Non-Hamiltonian monodromy

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Abstract

It has been proved by Maorong Zou and Nguyen Tien Zung that nontrivial monodromy occurs in every integrable two degrees of freedom Hamiltonian system which has a proper momentum mapping with a focus-focus singularity. In this paper we describe a generalization of this result which does not use the Hamiltonian structure of the system. This can be used to prove that nontrivial monodromy occurs in the two best known integrable cases of rolling bodies: the heavy solid of revolution rolling on a horizontal plane and the sphere rolling in a circular bowl with an upward dimple at the bottom.

1 Introduction

Because one of the main points of this paper is that monodromy occurs under seemingly quite weak assumptions, we begin with listing these hypotheses.

Assumptions *Let v and w be two smooth vector fields on a smooth four-dimensional manifold M. Let $p \in M$ and let f be a mapping from M to $\mathbf{R}^2$ which is smooth in $M \setminus \{p\}$, continuous at p, and satisfies $f(p) = 0$. For every $c \in \mathbf{R}^2$ we write $F_c = \{x \in M \mid f(x) = c\}$ for the fiber of f over c. We assume:*

a) *$v(p) = 0$ and $\mathrm{D}v(p)$ has no real eigenvalues. Moreover, one complex conjugate pair of eigenvalues of $\mathrm{D}v(p)$ has negative real part while the other has positive real part.*

b) *The vector fields v and w commute.*

c) *The derivative of f is equal to zero in the direction of both vector fields v and w.*

d) *At each point $x \in F_0 \setminus \{p\}$, the vectors $v(x)$ and $w(x)$ are linearly independent and the rank of $\mathrm{D}f(x)$ is equal to two.*

e) *The subset $F_0 = f^{-1}(\{0\})$ of M is both compact and connected.*

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Remark 1.1 If F_0 is not connected, but is equal to the union of two disjoint closed subsets K and L, where $p \in K$ and K is compact and connected, then one can replace M by an open neighborhood $\widetilde{M}$ of K such that $\widetilde{M} \cap F_0 = K$. One then requires that Assumptions a)–e) hold with M replaced by $\widetilde{M}$.

One says that v has a zero at p of *focus-focus type* if Assumption a) holds. $\oslash$

Before we list our conclusions in Theorem 1.2 below, we first discuss the literature about the Hamiltonian case and the applications which we have in mind for our non-Hamiltonian generalization. The *Hamiltonian case* of the theorem occurs when M is provided with a symplectic form ω and $f(x) = (g(x),\, h(x))$, where g and h are smooth real-valued functions on M such that $\{g,\, h\} = 0$. Here $\{g,\, h\}$ denotes the Poisson bracket-expression of g and h with respect to the symplectic form. If w and v are the Hamiltonian vector fields defined by g and h, respectively, then Assumptions b) and c) follow from $\{g,\, h\} = 0$. Assumption a) says that $\mathrm{d}h(p) = 0$ and that the eigenvalues of the infinitesimally symplectic transformation

$$\omega_p{}^{-1} \circ \mathrm{D}^2\, h(p) \;: \mathrm{T}_p\, M \to \mathrm{T}_p\, M$$

are neither real nor purely imaginary. Assumption d) holds if for every $x \in F_0 \setminus \{p\}$ the derivatives $\mathrm{d}g(x)$ and $\mathrm{d}h(x)$ are linearly independent.

In the Hamiltonian case, the monodromy has first been computed in a number of special cases, using the specific form of the equations. We mention the spherical pendulum, (Duistermaat [12], Cushman [6], Cushman and Bates [9, IV.5]), the Hamiltonian Hopf bifurcation, (van der Meer [15, Cor. 4.14, p.83], Duistermaat [13]), the Lagrange top (Cushman and Knörrer [10], Cushman and Bates [9, V, 7.3]), Cushman and van der Meer [11]), the champagne bottle (Bates [2]), and the magnetic spherical pendulum (Cushman and Bates [8]). In [12] the nontriviality of the monodromy is also presented as the coarsest obstruction to the existence of global action-angle variables. Cases of confluence of two focus-focus singularities have been studied by Bates and Maorong Zou [3].

The general Theorem 1.2, c) in the Hamiltonian case has been obtained by Maorong Zou [19, Th. 1.1], who ascribed to Flaschka the idea that the monodromy is determined by the local behaviour near the point p. The conclusion in [19, Lemma 3.1], which is used in the proof [19, Th. 1.1], is not quite correct since one can pinch off a small sphere from the torus. Because the pinching circle is trivial in the first homology group, the monodromy would be trivial in this case. However, without too much trouble, a version of [19, Lemma 3.1] can be given which leads to a correct proof of [19, Th. 1.1].

An independent proof of Theorem 1.2, c) in the Hamiltonian case has been obtained by Nguyen Tien Zung [18]. This paper also contains a Hamiltonian version of our Proposition 1.5 below, and includes the generalization to a cycle of n focus-focus points. The proofs in [18] make essential use of the Hamiltonian structure. The conclusion of [18], that in the Hamiltonian case the monodromy for a cycle of n focus-focus points is equal to the n-th power of the monodromy for $n = 1$, can also been seen as a consequence of our Proposition 1.7 below, which implies that the contributions to the monodromy of all focus-focus points have the same sign.

We got interested in the *non-Hamiltonian case* when one of us (R.C.) realized that the assumptions are satisfied in the case of the Routh sphere, a special case of a heavy solid of revolution which is rolling without slipping on a horizontal plane. The rolling condition is a nonholonomic constraint which destroys the Hamiltonian nature of the system, nevertheless we get the same monodromy as in the Hamiltonian case. For more details, see Cushman [7]. Using the equations found by Chaplygin [5], one can actually apply our Theorem 1.2 to a general solid of revolution which is rolling on a horizontal plane. We hope to be able to explain this at another occasion. Another application would be to a dynamically symmetric sphere which is rolling in a circular bowl with an upward dimple at the bottom. Here the proof can be given by extending the analysis of Hermans [14] to the case with the elevation at the center. The particle moving on the bottom of the champagne bottle, a Hamiltonian system for which the monodromy has been determined by Bates [2], can be considered as a limiting case of the rolling sphere when the radius of the sphere tends to zero.

Theorem 1.2 *The Assumptions lead to the following conclusions:*

a) *$F_0 \setminus \{p\}$ is diffeomorphic to the cylinder $(\mathbf{R}/2\pi\,\mathbf{Z}) \times \mathbf{R}$ and F_0 is homeomorphic to the one point compactification of this cylinder. Near p, F_0 is equal to the union two two-dimensional submanifolds of M which intersect transversally at the point p. If $S_\pm$ denotes the set of $x \in M$ such that $\phi^t(x) \to p$ as $t \to \pm\infty$, then $F_0 = S_+ = S_-$.*

b) *There is an open neighborhood $\widetilde{M}$ of F_0 in M and a simply connected open neighborhood U of 0 in $\mathbf{R}^2$ such that the restriction of f to $\widetilde{M} \setminus F_0$ defines a locally trivial 2-torus fibration over $U \setminus \{0\}$.*

c) *Let $\alpha : [0,\, 1] \to U \setminus \{0\}$ be a smooth closed curve in $U \setminus \{0\}$ which winds once around the origin in the positive direction. The 2-torus bundle $F_{\alpha(\varphi)}$, $\varphi \in [0,\, 1]$, over the loop α has monodromy $\mathcal{M} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ with respect to a suitable basis of generators of the two-dimensional lattice $\mathrm{H}_1 = \mathrm{H}_1\left(F_{\alpha(0)} \cap \widetilde{M},\, \mathbf{Z}\right)$.*

Remark 1.3 Because the cylinder is diffeomorphic to the two-dimensional sphere minus two points $p_\pm$, Theorem 1.2, a) implies that F_0 can be viewed as the sphere with p_+ and p_- identified with the point p, embedded into M in such a way that the tangent spaces of the sphere at the points $p_\pm$ are identified with complementary subspaces $T_\pm$ of T_pM. Alternatively, F_0 can also be described as a two-dimensional torus in which a generating circle is pinched to the point p.

The set S_+ and S_- is called the *stable and unstable manifold* of p for the vector field v, respectively. Because $S_+ = S_- = F_0$, the description of F_0 in Theorem 1.2, a) implies that $S_\pm$ can be viewed as an immersed smooth submanifold of M, with $\mathrm{T}_p\, S_\pm$ equal to the tangent space at p of one of the two manifolds which intersect at p. Actually, $\mathrm{T}_p\, S_+$ and $\mathrm{T}_p\, S_-$ is equal to the eigenspace of $\mathrm{T}_p\, v$ on which the real part of $\mathrm{T}_p\, v$ is negative and positive, respectively. $\oslash$

The proof of Theorem 1.2 is based on an analysis of the $\mathbf{R}^2$-action which is induced by the vector fields v and w. More precisely, to the conclusions of Theorem 1.2 we may add the following propositions:

Proposition 1.4 *Let ϕ^t and ψ^s denote the flow of v and w after time t and s, respectively. Then $(s, t) \mapsto \psi^s \circ \phi^t$ defines a smooth action of $\mathbf{R}^2$ on $\widetilde{M}$. For each $c \in U \setminus \{0\}$, $\widetilde{F}_c = F_c \cap \widetilde{M}$ is equal to an orbit of the $\mathbf{R}^2$-action. The stabilizer subgroup of the $\mathbf{R}^2$-action on $\widetilde{F}_c$ is a two-dimensional lattice Γ_c in $\mathbf{R}^2$ which depends smoothly on $c \in U \setminus \{0\}$. Under the canonical isomorphism of Γ_c with $\mathrm{H}_1\left(\widetilde{F}_c, \mathbf{Z}\right)$, the action of the monodromy operator $\mathcal{M}$ on H_1 corresponds to the automorphism of $\Gamma_{\alpha(0)}$ which one obtains by following the elements of $\Gamma_{\alpha(\varphi)}$ continuously as φ runs from 0 to 1. Finally, $F_0 \setminus \{p\}$ is equal to an orbit and its stabilizer subgroup is isomorphic to $\mathbf{Z}$.*

Proposition 1.5 *There exist unique smooth functions σ and τ on U such that $\sigma(0) > 0$ and the flow of the vector field $u = (\sigma \circ f)\, w + (\tau \circ f)\, v$ defines a free action of the circle group $\mathbf{R}/2\pi\mathbf{Z}$ on $\widetilde{M} \setminus \{p\}$.*

If $c \in U \setminus \{0\}$, then a u-circle in F_c defines a generator $\delta_1 = \delta_1(c)$ of the group of elements of $\mathrm{H}_1(F_c, \mathbf{Z})$ which are fixed under the monodromy operator $\mathcal{M} = \mathcal{M}_c$. For the second generator $\delta_2 = \delta_2(c)$ we can take a v-solution curve, starting and ending on a u-circle in F_c, followed by a part of the u-circle in order to close it up.

The linear transformation $\mathrm{D}u(p)$ in $\mathrm{T}_p\, M$ defines a complex structure in $\mathrm{T}_p\, M$, which in turn defines an orientation in $\mathrm{T}_p\, S_-$, the eigenspace of $\mathrm{T}_p\, v$ on which the real part of $\mathrm{T}_p\, v$ is positive. This orientation extends in a continuous fashion to an orientation of $\mathrm{T}_x\, M/\, \mathrm{T}_x\, S_+$, for $x \in S_+$, $x \neq p$.

Proposition 1.6 *If, for $x \in S_+$, $x \neq p$, the orientation of $\mathrm{T}_x\, M/\, \mathrm{T}_x\, S_+$ defined by the complex structure in $\mathrm{T}_p\, M$ agrees with the pullback of the orientation of $\mathbf{R}^2$ by means of $\mathrm{T}_x\, f$, then we can take (δ_1, δ_2) as the ordered basis of H_1 in Theorem 1.2, c). If the orientations do not agree, then we get the inverse monodromy matrix on this basis of H_1.*

Proposition 1.7 *In the Hamiltonian case, the orientations in Proposition 1.6 always agree.*

Remark 1.8 In the examples of the spherical pendulum [12] and the Hamiltonian Hopf bifurcation [13], which are Hamiltonian systems, the monodromy matrix indeed has the minus sign in the upper right corner if one goes around the origin in U in the positive, the counter-clockwise direction. In [12] the plus sign appeared in the upper right corner because the monodromy was taken over a clockwise loop around the origin in U.

In the non-Hamiltonian case the orientations in Proposition 1.6 need no longer agree, because the mapping f is no longer determined by the vector fields v and w as in the Hamiltonian case. We think that for a rolling solid of revolution one can have a cycle of focus-focus points where the monodromies cancel each other. Such a cycle occurs when both equilibrium positions of the body with the symmetry axis in the vertical direction are unstable, the heights of the center of mass are equal and larger than the height of the center of mass at all other positions. The heteroclinic orbits then consist of the body rolling over a half turn along a meridian. In the case that the body is symmetric with respect to the reflection about the center of mass, the orientations near the two equilibria can be compared by using a corresponding reflection in the phase space as a time reversing map. $\oslash$

Remark 1.9 It is understood that everywhere the word "smooth" may be replaced by "real analytic" or, in the other direction, by "of class C^k", where k is a given integer, $k \geq 1$. In the Hamiltonian case, the assumption that the vector fields v and w and the symplectic form are of class C^k would imply that the mapping f is of class C^{k+1}, so in this case the fibration has one higher degree of differentiability than the $\mathbf{R}^2$-action. $\oslash$

2 The Singular Fiber

2.1 We recall the notation ϕ^t and ψ^s for the flow of the vector field v and w after time t and s, respectively. We begin with collecting some facts about the stable and unstable manifold S_+ and S_-, defined as the set of $x \in M$ such that $\phi^t(x)$ converges to p when $t \to \pm\infty$.

Let us write $V = \mathrm{D}v(p)$, which is regarded as a linear transformations in $T = \mathrm{T}_p\, M$. Let T_+ and T_- be the V-invariant two-dimensional linear subspace of T such that the eigenvalues of the restriction V_+ and V_- of V to T_+ and T_- has negative and postive real part, respectively. If B is an open neighborhood of p in M, we denote by $S_\pm^B$ the set of $x \in M$ such that $\phi^t(x) \in B$ when $\pm t \geq 0$. The hyperbolicity of v at p implies that there exists an open neighborhood B of p in M such that $S_\pm^B$ is a smooth connected submanifold of M, with tangent space at p equal to $T_\pm$. In particular $S_\pm^B$ is two-dimensional and $S_\pm^B \setminus \{p\}$ is connected as well. It follows from the definition that $S_\pm$ is equal to union over all $t \in \mathbf{R}$ of the two-dimensional manifolds $\phi^t\left(S_\pm^B\right)$. In particular $S_\pm$ and $S'_\pm := S_\pm \setminus \{p\}$ are connected.

Because $f(x) = f(\phi^t(x))$ converges to zero when $x \in S_\pm$, $t \to \pm\infty$, we find that $S_\pm \subset F_0$.

Since F_0 is a compact subset of M, which is locally invariant under the v-flow, it is globally invariant in the sense that for every $t \in \mathbf{R}$ and $x \in F_0$, the flow $\phi^t(x)$ is well-defined and $\phi^t(x) \in F_0$.

In view of the rank condition for f in Assumption d), the set $F'_0 : F_0 \setminus \{p\}$ is a smooth two-dimensional submanifold of M. It follows that

$$\phi^t\left(S_\pm^B\right) \setminus \{p\}$$

is an open subset of F'_0 and therefore $S_\pm \setminus \{p\}$, the union of these over all $t \in \mathbf{R}$, is an open subset of F'_0 as well.

2.2 The compact set F_0 is also globally invariant under the w-flow. Because $[v,\, w] = 0$, $\phi^t \circ \psi^s = \psi^s \circ \phi^t$, which implies that $(s,\, t) \mapsto \psi^s \circ \phi^t$ defines an action of the additive group $\mathbf{R}^2$ on F_0. We also find that the common fixed point set Z of the ϕ^t is invariant under ψ^s. Because p is an isolated point of Z, it is a fixed point for the ψ^s, thus $w(p) = 0$. Moreover, if $x \in S_\pm$ then $\phi^t\left(\psi^s(x)\right) = \psi^s\left(\phi^t(x)\right)$ converges to $\psi^s(p) = p$ when $t \to \pm\infty$, so $S_\pm$ is not only invariant under the v-flow, but also under the w-flow. Because $\{p\}$ is a fixed point for both flows, $S'_\pm$ is invariant for the $\mathbf{R}^2$-action as well.

Because the vector fields v and w are linearly independent at each point of the 2-dimensional smooth manifold $F'_0 := F_0 \setminus \{p\}$, each orbit O of the $\mathbf{R}^2$-action in F'_0 is open in F'_0. Since the complement of O in F'_0 (which is equal to the union of the other orbits in F'_0) is also open, we deduce that O is a connected component of F'_0. In the same way the invariant open subset $S'_\pm$ of F'_0 is a union of orbits. Because $S'_\pm$ is connected, we conclude that $S'_\pm$ is equal to an orbit in F'_0.

Let O be an orbit in F'_0. It is open and closed in F'_0, which implies that it is open in $F_0 = F'_0 \cup \{p\}$. If p does not belong to the closure of O in M, then O is also closed in F_0 and hence is equal to F_0 because F_0 is connected. This is in contradiction with $p \in F_0$. Thus p belongs to the closure of O. On the other hand, $O \cup \{p\}$ is equal to the complement of the other orbits in F'_0, which are open in F_0. Hence $O \cup \{p\}$ is closed in F_0 and therefore closed in M. In other words, p is the unique limit point in $M \setminus O$ of the orbit O. We conclude that for any orbit O in F'_0, the closure of O in M is equal to $O \cup \{p\}$. This holds in particular for $O = S_\pm$.

2.3 In general, if O is an orbit of an $\mathbf{R}^2$-action and $x \in O$, then the mapping $(s,\, t) \mapsto \psi^s \circ \phi^t(x)$ induces a diffeomorphism from $\mathbf{R}^2/\Gamma_x$ onto O. Here $\Gamma_x := \{(s,\, t) \mid \psi^s \circ \phi^t(x) = x\}$ denotes the stabilizer subgroup of the point x. Γ_x is a closed additive subgroup of $\mathbf{R}^2$ and does not depend on the choice of $x \in O$. For this reason we write Γ_O instead of Γ_x.

Suppose now that O is an orbit in F'_0 and that $\Gamma_O \cap (\{0\} \times \mathbf{R}) \neq \emptyset$. Then the flow of v in O would be periodic, with a fixed common period $t_O > 0$. Because periodic solutions in O which start near p and leave a fixed neighborhood of p need arbitrily long time for this, we deduce that the v-solutions in O which start close to p remain close to p. However, the hyperbolicity of v at p implies that v does not have periodic solutions which remain close to p other than p. Thus we arrive at a contradiction.

Because $\Gamma_O \cap (\{0\} \times \mathbf{R}) = \emptyset$, for every $x \in O$ the integral curve $\phi^t(x)$ runs out of every compact subset of O when $|t| \to \infty$. Combined with the fact that p is the only limit point in $M \setminus O$ of O (and that O is contained in the compact subset F_0 of M), it follows that $\phi^t(x)$ actually converges to p as $t \to \infty$ and also as $t \to -\infty$. In other words, $x \in S_+$ and $x \in S_-$. Recalling that S'_+ and S'_- are orbits in F'_0, if follows that $O = S'_+ = S'_-$. Because this holds for every orbit O in F'_0, we deduce that $F'_0 = S'_+ = S'_-$ and $F_0 = S_+ = S_-$. We have proved the last statement in Theorem 1.2, a).

2.4 In order to obtain a better understanding of the structure of $S_\pm$ as an $\mathbf{R}^2$-orbit and also as a first step in the proof of Proposition 1.5, we take a closer look at the linear transformation $W = \mathrm{D}w(p)$ in $T = \mathrm{T}_p\, M$. Because $[v,\, w] = 0$ we have $[V,\, W] = 0$, which implies that $T_\pm$ is W-invariant. We write $W_\pm$ for the restriction of W to $T_\pm$. Note that

$W_\pm$ commutes with $V_\pm$.

Let $a,\, b \in \mathbf{R}$. We claim that under the Assumptions a), b), the following statements i)–iii) are equivalent.

i) $a\, W_\pm + b\, V_\pm = 0$.

ii) For each $x \in S_\pm \setminus \{p\}$ we have $a\, w(x) + b\, v(x) = 0$.

iii) There exists an $x \in S_\pm \setminus \{p\}$ such that $a\, w(x) + b\, v(x) = 0$.

In particular Assumption d) implies that $V_\pm$ and $W_\pm$ are linearly independent over $\mathbf{R}$.

Proof Because there are no resonances for $V_\pm$ in $T_\pm$, the theorem of Poincaré-Sternberg [17] yields the existence of a smooth coordinate system in $S_\pm^B$, where B is a sufficiently small neighborhood of p, in which the vector field v is linear. Because w commutes with v, we have

$$w(x) = \mathrm{e}^{-t\, V_\pm}\; w\left(\mathrm{e}^{t\, V_\pm}\right), \quad x \in S_\pm^B.$$

The differentiability of w at the origin impliess that the right hand side converges to $W_\pm\, x$ as $t \to \pm\infty$, so $w(x) = W_\pm\, x$, which means that w is linear in these coordinates as well. Using the fact that $S_\pm$ is equal to the union of the $\phi^t\left(S_\pm^B\right)$, $t \in \mathbf{R}$, we obtain the equivalence of i)–iii). $\square$

On $T_\pm$ there is a complex structure in terms of which $V_\pm$ is equal to multiplication by a complex number, which we will denote by the symbol $\widetilde{V_\pm}$. The fact that $W_\pm$ commutes with $V_\pm$ then implies that $W_\pm$ is also equal to multiplication by a complex number, denoted by $\widetilde{W_\pm}$. The linear independence over $\mathbf{R}$ of $V_\pm$ and $W_\pm$ implies that every complex number c can be written as $c = a\,\widetilde{W_\pm} + b\,\widetilde{V_\pm}$ for some $a,\, b \in \mathbf{R}$. In particular we can find $a,\, b \in \mathbf{R}$ such that the linear map $U = a\, W_\pm + b\, V_\pm$ has the property that U^2 is equal to minus the identity in $T_\pm$. Clearly $a \neq 0$ in this case and $a = a_\pm$ and $b = b_\pm$ are unique if we require that $a > 0$. The linear transformation $U_\pm = a_\pm\, W_\pm + b_\pm\, V_\pm$ is our choice of the aforementioned complex structure in $T_\pm$.

2.5 Consider the vector field $u_\pm = a_\pm\, w + b_\pm\, v$ on $S_\pm$. From the linearization argument in the proof of the equivalence of i)–iii) given above, we see that the $u_\pm$-solution curves are periodic with primitive period equal to 2π. In other words, if Γ denotes the stabilizer subgroup for the orbit $F_0' = S_+' = S_-'$, then both $(2\pi\, a_+,\, 2\pi\, b_+)$ and $(2\pi\, a_-,\, 2\pi\, b_-)$ belong to Γ and are not equal to $n\, T$ for some $T \in \Gamma$ and some integer $n \neq \pm 1$. Because $F_0' \simeq \mathbf{R}^2/\Gamma$ is not compact, the rank of the lattice Γ cannot be equal to two and because $\Gamma \neq \{0\}$ we conclude that $\Gamma \simeq \mathbf{Z}$. Because a_+ and a_- have the same sign, this implies that that $a_+ = a_-$ and $b_+ = b_-$; we will write $a = a_\pm$ and $b = b_\pm$ in the sequel. We have $\Gamma = \mathbf{Z}\, T$, where $T = 2\pi\,(a,\, b)$.

If $\chi^s = \psi^{a\, s} \circ \phi^{b\, s}$ denotes the u-flow after time s, then, for any $x \in F_0'$, the mapping $(s,\, t) \mapsto \chi^s \circ \phi^t(x)$ induces the diffeomorphism from the cylinder $(\mathbf{R}/2\pi\, \mathbf{Z}) \times \mathbf{R}$ onto F_0' which is mentioned in Theorem 1.2, a). We have proved all the statements in Theorem 1.2, a) and also the one concerning $F_0' = F_0 \setminus \{p\}$ in Proposition 1.4.

3 The Nearby Regular Fibers

3.1 Let B be an open ball around p, which is chosen small enough so that its boundary ∂B intersects F_0 in two circles: $S_+ \cap \partial B$ and $S_- \cap \partial B$. Using our knowledge about $V_\pm$ and $W_\pm$, we can also arrange that except at p the v, w-orbits in a neighborhood of the closure $\overline{B}$ of B in M are 2-dimensional and that the orbits through points close to p intersect ∂B in two circles close to $F_0 \cap \partial B$.

The set $F_0 \setminus B$ is a compact smooth 2-dimensional submanifold of M with the two circles $S_\pm \cap \partial B$ as its boundary. Thus $F_0 \setminus B$ is diffeomorphic to a compact cylinder. Using a suitable 2-dimensional fibration transversal to $F_0 \setminus B$, the fact that the rank of $Df(x)$ is equal to two for each $x \in F_0 \setminus B$, and the fact that $v(x)$, $w(x)$ are linearly independent, we obtain an open neighborhood N of $F_0 \setminus B$ in $M \setminus B$, an open neighborhood U of 0 in $\mathbf{R}^2$, which we can take to be simply connected, and a diffeomorphism Φ from N onto $(F_0 \setminus B) \times U$, such that:

a) The restriction f_N of f to N is equal to $\pi_2 \circ \Phi$ where π_2 denotes the projection from $(F_0 \setminus B) \times U$ onto the second factor U.

b) For each $x \in N$, $v(x)$ and $w(x)$ are linearly independent.

c) The fibers of f_N intersect ∂B in two circles, which are close to the two circles where the fiber $F_0 \setminus B$ over 0 intersects ∂B.

3.2 On the other hand, the orbits of the local $\mathbf{R}^2$-action in $\overline{B}$ through points in $(N \cap \partial B) \setminus F_0$ also intersect ∂B in two circles. These two circles coincide with the intersection of the fibers of f_N with ∂B, because f is constant on the orbits. We now define $\widetilde{M}$ as the union of N, the orbits in $\overline{B}$ which intersect $N \cap \partial B$, and $\{p\}$. The union of the orbits through $N \cap \partial B$ form an open subset of $\overline{B}$, which contains $V \setminus \{p\}$, where V is a sufficiently small open neighborhood of p in M. This follows because the v-solutions through points at a distance δ to p enter and leave B at points in ∂B which are at a distance $\mathcal{O}(\delta)$ to $\partial B \cap S_+$ and $\partial B \cap S_-$, respectively.

Consequently $\widetilde{M}$ is an open neighborhood of F_0 in M such that $F_c \cap \widetilde{M}$ is equal to the union of $F_c \cap N$ and the orbit in $\overline{B}$ through $F_c \cap \partial B$ for each $c \in U \setminus \{0\}$. From this description it is clear that the restriction of f to $\widetilde{M}$ is a proper mapping from $\widetilde{M}$ onto U and that its fibers are connected. The invariance of f under the action, combined with the fact that for every $x \in N \cap \partial B$ the rank of $Df(x)$ is equal to two, implies that the rank of $Df(x)$ is also equal to two for x on each orbit which intersects $N \cap \partial B$. Consequently, $Df(x)$ is surjective for every $x \in \widetilde{M} \setminus \{p\}$. Thus the restriction of f to $\widetilde{M} \setminus F_0$ is a locally trivial fibration over $U \setminus \{0\}$, with compact and connected fibers.

Since we have replaced M by $\widetilde{M}$, we will simplify our notation by writing F_c instead of $F_c \cap \widetilde{M}$ in the sequel. Because the flows of v and w leave the fibers F_c invariant, these flows are complete and define an action of $\mathbf{R}^2$ on F_c. Now let $c \in U \setminus \{0\}$. Because of ii) above, combined with the fact that the vectors $v(x)$ and $w(x)$ are linearly independent for each $x \in B \setminus \{p\}$, we deduce that each orbit O in F_c is open. Because F_c is connected, we conclude that $O = F_c$. Since O is diffeomorphic to $\mathbf{R}^2/\Gamma_O$ and $O = F_c$ is compact, Γ_O is a 2-dimensional lattice in $\mathbf{R}^2$, and $O = F_c$ is diffeomorphic to a 2-dimensional torus. In the sequel we will write Γ_c instead of Γ_O. We have now proved all the statements in Theorem 1.2, b).

3.4 Recall that there exists $a, b \in \mathbf{R}$, with $a > 0$, such that all solution curves of the vector field $a\,w + b\,v$ on $F_0' = S_-' = S_+'$ are periodic with primitive period equal to 2π. Let Y be a local smooth section of f at a point $y \in F_0'$. Applying the implicit function theorem to the equation $\phi^t \circ \psi^s(x) = x$, with unkowns (s, t) near $2\pi \cdot (a, b)$ and $x \in Y$ as a parameter, we obtain unique smooth functions σ and τ on an open neighborhood U' of 0 in U such that $\sigma(0) = a$, $\tau(0) = b$, and $\phi^t \circ \psi^s(x) = x$ when $x \in Y \cap f^{-1}(U')$, $s = 2\pi\,\sigma(f(x))$, $t = 2\pi\,\tau(f(x))$. In other words, all solutions of the vector field $u := (\sigma \circ f)\,w + (\tau \circ f)\,v$ which start in $Y \cap f^{-1}(U')$ are periodic with period equal to 2π. Becuase 2π is a primitive period for the solution starting at y, a continuity argument shows 2π is a primitive period for all solutions starting in $Y \cap f^{-1}(U')$ if we take U' sufficiently small. In other words, for each $c \in U'$, the element $T(c) = 2\pi \cdot (\sigma(c), \tau(c))$ is a primitive element of the lattice Γ_c.

We can take U' to be simply connected. We will write U and $\widetilde{M}$ instead of U' and $f^{-1}(U') \cap \widetilde{M}$. Using this notation, all the previous statements about $\widetilde{M}$ and U remain valid. In particular we have proved Proposition 1.5.

4 The Monodromy

The trivialization outside B is used in order to prove finally that the monodromy is determined by what happens inside B, and this leads to the description of the monodromy matrix as in Theorem 1.2, c) and Proposition 1.6.

4.1 The homology class of any u-circle in F_c can be taken as the first element $\delta_1(c)$ of a basis of $H_1 = H_1(F_c, \mathbf{Z})$. $\delta_1(c)$ depends smoothly and in a single-valued way on $c \in U \setminus \{0\}$. Thus when c encircles the origin in $U \setminus \{0\}$, we return to the same element of H_1. In other words, $\delta_1(c)$ is a fixed element for the monodromy operator in H_1.

If χ^s denotes the time s flow of u, then it will be convenient to work with the action $(s, t) \mapsto \phi^t \circ \chi^s$ of $\mathbf{R}^2$ on $\widetilde{M} \setminus F_0$, instead of the action $(s, t) \mapsto \phi^t \circ \psi^s$. Consider the mapping which assigns to each $(s, t) \in \Gamma_c$ the homology class of the solution curve of $s\,u + t\,v$ defined on a time interval of unit length, which is a closed curve. This mapping is an isomorphism from Γ_c onto $H_1(F_c, \mathbf{Z})$. We know that $(2\pi, 0) \in \Gamma_c$ corresponds to the previously defined fixed element $\delta_1(c)$ of the monodromy in $H_1(F_c, \mathbf{Z})$.

4.2 In order to find the second element $\delta_2(c)$ of a basis of $H_1(F_c, \mathbf{Z})$, we observe that the projection $(s, t) \mapsto t$ maps Γ_c onto a subset of the form $\mathbf{Z}\,t_c$, where t_c is a uniquely determined positive number. Any $(s, t_c) \in \Gamma_c$ corresponds to an element $\delta_2(c)$ such that $\delta_1(c)$ and $\delta_2(c)$ form a $\mathbf{Z}$-basis of $H_1(F_c, \mathbf{Z})$.

In terms of integral curves of u and v this means the following. Let C_c be any orbit in F_c of the circle action defined by the flow of u. The vector field v is transversal to C_c. Let γ_c be an integral curve of v in F_c such that $\gamma_c(0) \in C$. Then t_c is the smallest positive time

t such that $\gamma_c(t_c) \in C$. Moreover, there exists an $s \in \mathbf{R}$ such that $\chi^s(\gamma_c(t_c)) = \gamma_c(0)$. This corresponds to the condition that $(s,\, t_c) \in \Gamma_c$. We have proved the first part of Proposition 1.6.

4.3 In order to understand how the second generator $\delta_2(c)$ moves as a function of $c \in U \setminus \{0\}$, it is be convenient to use local coordinates around p in which the circle action of u is a one-parameter group of linear transformations. (It is a theorem of Bochner [4, Th. 1] that every compact group of smooth transformations can be linearized around a fixed point.) Furthermore, because the local stable and unstable manifold are u-invariant, one can perform a u-equivariant smooth coordinate transformation which maps the local stable and unstable manifold to open subsets of linear subspaces. In the new coordinates u is still linear.

In terms of the complex structures defined in Section 2.4, we can identify the neighborhood B of p in M with an open ball of small positive radius r around the origin in $\mathbf{C}^2$, such that $u(z) = i\, z$ and the local stable and unstable manifold correspond to an open neighborhood of the origin in $\mathbf{C} \times \{0\}$ and $\{0\} \times \mathbf{C}$. We write $z = (z_+,\, z_-) \in \mathbf{C}^2$.

Next we choose a fixed $0 < \epsilon < r$ and for every c close to 0 consider the u-circles $C_\pm(c)$ in F_c determined by the condition that $|z_\pm| = \epsilon$. As the initial point of the solution curve γ_c in F_c of the vector field v we take the point $z = z(c) \in C_-(c)$ such that $z_- = \epsilon$. Here we use that $z_+ \mapsto f(z_+,\, \epsilon)$ is a diffeomorphism from an open neighborhood of the origin in $\mathbf{C} = \mathbf{R}^2$ onto an open neighborhood of the origin in $\mathbf{R}^2$. Therefore $z_+ = z_+(c)$ is uniquely determined and depends smoothly on c when c is sufficiently close to the origin in $\mathbf{R}^2$. Because $z_+(0) = 0$, $z(c)$ is close to the local unstable manifold when c is small. Also note that $z_+(c)$ winds around the origin once if c does.

The solution curve γ_0 will leave $C_-(0)$ with growing $|\gamma_0(t)_-|$, while $\gamma_0(t)_+ = 0$. If B is sufficiently small, then γ_0 leaves B and reenters B with $\gamma_0(t)_- = 0$ and $|\gamma_0(t)_+| > \epsilon$, and lands on the circle $C_+(0)$ after some positive time. It follows that for small c there is a first positive time $t = t_1(c)$ for which $|\gamma_c(t)_+| = \epsilon$, and both $t_1(c)$ and $y(c) := \gamma_c(t_1(c))$ depend smoothly on c.

4.5 The following observation is crucial. For small c the point $y(c)$ remains close to the point $y(0) = (y(0)_+,\, 0)$ on the local stable manifold. In particular, $y(c)_+$ does not wind around the circle of radius ϵ in $\mathbf{C}$ when c winds around the origin in U. On the other hand, $y(c)_-$ winds around the origin once when c does.

For the proof, we write $y(c)_+ = \mathrm{e}^{i\,\alpha_c}\, y(0)_+$, in which $\alpha_c \in \mathbf{R}$ depends smoothly on c, $\alpha_0 = 0$. From the equation

$$\begin{aligned} c &= f(y(c)_+,\, y(c)_-) = f\left(y(0)_+,\, \mathrm{e}^{-i\,\alpha_c}\, y(c)_-\right) \\ &\sim \frac{\partial f(y(0)_+,\, z_-)}{\partial z_-}\Big|_{z_-=0}\, \mathrm{e}^{-i\,\alpha_c}\, y(c)_-, \quad |c| << 1, \end{aligned}$$

we find that $y(c)_-$ winds around the origin once when c does so. Writing $x = (y(0)_+,\, 0)$, we identify the z_--space with $\mathrm{T}_x\, M / \mathrm{T}_x\, S_+$. The winding number of $y(c)_-$, when c winds around the origin in $\mathbf{R}^2$ in the positive direction, is equal to $+1$ or -1 if the orientation of

$\mathrm{T}_x\, M / \mathrm{T}_x\, S_+$ defined by the complex structure in $\mathrm{T}_p\, M$ agrees or does not agree with the pullback of the orientation of $\mathbf{R}^2$ by means of $\mathrm{T}_x\, f$, respectively.

4.6 We now have to find the first positive time $t = t_2(c)$ such that

$$\gamma_c\,(t_1(c) + t) = \phi^t(y(c)) \in C_-(c).$$

Then $t_c = t_1(c) + t_2(c)$ and $\gamma_c\,(t_c) = \phi^{t_2(c)}(y(c))$.

We can write

$$\gamma_c\,(t_c)_- = \phi^{t_2(c)}(y(c))_- = \mathrm{e}^{i\,\theta_c}\,\epsilon,$$

where $\theta_c \in \mathbf{R}$ depends smoothly on c when c runs on a curve in $U \setminus \{p\}$. Therefore, if $\gamma_c(t)$, with t running from 0 to t_c, is followed by the curve $\mathrm{e}^{i\,\tau}$, with τ running from 0 to $-\theta_c$, we obtain a closed curve on F_c which represents $\delta_2(c)$, and $(-\theta_c,\, t_c) \in \Gamma_c$. From this we find that if k is the winding number of $c \mapsto \phi^{t_2(c)}(y(c))_-$ on the circle of radius ϵ as c winds once around the origin in $U \setminus \{0\}$, then $\mathcal{M}\,\delta_2 = \delta_2 - k\,\delta_1$. Note that the winding number k, and therefore the monodromy matrix $\mathcal{M}$, is completely determined by the behaviour of the solution curves of the vector field v in the ball B around p.

4.7 Write $v = V + R$ in which V is the linear part of v, $V(z) = (V_+\, z_+,\, V_-\, z_-)$, and R is the remainder term, which vanishes to at least second order at the origin. The condition that $z_- = 0$ and $z_+ = 0$ are the stable and unstable manifold of v, invariant under the flow of v, corresponds to the condition that $R\,(z_+,\, 0)_- \equiv 0$ and $R\,(0,\, z_-)_+ \equiv 0$. If we now replace v by $v_\delta = V + \delta\, R$ with $0 \leq \delta \leq 1$ then the winding number of ${\phi_\delta}^{t_2(c,\,\delta)}(y(c))_-$ is well defined and depends continuously on δ, and therefore does not depend on δ because it is an integer. Now we have

$${\phi_0}^t(z) = \left(\mathrm{e}^{V_+\,t}\; z_+,\; \mathrm{e}^{V_-\,t}\; z_-\right),$$

from which we see that $t = t_2(c,\, 0)$ is determined by the equation

$$\left|\mathrm{e}^{V_-\,t}\; y(c)_-\right| = \epsilon.$$

Writing $\rho_- = \mathrm{Re}\, V_- > 0$ and substituting the solution

$$t = (\rho_-)^{-1}\, \log\left(\epsilon / |y(c)_-|\right),$$

we obtain

$${\phi_0}^t(y(c))_- = \mathrm{e}^{(V_-/\rho_-)\,\log(\epsilon/|y(c)_-|)}\; y(c)_-,$$

the argument of which is equal to the sum of the single-valued function

$$(\mathrm{Im}\, V_- / \mathrm{Re}\, V_-)\, \log\left(\epsilon / |y(c)_-|\right)$$

of c and the argument of $y(c)_-$. It follows that the winding number of $\gamma_c\,(t_c)_-$ on the circle of radius ϵ, as c winds once around the origin in $U \setminus \{0\}$, is equal to the winding number of $y(c)_-$. Using the fact that $y(c)_+$ remains close to $y(0)_+$ on the circle of radius ϵ

and the fact that f is rotationally invariant, the latter winding number is equal to $+1$ or -1, according to whether the diffeomorphism $z_- \mapsto f(\epsilon, z_-)$ is orientation preserving or orientation reversing. The proof of Proposition 1.6 and Theorem 1.2, c) is complete.

4.8. Remark In passing we have seen that t_c is asymptotically proportional to $-\log|c| \to \infty$ as $c \to 0$. This is the way in which the 2-dimensional lattice Γ_c converges to the 1-dimensional lattice which is the stabilizer subgroup of the action of $\mathbf{R}^2$ on the manifold $F_0 \setminus \{p\} = S_+ \setminus \{p\} = S_- \setminus \{p\}$, which is diffeomorphic to a cylinder.

5 The Hamiltonian Case

5.1 For the proof of Proposition 1.7 we may replace w by u, which we may assume to be linear in our coordinate system around p. Furthermore, we may replace v by the vector field $z \mapsto (-z_+, z_-)$ and the Hamiltonian function h and g of u and v by the quadratic part of its Taylor expansion at the origin. We will give a basis which is adapted to the symplectic form Ω in $\mathrm{T}_p M$ and then determine h and g in terms of the coordinates with respect to this basis.

5.2 Because $V = \mathrm{D}v(p)$ is an infinitesimally symplectic transformation in $\mathrm{T}_p M$, we find for each $a, b \in \mathrm{T}_p M$ that

$$\Omega(a, b) = \Omega\left(\mathrm{e}^{tV}\, a,\, \mathrm{e}^{tV}\, b\right).$$

If $a, b \in T_\pm$ then, if we let t run to $\pm\infty$ in the right hand side, we get $\Omega(a, b) = 0$. In other words, T_+ and T_- are isotropic subspaces of $\mathrm{T}_p M$ with respect to the symplectic form Ω.

Because T is equal to the direct sum of T_+ and T_- and Ω is nondegenerate, we also obtain that

$$z_- \mapsto (z_+ \mapsto \Omega(z_+, z_-))$$

is a bijective linear mapping from T_- onto $(T_+)^*$, the space of linear forms on T_+.

5.3 Choose any nonzero vector $e_1 \in T_+$, and write $e_2 = U\, e_1$. Because U is our complex structure, $U^2 = -1$, we have $U\, e_2 = -e_1$.

The last statement in 5.2 implies that there are unique $\varepsilon_1, \varepsilon_2 \in T_-$, such that

$$\Omega(e_i, \varepsilon_j) = \delta_{ij}, \quad i, j = 1, 2.$$

Because U is a linear combination of the infinitesimally symplectic transformations $\mathrm{D}v(p)$ and $\mathrm{D}w(p)$ in $\mathrm{T}_p M$, it is infinitesimally symplectic as well. This implies that $\Omega(e_i, U\,\varepsilon_j) = -\Omega(U\, e_i, \varepsilon_j)$, which is equal to $-\Omega(e_2, U\,\varepsilon_j)$ and $\Omega(e_1, U\,\varepsilon_j)$ when $i = 1$ and $i = 2$, respectively. From this we see that $U\,\varepsilon_1 = \varepsilon_2$ and $U\,\varepsilon_2 = -\varepsilon_1$.

Using $\Omega(e_1, e_2) = 0$ and $\Omega(\varepsilon_1, \varepsilon_2) = 0$, we find that if

$$a = x_1\, e_1 + x_2\, e_2 + \xi_1\, \varepsilon_1 + \xi_2\, \varepsilon_2, \quad b = y_1\, e_1 + y_2\, e_2 + \eta_1\, \varepsilon_1 + \eta_2\, \varepsilon_2,$$

then

$$\Omega(a, b) = x_1\, \eta_1 + x_2\, \eta_2 - \xi_1\, y_1 - \xi_2\, y_2.$$

5.4 The Hamiltonian function g of the linear Hamiltonian vector field U is determined by the relation

$$\Omega\,(U(a),\,b) + \mathrm{d}g(a)(b) = 0,$$

which in terms of the notation in 5.3 is equivalent to

$$\frac{\partial g(a)}{\partial x_1} = -\xi_2,\ \frac{\partial g(a)}{\partial x_2} = \xi_1,\ \frac{\partial g(a)}{\partial \xi_1} = x_2,\ \frac{\partial g(a)}{\partial \xi_2} = -x_1.$$

Using the condition that $g(0) = 0$, this gives $g(a) = -x_1\,\xi_2 + x_2\,\xi_1$.

A similar calculation shows that the Hamiltonian function h of the linear Hamiltonian vector field

$$V : (x_1,\, x_2,\, \xi_1,\, \xi_2) \mapsto (-x_1,\, -x_2,\, \xi_1,\, \xi_2)$$

is given by $h(a) = x_1\,\xi_1 + x_2\,\xi_2$.

5.5 The matrix of the derivative of the mappng $a \mapsto (g(a),\, h(a))$ is equal to

$$\begin{pmatrix} -\xi_2 & \xi_1 & x_2 & -x_1 \\ \xi_1 & \xi_2 & x_1 & x_2 \end{pmatrix}.$$

In particular det $\frac{\partial f(a)}{\partial \xi} = {x_2}^2 + {x_1}^2 > 0$ when $a = (x,\,0) \in S_+,\ a \neq 0$. This completes the proof of Proposition 1.7.

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