

N=2 Supersymmetric Theories, Dyonic Charges and Instantons

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instantonen

(met een samenvatting in het Nederlands)

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Introduction

Three of the four fundamental forces of nature are described by the Standard Model. This is a quantum field theory, which presupposes all elementary particles to be point-like objects. The fourth force of nature is gravity. On large (classical) length scales it behaves according to the laws of General Relativity. On small enough length scales, or high enough energy scales, quantum effects become important and the classical theory should be modified. The typical energy scale at which these modifications are expected to be necessary is the Planck scale, $M_{Pl}c^2 = \sqrt{\hbar c^5/G_N} \sim 1,2 \times 10^{19} GeV$, where G_N is the gravitational coupling constant. It is important to realize that this scale is far beyond experimental reach. Present day particle accelerators can produce collisions in which energies up to order $1 TeV$ are involved, which is a factor 10^{16} away from the Planck scale. Quantum gravity therefore is a theoretical problem whose solution needs to be found without much help from experimental side.

The obvious first guess for a quantum gravity theory is a quantum field theory of General Relativity, set up along the same lines as the quantum theory of the other forces. This theory gives rise to infinities¹. By itself these do not need to be disastrous; also the Standard Model contains them. However, whereas in the latter case all infinities can be absorbed in the parameters of the theory and sensible physical predictions can be extracted, this turns out to be impossible for the quantized version of General Relativity. In other words, quantum General Relativity is *non-renormalizable*. Therefore a more drastic modification of the classical theory is called for. This is provided by *string theory*, which is no longer based on point-like particles, but on one-dimensional extended objects, called strings.

Strings come in two varieties. There are strings with endpoints (open strings) and strings without (closed strings). Both are described by a two-dimensional action, with the coordinates parameterizing a surface, called the worldsheet. This worldsheet is the surface swept out by the string in spacetime. Its embedding coordinates are functions of the two-dimensional worldsheet coordinates.

Quantizing such a - bosonic and relativistic - string yields interesting states: First of all, the closed string spectrum contains a massless spin-two state, which can be identified with the graviton, the massless particle that mediates the gravitational

¹In this respect we need to mention that recently discussion arose about a possible finiteness of $N = 8$ supergravity in four spacetime dimensions (see e.g. [1, 2]), which is the maximally supersymmetric extension of General Relativity.

force. One can therefore say that string theory not only describes quantum gravity, it even predicts it! Moreover, the spectrum of the open string incorporates massless spin-one states, which could play the role of vector gauge particles mediating the Standard Model forces. This makes string theory a candidate for being a unified description of the forces of nature. Bosonic string theory also contains tachyons, which are spinless objects with negative mass squared. They imply that the theory is unstable. We come back to this shortly.

String theory has two parameters. One of them is α' , which has dimension l^2 and sets the scale of the string length. The length of a string is taken about $10^{-35} m$, which is around the Planck scale (although alternative scenarios do exist, for instance based on [3]). The other parameter is g_s , the string coupling constant. It arises as the vacuum expectation value of the dilaton, which is another (scalar) field in the (closed) string spectrum. Perturbatively it controls the number of loops in a stringy Feynman diagram.

To obtain particles with half-integer spin, anticommuting fields are included on the worldsheet. The resulting theory is invariant under local Weyl transformations, diffeomorphisms and supersymmetry transformations. Furthermore, demanding spacetime supersymmetry (a symmetry interchanging bosonic and fermionic states) makes the theory free of tachyons. This leads to five possible superstring theories, which all live in ten spacetime dimensions.

As first conjectured by Witten, these five string theories are related [4]. They all arise as appropriate descriptions in special limits of one big theory, named M-theory. The different descriptions, i.e. the different string theories, are then connected by *duality* transformations.

Duality transformations by definition relate different descriptions of the same physical system. Three different types of them can be distinguished. First of all, there are dualities between descriptions that are based on different theories. Secondly, dualities may relate two descriptions that are based on the same theory, but with different values of the parameters involved. These are called selfdualities. Finally, there are dualities relating two descriptions that are based on the same theory, including equal values of the parameters. These latter dualities are invariances of the theories under consideration.

Electric/magnetic duality is an early discovered example of a (self)duality. In short, electric/magnetic duality comes with a rotation of elementary (electric) and solitonic (magnetic monopole) states, as well as an inversion of the coupling constant. Many aspects of the duality web connecting the five string theories are directly or indirectly related to dualities of this type.

A specific example of a duality in string theory is *T-duality*, which shows up as a selfduality in its purely bosonic (closed string) version. When there is a compact dimension, states in bosonic string theory are labelled by the (quantized) internal momenta and the number of winding modes (around the compact dimension). T-duality implies that there is an alternative description in which the momentum

(winding) modes of the original formulation are described as (winding) momentum modes. This duality involves an inversion of the radius of the compact dimension; it relates bosonic string theory in a spacetime background with a compact dimension with radius R to the same theory in a background with a compact dimension with radius $\frac{\alpha'}{R}$ (note that when the radius is taken to be $R = \sqrt{\alpha'}$ this T-duality is an invariance of the theory).

In the context of superstrings, there is a T-duality involving *type IIA* and *type IIB* string theory (which differ in the massless sector of their ten-dimensional spectra). Starting from a description based on one of these theories - in a background with a compact dimension - T-duality again implies the existence of an alternative description, in which the momentum modes are described as winding modes and vice versa. As before, this alternative description is based on a theory in which the radius of the compact dimension is the inverse of the radius of the compact dimension in the original theory. But, contrary to the bosonic case, this dual theory is *different* than the original one; T-duality relates type IIA string theory in a background with a compact dimension with radius R to type IIB string theory in a background with a compact dimension with radius $\frac{\alpha'}{R}$.

The T-duality between type IIA and type IIB string theory is an example of a perturbative duality (in g_s), which means that the perturbative region of one theory (i.e. the region of its parameter space where perturbation theory is applicable) is mapped to the perturbative region of another. This implies that (for finite R) type IIA and type IIB give useful descriptions of the same corner of M-theory. Some other dualities in the duality web are non-perturbative, in the sense that perturbative regions are mapped to non-perturbative regions (like the electric/magnetic dualities mentioned above). In these cases the domains of applicability of the two theories on both sides of the duality are different.

Coming back to T-duality, on open strings it affects the boundary conditions on the endpoints. Strings that have their endpoints fixed on a p -dimensional hypersurface are mapped to strings whose endpoints live on a $(p+1)/(p-1)$ -dimensional (depending on whether the compact dimension involved is part of the hypersurface or not) space.

The hypersurfaces appearing in the context of open strings are called *D-branes*. Polchinski discovered that non-perturbatively these D-branes become dynamical [5]. This can be understood best from the fact that their mass is proportional to the inverse of g_s . The type IIA and type IIB string theories (which are the ones mainly important for us) contain D-branes of even and odd dimensionality respectively (note the consistency with T-duality). These D-branes take over the role of strings as fundamental objects in the non-perturbative regime. The latter is consistent with the duality web, which involves for instance a class of $SL(2, \mathbb{Z})$ selfdualities of type IIB string theory, rotating the fundamental string and the D1-brane (a one-dimensional D-brane, the ‘D-string’) as fundamental objects [6]. This important work on the duality web and the role of D-branes has greatly

increased the understanding of string theory and certainly of its non-perturbative aspects. However, there remain areas to be explored. The central question in this respect is: What is M-theory? We know that in certain limits of the parameter space of this theory string theories appear as the appropriate descriptions. Away from these limits there is less clarity. What is known, is that the heart of M-theory can be reached from type IIA string theory by increasing its coupling constant to a value much bigger than one. As the string coupling of IIA string theory at low energies can be related to the radius of an extra dimension, it indicates that the appropriate theory should not be ten- but eleven-dimensional. Nowadays much research concentrates on this sector and on the full non-perturbative completion of string theory in general.

Another important area of research is concerned with the construction of phenomenologically interesting models out of string theory. As string theory necessarily lives in ten (or eleven) dimensions this involves first of all the assumption that six (or seven) of these parameterize a compact space. At energies low compared to the scale associated with the size of the compact space, four-dimensional theories then arise as appropriate effective descriptions.

What four-dimensional model is obtained (and how much supersymmetry is preserved) depends on the scheme chosen, involving e.g. the type of string theory and the properties of the six-dimensional compact space. As there are many options to choose from, many different four-dimensional models are obtainable. However, to get a four-dimensional model that can be related to our universe turns out to be difficult.

We point out two of the characteristics of our universe that are non-trivial to reproduce from string theory in particular. First of all there is the fact that no massless scalar fields are observed in nature, while string theory typically gives rise to massless moduli, associated with the geometry of the compact space. Secondly, the positive cosmological constant that our universe seems to have is not easy to obtain from string theory compactifications [7, 8].

The need to generate masses for the moduli of the internal manifold is usually referred to as the problem of *moduli stabilization*. It requires a potential for these fields in the four-dimensional theory. Such a potential can be provided by background *fluxes* (non-vanishing background values of the ten-dimensional fields) in the internal manifold.

A positive cosmological constant, it seems, can only be found in metastable string theory vacua (which are vacua that are unstable under tunneling effects, but with lifetimes large on cosmological scales) and requires the inclusion of non-perturbative effects in the string coupling constant g_s . The latter provides another motivation for investigations of non-perturbative g_s physics.

‘KKLT’ sketches a type IIB scenario in which it would be possible to stabilize all the four-dimensional moduli in a - metastable - de Sitter vacuum [9]. This scenario involves as ingredients background fluxes, three-dimensional (anti-)D-branes

transversal to the internal manifold and so-called instanton effects (to which we come back later on in this introduction). These days a lot of research focusses on realizing scenarios of this type in practice, either in type IIB or another type of string theory.

Above we gave a brief introduction to string theory and treated some relevant issues in the research concerning it. To summarize the latter, two important (and not unrelated) categories are investigations focussing on its non-perturbative (in g_s) completion and work on finding phenomenologically interesting solutions with all moduli stabilized and a positive cosmological constant. Let us now come a bit more to the point: What are the important issues in the material presented in this thesis and how do they fit in the framework outlined above?

One of our main results is a spacetime description of instanton solutions that can be related to D-branes and other non-perturbative string theory objects. The other main topic involves a construction of supersymmetric gauge theories with both electric and magnetic (*dyonic*) charges, which can be related to string theory through compactifications with background fluxes turned on. As we will explain in more detail later on, our study thus fits in both categories just mentioned.

The context of our work is formed by four-dimensional $N = 2$ supersymmetric models. All necessary properties and details of models of the latter type can be found in chapter 2. For now it is just important to know that these models are field theories (with or without a gravitational coupling), symmetric under the action of two independent supersymmetry generators. They contain models that descend from string theory at low energies. Shortly we will see how, but before going to that we first elaborate a bit on why they are interesting in the first place.

An important aspect of supersymmetry is the control it gives over a model. From a calculational point of view it is therefore preferable to consider systems which have (some) supersymmetry. $N = 2$ is an interesting amount of supersymmetry to have in this respect as it is just enough to have good control over calculations, while the amount is sufficiently low to allow for non-trivial (quantum) effects. Requiring ($N = 2$) supersymmetry has for instance proven to be useful in handling gauge theories. Of course these are theoretical rather than phenomenological arguments; they help to extract physics from a model, but the model and corresponding physics are not necessarily relevant for a description of the real world.

Nevertheless, regardless of the calculational ease it offers, there are also bottom-up motivations for considering supersymmetric models. One of these is the following. The energies associated with the heaviest Standard Model particles are of the order of 100 GeV . However, the natural cutoff of the Standard Model - the energy scale where it might lose its applicability - is the Planck scale ². A priori one would expect the masses involved to be of the latter order, but as we just saw, their typical scale is many orders of magnitude lower. This difference in scales

²Or the nearby grand unification scale ($\sim 10^{16} \text{ GeV}$) where the running couplings of the Standard Model seem to meet.

is referred to as the *hierarchy problem*. It is indicative of new physics just on or above levels reachable by present day particle accelerators (which is more or less the mass scale of the Standard Model particles). This new physics may correspond to a supersymmetric theory of (initially) massless fields. Supersymmetry then needs to be spontaneously broken at the appropriate scale such that the masses of the Standard Model particles arise accordingly. Whether this is realized in practice may be determined by upcoming LHC experiments ³. To have a scenario like described above, $N = 1$ supersymmetry suffices. In fact, since $N > 1$ models cannot accommodate chiral fermions, they are hard to relate to realistic models. $N = 2$ models are therefore not of direct value for phenomenology. However, either they could be useful in the exceptional scenario where $N = 2$ does give rise to a realistic non-supersymmetric theory or it might be hoped that they are relevant toy models for $N = 1$.

Besides the arguments given above, the arguably most intriguing aspect of a supersymmetric field theory is that its local version (with transformation parameters being free in their spacetime dependence) automatically includes gravity. Models exhibiting local $N = 2$ supersymmetry are therefore called $N = 2$ supergravity systems.

$N = 2$ supergravity theories arise as low energy effective actions from string theory. This can be understood as follows. At energies low compared to the Planck scale explicit knowledge of the behavior of the massive string states - which have Planck scale masses - is not important. So they can be integrated out, i.e. an effective (field theory) description in terms of the massless states suffices. When string theory is taken in a background with ten non-compact dimensions, supersymmetry completely fixes the form of the action at two-derivative level. In case of type II strings this leads to the maximally supersymmetric type IIA and type IIB supergravity. However, we want to end up with only four non-compact dimensions. As said above, this can be achieved by considering the remaining six dimensions to be a (small) compact space. The type of six-dimensional compact space then determines how many supersymmetry is left in four dimensions. In case of type II strings, to get a four-dimensional $N = 2$ supergravity model the compact space should be a so-called (compact) *Calabi-Yau* (CY) manifold ⁴.

As we will see in chapter 2, $N = 2$ supersymmetry has two important representations: The vector and the hypermultiplet. The bosonic part of a vector multiplet consists of a vector gauge field and a complex scalar, while its hypermultiplet counterpart has four real scalars. The scalars of the theory parameterize

³LHC stands for *Large Hadron Collider*. Currently under construction (in Geneva), it is supposed to become the world's highest energy particle accelerator.

⁴The same class of manifolds is found when demanding *heterotic* strings to descend to (phenomenologically interesting) four-dimensional models with $N = 1$ supersymmetry. $N = 1$ models can also be obtained from compactifying type II strings on Calabi-Yau orientifolds. These might be considered as arguments for hoping that low energy $N = 2$ type II theories are indeed relevant toy models.

a $(2n + 4(m + 1))$ -dimensional manifold (where n and $m + 1$ are the numbers of vector and hypermultiplets). This manifold is of a quite restricted type, due to the constraints of supersymmetry. To be precise, the $2n$ -dimensional manifold of the vector multiplet scalars should be *special Kähler* (SK) [10], while the $4(m + 1)$ -dimensional space parameterized by the scalars of the hypermultiplet sector necessarily is *quaternionic-Kähler* (QK) [11]. For more details we refer to chapter 2.

The number of multiplets emerging after compactifying type II strings on a CY is determined by the topological properties of the latter. Details of this can be found in chapter 3. For now the only relevant issue is the relation between the numbers of multiplets following from IIA and IIB compactifications. In case IIA gives n vector multiplets and $m + 1$ hypermultiplets, IIB compactified on the same CY yields $n' = m$ vector and $m' + 1 = n + 1$ hypermultiplets.

To obtain the precise form of the low energy effective $N = 2$ supergravity theory corresponding to type II string theory compactified on a CY⁵ would in principle require a full string theory calculation, which at present seems to be far too complicated. Fortunately one can get quite far without performing such a calculation, by doing a supergravity analysis.

Let us consider this supergravity analysis in more detail. First recall that the low energy effective action of type II superstrings in ten uncompact dimensions (with maximal supersymmetry) is known. This allows a compactification to be performed at supergravity level. It involves an expansion of the ten-dimensional fields in eigenfunctions of the CY wave operator. Keeping only the zero-modes, a classical four-dimensional $N = 2$ effective action in terms of massless fields is obtained (see for instance [12], where this supergravity compactification was explicitly performed in the context of type IIA strings). Importantly, the dilaton (whose vacuum expectation value, we recall, is the string coupling constant g_s) one always find back in a hypermultiplet.

All scalar fields together make up a $SK \times QK$ manifold in the four-dimensional Lagrangian. The precise form of the scalar geometry fixes the rest of the action as well. There only is an issue concerning isometries, which may be present in the geometry of the scalar manifolds. In compactifications where the background values of the fields other than the ten-dimensional metric are put to zero, they correspond to rigid invariances of the total action. However, when appropriate background fluxes are turned on, these invariances are local, i.e. the isometries are gauged. Gauged isometries in a $N = 2$ supersymmetric theory imply the presence of a scalar potential, which, we recall, is important for moduli stabilization.

The classical version of the scalar geometry as obtained from the CY compactification considered above receives corrections. These are of two types, (α') corrections

⁵When the numbers of vector and hypermultiplets are equal, the four-dimensional effective actions of type IIA and type IIB are the same. This is the result of *mirror symmetry* between the corresponding CY's.

associated with quantum effects on the worldsheet and corrections corresponding to spacetime quantum behavior. Here we only focus on the latter and just mention that the former are under some - and in a few cases complete - control [13, 14].

The spacetime quantum effects appear as g_s corrections (recall that perturbatively g_s controls the number of loops in a stringy Feynman diagram). As the dilaton lives in a hypermultiplet, the quaternion-Kähler manifold of the type II theories is modified in this way. The perturbative g_s corrections to the quaternion-Kähler space are fixed using some general knowledge about their properties and the constraints imposed by supersymmetry (see [15] and references therein). The non-perturbative g_s corrections, however, are not yet found, although some partial results were obtained in [16, 17].

Microscopically these non-perturbative g_s effects correspond to Euclidean p -branes wrapping $(p+1)$ -dimensional cycles in the CY. From a four-dimensional perspective these branes are points in Euclidean space and in the classical supergravity action they appear as *instanton* solutions.

Instantons are by definition solutions to Euclidean equations of motion with finite action. They are for example known from Yang-Mills theory where they are associated with tunneling effects between different classical vacua. The value of the action evaluated on instantons is typically of the form $S_{inst} = |q|/g$ (or with higher negative powers of g), where g is the coupling constant of the theory and q is some charge. Hence they give rise to non-perturbative ($e^{-|q|/g}$) contributions to the path integral.

In chapter 3 of this thesis instanton solutions are described that correspond to CY wrapping branes. More precisely, two classes of such solutions are determined in the general hypermultiplet model arising from type II strings on a CY with background fluxes turned to zero.

The first class is derived from known black hole solutions in the vector multiplet sector. In doing this the *c-map* [18, 19] is exploited, which involves a dimensional reduction of the (n) ungauged vector multiplet sector of type II CY compactifications. The resulting three-dimensional action can then be uplifted to the $(n+1)$ hypermultiplet sector of an $N=2$ supergravity theory of the same type. Note from what we said earlier that this basically is a map from type IIA to type IIB or vice versa. In fact, the underlying mechanism is the T-duality we described before, which says that type IIA compactified on a CY times a circle with radius R is equivalent to type IIB compactified on the same CY times a circle with radius $\frac{\alpha'}{R}$. The solutions found using the c-map on the black hole solutions are the D-brane instantons, which correspond to D-branes wrapping cycles in the CY.

The second class of instantons are obtained using a “Bogomol’nyi-bound-like” method, similar to the one described in [20]. These solutions arise from so-called *NS-fivebranes* (other higher-dimensional objects in string theory) wrapping the entire CY.

For both classes of instantons the value of the action is determined. This is important with respect to the corresponding deformation of the scalar manifold metric, since these corrections involve exponential factors with (minus) the instanton action appearing in the exponent.

Obviously this work concentrates on obtaining a better picture of non-perturbative string theory. Furthermore, as we already mentioned in the context of the KKLT scenario, understanding instantons and their effects is also important in relation to the construction of models with phenomenologically interesting aspects, such as a stabilization of moduli and a positive cosmological constant.

The latter can be understood from the fact that the scalar potential in a gauged $N = 2$ supersymmetric model depends on the geometrical properties of the scalar sigma manifold. With respect to the CY compactifications of type II strings it therefore matters if we take the sigma manifold with or without quantum corrections. [21] considered the case of one (the universal) hypermultiplet and the associated quaternion-Kähler manifold with g_s corrections corresponding to (membrane) instantons included ⁶. As it turns out, the corresponding scalar potential allows, contrary to its analog without g_s corrections, metastable de Sitter vacua with the moduli in the universal hypermultiplet stabilized. As said above, we analyzed the theory and corresponding instanton solutions that type II strings on an arbitrary CY give rise to. It would be interesting to see what scalar potential and corresponding vacua are generated from the associated higher-dimensional quaternion-Kähler manifold with the more general instanton corrections taken into account.

Let us then focus on the vector multiplets. An important feature of ungauged $N = 2$ supersymmetric actions based on n vector supermultiplets is the existence of the $Sp(2n, \mathbb{R})$ group of electric/magnetic duality transformations. Under these duality transformations the Lagrangian changes. Different Lagrangians related by a duality transformation belong to the same equivalence class, meaning that their sets of equations formed by equations of motion and Bianchi identities are equivalent. It may happen that the Lagrangian does not change under such a duality transformation (possibly up to redefinitions of the other fields), in which case one is dealing with an invariance of the theory. To appreciate this, it is important to note that the Lagrangian does not transform as a function under the duality transformation (although this may be the case for a restricted subgroup of the full invariance group). For a detailed treatment of this we refer to chapters 1 and 2.

Electric/magnetic duality transformations are realized by a constant rotation of the electric and magnetic field strengths. The new field strengths can then be solved in terms of new (dual) vector fields, which are not locally related to the

⁶A model with just one hypermultiplet can be realized in a geometric compactification of type IIA only.

original vector fields by a local field redefinition. The fact that electric/magnetic duality acts on the field strengths rather than the gauge fields is the reason why charges have to be absent when applying electric/magnetic duality, because they couple to the gauge fields. However, this does not preclude the possibility that one can describe a gauge theory with electric charges from a dual point of view.

One may wonder what is gained by using such a description. We mention two important advantages. First of all, when one is interested in gauging a certain subgroup of the rigid invariance group, the standard procedure is to first convert the theory to a suitable electric/magnetic duality frame, in which all the potential charges will appear as electric. This is a cumbersome procedure in general and it would be convenient if it could be avoided. Secondly, in the context of string theory the charges correspond to turning on fluxes in the internal manifold that emerges in a compactification to four spacetime dimensions. These fluxes are associated to background values of antisymmetric tensor fields on non-trivial cycles of the internal manifold, to background quantities associated with the geometry of the manifold itself or even to quantities associated with manifolds without a definite geometry (see [22] and references therein). In all known cases, the fluxes give rise to parameters in the four-dimensional theory that correspond to gauge charges⁷. When appropriate fluxes in the internal manifold are turned on, both electric and magnetic charges show up in four dimensions, giving rise to supergravity theories that are not of the canonical type (see for instance [23]). Moreover, the fluxes are subject to certain transformations defined in the internal manifold, which manifest themselves as electric/magnetic duality transformations in the four-dimensional theory. It is obviously advantageous to keep such symmetry aspects manifest where possible.

Recently, in a general - non-supersymmetric - context, a formalism was developed, which indeed allows the introduction of both electric and magnetic charges [25]⁸. It involves an extra set of magnetic gauge fields which couple to the magnetic charges, accompanied by a set of antisymmetric tensor fields. These new fields come with additional gauge transformations and, as a consequence, the total number of physical degrees of freedom remains unaltered. The electric and magnetic charges are contained in a so-called *embedding tensor*. This embedding tensor is treated as a spurionic quantity, which implies that it transforms non-trivially under the electric/magnetic dualities. In this way gauge theories are obtained that still contain the duality structure of the ungauged theories.

In the second part of this thesis we apply this formalism to $N = 2$ supersymmetric theories. We derive the supersymmetric Lagrangian and transformation rules for gaugings that involve both electric and magnetic charges (which we call dyonic gaugings). On the scalar fields the gauge symmetries are generated by

⁷For gauged supergravity to be a reliable low energy description, the fluxes should be chosen such that the backreaction on the internal geometry is negligible.

⁸The charges involved are mutually local, which means that an electric/magnetic frame exists in which they are all of the electric type.

isometries of the scalar sigma manifold of both the vector and the hypermultiplet sector. As one of the results, one finds a scalar potential that is independent of the electric/magnetic duality frame. In particular, in a subclass of our models, the potentials of [23] and [24] are reproduced.

Our work provides a unified description of whole classes of string theory models. It should facilitate the construction of phenomenologically interesting models, which may, for instance, lead to groundstates with a positive cosmological constant and to stabilization of the moduli.

This thesis is organized as follows. The first two chapters are meant as an introduction to the later chapters. Chapter 1 deals with electric/magnetic duality, studied in a wider context than $N = 2$ supersymmetric theories. In this chapter we also derive a new result concerning symmetries of Lagrangians that are subject to electric/magnetic duality transformations. The last section of the chapter is about related duality transformations between scalars and tensors. In chapter 2 we treat the relevant aspects of $N = 2$ supersymmetry. We introduce the so-called superconformal method, which is useful for the construction of the supergravity theory. The different multiplets (Weyl, vector and hyper) are considered and the c-map mentioned above is performed explicitly. We also give some new insights associated with electric/magnetic duality.

Then in chapter 3 we derive the instantons present in the hypermultiplet sector of $N = 2$ supergravity. First we treat the relatively simple case of the universal hypermultiplet, after which the D-brane and NS-fivebrane instantons of the general hypermultiplet theory - together with their action - are analyzed. We work in a formulation of the hypermultiplets where a set of scalars is dualized to tensors; the tensor multiplet formulation.

In chapter 4 we construct $N = 2$ supersymmetric gauge theories with both electric and magnetic charges. These gaugings are performed in the vector as well as the hypermultiplet sector of the theory. We derive the supersymmetric action and the supersymmetry transformation rules.

Several technical details of our work can be found in one of the Appendices. We refer to these when necessary.

1

Electric/magnetic duality

As is well-known, the eight equations that form the basis of all electromagnetic phenomena we observe in our universe are

$$\begin{aligned}\vec{\nabla} \cdot \vec{B} &= 0, & -\vec{\nabla} \times \vec{E} &= \frac{\partial \vec{B}}{\partial t}, \\ \vec{\nabla} \cdot \vec{D} &= \rho_e, & \vec{\nabla} \times \vec{H} &= \frac{\partial \vec{D}}{\partial t} + \vec{J}_e,\end{aligned}\tag{1.1}$$

as derived by J.C. Maxwell in 1864 [26]. Here \vec{E} and \vec{H} are the electric and magnetic fields, \vec{D} is the electric displacement and \vec{B} is the magnetic induction. \vec{D} and \vec{B} are related to \vec{E} and \vec{H} through the polarization \vec{P} and the magnetization \vec{M} of a material medium, via

$$\vec{D} = \vec{E} + \vec{P}, \quad \vec{B} = \vec{H} + \vec{M}.\tag{1.2}$$

ρ_e and \vec{J}_e are the electric charge and current density. We employ Heaviside-Lorentz units and put $c = 1$ (as we will do later on with \hbar as well).

Note the striking similarity between the way the magnetic (\vec{B} and \vec{H}) and electric (\vec{D} and \vec{E}) fields enter these equations. The only difference lies in the absence of sources in the equations of the first line, which corresponds to the fact that we do not observe magnetic monopoles in nature. Ignoring the latter fact and including a magnetic charge and current density nonetheless, we see that (1.1) remains equivalent under the transformations

$$\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{\vec{E}} \\ \tilde{\vec{H}} \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix},\tag{1.3}$$

accompanied by

$$\begin{pmatrix} \vec{P} \\ \vec{M} \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{\vec{P}} \\ \tilde{\vec{M}} \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \vec{P} \\ \vec{M} \end{pmatrix},\tag{1.4}$$

and ¹

$$\begin{aligned} \begin{pmatrix} \rho_e \\ \rho_m \end{pmatrix} &\longrightarrow \begin{pmatrix} \tilde{\rho}_e \\ \tilde{\rho}_m \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \rho_e \\ \rho_m \end{pmatrix}, \\ \begin{pmatrix} \vec{J}_e \\ \vec{J}_m \end{pmatrix} &\longrightarrow \begin{pmatrix} \tilde{\vec{J}}_e \\ \tilde{\vec{J}}_m \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \vec{J}_e \\ \vec{J}_m \end{pmatrix}. \end{aligned} \quad (1.5)$$

That (1.3) - (1.5) are equivalence transformations of (1.1) and (1.2) (with magnetic sources included) means by definition that they leave its space of solutions invariant. However, although, using $\tilde{\vec{D}} = \tilde{\vec{E}} + \tilde{\vec{P}}$ and $\tilde{\vec{B}} = \tilde{\vec{H}} + \tilde{\vec{M}}$, the new set of equations can be written in the same form as (1.1) and (1.2), (1.3) - (1.5) are generically *not* invariances of (1.1) and (1.2). This is due to the fact that there is input needed, contained in (\vec{P}, \vec{M}) , (ρ_e, ρ_m) and (J_e, J_m) , to solve these equations. This input is of fixed value, but forced to transform non-trivially via (1.4) and (1.5). Only when $(\rho_e, \rho_m) = 0$ and $(J_e, J_m) = 0$ and the polarization and magnetization of the medium are of the form $(\vec{P}, \vec{M}) \propto (\vec{E}, \vec{H})$ - which implies that the appropriate transformation of (\vec{P}, \vec{M}) is induced by the transformation of (\vec{E}, \vec{H}) - the transformations above are invariances of (1.1) and (1.2).

Notwithstanding the latter fact, the remarkable equivalence of Maxwell's equations under (1.3) - (1.5) seems to indicate that not only electric and magnetic phenomena are intimately related, but that a distinction between phenomena in this way has no intrinsic meaning. Whether a physical phenomenon is electric or magnetic just depends on which description is used. The transformation between the different descriptions is what is called electric/magnetic duality.

In the rest of this chapter we explore the status of electric/magnetic duality at a more fundamental level and in a more general context. We have to stress that as electric/magnetic duality is so big and diverse a subject, on which so much work is done, it is by far not possible to cover all aspects and certainly not in as much detail as they deserve. We mainly focus on those issues that are important for us in later chapters. They will be treated in some detail, embedded in a fairly qualitative discussion of the subject as a whole.

¹Note that we do not take charge quantization into account. We come back to this point in section 1.2.

1.1 Maxwell theory

Let us consider the relativistically covariant version of (1.1). The electric and magnetic fields then combine into the covariant tensor $F^{\mu\nu}$, which takes the form

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \quad (1.6)$$

The first line of (source-free) Maxwell equations (1.1) becomes $\partial_{[\mu}F_{\nu\rho]} = 0$, which is a Bianchi identity when we demand (locally) $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$. A_μ is called the gauge potential. This allows the second line of Maxwell's equations to be derived from the action ²

$$\mathcal{S} = -\frac{\pi}{g^2} \int d^4x F_{\mu\nu}F^{\mu\nu} - i\frac{\theta}{16\pi} \int d^4x \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}F_{\rho\sigma}, \quad (1.7)$$

using the variational principle. g is the coupling constant of the theory and θ is called the theta-angle. The second line of Maxwell's equations thus becomes a field equation. It can be formulated as $\partial_{[\mu}G_{\nu\rho]} = 0$, where we have defined

$$G_{\mu\nu} = i\varepsilon_{\mu\nu\rho\sigma} \frac{\delta\mathcal{L}}{\delta F_{\rho\sigma}}. \quad (1.8)$$

Note that for these classical considerations there does not seem to be a reason to include the second term in (1.7), as it is a total divergence. Furthermore, the coupling constant g can be scaled away. However, shortly we will see why it is useful to include them both.

It is convenient to rewrite (1.7) as

$$\mathcal{S} = -\frac{1}{4}i \int d^4x [\bar{\tau}F_{\mu\nu}^+F^{+\mu\nu} - \tau F_{\mu\nu}^-F^{-\mu\nu}], \quad (1.9)$$

where the complex parameter τ is given by

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}. \quad (1.10)$$

$F_{\mu\nu}^\pm$ is the (anti-)selfdual field strength, defined as $F_{\mu\nu}^\pm \equiv \frac{1}{2}(F_{\mu\nu} \pm \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma})$ (see Appendix A).

In terms of the (anti-)selfdual parts of $F_{\mu\nu}$ and $G_{\mu\nu}$, the set of source-free Maxwell equations takes the form

$$\partial_\mu \begin{pmatrix} F^{+\mu\nu} - F^{-\mu\nu} \\ G^{+\mu\nu} - G^{-\mu\nu} \end{pmatrix} = 0. \quad (1.11)$$

²Despite the factor i appearing in the prefactor, the second term in (1.7) is real. See Appendix A for our conventions.

Obviously this set of equations remains equivalent under

$$\begin{pmatrix} F_{\mu\nu}^{\pm} \\ G_{\mu\nu}^{\pm} \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{F}_{\mu\nu}^{\pm} \\ \tilde{G}_{\mu\nu}^{\pm} \end{pmatrix} = \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} F_{\mu\nu}^{\pm} \\ G_{\mu\nu}^{\pm} \end{pmatrix}, \quad (1.12)$$

where U , Z , W and V are real numbers satisfying $UV - WZ = 1$, i.e. form a matrix with determinant equal to one (this last condition excludes transformations that act as constant rescalings of the field strength and Lagrangian). Two-by-two matrices with determinant equal to one are elements of $Sp(2, \mathbb{R}) \sim SL(2, \mathbb{R})$. Note that (1.3) is included in (1.12) as the maximal compact subgroup of $Sp(2, \mathbb{R})$ ($U = V = \cos \alpha$ and $Z = -W = \sin \alpha$).

The effect of (1.12) is a rotation of the Bianchi identity of $F_{\mu\nu}$ and the field equation of A_{μ} , as following from (1.7). In other words, the equation $\partial_{\mu}(\tilde{F}^{+\mu\nu} - \tilde{F}^{-\mu\nu}) = 0$ is interpreted as the new Bianchi identity, while $\partial_{\mu}(\tilde{G}^{+\mu\nu} - \tilde{G}^{-\mu\nu}) = 0$ is the new field equation. From $\partial_{\mu}(\tilde{F}^{+\mu\nu} - \tilde{F}^{-\mu\nu}) = 0$ being a Bianchi identity it follows that (locally) $\tilde{F}_{\mu\nu} = 2\partial_{[\mu}\tilde{A}_{\nu]}$. \tilde{A}_{μ} is the new gauge potential, which is *not* locally related to the old gauge potential A_{μ} .

Since the transformed system of equations includes a new Bianchi identity and a new field equation that are not separately equivalent to their untransformed versions, the Lagrangian associated with the new system is non-trivially related to the original one. The expression for the transformed Lagrangian, $\tilde{\mathcal{L}}$, follows from $\tilde{G}_{\mu\nu} = i\varepsilon_{\mu\nu\rho\sigma} \frac{\delta \tilde{\mathcal{L}}}{\delta \tilde{F}_{\rho\sigma}}$. It can be written in the same form as (1.9),

$$\tilde{S} = -\frac{1}{4}i \int d^4x \tilde{\tau} \tilde{F}_{\mu\nu}^+ \tilde{F}^{+\mu\nu} + \text{h.c.}, \quad (1.13)$$

when we transform τ as

$$\tau \longrightarrow \tilde{\tau} = \frac{W + V\tau}{U + Z\tau}. \quad (1.14)$$

Observe that the transformations (1.14) contain an inversion of the coupling constant as a special case.

Despite the fact that we have written (1.13) in the same form as (1.9), (1.12) is *not* a symmetry of the action. A symmetry requires the Lagrangian to transform as an invariant function $\tilde{\mathcal{L}}(\tilde{F}(F)) = \mathcal{L}(F) = \mathcal{L}(\tilde{F}(F))$. However under (1.12) the Lagrangian does not transform as a function, $\tilde{\mathcal{L}}(\tilde{F}) \neq \mathcal{L}(F)$, nor is it invariant, $\tilde{\mathcal{L}}(\tilde{F}) \neq \mathcal{L}(\tilde{F})$. The latter refers to the fact that the value of the input parameter τ , which plays a similar role as \vec{P} and \vec{M} in (1.1) and (1.2), is different in (1.13) as compared to (1.9).

Note that in case $\tau = \pm i$, which is the analog of $\vec{P} = \vec{M} = 0$ in (1.1) and (1.2), the Lagrangian is invariant under the maximal compact subgroup of $Sp(2, \mathbb{R})$. This is consistent with the fact that (1.3) is an invariance of (1.1) and (1.2) in case $\vec{P} = \vec{M} = 0$ (and $(\rho_e, \rho_m) = (J_e, J_m) = 0$).

Next we include sources in the equations (1.11),

$$\partial_\mu \begin{pmatrix} F^{+\mu\nu} - F^{-\mu\nu} \\ G^{+\mu\nu} - G^{-\mu\nu} \end{pmatrix} = \begin{pmatrix} J_e^\nu \\ J_m^\nu \end{pmatrix}, \quad (1.15)$$

where (J_e^ν, J_m^ν) is the vector formed by the covariant electric and magnetic currents,

$$J_e^\nu = (\rho_e, \vec{J}_e), \quad J_m^\nu = (\rho_m, \vec{J}_m). \quad (1.16)$$

Obviously, to preserve electric/magnetic duality, (J_e^ν, J_m^ν) should transform as a symplectic vector.

So electric/magnetic equivalences are found of the classical set of Bianchi identities and equations of motion (1.15). This raises the question: What about full quantum theories, do they exhibit electric/magnetic duality as well? In the next section we briefly turn to this issue, basing ourselves on [27] and [28]. For a more extended review we refer to these papers.

1.2 The Montonen-Olive conjecture

The electric charges of (1.15) would appear dynamically when, in addition to the $U(1)$ gauge field, there are charged elementary fields contained in the model. Magnetic charges, on the other hand, correspond to (magnetically charged) *solitons*. Solitons are classical, localized, finite energy solutions, which typically travel undistorted in space with a uniform velocity and can therefore be seen as (classical) particles. They correspond to local minima of the potential. Often solitons are characterized by a topological index, which implies that they are stable.

Magnetically charged solitons - magnetic monopoles - were first found by 't Hooft and Polyakov in the Georgi-Glashow model. The latter is an $SU(2)$ gauge theory, broken to a $U(1)$ subgroup by a Higgs mechanism [30, 31]. The two massive vector fields emerging due to the spontaneous symmetry breaking are charged under the unbroken $U(1)$, with charges $\pm q_0$.

Later on, the Georgi-Glashow model turned out to contain dyons as well [32].

The mass of a gauge particle of the Georgi-Glashow model is

$$M(q_0, 0) = a|q_0|, \quad (1.17)$$

where a is the vacuum expectation value of the (triplet of) Higgs scalar fields and q_0 is the (electric) charge of the gauge particle. On the other hand, the mass of a (classical) dyon is bounded from below by

$$M(q, p) \geq a\sqrt{q^2 + p^2}, \quad (1.18)$$

with q and p the electric and magnetic charge of the dyon. (1.18) is called the ‘‘Bogomol’nyi bound’’ [33]. So, for dyons saturating the Bogomol’nyi bound, which

is realized in the so-called BPS - Bogomol'nyi, Prasad-Sommerfield - limit [34], (1.18) becomes

$$M(q, p) = a\sqrt{q^2 + p^2} . \quad (1.19)$$

Remarkably, the mass formula (1.19) is universal as it also applies to the gauge particles. Therefore the way a particle emerges (as the excitation of an elementary field or as a soliton) is irrelevant when computing its mass from its charges.

Let us now consider the corresponding quantum theory. First of all, we note that a semi-classical quantization around a local minimum of the potential associated with a stable soliton, can be performed in the same way as around the absolute minimum of the theory. The lowest lying state in the spectrum this gives rise to can be identified with the ground state of the soliton. For a review on this subject we refer to [29]. The charges of the quantum states then make up an integer lattice in the plane formed by electric and magnetic charges. Ignoring dyons (to which we come back shortly), the single particle states are associated to five points of the lattice. The Higgs field and the $U(1)$ gauge field are chargeless and so correspond to the origin, $(0, 0)$. The gauge particles have electric charges $\pm q_0$. They correspond to the points $(1, 0)$ and $(-1, 0)$. The soliton states come with magnetic charges $\pm p_0$ and so are associated to $(0, 1)$ and $(0, -1)$ (all in units of q_0 and p_0). Rotations over an angle of $\frac{\pi}{2}$ just rearrange these points. Furthermore, assuming the Bogomol'nyi bound is valid for quantum states as well, the masses involved in the spectrum remain the same.

The above led Montonen and Olive (1977) to the conjecture that there exists a dual or magnetic formulation of the theory under consideration, which is of the same form, but in which the elementary field excitations of the electric formulation should appear as solitons and vice versa. Furthermore, in view of the Dirac condition [35]

$$q^1 p^2 = 2\pi n , \quad n \in \mathbb{Z} , \quad (1.20)$$

(where q^1 is the electric charge of a purely electrically charged particle and p^2 is the magnetic charge of a magnetic monopole) they suggested that the coupling constant of the dual theory is the inverse of the electric coupling constant³.

However, this raises some questions. First of all, can the assumption that the Bogomol'nyi bound is preserved after quantization be justified? Secondly, the electrically charged states have unit spin. For the duality to hold this should be the same for the soliton states. How is this realized? And thirdly, the conjecture does not take dyons into account. How do the corresponding quantum states fit in the duality scheme?

³To be precise, (1.20) implies that the ‘‘magnetic fine structure constant’’ should be $\frac{p_0^2 c}{4\pi\hbar} = \frac{n_0^2}{4\alpha}$, where $\alpha = \frac{q_0^2}{4\pi\hbar c}$ is the electric fine structure constant and n_0 is an integer depending on the theory under consideration [36] (here we have temporarily reinstated \hbar and c).

As we explain below, these questions can all be answered in a satisfactory way when the Georgi-Glashow model is embedded in $N = 4$ supersymmetric Yang-Mills theory (SYM) with $SU(2)$ gauge group. Moreover, with the dyons taken into account the duality group becomes $SL(2, \mathbb{Z}) \sim Sp(2, \mathbb{Z})$.

In $N = 4$ SYM with $SU(2)$ gauge group the Bogomol'nyi bound is a consequence of the supersymmetry algebra and is therefore presumably quantum exact [37, 38]. When $M(q, p) = a\sqrt{q^2 + p^2}$ the structure of the supersymmetry algebra is such that the corresponding states fill out the massive version of a so-called short multiplet. As there is only one such multiplet in $N = 4$ SYM, the multiplets filled out by the electrically charged states corresponding to elementary field excitations and the multiplets associated with soliton states are necessarily isomorphic (this was made explicit in [38]). In particular, both have spin-one states as the states with highest spin. In addition to this, $N = 4$ SYM is quantum conformally invariant. Amongst other things this implies that the coupling constant does not renormalize, which in turn makes the question less pressing whether the Dirac quantization condition should be applied to the bare or renormalized coupling constant (as both are the same) [39].

When dyons are taken into account, we need to know what the allowed values of their charges are. This follows from the Schwinger-Zwanziger quantization condition [40, 41], which is the generalization of the Dirac condition (1.20) and reads as

$$q^1 p^2 - p^1 q^2 = 2\pi n, \quad n \in \mathbb{Z}. \quad (1.21)$$

Here (q^1, p^1) and (q^2, p^2) are the electric and magnetic charges of two dyons. Furthermore, we assume that the charges are conserved and that the TCP-theorem is valid, which implies that the set of allowed values of the charges must be closed under both addition and reversal of sign. Then the allowed values of the charges span a lattice. More precisely, they satisfy

$$q + ip = q_0(m\tau + n), \quad m, n \in \mathbb{Z}, \quad (1.22)$$

where ⁴

$$\tau = \frac{4\pi i}{q_0^2} + \frac{\theta}{2\pi}. \quad (1.23)$$

Here θ is a parameter of the theory under consideration. In the case of $N = 4$ SYM this parameter can be identified with the Yang-Mills theta-angle, which is the analog of the Maxwell theta-angle of the last section [42]. Note that this makes (1.23) the analog of the complex parameter (1.10) of Maxwell theory.

In a quantum theory, stable single particle states should correspond to the primitive vectors of the charge lattice (assuming single particle states obey (1.19)) ⁵. As

⁴We have used $n_0 = 2$ (see footnote below (1.20)), which holds for $N = 4$ SYM [27].

⁵A point of the lattice corresponds to a primitive vector when the line connecting it with the origin contains no other point of the lattice.

shown by Sen, this is indeed the case for $N = 4$ SYM with an $SU(2)$ gauge group broken to $U(1)$ [43]. The transformations that act as a rearrangement of the primitive vectors form the group $SL(2, \mathbb{Z})$.

In view of the above, $N = 4$ SYM with (spontaneously broken) $SU(2)$ gauge group is conjectured to have exact $SL(2, \mathbb{Z})$ electric/magnetic duality. Its set of quantum states, corresponding to both (electrically charged) gauge particles and (dyonic) solitons, rotates under the action of this group. Furthermore, the coupling constant and theta-angle of the theory transform as their Maxwell analogs,

$$\tau \longrightarrow \tilde{\tau} = \frac{W + V\tau}{U + Z\tau} . \quad (1.24)$$

As a non-trivial test of this conjecture, the partition function of a “twisted”, topological version of $N = 4$ SYM has been evaluated and indeed found to exhibit an $SL(2, \mathbb{Z})$ symmetry [44].

The $SL(2, \mathbb{Z})$ dualities of classical Maxwell theory (when charge quantization is taken into account) and of the full $N = 4$ SYM with $SU(2)$ gauge group are in fact related, since the former is (the gauge field part of) the low energy effective action of the latter.

Contrary to $N = 4$ SYM, $N = 2$ SYM with $SU(2)$ gauge group does not have exact electric/magnetic duality. However, in the Wilsonian effective abelian theory⁶ arising from it at low energies electric/magnetic equivalences of the Maxwell type do show up. Making use of the latter fact, Seiberg and Witten managed to solve this low energy theory completely [45]. This can be understood as follows. In these effective models the vacuum expectation values of the scalar fields parameterize a target-space manifold. Singularities in the moduli space parameterization signify the breakdown of the effective description due to additional degrees of freedom becoming massless. These additional degrees of freedom can be identified with magnetic monopoles or dyons of the non-abelian theory. Seiberg and Witten realized that a description of the exact low energy theory requires the use of electric/magnetic dual frames - and so dual parameterizations - in different regions of the moduli space. In this way the entire moduli space can be covered. Moreover, by determining the monodromies around the coordinate singularities and patching them together, the theory can be fixed completely.

The models of the next section we also treat as effective actions. They have an arbitrary number of gauge fields, come with an abelian gauge group and are coupled to gravity. These models contain the $N = 2$ supersymmetric systems we consider in the next chapter.

⁶The effective Wilsonian action is based on integrating out the massive degrees of freedom. It describes the correct physics for energies between appropriately chosen infrared and ultraviolet cutoffs.

1.3 Effective actions and e/m duality

The actions we study in this section are of the form

$$\begin{aligned} \mathcal{S} = \int d^4x e \left[& \left(-\frac{1}{4} i \bar{\tau}_{\Lambda\Sigma} F_{\mu\nu}^{+\Lambda} F^{+\mu\nu\Sigma} - \frac{1}{2} i F_{\mu\nu}^{+\Lambda} O^{+\mu\nu}{}_{\Lambda} \right. \right. \\ & \left. \left. + \frac{1}{8} ((Im\tau)^{-1})^{\Lambda\Sigma} O_{\mu\nu\Lambda}^+ O^{+\mu\nu}{}_{\Sigma} + \text{h.c.} \right) + \mathcal{L}' \right]. \end{aligned} \quad (1.25)$$

$F_{\mu\nu}^{\Lambda}$ are the field strengths of the vector gauge fields A_{μ}^{Λ} ($\Lambda = 1, \dots, n$). The couplings of these field strengths are encoded in the complex matrix $\tau_{\Lambda\Sigma}$, which can be field-dependent (typically it depends on scalar fields). Furthermore, we allow for a linear coupling of the field strengths to field dependent tensors $O_{\mu\nu\Lambda}$ (which are usually bilinear in spinor fields). \mathcal{L}' is arbitrary but independent of the vector gauge fields. Note that we could have absorbed the O^2 term in \mathcal{L}' . However, as we will later see, it is useful to include it explicitly.

The set of Bianchi identities and equations of motion of the gauge fields take the same form as in source-free Maxwell theory

$$D_{\mu} \begin{pmatrix} F^{+\mu\nu\Lambda} - F^{-\mu\nu\Lambda} \\ G^{+\mu\nu}{}_{\Lambda} - G^{-\mu\nu}{}_{\Lambda} \end{pmatrix} = 0, \quad (1.26)$$

where the derivatives D_{μ} are covariantized with respect to general coordinate transformations. The dual field strengths $G_{\mu\nu\Lambda} \equiv ie\epsilon_{\mu\nu\rho\sigma} \frac{\delta\mathcal{L}}{\delta F_{\rho\sigma}^{\Lambda}}$ are explicitly given by

$$G_{\mu\nu\Lambda}^+ = \bar{\tau}_{\Lambda\Sigma} \tilde{F}_{\mu\nu}^{+\Sigma} + O_{\mu\nu\Lambda}^+. \quad (1.27)$$

Obviously the set of equations (1.26) is invariant under

$$\begin{pmatrix} F_{\mu\nu}^{\Lambda} \\ G_{\mu\nu\Lambda} \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{F}_{\mu\nu}^{\Lambda} \\ \tilde{G}_{\mu\nu\Lambda} \end{pmatrix} = \begin{pmatrix} U^{\Lambda}{}_{\Sigma} & Z^{\Lambda\Sigma} \\ W_{\Lambda\Sigma} & V_{\Lambda}{}^{\Sigma} \end{pmatrix} \begin{pmatrix} F_{\mu\nu}^{\Sigma} \\ G_{\mu\nu\Sigma} \end{pmatrix}, \quad (1.28)$$

where U , Z , V and W are real-valued matrices. Similar to the Maxwell-case we interpret the equations $D_{\mu}(\tilde{F}^{+\mu\nu\Lambda} - \tilde{F}^{-\mu\nu\Lambda}) = 0$ as the new Bianchi identities, while the equations $D_{\mu}(\tilde{G}^{+\mu\nu}{}_{\Lambda} - \tilde{G}^{-\mu\nu}{}_{\Lambda}) = 0$ are the new field equations.

So we find again equivalence transformations of the set of equations formed by Bianchi identities and field equations of the gauge potentials. However, recall that the field strengths appearing in (1.25) are coupled to matter and gravity. These fields therefore take part in (1.28) as they appear in $G_{\mu\nu\Lambda}$. For this reason we also have to worry about their field equations as the transformations (1.28) are real duality transformations only when these field equations remain equivalent.

To determine what happens with the field equations of the matter and gravitational fields we have to turn to the Lagrangian and see how it transforms under (1.28). Doing this shows another feature of duality in more general models that was not

present in Maxwell theory: A dual Lagrangian can only be obtained for a special class of matrices U , Z , W and V .

This can be understood as follows. A new Lagrangian $\tilde{\mathcal{L}}(\tilde{F}, \tilde{G}(\tilde{F}))$ is implicitly given by $\tilde{G}_{\mu\nu\Lambda} = ie\varepsilon_{\mu\nu\rho\sigma} \frac{\delta\tilde{\mathcal{L}}}{\delta\tilde{F}_{\rho\sigma}{}^\Lambda}$, which can be rewritten as

$$\begin{aligned}
e \frac{\delta\tilde{\mathcal{L}}}{\delta F_{\eta\lambda}{}^\Sigma} &= -\frac{1}{4} i\varepsilon^{\rho\sigma\mu\nu} \frac{\delta\tilde{F}_{\rho\sigma}{}^\Lambda}{\delta F_{\eta\lambda}{}^\Sigma} \tilde{G}_{\mu\nu\Lambda} \\
&= -\frac{1}{4} i\varepsilon^{\eta\lambda\mu\nu} ((U^T W)_{\Sigma\Gamma} F_{\mu\nu}{}^\Gamma + (U^T V)_{\Sigma}{}^\Gamma G_{\mu\nu\Gamma}) \\
&\quad -\frac{1}{4} i\varepsilon^{\rho\sigma\mu\nu} (Z^T V)^{\Omega\Gamma} \frac{\delta G_{\rho\sigma\Omega}}{\delta F_{\eta\lambda}{}^\Sigma} G_{\mu\nu\Gamma} \\
&\quad -\frac{1}{4} i\varepsilon^{\rho\sigma\mu\nu} (W^T Z)_\Gamma{}^\Omega \frac{\delta G_{\rho\sigma\Omega}}{\delta F_{\eta\lambda}{}^\Sigma} F_{\mu\nu}{}^\Gamma .
\end{aligned} \tag{1.29}$$

To obtain $\tilde{\mathcal{L}}$, (1.29) should be integrated over $F_{\eta\lambda}{}^\Sigma$. In the case of Maxwell theory, when the U , Z , W and V are just numbers, this can always be done. In the general case, however, U , Z , W and V are $n \times n$ -matrices and having an integrable expression on the right-hand side of (1.29) is not automatically guaranteed.

Nevertheless, when we demand

$$\begin{aligned}
U^T V - W^T Z &= \alpha \mathbb{I} , \\
U^T W = W^T U, \quad Z^T V &= V^T Z ,
\end{aligned} \tag{1.30}$$

(1.29) becomes

$$\begin{aligned}
e \frac{\delta\tilde{\mathcal{L}}}{\delta F_{\eta\lambda}{}^\Sigma} &= -\frac{1}{8} i\varepsilon^{\rho\sigma\mu\nu} (U^T W)_{\Lambda\Gamma} \frac{\delta}{\delta F_{\eta\lambda}{}^\Sigma} (F_{\rho\sigma}{}^\Lambda F_{\mu\nu}{}^\Gamma) \\
&\quad -\frac{1}{8} i\varepsilon^{\rho\sigma\mu\nu} (Z^T V)^{\Lambda\Gamma} \frac{\delta}{\delta F_{\eta\lambda}{}^\Sigma} (G_{\rho\sigma\Lambda} G_{\mu\nu\Gamma}) \\
&\quad -\frac{1}{4} i\varepsilon^{\rho\sigma\mu\nu} (W^T Z)_\Lambda{}^\Gamma \frac{\delta}{\delta F_{\eta\lambda}{}^\Sigma} (F_{\rho\sigma}{}^\Lambda G_{\mu\nu\Gamma}) \\
&\quad -\frac{1}{4} i\varepsilon^{\eta\lambda\mu\nu} \alpha G_{\mu\nu\Sigma} ,
\end{aligned} \tag{1.31}$$

which is obviously integrable. Putting α to 1 (again leaving out total rescalings of the Lagrangian and field strengths), the conditions (1.30) can be written as

$$M^T \Omega M = \Omega , \tag{1.32}$$

with

$$M = \begin{pmatrix} U^\Lambda{}_\Sigma & Z^\Lambda{}_\Sigma \\ W_{\Lambda\Sigma} & V_\Lambda{}^\Sigma \end{pmatrix} , \quad \Omega = \begin{pmatrix} 0 & \delta^{\Lambda\Sigma} \\ -\delta_{\Lambda\Sigma} & 0 \end{pmatrix} . \tag{1.33}$$

So the matrices in (1.28) should be chosen such that they leave the skewsymmetric matrix Ω invariant. This means - by definition - that they are elements of the *symplectic* group $Sp(2n, \mathbb{R})$ (the connection between electric/magnetic duality and the symplectic group was first observed in [57]). We will call objects $(\alpha^\Lambda, \alpha_\Lambda)$ that transform as in (1.28) (so for example $(F_{\mu\nu}^\Lambda, G_{\mu\nu\Lambda})$) *symplectic vectors* from now on.

Restricting ourselves to transformations involving $Sp(2n, \mathbb{R})$ matrices we can integrate (1.31) to obtain

$$\tilde{S} = \int d^4x e \left[\mathcal{L} - \frac{1}{4} i (\tilde{F}_{\mu\nu}^{+\Lambda} \tilde{G}^{+\mu\nu}{}_\Lambda - F_{\mu\nu}^{+\Lambda} G^{+\mu\nu}{}_\Lambda - \text{h.c.}) + \hat{\mathcal{L}} \right]. \quad (1.34)$$

Note that as \tilde{S} is a functional of the new field strengths $\tilde{F}_{\mu\nu}^\Lambda$, the tensors $F_{\mu\nu}^\Lambda$, $G_{\mu\nu\Lambda}$ and $\tilde{G}_{\mu\nu\Lambda}$ should be understood as functions of the former. $\hat{\mathcal{L}}$ is an integration constant, i.e. an arbitrary functional of the matter fields.

Now that we have found the new Lagrangian we can determine what has happened with the field equations of the matter fields. We first vary (1.34) with respect to all the fields. This way we obtain

$$\begin{aligned} \delta \tilde{\mathcal{L}} &= \delta \mathcal{L} - i \delta \tilde{A}_\mu^\Lambda D_\nu (\tilde{G}^{+\mu\nu}{}_\Lambda - \tilde{G}^{-\mu\nu}{}_\Lambda) + i \delta A_\mu^\Lambda D_\nu (G^{+\mu\nu}{}_\Lambda - G^{-\mu\nu}{}_\Lambda) \\ &\quad + \delta \hat{\mathcal{L}}, \end{aligned} \quad (1.35)$$

where \tilde{A}_μ^Λ are the new gauge potentials, satisfying $\tilde{F}_{\mu\nu}^\Lambda = 2\partial_{[\mu} \tilde{A}_{\nu]}^\Lambda$. In deriving (1.35) we have used that $\tilde{F}_{\mu\nu}^{+\Lambda} \delta \tilde{G}^{+\mu\nu}{}_\Lambda - F_{\mu\nu}^{+\Lambda} \delta G^{+\mu\nu}{}_\Lambda = \delta \tilde{F}_{\mu\nu}^{+\Lambda} \tilde{G}^{+\mu\nu}{}_\Lambda - \delta F_{\mu\nu}^{+\Lambda} G^{+\mu\nu}{}_\Lambda$. Also we have thrown away a total divergence. From (1.35) we see that a variation of $\tilde{\mathcal{L}}$ with respect to the fields other than \tilde{A}_μ^Λ only differs from a variation of \mathcal{L} with respect to the fields other than A_μ^Λ by the term $\delta \hat{\mathcal{L}}$. This implies that the field equations of the matter and gravitational fields remain equivalent when $\hat{\mathcal{L}}$ vanishes. In other words, when we take $\hat{\mathcal{L}} = 0$ the set of transformations (1.28) are proper duality transformations.

Note that this whole derivation, involving the transformation of the Lagrangian, the appearance of the symplectic group and the equivalence of the equations of motion of the matter and gravitational fields does *not* depend on the explicit form of the original Lagrangian. All we need is that the gauge potentials only appear in the Lagrangian through their field strengths. This implies that the results (including the appearance of $Sp(2n, \mathbb{R})$) also hold for higher derivative theories in which higher powers of field strengths are involved. An example of such a theory is the Born-Infeld Lagrangian of non-linear electrodynamics.

Returning to the class of models given by (1.25), it turns out that the new action (1.34) (with $\hat{\mathcal{L}} = 0$) can be written back in the form (1.25),

$$\begin{aligned} \tilde{S} &= \int d^4x e \left[\left(-\frac{1}{4} i \tilde{\tau}_{\Lambda\Sigma} \tilde{F}_{\mu\nu}^{+\Lambda} \tilde{F}^{+\mu\nu\Sigma} - \frac{1}{2} i \tilde{F}_{\mu\nu}^{+\Lambda} \tilde{O}^{+\mu\nu}{}_\Lambda \right. \right. \\ &\quad \left. \left. + \frac{1}{8} ((Im\tilde{\tau})^{-1})^{\Lambda\Sigma} \tilde{O}_{\mu\nu\Lambda}^+ \tilde{O}^{+\mu\nu}{}_\Sigma + \text{h.c.} \right) + \mathcal{L}' \right]. \end{aligned} \quad (1.36)$$

i.e. the transformation of the Lagrangian is induced by a transformation of the objects appearing in (1.25), which is of the form

$$\begin{aligned}
F_{\mu\nu}^{+\Lambda} &\longrightarrow \tilde{F}_{\mu\nu}^{+\Lambda} , \\
\tau_{\Lambda\Sigma} &\longrightarrow \tilde{\tau}_{\Lambda\Sigma} \equiv ((W + V\bar{\tau})(U + Z\bar{\tau})^{-1})_{\Lambda\Sigma} , \\
O_{\mu\nu\Lambda}^+ &\longrightarrow \tilde{O}_{\mu\nu\Lambda}^+ \equiv ((U + Z\bar{\tau})^{-1})^\Sigma{}_\Lambda O_{\mu\nu\Sigma}^+ , \\
\mathcal{L}' &\longrightarrow \tilde{\mathcal{L}}' .
\end{aligned} \tag{1.37}$$

We now see why it was useful to start with the explicit O^2 term in (1.25).

Just as is the case in Maxwell theory, we stress that although the new action is of the same form as the original one, the electric/magnetic duality transformation involved is not an ordinary symmetry (we recall that a symmetry requires $\tilde{\mathcal{L}}(\tilde{F}) = \mathcal{L}(F) = \mathcal{L}(\tilde{F})$). From (1.34) it immediately follows that $\tilde{\mathcal{L}}(\tilde{F}) \neq \mathcal{L}(F)$. Generically we also have $\tilde{\mathcal{L}}(\tilde{F}) \neq \mathcal{L}(\tilde{F})$, implying that the electric/magnetic duality is a duality equivalence.

Only when the transformations of the objects in (1.37) are induced by transformations of fields on which they depend, the duality is a duality invariance,

$$\tilde{\mathcal{L}}(\tilde{F}, \tilde{\phi}) = \mathcal{L}(\tilde{F}, \tilde{\phi}) . \tag{1.38}$$

Here the matter fields are denoted by ϕ . In chapter 4, in the context of gaugings in $N = 2$ supersymmetric models, we are precisely interested in this subclass of electric/magnetic dualities.

The condition for having a duality invariance comes down to the requirement that the transformation of $\tau_{\Lambda\Sigma}$ and $O_{\mu\nu\Lambda}$ is induced by a transformation of the matter fields, combined with the demand that this transformation of the matter fields is a symmetry of \mathcal{L}' ,

$$\begin{aligned}
\tilde{\tau}_{\Lambda\Sigma}(\tilde{\phi}) &= \tau_{\Lambda\Sigma}(\tilde{\phi}) , \\
\tilde{O}_{\mu\nu\Lambda}(\tilde{\phi}) &= O_{\mu\nu\Lambda}(\tilde{\phi}) , \\
\mathcal{L}'(\tilde{\phi}) &= \mathcal{L}'(\phi) .
\end{aligned} \tag{1.39}$$

We stress that these relations do *not* imply that $\tau_{\Lambda\Sigma}$ and $O_{\mu\nu\Lambda}$ should transform as functions (which would mean $\tilde{\tau}_{\Lambda\Sigma}(\tilde{\phi}) = \tau_{\Lambda\Sigma}(\phi)$ and $\tilde{O}_{\mu\nu\Lambda}(\tilde{\phi}) = O_{\mu\nu\Lambda}(\phi)$). For continuous duality invariances (1.39) gives rise to the following identity

$$\begin{aligned}
C_{\Lambda\Sigma}(F_{\mu\nu}^{+\Lambda} F^{+\mu\nu\Sigma} - F_{\mu\nu}^{-\Lambda} F^{-\mu\nu\Sigma}) + D^{\Lambda\Sigma}(G_{\mu\nu\Lambda}^+ G^{+\mu\nu}{}_\Sigma - G_{\mu\nu\Lambda}^- G^{-\mu\nu}{}_\Sigma) \\
- 2B^\Sigma{}_\Lambda(F_{\mu\nu}^{+\Lambda} G^{+\mu\nu}{}_\Sigma - F_{\mu\nu}^{-\Lambda} G^{-\mu\nu}{}_\Sigma) = 2i \frac{\delta \mathcal{L}}{\delta \phi} \delta \phi ,
\end{aligned} \tag{1.40}$$

where the matrices B , C , D are defined by an expansion of a symplectic matrix around \mathbb{I} ,

$$\begin{pmatrix} U & Z \\ W & V \end{pmatrix} \approx \mathbb{I} + \begin{pmatrix} B & -D \\ C & -B^T \end{pmatrix} . \tag{1.41}$$

Here $C_{\Lambda\Sigma}$ and $D^{\Lambda\Sigma}$ are symmetric. (1.40) can be viewed as an equation determining (if possible) the transformation rules for the matter fields to obtain an invariance. Alternatively, when there are natural transformation rules for these matter fields - as will be the case in $N = 2$ supersymmetric systems - (1.40) is a condition on the matrices B , C and D .

Symmetries versus electric/magnetic duality

In the next chapter we will consider $N = 2$ supersymmetric systems. The vector multiplet sector of these models is of the form (1.25), so electric/magnetic duality is realized. A relevant question in this respect is: Do electric/magnetic duality transformations preserve the symmetries of the Lagrangian (and in particular supersymmetry)? Below we answer this question. Although our main interest is in $N = 2$ supersymmetric models with at most two derivatives, our treatment is also valid for general models with arbitrary powers of field strengths appearing. We start our exposition by considering a Lagrangian \mathcal{L} and an electric/magnetic dual of it, $\tilde{\mathcal{L}}$. We assume that both \mathcal{L} and $\tilde{\mathcal{L}}$ are invariant under a certain set of symmetry transformations, up to a total divergence and upon using the Bianchi identities of the field strengths. Leaving out the total divergences, the variations of the Lagrangians under the symmetry transformations can be denoted as

$$\begin{aligned}\delta\mathcal{L} &= -i\alpha_{\mu\Lambda}D_\nu(F^{+\mu\nu\Lambda} - F^{-\mu\nu\Lambda}), \\ \delta\tilde{\mathcal{L}} &= -i\tilde{\alpha}_{\mu\Lambda}D_\nu(\tilde{F}^{+\mu\nu\Lambda} - \tilde{F}^{-\mu\nu\Lambda}).\end{aligned}\quad (1.42)$$

The objects $\alpha_{\mu\Lambda}$ and $\tilde{\alpha}_{\mu\Lambda}$ are field dependent quantities. Note that they are defined up to total divergences.

In case the transformations of the fields other than the vector fields are the same for \mathcal{L} and $\tilde{\mathcal{L}}$, we also have

$$\delta\tilde{\mathcal{L}} + i\delta\tilde{A}_\mu{}^\Lambda D_\nu(\tilde{G}^{+\mu\nu}{}_\Lambda - \tilde{G}^{-\mu\nu}{}_\Lambda) = \delta\mathcal{L} + i\delta A_\mu{}^\Lambda D_\nu(G^{+\mu\nu}{}_\Lambda - G^{-\mu\nu}{}_\Lambda). \quad (1.43)$$

as follows directly from (1.35) (with $\hat{L} = 0$).

Combining (1.43) and (1.42) gives

$$\begin{aligned}& -i\tilde{\alpha}_{\mu\Lambda}D_\nu(\tilde{F}^{+\mu\nu\Lambda} - \tilde{F}^{-\mu\nu\Lambda}) + i\delta\tilde{A}_\mu{}^\Lambda D_\nu(\tilde{G}^{+\mu\nu}{}_\Lambda - \tilde{G}^{-\mu\nu}{}_\Lambda) \\ &= -i\alpha_{\mu\Lambda}D_\nu(F^{+\mu\nu\Lambda} - F^{-\mu\nu\Lambda}) + i\delta A_\mu{}^\Lambda D_\nu(G^{+\mu\nu}{}_\Lambda - G^{-\mu\nu}{}_\Lambda).\end{aligned}\quad (1.44)$$

We saw above that $(F_{\mu\nu}{}^\Lambda, G_{\mu\nu\Lambda})$ transforms as a symplectic vector. It then follows that we can solve (1.44) by demanding $\delta A_\mu{}^\Lambda$ and $\alpha_{\mu\Lambda}$ to form a symplectic vector as well (observe that both $\delta A_\mu{}^\Lambda$ and $\alpha_{\mu\Lambda}$ are defined up to a total divergence)⁷. In that case we see the appearance of a symplectic inner product on both sides of (1.44). This equation is then nothing more than the statement that a symplectic inner product transforms as a scalar under $Sp(2n, \mathbb{R})$.

⁷Note that this does imply that neither $\delta A_\mu{}^\Lambda$ nor $\alpha_{\mu\Lambda}$ is allowed to depend explicitly on $A_\mu{}^\Lambda$.

To summarize, we have found the following: When we start with a Lagrangian that is invariant under a certain set of transformations, the electric/magnetic dual Lagrangian is symmetric under a dual set of transformations. This set of transformations is the same for the fields other than the vector fields. The transformation rule for the dual vector field is found by a transformation of the symplectic vector formed by the symmetry transformation of the old gauge field and the prefactor of the Bianchi-identity-term to which the symmetry variation of the old Lagrangian gives rise. A large class of symmetries - amongst which is supersymmetry - is therefore indeed necessarily preserved under electric/magnetic duality transformations.

1.4 Scalar-tensor duality

In this last section of the first chapter we treat dualities involving scalar and tensor gauge fields rather than vector gauge fields. They also appear in four-dimensional field theories and low energy effective actions and are directly related to the electric/magnetic duality transformations described above.

To start with, we consider the simple model ⁸

$$\mathcal{S}^e = \int d^4x \left[\frac{1}{6} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{3} i \lambda \varepsilon^{\mu\nu\rho\sigma} \partial_\mu H_{\nu\rho\sigma} \right], \quad (1.45)$$

where λ is a scalar and $H_{\mu\nu\rho}$ is a tensor field of rank three. For future convenience we take a Euclidean setting.

From the action (1.45) two dual effective theories can be obtained. One by eliminating λ ,

$$\mathcal{S}^e = \int d^4x \frac{3}{2} \partial_{[\mu} B_{\nu\rho]} \partial^{[\mu} B^{\nu\rho]}, \quad (1.46)$$

(where $H_{\mu\nu\rho} = 3\partial_{[\mu} B_{\nu\rho]}$), the other by eliminating $H_{\mu\nu\rho}$,

$$\mathcal{S}^e = \int d^4x \partial_\mu \lambda \partial^\mu \lambda. \quad (1.47)$$

Depending on whether the theory to start with is viewed as fundamental or as an effective action, the elimination of the field should be done by solving its equation of motion or by integrating over it. How to do the latter is for example explained in [46]. Eliminating a field in (1.45) by solving its equation of motion either yields the Bianchi identity of $H_{\mu\nu\rho}$ (to give (1.46)) or the relation $i\varepsilon^{\mu\nu\rho\sigma} \partial_\mu \lambda = H^{\nu\rho\sigma}$ (to yield (1.47)).

⁸We take a flat background for notational convenience, but our analysis straightforwardly generalizes to arbitrary backgrounds.

This type of duality straightforwardly generalizes to more complicated actions depending either on tensor gauge fields or scalar fields. Also it can be realized in any dimension D , where it relates effective theories of rank p gauge potentials to effective theories with gauge potentials of rank $D - p - 2$.

(1.46) and (1.47) come with both a Bianchi identity and a field equation

$$\partial_\mu \begin{pmatrix} \varepsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma} \\ -H^{\mu\nu\rho} \end{pmatrix} = 0, \quad \partial_\mu \begin{pmatrix} -\varepsilon^{\mu\nu\rho\sigma} \partial_\sigma \lambda \\ 6 \partial^\mu \lambda \end{pmatrix} = 0. \quad (1.48)$$

Via (1.45) the Bianchi identity (field equation) of one theory is related to the field equation (Bianchi identity) of the other. This sounds very much like electric/magnetic duality. Indeed its (Minkowskian) version with $D = 4$ and $p = 1$,

$$\mathcal{S} = \int d^4x \left[\mathcal{L}(F) + \frac{1}{2} i \varepsilon^{\mu\nu\rho\sigma} \partial_\mu \tilde{A}_\nu F_{\rho\sigma} \right], \quad (1.49)$$

can be shown to effectuate the electric/magnetic duality rotation with rotation matrix M ,

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (1.50)$$

Put differently, the duality discussed here is the scalar-tensor analog of the special electric/magnetic duality transformation with matrix (1.50). This naturally brings up the question: What about electric/magnetic duality transformations with general matrix M , do they also have their scalar-tensor analogs?

We answer this question in a Euclidean context. One of the reasons to do so is that in chapter 3 we work with the Euclidean version of a ($N = 2$ supergravity) scalar-tensor model. There is also another reason, to which we come shortly.

Consider the Euclidean action of a model with both a scalar and a tensor gauge field.

$$\mathcal{S}^e = \frac{2\pi}{g^2} \int d^4x \left[\partial_\mu \chi \partial^\mu \chi + \frac{1}{6} H_{\mu\nu\rho} H^{\mu\nu\rho} \right] - i \frac{\theta}{12\pi} \int d^4x \varepsilon^{\mu\nu\rho\sigma} \partial_\mu \chi H_{\nu\rho\sigma}. \quad (1.51)$$

Its set of equations of motion and Bianchi identities we formulate as

$$\varepsilon^{\mu\nu\rho\sigma} \partial_\mu \begin{pmatrix} H_{\nu\rho\sigma} \\ G_{\nu\rho\sigma}^X \end{pmatrix} = 0, \quad \varepsilon^{\mu\nu\rho\sigma} \partial_\rho \begin{pmatrix} F_\sigma^X \\ G_\sigma^H \end{pmatrix} = 0, \quad (1.52)$$

where we have used

$$\begin{aligned} F_\mu^X &\equiv -i \partial_\mu \chi, \\ G_{\mu\nu\rho}^X &\equiv -\varepsilon_{\mu\nu\rho\sigma} \frac{\delta \mathcal{L}^e}{\delta F_\sigma^X}, \\ G_\mu^H &\equiv \varepsilon_{\mu\nu\rho\sigma} \frac{\delta \mathcal{L}^e}{\delta H_{\nu\rho\sigma}}. \end{aligned} \quad (1.53)$$

A source term in the first set of equations in (1.52) would involve a scalar charge density. This is different in a set of equations associated with vector gauge fields. Then the source term comes with a current vector, which is the natural covariant object describing the charge and current of a point particle. Likewise a current tensor of rank p naturally corresponds to a $p - 1$ -dimensional source. According to this, when the charge density is a scalar field ($p = 0$) (as here) the associated objects need to be localized in *all* directions. This naturally forces one to take a Euclidean setting, in which objects of the latter type make sense.

In both sets of (1.52) the first equation is a Bianchi identity, while the second one is an equation of motion. These sets therefore are very similar to the set of equations of Maxwell theory (1.11). Also just as in Maxwell theory we can perform the rotation

$$\begin{aligned} \begin{pmatrix} H_{\mu\nu\rho} \\ G_{\mu\nu\rho}^{\tilde{\chi}} \end{pmatrix} &\longrightarrow \begin{pmatrix} \tilde{H}_{\mu\nu\rho} \\ \tilde{G}_{\mu\nu\rho}^{\tilde{\chi}} \end{pmatrix} = \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} H_{\mu\nu\rho} \\ G_{\mu\nu\rho}^{\tilde{\chi}} \end{pmatrix}, \\ \begin{pmatrix} F_{\mu}^{\tilde{\chi}} \\ G_{\mu}^{\tilde{H}} \end{pmatrix} &\longrightarrow \begin{pmatrix} \tilde{F}_{\mu}^{\tilde{\chi}} \\ \tilde{G}_{\mu}^{\tilde{H}} \end{pmatrix} = \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} F_{\mu}^{\tilde{\chi}} \\ G_{\mu}^{\tilde{H}} \end{pmatrix}. \end{aligned} \quad (1.54)$$

Again we take $UV - WZ = 1$ to leave out total rescalings of the Lagrangian and field strengths, i.e. the matrix in (1.54) belongs to $Sp(2, \mathbb{R})$. We then interpret the first line of the sets of equations as the new Bianchi identities. So $\tilde{H}_{\mu\nu\rho} = 3\partial_{[\mu}\tilde{B}_{\nu\rho]}$ and $\tilde{F}_{\mu}^{\tilde{\chi}} = -i\partial_{\mu}\tilde{\chi}$ where $\tilde{B}_{\mu\nu}$ and $\tilde{\chi}$ are not locally related to $B_{\mu\nu}$ and χ . What we have effectively done is rotating the scalar (tensor) Bianchi identity and the tensor (scalar) equation of motion.

A dual Lagrangian is (consistently) defined by

$$\tilde{G}_{\mu\nu\rho}^{\tilde{\chi}} = -\varepsilon_{\mu\nu\rho\sigma} \frac{\delta\tilde{\mathcal{L}}^e}{\delta\tilde{F}_{\sigma}^{\tilde{\chi}}}, \quad \tilde{G}_{\mu}^{\tilde{H}} = \varepsilon_{\mu\nu\rho\sigma} \frac{\delta\tilde{\mathcal{L}}^e}{\delta\tilde{H}_{\nu\rho\sigma}}. \quad (1.55)$$

It takes the same form as (1.51),

$$\tilde{S}^e = \frac{2\pi}{\tilde{g}^2} \int d^4x \left[\partial_{\mu}\tilde{\chi}\partial^{\mu}\tilde{\chi} + \frac{1}{6}\tilde{H}_{\mu\nu\rho}\tilde{H}^{\mu\nu\rho} \right] - i\frac{\tilde{\theta}}{12\pi} \int d^4x \varepsilon^{\mu\nu\rho\sigma} \partial_{\mu}\tilde{\chi}\tilde{H}_{\nu\rho\sigma}, \quad (1.56)$$

where we used, just as in Maxwell theory,

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}, \quad \tau \longrightarrow \tilde{\tau} = \frac{W + V\tau}{U + Z\tau}. \quad (1.57)$$

We find that generically $\tilde{\mathcal{L}}(\tilde{F}^{\tilde{\chi}}, \tilde{H}) \neq \mathcal{L}(\tilde{F}^{\tilde{\chi}}, \tilde{H}) \neq \mathcal{L}(F^{\tilde{\chi}}, H)$, so (1.54) are duality equivalences. They can be viewed as the analogs of the electric/magnetic duality transformations of Maxwell theory with general matrix M . The question posed above is therefore found to have a positive answer. Note that the special transformation with matrix M given by (1.50) does not give back the duality of (1.45), but rather a doubled version of it.

The electric/magnetic duality of Maxwell theory and the duality of (1.51) are related. Namely, it is not difficult to show that the Maxwell action (1.7) and the action (1.51) yield the same three-dimensional theory after a dimensional reduction (for Maxwell theory this reduction should either be over time or be accompanied by a Wick rotation). This involves the identification of the dimensionally reduced version of the symplectic vectors $(F_{\mu\nu}, G_{\mu\nu})$ and the ones appearing in (1.54).

Like the electric/magnetic duality of Maxwell theory generalizes to the duality relations hidden in the (effective) action (1.25), the duality of (1.51) has a straightforward generalization to more complicated actions (with more scalar and tensor gauge fields and field dependent coupling matrices). In fact, these theories and their corresponding duality relations can be found via a dimensional reduction of the vector theory and an uplifting of the resulting three-dimensional theory to a scalar-tensor theory. In chapter 2 we will explicitly perform this map in the context of $N = 2$ supergravity, where it goes by the name *c-map*.

What is the relevance of these ‘general M ’ scalar-tensor dualities? In case the scalar-tensor theories under consideration are viewed as fundamental theories, which should be quantized, the ‘dualities’ we found in fact are only classical equivalences. To find out if they can be promoted to honest dualities the full quantum theories should be considered. When we have to do with effective actions, we can properly use the term dualities for the equivalence relations we found. However, then these dualities can be translated to the dual (in the sense of (1.45)) all-scalar theory, where they turn out to correspond to (ordinary) coordinate transformations on the sigma model manifold. So it is not so clear (yet) what the relevance of the scalar-tensor dualities is. Nevertheless, they are interesting equivalence relations between different scalar-tensor actions. Especially in the context of string theory, where quite often one naturally gets scalar-tensor theories instead of their all-scalar counterparts, they deserve some further study. For instance, it may be that in string theory four-dimensional scalar-tensor dualities are the low energy effective realizations of some string dualities, i.e. that they relate different string compactifications in the same way as the (generalized) electric/magnetic dualities in $N = 2$ supergravity do that we describe in chapter 4.

2

N=2 supersymmetry

In the introduction of this thesis we have motivated our use of $N = 2$ supersymmetric models. In short, the main reason for our interest is the realization of $N = 2$ supergravities as low energy effective actions of string theory, and in particular the controllable yet rich environment they offer for both doing (quasi-) string phenomenology and studying non-perturbative string effects.

In the present chapter our aim is to introduce and explain the aspects of $N = 2$ supersymmetric systems that are relevant in the later chapters. For a more extended review of the subject we refer for example to [47]. Our setup means that a lot of notation is introduced and therefore this chapter is a little on the technical side.

Supersymmetry transformations transform bosons into fermions and vice versa. Supersymmetric theories are invariant under transformations of this type. This can be summarized schematically as

$$\begin{aligned} B &\longrightarrow \tilde{B} \sim F \\ F &\longrightarrow \tilde{F} \sim B \\ T(B, F) &\longrightarrow T(\tilde{B}, \tilde{F}) \sim T(B, F) , \end{aligned} \tag{2.1}$$

where B/F denote the bosons/fermions of the theory $T(B, F)$. As bosons have integer spin and fermions half-integer spin it automatically follows that supersymmetry generators (Q) are fermions. Naturally they have spin one-half and therefore should transform in the spinor representation of the Lorentz group. In four dimensions the smallest spinor has four real components. It can be obtained from a Majorana constraint on a Dirac spinor, which has eight real components (our notation and conventions can be found in Appendix A). The number of four-component supersymmetry generators is denoted by N .

According to [48], supersymmetry generators have to obey the following anticommutation relation

$$\{Q_\alpha, \bar{Q}_\beta\} = 2(\gamma_\mu)_{\alpha\beta} P^\mu , \tag{2.2}$$

where P^μ are the generators of translations, which are part of the Poincaré group.

The group formed by supersymmetry and Poincaré generators is called the Poincaré supergroup.

Taking a field-theoretic setting, the Poincaré superalgebra should be represented on the fields. One can choose to either fix the spacetime dependence of the supersymmetry parameters or leave it unfixed. Supersymmetries of the former type are called ‘global’ or ‘rigid’. The latter type are the ‘local’ supersymmetries. (2.2) then implies that the translational symmetry is also local, which means that the theory is invariant under general coordinate transformations and therefore automatically includes gravity. Despite cancellations between Feynman diagrams induced by supersymmetry, $N = 2$ supergravity is not finite. We should therefore view it as an effective action of an underlying fundamental theory. As already explained, in our setup this fundamental theory is string theory.

An important tool in the construction of $N = 2$ supergravity is the so-called ‘superconformal method’ [49]. It involves an extension of the symmetry group from super Poincaré to the superconformal group. The $N = 2$ superconformal group contains, besides the symmetries of the super Poincaré group, additional symmetries, which include for instance dilatations. The corresponding superconformally invariant theories are *gauge equivalent* to Poincaré supergravity. This means that models of the latter type, which do not necessarily exhibit the extra superconformal symmetries, can be obtained by gauge fixing these symmetries or by writing the theory in terms of gauge invariant quantities. The fields disappearing in the process are called *compensating fields*. Let us clarify this using a model of pure gravity. The Lagrangian

$$\sqrt{g}\mathcal{L} \propto \sqrt{g} \left[g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{6} R \phi^2 \right], \quad (2.3)$$

is invariant under local dilatations with parameter $\Lambda(x)$: $\delta\phi = \Lambda\phi$, $\delta g_{\mu\nu} = -2\Lambda g_{\mu\nu}$. It is gauge equivalent to the Einstein-Hilbert Lagrangian, which itself does not exhibit local scale invariance. To make the equivalence manifest one either rewrites (2.3) in terms of a scale invariant metric $\phi^2 g_{\mu\nu}$, or one simply imposes the gauge condition and sets ϕ equal to a constant (ϕ therefore is a compensating field). For a further explanation of the principle of gauge equivalence we refer to [50] and references therein. In the context of supersymmetric systems, keeping the superconformal invariance manifest, one realizes a higher degree of symmetry, which facilitates the construction of supergravity Lagrangians and clarifies the geometrical features of the resulting theories.

The $N = 2$ superconformal algebra has three irreducible field representations that are important for us. These are the Weyl multiplet [51, 52] (containing for instance the graviton), the vector multiplet (containing a vector gauge field) and the hypermultiplet (with a bosonic sector consisting of scalar fields only) (the last two have also - non-conformal - rigidly supersymmetric versions). The first three sections of this chapter are devoted to the introduction of these three multiplets and the rigidly supersymmetric and superconformally invariant actions they give

rise to. In the fourth section we describe the procedure to go from a conformal model to Poincaré supergravity.

In this chapter, the gauge group associated with the gauge fields of the vector multiplets is abelian. Furthermore, we do not consider couplings of the gauge fields to the hypermultiplets (for both we refer to chapter 4). As a result the vector fields only appear through their field strengths. In turn it follows that, as already alluded to several times in the last chapter, the vector multiplet sector exhibits electric/magnetic duality¹. Involving these electric/magnetic dualities, we produce a new result concerning the behavior of the auxiliary field Y_{ij} .

As we found in chapter 1, between certain theories consisting of scalars and tensors there exist duality relations that are similar to electric-magnetic duality. In a subsector of the tensor formulation of the hypermultiplet part of $N = 2$ supergravity this scalar-tensor duality is realized. The duality structure in these tensor multiplet models can be understood from the existence of the c-map, which is a map from the vector multiplet sector to the hypermultiplet (or tensor multiplet) sector. In view of the use we will make of it in the next chapter, we treat this map in detail in section 2.5.

2.1 The $N=2$ superconformal group and the Weyl multiplet

Let us start by considering the different generators present in the $N = 2$ superconformal group. First of all there are the generators of the conformal group itself. These are the translations (P^a) and the Lorentz transformations (M^{ab}) of the Poincaré group together with the scale (D) and special conformal (K^a) transformations. Furthermore, the superconformal group contains two supersymmetry generators Q^i . The algebra of this set of generators does not close. As it turns out, two more fermionic generators (S^i , ‘special supersymmetries’) and the generators of $U(2)_R$ (to which we come back shortly) should be included. The algebra may furthermore contain a central charge, which would appear in the anticommutator of two different supersymmetry generators. However on the representations we consider it vanishes² and therefore we neglect it from now on.

With the latter taken into account, the full set of anticommutation relations be-

¹Some early references to electric/magnetic duality in supersymmetric theories are [18, 53, 54, 55, 56, 57, 58, 59].

²Notwithstanding this fact, a central charge is generated *dynamically* for field configurations that are electrically and/or magnetically charged [37]. The same holds for $N = 4$ supersymmetry, and it leads to the realization of the Bogomol’nyi bound at the level of the algebra mentioned in section 1.2.

tween the supersymmetry generators reads as ³

$$\{Q^i, \bar{Q}^j\} = 2\gamma^\mu P_\mu \delta^{ij} . \quad (2.4)$$

Observe that this expression is invariant under $U(2)$ transformations. In fact, these $U(2)$ transformations are an invariance of the whole $N = 2$ superconformal algebra and are therefore by definition part of its automorphism group. More precisely, they form the part of the automorphism group that commutes with the Lorentz group and are as such denoted by $U(2)_R$. They are included in the $N = 2$ superconformal group as we saw above. Note that for the $U(2)$ transformations to be compatible with the Majorana condition they should act in a chiral way: The chiral projections $Q_R^i = \frac{1}{2}(\mathbb{I} - \gamma_5)Q^i$ and $Q_L^i = \frac{1}{2}(\mathbb{I} + \gamma_5)Q^i$ transform in conjugate representations $\mathbf{2}$ and $\bar{\mathbf{2}}$ of $U(2)_R$ respectively. It is useful to introduce a notation for the chiral projections of the supercharges that also infers the representation of $U(2)_R$: $Q^i = Q_R^i$ and $Q_i = Q_L^i$ with the upper index transforming in the $\mathbf{2}$ and the lower index in the $\bar{\mathbf{2}}$ representation. Similarly, the chiral projections of other Majorana fermions are assigned upper or lower indices (see the tables in Appendix B for our conventions).

The gauge fields corresponding to each of the generators above form a representation of the $N = 2$ superconformal algebra. However, as such this is not yet a multiplet we are interested in. To have a theory including Einstein gravity, the local translations generated by P^a should be related to general coordinate transformations. This yields a set of constraints, which for instance makes the gauge field of local Lorentz transformations (ω_μ^{ab}), dependent on the other fields. Furthermore, also the gauge fields for special conformal (f_μ^a), and special supersymmetry transformations (ϕ_μ^i), turn out to be dependent quantities. Their expressions in terms of independent fields can be found in Appendix B. In addition to this, the constraints imply the inclusion of some new, auxiliary fields.

The representation arising in this way is called the Weyl multiplet. It contains the gauge fields of local translations (e_μ^a), Q -supersymmetry (ψ_μ^i), dilatations (b_μ), and the gauge fields of the $U(1)_R$ and $SU(2)_R$ -part (W_μ and \mathcal{V}_μ^{ij}) of $U(2)_R$. They are accompanied by the auxiliary fields T_{ab}^{ij} (antiselfdual, ij -antisymmetric) and D , which are both bosonic, and the fermionic χ^i . The superconformal transformation rules for all these fields can be found in Appendix B.

The algebra on the Weyl multiplet closes *off-shell*, which means that no field equations are needed to make it a genuine representation of the $N = 2$ superconformal algebra. As an off-shell representation of a supersymmetry algebra it necessarily has the same $(24 + 24)$ number of off-shell bosonic and fermionic degrees of freedom.

The Weyl multiplet provides the gauge fields needed to promote an action invariant under global $N = 2$ superconformal transformations to a locally $N = 2$ superconformally invariant action. The fact that the algebra closes off-shell on the Weyl

³The complete set of (anti-)commutation relations of the superconformal algebra can for instance be found in [47].

multiplet makes it relatively easy to construct such a coupling. In section 2.4, after we have considered the other relevant representations of the superconformal algebra and their rigidly superconformal actions, we explicitly show how this is done.

2.2 Vector multiplets

First we discuss the vector multiplet as a representation of rigid $N = 2$ supersymmetry. Later on we treat its superconformal version.

Rigid vector multiplets

The field content and rigid supersymmetry transformations of the vector multiplet are given by [49]

$$\begin{aligned}
\delta X &= \bar{\epsilon}^i \Omega_i , \\
\delta A_\mu &= \varepsilon^{ij} \bar{\epsilon}_i \gamma_\mu \Omega_j + \varepsilon_{ij} \bar{\epsilon}^i \gamma_\mu \Omega^j , \\
\delta \Omega_i &= 2 \not{\partial} X \epsilon_i + \frac{1}{2} \gamma_{\mu\nu} F^{-\mu\nu} \varepsilon_{ij} \epsilon^j + Y_{ij} \epsilon^j , \\
\delta Y_{ij} &= 2 \bar{\epsilon}_{(i} \not{\partial} \Omega_{j)} + 2 \varepsilon_{ik} \varepsilon_{jl} \bar{\epsilon}^{(k} \not{\partial} \Omega^{l)} .
\end{aligned} \tag{2.5}$$

Here X is complex scalar, A_μ a real vector and Y_{ij} (symmetric in i, j) a triplet of scalar fields, subject to the reality condition

$$Y_{ij} = \varepsilon_{ik} \varepsilon_{jl} Y^{kl} . \tag{2.6}$$

Ω_i are (the chiral projections of) two Majorana fermions. Observe that the transformation rules (2.5) are manifestly covariant under (global) $SU(2)_R$ transformations (acting as $\Omega_i \rightarrow \tilde{\Omega}_i = S_i^j \Omega_j$ and $Y_{ij} \rightarrow \tilde{Y}_{ij} = S_i^k S_j^l Y_{kl}$, where S_i^j is the generator of $SU(2)_R$ in the fundamental representation).

Also note that under supersymmetry the complex structure of the scalar sector and the chiral structure of the fermionic sector are related. This can be understood from the fact that the vector multiplet is a reduced $N = 2$ chiral multiplet. The Bianchi identity $\partial_{[\mu} F_{\nu\rho]} = 0$, implying $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$, and the reality condition on Y_{ij} both arise in the reduction (see for instance [47]). This turns out to be relevant later on.

Like the Weyl multiplet, the vector multiplet, consisting of the fields in (2.5), forms an off-shell representation of the algebra. This is consistent with the fact that the number of bosonic and fermionic off-shell degrees of freedom is the same ($8 + 8$). The rigidly $N = 2$ supersymmetric action of n of these abelian multiplets reads

[60],

$$\begin{aligned}
\mathcal{L} = & \left[i\partial_\mu F_\Lambda \partial^\mu \bar{X}^\Lambda + \frac{1}{2}iF_{\Lambda\Sigma} \bar{\Omega}_i^\Lambda \not{\partial} \Omega^{i\Sigma} + \frac{1}{4}iF_{\Lambda\Sigma} F_{\mu\nu}^{-\Lambda} F^{-\Sigma\mu\nu} \right. \\
& - \frac{1}{8}iF_{\Lambda\Sigma} Y_{ij}^\Lambda Y^{ij\Sigma} \\
& - \frac{1}{16}iF_{\Lambda\Sigma\Gamma} \bar{\Omega}_i^\Lambda \gamma^{\mu\nu} \Omega_j^\Gamma \varepsilon^{ij} F_{\mu\nu}^{-\Sigma} + \frac{1}{8}iF_{\Lambda\Sigma\Gamma} Y^{ij\Lambda} \bar{\Omega}_i^\Sigma \Omega_j^\Gamma \\
& \left. - \frac{1}{48}i\varepsilon^{ij} \varepsilon^{kl} F_{\Lambda\Sigma\Gamma\Xi} \bar{\Omega}_i^\Lambda \Omega_k^\Sigma \bar{\Omega}_j^\Gamma \Omega_l^\Xi + \text{h.c.} \right], \quad (2.7)
\end{aligned}$$

where Λ, Σ, \dots run from 1 to n . Observe that \mathcal{L} has a manifest global $SU(2)_R$ invariance.

We have used

$$F_{\Lambda_1 \dots \Lambda_q} \equiv \frac{\delta}{\delta X^{\Lambda_1}} \dots \frac{\delta}{\delta X^{\Lambda_q}} F(X), \quad (2.8)$$

and

$$N_{\Lambda\Sigma} \equiv -i(F_{\Lambda\Sigma} - \bar{F}_{\Lambda\Sigma}), \quad N^{\Lambda\Sigma} \equiv (N^{-1})^{\Lambda\Sigma}. \quad (2.9)$$

$F(X)$ is a holomorphic function of the scalar fields X^Λ . The precise form of this function fully fixes (2.12), i.e. *a choice of Lagrangian is equivalent to a choice of a holomorphic function $F(X)$* [61, 62].

The sigma model contained in (2.2) exhibits an interesting geometry. The complex scalars X^Λ parameterize an n -dimensional target-space with metric $g_{\Lambda\bar{\Sigma}} = N_{\Lambda\Sigma}$. This is a Kähler space: Its metric equals

$$g_{\Lambda\bar{\Sigma}} = \frac{\delta}{\delta X^\Lambda} \frac{\delta}{\delta \bar{X}^\Sigma} K(X, \bar{X}), \quad (2.10)$$

with Kähler potential

$$K(X, \bar{X}) = iX^\Lambda \bar{F}_\Lambda(\bar{X}) - i\bar{X}^\Lambda F_\Lambda(X). \quad (2.11)$$

The resulting geometry is known as (*rigid*) *special geometry*.

Note that the fields Y_{ij}^Λ only appear without derivatives acting on them. This implies that they are auxiliary fields, which can be eliminated from the Lagrangian by solving their field equations. The resulting vector multiplets, consisting of X^Λ , A_μ^Λ and Ω^Λ are *on-shell* representations of the supersymmetry algebra, which means that the algebra only closes when the field equations are satisfied. Correspondingly, only the number of on-shell degrees of freedom of the bosonic and fermion sector should be equal. This is indeed the case ($4 + 4$) as the use of their field equations reduces the number of degrees of freedom of the vectors A_μ^Λ from three to two and of the Majorana fermions from four to two.

As the gauge fields only appear through their field strengths, (2.7) obviously realizes electric/magnetic duality. Before considering this in detail, it is convenient to rewrite (2.7) as

$$\mathcal{L} = \mathcal{L}_{\text{vector}} + \mathcal{L}_{\text{matter}} + \mathcal{L}_{\Omega^4} + \mathcal{L}_Y , \quad (2.12)$$

where the different parts are given by

$$\begin{aligned} \mathcal{L}_{\text{vector}} = & \left[-\frac{1}{4}i\bar{F}_{\Lambda\Sigma}F_{\mu\nu}^{+\Sigma}F^{+\mu\nu\Lambda} + \frac{1}{16}i\bar{F}_{\Lambda\Sigma\Gamma}\bar{\Omega}^{i\Lambda}\gamma^{\mu\nu}\Omega^{j\Sigma}\varepsilon_{ij}F_{\mu\nu}^{+\Gamma} \right. \\ & + \frac{1}{256}iN^{\Delta\Omega}(\bar{F}_{\Delta\Lambda\Sigma}\bar{\Omega}^{i\Lambda}\gamma_{\mu\nu}\Omega^{j\Sigma}\varepsilon_{ij})(\bar{F}_{\Gamma\Xi\Omega}\bar{\Omega}^{k\Gamma}\gamma^{\mu\nu}\Omega^{l\Xi}\varepsilon_{kl}) \\ & \left. + \text{h.c.} \right] , \end{aligned} \quad (2.13)$$

$$\begin{aligned} \mathcal{L}_{\text{matter}} = & -N_{\Lambda\Sigma}\partial_\mu X^\Lambda\partial^\mu \bar{X}^\Sigma - \frac{1}{4}N_{\Lambda\Sigma}(\bar{\Omega}^{i\Lambda}\not{\partial}\Omega_i^\Sigma + \bar{\Omega}_i^\Lambda\not{\partial}\bar{\Omega}^{i\Sigma}) \\ & - \frac{1}{4}i(\bar{\Omega}_i^\Lambda\not{\partial}F_{\Lambda\Sigma}\Omega^{i\Sigma} - \bar{\Omega}^{i\Lambda}\not{\partial}\bar{F}_{\Lambda\Sigma}\Omega_i^\Sigma) , \end{aligned} \quad (2.14)$$

$$\begin{aligned} \mathcal{L}_{\Omega^4} = & -\frac{1}{384}i(\bar{F}_{\Lambda\Sigma\Gamma\Xi} - 3iN^{\Delta\Omega}\bar{F}_{\Delta(\Lambda\Gamma}\bar{F}_{\Sigma\Xi)\Omega})\bar{\Omega}^{i\Lambda}\gamma_{\mu\nu}\Omega^{j\Sigma}\varepsilon_{ij}\bar{\Omega}^{k\Gamma}\gamma^{\mu\nu}\Omega^{l\Xi}\varepsilon_{kl} \\ & + \text{h.c.} - \frac{1}{16}N^{\Delta\Omega}F_{\Delta\Lambda\Sigma}\bar{F}_{\Gamma\Xi\Omega}\bar{\Omega}^{i\Gamma}\Omega^{j\Xi}\bar{\Omega}_i^\Lambda\Omega_j^\Sigma , \end{aligned} \quad (2.15)$$

$$\begin{aligned} \mathcal{L}_Y = & \frac{1}{8}i\varepsilon_{ik}\varepsilon_{jl}\bar{F}_{\Lambda\Sigma}Y^{ij\Lambda}Y^{kl\Sigma} - \frac{1}{8}i\varepsilon^{ik}\varepsilon^{jl}F_{\Lambda\Sigma}Y_{ij}^\Lambda Y_{kl}^\Sigma \\ & - \frac{1}{8}i\varepsilon_{ik}\varepsilon_{jl}\bar{F}_{\Lambda\Sigma\Gamma}Y^{kl\Lambda}\bar{\Omega}^{i\Sigma}\Omega^{j\Gamma} + \frac{1}{8}i\varepsilon^{ik}\varepsilon^{jl}F_{\Lambda\Sigma\Gamma}Y_{kl}^\Lambda\bar{\Omega}_i^\Sigma\Omega_j^\Gamma \\ & - \frac{1}{32}\varepsilon_{ik}\varepsilon_{jl}N^{\Lambda\Sigma}\bar{F}_{\Lambda\Omega\Delta}\bar{F}_{\Sigma\Gamma\Xi}\bar{\Omega}^{i\Omega}\Omega^{j\Delta}\bar{\Omega}^{k\Gamma}\Omega^{l\Xi} + \text{h.c.} \\ & + \frac{1}{16}N^{\Lambda\Sigma}\bar{F}_{\Lambda\Omega\Delta}F_{\Sigma\Gamma\Xi}\bar{\Omega}^{i\Omega}\Omega^{j\Delta}\bar{\Omega}_i^\Gamma\Omega_j^\Xi . \end{aligned} \quad (2.16)$$

At first we restrict ourselves to the on-shell vector multiplets, so with Y_{ij}^Λ integrated out and $\mathcal{L}_Y = 0$, as the electric/magnetic duality relations contained in (2.7) are most clear for this version of the action. Later on we include the Y_{ij}^Λ in the electric/magnetic duality framework. Due to the reality condition (2.6) this is a non-trivial exercise. Nevertheless, as a new result, we show that the Y_{ij}^Λ can be incorporated in a natural way.

Observe that the Lagrangian $\mathcal{L} = \mathcal{L}_F + \mathcal{L}_{\text{matter}} + \mathcal{L}_{\Omega^4}$ is of the form (1.25) (except for the, in this respect irrelevant, lack of a gravitational coupling), with

$$\begin{aligned} \tau_{\Lambda\Sigma} &= F_{\Lambda\Sigma} , \\ O_{\mu\nu\Lambda}^+ &= -\frac{1}{8}\bar{F}_{\Lambda\Sigma\Gamma}\bar{\Omega}^{i\Sigma}\gamma_{\mu\nu}\Omega^{j\Gamma}\varepsilon_{ij} , \\ \mathcal{L}' &= \mathcal{L}_{\text{matter}} + \mathcal{L}_{\Omega^4} , \end{aligned} \quad (2.17)$$

and so $G_{\mu\nu\Lambda}^+$ is given by

$$G_{\mu\nu\Lambda}^+ = i\varepsilon_{\mu\nu\rho\sigma} \frac{\delta \mathcal{L}_{\text{vector}}}{F_{\rho\sigma}^{+\Lambda}} = \bar{F}_{\Lambda\Sigma} F_{\mu\nu}^{+\Sigma} - \frac{1}{8} \bar{F}_{\Lambda\Sigma\Gamma} \bar{\Omega}^{i\Sigma} \gamma_{\mu\nu} \Omega^{j\Gamma} \varepsilon_{ij} . \quad (2.18)$$

Recall that an electric/magnetic duality transformation of a Lagrangian of this type is induced by the transformations (1.37) on the objects in (2.17). Leaving the matter fields X^Λ and Ω^Λ as they are then gives a Lagrangian in which the matrix-components U, Z, W, V of the symplectic transformation explicitly appear. Since electric/magnetic duality preserves supersymmetry (as we saw in chapter 1), it should nevertheless be possible to rewrite the dual Lagrangian back in the form (2.12), so without the components of the symplectic matrix appearing. Note that the fact that we can write the dual Lagrangian in the form (2.12), implies the existence of a dual holomorphic function \tilde{F} , which may be a different one then the one started from.

To obtain the dual Lagrangian in the formulation (2.12), a suitable field redefinition of the matter fields X^Λ and Ω^Λ is required. To find out what these field redefinitions are, and which new function corresponds to the dual Lagrangian, we consider the transformed version of $F_{\Lambda\Sigma}$,

$$\tilde{F}_{\Lambda\Sigma}(\tilde{X}) = [W_{\Lambda\Gamma} + V_\Lambda{}^\Xi F_{\Xi\Gamma}] [\mathcal{S}^{-1}]^\Gamma{}_\Sigma , \quad (2.19)$$

where $\mathcal{S}^\Lambda{}_\Sigma = U^\Lambda{}_\Sigma + Z^{\Lambda\Gamma} F_{\Gamma\Sigma}$. It is easily seen that (2.19) is satisfied when $\tilde{F}_{\Lambda\Sigma}(\tilde{X})$ is taken to be the derivative of a new \tilde{F}_Λ with respect to a new \tilde{X}^Σ , with \tilde{F}_Λ and \tilde{X}^Σ given by

$$\begin{pmatrix} \tilde{X}^\Lambda \\ \tilde{F}_\Lambda \end{pmatrix} = \begin{pmatrix} U^\Lambda{}_\Sigma & Z^{\Lambda\Sigma} \\ W_{\Lambda\Sigma} & V_\Lambda{}^\Sigma \end{pmatrix} \begin{pmatrix} X^\Sigma \\ F_\Sigma \end{pmatrix} , \quad (2.20)$$

i.e. the scalars X^Λ and the derivatives of the function $F(X)$, $F_\Lambda(X)$, transform as a symplectic vector (X^Λ, F_Λ) . Using the X^Λ -dependence of F_Λ , the appropriate field redefinition of the scalars can be read of from the first line of (2.20). The second line of (2.20) contains the information about the new function $\tilde{F}(\tilde{X})$. This new function is defined (up to an irrelevant constant) by requiring \tilde{F}_Λ to be its derivative (with respect to \tilde{X}^Λ).

It is easily shown that (2.20) implies that

$$\begin{aligned} \tilde{F}_{\Lambda\Sigma\Gamma}(\tilde{X}) &= F_{\Xi\Delta\Omega} [\mathcal{S}^{-1}]^\Xi{}_\Lambda [\mathcal{S}^{-1}]^\Delta{}_\Sigma [\mathcal{S}^{-1}]^\Omega{}_\Gamma , \\ \tilde{N}_{\Lambda\Sigma}(\tilde{X}) &= N_{\Xi\Delta} [\mathcal{S}^{-1}]^\Xi{}_\Lambda [\mathcal{S}^{-1}]^\Delta{}_\Sigma . \end{aligned} \quad (2.21)$$

The correct field redefinition of Ω^Λ follows from a supersymmetry transformation of (2.20). We obtain

$$\begin{pmatrix} \tilde{\Omega}_i{}^\Lambda \\ \tilde{F}_{\Lambda\Gamma} \tilde{\Omega}_i{}^\Gamma \end{pmatrix} = \begin{pmatrix} U^\Lambda{}_\Sigma & Z^{\Lambda\Sigma} \\ W_{\Lambda\Sigma} & V_\Lambda{}^\Sigma \end{pmatrix} \begin{pmatrix} \Omega_i{}^\Sigma \\ F_{\Sigma\Xi} \Omega_i{}^\Xi \end{pmatrix} . \quad (2.22)$$

Note that all information is already contained in the first line; the second line is just consistent with the first (using (2.20)).

To convince oneself that (2.20) and (2.22) give indeed the right field redefinitions and the correct new function, one should take $\mathcal{L}_F + \mathcal{L}_{\text{matter}} + \mathcal{L}_{\Omega^4}$, perform an electric/magnetic duality transformation and write the result in terms of $\tilde{F}_{\mu\nu}^\Lambda$, $\tilde{\Omega}_i^\Lambda$, \tilde{X}^Λ , $\tilde{F}_{\Lambda\Sigma}$ and further derivatives of $\tilde{F}(\tilde{X})$. The ‘tilded’ version of $\mathcal{L}_F + \mathcal{L}_{\text{matter}} + \mathcal{L}_{\Omega^4}$ is then obtained.

Starting from a Lagrangian described by a holomorphic function $F(X)$ we thus find a new Lagrangian determined by a new function $\tilde{F}(\tilde{X})$, as defined by (2.20). The new and original function are related in the following way

$$\begin{aligned} F(X) \longrightarrow \tilde{F}(\tilde{X}) &= F(X) - \frac{1}{2}F_\Lambda(X)X^\Lambda + \frac{1}{2}\tilde{F}_\Lambda(\tilde{X})\tilde{X}^\Lambda \\ &= F(X) - \frac{1}{2}F_\Lambda(X)X^\Lambda + \frac{1}{2}(U^T W)_{\Lambda\Sigma}X^\Lambda X^\Sigma \\ &\quad + \frac{1}{2}(U^T V + W^T Z)_{\Lambda\Sigma}X^\Lambda F_\Sigma(X) \\ &\quad + \frac{1}{2}(Z^T V)^{\Lambda\Sigma}F_\Lambda(X)F_\Sigma(X), \end{aligned} \quad (2.23)$$

up to a (irrelevant) constant. Note the similarity between the symplectic behavior of the function $F(X)$ and the transformation property of the Lagrangian (1.34); The fact that the Lagrangian does not transform as a scalar $\tilde{\mathcal{L}}(\tilde{F}) \neq \mathcal{L}(F)$ translates into the statement that the function $F(X)$ does not do so, $\tilde{F}(\tilde{X}) \neq F(X)$.

Also the new function may be a different one than the original function, in the sense that $\tilde{F}(\tilde{X}) \neq F(\tilde{X})$. So different functions $F(X)$ and therefore different Lagrangians belong to the same equivalence class. The electric/magnetic duality transformations relating these different functions and Lagrangians are duality equivalences rather than duality invariances. However duality invariances ($\tilde{F}(\tilde{X}) = F(\tilde{X})$) do arise as special cases. In fact, these are of particular importance in chapter 4.

Electric/magnetic duality and Y_{ij}

So far we have considered the on-shell version of the vector multiplets, i.e. with the auxiliary fields Y_{ij}^Λ eliminated from the action by their field equations. Now we turn to the electric/magnetic duality behavior of off-shell vector multiplets, so in the presence of Y_{ij}^Λ .

Like we determined the symplectic behavior of Ω_i^Λ from a supersymmetry transformation on (X^Λ, F_Λ) , we can extract the appropriate transformation of Y_{ij}^Λ under symplectic transformations from a supersymmetry transformation on (2.22). The result is

$$\begin{pmatrix} Y_{ij}^\Lambda \\ Z_{ij\Lambda} \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{Y}_{ij}^\Lambda \\ \tilde{Z}_{ij\Lambda} \end{pmatrix} = \begin{pmatrix} U^\Lambda_\Sigma & Z^{\Lambda\Sigma} \\ W_{\Lambda\Sigma} & V_\Lambda^\Sigma \end{pmatrix} \begin{pmatrix} Y_{ij}^\Sigma \\ Z_{ij\Sigma} \end{pmatrix}, \quad (2.24)$$

where

$$Z_{ij\Lambda} = F_{\Lambda\Sigma} Y_{ij}{}^\Sigma - \frac{1}{2} F_{\Lambda\Sigma\Gamma} \bar{\Omega}_i{}^\Sigma \Omega_j{}^\Gamma . \quad (2.25)$$

We note that

$$Z_{ij\Lambda} = 4i\varepsilon_{ik}\varepsilon_{jl} \frac{\delta\mathcal{L}_Y}{\delta Y_{kl}{}^\Lambda} , \quad (2.26)$$

(with $Y_{ij}{}^\Lambda$ and $Y^{ij\Lambda}$ treated as independent fields) which becomes important shortly. One can check that the rotation (2.24) is consistent with a transformation of $Z_{ij\Lambda}$ as expressed in terms of the matter fields on which it depend, i.e. $\tilde{Z}_{ij\Lambda} = \tilde{F}_{\Lambda\Sigma} \tilde{Y}_{ij}{}^\Sigma - \frac{1}{2} \tilde{F}_{\Lambda\Sigma\Gamma} \tilde{\bar{\Omega}}_i{}^\Sigma \tilde{\Omega}_j{}^\Gamma$.

However, taking the complex conjugate of (2.24) and (2.25) gives

$$\begin{pmatrix} Y^{ij\Lambda} \\ Z^{ij}{}_\Lambda \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{Y}^{ij\Lambda} \\ \tilde{Z}^{ij}{}_\Lambda \end{pmatrix} = \begin{pmatrix} U^\Lambda{}_\Sigma & Z^{\Lambda\Sigma} \\ W_{\Lambda\Sigma} & V_\Lambda{}^\Sigma \end{pmatrix} \begin{pmatrix} Y^{ij\Sigma} \\ Z^{ij}{}_\Sigma \end{pmatrix} , \quad (2.27)$$

with

$$Z^{ij}{}_\Lambda = \bar{F}_{\Lambda\Sigma} Y^{ij\Sigma} - \frac{1}{2} \bar{F}_{\Lambda\Sigma\Gamma} \bar{\Omega}^{i\Sigma} \Omega^{j\Gamma} = -4i\varepsilon^{ik}\varepsilon^{jl} \frac{\delta\mathcal{L}_Y}{\delta Y^{kl\Lambda}} , \quad (2.28)$$

which is obviously incompatible with the reality conditions (2.6).

How to understand this? As mentioned earlier, similar to the Bianchi identity of $F_{\mu\nu}{}^\Lambda$, the reality condition on $Y_{ij}{}^\Lambda$ arises in the reduction of a chiral multiplet to a vector multiplet. As the Bianchi identity of $F_{\mu\nu}{}^\Lambda$ transforms under electric/magnetic duality, so does the embedding of the vector multiplet in the unreduced chiral multiplet, and therefore the reality condition on $Y_{ij}{}^\Lambda$ should transform as well.

In fact, the reality conditions on $Y_{ij}{}^\Lambda$ behave very similar to the Bianchi identities of $F_{\mu\nu}{}^\Lambda$. To show this, we write them in combination with the field equations of $Y_{ij}{}^\Lambda$ as

$$\begin{pmatrix} Y_{ij}{}^\Lambda \\ Z_{ij\Lambda} \end{pmatrix} - \varepsilon_{ip}\varepsilon_{jq} \begin{pmatrix} Y^{pq\Lambda} \\ Z^{pq}{}_\Lambda \end{pmatrix} = 0 . \quad (2.29)$$

From (2.29) it follows that the transformations (2.24) and (2.27) rotate the set of equations formed by the reality conditions on $Y_{ij}{}^\Lambda$ and the set of field equations of $Y_{ij}{}^\Lambda$, whereas the combined set remains equivalent. Observe the similarities with the action of electric/magnetic duality on the vector field strengths: The role of the Bianchi identities of $F_{\mu\nu}{}^\Lambda$ / equations of motion of $A_\mu{}^\Lambda$ in the gauge field sector, is played by the reality conditions on $Y_{ij}{}^\Lambda$ / equations of motion of $Y_{ij}{}^\Lambda$ in the $Y_{ij}{}^\Lambda$ -sector.

Also like in the gauge field sector, the dual version of \mathcal{L}_Y follows from $\tilde{Z}_{ij\Lambda} \equiv 4i\varepsilon_{ik}\varepsilon_{jl}\frac{\delta\tilde{\mathcal{L}}_Y}{\delta\tilde{Y}_{kl}^\Lambda}$. It takes the same form as (2.16), but then in terms of the fields \tilde{X}^Λ , $\tilde{\Omega}_i^\Lambda$ and \tilde{Y}_{ij}^Λ .

Superconformal vector multiplets

To obtain the vector multiplet as a representation of the $N = 2$ superconformal algebra, one needs to introduce appropriate transformation rules for X , A_μ , Ω_i and Y_{ij} under dilations, $U(1)_R$, special supersymmetry and special conformal transformations (recall that the $SU(2)_R$ transformation rules are already given below (2.6)).

The resulting weights of the fields under dilatations and chiral $U(1)_R$ can be found in Appendix B. The special supersymmetry transformations act on Ω_i only,

$$\delta\Omega_i = 2X\eta_i, \quad (2.30)$$

where η_i is the special supersymmetry parameter, while the special conformal transformations turn out to be trivial.

The Lagrangian corresponding to n (rigidly) superconformal vector multiplets is also given by (2.7). However, to have the dilatational and $U(1)_R$ symmetry realized, the function $F(X)$ should be a homogeneous function of second degree,

$$F(\lambda X) = \lambda^2 F(X). \quad (2.31)$$

This implies the following relations between F and its derivatives,

$$\begin{aligned} F(X) &= \frac{1}{2}F_\Lambda X^\Lambda, \\ F_\Lambda &= F_{\Lambda\Sigma}X^\Sigma, \\ F_{\Lambda\Sigma\Gamma}X^\Gamma &= 0, \\ F_{\Lambda\Sigma\Gamma} &= -F_{\Lambda\Sigma\Xi}X^\Xi. \end{aligned} \quad (2.32)$$

Using the fact that $F(X)$ is a homogeneous function of second degree, it can be shown that the Kähler space parameterized by the scalars admits a homothetic Killing vector of weight two, $(\chi^\Lambda, \bar{\chi}^\Lambda)$,

$$\chi_\Lambda \equiv \partial_\Lambda K = N_{\Lambda\Sigma}\bar{X}^\Sigma, \quad D_\Lambda\bar{\chi}_\Sigma + D_{\bar{\Sigma}}\chi_\Lambda = 2N_{\Lambda\Sigma}, \quad (2.33)$$

where the derivatives D_Λ contain the hermitian connection,

$$\Gamma_{\Sigma\Gamma}^\Lambda = N^{\Lambda\Xi}\partial_\Sigma N_{\Gamma\Xi} = -iN^{\Lambda\Xi}F_{\Sigma\Gamma\Xi}, \quad \Gamma_{\bar{\Sigma}\bar{\Gamma}}^{\bar{\Lambda}} = N^{\Lambda\Xi}\partial_{\bar{\Sigma}}N_{\bar{\Gamma}\Xi} = iN^{\Lambda\Xi}\bar{F}_{\Sigma\Gamma\Xi}. \quad (2.34)$$

The existence of the homothetic Killing vector implies that the sigma model is scale invariant, which is part of the superconformal invariance of the action as a whole.

Using the complex structure, $J^\Lambda_\Sigma = i\delta^\Lambda_\Sigma$, one can also construct the Killing vector $(k^\Lambda, \bar{k}^\Lambda)$,

$$\begin{aligned} k^\Lambda &= -J^\Lambda_\Sigma \chi^\Sigma = -iX^\Lambda, & \bar{k}^\Lambda &= -J^\Lambda_{\bar{\Sigma}} \chi^{\bar{\Sigma}} = i\bar{X}^\Lambda, \\ D_\Lambda \bar{k}^\Sigma + D_{\bar{\Sigma}} k_\Lambda &= 0, & D_\Lambda k_\Sigma + D_\Sigma k_\Lambda &= 0, \end{aligned} \quad (2.35)$$

which is associated to the chiral $U(1)_R$ transformations of the scalar fields and the $U(1)_R$ symmetry of the action.

In fact, the $2n$ -dimensional target-space of the scalar fields of n rigidly superconformal vector multiplets is a cone over a $(2n - 1)$ -dimensional so-called Sasakian space [63], which is itself a $U(1)$ fibration over a $(2n - 2)$ -dimensional *special Kähler* manifold. This special Kähler manifold is important in the context of Poincaré supergravity. The target-space of the vector multiplet scalars of the rigidly superconformal model is referred to as the *special Kähler cone*.

The constraints following from demanding superconformal invariance, have no influence on the electric/magnetic duality relations contained in (2.7). Only note that the symplectic vector of the complex scalars now is of the form $(X^\Lambda, F_\Lambda) = (X^\Lambda, F_{\Lambda\Sigma} X^\Sigma)$.

This concludes our treatment of the vector multiplet as a representation of $N = 2$ supersymmetry. We have analyzed both its rigidly supersymmetric and its rigidly superconformal version. Later on we consider its role in Poincaré supergravity. However, we first turn to another representation of $N = 2$ supersymmetry, the hypermultiplet.

2.3 Hypermultiplets

A hypermultiplet comprises four real scalar fields and two Majorana spinors. On-shell this adds up to $4 + 4$ degrees of freedom. Off-shell two Majorana spinors have 8 fermionic degrees of freedom, so at least four additional bosonic degrees of freedom would be needed to have an off-shell multiplet. However, as it turns out, there exists no unconstrained off-shell formulation of a hypermultiplet in terms of a finite number of degrees of freedom. Therefore we restrict ourselves to its on-shell version, following [64].

Rigid hypermultiplets

The supersymmetry transformation rules of the rigid hypermultiplet are given by

$$\begin{aligned} \delta\phi^A &= 2(\gamma_{i\bar{\alpha}}^A \bar{\epsilon}^i \zeta^{\bar{\alpha}} + \bar{\gamma}_\alpha^{Ai} \epsilon_i \zeta^\alpha), \\ \delta\zeta^\alpha &= V_{Ai}^\alpha \not{\partial}\phi^A \epsilon^i - \delta\phi^A \Gamma_A^\alpha{}_\beta \zeta^\beta, \\ \delta\zeta^{\bar{\alpha}} &= \bar{V}_A^{i\bar{\alpha}} \not{\partial}\phi^A \epsilon_i - \delta\phi^A \Gamma_A^{\bar{\alpha}}{}_\beta \zeta^{\bar{\beta}}. \end{aligned} \quad (2.36)$$

Here ϕ^A are the scalar fields, so A runs from 1 to $4m$ in case of a model with m hypermultiplets. ζ^α ($\alpha = 1, \dots, 2m$) are the negative-chiral parts of the Majorana spinors. Complex conjugation gives their positive chiral counterparts. The latter are denoted by $\zeta^{\bar{\alpha}}$. γ^A , V_A and $\Gamma_A^{\alpha\beta}$ are ϕ -dependent quantities. Their role becomes clear shortly.

The rigidly supersymmetric Lagrangian formed by the hypermultiplet degrees of freedom reads as

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2}g_{AB}\partial_\mu\phi^A\partial^\mu\phi^B \\ & -G_{\bar{\alpha}\beta}(\bar{\zeta}^{\bar{\alpha}}\mathcal{D}\zeta^\beta + \zeta^{\bar{\beta}}\mathcal{D}\zeta^{\bar{\alpha}}) - \frac{1}{4}W_{\bar{\alpha}\beta\gamma\delta}\bar{\zeta}^{\bar{\alpha}}\gamma_\mu\zeta^\beta\bar{\zeta}^{\bar{\gamma}}\gamma^\mu\zeta^\delta, \end{aligned} \quad (2.37)$$

where we used

$$\mathcal{D}_\mu\zeta^\alpha = \partial_\mu\zeta^\alpha + \partial_\mu\phi^A\Gamma_A^{\alpha\beta}\zeta^\beta. \quad (2.38)$$

The Lagrangian (2.37) and transformation rules (2.36) come with two sets of target-space equivalence transformations. These are the target-space diffeomorphisms $\phi \rightarrow \phi'(\phi)$ on the one hand and the reparameterizations of the fermion ‘frame’ $\zeta^\alpha \rightarrow S^\alpha_\beta(\phi)\zeta^\beta$ on the other hand (the latter are accompanied by appropriate redefinitions of other quantities carrying indices α or $\bar{\alpha}$, i.e for instance $G_{\bar{\alpha}\beta} \rightarrow [\bar{S}^{-1}]^{\bar{\gamma}\bar{\alpha}}[S^{-1}]^\delta_\beta G_{\bar{\gamma}\delta}$). This explains the appearance of the objects $\Gamma_A^{\alpha\beta}$: They are the connections associated with the reparameterizations of the fermion frame. Furthermore, observe that the Lagrangian is invariant under the $U(1)_R$ symmetry group, which acts by chiral transformations on the fermion fields, while the $SU(2)_R$ symmetry can only be realized when the target-space has an $SU(2)$ isometry.

The target-space metric of the non-linear sigma model parameterized by the scalars is a hyper-Kähler space [11, 65, 66], which, by definition, allows the existence of three anticommuting complex structures that are covariantly constant with respect to the Levi-Civita connection [67, 68, 69].

The tensor W is defined by

$$W_{\bar{\alpha}\beta\gamma\delta} = R_{AB}{}^{\bar{\epsilon}}{}_{\bar{\gamma}}\gamma_{i\bar{\alpha}}^{A-}{}_{\bar{\beta}}{}^{Bi}G_{\bar{\epsilon}\delta} = \frac{1}{2}R_{ABCD}\gamma_{i\bar{\alpha}}^{A-}{}_{\bar{\beta}}{}^{Bi}\gamma_{j\bar{\gamma}}^C{}_{\bar{\delta}}{}^{Dj}, \quad (2.39)$$

where $R_{AB}{}^{\alpha\beta}$ and R_{ABCD} are the curvatures corresponding to $\Gamma_A^{\alpha\beta}$ and the Levi-Civita connection $\Gamma_{AC}{}^B$. The curvature $R_{AB}{}^{\alpha\beta}$ takes its values in $Sp(m) \sim uSp(2m, \mathbb{C})$.

The target-space metric g_{AB} , the tensors γ^A , V_A and the fermionic hermitian metric $G_{\bar{\alpha}\beta}$ are all covariantly constant with respect to the Christoffel connection and the connections $\Gamma_A^{\alpha\beta}$ and $\Gamma_{AC}{}^B$. There are the following relations amongst them

$$\begin{aligned} \gamma_{i\bar{\alpha}}^A\bar{V}_B^{j\bar{\alpha}} + \bar{\gamma}_\alpha^{Aj}V_{Bi}^\alpha &= \delta_I^J\delta_B^A, \\ g_{AB}\gamma_{i\bar{\alpha}}^B &= G_{\bar{\alpha}\beta}V_{Ai}^\beta, \quad \bar{V}_A^{i\bar{\alpha}}\gamma_{j\bar{\beta}}^A = \delta_i^j\delta_{\bar{\beta}}^{\bar{\alpha}}. \end{aligned} \quad (2.40)$$

The complex structures of the hyper-Kähler target-space are spanned by the anti-symmetric covariantly constant target-space tensors

$$J_{AB}^{ij} = \gamma_{Ak\bar{\alpha}} \varepsilon^{k(i} \bar{V}_B^{j)\bar{\alpha}} , \quad (2.41)$$

which are symmetric in i, j and satisfy

$$(J_{ij})_{AB} \equiv (J_{AB}^{ij})^* = \varepsilon_{ik} \varepsilon_{jl} J_{AB}^{kl} , \quad J_A^{ijC} J_{CB}^{kl} = \frac{1}{2} \varepsilon^{i(k} \varepsilon^{l)j} g_{AB} + \varepsilon^{(i(k} J_{AB}^{l)j)} . \quad (2.42)$$

In addition we note the following useful identities,

$$\gamma_{Ai\bar{\alpha}} \bar{V}_B^{j\bar{\alpha}} = \varepsilon_{ik} J_{AB}^{kj} + \frac{1}{2} g_{AB} \delta_j^i , \quad J_{AB}^{ij} \gamma_{\bar{\alpha}k}^B = -\delta_k^{(i} \varepsilon^{j)l} \gamma_{Al\bar{\alpha}} . \quad (2.43)$$

Other important objects are the covariantly constant antisymmetric tensors

$$\Omega_{\bar{\alpha}\bar{\beta}} = \frac{1}{2} \varepsilon^{ij} g_{AB} \gamma_{i\bar{\alpha}}^A \gamma_{j\bar{\beta}}^B , \quad \bar{\Omega}^{\bar{\alpha}\bar{\beta}} = \frac{1}{2} \varepsilon_{ij} g^{AB} \bar{V}_A^{i\bar{\alpha}} \bar{V}_B^{j\bar{\beta}} . \quad (2.44)$$

Using these objects we can derive a reality condition on V and γ ,

$$\varepsilon_{ij} \Omega_{\bar{\alpha}\bar{\beta}} \bar{V}_A^{j\bar{\beta}} = g_{AB} \gamma_{i\bar{\alpha}}^B = G_{\bar{\alpha}\beta} V_{Ai}^\beta . \quad (2.45)$$

This leads to

$$g^{AB} V_{Ai}^\alpha V_{Bj}^\beta = \varepsilon_{ij} \Omega^{\alpha\beta} , \quad g_{AB} \gamma_{i\bar{\alpha}}^A \gamma_{j\bar{\beta}}^B = \varepsilon_{ij} \Omega_{\bar{\alpha}\bar{\beta}} , \quad (2.46)$$

and the relation,

$$\varepsilon_{ij} \Omega_{\bar{\alpha}\bar{\beta}} \bar{V}_A^{i\bar{\alpha}} \bar{V}_B^{j\bar{\beta}} = g_{AB} , \quad (2.47)$$

which makes that V_A can be interpreted as the quaternionic vielbein of the target-space, with γ^A being the inverse vielbein.

Superconformal hypermultiplets

Next we consider the superconformal version of the hypermultiplet. The dilatational and $U(1)$ weights of ϕ^A and ζ^α can be found in Appendix B. $SU(2)_R$ acts as a set of isometries of the target-space (as we will see below). Furthermore, the special conformal transformations are again trivial and special supersymmetry works as

$$\delta \zeta^\alpha = \chi^B(\phi) V_{Bi}^\alpha(\phi) \eta^i . \quad (2.48)$$

χ^B is a (real) homothetic Killing vector of the space parameterized by the scalar fields, as we will see shortly.

The corresponding Lagrangian is of the same form as in the rigidly supersymmetric case (2.37), however, the superconformal couplings imply that the manifold parameterized by the hypermultiplet scalars is a special type of hyper-Kähler manifold, a *hyper-Kähler cone*.

The (real) metric of a hyper-Kähler cone is the second derivative of a function χ ,

$$D_A \partial_B \chi = g_{AB} . \quad (2.49)$$

This function is sometimes called the hyper-Kähler potential. As mentioned above, the vector $\chi^A \equiv g^{AB} \partial_B \chi$ is a homothetic Killing vector of weight two,

$$D_A \chi_B + D_B \chi_A = 2g_{AB} , \quad (2.50)$$

implying that the associated sigma model is indeed invariant under dilatations. Furthermore, there exist three Killing vectors, satisfying

$$k_{ij}^A = J_{ij}^{AB} \chi_B , \quad (2.51)$$

which realize the $SU(2)_R$ symmetry of the model.

Altogether it can be shown that the $4m$ -dimensional hyper-Kähler manifold parameterized by the scalars of m superconformal hypermultiplets is a cone over a $(4m - 1)$ -dimensional 3-Sasakian manifold, which is itself a $Sp(1)$ fibration over a $(4m - 4)$ -dimensional so-called *quaternionic-Kähler* (QK) space [64]. As we will see in the next section, this latter space is the one parameterized by the scalars of the corresponding Poincaré supergravity theory.

Observe the similarities between the target-spaces of the vector multiplet and the hypermultiplet scalars. In both cases we find in the superconformal framework a cone over a fibration ($U(1)_R$ for the vector multiplet scalars and $Sp(1) \sim SU(2)_R$ for the hypermultiplet scalars) of the manifold that becomes relevant in the Poincaré supergravity context.

2.4 Poincaré supergravity

In the last two sections we considered the rigidly supersymmetric and the superconformal version of the vector and the hypermultiplet. The construction of the superconformal multiplets was only a first step towards the formulation of $N = 2$ Poincaré supergravity. How to finish this procedure is the subject of this section. First we gauge the superconformal symmetries present in the vector and hypermultiplet sectors by introducing couplings to the associated gauge fields of the Weyl multiplet. The vector and hypermultiplet transformation rules this gives rise to can be found in Appendix B. The corresponding action is given in [10] (the vector multiplet sector) and [64] (the hypermultiplet sector). Here we simplify matters

and take only the bosonic sector into account. This way we obtain [70]

$$\begin{aligned}
\mathcal{L} = & N_{\Lambda\Sigma}\mathbb{D}^\mu X^\Lambda\mathbb{D}_\mu\bar{X}^\Sigma + \frac{1}{2}g_{AB}\mathbb{D}^\mu\phi^A\mathbb{D}_\mu\phi^B \\
& -\frac{1}{6}KR - \frac{1}{6}\chi R + D(K - \frac{1}{2}\chi) \\
& + [\frac{1}{4}i\bar{F}_{\Lambda\Sigma}\mathcal{F}_{\mu\nu}^{+\Lambda}\mathcal{F}^{+\mu\nu\Sigma} \\
& + \frac{1}{8}i\bar{F}_\Lambda\mathcal{F}_{\mu\nu}^{+\Lambda}T_{ij}^{\mu\nu}\varepsilon^{ij} + \frac{1}{32}i\bar{F}T_{ij\mu\nu}T_{kl}^{\mu\nu}\varepsilon^{ij}\varepsilon^{kl} + \text{h.c.}] , \tag{2.52}
\end{aligned}$$

where

$$\mathcal{F}_{\mu\nu}^{+\Lambda} = 2\partial_{[\mu}A_{\nu]}^\Lambda - (\frac{1}{4}\varepsilon_{ij}\bar{X}^\Lambda T_{\mu\nu}^{ij} + \text{h.c.}) . \tag{2.53}$$

In (2.52) we omitted a term quadratic in Y_{ij}^Λ as it is irrelevant for the present discussion. Furthermore, we changed the overall sign for convenience. The covariant derivatives in (2.52) are given by

$$\begin{aligned}
\mathbb{D}_\mu X^\Lambda &= \partial_\mu X^\Lambda - b_\mu\chi^\Lambda - W_\mu k^\Lambda , \\
\mathbb{D}_\mu\phi^A &= \partial_\mu\phi^A - b_\mu\chi^A + \frac{1}{2}\mathcal{V}_\mu^i{}_k\varepsilon^{jk}k_{ij}^A . \tag{2.54}
\end{aligned}$$

We recall that b_μ , W_μ and \mathcal{V}_μ are the gauge fields corresponding to dilatations, $U(1)_R$ and $SU(2)_R$ transformations respectively, whereas χ^Λ and χ^A , k^Λ and k_{ij}^A are the homothetic Killing and Killing vectors associated with these symmetry transformations.

The locally superconformal theory (2.52) is gauge equivalent to $N=2$ Poincaré supergravity. To make this more explicit, the gauge fields of $U(1)_R$ and $SU(2)_R$ (whose generators are not part of the Poincaré supergroup), are eliminated⁴, as are the auxiliary fields $T_{\mu\nu}^{ij}$ and D . One then gets

$$\begin{aligned}
\mathcal{L} = & K\mathcal{M}_{\Lambda\Sigma}\partial_\mu X^\Lambda\partial^\mu\bar{X}^\Sigma + \frac{1}{2}\chi G_{AB}\partial_\mu\phi^A\partial^\mu\phi^B \\
& -K(\frac{1}{6}R - \frac{1}{4}(\partial_\mu\ln K)^2) - \chi(\frac{1}{6}R - \frac{1}{4}(\partial_\mu\ln\chi)^2) \\
& + (\frac{1}{4}i\mathcal{N}_{\Lambda\Sigma}F_{\mu\nu}^{+\Lambda}F^{+\mu\nu\Sigma} + \text{h.c.}) , \tag{2.55}
\end{aligned}$$

while the potentials of the hyper-Kähler cone and the special Kähler cone become equal,

$$\chi = 2K . \tag{2.56}$$

⁴Since the Lagrangian is invariant under special conformal transformations and the dilatational gauge field b_μ is the only field transforming non-trivially under this symmetry, the Lagrangian must be independent of b_μ .

$\mathcal{M}_{\Lambda\bar{\Sigma}}$, G_{AB} and $\mathcal{N}_{\Lambda\Sigma}$ are given by

$$\begin{aligned}\mathcal{M}_{\Lambda\bar{\Sigma}} &= \frac{1}{K}(N_{\Lambda\Sigma} - \frac{1}{2K}\chi_{\Lambda}\bar{\chi}_{\Sigma} - \frac{1}{2K}k_{\Lambda}\bar{k}_{\Sigma}), \\ G_{AB} &= \frac{1}{\chi}(g_{AB} - \frac{1}{2\chi}\chi_{A}\chi_{B} - \frac{1}{\chi}k_{Aij}k_{ij}^B), \\ \mathcal{N}_{\Lambda\Sigma} &= \bar{F}_{\Lambda\Sigma} + i\frac{N_{\Lambda\Gamma}X^{\Gamma}N_{\Sigma\Xi}X^{\Xi}}{N_{\Delta\Omega}X^{\Delta}X^{\Omega}}.\end{aligned}\tag{2.57}$$

The factor χ in (2.55) can be absorbed in the vierbein by the rescaling $e_{\mu}^a \rightarrow \sqrt{2/\chi}e_{\mu}^a$, such that a scale invariant metric is obtained. The Lagrangian then takes the Poincaré supergravity form

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}R + \mathcal{M}_{\Lambda\bar{\Sigma}}\partial_{\mu}X^{\Lambda}\partial^{\mu}\bar{X}^{\Sigma} + G_{AB}\partial_{\mu}\phi^A\partial^{\mu}\phi^B \\ &\quad + \frac{1}{4}i\mathcal{N}_{\Lambda\Sigma}F_{\mu\nu}^{+\Lambda}F^{+\mu\nu\Sigma} + \text{h.c.}.\end{aligned}\tag{2.58}$$

To have the correct signs of the kinetic terms, the special Kähler metric $\mathcal{M}_{\Lambda\bar{\Sigma}}$ and the QK metric G_{AB} should be negative definite.

The homothetic Killing and Killing vectors of the cones correspond to null-vectors of $\mathcal{M}_{\Lambda\bar{\Sigma}}$ and G_{AB} ,

$$\begin{aligned}\mathcal{M}_{\Lambda\bar{\Sigma}}\chi^{\Lambda} &= \mathcal{M}_{\Lambda\bar{\Sigma}}k^{\Lambda} = 0, \\ G_{AB}\chi^B &= G_{AB}k_{ij}^B = 0.\end{aligned}\tag{2.59}$$

Note that the vierbein rescaling implies that we have taken positive cone potentials. It follows that the cone metrics g_{AB} and $N_{\Lambda\Sigma}$ are mostly negative, but positive in the (homothetic) Killing directions. Furthermore, in [71] it is shown that in case $\mathcal{M}_{\Lambda\bar{\Sigma}}$ is negative definite the vector fields come with positive kinetic energy.

Starting from n superconformal vector multiplets $\mathcal{M}_{\Lambda\bar{\Sigma}}$ thus describes a $(2n - 2)$ -dimensional space. This is a special Kähler manifold (by definition). It can be parameterized in terms of $n - 1$ complex coordinates $z^A (A = 1, \dots, n - 1)$ by letting X^{Λ} be proportional to some holomorphic sections $Z^{\Lambda}(z)$ of the projective space PC^n [72, 73]. Similarly, from m superconformal hypermultiplets the metric G_{AB} arises, which corresponds to a $(4m - 4)$ -dimensional space being necessarily of the quaternionic-Kähler type. A procedure to describe this QK space in terms of $4m - 4$ coordinates was explained in [74]. It involves the fixing of the dilatational and $SU(2)_R$ symmetries.

Altogether we find that the resulting Poincaré supergravity model, descending from n vector and m hypermultiplets coupled to the Weyl multiplet, contains $2n$ (from the vector gauge fields) $+(2n - 2)$ (from the complex vector multiplet scalars) $+(4m - 4)$ (from the real hypermultiplet scalars) $+2$ (from the metric) $= 4(n + m - 1)$ bosonic degrees of freedom. A more complete analysis - including the fermions - shows that they constitute the bosonic sector of $n - 1$ vector, $m - 1$

hyper- and the supergravity multiplet of $N = 2$ Poincaré supergravity (the latter contains one of the vector fields, called the graviphoton).

The fields disappearing in the process of going from a rigidly superconformal model to Poincaré supergravity are compensators for the symmetries that are in the superconformal group, but not in the Poincaré supergroup.

To finish with, let us discuss the electric/magnetic duality properties of the Lagrangian (2.55). Being of the form (1.25) it obviously exhibits such a type of duality. In terms of the objects of (1.25) we have

$$\tau_{\Lambda\Sigma} = \tilde{\mathcal{N}}_{\Lambda\Sigma} , \quad (2.60)$$

whereas $O_{\mu\nu}^\Lambda$ vanishes in the absence of fermions.

Compared to the rigidly superconformal case the complex coupling matrix has acquired a non-holomorphic part, which is due to the coupling to, and elimination of, the auxiliary field $T_{\mu\nu}^{ij}$. This turns out to have some consequences.

First we recall from chapter 1 that $\tau_{\Lambda\Sigma}$ transforms as

$$\tau_{\Lambda\Sigma} \longrightarrow \tilde{\tau}_{\Lambda\Sigma} \equiv ((W + V\tau)(U + Z\tau)^{-1})_{\Lambda\Sigma} , \quad (2.61)$$

which implies that $(U + Z\tau)$ should be invertible to have a well-defined electric/magnetic duality transformation. In the superconformal case $\tau_{\Lambda\Sigma} = F_{\Lambda\Sigma}$, such that

$$U^\Lambda_\Sigma + Z^{\Lambda\Gamma} \tau_{\Gamma\Sigma} = U^\Lambda_\Sigma + Z^{\Lambda\Gamma} F_{\Gamma\Sigma} = \left(\frac{\delta \tilde{X}}{\delta X} \right)^\Lambda_\Sigma . \quad (2.62)$$

As invertibility of $\left(\frac{\delta \tilde{X}}{\delta X} \right)^\Lambda_\Sigma$ suffices (and is needed) to have a formulation of the dual theory in terms of a dual function $\tilde{F}(\tilde{X})$, we thus find in the superconformal case that all allowed electric/magnetic duality rotations yield models that can be formulated in terms of a dual function.

We then consider the Poincaré supergravity case, so with the matrix $\tau_{\Lambda\Sigma}$ given by (2.60). Also in this case electric/magnetic duality transformations give models determined by a dual function when the matrix $U^\Lambda_\Sigma + Z^{\Lambda\Gamma} F_{\Gamma\Sigma}$ is invertible. However, the condition to have a well-defined electric/magnetic duality rotation in supergravity is the requirement of invertibility of the *different* matrix $U^\Lambda_\Sigma + Z^{\Lambda\Gamma} \tilde{\mathcal{N}}_{\Gamma\Sigma}$. Furthermore,

$$[(U + ZF)^{-1}]^\Lambda_\Sigma \text{ exists} \rightarrow [(U + Z\tilde{\mathcal{N}})^{-1}]^\Lambda_\Sigma \text{ exists} . \quad (2.63)$$

whereas,

$$[(U + Z\tilde{\mathcal{N}})^{-1}]^\Lambda_\Sigma \text{ exists} \nrightarrow [(U + ZF)^{-1}]^\Lambda_\Sigma \text{ exists} . \quad (2.64)$$

From this it follows that in supergravity, starting from a theory fixed by a function $F(X)$, electric/magnetic duality rotations can be performed that give duality equivalent theories *not* determined by a dual function.

The above calls for a formulation of the general $N = 2$ Poincaré supergravity theory that does not presuppose the existence of a function $F(X)$. Such a framework indeed exists. It starts from the vector (X^Λ, F_Λ) instead of the function $F(X)$ [75]. We refrain from explaining this formulation in detail as it does not fit in the superconformal framework we adopted in this thesis. Moreover, it does not incorporate new physically inequivalent models as all models for which no function exist are electric/magnetically dual to systems that do have a formulation in terms of such a function.

2.5 The c-map

In section 1.4 we treated scalar-tensor theories and the accompanying dualities, which, we argued, could be related to theories of vector gauge fields and their electric/magnetic dualities through a dimensional reduction. In (ungauged) supergravity we have a vector and a hypermultiplet sector. When there are isometries in the manifold parameterized by the hypermultiplet scalars, the hypermultiplets can be dualized to tensor multiplets. As alluded to earlier, this tensor multiplet sector is indeed related to the vector multiplet sector via a dimensional reduction (although this is not true for every tensor multiplet theory). Furthermore, the electric/magnetic dualities on the vector side turn out to be related to similar dualities in the scalar-tensor sector. This map from the vector multiplet to the tensor multiplet sector of $N = 2$ supergravity is called the $(N = 2)$ c-map. From a string theory perspective it has its origin in the T-duality between type IIA and type IIB.

Besides the vector and tensor multiplets the c-map also includes the gravitational sector of the theory. In fact, in its most basic form it appears in a $(N = 1)$ model of a scalar and a tensor coupled to gravity (so without vector fields). To introduce the concept conveniently and to set the notation, we first consider this simple model. After that we perform the $N = 2$ supergravity c-map. In the latter, to prepare for the next chapter, we take a Euclidean setting.

2.5.1 A prototype model

We consider the following Lagrangian

$$\mathcal{L} = -R(e) - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}e^{2\phi}H_\mu H^\mu . \quad (2.65)$$

It appears as a sub-sector of $N = 1$ low-energy effective actions in which gravity is coupled to $N = 1$ tensor multiplets. In our case we have one tensor multiplet only, which, as seen from string theory, consists of the dilaton and the NS-NS tensor $B_{\mu\nu}$.

To perform the c-map, we dimensionally reduce the action (2.65) and assume that all the fields are independent of one coordinate. This can most conveniently be

done by first choosing an upper triangular form of the vierbein, in coordinates $(x^m, x^3 \equiv \tau)$, $m = 0, 1, 2$,

$$e_\mu^a = \begin{pmatrix} e^{-\tilde{\phi}/2} \hat{e}_m^i & e^{\tilde{\phi}/2} \tilde{B}_m \\ 0 & e^{\tilde{\phi}/2} \end{pmatrix}. \quad (2.66)$$

The metric then takes the form

$$ds^2 = e^{\tilde{\phi}} (d\tau + \tilde{B}_m dx^m)^2 + e^{-\tilde{\phi}} \hat{g}_{mn} dx^m dx^n, \quad (2.67)$$

and we demand $\tilde{\phi}$, \tilde{B}_m and \hat{g}_{mn} to be independent of τ . For the moment, τ is one of the spatial coordinates, but in the next subsection we will apply our results to the case when τ is the Euclidean time.

We get $e = e^{-\tilde{\phi}} \hat{e}$, and the scalar curvature decomposes as

$$-eR(e) = -\hat{e}R(\hat{e}) - \frac{1}{2} \hat{e} \partial_m \tilde{\phi} \partial^m \tilde{\phi} + \frac{1}{2} \hat{e} e^{2\tilde{\phi}} \tilde{H}_m \tilde{H}^m. \quad (2.68)$$

Similarly, we require the dilaton and the tensor to be independent of τ . The three-dimensional Lagrangian then is

$$\mathcal{L}_3 = -R(\hat{e}) - \frac{1}{2} \partial_m \tilde{\phi} \partial^m \tilde{\phi} + \frac{1}{2} e^{2\tilde{\phi}} \tilde{H}_m \tilde{H}^m - \frac{1}{2} \partial_m \phi \partial^m \phi + \frac{1}{2} e^{2\phi} H_m H^m, \quad (2.69)$$

where $H^m = -\frac{1}{2} i \varepsilon^{mnl} H_{nl} = -i \varepsilon^{mnl} \partial_n B_l$ and $B_l = B_{l\tau}$. In addition, there is an extra term in the Lagrangian,

$$\mathcal{L}_3^{aux} = \frac{1}{12} e^{2(\phi+\tilde{\phi})} (H_{mnl} - 3B_{[m} H_{nl]}) (H^{mnl} - 3B^{[m} H^{nl]}), \quad (2.70)$$

which plays no role in the three-dimensional theory. Being of rank three in three dimensions H_{mnl} is an auxiliary field. (2.70) can therefore trivially be eliminated by its field equation.

Note that the Lagrangian \mathcal{L}_3 has the symmetry

$$\phi \longleftrightarrow \tilde{\phi}, \quad B_m \longleftrightarrow \tilde{B}_m. \quad (2.71)$$

In fact, careful analysis shows that also \mathcal{L}_3^{aux} is invariant, provided we transform

$$B_{mn} \rightarrow \tilde{B}_{mn} \equiv B_{mn} - \tilde{B}_{[m} B_{n]}. \quad (2.72)$$

The resulting theory can now be reinterpreted as a dimensional reduction of a four-dimensional theory of gravity coupled to a scalar $\tilde{\phi}$ and a tensor $\tilde{B}_{\mu\nu}$ and with the vierbein given by

$$\tilde{e}_\mu^a = \begin{pmatrix} e^{-\phi/2} \hat{e}_m^i & e^{\phi/2} B_m \\ 0 & e^{\phi/2} \end{pmatrix}. \quad (2.73)$$

The map from ((2.65)) to the latter model is the c-map in its simplest form. The symmetry transformations involved, (2.71) and (2.72), are related to the Buscher rules for T-duality [76]. We here derived these rules from an effective action approach in Einstein frame, similar to [77].

In case of a Euclidean setting, the dimensional reduction is still based on the decomposition of the vierbein (2.66) with τ the Euclidean time. After dimensional reduction over τ , the Einstein-Hilbert term gives

$$eR(e) = \hat{e}R(\hat{e}) + \frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} + \frac{1}{2}e^{2\tilde{\phi}}\tilde{H}_\mu\tilde{H}^\mu, \quad (2.74)$$

such that the symmetry (2.71) still holds.

2.5.2 The c-map in N=2 supergravity

Having introduced the main idea in the last subsection we are now ready to consider the c-map in $N = 2$ supergravity. In view of the next chapter, we first treat the case of one double-tensor multiplet coupled to $N = 2$ supergravity, after which we perform the c-map on the $N = 2$ supergravity model with an arbitrary number of multiplets.

The double-tensor multiplet

The (Minkowskian) Lagrangian of one double-tensor multiplet coupled to $N = 2$ supergravity can be written as [20, 78]

$$\mathcal{L} = -R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}e^{-\phi}\partial_\mu\chi\partial^\mu\chi + \frac{1}{2}M_{ab}H_\mu^aH^{\mu b}. \quad (2.75)$$

The $N = 2$ pure supergravity sector contains the metric and the graviphoton field strength $F_{\mu\nu}$, whereas the matter sector consists of two scalars and a doublet of tensors, $H_\mu^a = -\frac{1}{2}i\varepsilon_{\mu\nu\rho\sigma}\partial^\nu B^{\rho\sigma a}$ (with $a = 1, 2$). The self-interactions in the double-tensor multiplet are encoded in the matrix

$$M(\phi, \chi) = e^\phi \begin{pmatrix} 1 & -\chi \\ -\chi & e^\phi + \chi^2 \end{pmatrix}. \quad (2.76)$$

(2.75) can be obtained (up to a factor 2) from (2.58) as a special example of the case of one hypermultiplet (with the appropriate isometries to dualize two scalars to tensors) and function $F(X) = \frac{1}{4}i(X^0)^2$ (which leads to $N_{00} = 1$, $\mathcal{M}_{00} = 0$ and $\mathcal{N}_{00} = \frac{i}{2}$). Note that (2.75) contains the model (2.65) as a subsector; we reobtain it when we put $F_{\mu\nu}$, χ and H_μ^1 to zero.

We then perform a standard Wick rotation (see Appendix A), use Euclidean metrics and dimensionally reduce over $\tau = it$. Doing so we decompose the vierbein as

in (2.66); this yields a three-dimensional metric, a vector \tilde{B} and a scalar $\tilde{\phi}$. The vector gauge potential decomposes in the standard way

$$A_\mu = (-\tilde{\chi}, \tilde{A}_m - \tilde{\chi}\tilde{B}_m) . \quad (2.77)$$

The result after dimensional reduction is ⁵

$$\begin{aligned} \mathcal{L}_3^e = & R(\hat{e}) + \frac{1}{2}\partial_m\phi\partial^m\phi + \frac{1}{2}e^{-\phi}\partial_m\chi\partial^m\chi + \frac{1}{2}M_{ab}(\phi, \chi)H_m^a H^{mb} \\ & + \frac{1}{2}\partial_m\tilde{\phi}\partial^m\tilde{\phi} + \frac{1}{2}e^{-\tilde{\phi}}\partial_m\tilde{\chi}\partial^m\tilde{\chi} + \frac{1}{2}M_{ab}(\tilde{\phi}, \tilde{\chi})\tilde{H}_m^a \tilde{H}^{mb} . \end{aligned} \quad (2.78)$$

Here we have combined the two vectors in a doublet $\tilde{B}_m^a = (\tilde{A}_m, \tilde{B}_m)$ that defines the (dual) field strengths \tilde{H}_m^a in three dimensions. The matrix multiplying their kinetic energy is exactly the same as in (2.76), but now with the tilde-fields. Therefore, the Lagrangian has the symmetry

$$\phi \longleftrightarrow \tilde{\phi}, \quad \chi \longleftrightarrow \tilde{\chi}, \quad B_m^a \longleftrightarrow \tilde{B}_m^a, \quad (2.79)$$

where $B_m^a = B_{m\tau}^a$. Similar to the last subsection, the symmetry (2.79) makes that (2.78) can be reinterpreted as the result of a dimensional reduction of a theory of the same form as (2.75), but with a double-tensor multiplet consisting of $\tilde{\phi}$, $\tilde{\chi}$ and \tilde{H}_μ^a , a vierbein given by

$$\tilde{e}_\mu^a = \begin{pmatrix} e^{-\phi/2}\hat{e}_m^i & e^{\phi/2}B_m^2 \\ 0 & e^{\phi/2} \end{pmatrix}, \quad (2.80)$$

and a graviphoton with components

$$A_\mu = (-\chi, B_m^1 - \chi B_m^2) . \quad (2.81)$$

The general model

We then consider general $N = 2$ supergravity systems of the form (2.58). Nevertheless, in the models we start from we suppress the hypermultiplets (in case these hypermultiplets allow for a tensor multiplet formulation, they could be taken into account, but for our purposes it suffices to neglect them). So our starting point is

$$\mathcal{L} = -R + 2\mathcal{M}_{\Lambda\Sigma}\partial_\mu X^\Lambda\partial^\mu \bar{X}^\Sigma + \frac{1}{4}ie^{-1}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^\Lambda G_{\rho\sigma\Lambda}, \quad (2.82)$$

where Λ runs from 0 to n , i.e. there are n vector multiplets involved. We put in a factor of 2 for convenience. $G_{\mu\nu\Lambda}$ is given by

$$G_{\mu\nu\Lambda} = -\frac{1}{2}ie\varepsilon_{\mu\nu\rho\sigma}\frac{\delta\mathcal{L}}{\delta F_{\rho\sigma}^\Lambda} = \frac{1}{2}ie\varepsilon_{\mu\nu\rho\sigma}\text{Im}\mathcal{N}_{\Lambda\Sigma}F^{\rho\sigma\Sigma} + \text{Re}\mathcal{N}_{\Lambda\Sigma}F_{\mu\nu}^{\Sigma}. \quad (2.83)$$

⁵We are suppressing here terms like (2.70), which are irrelevant for our purpose.

After a Wick rotation we perform a dimensional reduction over Euclidean time. The vielbein we parameterize as before,

$$e_{\mu}{}^a = \begin{pmatrix} e^{-\phi/2} \hat{e}_m{}^i & e^{\phi/2} B_m \\ 0 & e^{\phi/2} \end{pmatrix}, \quad (2.84)$$

while the vector gauge fields decompose as

$$A_{\mu}{}^{\Lambda} = (-\chi^{\Lambda}, B_m{}^{\Lambda} - \chi^{\Lambda} B_m). \quad (2.85)$$

This way we obtain

$$\begin{aligned} \mathcal{L}_3^e &= \hat{R} + \frac{1}{2} \partial_m \phi \partial^m \phi + \frac{1}{2} e^{2\phi} H_m H^m - 2\mathcal{M}_{\Lambda\bar{\Sigma}} \partial_m X^{\Lambda} \partial^m \bar{X}^{\Sigma} \\ &\quad - \frac{1}{2} i \hat{e}^{-1} \varepsilon^{mnl} G_{m\tau\Lambda} F_{nl}{}^{\Lambda} - \frac{1}{2} i \hat{e}^{-1} \varepsilon^{mnl} F_{m\tau}{}^{\Lambda} G_{nl\Lambda}, \end{aligned} \quad (2.86)$$

where $(iF_{m\tau}{}^{\Lambda}, iG_{m\tau\Lambda}) = (F_{mt}{}^{\Lambda}, G_{mt\Lambda})$ and $(F_{mn}{}^{\Lambda}, G_{mn\Lambda})$ are the components of $(F_{\mu\nu}{}^{\Lambda}, G_{\mu\nu\Lambda})$ with and without a time-index respectively. In terms of the three-dimensional fields in which we decomposed the vielbein and the vector fields, they read as

$$\begin{aligned} \begin{pmatrix} iF_{m\tau}{}^{\Lambda} \\ iG_{m\tau\Lambda} \end{pmatrix} &= \begin{pmatrix} -i\partial_m \chi^{\Lambda} \\ -\frac{1}{2} \hat{e} \varepsilon_{mnl} e^{\phi} Im \mathcal{N}_{\Lambda\Sigma} (H^{\Sigma} - \chi^{\Sigma} H)^{nl} - iRe \mathcal{N}_{\Lambda\Sigma} \partial_m \chi^{\Sigma} \end{pmatrix}, \\ \begin{pmatrix} F_{mn}{}^{\Lambda} \\ G_{mn\Lambda} \end{pmatrix} &= \begin{pmatrix} (H^{\Lambda} - \chi^{\Lambda} H)_{mn} \\ i\hat{e} \varepsilon_{mnl} e^{-\phi} Im \mathcal{N}_{\Lambda\Sigma} \partial^l \chi^{\Sigma} + Re \mathcal{N}_{\Lambda\Sigma} (H^{\Sigma} - \chi^{\Sigma} H)_{mn} \end{pmatrix} \\ &\quad + 2 \begin{pmatrix} F_{[m\tau}{}^{\Lambda} B_{n]} \\ G_{[m\tau\Lambda} B_{n]} \end{pmatrix} \\ &\equiv \begin{pmatrix} \hat{H}_{mn}{}^{\Lambda} \\ G_{mn\Lambda}^{\chi} \end{pmatrix} + 2 \begin{pmatrix} F_{[m\tau}{}^{\Lambda} B_{n]} \\ G_{[m\tau\Lambda} B_{n]} \end{pmatrix}. \end{aligned} \quad (2.87)$$

Here we used $H_{mn}{}^{\Lambda} = 2\partial_{[m} B_{n]}{}^{\Lambda}$ and $H_{mn} = 2\partial_{[m} B_{n]}$. Why we utilize the superscript χ in $G_{mn\Lambda}^{\chi}$ will become clear shortly. The second vector on the right hand side of $(F_{mn}{}^{\Lambda}, G_{mn\Lambda})$ in fact drops out when plugging in (2.87) in (2.86). So the vector fields $B_m{}^{\Lambda}$ and B_m only appear through their field strengths.

Compared with the model of subsection 2.5.1 and the double-tensor multiplet we considered earlier, no symmetries of the type (2.71) and (2.79) do appear. Nevertheless, similar to the case of (2.69) and (2.78), (2.86) can be uplifted to a new four-dimensional model. In case of (2.69) and (2.78), this yielded a four-dimensional model of the same form as the original one. Now the models on both sides are different: While we started with the vector multiplet sector we end up with the hypermultiplet sector of $N = 2$ supergravity.

More precisely, we obtain a four-dimensional (Euclidean) theory of n tensor multiplets and 1 double-tensor multiplet,

$$\begin{aligned} \mathcal{L}^e &= R + \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{1}{2} e^{2\phi} H_{\mu} H^{\mu} - 2\mathcal{M}_{\Lambda\bar{\Sigma}} \partial_{\mu} X^{\Lambda} \partial^{\mu} \bar{X}^{\Sigma} \\ &\quad - e^{-1} \hat{H}^{\mu\Lambda} G_{\mu\Lambda}^{\hat{H}} - e^{-1} G_{\Lambda}^{\chi\mu} F_{\mu}^{\chi\Lambda}, \end{aligned} \quad (2.88)$$

where $H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]}$, $H^\mu = \frac{1}{6}\varepsilon^{\mu\nu\rho\sigma}H_{\nu\rho\sigma}$ and $H_{\mu\nu\rho}^\Lambda = 3\partial_{[\mu}B_{\nu\rho]}^\Lambda$. Also $\hat{H}_{\mu\nu\rho}^\Lambda = H_{\mu\nu\rho}^\Lambda - \chi^\Lambda H_{\mu\nu\rho}$ and $\hat{H}^{\mu\Lambda} = \frac{1}{6}\varepsilon^{\mu\nu\rho\sigma}\hat{H}_{\nu\rho\sigma}^\Lambda$. Furthermore, $F_\mu^{\chi\Lambda} = -i\partial_\mu\chi^\Lambda$, while $G_{\mu\Lambda}^{\hat{H}}$ and $G_{\nu\rho\sigma\Lambda}^\chi$ are the four-dimensional versions of $iG_{m\tau\Lambda}$ and $G_{mn\Lambda}^\chi$, so given by (2.87) with H_{mn} and \hat{H}_{mn}^Λ replaced by $H_{\mu\nu\rho}$ and $\hat{H}_{\mu\nu\rho}^\Lambda$. Finally $G_\Lambda^{\chi\mu} = \frac{1}{6}\varepsilon^{\mu\nu\rho\sigma}G_{\nu\rho\sigma\Lambda}^\chi$. For future convenience we also give the Lagrangian in terms of the tensor multiplet fields

$$\begin{aligned} \mathcal{L}^e &= R + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}e^{2\phi}H_\mu H^\mu - 2\mathcal{M}_{\Lambda\bar{\Sigma}}\partial_\mu X^\Lambda\partial^\mu\bar{X}^\Sigma \\ &\quad + e^{-\phi}\text{Im}\mathcal{N}_{\Lambda\Sigma}\partial_\mu\chi^\Lambda\partial^\mu\chi^\Sigma + e^\phi\text{Im}\mathcal{N}_{\Lambda\Sigma}\hat{H}_\mu^\Lambda\hat{H}^{\mu\Sigma} \\ &\quad + 2ie^{-1}\text{Re}\mathcal{N}_{\Lambda\Sigma}\partial_\mu\chi^\Lambda\hat{H}^{\mu\Sigma}, \end{aligned} \quad (2.89)$$

where we implemented $F_\mu^{\chi\Lambda} = -i\partial_\mu\chi^\Lambda$ and eliminated $G_{\mu\Lambda}^{\hat{H}}$ and $G_{\mu\Lambda}^\chi$, using (the four-dimensional version of) (2.87).

Observe that (2.89) is invariant under complex rescalings of the X^Λ , which can be traced back to the dilatational and $U(1)_R$ gauge symmetries of the superconformal model (2.52).

Dimensionally reducing (2.89) would yield (2.86) when we identify

$$\begin{aligned} B_{m\tau} &= B_m, \\ B_{m\tau}^\Lambda &= B_m^\Lambda, \end{aligned} \quad (2.90)$$

leave out the kinetic terms for the fields descending from the metric components with one and two τ -indices ⁶ and neglect terms of the type (2.70).

Note that the tensor multiplet vectors $(F_\mu^{\chi\Lambda}, G_{\mu\Lambda}^{\hat{H}})$ and $(\hat{H}_\mu^\Lambda, G_{\mu\Lambda}^\chi)$ are directly related to the symplectic vector $(F_{\mu\nu}^\Lambda, G_{\mu\nu\Lambda})$ of the vector multiplet side. Actually, these tensor multiplet objects are symplectic vectors from a purely scalar-tensor perspective as well: They transform as such under symplectic scalar-tensor duality transformations of the type described in the last chapter. The latter follows from the fact that $G_{\mu\Lambda}^{\hat{H}}$ and $G_{\nu\rho\sigma\Lambda}^\chi$ are the functional derivatives of (2.88) with respect to $\hat{H}_{\mu\nu\rho}^\Lambda$ and $F_\mu^{\chi\Lambda}$ respectively,

$$\begin{aligned} G_{\mu\Lambda}^{\hat{H}} &= -\frac{1}{2}e\varepsilon_{\mu\nu\rho\sigma}\frac{\delta\mathcal{L}^e}{\delta\hat{H}_{\alpha\beta\gamma}^\Lambda}, \\ G_{\mu\nu\rho\Lambda}^\chi &= \frac{1}{2}e\varepsilon_{\mu\nu\rho\sigma}\frac{\delta\mathcal{L}^e}{\delta F_\sigma^{\chi\Lambda}}, \end{aligned} \quad (2.91)$$

which explains the use of the superscripts χ and \hat{H} in $G_{\mu\nu\rho\Lambda}^\chi$ and $G_{\mu\Lambda}^{\hat{H}}$. The set of

⁶Reinstalling these terms does not spoil the picture, but it would make the analysis more involved.

Bianchi identities and equations of motion of (2.88) then takes the form

$$\begin{aligned} \partial_\mu \begin{pmatrix} \hat{H}^{\mu\Lambda} \\ G^{\chi\mu} \end{pmatrix} &= -i \begin{pmatrix} F_\mu^{\chi\Lambda} \\ G_{\mu\Lambda}^{\hat{H}} \end{pmatrix} H^\mu, \\ \varepsilon^{\mu\nu\rho\sigma} \partial_\rho \begin{pmatrix} F_\sigma^\chi \\ G_\sigma^{\hat{H}} \end{pmatrix} &= 0, \end{aligned} \quad (2.92)$$

which is the $N = 2$ supergravity generalization of (1.52). (2.92) remains equivalent under the transformations

$$\begin{aligned} \begin{pmatrix} \hat{H}_\mu^\Lambda \\ G_{\mu\Lambda}^\chi \end{pmatrix} &\longrightarrow \begin{pmatrix} \tilde{\hat{H}}_\mu^\Lambda \\ \tilde{G}_{\mu\Lambda}^\chi \end{pmatrix} = \begin{pmatrix} U^\Lambda_\Sigma & Z^{\Lambda\Sigma} \\ W_{\Lambda\Sigma} & V_\Lambda^\Sigma \end{pmatrix} \begin{pmatrix} \hat{H}_\mu^\Sigma \\ G_{\mu\Sigma}^\chi \end{pmatrix}, \\ \begin{pmatrix} F_\mu^{\chi\Lambda} \\ G_{\mu\Lambda}^{\hat{H}} \end{pmatrix} &\longrightarrow \begin{pmatrix} \tilde{F}_\mu^{\chi\Lambda} \\ \tilde{G}_{\mu\Lambda}^{\hat{H}} \end{pmatrix} = \begin{pmatrix} U^\Lambda_\Sigma & Z^{\Lambda\Sigma} \\ W_{\Lambda\Sigma} & V_\Lambda^\Sigma \end{pmatrix} \begin{pmatrix} F_\mu^{\chi\Sigma} \\ G_{\mu\Sigma}^{\hat{H}} \end{pmatrix}, \end{aligned} \quad (2.93)$$

just as (1.52) does under (1.54). Note that the non-trivial right-hand side of the first set of equations (2.92) does not spoil the duality as it transforms consistently. The dual Lagrangian can be obtained from

$$\tilde{G}_{\mu\Lambda}^{\hat{H}} = -\frac{1}{2} e \varepsilon_{\mu\nu\rho\sigma} \frac{\delta \tilde{\mathcal{L}}^e}{\delta \tilde{H}_{\nu\rho\sigma}^\Lambda}, \quad \tilde{G}_{\mu\nu\rho\Lambda}^\chi = \frac{1}{2} e \varepsilon_{\mu\nu\rho\sigma} \frac{\delta \tilde{\mathcal{L}}^e}{\delta \tilde{F}_\sigma^{\chi\Lambda}}, \quad (2.94)$$

which has a consistent solution only when the matrix in (2.93) is an element of $Sp(2n, \mathbb{R})$. Transforming (X^Λ, F_Λ) as a symplectic vector as well then gives a dual theory of the same form as (2.88), completely similar to the vector multiplet sector.

3

Supergravity description of spacetime instantons

Black holes in superstring theory have both a macroscopic and microscopic description. On the macroscopic side, they can be described as solitonic solutions of the effective supergravity Lagrangian. Microscopically they can typically be constructed by wrapping p -branes over p -dimensional cycles in the manifold that the string theory is compactified on. The microscopic interpretation is best understood for BPS black holes.

Apart from this solitonic sector, string theory also contains instantons. Microscopically they arise as wrapped Euclidean p -branes over $p + 1$ -dimensional cycles of the internal manifold. The aim of this chapter (which is based on [79] and [80]) is to present a macroscopic picture of these instantons as solutions of the Euclidean equations of motion in the effective supergravity Lagrangian. We focus hereby on spacetime instantons, whose (non-perturbative) effects are inversely proportional to the string coupling constant g_s .

The models that we will study are type II string theories compactified on a Calabi-Yau (CY) threefold. The resulting four-dimensional effective action realizes $N = 2$ supergravity, whose structure we discussed in chapter 2. Both vector and tensor and hypermultiplets do appear. As explained before, the tensor multiplets can be dualized to hypermultiplets. The numbers of multiplets in the four-dimensional action depend on the topological properties of the CY. The latter are encoded in its Hodge numbers $h_{1,1}$ and $h_{1,2}$, which give the number of $(1, 1)$ and $(1, 2)$ cycles¹. Type IIA(B) string theory compactifications on a CY with Hodge numbers $h_{1,1}$ and $h_{1,2}$ yields $h_{1,1}$ ($h_{1,2}$) vector multiplets and $h_{1,2} + 1$ ($h_{1,1} + 1$) hypermultiplets. The fields in these multiplets include $h_{1,1}$ *Kähler* moduli, associated with deformations of the size of the CY and $h_{1,2}$ *complex structure* moduli, associated with its deformations in shape.

The geometry of the hypermultiplet moduli space - containing the dilaton - is

¹In the context of complex geometry, (p, q) cycles are dual to harmonic tensor fields of rank p, q , where p and q denote the number of holomorphic and antiholomorphic indices.

known to receive quantum corrections, both from string loops [15] and from instantons [81]. The instanton corrections are exponentially suppressed and are difficult to compute directly in string theory. Our results yield some progress in this direction, since within the supergravity description one finds explicit formulae for the instanton action ². Related work can also be found in [83, 84], but our results are somewhat different and contain several new extensions.

Interestingly, there is a relation between black hole solutions in type IIA/B and instanton solutions in type IIB/A. Microscopically, this can be understood from T-duality between IIA and IIB. Macroscopically, this follows from the c-map, discussed at the end of the last chapter. This makes that (BPS) solutions of the vector multiplet Lagrangian are mapped to (BPS) solutions of the tensor- or hypermultiplet Lagrangian. We will use this mapping in Euclidean spacetimes. Roughly speaking, there are two classes of solutions on the vector multiplet sector: (Euclidean) Black holes and Taub-NUT like solutions. These map to D-brane instantons and NS-fivebrane instantons respectively. The distinguishing feature is that the corresponding instanton actions are inversely proportional to g_s or g_s^2 respectively. For both type of instantons, we give the explicit solution and the precise value of the instanton action.

The D-brane instantons are found to be the solutions to the equations obtained from c-mapping the BPS equations of [85]. Their analysis contains also R^2 interactions, but they can be easily switched off. The BPS equations then obtained are similar, but not identical to the equations derived in [86]. The NS-fivebrane instantons are derived in a different way, not by using the c-map. This is because the BPS solutions in Euclidean supergravity coupled to vector multiplets are not fully classified. We therefore construct the NS-fivebrane instantons by extending the Bogomol'nyi-bound-formulation of [20].

As explained in the introduction, ultimately, we hope to get a better understanding of non-perturbative string theory. In particular, it is expected that instanton effects resolve conifold-like singularities in the hypermultiplet moduli space of Calabi-Yau compactifications, see e.g. [87]. These singularities are closely related - by the c-map - to the conifold singularities in the vector multiplet moduli space due to the appearance of massless black holes [88]. Moreover, in combination with the more recent relation between black holes and topological strings [89], it would be interesting to study if topological string theory captures some of the non-perturbative structure of the hypermultiplet moduli space. For some hints in this direction, see [90]. Finally, we recall that instantons may play an important role in the stabilization of moduli. For an example related to our discussion, we refer to [21].

This chapter is organized as follows: In section 3.1 we treat NS-fivebrane instantons

²Instanton actions can also be studied from worldvolume theories of D-branes. For a discussion on this in the context of our work, we refer to [82]. It would be interesting to find the precise relation to our analysis.

in the context of $N = 1$ supergravity. We use this simple setup to explain at a basic level various concepts we use in later sections. Section 3.2 is devoted to a review of instanton solutions in the universal hypermultiplet of $N = 2$ supergravity and their relation to gravitational solutions of pure $N = 2$ supergravity. Then in section 3.3 we consider instanton solutions to the theory obtained from arbitrary CY compactifications of type II superstrings.

3.1 NS-fivebrane instantons

In this section, we give the $N = 1$ supergravity description of the NS-fivebrane instanton. The main characteristic of this instanton is that the instanton action is inversely proportional to the square of the string coupling constant. In string theory, such instantons appear when Euclidean NS-fivebranes wrap six-cycles in the internal space, and therefore are completely localized in both space and (Euclidean) time.

It is well known that Euclidean NS-fivebranes in string theory are T-dual to Taub-NUT or more generally, ALF geometries [87] (see also [91]). We here re-derive these results from the perspective of four-dimensional (super-) gravity in a way that clarifies the methods used in this chapter.

3.1.1 A Bogomol'nyi bound

We start with the simple system of gravity coupled to a scalar and tensor in four spacetime dimensions given by (2.65), which we repeat for convenience,

$$\mathcal{L} = -R(e) - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}e^{2\phi}H_\mu H^\mu, \quad (3.1)$$

with

$$H^\mu = -\frac{1}{2}i\varepsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma}. \quad (3.2)$$

The instanton solution can be found by deriving a Bogomol'nyi bound on the Euclidean Lagrangian [92],

$$\mathcal{L}^e = \frac{1}{2}(e^\phi H_\mu \mp e^\phi \partial_\mu e^{-\phi})(e^\phi H^\mu \mp e^\phi \partial^\mu e^{-\phi}) \mp \frac{1}{6}ie^{-1}\partial_\mu(e^\phi H^\mu). \quad (3.3)$$

Here, we have left out the Einstein-Hilbert term. It is well known that this term is not positive definite, preventing us to derive a Bogomol'nyi bound including gravity. In most cases, our instanton solutions are purely in the matter sector, and spacetime will be taken flat. The Bogomol'nyi equation then is

$$H_\mu = \pm\partial_\mu e^{-\phi}. \quad (3.4)$$

This implies that $e^{-\phi}$ should be a harmonic function. The \pm solutions refer to instantons or anti-instantons. Notice that the surface term in (3.3) is topological

in the sense that it is independent on the spacetime metric. It is easy to check that the BPS configurations (3.4) have vanishing energy-momentum tensor, so that the Einstein equations are satisfied for any Ricci-flat metric.

One can now easily evaluate the instanton action on this solution. The only contribution comes from the surface term in (3.3). Defining the instanton charge as

$$\int_{S^3} d^3x \left(\frac{1}{6} \varepsilon^{mnl} H_{mnl} \right) = Q , \quad (3.5)$$

we find ³

$$S_{\text{inst}} = \frac{|Q|}{g_s^2} . \quad (3.6)$$

Here we have assumed that there is only a contribution from infinity, and not from a possible other boundary around the location of the instanton. It is easy to see this when spacetime is taken to be flat. In that case the single-centered solution for the dilaton is

$$e^{-\phi} = e^{-\phi_\infty} + \frac{|Q|}{4\pi^2 r^2} , \quad (3.7)$$

which is the standard harmonic function in flat space with the origin removed. We have furthermore related the string coupling constant to the asymptotic value of the dilaton by

$$g_s \equiv e^{-\phi_\infty/2} . \quad (3.8)$$

In our notation, this is the standard convention.

3.1.2 Taub-NUT geometries and NS-fivebrane instantons

As described in subsection 2.5.1, the model (3.1) realizes a prototype version of the c-map. We can make use of this map, i.e. the symmetry transformations (2.71), to generate scalar-tensor solutions from a (Euclidean) time independent solution of pure Einstein gravity. In other words, we do a T-duality over (Euclidean) time (this of course only makes sense as a solution-generating-technique). However, such a solution is not an instanton, since it is not localized in τ . We therefore have to uplift the solution to a τ -dependent solution in four dimensions. This is easy if the original solution is in terms of harmonic functions. In that case there is a natural uplifting scheme, which involves going from three- to four-dimensional harmonic functions.

We discuss now examples in the class of gravitational instantons [94]. These are vacuum solutions of the Euclidean Einstein equation, based on a three-dimensional

³In the tensor multiplet formulation, the instanton action has no imaginary theta-angle-like terms. They are produced after dualizing the tensor into an axionic scalar, by properly taking into account the constant mode of the axion. In the context of NS-fivebrane instantons, this was explained e.g. in [93].

harmonic function $V(\vec{x})$,

$$ds^2 = V^{-1}(d\tau + A_m dx^m)^2 + V d\vec{x} \cdot d\vec{x} , \quad (3.9)$$

with A_m satisfying $\frac{1}{2}\varepsilon_{mnl}\partial^n A^l = \pm\partial_m V$. The \pm solutions yields selfdual or anti-selfdual Riemann curvatures. In the notation of (2.67), we have that $A_m = \tilde{B}_m$, $\hat{e}_m{}^i = \delta_m{}^i$ and $e^{-\tilde{\phi}} = V$.

Multi-centered gravitational instantons correspond to harmonic functions of the form

$$V = V_0 + \sum_i \frac{m_i}{|\vec{x} - \vec{x}_i|} , \quad (3.10)$$

for some parameters V_0 and m_i . For non-zero V_0 , one can further rescale $\tau = V_0\tilde{\tau}$ and $m_i = 4V_0\tilde{m}_i$ such that one can effectively set $V_0 = 1$. The single-centered case corresponds to Taub-NUT geometries, or orbifolds thereof. For $V_0 = 0$ one obtains smooth resolutions of ALE spaces (like e.g. the Eguchi-Hanson metric for the two-centered solution). For more details, we refer to [95].

Before the c-map, the dilaton and the tensor are taken to be zero. The three-dimensional τ -independent solution after the symmetry transformations (2.71) is

$$e^\phi = V^{-1} , \quad H_m = \frac{1}{2}\varepsilon_{mnl}\partial^n B^l = \pm\partial_m V , \quad H_{mnl} = 0 , \quad (3.11)$$

and the metric is flat, $g_{mn} = \delta_{mn}$.

We now construct a four-dimensional τ -dependent solution by taking V a harmonic function in four dimensions. We take the four-dimensional metric to be flat and $H_{\mu\nu\rho}$ is determined by $H_\mu = \pm\partial_\mu V$.

That this is still a solution for (3.1) can directly be seen from the fact that the Bogomol'nyi equations (3.4) are satisfied. The instanton action is again given by (3.6). Notice further that the difference between instantons and anti-instantons for the fivebrane corresponds to selfdual and antiselfdual gravitational instantons. Due to our procedure, we are making certain aspects of T-duality not explicit. We have for instance suppressed any dependence on the radius of the compactified circle parameterized by τ . These aspects become important in order to dynamically realize the uplifting solution in terms of a decompactification limit after T-duality. It turns out that a proper T-duality of the Taub-NUT geometry, including worldsheet instanton corrections, produces a completely localized NS-fivebrane instanton based on the four-dimensional harmonic function given above. For more details, we refer to [91].

3.2 Membrane and fivebrane instantons

In the previous section, we have discussed aspects of NS-fivebrane instantons. Here, we will elaborate further on this, and also introduce membrane instantons.

These appear in M-theory or type IIA string theory compactifications, and we will be interested in four-dimensional effective theories with eight supercharges such as IIA strings compactified on Calabi-Yau manifolds. The main distinction with the previous section is the presence of RR fields, and these will play an important role in this section.

General CY compactifications of type IIA strings yield $N = 2$ supergravity theories with $h_{1,2} + 1$ hypermultiplets (or tensor multiplets), but in this section we will restrict ourselves to the case of the universal hypermultiplet only, leaving the general case for the next section. This situation occurs when the CY space is rigid, i.e. when $h_{1,2} = 0$ ⁴. Then there are only two three-cycles in the CY, around which the Euclidean membranes can wrap. These are the membrane instantons, and in this section we give their supergravity description. The $h_{1,1}$ vector multiplet fields can be truncated in our setup; it suffices to have pure supergravity coupled to the universal hypermultiplet.

3.2.1 Instantons in the double-tensor multiplet

We will describe the universal hypermultiplet in the double-tensor formulation, given by (2.75). In this formulation it contains two tensors and two scalars, which can be thought of as two $N = 1$ tensor multiplets coming from the NS-NS and RR sectors. Instantons in the double-tensor multiplet were already discussed in [20, 79] (see also [96]), and in the context of the c-map in [84]. In this section, we reproduce these results and show the correspondence between instantons and stationary gravitational solutions using the c-map.

The (Minkowskian) Lagrangian for the double-tensor multiplet, we recall, reads as

$$\mathcal{L} = -R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}e^{-\phi}\partial_\mu\chi\partial^\mu\chi + \frac{1}{2}M_{ab}H_\mu^aH^{\mu b}, \quad (3.12)$$

with

$$M(\phi, \chi) = e^\phi \begin{pmatrix} 1 & -\chi \\ -\chi & e^\phi + \chi^2 \end{pmatrix}. \quad (3.13)$$

From a string theory point of view, the metric, ϕ and H_μ^2 come from the NS-NS sector, while the graviphoton, χ and H_μ^1 descend from the RR sector in type IIA strings. When we truncate to the NS-NS sector, we get the Lagrangian (3.1), so the results obtained there are still valid here.

As we are interested in instanton solutions, we consider the Euclidean version of (3.12), which can be obtained by doing a standard Wick rotation (see Appendix A) and using Euclidean metrics. The form of the Lagrangian is still given by (3.12), but now the matter Lagrangian is positive definite. In [20] and [79], Bogomol'nyi equations were derived and solved for the double-tensor multiplet coupled to pure

⁴The universal hypermultiplet cannot be obtained from a geometric compactification of type IIB strings as there exists no CY with $h_{1,1} = 0$.

$N = 2$ supergravity with vanishing graviphoton field strength and Ricci tensor. The solutions of these equations preserving half of the supersymmetry can be recasted into the following compact form,

$$\begin{aligned}
g_{\mu\nu} &= \delta_{\mu\nu} , \\
e^{-\phi} &= \frac{1}{4}(h^2 - p^2) , \\
H_\mu^2 &= \frac{1}{2}(h\partial_\mu p - p\partial_\mu h) , \\
\chi &= -e^\phi p + \chi_c , \\
H_\mu^1 - \chi_c H_\mu^2 &= -\partial_\mu h ,
\end{aligned} \tag{3.14}$$

with h and p four-dimensional harmonic functions (satisfying $|h| \geq |p|$) and χ_c an arbitrary constant. The cases where h is negative or positive correspond to instantons or anti-instantons respectively. We have written here a flat metric $g_{\mu\nu}$, but it is easy to generalize this to any Ricci flat metric, as long as it admits harmonic functions. Non-trivial h and p can be obtained when one or more points are taken out of four-dimensional flat space. The solution is then of the form

$$h = h_\infty + \frac{Q_h}{4\pi^2|\vec{x} - \vec{x}_0|^2} , \quad p = p_\infty + \frac{Q_p}{4\pi^2|\vec{x} - \vec{x}_0|^2} , \tag{3.15}$$

or multi-centered versions thereof. It can be easily seen that a pole in p corresponds to a source with (electric) charge in the field equation of χ . Similarly a pole in h corresponds to a source with (magnetic) charge in the Bianchi identity of $H_\mu^1 - \chi_c H_\mu^2$. For single-centered solutions, there are five independent parameters, two for each harmonic function, together with χ_c .

NS-fivebrane instantons with RR background fields

The general solution in (3.14) falls into two classes, depending on the asymptotic behavior of the dilaton at the origin. The first class fits into the category of NS-fivebrane instantons. The solution is characterized by

$$p = \pm(h - \alpha) , \tag{3.16}$$

with α an arbitrary constant. In terms of (3.15), this condition is equivalent to

$$Q_h = \pm Q_p , \tag{3.17}$$

such that the solution only has four independent parameters. This implies that the dilaton behaves at the origin like

$$e^{-\phi} \rightarrow \mathcal{O}\left(\frac{1}{r^2}\right) . \tag{3.18}$$

The condition (3.16) implies that

$$H_\mu^2 = \pm \partial_\mu e^{-\phi} , \quad H_\mu^1 = \pm \partial_\mu (e^{-\phi} \chi) , \quad (3.19)$$

with $e^{-\phi}$ the harmonic function

$$e^{-\phi} = \frac{1}{2} \alpha h - \frac{1}{4} \alpha^2 , \quad (3.20)$$

and χ is fixed in terms of h via (3.14). In this form, we get back the results of [20] and [79]. However, as we will see shortly, the equations (3.19) are not completely equivalent to (3.14) and (3.16).

The prototype example for $e^{-\phi}$ is of the form

$$e^{-\phi} = g_s^2 + \sum_i \frac{|Q_i|}{4\pi^2 |\vec{x} - \vec{x}_i|^2} . \quad (3.21)$$

It is easy to check that, whereas the dilaton diverges, the RR field χ remains finite at the excised points. When taking (3.19) as the starting point, the values of χ at the points \vec{x}_i are allowed to be different. However, as pointed out in [93], for the solutions to preserve half of the supersymmetry, χ should take equal values (χ_0) at these points. As the difference with (3.19) we just alluded to, the equations (3.14) and (3.16) *do* include this restriction from supersymmetry.

The solution is characterized by the parameters α , χ_c , g_s and the charges

$$Q \equiv \int_{S_\infty^3} d^3x \left(\frac{1}{6} \varepsilon^{mnl} H_{mnl}^2 \right) = \mp \sum_i |Q_i| . \quad (3.22)$$

The parameters α and χ_c can be traded for the boundary values of the RR field, χ_∞ and χ_0 . The action of the multi-centered instanton was calculated in [20, 79], and the result is

$$S_{\text{inst}} = |Q| \left(\frac{1}{g_s^2} + \frac{1}{2} (\Delta\chi)^2 \right) , \quad (3.23)$$

with $\Delta\chi \equiv \chi_\infty - \chi_0$.

The solution above describes a generalization of the NS-fivebrane instanton discussed in section 3.1. Notice that the first term in the instanton action is inversely proportional to the square of the string coupling constant, as is common for NS-fivebrane instantons. The second term is the contribution from the RR background field. Only for constant χ does one obtain a local minimum of the action ⁵.

⁵Solutions with constant χ can be obtained from (3.14) and (3.16) by taking the limit $\alpha \rightarrow 0$ while both h_∞ and $Q_h \rightarrow \infty$ in such a way that αh is kept fixed. Such solutions follow more directly from the Bogomol'nyi equations considered in [79].

Membrane instantons

The remaining solutions, other than (3.17), are given by (3.14) with $Q_h \neq Q_p$. One can see that the asymptotic behavior of the dilaton around the origin is now

$$e^{-\phi} \rightarrow \mathcal{O}\left(\frac{1}{r^4}\right). \quad (3.24)$$

Compared to the fivebrane instanton case, this behavior is more singular. However, the instanton action is still finite. As was shown in [20], the action reduces to a surface term, and the only contribution comes from infinity. One way of writing the instanton action is [79]

$$S_{\text{inst}} = \sqrt{\frac{4}{g_s^2} + (\Delta\chi)^2} \left(|Q_h| \pm \frac{1}{2} \Delta\chi Q \right), \quad (3.25)$$

with the same convention as for fivebranes, i.e.

$$\Delta\chi \equiv \chi_\infty - \chi_0 = -\frac{p_\infty}{g_s^2}, \quad (3.26)$$

and Q still defined by

$$Q \equiv \int_{S_\infty^3} d^3x \left(\frac{1}{6} \varepsilon^{mnl} H_{mnl}^2 \right) = -\frac{1}{2} (h_\infty Q_p - p_\infty Q_h). \quad (3.27)$$

The plus and minus sign in (3.25) refer to instanton and anti-instanton respectively. Using the relations given above and $g_s^2 = \frac{1}{4}(h_\infty^2 - p_\infty^2)$ one can show that (3.25) is always positive, as it should be.

Notice that the instanton action contains both the fivebrane charge Q and Q_h , which we identify with a membrane charge. For pure membrane instantons, which have vanishing NS-NS field, the second term in (3.25) vanishes. When we put H_μ^2 (and its BPS equation) to zero from the start, we can dualize χ to a tensor and obtain a ‘‘tensor-tensor’’ theory. To perform this dualization we have to replace $-i\partial_\mu\chi$ by the vector F_μ^χ in the Euclideanized (H_μ^2 -less) version of (3.12) and add a Lagrange multiplier term

$$\mathcal{L}^e(\chi) \longrightarrow \mathcal{L}^e(F) + e^{-1} \varepsilon^{\mu\nu\rho\sigma} B_{\mu\nu\chi} \partial_\rho F_\sigma^\chi. \quad (3.28)$$

Integrating out $B_{\mu\nu\chi}$ enforces $\partial_{[\mu} F_{\nu]}^\chi = 0$ and locally $F_\mu^\chi = -i\partial_\mu\chi$ again. Subtracting the total derivative $e^{-1} \varepsilon^{\mu\nu\rho\sigma} \partial_\mu (B_{\nu\rho\chi} F_\sigma^\chi)$ and integrating out F_μ^χ yields the tensor-tensor theory. Using this action to evaluate the pure membrane instantons on gives

$$\begin{aligned} S'_{\text{inst}} &= S_{\text{inst}} + \Delta\chi Q_p \\ &= \frac{2}{g_s} \sqrt{Q_h^2 - Q_p^2}. \end{aligned} \quad (3.29)$$

The appearance of the second term in the first line is a result of the subtraction of the boundary term in the dualization procedure. In going from the first to the second line we used the fact that $Q = 0$, which allowed us to express $\Delta\chi$ in terms of the charges Q_h and Q_p and g_s ,

$$\Delta\chi = -\frac{2}{g_s} \frac{Q_p}{\sqrt{Q_h^2 - Q_p^2}} . \quad (3.30)$$

The $\frac{1}{g_s}$ dependence in the instanton action is typical for D-brane instantons that arise after wrapping Euclidean D-branes over supersymmetric cycles in the Calabi-Yau [81].

The microscopic interpretation of the general solution is not so clear. In the next subsection, we will see how these solutions are generated from the c-map. In this way, one can give a natural interpretation in terms of black holes and gravitational instantons.

The form of the instanton action for both fivebrane (3.23) and membrane instantons (3.25) and (3.29) was recently re-derived by solving the constraints from supersymmetry of the effective action [16]. This provides an alternative derivation of the formulas in this section and confirms that the supergravity method for computing the instanton action is correct.

3.2.2 Instantons and gravitational solutions

In this subsection, we show that our membrane and fivebrane instanton solutions naturally follow from the c-map.

As considered in subsection 2.5.2, a dimensional reduction of (3.12) gives rise to the symmetry transformations (2.79). Using these symmetry transformations, the general BPS instanton, given by (3.14), can be translated back to stationary BPS solutions of pure $N = 2$ supergravity. This involves replacing four-dimensional by three-dimensional harmonic functions.

Starting from (3.14), one thus obtains solutions to the equations of motion in Euclidean space. Stationary solutions of the Einstein-Maxwell Lagrangian can however easily be continued from Minkowski to Euclidean space, and vice versa. If we make the following decomposition for the metric and graviphoton vector field in Minkowski space

$$\begin{aligned} g_{\mu\nu} dx^\mu dx^\nu &= -e^{\tilde{\phi}} (dt + \omega_m dx^m)^2 + e^{-\tilde{\phi}} \hat{g}_{mn} dx^m dx^n , \\ A_\mu &= (-\tilde{\chi}', \tilde{A}_m - \tilde{\chi}' \omega_m) , \end{aligned} \quad (3.31)$$

then we can analytically continue to Euclidean space by identifying

$$\omega_m = -i\tilde{B}_m , \quad \tilde{\chi}' = i\tilde{\chi} . \quad (3.32)$$

Using the inverse of this we generate the following BPS equations for pure $N = 2$ supergravity,

$$\begin{aligned}
\hat{g}_{mn} &= \delta_{mn} , \\
e^{-\tilde{\phi}} &= \frac{1}{4}(h^2 + q^2) , \\
\frac{1}{2}\varepsilon_{mnl}\partial^n\omega^l &= -\frac{1}{2}(h\partial_m q - q\partial_m h) , \\
\tilde{\chi}' &= -e^{\tilde{\phi}}q + \chi'_c , \\
\tilde{H}_m^1 - \varepsilon_{mnl}\chi'_c\partial^n\omega^l &= -\partial_m h ,
\end{aligned} \tag{3.33}$$

with h, q three-dimensional flat space harmonic functions, $\tilde{H}_m^1 = \varepsilon_{mnl}\partial^n\tilde{A}^l$ and χ'_c an arbitrary constant.

BPS equations of pure $N = 2$ supergravity were studied in [85], [86], [97] and [98]. (3.33) can be shown to reproduce the results of [85].

The line element of (3.33) falls into the general class of Israel-Wilson-Perjés (IWP) metrics [99, 100],

$$ds^2 = -|U|^{-2}(dt + \omega_m dx^m)^2 + |U|^2 d\vec{x} \cdot d\vec{x} , \tag{3.34}$$

where U is any complex solution to the three-dimensional Laplace equation. Comparing to (3.33) and (3.31), we have that

$$U = \frac{1}{2}(h + iq) . \tag{3.35}$$

Let $F_{\mu\nu}$ be the field strength of the four-dimensional gauge field and $G_{\mu\nu}$ is its dual,

$$G_{\mu\nu} \equiv -\frac{1}{2}ie\varepsilon_{\mu\nu\rho\sigma}\frac{\delta\mathcal{L}}{\delta F_{\rho\sigma}} . \tag{3.36}$$

Then the components of $F_{\mu\nu}$ with a time-index are

$$F_{mt} = -\partial_m\tilde{\chi}' , \quad G_{mt} = -\frac{1}{4}e^{\tilde{\phi}}\varepsilon_{mnl}(2\omega^n F^{lt} + F^{nl}) . \tag{3.37}$$

To derive the second equation in (3.37) one needs to decompose the component of (3.36) with a time-index, $G_{mt} = -\frac{1}{4}i\varepsilon_{mntl}g^{n\mu}g^{l\nu}F_{\mu\nu}$, using the metric parameterization (3.31). The last two equations in (3.33) can now elegantly be rewritten as

$$F_{mt} = \partial_m(e^{\tilde{\phi}}q) , \quad G_{mt} = -\frac{1}{2}\partial_m(e^{\tilde{\phi}}h) . \tag{3.38}$$

In fact, in [85] solutions were given in terms of these objects. This will become important in the next section.

The class of IWP metrics contains many interesting examples, some of which we discuss now.

Pure membrane instantons and black holes

We consider here solutions to (3.14) with vanishing NS-NS tensor,

$$H_\mu^2 = 0 . \quad (3.39)$$

These were the solutions that lead to the pure membrane instantons. The vanishing of H_μ^2 implies that the two harmonic functions h and p are proportional to each other,

$$p = c h , \quad (3.40)$$

for some real constant c . We take h of the form

$$h = h_\infty + \sum_i \frac{Q_{h,i}}{4\pi^2 |\vec{x} - \vec{x}_i|^2} , \quad Q_{p,i} = c Q_{h,i} . \quad (3.41)$$

This membrane instanton is in the image of the c-map. The dual (Minkowskian) gravitational solution is static,

$$\partial_{[m} \omega_{n]} = 0 , \quad (3.42)$$

and has $q = c' h$. The IWP metric now becomes of the Majumdar-Papapetrou type. These are multi-centered versions of the extreme Reissner-Nordström black hole. Our solutions describe the outer horizon part of spacetime in isotropic coordinates,

$$\begin{aligned} ds^2 = & - \left(\gamma + \sum_i \frac{M_i}{4\pi |\vec{x} - \vec{x}_i|} \right)^{-2} dt^2 \\ & + \left(\gamma + \sum_i \frac{M_i}{4\pi |\vec{x} - \vec{x}_i|} \right)^2 (dr^2 + r^2 d\Omega^2) , \end{aligned} \quad (3.43)$$

with

$$M_i = \frac{1}{2} \sqrt{(Q_{h,i})^2 + (Q_{q,i})^2} , \quad \gamma = \frac{1}{2} \sqrt{1 + c'^2} h_\infty , \quad (3.44)$$

and $Q_{q,i} = c' Q_{h,i}$ for each charge labelled by i . Note that in the parameterization (3.43) the event horizons are located at $\vec{x} = \vec{x}_i$. The metric can be made asymptotically Minkowski by a rescaling of the coordinates

$$t = \gamma t' , \quad r = \frac{r'}{\gamma} . \quad (3.45)$$

NS-fivebrane instantons and Taub-NUT with selfdual graviphoton

Here we consider the NS-fivebrane instantons with RR background fields. This solution was specified by equations (3.16), (3.19) and (3.20). Using the inverse

c-map, we can relate it to a BPS solution of pure N=2 supergravity, based on the three-dimensional harmonic function

$$e^{-\tilde{\phi}} = V \equiv v + \sum_i \frac{Q_i}{4\pi^2 |\vec{x} - \vec{x}_i|^2} . \quad (3.46)$$

The metric solution of the Taub-NUT geometry (3.9) then reappears,

$$ds^2 = V^{-1} (d\tau + \tilde{B}_m dx^m)^2 + V d\vec{x} \cdot d\vec{x} , \quad (3.47)$$

with

$$2\partial_{[m} \tilde{B}_{n]} = \pm \varepsilon_{mnl} \partial^l V . \quad (3.48)$$

Analogously to the NS-fivebrane instanton supporting a non-trivial χ , the Taub-NUT metric (3.47) supports a non-trivial graviphoton,

$$\begin{aligned} F_{m\tau} &= \pm \frac{1}{2} \alpha V^{-2} \partial_m V , & -\frac{1}{2} e \varepsilon_{m\tau\mu\nu} F^{\mu\nu} &= \frac{1}{2} \alpha V^{-2} \partial_m V , \\ F_{mn} &= \mp \alpha \partial_{[m} (V^{-1} \tilde{B}_{n]}) , & -\frac{1}{2} e \varepsilon_{mn\mu\nu} F^{\mu\nu} &= -\alpha \partial_{[m} (V^{-1} \tilde{B}_{n]}) . \end{aligned} \quad (3.49)$$

The solution (3.49) is (anti)selfdual, $F_{\mu\nu} = \mp \frac{1}{2} e \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$. In fact, it is precisely the one found in [101] (see equation (4.15) in that reference).

The fact that the graviphoton is (anti)selfdual implies that it has vanishing energy-momentum, which is consistent with the fact that the Taub-NUT solution is Ricci-flat. Taub-NUT solutions with (anti)selfdual graviphoton and their T-duality relation with NS-fivebranes played an important role in a study of the partition sum of the NS-fivebrane [102].

3.3 Instantons in matter coupled N=2 supergravity

In the last section we considered instantons in the double-tensor multiplet coupled to $N = 2$ supergravity. Now we are interested in instanton solutions of the general four-dimensional low energy effective action which type II superstrings compactified on a Calabi-Yau gives rise to. In the absence of fluxes, this yields (ungauged) $N = 2$ supergravity coupled to vector and tensor multiplets (or their dual hypermultiplets). We recall that in type IIA(B) string theory the number of vector multiplets is $h_{1,1}$ ($h_{1,2}$) and the number of tensor multiplets is $h_{2,1} + 1$ ($h_{1,1} + 1$) (where $h_{1,1}$ and $h_{1,2}$ are Hodge numbers of the Calabi-Yau).

In section 2.5 we generated the tensor multiplet model from the c-map on the gravitational and vector multiplet sector. This way we obtained from n vector multiplets coupled to $N = 2$ supergravity a model of one double-tensor and n tensor multiplets. These tensor multiplets can be dualized further to hypermultiplets,

but similarly to the previous section, we will not carry out this dualization. This turns out to be the most convenient way to describe instanton solutions, i.e. they are naturally described in the tensor multiplet formulation.

In this section we use the c-map once more, to map the BPS equations for the vector multiplets as found in [85] (with the R^2 -interactions which are present in there switched off) to instantonic BPS equations for the tensor multiplet theory ⁶. The picture that emerges is that all BPS black hole solutions have their corresponding instantonic description after the (Euclidean) c-map. For a generic tensor multiplet theory these solutions all carry some RR-charge, and the instanton action is inversely proportional to the string coupling. There should also be NS-fivebrane instantons whose action is proportional to $\frac{1}{g_s^2}$. However it is not clear for a generic tensor multiplet theory how to get these from the Euclidean c-map. Therefore we derive them in a way independent of the c-map.

3.3.1 The tensor multiplet theory

We first discuss the tensor multiplet Lagrangian obtained after the c-map. Details of the derivation we gave in section 2.5. The result is $N = 2$ supergravity coupled to a double-tensor multiplet and n tensor multiplets, with $n = h_{1,1}$ or $h_{1,2}$ when starting from the type IIA or IIB vector multiplet sector respectively. We recall that the bosonic Lagrangian, in Euclidean space, reads

$$\begin{aligned} \mathcal{L}^e = & R + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}e^{2\phi}H_\mu H^\mu - 2\mathcal{M}_{\Lambda\Sigma}\partial_\mu X^\Lambda\partial^\mu\bar{X}^\Sigma \\ & + e^{-\phi}Im\mathcal{N}_{\Lambda\Sigma}\partial_\mu\chi^\Lambda\partial^\mu\chi^\Sigma + e^\phi Im\mathcal{N}_{\Lambda\Sigma}(H_\mu^\Lambda - \chi^\Lambda H_\mu)(H^{\mu\Sigma} - \chi^\Sigma H^\mu) \\ & + 2ie^{-1}Re\mathcal{N}_{\Lambda\Sigma}\partial_\mu\chi^\Lambda(H^{\mu\Sigma} - \chi^\Sigma H^\mu), \end{aligned} \quad (3.50)$$

as given before in (2.89). We have left out the vector multiplet sector including the graviphoton as it is not relevant for our purposes. This sector can be easily reinstalled. The NS-NS part of the bosonic sector of the (double-)tensor multiplets consists of the dilaton, ϕ , the tensor $B_{\mu\nu}$ ($H_{\mu\nu\rho} \equiv 3\partial_{[\mu}B_{\nu\rho]}$, $H^\mu = \frac{1}{6}\varepsilon^{\mu\nu\rho\sigma}H_{\nu\rho\sigma}$) and the complex scalars X^Λ ($\Lambda = 0, 1, \dots, n$). The RR part of the bosonic sector of the (double-)tensor multiplets is formed by the (real) scalars χ^Λ and the tensors $B_{\mu\nu}^\Lambda$ ($H_{\mu\nu\rho}^\Lambda \equiv 3\partial_{[\mu}B_{\nu\rho]}^\Lambda$, $H^{\mu\Lambda} = \frac{1}{6}\varepsilon^{\mu\nu\rho\sigma}H_{\nu\rho\sigma}^\Lambda$).

The metric $M_{\Lambda\bar{\Sigma}}$ of the manifold parameterized by the complex scalars X^Λ (given by (2.57)) can be written as

$$\mathcal{M}_{\Lambda\bar{\Sigma}} \equiv N_{\Lambda\Sigma} - \frac{N_{\Lambda\Gamma}N_{\Sigma\Xi}\bar{X}^\Gamma X^\Xi}{N_{\Omega\Delta}X^\Omega\bar{X}^\Delta}. \quad (3.51)$$

⁶For some earlier work on vector multiplet BPS equations see [86, 103] and references therein.

The matrix $Im\mathcal{N}_{\Lambda\Sigma}$ appearing in the quadratic terms of (3.50) we recall is determined by

$$\mathcal{N}_{\Lambda\Sigma} \equiv \bar{F}_{\Lambda\Sigma} + i \frac{N_{\Lambda\Gamma} X^\Gamma N_{\Sigma\Xi} X^\Xi}{N_{\Omega\Delta} X^\Omega X^\Delta} . \quad (3.52)$$

Notice that the last term in (3.50) is imaginary, similar to a theta-angle-like term. It will therefore be difficult to find a Bogomol'nyi bound on the action. We will return to the issue of a BPS bound in the last subsection. In fact, as we will see, we need to drop the reality conditions on the fields, as not all solutions we discuss below respect these reality conditions. For the moment, we will simply complexify all the fields ⁷, and discuss below which instanton solutions respect which reality conditions.

In the next subsection we derive BPS equations for (3.50) by c-mapping BPS equations for stationary solutions of its vector multiplet counterpart. These latter equations are naturally formulated in terms of symplectic vectors. Therefore it is useful to write (3.50) in terms of symplectic vectors as well. To do this we can make use of the results obtained in section 2.5. We get

$$\begin{aligned} \mathcal{L}^e &= R + \frac{1}{2} e^{2\phi} H_\mu H^\mu \\ &+ \frac{1}{2} e^{2\phi} (F_\Lambda \partial_\mu \bar{Y}^\Lambda - Y^\Lambda \partial_\mu \bar{F}_\Lambda + c.c.) (F_\Lambda \partial^\mu \bar{Y}^\Lambda - Y^\Lambda \partial^\mu \bar{F}_\Lambda + c.c.) \\ &- c^{\mu\Lambda} d_{\mu\Lambda} + c^\mu{}_\Lambda d_\mu{}^\Lambda \\ &- e^{-1} \hat{H}^{\mu\Lambda} G_{\mu\Lambda}^{\hat{H}} - e^{-1} G_\Lambda^{\chi\mu} F_\mu^{\chi\Lambda} . \end{aligned} \quad (3.53)$$

Here and below, by *c.c.* we mean taking the complex conjugate before dropping the reality conditions, and then treating X^Λ and \bar{X}^Λ as independent complex fields. To obtain (3.53) we demanded

$$N_{\Lambda\Sigma} \bar{X}^\Lambda X^\Sigma = 1 , \quad (3.54)$$

which fixes the norm of the scalar fields X^Λ (recall from chapter 2 that (3.50) is invariant under complex rescalings of X^Λ). Furthermore, we introduced the $U(1)_R$ invariant variables

$$Y^\Lambda \equiv e^{-\frac{1}{2}\phi} \bar{h} X^\Lambda , \quad \bar{Y}^\Lambda \equiv e^{-\frac{1}{2}\phi} h \bar{X}^\Lambda . \quad (3.55)$$

Here h is an arbitrary (space-dependent) phase factor, which drops out when plugging in (3.55) in the action. As a consequence of (3.54) and (3.55), $e^{-\phi}$ should be understood as a function of Y^Λ and \bar{Y}^Λ ,

$$e^{-\phi} = i(Y^\Lambda \bar{F}_\Lambda(\bar{Y}) - \bar{Y}^\Lambda F_\Lambda(Y)) . \quad (3.56)$$

⁷For instance, this means that we treat X^Λ and \bar{X}^Λ as independent complex fields. The action then only depends on X^Λ and \bar{X}^Λ in a holomorphic way.

$(c_\mu^\Lambda, c_{\mu\Lambda})$ and $(d_\mu^\Lambda, d_{\mu\Lambda})$ are symplectic vectors belonging to the NS-NS sector. They are defined as

$$\begin{aligned} \begin{pmatrix} c_\mu^\Lambda \\ c_{\mu\Lambda} \end{pmatrix} &\equiv \begin{pmatrix} +i\partial_\mu(Y^\Lambda - \bar{Y}^\Lambda) \\ +i\partial_\mu(F_\Lambda - \bar{F}_\Lambda) \end{pmatrix}, \\ \begin{pmatrix} d_\mu^\Lambda \\ d_{\mu\Lambda} \end{pmatrix} &\equiv \begin{pmatrix} \partial_\mu(e^\phi(Y^\Lambda + \bar{Y}^\Lambda)) \\ \partial_\mu(e^\phi(F_\Lambda + \bar{F}_\Lambda)) \end{pmatrix}. \end{aligned} \quad (3.57)$$

Furthermore, we repeat that the symplectic vectors from the RR part of the theory, $(\hat{H}_\mu^\Lambda, G_{\mu\Lambda}^\chi)$ and $(F_\mu^{\chi\Lambda}, G_{\mu\Lambda}^{\hat{H}})$, are given by

$$\begin{aligned} \begin{pmatrix} \hat{H}_\mu^\Lambda \\ G_{\mu\Lambda}^\chi \end{pmatrix} &\equiv \begin{pmatrix} H_\mu^\Lambda - \chi^\Lambda H_\mu \\ -iee^{-\phi} \text{Im} \mathcal{N}_{\Lambda\Sigma} \partial_\mu \chi^\Sigma + \text{Re} \mathcal{N}_{\Lambda\Sigma} (H_\mu^\Sigma - \chi^\Sigma H_\mu) \end{pmatrix}, \\ \begin{pmatrix} F_\mu^{\chi\Lambda} \\ G_{\mu\Lambda}^{\hat{H}} \end{pmatrix} &\equiv \begin{pmatrix} -i\partial_\mu \chi^\Lambda \\ -ee^\phi \text{Im} \mathcal{N}_{\Lambda\Sigma} (H_\mu^\Sigma - \chi^\Sigma H_\mu) - i \text{Re} \mathcal{N}_{\Lambda\Sigma} \partial_\mu \chi^\Sigma \end{pmatrix}. \end{aligned} \quad (3.58)$$

$G_{\mu\Lambda}^\chi$ and $G_{\mu\Lambda}^{\hat{H}}$ are the functional derivatives of the Lagrangian with respect to $F_\mu^{\chi\Lambda}$ and \hat{H}_μ^Λ as given by (2.91).

3.3.2 BPS equations from the c-map

In this section we use the c-map to obtain BPS instanton equations of a $n + 1$ tensor multiplet theory from the BPS equations of a n vector multiplet theory. As said before, the latter equations are known and we use the results of [85]. In here equations were constructed for stationary solutions preserving half of the supersymmetry, with parameters satisfying

$$h\epsilon_i = \varepsilon_{ij}\gamma_0\epsilon^j. \quad (3.59)$$

We remind that h is the phase factor appearing in (3.55).

The metric components, given by

$$g_{\mu\nu} dx^\mu dx^\nu = -e^\phi (dt + \omega_m dx^m)^2 + e^{-\phi} \hat{g}_{mn} dx^m dx^n, \quad (3.60)$$

were found to be related to the complex scalars Y^Λ in the following way

$$\begin{aligned} e^{-\phi} &= i(Y^\Lambda \bar{F}_\Lambda - \bar{Y}^\Lambda F_\Lambda), \\ \hat{g}_{mn} &= \delta_{mn}, \\ \varepsilon_{mnl} \partial^n \omega^l &= \bar{F}_\Lambda \partial_m Y^\Lambda + \bar{Y}^\Lambda \partial_m F_\Lambda + c.c.. \end{aligned} \quad (3.61)$$

Furthermore, $-i(Y^\Lambda - \bar{Y}^\Lambda)$ and $-i(F_\Lambda - \bar{F}_\Lambda)$ are three-dimensional harmonic functions. This fixes the NS-NS sector completely. Recall that the equation for $e^{-\phi}$ is identically true, as follows from the definition of Y^Λ and the condition (3.54).

For the RR fields the BPS equations of [85] are

$$\begin{pmatrix} F_{mt}{}^\Lambda \\ G_{mt\Lambda} \end{pmatrix} = \begin{pmatrix} d_m{}^\Lambda \\ d_{m\Lambda} \end{pmatrix}. \quad (3.62)$$

These equations can be equivalently formulated as

$$\begin{pmatrix} F_{mn}{}^\Lambda \\ G_{mn\Lambda} \end{pmatrix} = \varepsilon_{mnl} \begin{pmatrix} c^{l\Lambda} \\ c^l{}_\Lambda \end{pmatrix} + 2\partial_{[m} \begin{pmatrix} e^\phi(Y^\Lambda + \bar{Y}^\Lambda)\omega_n|_{BPS} \\ e^\phi(F_\Lambda + \bar{F}_\Lambda)\omega_n|_{BPS} \end{pmatrix}, \quad (3.63)$$

where $\omega_n|_{BPS}$ is the BPS solution of ω_n . (3.63) is the form in which the equations for the RR fields in [86] are written, however they do not have the second term on the right-hand side. Both sets of equations (3.62) and (3.63) fix the RR fields completely in terms of the complex scalars Y^Λ .

By construction the equations above only have stationary solutions. When $\omega_n = 0$ one gets static extremal black holes. This works similar as in the pure supergravity case discussed in the last section. However there is a difference between the generic case and pure supergravity, which will become important later on in the context of NS-fivebrane instantons. We saw in the last section that the pure $N = 2$ supergravity BPS equations, after an analytic continuation to Euclidean space, gave rise to Taub-NUT solutions as well. In contrast to this, for generic functions $F(X)$ it is far from clear if, and if yes how, this kind of solutions is contained in the general solution.

Using the c-map treated in subsection 2.5.2, the equations above can be mapped quite easily to instanton equations of the Euclidean tensor sector. It requires an analytic continuation, involving $\omega_m = -iB_m$, and a replacement of three-dimensional by four-dimensional harmonic functions.

We thus find as instanton equations for the NS-NS fields

$$\begin{aligned} e^{-\phi} &= i(Y^\Lambda \bar{F}_\Lambda - \bar{Y}^\Lambda F_\Lambda), \\ g_{\mu\nu} &= \delta_{\mu\nu}, \\ H_\mu &= i(\bar{F}_\Lambda \partial_\mu Y^\Lambda - \bar{Y}^\Lambda \partial_\mu F_\Lambda + c.c.), \end{aligned} \quad (3.64)$$

while $-i(Y^\Lambda - \bar{Y}^\Lambda)$ and $-i(F_\Lambda - \bar{F}_\Lambda)$ are now four-dimensional harmonic functions. The instanton equations for the RR fields are

$$\begin{pmatrix} F_\mu^{\chi\Lambda} \\ G_{\mu\Lambda}^{\hat{H}} \end{pmatrix} = \begin{pmatrix} d_\mu{}^\Lambda \\ d_{\mu\Lambda} \end{pmatrix}, \quad (3.65)$$

or

$$\begin{pmatrix} \hat{H}_\mu{}^\Lambda \\ G_{\mu\Lambda}^\chi \end{pmatrix} = \begin{pmatrix} c_\mu{}^\Lambda \\ c_{\mu\Lambda} \end{pmatrix} - i \begin{pmatrix} e^\phi(Y^\Lambda + \bar{Y}^\Lambda) \\ e^\phi(F_\Lambda + \bar{F}_\Lambda) \end{pmatrix} H_\mu|_{\text{inst}}. \quad (3.66)$$

Just as on the vector multiplet side both (3.65) and (3.66) fix the RR fields completely in terms of the complex scalars Y^Λ . For the fields appearing in (3.50) the

equations take the form

$$\begin{aligned}\chi^\Lambda &= ie^\phi(Y^\Lambda + \bar{Y}^\Lambda) + \chi_c^\Lambda, \\ H_\mu^\Lambda - \chi_c^\Lambda H_\mu &= i\partial_\mu(Y^\Lambda - \bar{Y}^\Lambda),\end{aligned}\tag{3.67}$$

where χ_c^Λ are arbitrary constants.

Recall from subsection (3.3.1) that all fields are complex. However, when we take $-i(Y^\Lambda - \bar{Y}^\Lambda)$ and $-i(F_\Lambda - \bar{F}_\Lambda)$ to be real, then the solutions for the dilaton and H^Λ are real whereas χ^Λ and H become imaginary.

Let us make contact with the results of section 3.2. When we take the function $F(Y)$ to be $F(Y) = \frac{1}{4}i(Y^0)^2$, we see that (3.50) reduces to the Euclidean version of (3.12). We then get

$$Y^0 + \bar{Y}^0 = -2i(F_0 - \bar{F}_0), \quad F_0 + \bar{F}_0 = \frac{1}{2}i(Y^0 - \bar{Y}^0).\tag{3.68}$$

Now we make the following identification of the harmonic functions $-i(Y^0 - \bar{Y}^0)$ and $-i(F_0 - \bar{F}_0)$ and the harmonic functions h and p which appeared in the BPS equations of section 3.2,

$$-i(Y^0 - \bar{Y}^0) = h, \quad -i(F_0 - \bar{F}_0) = \frac{1}{2}ip.\tag{3.69}$$

Equations (3.14) then follow directly. We can in this case obtain a real solution for χ and H if we impose that $-i(F_0 - \bar{F}_0)$ is imaginary, such that the harmonic function p is real.

3.3.3 D-brane instantons

We now discuss the different types of solutions to the equations we obtained above. Clearly the general solution is a function of $2n + 2$ harmonic functions. In the following we take them single-centered

$$\begin{aligned}-i(Y^\Lambda - \bar{Y}^\Lambda) &= -i(Y^\Lambda - \bar{Y}^\Lambda)_\infty + \frac{\hat{Q}^\Lambda}{4\pi^2|\vec{x} - \vec{x}_0|^2}, \\ -i(F_\Lambda - \bar{F}_\Lambda) &= -i(F_\Lambda - \bar{F}_\Lambda)_\infty + \frac{Q_\Lambda}{4\pi^2|\vec{x} - \vec{x}_0|^2}.\end{aligned}\tag{3.70}$$

However our results are easily generalized to multi-centered versions of (3.70).

In section 3.2 we saw that the two different types of solutions to the BPS equations for the double-tensor multiplet, membrane and the NS-fivebrane instantons, have different behavior of $e^{-\phi}$. For membrane instantons (having non-zero RR-charge) the dilaton behaves towards the excised point(s) as $e^{-\phi} \rightarrow \mathcal{O}\left(\frac{1}{|\vec{x} - \vec{x}_0|^4}\right)$. For NS-fivebrane instantons (having non-zero NS-NS-, but vanishing RR-charge) $e^{-\phi}$ is a

harmonic function, which implies that towards the excised point(s) the behavior of the dilaton is $e^{-\phi} \rightarrow \mathcal{O}\left(\frac{1}{|\vec{x}-\vec{x}_0|^2}\right)$. The different behavior of the dilaton in both types of solutions is reflected in a different dependence of the instanton action on the string coupling.

Let us now consider solutions to the general equations of last subsection (i.e. for general functions $F(Y)$). The above seems to indicate that for a study of the characteristics of the instanton solutions it is good to start by analyzing the behavior of the dilaton towards the excised point(s). Doing this analysis we find that to leading order in $\frac{1}{|\vec{x}-\vec{x}_0|}$

$$e^{-\phi}|_{\vec{x}\rightarrow\vec{x}_0} = \frac{|Z_0|^2}{16\pi^4|\vec{x}-\vec{x}_0|^4}, \quad (3.71)$$

which is as singular as the membrane instanton of section 3.2. Here Z_0 is defined as

$$Z_0 \equiv (\hat{Q}^\Lambda F_\Lambda(X) - Q_\Lambda X^\Lambda)|_{\vec{x}\rightarrow\vec{x}_0}. \quad (3.72)$$

As seen from the c-map, the function $Z = Q^\Lambda F_\Lambda(X) - Q_\Lambda X^\Lambda$ is the dual of the central charge function of the vector multiplet theory ⁸.

$Z_0 \neq 0$

We first consider the case $Z_0 \neq 0$. Generic single-centered solutions consist of $5n + 5$ parameters, 2 for each harmonic function and the $n + 1$ constants χ_c^Λ . The RR scalars χ^Λ take the values χ_c^Λ at $\vec{x} = \vec{x}_0$. The constants \hat{Q}^Λ and Q_Λ appearing in (3.70) can be identified with magnetic and electric charges of sources appearing in Bianchi identities and field equations respectively. \hat{Q}^Λ is equal to the charge of the source in the Bianchi identity of $(H^\Lambda - \chi_c^\Lambda H)_\mu$,

$$\hat{Q}^\Lambda = \int_{S_\infty^3} d^3x \left(\frac{1}{6} \varepsilon^{mnl} (H_{mnl}{}^\Lambda - \chi_c^\Lambda H_{mnl}) \right). \quad (3.73)$$

Q_Λ is up to a factor $-2i$ the charge of the source in the field equation of χ^Λ ,

$$-2iQ_\Lambda = \int_{\mathbb{R}^4} d^4x e \left(\frac{\delta \mathcal{L}}{\delta \chi^\Lambda} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \chi^\Lambda} \right). \quad (3.74)$$

Notice that this is consistent with the fact that the solutions for χ^Λ are imaginary. As there are non-vanishing RR charges we can identify these solutions as D-brane instantons, generalizing the membrane instantons found in section 3.2. Also the Bianchi identity of H_μ is sourced. The corresponding charge can be expressed in terms of the parameters appearing in the $2n + 2$ harmonic functions,

$$Q \equiv \int_{S_\infty^3} d^3x \left(\frac{1}{6} \varepsilon^{mnl} H_{mnl} \right) = (Y^\Lambda - \bar{Y}^\Lambda)_\infty Q_\Lambda - (F_\Lambda - \bar{F}_\Lambda)_\infty \hat{Q}^\Lambda. \quad (3.75)$$

⁸ $Z_\infty = (Q^\Lambda F_\Lambda(X) - Q_\Lambda X^\Lambda)|_\infty$ is the dynamically generated central charge mentioned in the footnote at the beginning of section 2.1.

Evaluating (3.53) on these instantons gives

$$\begin{aligned}
S_{\text{inst}} &= \int_{\mathbb{R}^4} d^4x (-c^{\mu\Lambda} d_{\mu\Lambda} + i(e^\phi(Y^\Lambda + \bar{Y}^\Lambda) d_{\mu\Lambda} \\
&\quad + e^\phi(F_\Lambda(Y) + \bar{F}_\Lambda(\bar{Y})) d_{\mu}{}^\Lambda) H^\mu)|_{BPS} \\
&= -\frac{2}{g_s^2} (F_\Lambda(Y) + \bar{F}_\Lambda(\bar{Y}))_\infty \hat{Q}^\Lambda \\
&\quad + \frac{i}{g_s^4} (F_\Lambda(Y) + \bar{F}_\Lambda(\bar{Y}))_\infty (Y^\Lambda + \bar{Y}^\Lambda)_\infty Q .
\end{aligned} \tag{3.76}$$

Applying (3.76) to the double-tensor multiplet theory of section 3.2, we have to take again $F(Y) = \frac{1}{4}i(Y^0)^2$. Then using (3.68), (3.69), (3.26) and the double-tensor multiplet relation $g_s^2 = \frac{1}{4}(h_\infty^2 - p_\infty^2)$ we re-obtain (3.25).

Defining $\Delta\varphi_\Lambda \equiv \frac{i}{g_s^2} (F_\Lambda(Y) + \bar{F}_\Lambda(\bar{Y}))_\infty$ and $\Delta\sigma \equiv \frac{1}{2} \frac{1}{g_s^4} (Y^\Lambda + \bar{Y}^\Lambda)_\infty (F_\Lambda(Y) + \bar{F}_\Lambda(\bar{Y}))_\infty$, we can rewrite (3.80) as

$$S_{\text{inst}} = 2i\Delta\varphi_\Lambda \hat{Q}^\Lambda + 2i\Delta\sigma Q . \tag{3.77}$$

In fact, one can show that $\Delta\varphi_\Lambda = \varphi_{\Lambda\infty} - \varphi_{\Lambda 0}$, and $\Delta\sigma = \sigma_\infty - \sigma_0$, where φ_Λ is the dual (RR) scalar of \hat{H}_μ^Λ and σ is the dual (NS-NS) scalar of H_μ and $\varphi_{\Lambda\infty}$, $\varphi_{\Lambda 0}$, σ_∞ , and σ_0 are the asymptotic values of φ_Λ and σ evaluated on the BPS solution

$$\begin{aligned}
\varphi_\Lambda &= i e^\phi (F_\Lambda + \bar{F}_\Lambda) + \varphi_{\Lambda c} , \\
\sigma &= \frac{1}{2} e^{2\phi} (Y^\Lambda + \bar{Y}^\Lambda) (F_\Lambda + \bar{F}_\Lambda) + \sigma_c .
\end{aligned} \tag{3.78}$$

Here $\varphi_{\Lambda c}$ and σ_c are integration constants, which coincide with $\varphi_{\Lambda 0}$ and σ_0 , the values of φ_Λ and σ at the point $\vec{x} = \vec{x}_0$.

The BPS equation for φ_Λ is in fact implicitly stated already in the bottom equation in (3.65). Observe furthermore that the BPS solutions for χ^Λ , as in (3.67), and φ_Λ are consistent with symplectic transformations, so we can write

$$\begin{pmatrix} \chi^\Lambda \\ \varphi_\Lambda \end{pmatrix} = i e^\phi \begin{pmatrix} Y^\Lambda + \bar{Y}^\Lambda \\ F_\Lambda + \bar{F}_\Lambda \end{pmatrix} + \begin{pmatrix} \chi_c^\Lambda \\ \varphi_{\Lambda c} \end{pmatrix} . \tag{3.79}$$

For $Q = 0$ the second term in (3.76) vanishes and we find

$$S_{\text{inst}} = 2i\Delta\varphi_\Lambda \hat{Q}^\Lambda = -\frac{2}{g_s} (\bar{h}F_\Lambda(X) + h\bar{F}_\Lambda(\bar{X}))_\infty \hat{Q}^\Lambda . \tag{3.80}$$

This is the action for pure D-brane instantons of which the pure membrane instanton of section 3.2 is a specific example. We have reintroduced the variables X^Λ to make explicit the typical $\frac{1}{g_s}$ dependence of D-brane instanton actions. From the c-map point of view pure D-brane instantons are the duals of static BPS black holes living in the vector multiplet sector. Microscopically D-brane instantons

come from wrapping even/odd branes over odd/even cycles in the Calabi-Yau in type IIA/B string theory.

Like in section 3.2, when we put H_μ (and its BPS equation) to zero from the start, we can dualize all RR-scalars to tensors. This way we obtain a formulation of the theory consisting of $2n + 2$ tensors, the “tensor-tensor” theory. The dualization procedure works similar as the one described in section (3.2). First we write $F_\mu^{\chi^\Lambda}$ instead of $-i\partial_\mu\chi^\Lambda$ in (3.50) (without H_μ) and add a Lagrange multiplier term

$$\mathcal{L}^e(\chi) \longrightarrow \mathcal{L}^e(F) + e^{-1}\varepsilon^{\mu\nu\rho\sigma}B_{\mu\nu\Lambda}\partial_\rho F_\sigma^{\chi^\Lambda} . \quad (3.81)$$

Integrating out $B_{\mu\nu\Lambda}$ enforces $\partial_{[\mu}F_{\nu]}^{\chi^\Lambda} = 0$, giving back (locally) $F_\mu^{\chi^\Lambda} = -i\partial_\mu\chi^\Lambda$. Subtracting the total derivative $\varepsilon^{\mu\nu\rho\sigma}\partial_\mu(B_{\nu\rho\Lambda}F_\sigma^{\chi^\Lambda})$ and integrating out $F_\mu^{\chi^\Lambda}$ yields the tensor-tensor theory. When we evaluate this action on the pure D-brane instantons we get

$$\begin{aligned} S'_{\text{inst}} &= 2i\Delta\varphi_\Lambda Q^\Lambda - 2i\Delta\chi^\Lambda Q_\Lambda \\ &= -\frac{2}{g_s}(\bar{h}F_\Lambda(X) + h\bar{F}_\Lambda(\bar{X}))_\infty \hat{Q}^\Lambda + \frac{2}{g_s}(\bar{h}X^\Lambda + h\bar{X}^\Lambda)_\infty Q_\Lambda \\ &= \frac{4}{g_s}|Z|_\infty . \end{aligned} \quad (3.82)$$

The second term in the first line is due to the subtraction of the boundary term in the dualization procedure. To arrive at the last line we have used that $Q = 0$, which is a consequence of the fact that we have put H_μ to zero. The expression in the last line is (up to a factor of 4) the value of the real part of the pure D-brane instanton action as suggested in (a five-dimensional context) in [83].

$Z_0 = 0$

Since the behavior of the dilaton is different, the case $Z_0 = 0$ needs to be analyzed separately. For the double-tensor multiplet of section 3.2 it yields NS-fivebrane instantons, which have a harmonic $e^{-\phi}$. This can most easily be understood from the fact that in the double-tensor multiplet case we have $|Z_0| = \frac{1}{2}\sqrt{Q_h^2 - Q_p^2}$ (for single-centered instantons, as can be derived using (3.55), (3.68) and (3.69)). Requiring $|Z_0|$ to vanish then gives the NS-fivebrane relation (3.17).

However, for generic functions $F(X)$ NS-fivebrane instantons do not arise from taking $Z_0 = 0$. In fact, in these cases the $Z_0 = 0$ solution only differs qualitatively from the $Z_0 \neq 0$ solution close to the excised points, which is directly related to the fact that only the asymptotic behavior of the dilaton is different. Now recall that Z is the dual of the central charge function of the vector multiplet theory. So $Z_0 = 0$ solutions are the duals of vector multiplet solutions with vanishing central charge function at $\vec{x} = \vec{x}_0$. In case $Q = 0$ these are zero-horizon black holes. Just as higher derivative corrections lift zero-horizon black holes at the two-derivative-level to finite horizon black holes [104], we expect that for $Z_0 = 0$ -instantons higher

derivative corrections have a qualitative effect on the behavior of $e^{-\phi}$ in the limit $\vec{x} \rightarrow \vec{x}_0$. If this is the case the (two-derivative) differences between these solutions and the $Z \neq 0$ instanton have no real physical significance.

3.3.4 NS-fivebrane instantons

In the previous section, we have discussed D-brane instantons. These were obtained from the c-map of the BPS solutions of [45], analytically continued to Euclidean space. We also saw that for generic functions $F(X)$ NS-fivebrane instantons did not appear as a limiting case in a similar way as in the double-tensor multiplet theory of section 3.2. In fact, it is not clear if they are contained at all in the general solution to the equations in subsection 3.3.2, just as was the case for their supposedly dual Taub-NUT solutions on the vector multiplet side.

However, we expect there to be (BPS) NS-fivebrane instantons in the general theory as well. That we have missed them so far could be understood from the fact that not all solutions in the Euclidean theory can be obtained from Wick rotating real solutions in the Lorentzian theory. Therefore, we will follow a different strategy and work directly in the Euclidean tensor multiplet Lagrangian, using a similar method as in [20]. This way we indeed find a class of NS-fivebrane instanton solutions.

We first write (3.50) as

$$\begin{aligned} \mathcal{L}^e &= R - 2\mathcal{M}_{\Lambda\Sigma}\partial_\mu X^\Lambda\partial^\mu \bar{X}^\Sigma \\ &\quad + (N\mathcal{H}_\mu + OE_\mu)A(N\mathcal{H}^\mu + OE^\mu) - 2e^{-1}\mathcal{H}_\mu^t N^t AOE^\mu \\ &\quad + 2ie^{-1}Re\mathcal{N}_{\Lambda\Sigma}\partial_\mu\chi^\Lambda(H^{\mu\Lambda} - \chi^\Lambda H^\mu). \end{aligned} \quad (3.83)$$

Here we have defined the vectors

$$\mathcal{H}_\mu = \begin{pmatrix} H_\mu \\ H_\mu^\Lambda \end{pmatrix}, \quad E_\mu = \begin{pmatrix} \partial_\mu\phi \\ e^{-\frac{\phi}{2}}\partial_\mu\chi^\Lambda \end{pmatrix}, \quad (3.84)$$

and the matrices

$$N = e^{\frac{\phi}{2}} \begin{pmatrix} e^{\frac{\phi}{2}} & 0 \\ -\chi^\Lambda & \delta^\Lambda_\Sigma \end{pmatrix}, \quad A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & Im\mathcal{N}_{\Lambda\Sigma} \end{pmatrix}, \quad (3.85)$$

O is a matrix as well, satisfying $O^t A O = A$. When all fields are taken real, clearly the real part of (3.83) is bounded from below by

$$\begin{aligned} Re\mathcal{L}^e &\geq R - 2\mathcal{M}_{\Lambda\Sigma}\partial_\mu X^\Lambda\partial^\mu \bar{X}^\Sigma \\ &\quad - 2e^{-1}\mathcal{H}_\mu^t N^t AOE^\mu. \end{aligned} \quad (3.86)$$

Next we take the matrix O to be

$$O_{1,2} = \pm \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}, \quad (3.87)$$

with $\epsilon = \delta^\Lambda_\Sigma$ in the O_1 -case and $\epsilon = -\delta^\Lambda_\Sigma$ in the O_2 -case. The plus and minus signs refer to the instanton and the anti-instanton respectively.

We now consider configurations for which the square in (3.83) is zero (i.e that saturate the bound (3.86) in case all fields are taken real). It is easy to show that for constant X^Λ these configurations satisfy the field equations of ϕ , χ^Λ and the tensors (to the field equations of X^Λ we come back at a later stage). Furthermore, these configurations can be shown to have vanishing energy-momentum. Therefore the gravitational background should be flat and (3.83) reduces to a total derivative. In the following we need the explicit form of this total derivative in the O_2 -case

$$\begin{aligned}\mathcal{L}_2^i &= -e^{-1}\partial_\mu\left(e^\phi H^\mu\right) + 2ie^{-1}\partial_\mu\left(\bar{\mathcal{N}}_{\Lambda\Sigma}\chi^\Lambda\left(H^{\mu\Sigma} - \frac{1}{2}\chi^\Sigma H^\mu\right)\right), \\ \mathcal{L}_2^{a.i} &= +e^{-1}\partial_\mu\left(e^\phi H^\mu\right) + 2ie^{-1}\partial_\mu\left(\mathcal{N}_{\Lambda\Sigma}\chi^\Lambda\left(H^{\mu\Sigma} - \frac{1}{2}\chi^\Sigma H^\mu\right)\right),\end{aligned}\quad (3.88)$$

where the upper equation corresponds to the instanton and the lower one to the anti-instanton. In the O_1 -case we get similar expressions.

Again we can make contact with the double-tensor multiplet theory by taking the function F to be $F(X) = +\frac{1}{4}i(X^0)^2$. The analysis above then reduces to the analysis of [20], with the matrices $O_{1,2}$ corresponding to their matrices $O_{1,2}$. The instantons related to these matrices are the NS-fivebrane instantons discussed in section 3.2.

Let us now consider the conditions which follow from requiring the square in (3.83) to vanish. Firstly, the O_1 matrix gives

$$\mathcal{H}_\mu = \pm \begin{pmatrix} \partial_\mu e^{-\phi} \\ \chi^\Lambda \partial_\mu e^{-\phi} - e^{-\phi} \partial_\mu \chi^\Lambda \end{pmatrix}. \quad (3.89)$$

These equations are very similar to the O_1 equations of [20]. Note in particular the relation $H_\mu = \pm \partial_\mu e^{-\phi}$, which is contained in both. Similarly to [20] we find that the finite-action-solution to (3.89) has a harmonic $e^{-\phi}$ and constant $\chi^\Lambda = \chi_0^\Lambda = \frac{Q^\Lambda}{Q}$. Here

$$Q^\Lambda = \int_{S_\infty^3} d^3x \left(\frac{1}{6}\varepsilon^{mnl} H_{mnl}^\Lambda\right), \quad Q = \int_{S_\infty^3} d^3x \left(\frac{1}{6}\varepsilon^{mnl} H_{mnl}\right), \quad (3.90)$$

consistent with the notation we used in our treatment of D-brane instantons.

The conditions following from taking the matrix O_2 in (3.83) are

$$\mathcal{H}_\mu = \pm \partial_\mu \begin{pmatrix} e^{-\phi} \\ e^{-\phi} \chi^\Lambda \end{pmatrix}. \quad (3.91)$$

These equations are very similar to the O_2 equations of [20], with once more $H_\mu = \pm \partial_\mu e^{-\phi}$ contained in both sets. The latter equation implies that $e^{-\phi}$ is

again harmonic. The remaining equations in (3.91) tell us that the same is true for $e^{-\phi}\chi^\Lambda$. For single-centered solutions this allows us to write χ^Λ as

$$\chi^\Lambda = \chi_1^\Lambda e^\phi + \chi_0^\Lambda, \quad (3.92)$$

where the χ_1^Λ are arbitrary constants. Note that χ_0^Λ is the value χ^Λ takes at the excised point(s). Putting (3.92) back into (3.91) we find again $\chi_0^\Lambda = \frac{Q^\Lambda}{Q}$. Observe that the finite-action O_1 -solution is contained in this O_2 -solution; we re-obtain it when we put χ_1^Λ to zero. The action becomes for the single-centered instanton

$$S_{\text{inst}} = \frac{|Q|}{g_s^2} - i\bar{\mathcal{N}}_{\Lambda\Sigma}^c \Delta\chi^\Lambda \Delta\chi^\Sigma Q, \quad (3.93)$$

where we have (again) defined $\Delta\chi^\Lambda \equiv \chi_\infty^\Lambda - \chi_0^\Lambda$. $\bar{\mathcal{N}}_{\Lambda\Sigma}^c$ is the (constant) solution of $\bar{\mathcal{N}}_{\Lambda\Sigma}$. For the anti-instanton $\bar{\mathcal{N}}^c$ should be replaced by \mathcal{N}^c .

Similarly to the case of D-brane instantons, we can rewrite (3.93) as

$$S_{\text{inst}} = 2i\Delta\sigma Q. \quad (3.94)$$

$\Delta\sigma$ is now defined as $\Delta\sigma \equiv \frac{1}{2}\frac{i}{g_s^2} - \frac{1}{2}\bar{\mathcal{N}}_{\Lambda\Sigma}^c \Delta\chi^\Lambda \Delta\chi^\Sigma$ and satisfies $\Delta\sigma = \sigma_\infty - \sigma_0$, where σ is the dual scalar of H and σ_∞ and σ_0 are the asymptotic values of its solution

$$\sigma = \frac{1}{2}ie^\phi - \frac{1}{2}\bar{\mathcal{N}}_{\Lambda\Sigma}(\chi^\Lambda - \chi_0^\Lambda)(\chi^\Sigma - \chi_0^\Sigma) + \sigma_c. \quad (3.95)$$

σ_c is an integration constant, which coincides with σ_0 , the value of the solution of σ at the excised point.

The equations of motion of X^Λ are not automatically satisfied. Requiring this gives the extra condition that the last term in (3.93) should be extremized with respect to (the constants) X^Λ . Consequently the χ_1^Λ and the X^Λ become related, unless $\bar{\mathcal{N}}_{\Lambda\Sigma}$ is a constant matrix. The latter is for example the case in the double-tensor multiplet theory, for which we have $\mathcal{N}_{00} = \frac{i}{2}$ (in that case (3.93) can be seen to reduce to (3.23)). The precise relations between χ_1^Λ and the X^Λ depend on the function $F(X)$. This implies that there is no general prescription for obtaining real solutions.

From $\chi_0^\Lambda = \frac{Q^\Lambda}{Q}$ it directly follows that the charges $\hat{Q}^\Lambda \equiv Q^\Lambda - \chi_0^\Lambda Q$ are zero. Furthermore, one can show that there are no sources in the field equations of χ^Λ . So there are no RR charges at all in the solutions. This means that they can be identified as (generalized) NS-fivebrane instantons. On the basis of what we know about NS-fivebrane instantons in the double-tensor multiplet [93] we expect these solutions (or at least all single-centered ones) to preserve half of the supersymmetry.

Let us finish our treatment of (generalized) NS-fivebrane instantons by considering its image under the (inverse) c-map. We find that this is a Taub-NUT geometry with n (anti)selfdual vector fields, all of the form (3.49). It would be interesting to find out if there are more general Euclidean BPS solutions of this type. We leave this for further study.

4

N=2 supersymmetric gaugings with dyonic charges

In chapter 2 we introduced $N = 2$ supersymmetric theories based on a variety of supermultiplets. The vector multiplets contain gauge fields which can be associated with a certain gauge group. So far we only considered gauge groups that are abelian. In this chapter (which is based on [105]) we present the extension to non-abelian gauge groups. Because of supersymmetry, all the fields of the vector supermultiplet will transform under this non-abelian group. In particular the target-space parameterized by the vector multiplet scalars must possess associated isometries. In addition, we study models in which a subgroup of the isometries of the hypermultiplet target-space is associated with the local gauge group (which can be both abelian and non-abelian). This introduces a coupling between vector and hypermultiplets. In principle, these $N = 2$ supersymmetric gauge theories are well known for the case where all the charges are electric. A distinct feature of the approach followed in this chapter is that we consider theories that may have both electric *and* magnetic (dyonic) charges.

As explained in chapter 2, the rigid invariance group of the abelian supersymmetric gauge theories is not necessarily an invariance of the action (in the sense that the action does not need to transform as an invariant function) but of the set of Bianchi identities and equations of motion. The group that can be realized on this set is a subgroup of the electric/magnetic duality group, $Sp(2n, \mathbb{R})$, where n denotes the number of independent vector multiplets. Because this group rotates electric into magnetic fields and vice versa, there is, for a given Lagrangian, the option of having both electric and magnetic charges. Because vector multiplets only provide the gauge fields that couple to electric charges, the standard approach is therefore to apply an electric/magnetic duality transformation to the ungauged Lagrangian, so that all the charges that one intends to introduce will be electric. In other words, one first converts the Lagrangian to a suitable electric/magnetic duality frame after which one switches on purely electric charges corresponding to a certain gauge group.

This approach is somewhat inconvenient. To set up a more general framework requires the introduction of electric and magnetic gauge fields on a par. Such a framework has been proposed in [25] and it allows to introduce a gauging irrespective of the choice of the duality frame. It incorporates both electric and magnetic charges and their corresponding gauge fields. The former are encoded in terms of a so-called embedding tensor, which determines the embedding of the gauge group into the full rigid invariance group. This embedding tensor is treated as a spurionic object, so that the electric/magnetic duality structure of the ungauged theory is preserved after charges are turned on. Besides introducing a set of dual magnetic gauge fields, the framework requires the introduction of a number of tensor fields, transforming in the adjoint representation of the rigid invariance group. These extra fields carry additional off-shell degrees of freedom. The number of physical degrees of freedom remains the same, owing to extra gauge transformations that are associated with the tensor fields.

Besides avoiding the need for performing duality transformations of the Lagrangian prior to switching on the gauging, the more general framework is important for a variety of other reasons. For instance, the scalar potential (and other, masslike, terms) that accompany the gaugings can be formulated in a way that is independent of the electric/magnetic duality frame. By introducing both electric and magnetic charges the potential will thus fully exhibit the duality invariances. This is of interest, when studying flux compactifications in string theory, because the underlying fluxes are usually subject to integer-valued rotations associated to the non-trivial cycles of the underlying internal manifold. Furthermore, the fact that tensor gauge fields are involved in the procedure relates to earlier examples of more general gaugings (see for instance [23]).

In this chapter we show explicitly how to apply the formalism of [25] to $N = 2$ gauge theories based on vector multiplets and hypermultiplets. In other words, by introducing dyonic charges, we gauge the isometries in a symplectically covariant way in both the special Kähler and the hyper-Kähler sector of the target-space parameterized by the scalar fields associated with the vector multiplets and hypermultiplets.

The supersymmetric Lagrangians we will derive in sections 4.3 and 4.4 introduce gaugings in both the vector and hypermultiplet sectors. Although the vector multiplets are off-shell multiplets, the presence of the magnetic charges introduces a breakdown of off-shell supersymmetry. The hypermultiplets are also sensitive to this, but they are not based on an off-shell representation of the supersymmetry algebra prior to introducing the charges. It is an interesting question whether the results of this chapter can be reformulated in an off-shell form and we will reflect on this in section 4.5.

In section 4.5 we also discuss some applications of our results, concerning for instance Fayet-Iliopoulos terms. Other possible applications are mainly in supergravity, where our work may be useful in constructing low energy effective actions

corresponding to string theory flux compactifications.

This chapter is organized as follows. In section 4.1 we repeat the relevant features of $N = 2$ vector multiplets and their behavior under electric/magnetic duality. Furthermore, we explain how dyonic non-abelian charges are introduced in models of this type. In section 4.2 we discuss how the gauge group is embedded into the rigid invariance group by means of the embedding tensor. Section 4.3 deals with the restoration of supersymmetry in vector multiplet models after gauging, while section 4.4 gives the extension with hypermultiplets. In section 4.5 we summarize the results obtained, and as we mentioned, indicate some of their applications and discuss some features related to the off-shell structure of these theories.

4.1 Vector multiplets and non-abelian charges

As explained in section 2.2, an off-shell $N = 2$ vector multiplet consists of a vector gauge field, A_μ , a complex scalar X , two Majorana fermions, Ω_i , and an auxiliary bosonic field Y_{ij} , satisfying the reality condition $(Y_{ij})^* = \varepsilon^{ik}\varepsilon^{jl}Y_{kl}$. The supersymmetric Lagrangian of n of such multiplets is encoded in terms of a function $F(X)$. Its rigid version is given by (2.7).

The Lagrangian of this model is subject to electric/magnetic duality transformations, which, we remind, act as

$$\begin{pmatrix} F_{\mu\nu}^{\pm\Lambda} \\ G_{\mu\nu\Lambda}^{\pm} \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{F}_{\mu\nu}^{\pm\Lambda} \\ \tilde{G}_{\mu\nu\Lambda}^{\pm} \end{pmatrix} = \begin{pmatrix} U_{\Sigma}^{\Lambda} & Z^{\Lambda\Sigma} \\ W_{\Lambda\Sigma} & V_{\Lambda}^{\Sigma} \end{pmatrix} \begin{pmatrix} F_{\mu\nu}^{\pm\Sigma} \\ G_{\mu\nu\Sigma}^{\pm} \end{pmatrix}, \quad (4.1)$$

where the matrix involved is from the group $Sp(2n, \mathbb{R})$. Here

$$G_{\mu\nu\Lambda}^+ = i\varepsilon_{\mu\nu\rho\sigma} \frac{\delta \mathcal{L}_{\text{vector}}}{\delta F_{\rho\sigma}^{+\Lambda}}. \quad (4.2)$$

Except for the Y_{ij}^Λ -sector¹, the Lagrangian obtained after such a duality transformation can be written back in the form (2.7), using the function \tilde{F} , the scalars \tilde{X}^Λ and the fermions $\tilde{\Omega}_i^\Lambda$, as obtained from a similar symplectic transformation on (X^Λ, F_Λ) and $(\Omega_i^\Lambda, F_{\Lambda\Sigma}\Omega_i^\Sigma)$.

In the following we find it convenient to use the notation $\alpha^M = (\alpha^\Lambda, \alpha_\Lambda)$ for symplectic vectors. So we get

$$\begin{aligned} G_{\mu\nu}^{\pm M} &= (F_{\mu\nu}^{\pm\Lambda}, G_{\mu\nu\Lambda}^{\pm}), \\ X^M &= (X^\Lambda, F_\Lambda), \\ \Omega_i^M &= (\Omega_i^\Lambda, F_{\Lambda\Sigma}\Omega_i^\Sigma). \end{aligned} \quad (4.3)$$

¹As we saw in section 2.2, to reobtain an expression of the form (2.7) including the Y_{ij}^Λ -sector, $(Y_{ij}^\Lambda, Z_{ij\Lambda})$ and $(Y^{ij\Lambda}, Z^{ij}_\Lambda)$ (with $Z_{ij\Lambda}$ and Z^{ij}_Λ given by (2.25) and (2.28)) should transform as symplectic vectors as well. This comes down to a rotation of the reality conditions on Y_{ij}^Λ and the equations of motion of these fields.

Likewise we use vectors with lower indices, $\beta_M = (\beta_\Lambda, \beta^\Lambda)$, transforming according to the conjugate representation such that $\alpha^M \beta_M$ is invariant.

We are especially interested in the subgroup of electric/magnetic duality transformations that leaves the function F (and therefore the Lagrangian) invariant,

$$\tilde{F}(\tilde{X}) = F(\tilde{X}) , \quad (4.4)$$

as these are the ones that can be gauged. We note that for such an invariance one gets

$$\begin{aligned} F_\Lambda(\tilde{X}) &= V_\Lambda^\Sigma F_\Sigma(X) + W_{\Lambda\Sigma} X^\Sigma , \\ F_{\Lambda\Sigma}(\tilde{X}) &= (V_\Lambda^\Gamma F_{\Gamma\Sigma} + W_{\Lambda\Sigma}) [\mathcal{S}^{-1}]^\Xi_\Sigma , \\ F_{\Lambda\Sigma\Gamma}(\tilde{X}) &= F_{\Xi\Delta\Omega} [\mathcal{S}^{-1}]^\Xi_\Lambda [\mathcal{S}^{-1}]^\Delta_\Sigma [\mathcal{S}^{-1}]^\Omega_\Gamma . \end{aligned} \quad (4.5)$$

where we recall that $\mathcal{S}^\Lambda_\Sigma = \partial\tilde{X}^\Lambda/\partial X^\Sigma = U^\Lambda_\Sigma + Z^{\Lambda\Gamma} F_{\Gamma\Sigma}$.

We elucidate these invariances for the subgroup that acts linearly on the gauge fields A_μ^Λ . These symmetries are characterized by the fact that the matrix in (4.1) has a block-triangular form with $V = [U^T]^{-1}$ and $Z = 0$. Hence this is not a general duality as the Lagrangian is still based on the same gauge fields, up to the linear transformation $A_\mu^\Lambda \rightarrow \tilde{A}_\mu^\Lambda = U^\Lambda_\Sigma A_\mu^\Sigma$. All fields in the Lagrangian (2.7) carry upper indices and are thus subject to the same linear transformation. The function $F(X)$ changes with an additive term which is a quadratic polynomial with real coefficients.

$$\tilde{F}(\tilde{X}) = F(U^\Lambda_\Sigma X^\Sigma) = F(X) + \frac{1}{2} (U^T W)_{\Lambda\Sigma} X^\Lambda X^\Sigma . \quad (4.6)$$

This term induces a total derivative term in the Lagrangian, equal to

$$\mathcal{L} \rightarrow \mathcal{L} - \frac{1}{8} i \varepsilon^{\mu\nu\rho\sigma} (U^T W)_{\Lambda\Sigma} F_{\mu\nu}^\Lambda F_{\rho\sigma}^\Sigma . \quad (4.7)$$

4.1.1 Gauge transformations

Non-abelian gauge groups will act non-trivially on the vector fields and must therefore involve a subgroup of the duality group. The electric gauge fields A_μ^Λ associated with this gauge group are provided by vector multiplets. Because the duality group acts on both electric and magnetic charges, in view of the fact that it mixes field strengths with dual field strengths as shown by (4.1), we will eventually have to introduce magnetic gauge fields $A_{\mu\Lambda}$ as well, following the procedure explained in [25]. The $2n$ gauge fields A_μ^M will then comprise both type of fields, $A_\mu^M = (A_\mu^\Lambda, A_{\mu\Lambda})$. The role played by the magnetic gauge fields will be clarified later. For the moment one may associate $A_{\mu\Lambda}$ with the dual field strengths $G_{\mu\nu\Lambda}$, by writing $G_{\mu\nu\Lambda} \equiv 2 \partial_{[\mu} A_{\nu]\Lambda}$.

The generators (as far as their embedding in the duality group is concerned) are defined as follows. The generators of the subgroup that is gauged, are $2n$ -by- $2n$

matrices T_M , where we are assuming the presence of both electric and magnetic gauge fields, so that the generators decompose according to $T_M = (T_\Lambda, T^\Lambda)$. Obviously $T_{\Lambda N}^P$ and T^Λ_{NP} can be decomposed into the generators of the duality group and are thus of the form specified in (4.1). Denoting the gauge group parameters by $\Lambda^M(x) = (\Lambda^\Lambda(x), \Lambda_\Lambda(x))$, $2n$ -dimensional $\text{Sp}(2n; \mathbb{R})$ vectors α^M and β_M transform according to

$$\delta\alpha^M = -g\Lambda^N T_{NP}^M \alpha^P, \quad \delta\beta_M = g\Lambda^N T_{NM}^P \beta_P, \quad (4.8)$$

where g denotes a universal gauge coupling constant. Covariant derivatives thus take the form,

$$\begin{aligned} D_\mu\alpha^M &= \partial_\mu\alpha^M + gA_\mu^N T_{NP}^M \alpha^P \\ &= \partial_\mu\alpha^M + gA_\mu^\Lambda T_{\Lambda P}^M \alpha^P + gA_{\mu\Lambda} T^\Lambda_{P^M} \alpha^P, \end{aligned} \quad (4.9)$$

and similarly for $D_\mu\beta_M$. The gauge fields then transform according to

$$\delta A_\mu^M = \partial_\mu\Lambda^M + g T_{PQ}^M A_\mu^P \Lambda^Q. \quad (4.10)$$

Electric charges

For clarity we first consider electric gaugings where the gauge transformations have a block-triangular form and there are only electric gauge fields. Hence we ignore the fields $A_{\mu\Lambda}$ and assume $T^\Lambda_{NP} = 0$ and $T_{\Lambda\Sigma}^\Gamma = 0$. All the fields in the Lagrangian carry upper indices, so that they will transform as in $\delta X^\Lambda = -g\Lambda^\Gamma T_{\Gamma\Sigma}^\Lambda X^\Sigma$. The transformation rule for A_μ^Λ given above is in accord with this expression, provided we assume that $T_{\Gamma\Sigma}^\Lambda$ is antisymmetric in Γ and Σ . This has to be the case here as consistency requires that the $T_{\Gamma\Sigma}^\Lambda$ are structure constants of the non-abelian group. In the more general situation discussed in later sections, this is not necessarily the case. The embedding into $\text{Sp}(2n, \mathbb{R})$ implies furthermore that $T_{\Lambda\Sigma}^\Gamma = -T_{\Lambda^\Sigma}^\Gamma$, while the non-vanishing left-lower block $T_{\Lambda\Sigma\Gamma}$ is symmetric in Σ and Γ . Furthermore, we note that (4.6) implies

$$F_\Lambda(X) \delta X^\Lambda = -g\Lambda^\Gamma T_{\Gamma\Sigma}^\Lambda F_\Lambda(X) X^\Sigma = -\frac{1}{2}g\Lambda^\Lambda T_{\Lambda\Sigma\Gamma} X^\Sigma X^\Gamma. \quad (4.11)$$

Upon replacing Λ^Λ with X^Λ we conclude that the fully symmetric part of $T_{\Lambda\Sigma\Gamma}$ vanishes. This, and the closure of the gauge group, leads to the following three equations,

$$\begin{aligned} T_{(\Lambda\Sigma\Gamma)} &= 0, \\ T_{[\Lambda\Sigma}^\Delta T_{\Gamma]\Delta}^\Xi &= 0, \\ 4 T_{(\Gamma[\Lambda}^\Delta T_{\Sigma]\Xi)\Delta} - T_{\Lambda\Sigma}^\Delta T_{\Delta\Gamma\Xi} &= 0. \end{aligned} \quad (4.12)$$

The variation of the Lagrangian under gauge transformations now takes the form

$$\mathcal{L} \rightarrow \mathcal{L} + \frac{1}{8} i \varepsilon^{\mu\nu\rho\sigma} \Lambda^\Lambda T_{\Lambda\Sigma\Gamma} \mathcal{F}_{\mu\nu}{}^\Sigma \mathcal{F}_{\rho\sigma}{}^\Gamma . \quad (4.13)$$

where the tensors $\mathcal{F}_{\mu\nu}{}^\Lambda$ denote the non-abelian field strengths,

$$\mathcal{F}_{\mu\nu}{}^\Lambda = \partial_\mu A_\nu{}^\Lambda - \partial_\nu A_\mu{}^\Lambda + g T_{\Sigma\Gamma}{}^\Lambda A_\mu{}^\Sigma A_\nu{}^\Gamma . \quad (4.14)$$

This result implies that (4.13) no longer constitutes a total derivative in view of the spacetime dependent transformation parameters $\Lambda^\Lambda(x)$. Therefore its cancellation requires to add a new type of term [10],

$$\mathcal{L} = \frac{1}{3} i g \varepsilon^{\mu\nu\rho\sigma} T_{\Lambda\Sigma\Gamma} A_\mu{}^\Lambda A_\nu{}^\Sigma (\partial_\rho A_\sigma{}^\Gamma + \frac{3}{8} g T_{\Xi\Delta}{}^\Gamma A_\rho{}^\Xi A_\sigma{}^\Delta) . \quad (4.15)$$

No other terms in the action will depend on $T_{\Lambda\Sigma\Gamma}$. At this point we should remind the reader that the gauging breaks supersymmetry, unless one adds the standard masslike and potential terms to the Lagrangian (2.7), which involve the $T_{\Lambda\Sigma}{}^\Gamma$. We present them below for completeness,

$$\begin{aligned} \mathcal{L}_g &= -\frac{1}{2} g N_{\Lambda\Sigma} T_{\Gamma\Xi}{}^\Sigma \left[\varepsilon^{ij} \bar{\Omega}_i{}^\Lambda \Omega_j{}^\Gamma \bar{X}^\Xi + \varepsilon_{ij} \bar{\Omega}^{i\Lambda} \Omega^{j\Gamma} X^\Xi \right] , \\ \mathcal{L}_{g^2} &= g^2 N_{\Lambda\Sigma} T_{\Gamma\Xi}{}^\Lambda \bar{X}^\Gamma X^\Xi T_{\Delta\Omega}{}^\Sigma \bar{X}^\Delta X^\Omega . \end{aligned} \quad (4.16)$$

In later sections we will exhibit the generalization of these terms to the case where both electric and magnetic charges are present.

Electric and magnetic charges

We now consider more general gauge groups without restricting ourselves to electric charges. Therefore we have to include both electric gauge fields $A_\mu{}^\Lambda$ and magnetic gauge fields $A_{\mu\Lambda}$. Only a subset of these fields is usually involved in the gauging, but the additional magnetic gauge fields could conceivably lead to new propagating degrees of freedom. We will discuss in due course how this is avoided. In the remainder of this section we will consider the scalar and spinor fields. The treatment of the vector fields is more involved and is explained in section 4.2.

The charges $T_{MN}{}^P$ correspond to a more general subgroup of the duality group. Hence they must take values in the Lie algebra associated with $\text{Sp}(2n, \mathbb{R})$, which implies,

$$T_{M[N}{}^Q \Omega_{P]Q} = 0 . \quad (4.17)$$

Combining the two equations (2.23) and (4.4) leads to the condition [10],

$$T_{MN}{}^Q \Omega_{PQ} X^N X^P = T_{M\Lambda\Sigma} X^\Lambda X^\Sigma - 2 T_{M\Lambda}{}^\Sigma X^\Lambda F_\Sigma - T_M{}^{\Lambda\Sigma} F_\Lambda F_\Sigma = 0 . \quad (4.18)$$

This result can also be written as

$$F_\Lambda \delta X^\Lambda = -\frac{1}{2} \Lambda^M \left(T_{M\Lambda\Sigma} X^\Lambda X^\Sigma + T_M^{\Lambda\Sigma} F_\Lambda F_\Sigma \right), \quad (4.19)$$

which generalizes (4.11). Furthermore, we impose the so-called representation constraint [25], which implies that we suppress a representation of the rigid symmetry group in T_{MN}^P ,

$$T_{(MN}^Q \Omega_{P)Q} = 0 \implies \begin{cases} T^{(\Lambda\Sigma\Gamma)} = 0, \\ 2T_{(\Gamma\Lambda)\Sigma} = T_\Sigma^{\Lambda\Gamma}, \\ T_{(\Lambda\Sigma\Gamma)} = 0, \\ 2T_{(\Gamma\Lambda)^\Sigma} = T_{\Lambda\Gamma}^\Sigma. \end{cases} \quad (4.20)$$

This constraint is a generalization of the first equation (4.12). Observe that the generators $T_{\Lambda\Sigma}^\Gamma$ are no longer antisymmetric in Λ and Σ , a feature that we will discuss in more detail in the following section.

Using (4.3) we can rewrite the Lagrangian (2.2) in a compact form,

$$\mathcal{L}_{\text{matter}} = -i\Omega_{MN} \partial_\mu X^M \partial^\mu \bar{X}^N + \frac{1}{4} i\Omega_{MN} \left[\bar{\Omega}^{iM} \not{\partial} \Omega_i^N - \bar{\Omega}_i^M \not{\partial} \Omega^{iN} \right]. \quad (4.21)$$

In the expressions on the right-hand side it is straightforward to replace the ordinary derivatives by the covariant ones defined in (4.9), i.e.,

$$\begin{aligned} D_\mu X^M &= \partial_\mu X^M + g A_\mu^N T_{NP}^M X^P, \\ D_\mu \Omega_i^M &= \partial_\mu \Omega_i^M + g A_\mu^N T_{NP}^M \Omega_i^P, \end{aligned} \quad (4.22)$$

and evaluate the gauge couplings. In particular we can then compare to the results of subsection 4.1.1, where we considered only electric gauge fields with charges restricted by $T_\Lambda^{\Sigma\Gamma} = 0$. To do this systematically we note the identity,

$$T_{MN\Lambda} X^N - F_{\Lambda\Sigma} T_{MN}^\Sigma X^N = 0. \quad (4.23)$$

This equation can also be written as $F_{\Lambda\Sigma} \delta X^\Sigma = -\Lambda^M T_{MN\Lambda} X^N$, which is the infinitesimal form of the first equation (4.5). Alternatively it can be derived from (4.18) upon differentiation with respect to X^Λ .

It is possible to cast (4.23) in a symplectic covariant form by introducing a vector $U^M = (U^\Lambda, F_{\Sigma\Gamma} U^\Gamma)$, so that

$$\Omega_{MQ} T_{NP}^Q X^P U^M = 0, \quad (4.24)$$

for any such vector U^M . This form is convenient in calculations presented later. From (4.23) one easily derives that $D_\mu X_\Lambda = D_\mu F_\Lambda = F_{\Lambda\Sigma} D_\mu X^\Sigma$, which enables one to derive

$$-i\Omega_{MN} D_\mu X^M D^\mu \bar{X}^N = -N_{\Lambda\Sigma} D_\mu X^\Lambda D^\mu \bar{X}^\Sigma, \quad (4.25)$$

This result shows that the generators $T_{M\Lambda\Sigma}$ are absent, in accord with what was found in subsection 4.1.1.

Next we consider the gauge field interactions with the fermions. It is convenient to first derive an additional identity, which follows from taking a supersymmetry variation of (4.23),

$$T_{MNA}\Omega_i^N = F_{\Lambda\Sigma} T_{MN}{}^\Sigma\Omega_i^N + F_{\Lambda\Sigma\Gamma}\Omega_i^\Sigma T_{MN}{}^\Gamma X^N . \quad (4.26)$$

This result can be obtained from the infinitesimal form of the third equation of (4.5). Using this equation one verifies that $D_\mu\Omega_{i\Lambda} = F_{\Lambda\Sigma} D_\mu\Omega_i^\Sigma + F_{\Lambda\Sigma\Gamma}\Omega_i^\Gamma D_\mu X^\Sigma$, which leads to

$$\begin{aligned} \frac{1}{4}i\Omega_{MN}\left[\bar{\Omega}^{iM}\not{D}\Omega_i^N - \bar{\Omega}_i^M\not{D}\Omega^{iN}\right] &= -\frac{1}{4}N_{\Lambda\Sigma}\left(\bar{\Omega}^{i\Lambda}\not{D}\Omega_i^\Sigma + \bar{\Omega}_i^\Lambda\not{D}\Omega^{i\Sigma}\right) \\ &\quad -\frac{1}{4}i\left(F_{\Lambda\Sigma\Gamma}\bar{\Omega}_i^\Lambda\not{D}X^\Sigma\Omega^{i\Gamma}\right. \\ &\quad \left.-\bar{F}_{\Lambda\Sigma\Gamma}\bar{\Omega}^{i\Lambda}\not{D}\bar{X}^\Sigma\Omega_i^\Gamma\right) . \end{aligned} \quad (4.27)$$

Again the generator $T_{M\Lambda\Sigma}$ is absent in the expression above. The results of this subsection explain how to introduce the electric and magnetic charges, but in no way ensure the gauge invariance or the supersymmetry of the Lagrangian. To obtain such a result we first need to explain some more general features of theories with both electric and magnetic gauge fields in four spacetime dimensions. This is the topic of the following section.

As a side remark we note that the moment map associated with the isometries considered above, takes the form,

$$\nu_M = T_{MN}{}^Q\Omega_{PQ}\bar{X}^N X^P . \quad (4.28)$$

Indeed, making use again of (4.23), one straightforwardly derives $\partial_\Lambda\nu_M = iN_{\Lambda\Sigma}\delta\bar{X}^\Sigma$.

4.2 The gauge group and the embedding tensor

Here we follow [25] and discuss the embedding of possible gauge groups into the rigid invariance group G_{rigid} of the theory. In the context of our work, the latter is often a product group as the vector multiplets and the hypermultiplets are invariant under independent symmetry groups. As explained in the previous section the non-abelian gauge transformations on the vector multiplets are necessarily embedded into the electric/magnetic duality group.

It is convenient to discuss group embeddings in terms of a so-called embedding tensor Θ_M^a which specifies the decomposition of the gauge group generators T_M into the generators associated with the full rigid invariance group G_{rigid} ,

$$T_M = \Theta_M^a t_a . \quad (4.29)$$

Not all the gauge fields have to be involved in the gauging, so generically the embedding tensor projects out certain combinations of gauge fields; the rank of the tensor determines the dimension of the gauge group, up to central extensions associated with abelian factors. Decomposing the embedding tensor as $\Theta_M^a = (\Theta_\Lambda^a, \Theta^{\Lambda a})$, covariant derivatives take the form,

$$D_\mu \equiv \partial_\mu - gA_\mu^M T_M = \partial_\mu - gA_\mu^\Lambda \Theta_\Lambda^a t_a - gA_{\mu\Lambda} \Theta^{\Lambda a} t_a . \quad (4.30)$$

The embedding tensor will be regarded as a spurionic object which can be assigned to a (not necessarily irreducible) representation of the rigid invariance group G_{rigid} . It is known that a number of (G_{rigid} -covariant) constraints must be imposed on the embedding tensor. We already encountered the representation constraint (4.20), which is linear in the embedding tensor. Two other constraints are quadratic in the embedding tensor and read,

$$f_{ab}{}^c \Theta_M^a \Theta_N^b + (t_a)_N{}^P \Theta_M^a \Theta_P^c = 0 , \quad (4.31)$$

$$\Omega^{MN} \Theta_M^a \Theta_N^b = 0 \iff \Theta^{\Lambda [a} \Theta_{\Lambda}^{b]} = 0 , \quad (4.32)$$

where the $f_{ab}{}^\gamma$ are the structure constants associated with the group G . The first constraint is required by the closure of the gauge group generators. Indeed, from (4.31) it follows that the gauge algebra generators close according to

$$[T_M, T_N] = -T_{MN}{}^P T_P , \quad (4.33)$$

where the structure constants of the gauge group coincide with $T_{MN}{}^P \equiv \Theta_M^a (t_a)_N{}^P$ up to terms that vanish upon contraction with the embedding tensor Θ_P^a . We recall that the $T_{MN}{}^P$ generate a subgroup of $\text{Sp}(2n, \mathbb{R})$ in the $(2n)$ -dimensional representation, so that they are subject to the condition (4.17). In electric/magnetic components the latter condition corresponds to $T_{M\Lambda}{}^\Sigma = -T_M{}^\Sigma{}_\Lambda$, $T_{M\Lambda\Sigma} = T_{M\Sigma\Lambda}$ and $T_M{}^{\Lambda\Sigma} = T_M{}^{\Sigma\Lambda}$.

Note that (4.31) implies that the embedding tensor is gauge invariant, while the second quadratic constraint (4.32) implies that the charges are mutually local, so that an electric/magnetic duality exists that converts all the charges to electric ones. These two quadratic constraints are not completely independent, as can be seen from symmetrizing the constraint (4.31) in (MN) and making use of the linear conditions (4.20) and (4.17). This leads to

$$\Omega^{MN} \Theta_M^a \Theta_N^b (t_b)_P{}^Q = 0 . \quad (4.34)$$

This shows that, for non-vanishing $(t_b)_P{}^Q$, the second quadratic constraint (4.32) is in fact a consequence of the other constraints. The constraint (4.32) is only an independent constraint when \mathbf{a} and \mathbf{b} do not refer to generators that act on the vector multiplets. This issue is relevant here as G_{rigid} may contain independent generators that act exclusively in the matter (i.e., hypermultiplet) sector.

A further consequence of (4.20) is the equation

$$T_{(MN)}{}^P = Z^{P,a} d_{aMN} , \quad (4.35)$$

with

$$\begin{aligned} d_{aMN} &\equiv (t_a)_M{}^P \Omega_{NP} , \\ Z^{M,a} &\equiv \frac{1}{2} \Omega^{MN} \Theta_N{}^a \implies \begin{cases} Z^{\Lambda a} = \frac{1}{2} \Theta^{\Lambda a} , \\ Z_{\Lambda}{}^a = -\frac{1}{2} \Theta_{\Lambda}{}^a , \end{cases} \end{aligned} \quad (4.36)$$

so that d_{aMN} defines a G_{rigid} -invariant tensor symmetric in (MN) . The gauge invariant tensor $Z^{M,a}$ will serve as a projector on the tensor fields to be introduced below [106]. We note that the constraint (4.32) can now be written as,

$$Z^{M,a} \Theta_M{}^b = 0 . \quad (4.37)$$

Let us return to the closure relation (4.33). Although the left-hand side is antisymmetric in M and N , this does not imply that $T_{MN}{}^P$ is antisymmetric as well, but only that its symmetric part vanishes upon contraction with the embedding tensor. Indeed, this is reflected by (4.35) and (4.37). Consequently, the Jacobi identity holds only modulo terms that vanish upon contraction with the embedding tensor, as is shown explicitly by

$$T_{[MN]}{}^P T_{[QP]}{}^R + T_{[QM]}{}^P T_{[NP]}{}^R + T_{[NQ]}{}^P T_{[MP]}{}^R = -Z^{R,a} d_{aP[Q} T_{MN]}{}^P . \quad (4.38)$$

To compensate for this lack of closure and, at the same time, to avoid unwanted degrees of freedom, we introduce an extra gauge invariance for the gauge fields, in addition to the usual non-abelian gauge transformations,

$$\delta A_{\mu}{}^M = D_{\mu} \Lambda^M - g Z^{M,a} \Xi_{\mu a} , \quad (4.39)$$

where the Λ^M are the gauge transformation parameters and the covariant derivative reads, $D_{\mu} \Lambda^M = \partial_{\mu} \Lambda^M + g T_{PQ}{}^M A_{\mu}{}^P \Lambda^Q$. The transformations proportional to $\Xi_{\mu a}$ enable one to gauge away those vector fields that are in the sector of the gauge generators $T_{MN}{}^P$ where the Jacobi identity is not satisfied (this sector is perpendicular to the embedding tensor by virtue of (4.37)). Note that the covariant derivative is invariant under the transformations parameterized by $\Xi_{\mu a}$, because of the contraction of the gauge fields $A_{\mu}{}^M$ with the generators T_M . The gauge symmetries parameterized by the functions $\Lambda^M(x)$ and $\Xi_{a\mu}(x)$ form a group, as follows from the commutation relations,

$$\begin{aligned} [\delta(\Lambda_1), \delta(\Lambda_2)] &= \delta(\Lambda_3) + \delta(\Xi_3) , \\ [\delta(\Lambda), \delta(\Xi)] &= \delta(\tilde{\Xi}) , \end{aligned} \quad (4.40)$$

where

$$\begin{aligned}\Lambda_3^M &= g T_{[NP]}^M \Lambda_2^N \Lambda_1^P, \\ \Xi_{3\mu a} &= d_{aNP} (\Lambda_1^N D_\mu \Lambda_2^P - \Lambda_2^N D_\mu \Lambda_1^P), \\ \tilde{\Xi}_{\mu a} &= g \Lambda^P (T_{Pa}{}^b + 2d_{aPN} Z^{N;b}) \Xi_{\mu b}.\end{aligned}\quad (4.41)$$

The field strengths follow from the Ricci identity, $[D_\mu, D_\nu] = -g \mathcal{F}_{\mu\nu}^M T_M$, and depend only on the antisymmetric part of T_{MN}^P ,

$$\mathcal{F}_{\mu\nu}^M = \partial_\mu A_\nu^M - \partial_\nu A_\mu^M + g T_{[NP]}^M A_\mu^N A_\nu^P. \quad (4.42)$$

Because of the lack of closure expressed by (4.38), they do not satisfy the Palatini identity,

$$\delta \mathcal{F}_{\mu\nu}^M = 2 D_{[\mu} \delta A_{\nu]}^M - 2g T_{(PQ)}^M A_{[\mu}^P \delta A_{\nu]}^Q, \quad (4.43)$$

under arbitrary variations δA_μ^M . Note that the last term cancels upon multiplication with the generators T_M . The result (4.43) shows that $\mathcal{F}_{\mu\nu}^M$ transforms under gauge transformations as

$$\delta \mathcal{F}_{\mu\nu}^M = g \Lambda^P T_{NP}^M \mathcal{F}_{\mu\nu}^N - 2g Z^{M;a} (D_{[\mu} \Xi_{\nu]a} + d_{aPQ} A_{[\mu}^P \delta A_{\nu]}^Q), \quad (4.44)$$

and is therefore not covariant. The standard strategy is therefore to define modified field strengths,

$$\mathcal{H}_{\mu\nu}^M = \mathcal{F}_{\mu\nu}^M + g Z^{M;a} B_{\mu\nu a}, \quad (4.45)$$

by introducing new tensor fields $B_{\mu\nu a}$ with suitably chosen gauge transformation rules, so that covariant results can be obtained.

At this point we remind the reader that the invariance transformations in the rigid case implied that the field strengths $G_{\mu\nu}^M$ transform under a subgroup of $\text{Sp}(2n, \mathbb{R})$ (c.f. (4.1)). Our aim is to find a similar symplectic vector of field strengths so that these transformations are generated in the non-abelian case as well. This is not possible based on the variations of the vector fields A_μ^M , which will never generate the type of fermionic terms contained in $G_{\mu\nu\Lambda}$. However, the presence of the tensor fields enables us to achieve our objectives, at least in part. Just as in the abelian case, we define an $\text{Sp}(2n, \mathbb{R})$ vector of field strengths $\mathcal{G}_{\mu\nu}^M$ by

$$\begin{aligned}\mathcal{G}_{\mu\nu}^{-\Lambda} &= \mathcal{H}_{\mu\nu}^{-\Lambda}, \\ \mathcal{G}_{\mu\nu\Lambda}^{-} &= F_{\Lambda\Sigma} \mathcal{H}_{\mu\nu}^{-\Sigma} - \frac{1}{8} F_{\Lambda\Sigma\Gamma} \bar{\Omega}_i{}^\Sigma \gamma_{\mu\nu} \Omega_j{}^\Gamma \varepsilon^{ij}.\end{aligned}\quad (4.46)$$

Note that the expression for $\mathcal{G}_{\mu\nu\Lambda}$ is the analogue of (2.18), with $F_{\mu\nu}^\Lambda$ replaced by $\mathcal{H}_{\mu\nu}^\Lambda$.

Following [25] we introduce the following transformation rule for $B_{\mu\nu a}$ (contracted with $Z^{M;a}$, because only these combinations will appear in the Lagrangian),

$$Z^{M;a} \delta B_{\mu\nu a} = 2 Z^{M;a} (D_{[\mu} \Xi_{\nu]a} + d_{aNP} A_{[\mu}^N \delta A_{\nu]}^P) - 2 T_{(NP)}^M \Lambda^P \mathcal{G}_{\mu\nu}^N, \quad (4.47)$$

where $D_\mu \Xi_{\nu a} = \partial_\mu \Xi_{\nu a} - g A_\mu^M T_{Ma}^b \Xi_{\nu b}$ with $T_{Ma}^b = -\Theta_M^\gamma f_{\gamma a}^b$ the gauge group generator in the adjoint representation of G_{rigid} . With this variation the modified field strengths (4.45) are invariant under tensor gauge transformations. Under the vector gauge transformations we derive the following result,

$$\begin{aligned}\delta \mathcal{G}_{\mu\nu}^{-\Lambda} &= -g \Lambda^P T_{PN}^\Lambda \mathcal{G}_{\mu\nu}^{-N} - g \Lambda^P T_{P\Lambda}^\Gamma (\mathcal{G}_{\mu\nu}^- - \mathcal{H}_{\mu\nu}^-)_\Gamma, \\ \delta \mathcal{G}_{\mu\nu\Lambda}^- &= -g \Lambda^P T_{PN\Lambda} \mathcal{G}_{\mu\nu}^{-N} - g F_{\Lambda\Sigma} \Lambda^P T_{P\Sigma}^\Gamma (\mathcal{G}_{\mu\nu}^- - \mathcal{H}_{\mu\nu}^-)_\Gamma, \\ \delta (\mathcal{G}_{\mu\nu}^- - \mathcal{H}_{\mu\nu}^-)_\Lambda &= g \Lambda^P (T_{P\Lambda}^\Gamma - T_{P\Sigma}^\Gamma F_{\Sigma\Lambda}) (\mathcal{G}_{\mu\nu}^- - \mathcal{H}_{\mu\nu}^-)_\Gamma.\end{aligned}\quad (4.48)$$

Hence $\delta \mathcal{G}_{\mu\nu}^M = -g \Lambda^P T_{PN}^M \mathcal{G}_{\mu\nu}^N$, just as the variation of the abelian field strengths $G_{\mu\nu}^M$ in the absence of charges, up to terms that are proportional to $\Theta^{\Lambda a} (\mathcal{G}_{\mu\nu}^- - \mathcal{H}_{\mu\nu}^-)_\Lambda$. According to [25], the latter terms represent a set of field equations. In that case the last equation of (4.48) expresses the well-known fact that, under a symmetry, field equations transform into field equations. As a result the gauge algebra on these tensors closes according to (4.40), up to the same field equations. In order that the Lagrangian becomes invariant under the vector and tensor gauge transformations, we have to make a number of changes. First of all, we replace the abelian field strengths $F_{\mu\nu}^\Lambda$ in (2.13) by $\mathcal{H}_{\mu\nu}^\Lambda$, so that

$$\mathcal{G}_{\mu\nu\Lambda} = i \varepsilon_{\mu\nu\rho\sigma} \frac{\delta \mathcal{L}_{\text{vector}}}{\delta \mathcal{H}_{\rho\sigma}^\Lambda}. \quad (4.49)$$

Under general variations of the vector and tensor fields we then obtain the result,

$$\delta \mathcal{L}_{\text{vector}} = -i \mathcal{G}^{+\mu\nu}{}_\Lambda \left[D_\mu \delta A_\nu^\Lambda + \frac{1}{4} g \Theta^{\Lambda a} (\delta B_{\mu\nu a} - 2 d_{aPQ} A_\mu^P \delta A_\nu^Q) \right] + \text{h.c.} \quad (4.50)$$

The reader can check that the Lagrangian (2.13) (with $F_{\mu\nu}^\Lambda$ replaced by $\mathcal{H}_{\mu\nu}^\Lambda$) is indeed invariant under the tensor gauge transformations. Even when we include the transformations of the scalar and spinor fields, the Lagrangian is, however, not yet invariant under the vector gauge transformations. For that it is necessary to introduce the following universal terms to the Lagrangian [25],

$$\begin{aligned}\mathcal{L}_{\text{top}} &= \frac{1}{8} i g \varepsilon^{\mu\nu\rho\sigma} \Theta^{\Lambda a} B_{\mu\nu a} \left(2 \partial_\rho A_{\sigma\Lambda} + g T_{MN\Lambda} A_\rho^M A_\sigma^N - \frac{1}{4} g \Theta_\Lambda^b B_{\rho\sigma b} \right) \\ &+ \frac{1}{3} i g \varepsilon^{\mu\nu\rho\sigma} T_{MN\Lambda} A_\mu^M A_\nu^N \left(\partial_\rho A_\sigma^\Lambda + \frac{1}{4} g T_{PQ}^\Lambda A_\rho^P A_\sigma^Q \right) \\ &+ \frac{1}{6} i g \varepsilon^{\mu\nu\rho\sigma} T_{MN}^\Lambda A_\mu^M A_\nu^N \left(\partial_\rho A_{\sigma\Lambda} + \frac{1}{4} g T_{PQ\Lambda} A_\rho^P A_\sigma^Q \right).\end{aligned}\quad (4.51)$$

The first term represents a topological coupling of the antisymmetric tensor fields with the magnetic gauge fields, and the last two terms are a generalization of the Chern-Simons-like terms (4.15) that we encountered in subsection 4.1.1. Under

variations of the vector and tensor fields, this Lagrangian varies into (up to total derivative terms)

$$\delta\mathcal{L}_{\text{top}} = i\mathcal{H}^{+\mu\nu\Lambda} D_\mu\delta A_{\nu\Lambda} + \frac{1}{4}ig\mathcal{H}^{+\mu\nu}{}_\Lambda\Theta^{\Lambda a}(\delta B_{\mu\nu a} - 2d_{aPQ}A_\mu{}^P\delta A_\nu{}^Q) + \text{h.c.} . \quad (4.52)$$

Under the gauge transformations associated with the tensor fields $B_{\mu\nu a}$ this variation becomes equal to $(ig\mathcal{H}^{+\mu\nu M}\Theta_M{}^a D_\mu\Xi_{\nu a} + \text{h.c.})$. This expression equals a total derivative by virtue of the invariance of the embedding tensor, the Bianchi identity - which reads $D_{[\mu}\mathcal{H}_{\nu\rho]}{}^M = \frac{1}{3}gZ^{M,a}\mathcal{H}_{\mu\nu\rho a}$ - and (4.37).

In the Bianchi identity mentioned above, $D_\mu\mathcal{H}_{\nu\rho}{}^M = \partial_\mu\mathcal{H}_{\nu\rho}{}^M + gA_\mu{}^PT_{PN}{}^M\mathcal{H}_{\nu\rho}{}^N$ and $\mathcal{H}_{\mu\nu\rho a}$ denotes a field strength associated with the tensor fields. The expression for the Bianchi identity given above is suitable for our purpose here, but we note that it is not covariant in this form, in view of the fact that the fully covariant derivative of $\mathcal{H}_{\mu\nu}{}^M$ reads,

$$\mathcal{D}_\rho\mathcal{H}_{\mu\nu}{}^M = \partial_\rho\mathcal{H}_{\mu\nu}{}^M + gA_\rho{}^PT_{PN}{}^M\mathcal{G}_{\mu\nu}{}^N + gA_\rho{}^PT_{NP}{}^M(\mathcal{G}_{\mu\nu} - \mathcal{H}_{\mu\nu})_N , \quad (4.53)$$

and the covariant field strength of the tensor fields equals

$$\begin{aligned} \mathcal{H}_{\mu\nu\rho a} \equiv & 3D_{[\mu}B_{\nu\rho]a} + 6d_{aMN}A_{[\mu}{}^M(\partial_{\nu}A_{\rho]}{}^N \\ & + \frac{1}{3}gT_{[RS]}{}^NA_\nu{}^RA_{\rho]}{}^S + \mathcal{G}_{\nu\rho]}{}^N - \mathcal{H}_{\nu\rho]}{}^N) , \end{aligned} \quad (4.54)$$

where $D_\rho B_{\mu\nu a} = \partial_\rho B_{\mu\nu a} - gA_\rho{}^MT_{Ma}{}^bB_{\mu\nu b}$. With these definitions the covariant form of the Bianchi identity holds,

$$\mathcal{D}_{[\mu}\mathcal{H}_{\nu\rho]}{}^M = \frac{1}{3}gZ^{M,a}\mathcal{H}_{\mu\nu\rho a} . \quad (4.55)$$

These modifications ensure the gauge invariance of the total Lagrangian $\mathcal{L}_{\text{vector}} + \mathcal{L}_{\text{top}}$, provided we include the gauge transformations of the scalar and spinor fields [25]. Furthermore, variation of the tensor fields yields the field equations identified above,

$$\delta\mathcal{L}_{\text{vector}} + \delta\mathcal{L}_{\text{top}} = -\frac{1}{4}ig\delta B_{\mu\nu a}\Theta^{\Lambda a}\left[(\mathcal{G}^{+\mu\nu} - \mathcal{H}^{+\mu\nu})_\Lambda - (\mathcal{G}^{-\mu\nu} - \mathcal{H}^{-\mu\nu})_\Lambda\right] . \quad (4.56)$$

In spite of the modifications above, supersymmetry will be broken by the gauging. In the next section we show it can be restored.

4.3 Restoring supersymmetry for non-abelian vector multiplets

In this section we show how the supersymmetry can be restored in the presence of a gauging. In this way we will find the generalizations of the masslike and potential

terms of order g and g^2 , respectively, which was already exhibited in (4.16) for the case of electric charges. In addition we determine the corresponding changes in the transformation rules.

The supersymmetry transformations that leave the ungauged action (2.7) invariant, we recall, are given by

$$\begin{aligned}
\delta X^\Lambda &= \bar{\epsilon}^i \Omega_i^\Lambda, \\
\delta A_\mu^\Lambda &= \varepsilon^{ij} \bar{\epsilon}_i \gamma_\mu \Omega_j^\Lambda + \varepsilon_{ij} \bar{\epsilon}^i \gamma_\mu \Omega^j{}^\Lambda, \\
\delta \Omega_i^\Lambda &= 2\bar{\not{\partial}} X^\Lambda \epsilon_i + \frac{1}{2} \gamma^{\mu\nu} F_{\mu\nu}{}^{-\Lambda} \varepsilon_{ij} \epsilon^j + Y_{ij}{}^\Lambda \epsilon^j, \\
\delta Y_{ij}{}^\Lambda &= 2\bar{\epsilon}_{(i} \not{\partial} \Omega_{j)}^\Lambda + 2\varepsilon_{ik} \varepsilon_{jl} \bar{\epsilon}^{(k} \not{\partial} \Omega^{l)\Lambda}.
\end{aligned} \tag{4.57}$$

The extension of these transformations in the presence of electric charges is known [51]. Therefore we will now proceed and consider the case of electric and/or magnetic charges.

Introducing the charges, with a uniform gauge coupling constant g as before, we have already discussed some universal changes of the Lagrangian in the previous section. In $\mathcal{L}_{\text{matter}}$ we have to covariantize the derivatives as already discussed in section 4.1.1. It is convenient to use the representation (4.21). With the covariantizations included we thus have

$$\mathcal{L}_{\text{matter}} = -i\Omega_{MN} D_\mu X^M D^\mu \bar{X}^N + \frac{1}{4} i\Omega_{MN} \left[\bar{\Omega}^{iM} \not{D} \Omega_i{}^N - \bar{\Omega}_i{}^M \not{D} \Omega^{iN} \right]. \tag{4.58}$$

In $\mathcal{L}_{\text{vector}}$ we must replace the abelian field strengths $F_{\mu\nu}{}^\Lambda$ by the modified field strengths $\mathcal{H}_{\mu\nu}{}^\Lambda$, defined in (4.45). Therefore we replace (2.13) by

$$\begin{aligned}
\mathcal{L}_{\text{vector}} &= \left(\frac{1}{4} i F_{\Lambda\Sigma} \mathcal{H}_{\mu\nu}{}^{-\Lambda} \mathcal{H}^{-\mu\nu\Sigma} - \frac{1}{16} i F_{\Lambda\Sigma\Gamma} \bar{\Omega}_i{}^\Lambda \gamma^{\mu\nu} \mathcal{H}_{\mu\nu}{}^{-\Sigma} \Omega_j{}^\Gamma \varepsilon^{ij} \right. \\
&\quad \left. - \frac{1}{256} i N^{\Delta\Omega} \left(F_{\Delta\Lambda\Sigma} \bar{\Omega}_i{}^\Lambda \gamma_{\mu\nu} \Omega_j{}^\Sigma \varepsilon^{ij} \right) \left(F_{\Gamma\Xi\Omega} \bar{\Omega}_k{}^\Gamma \gamma^{\mu\nu} \Omega_l{}^\Xi \varepsilon^{kl} \right) + \text{h.c.} \right).
\end{aligned} \tag{4.59}$$

Furthermore, one includes the Lagrangians L_{Ω^4} (2.15), L_Y (2.16) (which remain unaltered) and (4.51). Up to an extension of (4.16), whose form we will establish in this section, we do not expect further modifications.

Also the supersymmetry transformation rules acquire a number of modifications, extending spacetime derivatives and field strengths to covariant ones. Furthermore, one has to take account of the presence of the new magnetic gauge fields and the tensor fields. However, one also needs a few additional terms in the transformation rules, whose form will be established in due course. For the moment we use the following modified transformation rules, where we also include the variations of

the magnetic gauge fields, which we denote by δ_0 ,

$$\begin{aligned}
\delta_0 X^\Lambda &= \bar{\epsilon}^i \Omega_i^\Lambda, \\
\delta_0 A_\mu^\Lambda &= \varepsilon^{ij} \bar{\epsilon}_i \gamma_\mu \Omega_j^\Lambda + \varepsilon_{ij} \bar{\epsilon}^i \gamma_\mu \Omega^{j\Lambda}, \\
\delta_0 A_{\mu\Lambda} &= F_{\Lambda\Sigma} \varepsilon^{ij} \bar{\epsilon}_i \gamma_\mu \Omega_j^\Sigma + \bar{F}_{\Lambda\Sigma} \varepsilon_{ij} \bar{\epsilon}^i \gamma_\mu \Omega^{j\Sigma}, \\
\delta_0 \Omega_i^\Lambda &= 2\mathcal{D}X^\Lambda \epsilon_i + \frac{1}{2} \gamma^{\mu\nu} \mathcal{H}_{\mu\nu}^- \varepsilon_{ij} \epsilon^j + Y_{ij}^\Lambda \epsilon^j, \\
\delta_0 Y_{ij}^\Lambda &= 2\bar{\epsilon}_{(i} \mathcal{D}\Omega_{j)}^\Lambda + 2\varepsilon_{ik} \varepsilon_{jl} \bar{\epsilon}^{(k} \mathcal{D}\Omega^{l)\Lambda}.
\end{aligned} \tag{4.60}$$

At this point it is convenient to note that the supersymmetry variations of the scalar, spinor and vector fields can be written in the form,

$$\begin{aligned}
\delta_0 X^M &= \bar{\epsilon}^i \Omega_i^M, \\
\delta_0 A_\mu^M &= \varepsilon^{ij} \bar{\epsilon}_i \gamma_\mu \Omega_j^M + \varepsilon_{ij} \bar{\epsilon}^i \gamma_\mu \Omega^{jM}, \\
\delta_0 \Omega_i^M &= 2\mathcal{D}X^M \epsilon_i + \frac{1}{2} \gamma^{\mu\nu} \mathcal{G}_{\mu\nu}^- \varepsilon_{ij} \epsilon^j + \dots,
\end{aligned} \tag{4.61}$$

where the fermions Ω_i^M and the field strengths $\mathcal{G}_{\mu\nu}^M$ were defined in (4.3) and (4.46), respectively. The suppressed terms in $\delta\Omega_i^M$ are proportional to Y_{ij}^Λ and/or terms quadratic in the spinor fields and are not of immediate interest here.

Most of the cancellations required for demonstrating the supersymmetry of the Lagrangian will still take place when derivatives are replaced by covariant derivatives. A clear exception arises when dealing with the commutator of two derivatives, because it will lead to a field strength upon using the Ricci identity. This situation occurs for the variations of the fermion kinetic term. Furthermore, when establishing supersymmetry for the more conventional Lagrangians, one makes use of the Bianchi identity for the field strengths, which no longer applies to the new field strengths. Of course, the gauge fields in the covariant derivatives will also lead to new variations. To investigate these issues, we first determine the supersymmetry variation of $\mathcal{L}_{\text{matter}}$ under the transformations given above (up to total derivatives),

$$\begin{aligned}
\delta_0 \mathcal{L}_{\text{matter}} &= ig \Omega_{MQ} T_{PN}^Q \left[D^\mu \bar{X}^M X^N - \bar{X}^M D_\mu X^N + \frac{1}{2} \bar{\Omega}^{iM} \gamma^\mu \Omega_i^N \right] \delta A_\mu^P \\
&\quad - \frac{1}{2} ig \Omega_{MQ} T_{PN}^Q \left[\bar{X}^M \bar{\Omega}_i^N \gamma^{\mu\nu} \epsilon^i \mathcal{H}_{\mu\nu}^- - \text{h.c.} \right] \\
&\quad + i \Omega_{MN} \left[\bar{\Omega}^{iM} \gamma_\nu \epsilon^j \varepsilon_{ij} D_\mu \mathcal{G}^{-\mu\nu N} - \text{h.c.} \right],
\end{aligned} \tag{4.62}$$

where we suppressed variations that involve neither the gauge coupling constant g nor the modified field strengths. These variations will cancel as before.

It is now easy to verify that the term of order g^0 can be combined with the result from the variation of $\mathcal{L}_{\text{vector}} + \mathcal{L}_{\text{top}}$ (c.f. (4.50) and (4.52)),

$$\delta_0 (\mathcal{L}_{\text{vector}} + \mathcal{L}_{\text{top}}) = i \Omega_{MN} \mathcal{G}^{+\mu\nu M} D_\mu \delta A_\nu^N + \text{h.c.} + \dots \tag{4.63}$$

The combined result thus leads to a total derivative plus terms proportional to $D_\mu F_{\Lambda\Sigma}$ and terms cubic in the fermions. These terms cancel for the abelian theory with an ordinary derivative and the cancellation proceeds identically when ordinary derivatives are replaced by covariant ones. Note that nowhere one needs to use the Bianchi identity. This calculation confirms the correctness of the transformation rule for the magnetic gauge fields. Hence we can now concentrate on the remaining terms of (4.62), which are the only variations left, up to terms induced by the variation of the tensor fields which we will need in due course.

To cancel the order- g terms in (4.62) we need to add new terms in the transformation rules of Ω_i^Λ and Y_{ij}^Λ . Furthermore, new terms to the Lagrangian are required. For the case of purely electric charges these terms are known and the easiest strategy is to simply generalize these terms. This leads to the expressions,

$$\begin{aligned}\delta_g \Omega_i^\Lambda &= -2g T_{MN}^\Lambda \bar{X}^M X^N \varepsilon_{ij} \epsilon^j, \\ \delta_g Y_{ij}^\Lambda &= -4g T_{MN}^\Lambda [\bar{\Omega}_{(i}^M \epsilon^k \varepsilon_{j)k} \bar{X}^N - \bar{\Omega}^{kM} \varepsilon_{(i} \varepsilon_{j)k} X^N], \\ \mathcal{L}_g &= \frac{1}{4} ig \Omega_{MQ} T_{PN}^Q [\varepsilon^{ij} \bar{\Omega}_i^M \Omega_j^N \bar{X}^P - \varepsilon_{ij} \bar{\Omega}^{iM} \Omega^{jN} X^P].\end{aligned}\quad (4.64)$$

In the case of purely electric charges the expression for \mathcal{L}_g reduces to the first expression of (4.16) upon using (4.23).

Collecting the new variations proportional to the field strengths that arise as a result of (4.64), we find, using (4.46), (4.26) and (4.20),

$$\delta_g \mathcal{L}_{\text{vector}} + \delta_0 \mathcal{L}_g = \frac{1}{2} ig \Omega_{MQ} T_{PN}^Q \bar{X}^M \bar{\Omega}_i^N \gamma^{\mu\nu} \epsilon^i \mathcal{G}_{\mu\nu}^{-P} + \text{h.c.} . \quad (4.65)$$

This term is almost identical to the second term of (4.62) except that it is proportional to $\mathcal{G}_{\mu\nu}^M$ rather than to $\mathcal{H}_{\mu\nu}^M$. However, the combination of these two terms is cancelled by assigning the following variation to the tensor fields,

$$\delta B_{\mu\nu a} = -2t_{aM}^P \Omega_{PN} (A_{[\mu}^M \delta A_{\nu]}^N - \bar{X}^M \bar{\Omega}_i^N \gamma_{\mu\nu} \epsilon^i - X^M \bar{\Omega}^{iN} \gamma_{\mu\nu} \epsilon_i) . \quad (4.66)$$

At this point one can verify that all supersymmetry variations linear in the gauge coupling constant g vanish. Here one makes use of the various results derived in section 4.1, and in particular of (4.24). What remains are the order- g^2 interactions induced by the order- g transformations of the spinors, which can be written as,

$$\delta \Omega_i^M = -2g T_{NP}^M \bar{X}^N X^P \varepsilon_{ij} \epsilon^j . \quad (4.67)$$

The order- g^2 variation follows from $\delta_g \mathcal{L}_g$, and can be written proportional to the supersymmetry variation δX^M given in (4.61),

$$\delta_g \mathcal{L}_g = -2ig^2 \Omega_{MQ} T_{NP}^Q \bar{X}^P \delta X^{[M} T_{RS}^{N]} \bar{X}^R X^S + \text{h.c.} . \quad (4.68)$$

Using the Lie algebra relation (4.33), as well as the relation (4.24), we can write this in a form that can be integrated. This reveals that these variations can be cancelled by the variation of a scalar potential, corresponding to

$$\mathcal{L}_{g^2} = ig^2 \Omega_{MN} T_{PQ}^M X^P \bar{X}^Q T_{RS}^N \bar{X}^R X^S . \quad (4.69)$$

This expression reduces to (4.16) for purely electric gaugings upon using (4.23). Observe that the charges $T_{\Lambda\Sigma\Gamma}$ do not contribute to (4.69), as is well known from previous constructions.

This concludes the derivation of supersymmetric vector multiplet Lagrangians with electric and magnetic gauge charges. In the following section we will consider the coupling to matter by introducing hypermultiplets. This will lead to a second scalar potential.

4.4 Hypermultiplets

In this section we give a brief description of the possible gaugings of isometries in the hyper-Kähler space parameterized by the hypermultiplet scalars, following the framework of [64].

As we saw in section 2.3, n_{H} hypermultiplets are described by $4n_{\text{H}}$ real scalars ϕ^A , $2n_{\text{H}}$ positive-chirality spinors $\zeta^{\bar{\alpha}}$ and $2n_{\text{H}}$ negative-chirality spinors ζ^{α} . Their (rigid) supersymmetry transformations and invariant Lagrangian are given by (2.36) and (2.37).

The equivalence transformations of the fermions and the target-space diffeomorphisms associated with this Lagrangian do not constitute invariances of the theory, unless they leave the metric g_{AB} and the $\text{Sp}(n_{\text{H}}) \times \text{Sp}(1)$ one-form V_i^α (and thus the related geometric quantities) invariant. Therefore invariances are related to isometries of the hyper-Kähler space. A subset of them can be elevated to a group of local (i.e. spacetime-dependent) transformations, which require a coupling to corresponding vector multiplets. Such gauged isometries have been studied in the literature [70, 73, 107, 108, 109, 110] but only for electric charges.

Infinitesimal isometries are characterized by Killing vectors and the ones associated to local transformations will be labelled by the same index M that labels the electric and magnetic gauge fields of the previous sections. In principle, the gauged isometries constitute a subgroup of the full group of isometries, defined by the embedding tensor. Hence the corresponding Killing vectors are proportional to the embedding matrix, $k^A_M = \Theta_M^a k^A_a$, and (4.37) implies,

$$Z^{M,a} k^A_M = 0 . \quad (4.70)$$

Without gauge interactions, the hypermultiplets do not couple to the vector multiplets, so that the full group of invariances factorizes into separate invariance groups of the vector multiplet Lagrangian and of the hypermultiplet Lagrangian. The index a refers to all these symmetries, and therefore k^A_a will vanish whenever the index a refers to a generator acting exclusively on the vector multiplets.

The local gauge transformations are thus generated by the Killing vectors $k^A_M(\phi) = (k^A_\Lambda(\phi), k^{A\Lambda}(\phi))$, with parameters Λ^M . Under infinitesimal transformations we have

$$\delta\phi^A = g \Lambda^M k^A_M(\phi) , \quad (4.71)$$

where g is the coupling constant and the $k^A_M(\phi)$ satisfy the Killing equation,

$$D_A k_{BM} + D_B k_{AM} = 0 . \quad (4.72)$$

Higher derivatives of Killing vector are not independent, as is shown by

$$D_A D_B k_{CM} = R_{BCAE} k^E_M . \quad (4.73)$$

The isometries close under commutation,

$$k^B_M \partial_B k^A_N - k^B_N \partial_B k^A_M = T_{MN}{}^P k^A_P , \quad (4.74)$$

where, as before, the antisymmetry in $[MN]$ on the right-hand side is ensured by (4.70).

The invariances associated with the target-space isometries act on the fermions by field dependent matrices, which satisfy the relation

$$(t_M)^\alpha{}_\beta V_{Ai}^\beta = D_A k^B_M V_{Bi}^\alpha , \quad (4.75)$$

leading to

$$(t_M)^\alpha{}_\beta = \frac{1}{2} V_{Ai}^\alpha \bar{\gamma}^{Bi} D_B k^A_M . \quad (4.76)$$

The result (4.75) was derived by requiring that the tensor V_{Ai}^α is invariant under the isometries, up to a rotation on the indices α . The invariance implies that target-space scalars satisfy algebraic identities such as

$$\bar{t}_M \bar{\gamma}_{\bar{\alpha}} G_{\bar{\gamma}\beta} + t_M^{\gamma\beta} G_{\bar{\alpha}\gamma} = t_M \bar{\gamma}_{[\bar{\alpha}} \Omega_{\bar{\beta}]\bar{\gamma}} = 0 , \quad (4.77)$$

which establishes that the matrices $t_M^\alpha{}_\beta$ take values in $\mathfrak{sp}(n_H)$. From (4.74) and (4.73), one may derive

$$D_A t_M^\alpha{}_\beta = R_{AB}{}^\alpha{}_\beta k^B_M , \quad (4.78)$$

for any infinitesimal isometry. From the group property of the isometries it follows that the matrices t_M satisfy the commutation relations,

$$[t_M, t_N]^\alpha{}_\beta = -T_{MN}{}^P (t_P)^\alpha{}_\beta + k^A_M k^B_N R_{AB}{}^\alpha{}_\beta , \quad (4.79)$$

which takes values in $\mathfrak{sp}(n_H)$. This result is consistent with the Jacobi identity.

The previous results imply that the complex structures J_{AB}^{ij} are invariant under the isometries,

$$k^C_M \partial_C J_{AB}^{ij} - 2\partial_{[A} k^C_M J_{B]C}^{ij} = 0 , \quad (4.80)$$

implying that the isometries are *tri-holomorphic*. From (4.80) one shows that $\partial_A (J_{BC}^{ij} k^C_M) - \partial_B (J_{AC}^{ij} k^C_M) = 0$, so that, locally, one can associate three Killing potentials (or moment maps) μ^{ij}_M to every Killing vector, according to

$$\partial_A \mu^{ij}_M = J_{AB}^{ij} k^B_M , \quad (4.81)$$

which determines μ^{ij}_M up to a constant. These constants correspond to Fayet-Iliopoulos terms. Up to such constants one derives the equivariance condition,

$$J_{AB}^{ij} k^A_M k^B_N = T_{MN}{}^P \mu^{ij}_P, \quad (4.82)$$

which implies that the Killing potentials transform covariantly under the isometries,

$$\delta\mu^{ij}_M = \Lambda^N k^A_N \partial_A \mu^{ij}_M = \Lambda^N T_{NM}{}^P \mu^{ij}_P. \quad (4.83)$$

Subsequently we consider the consequences of realizing the isometry (sub)group generated by the k^A_M as local gauge group. The latter acts on the hypermultiplet fields in the following way,

$$\delta\phi = g \Lambda^M k^A_M, \quad \delta\zeta^\alpha = g \Lambda^M t_{M\beta}{}^\alpha \zeta^\beta - \delta\phi^A \Gamma_{A\beta}{}^\alpha \zeta^\beta, \quad (4.84)$$

where the parameters Λ^M are functions of x^μ . The relevant covariant derivatives are equal to,

$$\mathcal{D}_\mu \phi^A = \partial_\mu \phi^A - g A_\mu{}^M k^A_M, \quad \mathcal{D}_\mu \zeta^\alpha = \partial_\mu \zeta^\alpha + \partial_\mu \phi^A \Gamma_{A\beta}{}^\alpha \zeta^\beta - g A_\mu{}^M t_{M\beta}{}^\alpha \zeta^\beta. \quad (4.85)$$

These covariant derivatives must be substituted into the transformation rules (2.36) and the Lagrangian (2.37). The covariance of $\mathcal{D}_\mu \zeta^\alpha$,

$$\delta \mathcal{D}_\mu \zeta^\alpha = g \Lambda^M t_{M\beta}{}^\alpha \mathcal{D}_\mu \zeta^\beta - \delta\phi^A \Gamma_{A\beta}{}^\alpha \mathcal{D}_\mu \zeta^\beta, \quad (4.86)$$

follows from (4.78) and (4.79).

Just as for the vector multiplets, the introduction of the gauge covariant derivatives to the Lagrangian breaks the supersymmetry of the Lagrangian. To restore supersymmetry we follow the same procedure as in section 4.3. But in this case the situation is somewhat simpler because the electric and magnetic gauge fields couple to standard hypermultiplet isometries. This means that the initial results will coincide with those obtained for electric gaugings.

Let us first present the variations of the Lagrangian (2.37) with the proper gauge covariantizations and determine the supersymmetry variation linear in the gauge coupling constant g and linear in the fermion fields,

$$\delta \mathcal{L}_0 = g k_{AM} \left[\gamma_{i\bar{\alpha}}^A \bar{\zeta}^{\bar{\alpha}} \gamma^{\mu\nu} \epsilon^i \mathcal{F}_{\mu\nu}{}^M + \varepsilon^{ij} \bar{\Omega}_i{}^M \mathcal{D} \phi^A \epsilon_j + \text{h.c.} \right]. \quad (4.87)$$

The first term originates from the fact that the commutator of two covariant derivatives acquires an extra field strength in the presence of the gauging, whereas the second term originates from the variation of the gauge fields in the covariant derivatives of the scalars. The first term can be cancelled by a supersymmetry variation of the following new term,

$$\mathcal{L}_g^{(1)} = 2g k_{AM} \left[\bar{\gamma}_\alpha^{Ai} \varepsilon_{ij} \bar{\zeta}^\alpha \Omega_j{}^M + \gamma_{i\bar{\alpha}}^A \varepsilon^{ij} \bar{\zeta}^{\bar{\alpha}} \Omega_j{}^M \right]. \quad (4.88)$$

The variations of this term proportional to the field strength $\mathcal{G}_{\mu\nu}^M$ cancel against the term proportional to $\mathcal{H}_{\mu\nu}^M$ (the field strength $\mathcal{F}_{\mu\nu}^M$ can be replaced by $\mathcal{H}_{\mu\nu}^M$ by virtue of (4.70)) by introducing a new term to the variation of the tensor fields $B_{\mu\nu a}$ (4.66),

$$\delta B_{\mu\nu a} = -4ik^A{}_a [\gamma_{A\bar{\alpha}} \bar{\zeta}^{\bar{\alpha}} \gamma_{\mu\nu} \epsilon^i - \bar{\gamma}_{A\alpha}^i \bar{\zeta}^{\bar{\alpha}} \gamma_{\mu\nu} \epsilon_i] . \quad (4.89)$$

Another term in the variation of (4.88) is proportional to X^M and its complex conjugate. Their cancellation requires the following extra variations of the hypermultiplet spinors,

$$\delta\zeta^\alpha = 2gX^M k^A{}_M V_{A\bar{i}}^\alpha \varepsilon^{ij} \epsilon_j , \quad \delta\bar{\zeta}^{\bar{\alpha}} = 2g\bar{X}^M k^A{}_M \bar{V}_A^{\bar{\alpha}i} \varepsilon_{ij} \epsilon^j , \quad (4.90)$$

and an extra term in the Lagrangian equal to

$$\mathcal{L}_g^{(2)} = 2g \left[\bar{X}^M t_M \gamma_\alpha \bar{\Omega}_{\beta\gamma} \bar{\zeta}^\alpha \zeta^\beta + X^M t_M \bar{\gamma}_{\bar{\alpha}} \Omega_{\bar{\beta}\bar{\gamma}} \bar{\zeta}^{\bar{\alpha}} \zeta^{\bar{\beta}} \right] . \quad (4.91)$$

The remaining variations then take the following form.

$$\begin{aligned} \delta\mathcal{L}_0 + \delta\mathcal{L}_g^{(1)} + \delta\mathcal{L}_g^{(2)} &= -2g \partial_A \mu^{ij}{}_M \bar{\Omega}_i{}^M \mathcal{P} \phi^A \epsilon_j - 2g \partial_A \mu_{ij}{}_M \bar{\Omega}^{iM} \mathcal{P} \phi^A \epsilon^j \\ &\quad - 2g \left[\partial_A \mu_{ij\Lambda} Y^{ij\Lambda} + \partial_A \mu_{ij}{}^\Lambda \bar{F}_{\Lambda\Sigma} Y^{ij\Sigma} \right] \bar{\gamma}_\alpha^{Ak} \bar{\epsilon}_k \zeta^\alpha \\ &\quad - 2g \left[\partial_A \mu_{ij\Lambda} Y^{ij\Lambda} + \partial_A \mu_{ij}{}^\Lambda F_{\Lambda\Sigma} Y^{ij\Sigma} \right] \gamma_{k\bar{\alpha}}^A \bar{\epsilon}^k \zeta^{\bar{\alpha}} , \end{aligned} \quad (4.92)$$

where we restricted ourselves to variations linear in the fermion fields and linear in g .

To cancel these variations we must include the following new term in the Lagrangian,

$$\begin{aligned} \mathcal{L}_g^{(3)} &= g Y^{ij\Lambda} \left[\mu_{ij\Lambda} + \frac{1}{2} (F_{\Lambda\Sigma} + \bar{F}_{\Lambda\Sigma}) \mu_{ij}{}^\Sigma \right] \\ &\quad - \frac{1}{4} g \left[F_{\Lambda\Sigma\Gamma} \mu^{ij\Lambda} \bar{\Omega}_i{}^\Sigma \Omega_j{}^\Gamma + \bar{F}_{\Lambda\Sigma\Gamma} \mu_{ij}{}^\Lambda \bar{\Omega}^{i\Sigma} \Omega^{j\Gamma} \right] , \end{aligned} \quad (4.93)$$

as well as assign new variations of the fields $\Omega_i{}^\Lambda$ and $Y_{ij}{}^\Lambda$ of the vector multiplet,

$$\begin{aligned} \delta_g \Omega_i{}^\Lambda &= 2ig \mu_{ij}{}^\Lambda \epsilon^j , \\ \delta Y_{ij}{}^\Lambda &= 4ig k^{A\Lambda} \left[\varepsilon_{k(i} \gamma_{j)\bar{\alpha}A} \bar{\epsilon}^k \zeta^{\bar{\alpha}} + \varepsilon_{k(i} \bar{\epsilon}_{j)} \zeta^\alpha \bar{\gamma}_{\alpha A}^k \right] . \end{aligned} \quad (4.94)$$

This completes the discussion of all the variations linear in g and in the fermion fields. The result remains valid for the cubic fermion variations as well. However, new variations arise in second order in g , by the order- g variations in the new order- g terms in the Lagrangian. These variations cancel against the variation of a scalar potential, corresponding to

$$\mathcal{L}_{g^2} = -2g^2 k^A{}_M k^B{}_N g_{AB} X^M \bar{X}^N - \frac{1}{2} g^2 N_{\Lambda\Sigma} \mu_{ij}{}^\Lambda \mu^{ij\Sigma} . \quad (4.95)$$

To prove (4.95), one has to make use of the equivariance condition (4.82). Actually, gauge invariance, which is prerequisite to supersymmetry, already depends on (4.83).

4.5 Summary and discussion

In this chapter we presented Lagrangians and supersymmetry transformations for a general supersymmetric system of vector multiplets and hypermultiplets in the presence of both electric and magnetic charges. The results were verified to all orders and are consistent with results known in the literature that are based on purely electric charges. We have also verified the closure of the supersymmetry algebra, which holds up to the field equations associated with the fields $A_{\mu\Lambda}$, $B_{\mu\nu a}$, Y_{ij}^Λ and the hypermultiplet spinors ζ^α . In the absence of magnetic charges, the supersymmetry algebra closes on the vector multiplets without the need for imposing the field equations. We return to this issue later in subsection 4.5.2.

Before discussing possible implications of these results, let us first summarize the terms induced by the gauging. We first present the combined supersymmetry variations. First of all, we have the original transformations in the absence of the gauging, where spacetime derivatives are replaced by gauge-covariant derivatives and where the abelian field strengths $F_{\mu\nu}^\Lambda$ are replaced by the covariant field strengths $\mathcal{H}_{\mu\nu}^\Lambda$. We will not repeat the corresponding expressions here, but we present the other terms in the transformation rules that are induced by the gauging. They read as follows,

$$\begin{aligned}
\delta_g \Omega_i^\Lambda &= -2g T_{NP}^\Lambda \bar{X}^N X^P \varepsilon_{ij} \epsilon^j + 2ig \mu_{ij}^\Lambda \epsilon^j , \\
\delta_g \zeta^\alpha &= 2g X^M k^A{}_M V_{Ai}^\alpha \varepsilon^{ij} \epsilon_j , \\
\delta_g Y_{ij}^\Lambda &= -4g T_{MN}^\Lambda [\bar{\Omega}_{(i}^M \epsilon^k \varepsilon_{j)k} \bar{X}^N - \bar{\Omega}^{kM} \varepsilon_{(i} \varepsilon_{j)k} X^N] \\
&\quad + 4ig k^{A\Lambda} [\varepsilon_{k(i} \gamma_{j)\bar{\alpha}A} \bar{\epsilon}^k \zeta^{\bar{\alpha}} + \varepsilon_{k(i} \bar{\epsilon}_{j)} \zeta^\alpha \bar{\gamma}_{\alpha A}^k] , \\
\delta B_{\mu\nu a} &= -2t_{aM}^P \Omega_{PN} (A_{[\mu}^M \delta A_{\nu]}^N - \bar{X}^M \bar{\Omega}_i^N \gamma_{\mu\nu} \epsilon^i - X^M \bar{\Omega}^{iN} \gamma_{\mu\nu} \epsilon_i) \\
&\quad - 4ik^A{}_a [\gamma_{Ai\bar{\alpha}} \bar{\zeta}^{\bar{\alpha}} \gamma_{\mu\nu} \epsilon^i - \bar{\gamma}_{A\alpha}^i \bar{\zeta}^\alpha \gamma_{\mu\nu} \epsilon_i] .
\end{aligned} \tag{4.96}$$

Likewise we will not repeat the original Lagrangians (2.7) and (2.37) for the vector multiplets and hypermultiplets, respectively, which are only modified by replacing spacetime derivatives by gauge-covariant ones, and field strengths by the covariant field strengths $\mathcal{H}_{\mu\nu}^\Lambda$. The Lagrangian (4.51) remains unchanged. The additional terms induced by the gauging that are linear in g take the following form,

$$\begin{aligned}
\mathcal{L}_g &= \frac{1}{4} ig \Omega_{MQ} T_{PN}^Q [\varepsilon^{ij} \bar{\Omega}_i^M \Omega_j^N \bar{X}^P - \varepsilon_{ij} \bar{\Omega}^{iM} \Omega^{jN} X^P] \\
&\quad - \frac{1}{4} g [F_{\Lambda\Sigma\Gamma} \mu^{ij\Lambda} \bar{\Omega}_i^\Sigma \Omega_j^\Gamma + \bar{F}_{\Lambda\Sigma\Gamma} \mu_{ij}^\Lambda \bar{\Omega}^{i\Sigma} \Omega^{j\Gamma}] \\
&\quad + 2g k_{AM} [\bar{\gamma}_\alpha^{Ai} \varepsilon_{ij} \bar{\zeta}^\alpha \Omega^{jM} + \gamma_{i\bar{\alpha}}^A \varepsilon^{ij} \bar{\zeta}^{\bar{\alpha}} \Omega_j^M] \\
&\quad + 2g [\bar{X}^M t_M^\gamma \bar{\Omega}_{\beta\gamma} \bar{\zeta}^\alpha \zeta^\beta + X^M t_M^{\bar{\gamma}\bar{\alpha}} \Omega_{\bar{\beta}\bar{\gamma}} \bar{\zeta}^{\bar{\alpha}} \zeta^{\bar{\beta}}] \\
&\quad + g Y^{ij\Lambda} \left[\mu_{ij\Lambda} + \frac{1}{2} (F_{\Lambda\Sigma} + \bar{F}_{\Lambda\Sigma}) \mu_{ij}^\Sigma \right] .
\end{aligned} \tag{4.97}$$

The terms of order g^2 correspond to a scalar potential proportional to g^2 and are given by

$$\begin{aligned} \mathcal{L}_{g^2} = & ig^2 \Omega_{MN} T_{PQ}{}^M X^P \bar{X}^Q T_{RS}{}^N \bar{X}^R X^S \\ & - 2g^2 k^A{}_M k^B{}_N g_{AB} X^M \bar{X}^N - \frac{1}{2}g^2 N_{\Lambda\Sigma} \mu_{ij}{}^\Lambda \mu^{ij\Sigma} . \end{aligned} \quad (4.98)$$

4.5.1 Applications

The above results have many applications. A relatively simple one concerns the Fayet-Iliopoulos terms, which are the integration constants of the Killing potentials $\mu^{ij}{}_M$. This enables us to truncate the above expressions by setting the embedding tensor to zero, while still retaining the constants $g\mu^{ij}{}_M$. In that case all effects of the gauging are suppressed and one is left with a potential accompanied by fermionic masslike terms,

$$\begin{aligned} \mathcal{L}_{\text{FI}} = & \frac{1}{8}N^{\Lambda\Sigma} \left(N_{\Lambda\Gamma} Y_{ij}{}^\Gamma + \frac{1}{2}i(F_{\Lambda\Gamma\Omega} \bar{\Omega}_i{}^\Gamma \Omega_j{}^\Omega - \bar{F}_{\Lambda\Gamma\Omega} \bar{\Omega}^{k\Gamma} \Omega^{l\Omega} \varepsilon_{ik} \varepsilon_{jl}) \right) \\ & \times \left(N_{\Sigma\Xi} Y^{ij\Xi} + \frac{1}{2}i(F_{\Sigma\Xi\Delta} \bar{\Omega}_m{}^\Xi \Omega_n{}^\Delta \varepsilon^{im} \varepsilon^{jn} - \bar{F}_{\Sigma\Xi\Delta} \bar{\Omega}^{i\Xi} \Omega^{j\Delta}) \right) \\ & - \frac{1}{2}g^2 N_{\Lambda\Sigma} \mu_{ij}{}^\Lambda \mu^{ij\Sigma} + g Y^{ij\Lambda} \left[\mu_{ij\Lambda} + \frac{1}{2}(F_{\Lambda\Sigma} + \bar{F}_{\Lambda\Sigma}) \mu_{ij}{}^\Sigma \right] \\ & - \frac{1}{4}g \left[F_{\Lambda\Sigma\Gamma} \mu^{ij\Lambda} \bar{\Omega}_i{}^\Sigma \Omega_j{}^\Gamma + \bar{F}_{\Lambda\Sigma\Gamma} \mu_{ij}{}^\Lambda \bar{\Omega}^{i\Sigma} \Omega^{j\Gamma} \right] . \end{aligned} \quad (4.99)$$

Eliminating the auxiliary fields $Y_{ij}{}^\Lambda$ gives rise to the following expression,

$$\begin{aligned} \mathcal{L}_{\text{FI}} = & -\frac{1}{2}iN^{\Lambda\Sigma} F_{\Sigma\Gamma\Xi} \bar{\Omega}_i{}^\Gamma \Omega_j{}^\Xi \left[\mu^{ij}{}_\Lambda + \bar{F}_{\Lambda\Delta} \mu^{ij\Delta} \right] \\ & + \frac{1}{2}iN^{\Lambda\Sigma} \bar{F}_{\Sigma\Gamma\Xi} \Omega^{i\Gamma} \Omega^{j\Xi} \left[\mu_{ij\Lambda} + F_{\Lambda\Delta} \mu_{ij}{}^\Delta \right] \\ & - 2g^2 \left[\mu^{ij}{}_\Lambda + F_{\Lambda\Gamma} \mu^{ij\Gamma} \right] N^{\Lambda\Sigma} \left[\mu_{ij\Sigma} + \bar{F}_{\Sigma\Xi} \mu_{ij}{}^\Xi \right] . \end{aligned} \quad (4.100)$$

The above expression transforms as a function under electric/magnetic duality provided that the $\mu^{ij}{}_M$ are treated as spurionic quantities. The last term in (4.100) corresponds to minus the potential, which is positive definite (assuming positive $N_{\Lambda\Sigma}$). The Lagrangian is a generalization of the Lagrangian presented in [111], where it was also shown how the potential can lead to spontaneous partial supersymmetry breaking when $\mu_{ij}{}^\Lambda \neq 0$. Note that the hypermultiplets play only an ancillary role here, as they decouple from the vector multiplets.

Most of the possible applications can be found in the context of supergravity, where they will be useful for constructing low-energy effective actions associated with string compactifications in the presence of fluxes. In principle it is straightforward to extend our results to the case of local supersymmetry. The target-space of

the vector multiplets should then be restricted to a special Kähler cone (as we saw in chapter 2, this requires that $F(X)$ be a homogeneous function of second degree), and the hypermultiplet scalars should coordinatize a hyper-Kähler cone. Furthermore, the various formulae for the action and the supersymmetry transformation rules should be evaluated in the presence of a superconformal background, so that the action and transformation rules will also involve the superconformal fields. This has not yet been worked out in detail for $N = 2$ supergravity, although it is in principle straightforward. In view of the fact that gaugings of $N = 4$ and $N = 8$ supergravity have already been worked out using the same formalism as in our work [112, 113], no complications are expected. Note that Fayet-Iliopoulos terms do not exist in $N = 2$ supergravity, because the Killing potentials cannot contain arbitrary constants as those would break the scale invariance of the hyper-Kähler cone.

The potential is rather independent of all these details, although it must be rewritten in terms of the proper quantities, as was for instance demonstrated in [70]. It was already shown in [25] that the theory simplifies considerably for abelian gaugings where $T_{MN}{}^P = 0$ and where the potential is exclusively generated by the hypermultiplet charges. Making use of the steps described in [70], it is rather straightforward to derive the potential (as was already foreseen in [25]), which takes precisely the form conjectured quite some time ago in [24]. The results can also be compared to the work of [114].

4.5.2 Off-shell structure

In the absence of magnetic charges, the vector multiplets constitute off-shell representations of the supersymmetry algebra. On the hypermultiplets the supersymmetry algebra is only realized up to the fermionic field equations. The tensor fields decouple from the theory. However, when magnetic charges are present, there are no longer any off-shell multiplets and the supersymmetry algebra is only realized when the fields satisfy the field equations of the hypermultiplet spinors and of the fields $A_{\mu\Lambda}$, $Y_{ij}{}^\Lambda$ and $B_{\mu\nu a}$. In this subsection we discuss how the off-shell closure can possibly be regained for the vector multiplets when magnetic charges are switched on.

We start by introducing $2n$ independent vector multiplets, associated with the electric and magnetic gauge fields, $A_\mu{}^\Lambda$ and $A_{\mu\Lambda}$, and collectively denoted by $A_\mu{}^M$. In the absence of charges, these fields are subject to the standard off-shell transformation rules,

$$\begin{aligned}
\delta X^M &= \bar{\epsilon}^i \Omega_i{}^M, \\
\delta A_\mu{}^M &= \varepsilon^{ij} \bar{\epsilon}_i \gamma_\mu \Omega_j{}^M + \varepsilon_{ij} \bar{\epsilon}^i \gamma_\mu \Omega^j{}^M, \\
\delta \Omega_i{}^M &= 2\bar{\not{\partial}} X^M \epsilon_i + \frac{1}{2} \gamma^{\mu\nu} F_{\mu\nu}{}^{-M} \varepsilon_{ij} \epsilon^j + Y_{ij}{}^M \epsilon^j, \\
\delta Y_{ij}{}^M &= 2\bar{\epsilon}_{(i} \not{\partial} \Omega_{j)}{}^M + 2\varepsilon_{ik} \varepsilon_{jl} \bar{\epsilon}^{(k} \not{\partial} \Omega^{l)M}.
\end{aligned} \tag{4.101}$$

We stress once more that, unlike previously, the $2n$ vector multiplets are independent. In due course we shall see how to make contact with the previous description. We also introduce p off-shell tensor multiplets,

$$\begin{aligned}
\delta G_a &= -2\bar{\epsilon}_i \not{\partial} \varphi_a^i, \\
\delta B_{\mu\nu a} &= \frac{1}{2} (i\bar{\epsilon}^i \gamma_{\mu\nu} \varphi_a^j \varepsilon_{ij} - i\bar{\epsilon}_i \gamma_{\mu\nu} \varphi_{ja} \varepsilon^{ij}), \\
\delta \varphi_a^i &= \not{\partial} (l_{ij}^i{}_a + \varepsilon^{ik} \varepsilon^{jl} l_{kla}) \epsilon_j + 2\varepsilon^{ij} \mathcal{H}_a \epsilon_j - G_a \epsilon^i, \\
\delta l_{ija} &= 2\varepsilon_{ik} \varepsilon_{jl} \bar{\epsilon}^{(k} \varphi^{l)}{}_a,
\end{aligned} \tag{4.102}$$

where $\mathbf{a} = 1, \dots, p$. $\mathcal{H}_{\mu\mathbf{a}} = \frac{1}{2} i \varepsilon_{\mu\nu\rho\sigma} \mathcal{H}^{\nu\rho\sigma}{}_{\mathbf{a}}$ and $\mathcal{H}_{\mu\nu\rho\mathbf{a}} = 3\partial_{[\mu} B_{\nu\rho]\mathbf{a}}$. The G_a are complex scalar fields, the l_{ija} triplets of complex scalar fields and the φ_a^i are the right-handed parts of sets of two Majorana spinors. $B_{\mu\nu a}$ and l_{ija} are subject to the gauge transformations

$$\delta B_{\mu\nu a} = 2\partial_{[\mu} \Xi_{\nu]\mathbf{a}}, \quad \delta l_{ija} = \gamma_{ija}, \tag{4.103}$$

where $\Xi_{\mu\mathbf{a}}$ is real and γ_{ija} is imaginary (in the sense that $(\gamma_{ija})^* = -\varepsilon^{ik} \varepsilon^{jl} \gamma_{kla}$). When $l_{ija} - \varepsilon_{ik} \varepsilon_{jl} l^{kl}{}_a = 0$ (4.102) reduces to the set of transformation rules for off-shell tensor multiplets of [49].

Next charges are turned on. We restrict ourselves to abelian gauge groups and leave non-abelian gaugings for further study. In the presence of charges the tensor multiplets appear in the supersymmetry transformations of the vector multiplets. As before, the field strengths $F_{\mu\nu}^M$ in (4.101) are replaced by the covariant field strengths $\mathcal{H}_{\mu\nu}^M = F_{\mu\nu}^M + gZ^{M,\mathbf{a}} B_{\mu\nu\mathbf{a}}$. Furthermore, in view of the adjustment required for hypermultiplet gaugings (4.94), the real auxiliary fields Y_{ij}^M are replaced by the complex fields

$$\mathcal{Y}_{ij}^M = Y_{ij}^M - igZ^{M,\mathbf{a}} l_{ija}. \tag{4.104}$$

One thus gets

$$\begin{aligned}
\delta X^M &= \bar{\epsilon}^i \Omega_i^M, \\
\delta A_\mu^M &= \varepsilon^{ij} \bar{\epsilon}_i \gamma_\mu \Omega_j^M + \varepsilon_{ij} \bar{\epsilon}^i \gamma_\mu \Omega^{jM}, \\
\delta \Omega_i^M &= 2\not{\partial} X^M \epsilon_i + \frac{1}{2} \gamma^{\mu\nu} \mathcal{H}_{\mu\nu}^{-M} \varepsilon_{ij} \epsilon^j + \mathcal{Y}_{ij}^M \epsilon^j, \\
\delta Y_{ij}^M &= 2\bar{\epsilon}_i \not{\partial} \Omega_j^M + 2\varepsilon_{ik} \varepsilon_{jl} \bar{\epsilon}^{(k} \not{\partial} \Omega^{l)M},
\end{aligned} \tag{4.105}$$

and the gauge transformations (4.103) now also act on the vector multiplets,

$$\delta A_\mu^M = -gZ^{M,\mathbf{a}} \Xi_{\mu\mathbf{a}}, \quad \delta Y_{ij}^M = igZ^{M,\mathbf{a}} \gamma_{ija}. \tag{4.106}$$

It can be shown that (4.105) are still off-shell multiplets. The supersymmetry algebra closes as on (4.101), up to additional gauge transformations of the type

(4.106), with parameters

$$\begin{aligned}\Xi'_{\mu a} &= 4\bar{\epsilon}_{i[1}\gamma^\sigma\epsilon_2^i]B_{\sigma\mu a} + 2i\varepsilon^{ij}\bar{\epsilon}_{j[1}\gamma_\mu\epsilon_2^k]l_{ika} - 2i\varepsilon_{ij}\bar{\epsilon}_{[1}^j\gamma_\mu\epsilon_2]k]l^{ik}{}_a, \\ \gamma'_{ija} &= 4(\bar{\epsilon}_{i[1}\gamma^\rho\epsilon_2^k]\partial_\rho l_{j)ka} - \varepsilon_{ik}\varepsilon_{jl}\bar{\epsilon}_{[1}^k\gamma^\rho\epsilon_2]p] \partial_\rho l^l{}_p{}_a) \\ &\quad - 4i\varepsilon_{k(i}\bar{\epsilon}_{[1}^k\gamma^\rho\epsilon_2]j)\varepsilon_{\rho\alpha\beta\gamma}\partial^\alpha B^{\beta\gamma}{}_a.\end{aligned}\quad (4.107)$$

Here ϵ_1 and ϵ_2 are the parameters of the supersymmetry transformations, as appearing in the commutator $[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)]$.

Then we construct supersymmetric Lagrangians in terms of the off-shell multiplets (4.102) and (4.105).

Since just the A_μ^Λ are meant to play the role of electric gauge fields, we introduce a Lagrangian of the form (2.7) for the vector multiplets with upper indices Λ only. We replace $F_{\mu\nu}^\Lambda$ by $\mathcal{H}_{\mu\nu}^\Lambda$ and Y_{ij}^Λ by \mathcal{Y}_{ij}^Λ and, using the results we obtained before, add the topological term

$$\mathcal{L}_{\text{top}} = \frac{1}{8}ig\varepsilon^{\mu\nu\rho\sigma}\Theta^{\Lambda a}B_{\mu\nu a}\left(2\partial_\rho A_{\sigma\Lambda} - \frac{1}{4}g\Theta_\Lambda{}^b B_{\rho\sigma b}\right). \quad (4.108)$$

Recall that the equations of motion of the tensor fields $B_{\mu\nu a}$ relate the field strengths associated to the magnetic vector gauge fields, $\mathcal{H}_{\mu\nu\Lambda}$, to the dual field strengths, $\mathcal{G}_{\mu\nu\Lambda}$,

$$-\frac{1}{8}ig\varepsilon^{\mu\nu\rho\sigma}\Theta^{\Lambda a}(\mathcal{G}_{\rho\sigma} - \mathcal{H}_{\rho\sigma})_\Lambda = 0. \quad (4.109)$$

Since the l_{ija} enter the vector multiplets in a way similar to $B_{\mu\nu a}$ we expect that the equations of motion of l_{ija} , when following from a supersymmetric Lagrangian, relate $\mathcal{Y}_{ij\Lambda}$ and the duals of \mathcal{Y}_{ij}^Λ in the same way as (4.109) does with $\mathcal{H}_{\mu\nu\Lambda}$ and $\mathcal{G}_{\mu\nu\Lambda}$. To realize this, the term

$$\mathcal{L}'_g = \frac{1}{8}g\varepsilon^{ik}\varepsilon^{jl}\Theta^{\Lambda a}l_{ija}(Y_{kl\Lambda} + \frac{1}{4}ig\Theta_\Lambda{}^b l_{klb}) + h.c., \quad (4.110)$$

is added to the Lagrangian. As required, the equations of motion of the l_{ija} then become

$$-\frac{1}{8}\varepsilon^{ik}\varepsilon^{jl}\Theta^{\Lambda a}(\mathcal{Z}_{ij} - \mathcal{Y}_{ij})_\Lambda = 0, \quad (4.111)$$

where

$$\mathcal{Z}_{ij\Lambda} = F_{\Lambda\Sigma}\mathcal{Y}_{ij}^\Sigma - \frac{1}{2}F_{\Lambda\Sigma\Gamma}\bar{\Omega}_i{}^\Sigma\Omega_j{}^\Gamma, \quad (4.112)$$

is the analog of $Z_{ij\Lambda}$ (which is defined in (2.25)).

Similarly, the equations of motion of the tensor multiplet fermions φ_a^i and scalars G_a should relate the vector multiplet fermions $F_{\Lambda\Sigma}\Omega_i^\Sigma$ and $\Omega_{i\Lambda}$ and scalars F_Λ and X_Λ . This requires the inclusion of the terms

$$\begin{aligned} \mathcal{L}_g'' &= \frac{1}{4}g\Theta^{\Lambda a}\bar{\varphi}_a^i(F_{\Lambda\Sigma}\Omega_i^\Sigma - \Omega_{i\Lambda}) + \text{h.c.} \\ &+ \frac{1}{4}g\Theta^{\Lambda a}G_a(F_\Lambda - X_\Lambda) + \text{h.c.} , \end{aligned} \quad (4.113)$$

in the Lagrangian. It is straightforward to check that the model thus obtained is indeed supersymmetric.

The abelian gauge group can be embedded in the rigid invariance group associated with the hyper-Kähler space parameterized by the hypermultiplet scalars. Since all $2n$ gauge fields come with complete vector multiplets, gaugings of this type are similar to gaugings with $2n$ electric charges. The corresponding supersymmetric hypermultiplet Lagrangian therefore follows from the results of section 4.4 in the case of vanishing magnetic charges. It reads

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_0 + 2g\bar{X}^M t_M^\gamma \bar{\Omega}_{\beta\gamma} \bar{\zeta}^\alpha \zeta^\beta + \text{h.c.} \\ &+ 2gk^A{}_M \bar{\gamma}_{A\alpha}^i \varepsilon_{ij} \bar{\zeta}^{\bar{\alpha}} \Omega^{jM} + \text{h.c.} \\ &+ g\mu^{ij}{}_M Y_{ij}{}^M \\ &- 2g^2 k^A{}_M k^B{}_N g_{AB} X^M \bar{X}^N , \end{aligned} \quad (4.114)$$

(note that, due to (4.37), the terms in the supersymmetry transformation of Ω^M that are proportional to $B_{\mu\nu a}$ and l_{ija} play no role), while the order- g variations of the hypermultiplet spinors are

$$\delta\zeta^\alpha = 2gX^M k^A{}_M V_{Ai}^\alpha \varepsilon^{ij} \epsilon_j , \quad \delta\bar{\zeta}^{\bar{\alpha}} = 2g\bar{X}^M k^A{}_M \bar{V}_A^{\bar{\alpha}i} \varepsilon_{ij} \epsilon^j . \quad (4.115)$$

The complete Lagrangian is then given by (4.114) supplemented with a Lagrangian of the form (2.7) for the electric vector multiplets and the terms (4.108), (4.110) and (4.113).

Eliminating $Y_{ij\Lambda}$, $\Omega_{i\Lambda}$ and X_Λ (and absorbing the real part of $-\frac{1}{2}ig\Theta^{\Lambda a}l_{ija}$ in $Y_{ij}{}^\Lambda$) reproduces the results of (4.4) for the subclass of abelian gaugings.

A

Notation and conventions

We use μ, ν, \dots (m, n, \dots) for four- (three-) dimensional spacetime indices. a, b, \dots are four-dimensional Lorentz indices. In the context of $N = 2$ supersymmetry, i, j, \dots are usually $SU(2)_R$ indices.

Our conventions for (anti-)symmetrization are

$$[ab] = \frac{1}{2}[ab - ba] , \quad (ab) = \frac{1}{2}(ab + ba) . \quad (\text{A.1})$$

Gamma matrices we take, such that

$$\gamma_a \gamma_b = \eta_{ab} + \gamma_{ab} , \quad \gamma_5 = i\gamma_0 \gamma_1 \gamma_2 \gamma_3 , \quad (\text{A.2})$$

where we use $\eta_{ab} = (- + + +)$. A charge conjugation matrix C is defined, such that

$$-\gamma_\mu^T = C\gamma_\mu C^{-1} , \quad \gamma_5^T = C\gamma_5 C^{-1} , \quad C^T = -C . \quad (\text{A.3})$$

In four dimensions the fully antisymmetric tensor reads

$$\varepsilon^{abcd} = e^{-1} \varepsilon^{\mu\nu\lambda\sigma} e_\mu^a e_\nu^b e_\lambda^c e_\sigma^d , \quad \varepsilon^{0123} = i , \quad \varepsilon^{1234} = 1 , \quad (\text{A.4})$$

where ε^{1234} is a Euclidean component. The fully antisymmetric tensor in three dimensions is defined analogously.

The dual of an antisymmetric tensor field F_{ab} (in Minkowski space) is defined by

$$\tilde{F}_{ab} = \frac{1}{2} \varepsilon_{abcd} F^{cd} , \quad (\text{A.5})$$

such that its (anti-)selfdual part is given by

$$F_{ab}^\pm = \frac{1}{2} (F_{ab} \pm \tilde{F}_{ab}) . \quad (\text{A.6})$$

Under hermitian conjugation selfdual becomes antiselfdual and vice versa. In the context of $N = 2$, our conventions are such that $SU(2)_R$ indices change place under complex conjugation.

The Dirac conjugate $\bar{\psi}$ of a Dirac spinor ψ is defined by

$$\bar{\psi} = \psi^\dagger \gamma_0 . \quad (\text{A.7})$$

The (pseudo) reality condition

$$\bar{\psi} = \psi^T C , \quad (\text{A.8})$$

defines a Majorana spinor.

Under complex conjugation there are the following identities,

$$\bar{\psi} \gamma_a \phi = -\bar{\phi} \gamma_a \psi , \quad \psi \phi = \bar{\phi} \psi , \quad (\text{A.9})$$

for any two spinors ψ, ϕ . From (A.9) similar identities for other bilinears can be derived. Furthermore, *Majorana* spinors ψ and ϕ satisfy

$$\bar{\psi} \gamma_a \phi = -\bar{\phi} \gamma_a \psi , \quad \bar{\psi} \phi = \bar{\phi} \psi . \quad (\text{A.10})$$

If two spinors ψ and ϕ do not form a bilinear, their product can be decomposed on a basis of four-by-four matrices by means of a Fierz rearrangement,

$$\begin{aligned} \phi \bar{\psi} = & -\frac{1}{4}(\bar{\psi} \phi) \mathbb{I} - \frac{1}{4}(\bar{\psi} \gamma^a \phi) \gamma_a - \frac{1}{4}(\bar{\psi} \gamma_5 \phi) \gamma_5 + \frac{1}{4}(\bar{\psi} \gamma^a \gamma_5 \phi) \gamma_a \gamma_5 \\ & + \frac{1}{8}(\bar{\psi} \gamma^{ab} \phi) \gamma_{ab} . \end{aligned} \quad (\text{A.11})$$

Finally, we note the identities,

$$\begin{aligned} \gamma_{ab} &= -\frac{1}{2} \varepsilon_{abcd} \gamma^{cd} \gamma_5 , & \gamma^b \gamma_a \gamma_b &= -2 \gamma_a , \\ \gamma^{ab} \gamma_{ab} &= -12 , & \gamma^{cd} \gamma_{ab} \gamma_{cd} &= 4 \gamma_{ab} , \\ \gamma^c \gamma_{ab} \gamma_c &= 0 , & \gamma^{bc} \gamma_a \gamma_{bc} &= 0 , \\ [\gamma^c, \gamma_{ab}] &= 4 \delta_{[a}^c \gamma_{b]} , & \{\gamma^c, \gamma_{ab}\} &= 2 \varepsilon_{ab}{}^{cd} \gamma_5 \gamma_d , \\ [\gamma_{ab}, \gamma^{cd}] &= -8 \delta_{[a}^c \gamma_{b]}^d , & \{\gamma_{ab}, \gamma^{cd}\} &= -4 \delta_{[a}^c \delta_{b]}^d + 2 \varepsilon_{ab}{}^{cd} \gamma_5 . \end{aligned} \quad (\text{A.12})$$

Wick rotation

The standard Wick rotation

$$t = -i\tau , \quad (\text{A.13})$$

defines Euclidean Lagrangians

$$\mathcal{L}_m = i\mathcal{L}_e . \quad (\text{A.14})$$

The Wick rotation on tensors is

$$B_{tm} \rightarrow iB_{\tau m} , \quad B_{mn} \rightarrow B_{mn} , \quad (\text{A.15})$$

and similarly for vectors.

B

Superconformal multiplets

The Weyl multiplet

The *dependent* fields of the Weyl multiplet:

$$\begin{aligned}
\omega_\mu^{ab} &= -2e^{\nu[a}\partial_{[\mu}e_{\nu]}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\partial_\sigma e_\nu{}^c - 2e_\mu{}^{[a}e^{b]\nu}b_\nu \\
&\quad - \frac{1}{4}(2\bar{\psi}_\mu{}^i\gamma^{[a}\psi^{b]}{}_i + \bar{\psi}{}^{ai}\gamma_\mu\psi^b{}_i + \text{h.c.}) , \\
\phi_\mu{}^i &= \left(\frac{1}{2}\gamma^{\rho\sigma}\gamma_\mu - \frac{1}{6}\gamma_\mu\gamma^{\rho\sigma}\right)(\mathbb{D}_\rho\psi_{\sigma i} - \frac{1}{16}\gamma^{\eta\lambda}T_{\eta\lambda}{}^{ij}\gamma_\rho\psi_{\sigma j} + \frac{1}{4}\gamma_{\rho\sigma}\chi^i) , \\
f_\mu{}^\mu &= \frac{1}{6}R - D - \left(\frac{1}{12}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu{}^i\gamma_\nu\mathbb{D}_\rho\psi_{\sigma i} \right. \\
&\quad \left. - \frac{1}{12}\bar{\psi}_\mu{}^i\psi_\nu{}^jT_{ij}{}^{\mu\nu} - \frac{1}{4}\bar{\psi}_\mu{}^i\gamma^\mu\chi_i + \text{h.c.}\right) . \tag{B.1}
\end{aligned}$$

The Q -supersymmetry, special supersymmetry and special conformal transformation rules of the *independent* fields of the Weyl multiplet:

$$\begin{aligned}
\delta e_\mu{}^a &= \bar{\epsilon}^i\gamma^a\psi_{\mu i} + \text{h.c.} , \\
\delta\psi_\mu{}^i &= 2\mathbb{D}_\mu\epsilon^i - \frac{1}{8}\gamma^{\rho\sigma}T_{\rho\sigma}{}^{ij}\gamma_\mu\epsilon_j - \gamma_\mu\eta^i , \\
\delta b_\mu &= \frac{1}{2}\bar{\epsilon}^i\phi_{\mu i} - \frac{3}{4}\bar{\epsilon}^i\gamma_\mu\chi_i - \frac{1}{2}\bar{\eta}^i\psi_{\mu i} + \text{h.c.} + \Lambda_K^a e_{\mu a} , \\
\delta W_\mu &= \frac{1}{2}i\bar{\epsilon}^i\phi_{\mu i} + \frac{3}{4}i\bar{\epsilon}^i\gamma_\mu\chi_i + \frac{1}{2}i\bar{\eta}^i\psi_{\mu i} + \text{h.c.} , \\
\delta\mathcal{V}_\mu{}^i{}_j &= 2\bar{\epsilon}_j\phi_\mu{}^i - 3\bar{\epsilon}_j\gamma_\mu\chi^i + 2\bar{\eta}_j\psi_\mu{}^i - (\text{h.c.}; \text{traceless}) , \\
\delta T_{ab}{}^{ij} &= 8\bar{\epsilon}^{[i}\hat{R}_{ab}(Q)^{j]} , \\
\delta\chi^i &= -\frac{1}{12}\gamma^{ab}\not{D}T_{ab}{}^{ij}\epsilon_j + \frac{1}{6}\hat{R}_{\mu\nu}(SU(2))^i{}_j\gamma^{\mu\nu}\epsilon^j - \frac{1}{3}i\hat{R}_{\mu\nu}(U(1))\gamma^{\mu\nu}\epsilon^i \\
&\quad + D\epsilon^i + \frac{1}{12}\gamma^{\mu\nu}T_{\mu\nu}{}^{ij}\eta_j , \\
\delta D &= \bar{\epsilon}^i\not{D}\chi_i + \text{h.c.} , \tag{B.2}
\end{aligned}$$

where Λ_K^a is the parameter associated with special conformal transformations.

The derivatives D_μ appearing above are covariantized with respect to all superconformal transformations, while \mathbb{D}_μ denotes a derivative covariantized with respect to Lorentz transformations, dilatations and $U(2)_R$ transformations only (see below). Explicit expressions for the curvatures $\hat{R}(SU(2))$, $\hat{R}(U(1))$ and $\hat{R}(Q)$ will also be given below.

Vector multiplets

The Q -supersymmetry and special supersymmetry transformation rules of abelian vector multiplets:

$$\begin{aligned}
\delta X^\Lambda &= \bar{\epsilon}^i \Omega_i^\Lambda, \\
\delta A_\mu^\Lambda &= \varepsilon^{ij} \bar{\epsilon}_i \gamma_\mu \Omega_j^\Lambda + \varepsilon_{ij} \bar{\epsilon}^i \gamma_\mu \Omega^{j\Lambda} + 2\bar{X}^\Lambda \varepsilon_{ij} \bar{\epsilon}^i \psi_\mu^j + 2X^\Lambda \varepsilon^{ij} \bar{\epsilon}_i \psi_{j\mu}, \\
\delta \Omega_i^\Lambda &= 2\mathcal{D}X^\Lambda \epsilon_i + \frac{1}{2} \gamma_{\mu\nu} \mathcal{F}^{-\mu\nu\Lambda} \varepsilon_{ij} \epsilon^j + Y_{ij}^\Lambda \epsilon^j + 2X^\Lambda \eta_i, \\
\delta Y_{ij}^\Lambda &= 2\bar{\epsilon}_{(i} \mathcal{D} \Omega_{j)}^\Lambda + 2\varepsilon_{ik} \varepsilon_{jl} \bar{\epsilon}^{(k} \mathcal{D} \Omega^{l)\Lambda}.
\end{aligned} \tag{B.3}$$

Here

$$\mathcal{F}_{\mu\nu}^\Lambda = 2\partial_{[\mu} A_{\nu]}^\Lambda - (\varepsilon_{ij} \bar{\psi}_{[\mu}^i \gamma_{\nu]} \Omega^{j\Lambda} + \varepsilon_{ij} \bar{X}^\Lambda \bar{\psi}_\mu^i \psi_\nu^j + \frac{1}{4} \varepsilon_{ij} \bar{X}^\Lambda T_{\mu\nu}^{ij} + \text{h.c.}), \tag{B.4}$$

such that

$$\delta \mathcal{F}_{\mu\nu}^\Lambda = 2\varepsilon^{ij} \bar{\epsilon}_i \gamma_{[\mu} D_{\nu]} \Omega_j^\Lambda + \text{h.c.} . \tag{B.5}$$

Hypermultiplets

The Q -supersymmetry and special supersymmetry transformation rules of hypermultiplets (without a coupling to vector multiplets):

$$\begin{aligned}
\delta \phi^A &= 2(\gamma_{i\bar{\alpha}}^A \bar{\epsilon}^i \zeta^{\bar{\alpha}} + \bar{\gamma}_{\bar{\alpha}}^{Ai} \bar{\epsilon}_i \zeta^\alpha), \\
\delta \zeta^\alpha &= \mathcal{D} A_i^\alpha \epsilon^i - \delta_Q \phi^B \Gamma_B^\alpha{}_\beta \zeta^\beta + A_i^\alpha \eta^i,
\end{aligned} \tag{B.6}$$

where

$$A_i^\alpha = \chi^B V_{Bi}^\alpha, \quad \mathcal{D}_\mu A_i^\alpha = D_\mu A_i^\alpha + \partial_\mu \phi^A \Gamma_A^\alpha{}_\beta A_i^\beta, \tag{B.7}$$

with D_μ the derivative covariantized with respect to all superconformal transformations.

Weights and chirality

Weyl multiplet											
field	e_μ^a	ψ_μ^i	b_μ	W_μ	\mathcal{V}_μ^i	T_{ab}^{ij}	χ^i	D	ω_μ^{ab}	f_μ^a	ϕ_μ^i
w	-1	$-\frac{1}{2}$	0	0	0	1	$\frac{3}{2}$	2	0	1	$\frac{1}{2}$
c	0	$-\frac{1}{2}$	0	0	0	-1	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$
γ_5		+					+				-

Table B.I: Dilatational and $U(1)_R$ weights (w and c , respectively) and fermion chirality (γ_5) of the Weyl multiplet component fields.

vector multiplet				hypermultiplet		parameters		
field	X^Λ	Ω_i^Λ	A_μ^Λ	Y_{ij}^Λ	ϕ^A	ζ^α	ϵ^i	η^i
w	1	$\frac{3}{2}$	0	2	1	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
c	-1	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
γ_5		+				-	+	-

Table B.II: Dilatational and $U(1)_R$ weights (w and c , respectively) and fermion chirality (γ_5) of the vector and hypermultiplet component fields and of the supersymmetry transformation parameters.

Covariant derivatives

When a derivative is covariantized with respect to a set of transformations δ_A , it is given by

$$D_\mu = \partial_\mu - \sum_A \delta_A(h_\mu(A)) , \tag{B.8}$$

where $h_\mu(A)$ is the gauge field associated with δ_A . For the superconformal trans-

formations, the gauge fields are normalized like in [60],

$$\begin{aligned}
h_\mu^{ab}(M) &= \omega_\mu^{ab} , \\
h_\mu(D) &= b_\mu , \\
h_\mu(U(1)_R) &= W_\mu , \\
h_\mu^i{}_j(SU(2)_R) &= -\frac{1}{2}\mathcal{V}_\mu^i{}_j , \\
h_\mu^i(Q) &= \frac{1}{2}\psi_\mu^i , \\
h_\mu^i(S) &= \frac{1}{2}\phi_\mu^i , \\
h_\mu^a(K) &= f_\mu^a .
\end{aligned} \tag{B.9}$$

Supercovariant curvatures

$$\begin{aligned}
\hat{R}_{\mu\nu}(Q)^i &= 2\mathbb{D}_{[\mu}\psi_{\nu]}^i - \gamma_{[\mu}\phi_{\nu]}^i - \frac{1}{8}\gamma^{\eta\lambda}T_{\eta\lambda}^{ij}\gamma_{[\mu}\psi_{\nu]}j , \\
\hat{R}_{\mu\nu}(U(1)) &= 2\partial_{[\mu}A_{\nu]} - i\left(\frac{1}{2}\bar{\psi}_{[\mu}^i\phi_{\nu]}i + \frac{3}{4}\bar{\psi}_{[\mu}^i\gamma_{\nu]}\chi_i - \text{h.c.}\right) , \\
\hat{R}_{\mu\nu}(SU(2))^i{}_j &= 2\partial_{[\mu}\mathcal{V}_{\nu]}^i{}_j + \mathcal{V}_{[\mu}^i{}_k\mathcal{V}_{\nu]}^k{}_j \\
&\quad + (2\bar{\psi}_{[\mu}^i\phi_{\nu]}j - 3\bar{\psi}_{[\mu}^i\gamma_{\nu]}\chi_j - (\text{h.c.; traceless})) .
\end{aligned} \tag{B.10}$$

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Samenvatting

De wereld om ons heen (inclusief wij zelf) is opgebouwd uit atomen. Dit zijn zeer kleine objecten; in een druppel water zitten er al gauw 100000000000000000000! Toch bestaat een atoom zelf ook weer uit kleinere deeltjes. Het heeft een kern bestaande uit protonen en neutronen, waaromheen een wolk van elektronen hangt. Elektronen vallen niet verder op te delen; het zijn fundamentele deeltjes. Protonen en neutronen daarentegen, bestaan op hun beurt weer uit quarks, een ander type fundamenteel deeltje.

Welke krachten zijn werkzaam in een atoom? Om te beginnen worden de quarks in een proton of neutron bijeengehouden door de sterke kernkracht. Deze kracht zorgt er tevens voor dat de protonen en neutronen samenklonteren tot de atoomkern. Behalve de sterke kernkracht is binnen een atoomkern ook nog de zwakke kernkracht actief. Onder bepaalde omstandigheden wordt onder invloed van deze kracht een proton omgezet in een neutron of omgekeerd. Dit gaat dan gepaard met radioactieve straling. De atoomkern en de elektronenwolk die om hem heen hangt, blijven bij elkaar door de werking van de elektromagnetische kracht. De atoomkern is positief geladen en de elektronen negatief met als gevolg dat beide elkaar aantrekken.

Zo zien we binnen een atoom drie krachten aan het werk: de sterke kernkracht, de zwakke kernkracht en de elektromagnetische kracht. Dit zijn alle drie fundamentele natuurkrachten. De werking van deze drie krachten op het kleinste niveau wordt heel nauwkeurig beschreven door het zogeheten Standaard Model. Een belangrijk aspect van dit model is dat het fundamentele deeltjes behandelt alsof het perfecte puntjes zijn, zonder enige afmeting. Met behulp van het Standaard Model kun je zeer precieze voorspellingen doen over de uitkomsten van experimenten waarbij de allerkleinste deeltjes betrokken zijn. Hierbij is het van belang je te realiseren dat het gedrag van de natuur als je inzoomt op heel kleine schalen, heel anders is dan wat we gewend zijn op alledaags niveau. Op heel klein niveau wordt de natuur namelijk geregeerd door de wetten van de quantummechanica. Dit heeft allerlei effecten tot gevolg die ingaan tegen onze intuïtie. Zo heeft een heel klein deeltje de kans door een barrière heen te breken die op grond van klassieke (allegaagse) wetten eigenlijk te groot voor hem is ¹. Dit is alsof wij, als we tegen een muur op lopen, de kans hebben er door heen te gaan! Het Standaard Model kan zo succesvol zijn omdat het voldoet aan de quantummechanische wetten.

Naast de drie reeds genoemde krachten is er nog een vierde fundamentele natuurkracht. Dat is de zwaartekracht. Deze zorgt ervoor dat alles wat massa heeft elkaar aantrekt. De reden dat we deze kracht niet tegengekomen zijn in de behandeling van het atoom is dat de zwaartekracht ontzettend zwak is vergeleken met de andere natuurkrachten. Dit mag misschien raar klinken, maar valt als volgt te begrijpen. De sterkte van de zwaartekracht

¹Deze effecten zijn gerelateerd aan zogeheten instantonen. Zoals we verderop zullen zien zijn instantonen ook van belang in dit proefschrift.

die elk object uitoefent, wordt bepaald door zijn massa: hoe zwaarder een object, hoe sterker zijn zwaartekracht. Wij ervaren de zwaartekracht van de aarde. En de aarde is zwaar (zo'n miljoen maal een miljard maal een miljard kilo). Binnen een atoom is echter de massa van de kleine deeltjes van belang. Om te laten zien dat de corresponderende zwaartekracht dan te verwaarlozen is, doen we het volgende gedachtenexperiment. Twee losse elektronen bevinden zich op een kleine afstand van elkaar. Aangezien elektronen altijd hetzelfde type (nl. negatieve) elektrische lading hebben zullen ze elkaar afstoten. Echter, omdat elektronen niet alleen een lading maar tevens een massa hebben zullen ze elkaar ook, via de zwaartekracht, aantrekken. Er zijn dus twee tegengesteld gerichte krachten aan het werk. De vraag die we ons vervolgens stellen is: welk van de twee wint? Uit het antwoord blijkt de zwakte van de zwaartekracht. Ze is geen partij voor de elektromagnetische kracht. Als we namelijk op de elektronen de formules toepassen die de sterkte van de zwaartekracht en de elektromagnetische kracht berekenen op grond van massa en lading, vinden we dat de elektromagnetische kracht niet een beetje, maar ruwweg 10^{40} (een 1 met veertig nullen) maal sterker is! De elektronen zullen dus van elkaar af bewegen alsof de zwaartekracht niet bestaat.

Als we te maken zouden hebben met veel zwaardere kleine deeltjes, dan zou de zwaartekracht tussen die deeltjes ook veel sterker zijn. Deeltjes die zwaar genoeg zijn om de zwaartekracht een voorname rol te laten spelen, kennen we echter niet. Desondanks kan de zwaartekracht tussen elementaire deeltjes toch van belang worden. Via misschien wel de beroemdste natuurkundige formule die er is, $E = mc^2$, heeft Einstein ons geleerd dat massa en energie eigenlijk dezelfde dingen zijn (de formule zegt dat de energie E van een deeltje gelijk is aan zijn massa m vermenigvuldigd met de snelheid van het licht c in het kwadraat). Wanneer in een experiment twee deeltjes met grote snelheid (= veel energie) op elkaar afgeschoten worden, zal dit aanleiding geven tot een botsing waarbij een grote hoeveelheid energie betrokken is. Aangezien energie, zoals we gezien hebben, equivalent is aan massa, zal de zwaartekracht tussen die botsende deeltjes dan toch belangrijk worden.

Dit roept de vraag op: hoeveel energie moet aan de kleinste deeltjes meegegeven worden om de zwaartekracht van hetzelfde belang te maken als de andere natuurkrachten? Het antwoord is: ongeveer een miljard maal een miljard keer meer dan mogelijk is met hedendaagse deeltjesversnellers ². In de praktijk kunnen dit soort experimenten daarom (in elk geval voorlopig) niet gedaan worden. Dit neemt niet weg dat het interessant blijft wat er zou gebeuren mocht zo'n experiment wel uitgevoerd kunnen worden. Om hier een voorspelling over te kunnen doen is een theorie voor de zwaartekracht nodig die toepasbaar is op kleine deeltjes, dus die werkt op heel kleine schalen. Een zwaartekrachtstheorie voor de wereld van alledag is er. Deze theorie is ontwikkeld door Newton en later aangepast door Einstein. Een zwaartekrachtstheorie voor kleine schalen

²In Genève wordt momenteel gebouwd aan de grootste deeltjesversneller ter wereld, LHC genoemd. Als deze gereed is (waarschijnlijk eind dit jaar), zullen in een ondergrondse, cirkelvormige tunnel met een omtrek van 27 kilometer(!) zowel linksom als rechtsom kleine deeltjes opgestuwd worden tot hele grote snelheden (zodat ze in één seconde die tunnel wel 1000 keer rond gaan!). Bij botsingen tussen linksomgaande en rechtsomgaande deeltjes zijn dan enorme energiehoeveelheden betrokken. Enkele, nog niet onderzochte aspecten van het Standaard Model kunnen zo getest worden. Echter, zelfs deze botsingen zullen bij lange na niet krachtig genoeg zijn om zoveel energie op te wekken dat de zwaartekracht tussen de deeltjes belangrijk wordt... (hoewel, we moeten toegeven dat we dit niet helemaal zeker weten, het blijft theorie...).

is echter een ander verhaal. Want zoals we eerder al aangaven, op het niveau van de kleinste deeltjes gedraagt de natuur zich volgens de wetten van de quantummechanica. Een theorie die voorspellingen doet over het gedrag van hele kleine deeltjes onder invloed van de zwaartekracht moet dus een quantumzwaartekrachtstheorie zijn.

Je kunt je afvragen: wat hebben we nog meer aan zo'n theorie? Is ze ook van nut voor andere zaken dan het voorspellen wat er gebeurt als je kleine deeltjes met enorme snelheden op elkaar af schiet? Ja, dit is het geval. Zo zou een quantumzwaartekrachtstheorie ook meer inzicht kunnen geven in zwarte gaten. Zwarte gaten ontstaan als zware sterren opgebrand zijn en ineenstorten. Alle massa trekt dan samen en vormt een object met een minuscule omvang (kleiner dan dat van een atoom!). Dit object is het centrum van een zwart gat. Deze naam wordt gebruikt omdat de enorme zwaartekracht van een zwart gat niet toestaat dat er ook maar iets vandaan ontsnapt, zelfs lichtstralen niet. De reden dat een quantumzwaartekrachtstheorie nodig is om zwarte gaten goed te beschrijven is dat ook hier, net als bij botsende deeltjes met extreem hoge energieën, heel veel massa/energie (wat nodig is om de zwaartekracht belangrijk te maken) zich bevindt in een heel kleine ruimte (waardoor quantummechanische aspecten relevant worden). Hetzelfde was het geval tijdens de 'oerknal'. Uit de oerknal ontstond zo'n 14 miljard jaar geleden ons universum. Dit gebeurde door de explosie van een heel klein pakketje met een onwaarschijnlijk hoge energie (het bevatte in principe alles wat er zich nu in ons universum bevindt!). Dus een quantumzwaartekrachtstheorie zou zelfs van nut zijn voor het beter begrijpen van het ontstaan van ons universum.

In eerste instantie ging men er vanuit dat de werking van de quantumzwaartekracht vergelijkbaar is met de werking, op kleine schalen, van de drie andere fundamentele natuurkrachten. Een eerste poging voor een quantumzwaartekrachtstheorie was dan ook een model dat opgezet is op dezelfde manier als het Standaard Model. Zo'n theorie bleek echter niet te werken. Ze doet geen zinnige voorspellingen (heel in het kort uitgelegd: sommige berekeningen geven als uitkomst $\frac{1}{0}$ en zoals we op de middelbare school hebben geleerd: "delen door nul is flauwekul!"). Dus moest er in een andere richting worden gezocht. Op zoek naar alternatieve quantumzwaartekrachtstheorieën kwamen natuurkundigen uit bij de snaartheorie. Het idee achter de snaartheorie is het volgende: wat als we aannemen dat de elementaire deeltjes geen perfecte puntjes zijn (zoals in het Standaard Model is aangenomen), maar minuscule snaartjes. Kunnen we dan wél een goede quantumzwaartekrachtstheorie opstellen?

Het antwoord luidt ja. De snaartheorie leidt zelfs tot spectaculaire resultaten. Niet alleen doet ze zinnige voorspellingen over de uitkomst van experimenten met kleine deeltjes waarbij zwaartekracht betrokken is, ze voorspelt zelfs het bestaan van de zwaartekracht zelf! ³ En sterker nog, de snaartheorie voorspelt ook het bestaan van een aantal andere krachten. Deze krachten zouden heel goed de ons bekende elektromagnetische kracht, zwakke kernkracht en sterke kernkracht kunnen zijn. De snaartheorie is dus niet alleen een kandidaat quantumzwaartekrachtstheorie, maar zou zelfs een overkoepelende beschrijving kunnen geven van alle natuurkrachten. Ofwel ze is een kandidaat Theorie van Alles.

De snaartheorie heeft echter wel een rare eigenschap: ze voorspelt een universum met

³Natuurlijk wisten we al wel dat zwaartekracht bestond en is het dus eerder een *naspelling*, maar het feit dat een theorie uit zich zelf op het bestaan van de zwaartekracht uitkomt, zonder dat je het er in hebt gestopt, is zonder meer een reden om de theorie extra te waarderen. . .

negen dimensies. Dit lijkt misschien absurd en in tegenspraak met de drie dimensies die wij waarnemen ⁴. Toch maakt het de theorie niet direct onbruikbaar. Ten eerste, het mag mogelijk absurd lijken, maar wiskundig is een hoger-dimensionale ruimte geen probleem. Dat wij haar ons niet voor kunnen stellen doet daar niks aan af. Ten tweede, de snaartheorie staat toe dat een deel van de ruimte opgerold en als zodanig onzichtbaar is. Om duidelijk te maken wat we hiermee bedoelen vergelijken we de gehele ruimte voor het gemak met een blad papier. Zo'n blad is praktisch gezien twee-dimensionaal. Wanneer het papier echter strak opgerold is, vormt het een dunne koker. Als deze koker dun genoeg is, zal het net lijken alsof het maar één richting kent; het twee-dimensionale blad papier is een één-dimensionale papierspriet geworden. Op een vergelijkbare wijze kunnen zes van de negen dimensies die de snaartheorie voorspelt 'verdwenen' zijn. Alsof een negen-dimensionaal blaadje papier is opgerold tot een drie-dimensionale spriet. Probeer je hier geen voorstelling van te maken (dat is namelijk niet mogelijk). Besef alleen dat het kwalitatief hetzelfde is als een gewoon twee-dimensionaal blaadje papier dat opgerold is tot een één-dimensionale spriet.

De zes extra dimensies die de snaartheorie voorspelt kunnen op diverse manieren zijn opgerold. Dit geeft dan aanleiding tot verschillende drie-dimensionale universa. Dit roept natuurlijk de vraag op: zit ons universum daarbij? Ofwel, is er een scenario waarin de zes dimensies zo zijn opgerold, dat ons drie-dimensionale universum tevoorschijn komt? Het antwoord op deze vraag is van groot belang. De snaartheorie mag dan wel quantumzwaartekracht beschrijven (of zelfs voorspellen) en een kandidaat Theorie van Alles zijn, ze heeft alleen waarde als ze werkt voor ons universum. Het vinden van een scenario waarin de zes extra dimensies zo zijn opgerold dat ons drie-dimensionale universum verschijnt, is echter geen eenvoudige opgave. Hier wordt momenteel veel onderzoek naar verricht.

Ook is er nog een ander probleem in de snaartheorie: de theorie is weliswaar ontwikkeld vanuit het beschrijven van (één-dimensionale) snaartjes, inmiddels is duidelijk dat ze ook het bestaan van hoger-dimensionale deeltjes voorspelt. Dit betreft bijvoorbeeld twee-dimensionale objecten, genoemd membranen, maar ook objecten van een nog hogere dimensie. Al deze hoger-dimensionale deeltjes (met een algemene term 'branen' genoemd) zouden hun plaats moeten krijgen in de theorie. Welke rol ze spelen is echter nog allerminst duidelijk.

Met onze studie, waarvan de resultaten beschreven staan in dit proefschrift, hebben we een klein steentje bijgedragen aan het oplossen van bovenstaande problemen in de snaartheorie. Hieronder leggen we globaal uit wat de inhoud was van ons onderzoek.

Wij hebben gewerkt aan $N = 2$ supersymmetrische modellen. Deze vormen een bepaalde klasse van drie-dimensionale modellen die volgen wanneer de zes extra dimensies opgerold zijn. Ze zijn niet direct geschikt om ons universum mee te beschrijven, aangezien ons universum niet $N = 2$ supersymmetrisch is. Om te begrijpen wat supersymmetrie is en wat $N = 2$ supersymmetrische modellen zijn, is het noodzakelijk om het volgende te weten. Elk deeltje behoort òf tot de klasse van fermionen òf tot de klasse van bosonen. Zo zijn elektronen en quarks fermionen, maar is een lichtdeeltje een boson. Fermionen en bosonen verschillen in hun quantummechanische eigenschappen (het voert te ver pre-

⁴Een dimensie correspondeert met een richting waarin je kan gaan. Wij kennen drie basisrichtingen: naar boven/onder, naar links/rechts en naar voor/achter. Alle andere richtingen vormen een combinatie van deze drie basisrichtingen.

cies uit te leggen wat die verschillen zijn). Supersymmetrische modellen zijn modellen die onveranderd blijven wanneer je de wiskundige symbolen die fermionische deeltjes beschrijven en de wiskundige symbolen die bosonische deeltjes beschrijven verwisselt. De toevoeging $N = 2$ geeft aan dat modellen van dit type onveranderd blijven onder twee van dit soort verwisselingen ⁵. ($N = 2$) Supersymmetrische modellen hebben een eenvoudiger vorm dan modellen die niet supersymmetrisch zijn. Zo zijn de massa en lading van de fermionen en bosonen van welke de wiskundige symbolen verwisseld worden noodzakelijkerwijs hetzelfde (anders verkrijg je na verwisseling een beschrijving van fermionen/bosonen die een andere massa en lading hebben dan de oorspronkelijke fermionen/bosonen en is het model in kwestie dus wèl veranderd). Deze eenvoudige vorm zorgt er voor dat berekeningen in dit type modellen minder ingewikkeld zijn dan berekeningen in niet-supersymmetrische modellen. Daardoor is er makkelijker informatie uit te halen. Zo kan meer inzicht in de snaartheorie verkregen worden. In het bijzonder kan het bestuderen van $N = 2$ supersymmetrische modellen licht werpen op de relatie tussen de wijze van opgerold zijn van de zes dimensies en de resulterende drie-dimensionale universa. De hoop is dat deze inzichten uiteindelijk tevens van nut blijken voor het construeren van een (niet-supersymmetrisch) model dat ons universum beschrijft. Buiten dat zijn $N = 2$ supersymmetrische modellen van nut om het gedrag van branen (de hoger-dimensionale objecten in de snaartheorie) te onderzoeken.

Eén van de zaken die beschreven staan in dit proefschrift betreft de constructie van nieuwe modellen met $N = 2$ supersymmetrie. Deze modellen vormen een algemenere versie van al bestaande modellen. Waar de bestaande $N = 2$ supersymmetrische modellen alleen deeltjes met puur elektrische lading beschrijven, bevatten onze modellen deeltjes die zowel een elektrische als een magnetische lading hebben ⁶ (een combinatie van beide ladingen noemen we een ‘dyonische’ lading). Ons resultaat is in eerste instantie van belang voor het begrijpen van $N = 2$ supersymmetrische modellen op zich en hoe die volgen uit het opgerold zijn van de zes extra dimensies. Het vermoeden bestaat echter ook dat de klasse van modellen die wij ontwikkeld hebben universa kan beschrijven die in enkele opzichten vergelijkbaar zijn met het onze. Vervolgonderzoek zal uit moeten wijzen of dat vermoeden gerechtvaardigd is. Zo ja, dan kan het begrijpen van de samenhang tussen deze eigenschappen en het oprollen van de zes extra dimensies mogelijk wat meer inzicht leveren in hoe een model van ons universum te verkrijgen is uit de snaartheorie. Ook hebben we ons bezig gehouden met onderzoek naar zogenoemde instantonen in bestaande $N = 2$ supersymmetrische modellen. Zoals we al aangaven in de eerste voetnoot, beschrijven instantonen in het algemeen quantummechanische effecten die gerelateerd zijn aan het door barrières heen breken van kleine deeltjes. Instantonen verschijnen als oplossingen van wiskundige vergelijkingen van $N = 2$ supersymmetrische modellen. Ze corresponderen met branen (de hoger-dimensionale objecten in de snaartheorie) die zich bevinden in de opgerolde zes extra dimensies. Inzicht in deze instantonen en de

⁵Neem bijvoorbeeld een model met twee fermionen, F_1 en F_2 , en twee bosonen, B_1 en B_2 . Stel het model blijft onveranderd onder de verwisseling $F_1 \leftrightarrow B_1$, $F_2 \leftrightarrow B_2$ en ook onder de verwisseling $F_1 \leftrightarrow B_2$, $F_2 \leftrightarrow B_1$. Dan is het model in kwestie $N = 2$ supersymmetrisch.

⁶Magnetische lading heeft vergelijkbare effecten op deeltjes als elektrische lading: deeltjes met positieve en negatieve magnetische lading trekken elkaar aan, terwijl deeltjes met een lading van hetzelfde teken elkaar afstoten. In feite is het zelfs zo dat het alleen duidelijk is dat er twee typen lading zijn. Welk type lading elektrisch en welk type magnetisch is hangt af van het precieze model dat gebruikt wordt. Dit heet elektrisch/magnetische dualiteit.

effecten die ze beschrijven is om twee redenen van belang. Ten eerste werpt het licht op een deel van de rol die branen spelen in de snaartheorie. Ten tweede is het noodzakelijk instantoneffecten te begrijpen, om in te zien wat de modellen precies zijn die volgen uit het opgerold zijn van de zes extra dimensies. Dit laatste is weer van belang om de link te kunnen maken tussen de snaartheorie en een model dat ons universum beschrijft. In ons onderzoek hebben we twee verschillende klassen van instantonen beschreven. Deze corresponderen met verschillende type branen in de zes extra dimensies. Voor beide klassen hebben we een maat bepaald voor de sterkte van hun effect in de ($N = 2$ supersymmetrische) drie-dimensionale universa. Onze resultaten vormen een eerste aanzet op weg naar het bepalen van de precieze vorm van door branen veroorzaakte instantoneffecten.

Ons werk heeft (uiteraard) niet de bovengenoemde problemen in de snaartheorie volledig opgelost. Het vormt slechts een klein stapje in de goede richting. Voordat de snaartheorie volledig begrepen is (inclusief de precieze rol van de branen) en duidelijk is hoe ze gerelateerd is aan ons universum zal nog veel onderzoek nodig zijn. Theoretisch natuurkundigen over de hele wereld zijn hier intensief mee bezig.

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Curriculum Vitae

Ik ben geboren op 2 november 1976, te Groningen. Mijn VWO-opleiding genoot ik op het Stellingwerf College te Oosterwolde, waar ik in 1994 mijn diploma behaalde. Van 1994 tot 1997 en van 1998 tot 2003 was ik student aan de Universiteit Utrecht. Ik studeerde er Natuurkunde (zowel Meteorologie & Fysische Oceanografie als Theoretische Natuurkunde) en Geschiedenis (propaedeuse behaald in 1999). In 2003 studeerde ik af in de Theoretische Natuurkunde.

Van juni 2003 tot juni 2007 deed ik, in dienst van de stichting FOM, promotieonderzoek aan de Universiteit Utrecht. Mijn begeleiders waren dr. S. Vandoren (copromotor) en prof. dr. B. de Wit (promotor).

Tijdens mijn onderzoek heb ik geassisteerd bij het geven van onderwijs aan studenten. Ook heb ik verschillende onderzoeksscholen en conferenties bezocht en heb ik mijn werk uitgedragen middels het geven van voordrachten en posterpresentaties. De resultaten van mijn onderzoek zijn terug te vinden in dit proefschrift.

