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Abstract. We provide a proof of the following "folk" theorem in (three-dimensional) VLSI-theory: given a $v \times w \times l$ rectilinear block of the three-dimensional grid ($v \leq w \leq l$), its minimum bisection width is at least vw.

Keywords and phrases: VLSI-theory, minimum bisection width. A.K. Lenstra.

1. Introduction. A VLSI-circuit can be defined as a finite graph of nodes (gates) and edges (wires) that satisfy certain additional constraints. Thompson's model of a 2-layered VLSI-chip assumes that circuits are laid out in a 2-dimensional grid, with nodes occupying single cells of the grid and wires running horizontally or vertically (ninety-degree bends allowed). At most two wires can cross in a single cell, and only "orthogonal" crossings are allowed. For details see Thompson [2]. The model is easily extended for multilayered ("three-dimensional") VLSI-chips, leading to embeddings in the 3-dimensional grid. For details see e.g. Rosenberg [1].

Lowerbound proofs for VLSI-circuits typically make use of the following argument. Consider a cutset D consisting of (say) d edges, and let it dissect the circuit in two parts Y and B. {This means that every path between a Y-node and a B-node must contain an edge in D.} If one can determine or estimate the information transfer I (in bits) required for Y and B in the course of the computation, then under reasonable assumptions $^{I}/d$ will be a lowerbound on the computation time of the circuit. Traditionally one tries to find a cutset of a smallest possible number of edges that dissects the circuit in two balanced halves.

<u>Definition</u>. Let $G = \langle V,E \rangle$ be a finite (undirected) graph and $S \subseteq V$. We say that $D \subseteq E$ bisects S in G if there are $Y,B \subseteq V$ such that

(i) $Y \cup B = S$ and $Y \cap B = \phi$,

(ii) $|Y| \leq |B| \leq |Y|+1$,

(iii) every path between a Y-node and a B-node contains an edge of D.

The minimum bisection width of S in G, notation: mbw(G,S), is the number of edges in the smallest D that bisects S in G.

If $S \equiv V$ we speak of the minimum bisection width of G, notation: mbw(G). The nodes of Y will be called "yellow" and of B "blue". Necessarily $|Y| = \lfloor \frac{1}{2}s \rfloor$ and $|B| = \lceil \frac{1}{2}s \rceil$, with $s = |S|$.

Thompson [2], p.52 notes that the mbw of a w x w square block of the 2-dimensional grid is w + (w mod 2) but gives no proof of this fact. As the result is intuitively obvious it has become a folk theorem in this part of VLSI-theory. Yet we feel that a rigourous proof is required. In this note we show the following more general result. Let G be a v x w x l rectilinear block of the 3-dimensional grid with $v \leq w \leq l$, then $mbw(G) \geq vw$.

<u>Definition</u>. $G(v,w,l) = \langle V,E \rangle$ is the graph $\subseteq \mathbb{Z}^3$ with node-set $V = \{(i,j,k) \mid 1 \leq i \leq v \ \& \ 1 \leq j \leq w \ \& \ 1 \leq k \leq l)\}$ and an edge between every two nodes that differ by 1 in exactly one coordinate.

2. <u>The folk theorem</u>. Consider a minimum bisection of G(v,w,l) into parts Y and B, using a set $D \subseteq E$ of d edges ($d \equiv$ the minimum bisection width). Let nodes in Y and B be called "yellow" and "blue" respectively. An edge will be called yellow or blue if it connects two yellow or two blue nodes, respectively. An edge will be called "green" if it connects a yellow node and a blue node. {By definition every path from a Y-node to a B-node must contain a green edge. It also follows that all green edges belong to D.} We will estimate the number of green edges in G(v,w,l). View G as a collection of v "planes" of size w x l, with $w \leq l$. {Thus a plane is a w x l rectangle of the two-dimensional grid.}

Lemma 2.1. Suppose a plane contains at least x nodes from Y and x nodes from B. Then it contains $\geq \min\{w, 2\sqrt{x}\}$ green edges.

Proof.

Let the plane be P. One can view P as a collection of w or l "parallel" lines. A line will be called yellow or blue if it contains yellow or blue edges only. A line will be called green if it contains at least one green edge. Consider the lines in one direction. If all lines are green, then we have at least as many green edges and the estimate of $\min\{w,l\} = w$ follows. Suppose in both directions there is at least one yellow or blue line. Because the lines intersect they must both be yellow or both be blue. Assume they are both yellow (without loss of generality), which implies that there can be no blue lines. It follows that every blue node must lie on the intersection of two green lines. Because there are at least x blue nodes there are at least as many intersections of green lines. Suppose there are x_1 green lines in one direction and x_2 in the other. The number of green edges will be $\geq x_1+x_2$, whereas for the number of intersections $x_1x_2 \geq x$. Under the latter constraint $x_1+x_2 \geq 2\sqrt{x}$. □

Proposition 2.2. The minimum bisection width of a w x l rectilinear block of the two-dimensional grid with $w \leq l$ is $w + (l \bmod 2)$, for $w > 1$.

Proof.

Let the block be G. Clearly G can be bisected by a "vertical" cut using a number of edges as stated. Only when l is odd there will be one "zig zag" in the vertical cut, accounting for the one extra cutting edge. For a lowerbound apply lemma 2.1. with $x = \lfloor \frac{1}{2}wl \rfloor$. When l is even (hence $l \geq 2$) we have $x = \frac{1}{2}l.w$ and the number of green edges is bounded by $\min\{w, 2\sqrt{\frac{1}{2}lw}\} = w$, using that $l \geq w$. Thus the upper- and lowerbound on the mbw match. Let l be odd. The same argument would still result in a lowerbound of w, although $\lceil 2\sqrt{x} \rceil = \lceil 2\sqrt{\lfloor \frac{1}{2}wl \rfloor} \rceil \geq w+1$ in this case. {We exclude the degenerate case that w equals 1.} Consider the first part of the proof of lemma 2.1. again. Distinguish the following cases:

(i) all w lines in one direction are green. This leads to at least w green edges in (say) the horizontal direction. The l vertical lines cannot be all yellow or blue, or else we would not have a proper bisection (l odd). Thus at least one vertical line is green, and the estimate of the number of green edges becomes w + 1.

(ii) all l lines in the other direction are green. This leads to at least l green lines in the vertical direction. Either w is even and the estimate becomes $l \geq w + 1$, or w is odd and the same argument as before shows that there must be at least one extra green line in the horizontal direction and (hence) at least l + 1 green edges total. Thus at least w + 1 green edges are required when l is odd, and again the upper- and lowerbound on the mbw match. □

Observe that proposition 2.2. proves Thompson's remark that the minimum bisection width of a w x w block is w + (w mod 2) ([2], p.52).

Theorem 2.3. The minimum bisection width of a v x w x l rectilinear block of the three-dimensional grid with $v \leq w \leq l$ is at least vw.

Proof.

View G(v,w,l) as a stack of v planes $P_1,\dots,P_v$ of size w x l each. Let P_i $(1 \leq i \leq v)$ contain y_i yellow nodes and hence $wl - y_i$ blue nodes. Let t be a threshold value with $0 < t \leq wl$, t not necessarily integer. We will fix t at $t = \frac{1}{4}w^2$. Divide the planes into the following three categories:

$X_1 = \{P_i \mid 0 \leq y_i \leq t\}$
$X_2 = \{P_i \mid t < y_i < wl-t\}$
$X_3 = \{P_i \mid wl-t \leq y_i \leq wl\}$

and let $x_i = |X_i|$ $(1 \leq i \leq 3)$. Clearly $x_1+x_2+x_3 = v$. Also note that the division of the planes into categories based on the number of blue nodes is symmetric to the current one. Define $\bar{\alpha} = \min \{y_i \mid P_i \in X_1\}$ and $\bar{\bar{\alpha}} = \max \{y_i \mid P_i \in X_1\}$, $\bar{\beta} = \min \{y_i \mid P_i \in X_2\}$ and $\bar{\bar{\beta}} = \max \{y_i \mid P_i \in X_2\}$, and $\bar{\gamma} = \min \{y_i \mid P_i \in X_3\}$ and $\bar{\bar{\gamma}} = \max \{y_i \mid P_i \in X_3\}$. We estimate the number of green edges in the graph. Consider the planes in X_1, X_2, and X_3.

Planes in X_1:

There are at least $\bar{\alpha}$ yellow nodes and $wl - \bar{\bar{\alpha}}$ blue nodes in every plane in X_1. Because $\bar{\alpha},\bar{\bar{\alpha}} \leq t$ and $t \leq \frac{1}{2}wl$ it follows that every plane contains at least $\bar{\alpha}$ yellow nodes and $\bar{\alpha}$ blue nodes, and lemma 2.1. can be applied. This leads to an estimate of min $\{w, 2\sqrt{\bar{\alpha}}\} = 2\sqrt{\bar{\alpha}}$ green edges in every plane of X_1 (using that $2\sqrt{\bar{\alpha}} \leq 2\sqrt{t} = w$).

Planes in X_2:

By their very definition planes in X_2 contain at least t yellow nodes and t blue nodes each. By lemma 2.1. it follows that every plane in X_2 contains at least min $\{w, 2\sqrt{t}\} = w$ green edges.

Planes in X_3:

There are at least $\bar{\gamma}$ yellow nodes and $wl - \bar{\bar{\gamma}}$ blue nodes in every plane of X_3, hence at least $wl - \bar{\bar{\gamma}}$ nodes of each kind. By lemma 2.1. it follows that every plane in X_3 contains at least min $\{w, 2\sqrt{wl-\bar{\bar{\gamma}}}\} = 2\sqrt{wl-\bar{\bar{\gamma}}}$ green edges (using that $wl - \bar{\bar{\gamma}} \leq t = \frac{1}{4}w^2$).

The remainder of the argument is tedious. If $x_1 = x_3 = 0$ then $x_2 = v$ and the bound of vw green edges follows (because the planes in X_2 contribute w edges each). Thus we may assume that $x_1 \neq 0$ or $x_3 \neq 0$, and without loss of generality we can take $x_1 \neq 0$. We distinguish the following three cases.

Case I: $x_1 \geq \frac{1}{2}w$.

If all planes would belong to X_1 (i.e., $x_1 = v$) then the number of yellow nodes would be bounded by $\frac{1}{4}vw^2$ and be less than $|Y| = \lfloor \frac{1}{2}vwl \rfloor$, a contradiction. Thus $x_1 < v$, and we can define $\bar{\bar{\delta}} = \max \{y_i | P_i \in X_2 \cup X_3\}$ $(\neq 0)$. By slightly extending the argument above, a plane $P_i \in X_1$ contains at least $2\sqrt{y_i}$ green edges. Because $y_i \leq \frac{1}{4}w^2$ we can estimate this at $\frac{4}{w}y_i$ green edges per plane. Considering the wl lines perpendicular to the planes it can be noted that at least $\bar{\bar{\delta}} - \bar{\alpha}$ must be green and thus account for as many additional green edges. Now let $Y_1 = \sum_{P_i \in X_1} y_i$, $Y_2 = \sum_{P_i \in X_2} y_i$, and $Y_3 = \sum_{P_i \in X_3} y_i$, and observe that $\bar{\bar{\delta}} \geq \frac{(Y_2+Y_3)}{v-x_1} \geq \frac{2}{w}(Y_2+Y_3)$ by using the assumed bound on x_1. The total number of green edges can be estimated at:

$$\sum_{P_i \in X_1} \frac{4}{w} y_i + \bar{\bar{\delta}} - \bar{\alpha} \geq \frac{4}{w} Y_1 + \frac{2}{w}(Y_2+Y_3) - \bar{\alpha} =$$

$$= \frac{2}{w}(Y_1+Y_2+Y_3) + \frac{2}{w} Y_1 - \bar{\alpha} = \frac{2}{w} \lfloor \tfrac{1}{2}vwl \rfloor + \frac{2}{w} Y_1 - \bar{\alpha}.$$

Note that $\frac{2}{w} \lfloor \frac{1}{2}vwl \rfloor \geq vl - \frac{1}{w}$, and that $\frac{2}{w} Y_1 - \bar{\alpha} \geq \frac{2}{w} . x_1 \bar{\alpha} - \bar{\alpha} \geq 0$.

Thus the bound becomes vl ($\geq$vw) green edges, using that the bound is integer.

<u>Case</u> II: $x_1 < \frac{1}{2}w$ (but $x_1 \neq 0$) and $x_3 = 0$.

As before it can be argued that $x_1 < v$ and (hence) that $x_2 > 0$. The planes of X_1 contain at least $2\sqrt{\bar\alpha} \geq \frac{4}{w}\bar\alpha$ green edges each, and the planes of X_2 at least w green edges each. The lines perpendicular to the planes account for an additional $\bar{\bar\beta} - \bar\alpha$ green edges. Observe that $\bar{\bar\beta} \geq \frac{1}{v}\lfloor \frac{1}{2}vwl \rfloor \geq \frac{1}{2}wl - \frac{1}{2v}$ and (hence) that $\bar{\bar\beta} \geq \frac{1}{2}wl$. {Clearly we can exclude the case v = 1.} The number of green edges can be estimated at

$$\begin{aligned}\frac{4}{w}\bar\alpha x_1 + x_2 w + \bar{\bar\beta} - \bar\alpha &= x_1\left(\frac{4}{w}\bar\alpha + \frac{\frac{1}{2}wl-\bar\alpha}{x_1}\right) + x_2 w + \bar{\bar\beta} - \tfrac{1}{2}wl \geq \\ &> x_1\left(\frac{4}{w}\bar\alpha + 1 - \frac{2}{w}\bar\alpha\right) + x_2 w + \bar{\bar\beta} - \tfrac{1}{2}wl \geq \\ &\geq x_1 l + x_2 w \geq \\ &\geq vw\end{aligned}$$

(using that $x_1 + x_2 = v$).

<u>Case III</u>: $x_1 < \frac{1}{2}w$ (but $x_1 \neq 0$) and $x_3 \neq 0$.

The planes of X_1 contain at least $2\sqrt{\bar\alpha} \geq \frac{4}{w}\bar\alpha$ green edges each, the planes of X_2 at least w green edges each, and the planes of X_3 at least $2\sqrt{wl-\bar{\bar\gamma}} \geq \frac{4}{w}(wl-\bar{\bar\gamma}) = 4l - \frac{4}{w}\bar{\bar\gamma}$ green edges each. The lines perpendicular to the planes account for an additional $\bar{\bar\gamma} - \bar\alpha$ green edges. By symmetry we may assume that $x_3 < \frac{1}{2}w$. The number of green edges can be estimated as follows

$$\begin{aligned}&\frac{4}{w}\bar\alpha x_1 + x_2 w + x_3\left(4l - \frac{4}{w}\bar{\bar\gamma}\right) + \bar{\bar\gamma} - \bar\alpha = \\ &= x_1\left(\frac{4}{w}\bar\alpha + \frac{\frac{1}{2}wl-\bar\alpha}{x_1}\right) + x_2 w + x_3\left(4l - \frac{4}{w}\bar{\bar\gamma} + \frac{\bar{\bar\gamma} - \frac{1}{2}wl}{x_3}\right) \geq \\ &\geq x_1\left(\frac{4}{w}\bar\alpha + 1 - \frac{2}{w}\bar\alpha\right) + x_2 w + x_3\left(4l - \frac{4}{w}\bar{\bar\gamma} + \frac{2}{w}\bar{\bar\gamma} - l\right) \geq \\ &\geq x_1 . l + x_2 w + x_3\left(3l - \frac{2}{w}\bar{\bar\gamma}\right) \geq \\ &\geq x_1 l + x_2 w + x_3 l \geq \\ &\geq vw.\end{aligned}$$

It follows that in all cases a lowerbound of vw green edges can be shown and thus that the minimum bisection width of G(v,w,l) is at least vw($v \leq w \leq l$). □

It follows in particular that for l even the minimum bisection width of a v x w x l block ($v \leq w \leq l$) is precisely vw.

3. References.

[1] Rosenberg, A.L., Three-dimensional VLSI: a case study, J. ACM 30 (1983) 397 - 416.

[2] Thompson, C.D., A complexity theory for VLSI, Ph.D. Thesis, Techn. Rep. CMU-CS-80-140, Dept. of Computer Science, Carnegie-Mellon University, Pittsburgh, 1980.