Notice that throughout this paper we take the metric tensor to be constant, which is a good approximation in the cases we are interested in at the moment.

(62), we get

$$g_{\mu\nu}[X^\nu(\tilde\sigma), X^\alpha(\tilde\sigma')] \stackrel{?}{=} -i\ell_{\rm Pl}^2\, g_\mu^\alpha f(\tilde\sigma-\tilde\sigma') \tag{63}$$

and thus, contracting with $g^{\mu\beta}$,

$$[X^\beta(\tilde\sigma), X^\alpha(\tilde\sigma')] \stackrel{?}{=} -i\ell_{\rm Pl}^2\, g^{\alpha\beta} f(\tilde\sigma-\tilde\sigma'), \tag{64}$$

which obviously cannot be true, independently of what redefinitions we take or what coordinate system we choose, because the metric tensor is symmetric and the commutator antisymmetric with respect to the indices μ,ν. In particular, choosing light-cone coordinates and redefining momenta with a minus sign will not help; so the conclusion is that the above action is not consistent with covariant quantization. Remarkably enough, for 't Hooft's model (equation (1)) *was* consistent with it. This can only mean that the action was not generalized in the right way.

We will now choose a particular gauge to show where the discrepancy with 't Hooft's model comes from. In the Rindler gauge, (60) and (61), the action becomes

$$S \stackrel{?}{=} \frac{T}{2}\int {\rm d}^2\tilde\sigma\,(2U\triangle V + X\triangle X + Y\triangle Y) + \int {\rm d}^2\tilde\sigma\,(P_{\rm U}U + P_{\rm V}V + P_{\rm X}X + P_{\rm Y}Y)\,, \tag{65}$$

where $\triangle \equiv \partial_\sigma^2 + \partial_\tau^2$. Varying with respect to the four fields U, V, X and Y, we find the equations of motion

$$\begin{aligned} P_{\rm U}(\tilde\sigma) &\stackrel{?}{=} -T\,\triangle V(\tilde\sigma) \\ P_{\rm V}(\tilde\sigma) &\stackrel{?}{=} -T\,\triangle U(\tilde\sigma), \end{aligned} \tag{66}$$

and for the transverse fields

$$\begin{aligned} P_{\rm X}(\tilde\sigma) &\stackrel{?}{=} -\frac{T}{2}\,\triangle X(\tilde\sigma) \\ P_{\rm Y}(\tilde\sigma) &\stackrel{?}{=} -\frac{T}{2}\,\triangle Y(\tilde\sigma). \end{aligned} \tag{67}$$

This looks very much like a covariant generalization of 't Hooft's shift equations because the particles are allowed to have a transverse momentum —but it is not! Notice that $P_{\rm X}$ is proportional to X and $P_{\rm Y}$ proportional to Y. In the original derivation of [2], the shifts on the outgoing particle were in the direction perpendicular to the direction of the ingoing particle (in the U-V plane), as above. So if we give a particle a momentum in the X direction, the other particle must be shifted in the Y direction[17], having a relation like $P_{\rm X} \sim \triangle Y$ and $P_{\rm Y} \sim \triangle X$, not as in equation (67). Next we present an argument to see that our Lagrangian is indeed inconsistent with covariant quantization, and that the coupling between the directions U and V in the equations of motion (66) is not an artifact of the light-cone gauge. Following 't Hooft [9], we promote the p's, which come from Fourier transforming the X^μ-fields in the S-matrix, to operators that are canonically conjugated to X^μ. We are allowed to do so, because the way they came in the calculation was as a parameter that determines the momentum of the particle. Therefore they must

[17] Actually, the shift will be in all the perpendicular directions, but we consider one single plane for simplicity. This has been pointed out in section 2.

be conjugate to the position operators and satisfy (10). But one directly sees that the relation (10) cannot coexist with (66) and (67).

To understand this, let us adopt the covariant notation [9]. Up to an irrelevant overall minus sign, the only possibility is to define:

$$\begin{aligned} X^+ &= V \\ X^- &= U \end{aligned} \tag{68}$$

and

$$\begin{aligned} P^- &= P_+ = P_{\rm V} \\ P^+ &= P_- = -P_{\rm U}. \end{aligned} \tag{69}$$

Like in field theory, we have taken outgoing momentum with a minus sign, so that all momenta are ingoing. This was needed in order to get non-vanishing commutators. But then $[P_{\rm U}, U] = -i$, $[P_{\rm V}, V] = -i$, gives us

$$\begin{aligned} [P_+(\tilde\sigma), X^+(\tilde\sigma)] &\stackrel{?}{=} -i\,\delta(\tilde\sigma - \tilde\sigma') \\ [P_-(\tilde\sigma), X^-(\tilde\sigma)] &\stackrel{?}{=} +i\,\delta(\tilde\sigma - \tilde\sigma'), \end{aligned} \tag{70}$$

which is inconsistent with (10)! Otherwise we are forced to set $P_{\rm U} = U = P_{\rm V} = V = 0$. The same happens if we try to quantize the operators in the X, Y-directions, the inconsistency being much more clear because of the proportionality $P_{\rm X} \sim X$, $P_{\rm Y} \sim Y$. We then simply get $X = P_{\rm X} = Y = P_{\rm Y} = 0$, so that no transverse momentum is allowed.

Up to overall minus signs, the definition (69) is the only possibility[18]. So the action (1) *is* consistent with quantization, but if we go over to the covariant notation, the generalization is not (59).

Now the correct equations of motion are

$$\begin{aligned} P_{\rm U} &= +T\,\triangle V \\ P_{\rm V} &= -T\,\triangle U, \end{aligned} \tag{71}$$

which are essentially different from (66). Covariant quantization is then reached defining

$$\begin{aligned} X^+ &= V \\ X^- &= U \\ P_+ &= P_{\rm V} \\ P_- &= P_{\rm U}, \end{aligned} \tag{72}$$

so that we obtain

$$[U(\tilde\sigma), V(\tilde\sigma')] = -i\ell_{\rm Pl}^2\, f(\tilde\sigma - \tilde\sigma'). \tag{73}$$

This is, indeed, what is obtained if one uses the definitions of section 1.

[18] We indeed see that if we redefine U with a minus sign, we also have to modify the conjugate momentum $P_{\rm U}$ (and *vice versa*).

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