

LETTER TO THE EDITOR

GEOMETRY OF THE ROOTS OF HYDRODYNAMIC MATRICES  
IN THE DISSIPATION-FREE LIMIT

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It is shown that a hydrodynamic matrix describing the coupling between  $n$  variables that are even under time reversal and  $m$  variables that are odd has in the dissipation-free limit  $2n$  or  $2m$  (whichever is smaller) purely imaginary roots and  $|n - m|$  roots that are zero.

The problem of the maximum number of complex eigenvalues of hydrodynamic matrices has been studied by Lekkerkerker and Laidlaw<sup>1,2</sup>) and by McLennan<sup>3</sup>). Using the Onsager–Casimir symmetry relations for linear irreversible processes it has been shown that the maximum number of complex roots of a hydrodynamic matrix describing the coupling between  $n$  variables that are even under time reversal and  $m$  variables that are odd is  $2n$  or  $2m$ , whichever is smaller. The implications of this result are especially important in light-scattering spectroscopy. Indeed, to every distinct complex root there corresponds a shifted line in the spectrum of scattered light (unless of course the hydrodynamic mode that corresponds to that particular eigenvalue does not give rise to variations in the dielectric constant).

In this letter we consider the distribution of the roots of hydrodynamic matrices in the dissipation-free limit. Although an idealization, the dissipation-free limit is often considered as a first step in the study of systems that are described by complicated set of hydrodynamic equations. Recent examples of such a procedure may be found in the work of De Gennes<sup>4</sup>) on the hydrodynamic modes in Smectics-A and in the work of Balescu and Veretennicoff<sup>5</sup>) on the plasma-dynamical modes in the two-fluid model.

The linearized hydrodynamic equations for the spatially Fourier transformed variables are denoted as follows

$$\frac{\partial}{\partial t} \alpha(\mathbf{k}, t) = -\mathbf{M}(\mathbf{k}) \alpha(\mathbf{k}, t). \quad (1)$$

Here  $\alpha(\mathbf{k}, t)$  is a column vector containing the  $k$ th spatial Fourier components of the fluctuations in the variables  $\{A_i\}$  that describe the hydrodynamic state of the system. The matrix  $\mathbf{M}(\mathbf{k})$  is commonly referred to as the hydrodynamic matrix. The Onsager-Casimir symmetry relations can be written as<sup>6)</sup>

$$[\mathbf{M}(\mathbf{k}) \chi(\mathbf{k})]^\dagger = \mathbf{E} \mathbf{M}(\mathbf{k}) \chi(\mathbf{k}) \mathbf{E}. \quad (2)$$

Here  $\mathbf{E}$ , the so-called signature matrix, is a diagonal matrix with elements  $E_j$  equal to  $+1$  if the variable  $A_j$  is even under time reversal and equal to  $-1$  if  $A_j$  is odd. Further the matrix  $\chi(\mathbf{k})$  describes the mean squares of the thermal fluctuations

$$\chi_{ij}(\mathbf{k}) = \langle \alpha_i(\mathbf{k}) \alpha_j(\mathbf{k})^* \rangle. \quad (3)$$

In order to utilize the Onsager-Casimir symmetry relations to determine the distribution of the roots of the hydrodynamic matrix we work with a hydrodynamic matrix for statistically independent and normalized variables for which  $\chi(\mathbf{k})$  is a constant times the unit matrix. This does not involve any loss of generality as the hydrodynamic matrix so obtained is related via a similarity transformation to the hydrodynamic matrix for an arbitrary set of variables and thus these matrices have the same eigenvalues<sup>1,2)</sup>. If  $\chi(\mathbf{k})$  is a constant matrix then the hydrodynamic matrix has the following symmetry

$$\mathbf{M}(\mathbf{k})^\dagger = \mathbf{E} \mathbf{M}(\mathbf{k}) \mathbf{E}. \quad (4)$$

In the dissipation-free limit there are no coupling coefficients in the hydrodynamic matrix between variables with the same signature under time reversal<sup>7)</sup>. Taking this into account together with the Onsager-Casimir symmetry it follows that the hydrodynamic matrix of a system described by  $n$  variables that are even under time reversal and  $m$  that are odd, has the form

$$\mathbf{M}(\mathbf{k}) = \begin{bmatrix} 0 & \cdots & 0 & M_{1,n+1} & \cdots & M_{1,n+m} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & M_{n,n+1} & \cdots & M_{n,n+m} \\ -M_{1,n+1}^* & \cdots & -M_{n,n+1}^* & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ -M_{1,n+m}^* & \cdots & -M_{n,n+m}^* & 0 & \cdots & 0 \end{bmatrix} \quad (5)$$

where we have taken the first  $n$  variables to be even and the remaining  $m$  variables to be odd. We will now show that the skew-hermitian matrix given by eq. (5) has  $2n$  or  $2m$  (whichever is smaller) purely imaginary roots and  $|n - m|$  roots that

\* The superscript  $\dagger$  denotes the hermitian conjugate.

are zero. In the following proof we assume that  $n \geq m$ . Obviously the proof for the case  $n < m$  goes along the same lines.

The characteristic polynomial of  $\mathbf{M}$  has the following general form

$$p(\lambda) = \lambda^{n+m} - A_1 \lambda^{n+m-1} + A_2 \lambda^{n+m-2} \dots + (-1)^{n+m} A_{n+m}, \quad (6)$$

where the coefficients  $A_i$ ,  $i = 1, \dots, n+m$  are the sum of the principal minors of order  $i$  of  $\mathbf{M}$ .

The principal minor of order  $i$  consists of terms of the form

$$C_{j,k}^i = \det \left[ \begin{array}{c|c} \bigcirc & \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \\ \hline - & \bigcirc' \end{array} \right]^{j+k=i}, \quad (7)$$

$\xleftarrow{k} \quad \xrightarrow{j}$

where  $\bigcirc$  and  $\bigcirc'$  are square zero blocks of dimension  $k \leq n$  and  $j \leq m$  respectively and the  $k \times j$  block  $\begin{smallmatrix} \diagup & \diagdown \\ \diagdown & \diagup \end{smallmatrix}$  can be obtained by deleting  $m-j$  columns and  $n-k$  rows from

$$\begin{bmatrix} M_{1,n+1} & \dots & M_{j,n+m} \\ \vdots & & \vdots \\ M_{n,n+1} & \dots & M_{n,n+m} \end{bmatrix}.$$

The terms  $C_{j,k}^i$  are only non-zero when  $k = j$ , since otherwise  $C_{j,k}^i$  is the determinant of a matrix with linearly dependant rows or columns. The condition  $k = j$  can only be fulfilled for  $i \leq 2m$  (since  $j \leq m$  and  $k + j = i$ ) and  $i$  even. Therefore only the coefficients  $A_{2l}$ ,  $l = 1, \dots, m$  are non-zero and the characteristic polynomial takes the form

$$p(\lambda) = \lambda^{n-m} (\lambda^{2m} + A_2 \lambda^{2m-2} + \dots + A_{2m}). \quad (8)$$

From eq. (8) it follows immediately that there are  $n-m$  roots equal to zero. Further, since the real parts of the eigenvalues of a skew-hermitian matrix are zero, the remaining  $2m$  eigenvalues of  $\mathbf{M}$  are purely imaginary or zero. In addition, due to the special form of the characteristic polynomial, imaginary roots must occur in complex conjugated pairs.

We will now show that if the  $m$  odd variables are irreducibly coupled to the  $n$  even variables (*i.e.* there does not exist a transformation among the  $m$  odd variables decoupling one or more odd variables from the set of  $n+m$  variables) that there are no zero's among the remaining  $2m$  eigenvalues. To show this we assume that there is at least one zero among these remaining  $2m$  roots. This will happen

when  $A_{2m} = 0$  which in turn means that the rank of

$$\begin{bmatrix} M_{1,n+1} & \cdots & M_{1,n+m} \\ \vdots & & \vdots \\ M_{n,n+1} & \cdots & M_{n,n+m} \end{bmatrix}$$

is equal to or less than  $m - 1$ . In that case one can obtain from  $\mathbf{M}$  by a similarity transformation a new matrix  $\mathbf{M}'$  of the following form

$$\mathbf{M}'(k) = \begin{bmatrix} 0 & \cdots & 0 & M'_{1,n+1} & \cdots & M'_{1,n+m-1} & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & M'_{n,n+1} & \cdots & M'_{n,n+m-1} & 0 \\ -M'^*_{1,n+1} & \cdots & -M'^*_{n,n+1} & 0 & \cdots & 0 & \\ \vdots & & \vdots & \vdots & & \vdots & \\ -M'^*_{1,n+m-1} & \cdots & -M'^*_{n,n+m-1} & \vdots & & \vdots & \\ & 0 \cdots 0 & & 0 & \cdots & 0 & \end{bmatrix}. \quad (9)$$

Clearly  $\mathbf{M}'$  describes a situation in which one of the odd variables is completely decoupled from the remaining  $n + m - 1$  variables. This is in contradiction with our initial assumption that the  $m$  odd variables are irreducibly coupled to the  $n$  even variables.

Thus we may conclude that if the  $m$  odd variables are irreducibly coupled to the  $n$  even variables, the hydrodynamic matrix in the dissipation-free limit has  $2m$  purely imaginary, complex conjugated roots and  $(n - m)$  roots equal to zero. Similarly if  $n < m$  and the  $n$  even variables are irreducibly coupled to the  $m$  odd variables the hydrodynamic matrix in the dissipation-free has  $2m$  purely imaginary roots and  $(m - n)$  roots equal to zero.

In reality there is dissipation and the hydrodynamic matrix takes the form

$$\mathbf{M}(k) = \left[ \begin{array}{ccc|ccc} M_{11} & M_{12} & \cdots & M_{1n} & M_{1,n+1} & \cdots & M_{1,n+m} \\ M_{12}^* & M_{22} & \cdots & M_{2n} & \vdots & & \vdots \\ \vdots & & \ddots & \vdots & \vdots & & \vdots \\ M_{1n}^* & \cdots & & M_{nn} & M_{n,n+1} & \cdots & M_{n,n+m} \\ \hline -M_{1,n+1}^* & \cdots & -M_{n,n+1}^* & & M_{n+1,n+1} & \cdots & M_{n+1,n+m} \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ -M_{1,n+m}^* & \cdots & -M_{n,n+m}^* & & M_{n+1,n+m}^* & \cdots & M_{n+m,n+m} \end{array} \right]. \quad (10)$$

As we have shown above in the dissipation-free limit there are  $2n$  or  $2m$  (whichever is smaller) purely imaginary roots. In the purely dissipative limit ( $M_{ij} = 0$ ,  $i = 1, \dots, n$ ,  $j = n + 1, \dots, n + m$ ) the hydrodynamic matrix is hermitian and

has only real roots. Finally in the general case one can show<sup>2,3</sup>) that the maximum number of complex roots is  $2n$  or  $2m$  (whichever is smaller). Knowing the limit situations (dissipation-free limit and purely dissipative) it would be of interest to find the threshold conditions at which a pair of complex conjugated roots changes into real roots.

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