THE PHYSICAL MECHANISM OF OSCILLATORY AND FINITE AMPLITUDE INSTABILITIES IN SYSTEMS WITH COMPETING EFFECTS

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I. INTRODUCTION

The onset of convection in a horizontal layer of fluid heated from below, the so-called Rayleigh-Bénard instability, has been extensively investigated for a long time [1,2]. In recent years there has been considerable interest in the new phenomena that occur in systems in which two competing stability influencing effects with different relaxation times are present. The first example of such systems that was studied in detail is the case of thermal convection in binary mixtures.[3,4] More recently it has been pointed out that thermal convection in nematic liquid crystals can also be advantageously considered from the same point of view [5]. It has been found that under certain conditions both binary mixtures [3,4] and nematic liquid crystals [6,7], when heated from below, show oscillatory and finite amplitude instabilities. In this contribution I will discuss the common physical mechanism underlying these phenomena from the point of view of the energy balance of convective disturbances.

II. OSCILLATORY INSTABILITY

According to Chandrasekhar [8] "The onset of thermal instability will be as overstable oscillations if it is possible (at a lower adverse temperature gradient than necessary for the onset of stationary convection) to balance in a synchronous manner the periodically varying amounts of kinetic and other forms of energy with similarly varying rates of dissipation and liberation of energy". Applied to binary mixtures the above principle implies that oscillatory convection is possible if the following balance can be maintained

$$\dot{F}^{v}_{kin} + \dot{F}^{g}_{kin} = \dot{F}^{o}_{kin} \quad (1)$$

Here $\dot{F}^{v}_{kin}$ is the rate of viscous dissipation of kinetic flow energy $\dot{F}^{g}_{kin}$ is the rate of production of kinetic flow energy by the buoyancy force and $\dot{F}^{o}_{kin}$ is the rate of change of kinetic energy due to the oscillatory nature of the velocity field. For the oscillatory convective disturbance

$$v_x(\vec{r}) = -2\frac{q_z}{q_x} v \cos\omega t \sin q_x x \cos q_z z$$

$$v_y(\vec{r}) = 0 \quad (2)$$

$$v_z(\vec{r}) = 2 v \cos\omega t \cos q_x x \sin q_z z$$

($q_z = \frac{\pi}{d}$, d : thickness layer)
with frequency ω such that

$$Dq^2 \ll \omega < \chi q^2$$

(D : mass diffusion constant, χ : heat diffusivity)
one obtains [9] (per unit volume)

$$\dot{F}^{v}_{kin} = -\eta \frac{q^4}{q_x^2} v^2 \cos^2 \omega t \quad (3)$$

and

$$F^{g}_{kin} = g\rho\alpha\beta v^2 \left[\frac{1}{\chi q^2} \cos^2 \omega t + \frac{\omega}{(\chi q^2)^2} \cos\omega t \sin\omega t + \frac{S}{\omega} \cos\omega t \sin\omega t \right] \quad (4)$$

(η : viscosity, ρ : density, α : thermal expansivity, $\beta = -\frac{\partial T}{\partial z}$)
The dimension less parameter S which is defined as

$$S = \frac{k_T}{T}\frac{\alpha'}{\alpha} \quad (5)$$

(k_T : thermal diffusion ratio, α' : solutal expansivity) is assumed to be negative, meaning that due to the Soret effect the more dense component moves to the warm lower boundary and thus has a stabilizing effect. The first two terms in square brackets on the righ hand side of Eq. (4) are due to the presence of the temperature gradient where as the third term is due to the presence of the Soret driven concentration gradient. It can be shown that $\dot{F}^{o}_{kin}$ is negligible compared to $\dot{F}^{v}_{kin}$ and $\dot{F}^{g}_{kin}$. Taking this into account it follows from Eqs (1), (3) and (4) that oscillatory convection is possible for $\beta > \beta^{o}_{c}$ where

$$\frac{g\alpha\beta^{o}_{c}d}{\chi\nu} = \frac{q^6 d^4}{q_x^2} \tag{6}$$

The oscillation frequency at the critical point is given by

$$\omega_c = \chi q^2 \sqrt{-S} \tag{7}$$

We see that the $\cos^2\omega t$ term in $\dot{F}^{g}_{kin}$ serves to balance the viscous dissipation, whereas the required oscillation frequency is determined by the balancing of the $\cos\omega t \sin\omega t$ terms in $\dot{F}^{g}_{kin}$. Physically this can be interpreted that due to the oscillatory character of the convective disturbance the stabilizing effect of the concentration gradient is virtually eliminated whereas the destabilizing effect of the temperature gradient is retained. Note in this connection that the threshold given by Eq. (6) is the same as for the onset of stationary convection in an one-component liquid.

Homeotropic nematic liquid crystals heated from below [6,7] show similar characteristics as binary mixtures in which the Soret effect drives the more dense component to the warm lower boundary. Like in the case of binary mixtures the oscillatory instability in homeotropic nematic liquid crystals can be explained on the basis of the competition between a destabilizing effect and a stabilizing effect [10]. The destabilizing effect here is again the temperature gradient whereas the stabilizing effect this time is the heat focusing effect arising from a combination of the shear-director coupling and the anisotropic heat conductivity. Again applying the synchronous energy balance principle one obtains [10]

$$\frac{g\alpha\beta^{o}_{c}\ d^4}{\chi\nu} = \frac{q^6 d^4}{q_x^2} \tag{8}$$

and

$$\omega_c = \chi q^2 \sqrt{A} \tag{9}$$

The dimensionless parameter A is given by

$$A = \frac{C\chi_a}{\chi} \tag{10}$$

where C is the shear-director coupling constant, χ_a is the difference between the heat conductivity parallel and perpendicular to the director and χ is a suitable average of these quantities. Note that the threshold given by Eq. (8) is again the same as for the onset of stationary convection in an one-component liquid. Further note that the parameter A here plays the same role as the parameter S for binary mixtures.

III. FINITE AMPLITUDE INSTABILITY

Finite amplitude instabilities arise when the modification of the basic state (basic flow, basic temperature gradient) by a finite amplitude convective disturbance facilitates the onset of instability. In terms of an energy balance treatment it means that if one expands the rate of change of the energy of a (stationary) convective disturbance in terms of its amplitude

$$\dot{F} = av^2 + bv^4 + \ldots \tag{11}$$

that the coefficient b is positive.

That the coefficient b can be positive for a binary mixture, where the Soret effect drives the more dense component to the warm lower boundary, is due to the fact that convection modifies the temperature and concentration gradient to a different extent [11]. For the velocity field given by Eq. (2) with $\omega = 0$ one obtains for the mean steady-state temperature and concentration distribution

$$\frac{d\langle T\rangle}{dz} = -\beta\left[1 + \frac{v^2}{\chi^2 q^2}\cos 2q_z z\right] \tag{12}$$

$$\frac{d\langle c\rangle}{dz} = \frac{k_T}{T}\beta\left[1 - \frac{2v^2}{D^2 q^2}\sin^2 q_z z\right] \tag{13}$$

Since $D \ll \chi$ it follows from the above equations that convection has a much stronger influence on the concentration gradient than on the temperature gradient. The result is that finite amplitude convection leads to a strong reduction of the stabilizing concentration gradient while the destabilizing temperature gradient is hardly affected. Taking into account the modification of the temperature and concentration gradient one obtains for the energy balance

$$\dot{F} = \dot{F}^{v}_{kin} + \dot{F}^{g}_{kin}$$

$$= \frac{g\rho\alpha\beta}{\chi q^2} (1 + S \frac{\chi}{D}) v^2 - \frac{g\rho\alpha\beta}{2\chi^3 q^4} (1 + 3S \frac{\chi^3}{D^3}) v^4 \qquad (14)$$

Thus for

$$S < - \frac{1}{3} \frac{D^3}{\chi^3}$$

the coefficient of the v^4 term becomes positive and finite amplitude instability becomes possible.

So far no similar treatment for homeotropic liquid crystals has been worked out. The situation is complicated here by the fact that the steady state temperature distribution will not only be affected by convection but also by the effect of non-linearities on the orientation of the director [12].

IV. CONCLUDING REMARKS

The analysis of the foregoing sections allows us to state that the physical mechanism underlying both oscillatory instability and finite amplitude instability in thermal convection is the competition between a stabilizing effect with a long relaxation time and a destabilizing effect with a short relaxation time.

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