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PROCESS ALGEBRA SEMANTICS FOR QUEUES

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An unbounded queue over a finite set of data values is a process Q in A^{∞} defined by an infinite system of guarded equations. The aim of this paper is to show that no finite system of guarded equations is capable of defining Q.

INTRODUCTION

An *unbounded queue*, (or buffer working in FIFO mode) is a device able to sequentially receive data values from a domain D, to store them and to deliver them in the order in which they were received,

In order to describe the behaviour of queue Q it is assumed that D is finite, moreover the input actions and output actions together form an alphabet of actions for Q. These actions exclude one another in time. In particular, for each $d \in D$ there are these actions:

d : input d
$\underline{d}$: output d

Sets $\underline{D}$ and A are defined by $\underline{D} = \{\underline{d} \mid \in D\}$, $A = D \cup \underline{D}$. Then Q can be semantically defined by the following infinite system of <u>guarded</u> equations.

$$\pi(\emptyset) = \sum_{d \in D} d.\,\pi(d)$$

$$\pi(s * d) = \underline{d}.\,\pi(s) + \sum_{d \in D} e.\pi(e * s * d),$$

where $S \in D^*$, $d \in D$ and $*$ denotes the concatenation of strings. $\emptyset \in D*$ denotes the empty string.

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In the above system of equations π (s), for $S \in D*$, can be thought as a process in A^∞ whose behaviour is that of a queue which initially contains a sequence s of data values. Assuming that Q is initially empty its behaviour is given by $\pi(\emptyset) = Q$.

Working in the two sorted system, containing both A^∞ and D^* as sorts as well as the auxiliary operator π, the above equations provide a finite equational specification of Q.

The aim of this paper is to show that no finite system of guarded equations in the standard model $A^\infty = (A^\infty, +, \cdot, \|, \lfloor\!\lfloor)$ of Process Algebra is capable of defining Q. This result is proved in Section 2. It turns out, however, that Q can be defined by a finite system of guarded equations in the signature containing the so-called <u>renaming operators</u>, cf. [1].

1. PROCESS ALGEBRA

Let A be a finite set of atomic actions. Processes are configurations of actions of A. Composition tools for processes are:

- $+$ alternative composition
- $.$ sequential composition
- $\|$ parallel composition (merge)
- $\lfloor\!\lfloor$ left merge.

The axioms of PA below, taken from [1] describe the operators; a varies over A.

$X + Y = Y + X$	(A1)
$X + (Y+Z) = (X+Y) + Z$	(A2)
$X + X = X$	(A3)
$(X+Y).Z = X.Z + Y.Z$	(A4)
$(X.Y).Z = X.(Y.Z)$	(A5)
$X\|Y = X\lfloor\!\lfloor Y + Y\lfloor\!\lfloor X$	(CM1)
$a\lfloor\!\lfloor X = a.X$	(CM2)
$(a.X)\lfloor\!\lfloor Y = a(X\|Y)$	(CM3)
$(X+Y)\lfloor\!\lfloor Z = X\lfloor\!\lfloor Z + Y\lfloor\!\lfloor Z$	(CM4)

Because A is finite PA is finite too. As an equational specification it has an initial algebra, called A_ω. As original references for processes and term models we mention [7] and [6].

For each n one may identify processes which coincide until depth n, thus obtaining a congruence $\equiv_n$ on A_ω. $A_n = A_\omega/\equiv_n$ is a model of PA as well.

The structures $A_n (n \in \omega)$ have a projective limit A^∞ which contains A_ω as a proper substructure. (A^∞ was introduced in [4] and is in fact an algebraic reconstruction of the topological process semantics in [2,3].) A^∞ serves us as a standard model for processes.

For processes $X \in A_\omega$ one defines projections $(X)_n$ as follows

$$(a)_n = a$$
$$(aX)_1 = a$$
$$(aX)_{n+1} = a\,(X)_n$$
$$(X+Y)_n = (X)_n + (Y)_n$$

The congruence $\equiv_n$ can be formally defined by $X \equiv_n Y \Leftrightarrow (X)_n = (Y)_n$.
An element of A^∞ is just a sequence

$$(P_1, P_2, P_3, \ldots)$$

with $P_n \in A_n$ (i.e. $(P_n)_n = P_n$) and for all n: $(P_{n+1})_n = P_n$. The operations $+, \cdot, \|$ and $\lfloor\!\lfloor$ are defined componentwise.

An equation $Y = T(X_1,\ldots,X_k)$ over A^∞ is guarded if every occurence of X_i in T is preceded in T by some atomic action.

A system of *guarded* fixed point equations $X_i = T_i(X_1,\ldots,X_k)$ $i = 1,\ldots,k$ always has a unique solution $\underline{X}_1,\ldots,\underline{X}_k$ in A^∞.

P is called *recursively definable* if there exists a (finite) system of guarded fixed point equations with solutions $\underline{X}_1,\ldots,\underline{X}_k$ such that $\underline{X}_1 = p$.

Recursive definitions are the most appropriate specification method in process algebra.

2. QUEUE CANNOT BE RECURSIVELY DEFINED IN $A^\infty(+,.,\|,\lfloor\!\lfloor)$

In this section we are going to prove the above statement. We assume that D has two different input actions a and b. Thus $A = \{a,b,\underline{a},\underline{b}\}$. A queue over a one-element set of input actions is just a "bag" and it is definable in $A^\infty(+,\cdot,\|,\lfloor\!\lfloor)$ by the following recursive equation $X = a(X\|\underline{a})$. (see [4]).

This section is organized as follows. We start with preliminary definitions in section 2.1, and prove some auxiliary results in section 2.2.
In section 2.3 the problem in first reduced to the same problem without $\|$ and $\lfloor\!\lfloor$. Then the latter question is settled in the negative.

2.1 Preliminaries

2.1.1 Definition.

Let $p \in A^{\infty}$. By the set of *states* of p we mean the least set $ST(p)$ satisfying the following conditions

(1) $p \in ST(p)$

(2) If $c.q + r \in ST(p)$, then $q \in ST(p)$, where $c \in A$, $q, r \in A^{\infty}$.

2.1.2 Definition.

The set of all *semistates* of p, $SST(p)$, is the least set satisfying these conditions

(3) $ST(p) \supseteq SST(p)$

(4) If $q + r \in SST(p)$, then $q \in SST(p)$ where $q,r \in A^{\infty}$.

2.1.3 Definition.

We say that a process h is a *factor* of a process p if for some $q \in A^{\infty}$, $h.q = p$.

In the context of this proof, however, a factor will be any process which is a factor of a semistate of Q. Let F(Q) denote the set of all factors.

A *trivial factor* is a factor in $F(Q) \cap A$. All other factors will be called *nontrivial.*

The states of Q are just all processes $\pi(s)$ with s ranging over D^*. Examples of semistates of Q which are not states are: $\underline{d}\pi(s)$, $d\pi(d*s)$, $\underline{d}_1\pi(s) + d_2\pi(d_2*s)$, where $d, d_1, d_2 \in D$, and $s \in D^*$. Examples of factors which are not semistates of Q are: a, b, $a + b\pi(b)$, $b(a\pi(ab) + b\pi(bb) + \underline{b})$.

The following inclusions are obvious: $ST(Q) \subseteq SST(Q) \subseteq F(Q)$.
The second inclusion does not hold in general. However, in the case of Q, for every $p \in SST(Q)$ and for every $q \in A^{\infty}$, $p.q = p$, which yields the required inclusion.

2.1.4 Let $\sigma \in A^*$, $p, q \in A^{\infty}$. We are going to define the relation $\sigma: p \to q$ which intuitively means that σ is a *path* in p which leads to q.

If $\phi \in A^*$ is the empty word, then $\phi: p \to q$ iff there exists $r \in A^{\infty}$ such that $p = q+r$.

Let $c \in A$, then $c: p \to q$ iff $p = c(q+r_1)+r_2$ for some $r_1, r_2 \in A^{\infty}$.

Finally, $\sigma c: p \to q$ iff for some $r \in A^{\infty}$ $\sigma: p \to r$, and $c: r \to q$.
The following fact is easy to prove by induction on the length of σ.

2.1.5 Fact.

(i) For every $\sigma \in A^*$ and for every $p,q \in A^{\infty}$, if $\sigma: p \to q$, then $q \in SST(p)$.

(ii) Moreover, for every $q \in SST(p)$ there exists $\sigma \in A^*$ such that $\sigma: p \to q$.
In the case of Q we can deduce more.

2.1.6 Fact.

(i) For every $q \in SST(Q)$ there exists a unique $\sigma \in \{a,b\}^*$ such that $\sigma: Q \to q$.

(ii) For every $q \in ST(Q)$ and for every $\sigma \in A^*$, $\sigma: Q \to q$ iff $\pi(\sigma) = q$.
The proof of this fact is easy and we leave it for the reader.

2.1.7 Now we are going to define the notion of a trace of a process $p \in A^{\infty}$.

A finite word $\sigma \in A^*$ is a *trace* if there exist $c \in A$ and $\tau \in A^*$ such that $\sigma = \tau c$ and $\tau : p \to c$. An infinite word $V \in A^{\omega}$ is a *trace* of p if there exists a sequence of initial fragments of V, $\sigma_1 < \sigma_2 < ... < \sigma_n < ...$
and a sequence of semistates of p, $q_1, q_2, ..., q_n, ...$
such that for every $n \in \omega$, $\sigma_n : p \to q_n$ and $q_{n+1} \in SST(q_n)$.

Let $tr(p)$ denote the set containing all finite and infinite traces of p.

A process is called *perpetual* iff it contains no finite traces. Clearly every semistate of Q is a perpetual process. The following fact is obvious.

2.1.8 Fact.

Let $\sigma \in tr(p) \cap A^*$. Then for every $q \in A^{\infty}$, $\sigma: pq \to q$.

Another general property of A^{∞} which we will use later has a straightforward proof as well.

2.1.9 Fact.

For all $q,r,p \in A^{\infty}$ and $c \in A$, if $q + r = cp$, then $q = r$.

2.1.10 We introduce two functions $O, I: A^{\omega} \to \{a,b\}^* \cup \{a,b\}^{\omega}$.

They are uniquely determined by the following properties

If $V \in \{a,b\}^{\omega}$, then $O(V) = \emptyset$

If $V \in \{\underline{a},\underline{b}\}^{\omega}$, then $I(V) = \emptyset$

If $c \in \{a,b\}$ and $V \in A^{\omega}$, then $I(cV) = cI(V)$, and $O(cV) = O(V)$.

If $\underline{c} \in \{\underline{a},\underline{b}\}$ and $V \in A^{\omega}$, then $I(\underline{c}V) = I(V)$, and $O(\underline{c}V) = cO(V)$.

Intuitively $I(V)$ (respectively $O(V)$) is the sequence of all input (output) actions of V in the order in which they occur in V.

Call $V \in A^{\omega}$ *input periodic* if there exists $\sigma \in \{a,b\}^+$ such that $I(V) = \sigma^{\omega}$. Otherwise V will be called input nonperiodic.

$V \in A^{\omega}$ is said to have infinitely many output actions if $O(V) \in \{a,b\}^{\omega}$.

The last result of this subsection is the following lemma.

2.1.11 Lemma.

Let $p_0, p_1 \in SST(Q)$ *be such that* $tr(p_0) \cap tr(p_1)$ *contains an input nonperiodic trace with infinitely many output actions. Then for some* $\sigma \in \{a,b\}^*$, $\sigma: Q \to p_0$ *and* $\sigma: Q \to p_1$.

Proof.

Let $\sigma_0: Q \to p_0$ and $\sigma_1: Q \to p_1$ for some $\sigma_0, \sigma_1 \in \{a,b\}^*$. Assume that $\sigma_0 \neq \sigma_1$. Let $V \in tr(p_0) \cap tr(p_1)$ be a trace with infinitely many output actions. We are going to show that V is input periodic.

Since $\sigma_i: Q \to p_i$, it follows that $\sigma_i \leq O(V)$ for $i = 0, 1$. Therefore either $\sigma_0 < \sigma_1$ or $\sigma_1 < \sigma_0$. We may assume without loss of generality that $\sigma_0 < \sigma_1$. Let $\tau \in \{a,b\}^+$ be such that $\sigma_1 = \sigma_0.\tau$.

Let V_0 be a sequence which results from V by removing from it the first $|\sigma_0|$ output actions. Since $V \in tr(p_0)$, it follows that $V_0 \in tr(Q)$.

Let V_1 be a sequence which results from V by removing from it the first $|\sigma_1|$ output actions. Again we have $V_1 \in tr(Q)$. Moreover we have the following relationships

(1) $I(V_0) = I(V_1) = I(V)$,

(2) $O(V) = \sigma_0\, O(V_0)$,

(3) $O(V) = \sigma_0\tau\, O(V_1)$.

Since V_0 is a trace of Q with infinitely many output actions, it follows that $O(V_0) = I(V_0)$. The same holds for V_1. Hence by (1), $O(V_0) = O(V_1)$ and by (2), (3) we obtain $O(V_0) = \tau\, O(V_0)$. Thus $I(V_0) = O(V_0) = \tau^{\omega}$. This, together with (1) proves the lemma.

2.2 Auxiliary results.

Let us start with the following result.

2.2.1 Lemma.

Let h *be a nontrivial factor. Then*

(i) h *has an input-nonperiodic trace with infinitely many output actions.*

(ii) *there is a unique* $p \in SST(Q)$ *such that* h *is a factor of* p.

Proof.

We prove (i) first. Let h be a nontrivial factor and let $q \in A^{\infty}$ be such that $hq \in SST(Q)$. If h is perpetual, then $hq = h$ and h is a semistate of Q. Then h obviously satisfies (i).

Suppose now that h is not perpetual and let σ be a finite trace of h. By Fact 2.1.8, $\sigma: hq \to q$. Therefore, by Fact 2.1.5 $q \in SST(Q)$. Let $\tau, \rho \in \{a,b\}^*$ be such that $\tau: Q \to q$, and $\rho: Q \to hq$ (cf. Fact 2.1.6).

Since hq is a semistate of Q it can be uniquely presented in the following form

(*) $hq = \Sigma_{c \in C} c.p_c$,

where $C \subseteq \{a,b,\underline{a},\underline{b}\}$, and $p_c \in A^{\infty}$ for every $c \in C$.

Consider these two cases

(1) There exists $c \in C$ such that *non* $\rho c: Q \to q$.

(2) For all $c \in C$, $\rho c: Q \to q$.

Notice that (2) may happen only if C is a one element set.

Suppose (1) holds. We construct a $V \in A^{\omega}$ such that

(3) $V \in tr(\pi(\rho c))$ is input-nonperiodic

(4) for every initial segment α of V, *non* $\rho c\alpha: Q \to q$.

(5) V has infinitely many output actions.

To see that such V exists take $\rho' \in \{a,b\}^*$ such that $\pi(\rho') = \pi(\rho c)$. If ρ' is not an initial subword of τ, then we may take as V any trace of Q which is input-nonperiodic and which has infinitely many output actions. If, however, ρ' is an initial subword of τ, then it follows from (1) that $\rho' \neq \tau$. Then it is enough to take as V eW, where $e \in \{a,b\}$ is such that $\rho' e$ is not an initial subword of τ, and $W \in tr(Q)$ is any input-nonperiodic trace with infinitely many output actions.

Let V be a trace satisfying (3)-(5). By (*) and (3), $cV \in tr(hq)$, and by (4) we obtain that cV must be a trace of h.

Suppose now that (2) holds. As we noticed this may happen only if C is a one element set, say $C = \{c\}$. Since $hq = cp_c$ and since h is nontrivial, there exist $h_1, h_2 \in A^{\infty}$ such that $h = ch_1 + h_2$.

Therefore, $ch_1q + h_2q = cp_c$.

By fact 2.1.9, $ch_1q = h_2q$, and we obtain $ch_1q = cp_c$.

The latter equality can be simplified to $h_1q = p_c$.

Since h_1q is a state of Q, it follows that in the decomposition (*) for h_1q the set C has at least two elements. Thus the argument of case (1) is applicable to h_1, and we may conclude that h_1 contains an input-nonperiodic trace with infinitely many output actions. Therefore h contains such a trace as well. This completes the proof of (i).

Now we prove (ii). Let $p_0, p_1 \in SST(Q)$ be such that for some $q_0, q_1 \in A^{\infty}$,

(6) $h.q_i = p_i$ for $i = 0, 1$.

By (i) h has an input-nonperiodic trace with inifitely many out actions. Therefore, by (6), $tr(p_0) \cap tr(p_1)$ contains such a trace and by Lemma 2.1.11 for some $\sigma \in \{a,b\}^*$,

(7) $\sigma: q \to p_i$, for $i = 0, 1$.

If $p_0 \neq p_1$, then by (7) this is only possible if for some $c \in A$ and $i_0 \in \{0, 1\}$, p_{i0} has a trace starting with c and in p_{1-i0} all traces start with a symbol different from c. Since $hq_{i0} = p_{i0}$, h must have a trace as well. Obtained contradiction proves (ii), and the proof of Lemma 2.2.1 is completed.

2.2.2 Lemma.

(i) *if* $h_1 + h_2 \in F(Q)$ *then* $h_1, h_2 \in F(Q)$

(ii) *if* $h_1 . h_2 \in F(Q)$ *then* $h_1 \in F(Q)$, *moreover if* h_1 *is not perpetual then* $h_2 \in F(Q)$ *as well*

(iii) $h_1 \parallel h_2 \notin F(Q)$

(iv) $h_1 \lfloor\!\lfloor h_2 \notin F(Q)$ *provided* h_1 *is not a sum of atoms.*

Proof.

(i) Suppose $(h_1+h_2).r = q$, $q \in SST(Q)$, then $h_1.r + h_2.r = q$ thus $h_1.r$, $h_2.r \in SST(Q)$ whence by definition h_1, $h_2 \in F(Q)$.

(ii) Suppose $(h_1.h_2).r = q, q \in SST(Q)$, then $h_1(h_2.r) = q$ so $h_1 \in F(q)$. If σ is a finite trace of h_1 then by 2.1.8 σ: $h_1(h_2.r) \to h_2.r$, and also σ: $q \to h_2.r$. So $h_2.r \in ST(Q)$ by fact 2.1.5 and by definition $h_2 \in F(Q)$.

(iii) Suppose $h_1 \parallel h_2 \in F(q)$. Let $q \in A^\infty$ be such that $(h_1 \parallel h_2).q \in SST(Q)$, $h_1 \parallel h_2$ cannot be a,b,$\underline{a}$, or $\underline{b}$ so $h_1 \parallel h_2$ is a nontrivial factor. In view of Lemma 2.2.1 $h_1 \parallel h_2$ has an input nonperiodic infinite trace V with infinitely many output actions. V must have both infinitely many $\underline{a}$'s and $\underline{b}$'s. Let V_1 and V_2 be traces of h_1 resp. h_2 such that V can be obtained by merging V_1 and V_2.

Choose $\sigma \in \{a,b\}^*$ such that σ: $Q \to (h_1 \parallel h_2).q$. Then for every sequence U obtained by merging V_1 and V_2 $\sigma U \in tr(Q)$. We will manufacture a contradiction from this situation. First of all we notice that either V_1 or V_2 contains no output actions, otherwise V_1 must contain an action $\underline{a}$ and V_2 an action $\underline{b}$ or conversely. Let us assume that $V_1 = \sigma_1 \underline{a} U_1$, $V_2 = \sigma_2 \underline{b} U_2$ then $\sigma\sigma_1\sigma_2\underline{a}U_1$ and $\sigma\sigma_1\sigma_2\underline{b}U_1$ are both traces of Q (because V_1 is infinite U_1 is infinite and both are σ followed by a merge of V_1 and V_2). Now this is impossible because after $\sigma\sigma_1\sigma_2$ at most one output is possible. So suppose that V_1 contains infinitely many $\underline{a}$'s and $\underline{b}$'s and V_2 contains only a's and b's; σV_1 is a trace of Q, inserting the first action, say a, from V_2 in σV_1 at some position after σ must also produce a trace of Q. However choose ρ and W such that $V_1 = \rho \underline{b} W$ then $\sigma\rho a \underline{b} W$ cannot be a trace of Q because the output action in V_2 that corresponds to the displayed input b has now become incorrect. Thus we have obtained a contradiction thereby proving (iii) of the Lemma.

(iv) the case for $\lfloor\!\lfloor$ is similar to the previous one.

2.3 Proof of the main result.

Let us now consider a recursive definition of Q: $X_i = T_i(X_1,...,X_n)$ $i = 1,...n$ with solutions $\underline{X}_1,...,\underline{X}_n$, and $\underline{X}_1 = Q$.

Without loss of generality we may assume that the system has the following properties:

(α1) All $\underline{X}_i$ are infinite (otherwise they can be eliminated by substitution).

(α2) All $\underline{X}_i$ are "used" (i.e. no proper sybsystem defines $\underline{X}_1 = Q$ as well).

(α3) None of the T_i has a subterm of the form $(t_2+t_2).t_3$. (Such subterms are eliminated using the equation A4).

(α4) None of the T_i has a subterm $t_1 t_2$ with $t_1(\underline{X}_1,...,\underline{X}_n)$ perpetual (in such cases t_2 can just be omitted and the $\underline{X}_1,...,\underline{X}_n$ still constitute a unique solution).

(α5) In subterms of the form $t_1 \lfloor\!\lfloor t_2$, t_1 is not a sum of atoms. (Otherwise use CM4 and CM3).

We can now start the actual proof of the main result of this paper with the following lemma.

2.3.1 Lemma.

If Q *can be recursively defined in* $A^\infty(+,\cdot,\parallel,\lfloor\!\lfloor)$ *then it can be recursively defined in* $A^\infty(+,\cdot)$.

Proof.

We assume that the system

$$X_i = T_i(X_1,...,X_n) \quad i = 1,...,n$$

satisfies the requirements (α1)-(α5) above and defines Q. Let K be the collection of all subterms of the $T_i(X_1,...,X_n)$. We can define a distance $d(\cdot,\cdot)$ between elements of K. d is not symmetric, however:

(i) $d(t,t) = 0$

(ii) if t' is an immediate subterm of t then $d(t,t') = 1$

(iii) $d(t_1,t_2) = \min\{d(t_1,t')+d(t',t_2) \mid t' \in K\}$.

Now it follows that for each $t \in K$, $d(X_1,t)$ is defined. With induction on $d(X_1,t)$ one shows using Lemma 2.2.2 that

(*) for each $t(X_1,...,X_n) \in K$, $t(\underline{X}_1,...,\underline{X}_n)$ is in F(Q).

Moreover by 2.1.1 we conclude that $\parallel$ and $\lfloor\!\lfloor$ do not occur in any of the $t \in K$. This proves the lemma.

2.3.2 Lemma.

If Q *has a recursive definition in* $A^\infty(+,\cdot)$ *then* ST(Q) *is generated (in* $A^\infty(+,\cdot)$) *by finitely many states* $\pi(\sigma_1),...,\pi(\sigma_K) \in ST(Q)$.

Proof.

According to [7] it is in general the case that the solutions $\underline{X}_1,...,\underline{X}_n$ of a system of recursive equations generate all states of $\underline{X}_1,...,\underline{X}_n$ and in particular of $\underline{X}_1$.

So let $X_i = T_i(X_1,...,X_n)$, $i = 1,...,n$, be a recursive definition in $A^\infty(+,\cdot)$ again satisfying (α1)-(α5) above, with solution $\underline{X}_1,...,\underline{X}_n$ and $\underline{X}_1 = Q$. Then for each $q \in ST(Q)$ there is a term $t(X_1,...,X_n)$ in $(A,+,.)$ such that $t(\underline{X}_1,...,\underline{X}_n) = \dot{q}$.

Let ϕ: $F(Q) \to SST(Q)$ be the mapping which assigns to each $h \in F(Q)$ the

unique (in view of Lemma 2.2.1 (ii) $q \in SST(Q)$ such that for some $p \in A^{\infty}$ $h.p = q$.
It follows from (*) in the proof of Lemma 2.3.1 that for every $i = 1,...n$, $X_i \in F(Q)$.

Claim.

For every term $t(X_1,...,X_n)$ in $(A, +, .)$, if $t(\underline{X}_1,...,\underline{X}_n) \in SST(Q)$ then $t(\underline{X}_1,...,\underline{X}_n = t(\phi(\underline{X}_1),...,\phi(\underline{X}_n))$.

Using this claim one finds that every $q \in ST(Q)$ is generated by the semistates $\phi(\underline{X}_1),...,\phi(\underline{X}_n)$.

Now, by the special definition of Q and since D has two elements, each semistate $\phi(\underline{X}_i)$ can be written as $c_1{}^i\pi(\sigma_1{}^i) + c_2{}^i\pi(\sigma_2{}^i) + c_3{}^i\pi(\sigma_3{}^i)$,
for appropriate $c_j{}^i \in A$ and $\sigma_j{}^i \in \{a,b\}^*$.
It follows that the subset $\{\pi(\sigma_j{}^i) \mid 1 \le i \le n, 1 \le j \le 3\}$
of $ST(Q)$ generates all of $ST(Q)$, thus proving the lemma.

Proof (of the Claim).

Let $L = L(X_1,...,X_n)$ be the following inductively defined collection of terms:

(i) $X_i \in L$, for $i = 1,...,n$
(ii) $c.t \in L$, for $c \in A$, $t \in L$
(iii) $X_i.t \in L$, if $t \in L$
(iv) $t_1 + t_2 \in L$, if $t_1, t_2 \in L$
(v) $c \in L$, for $c \in A$.

Now each term t over $+,.,A,X_1,...,X_n$ is equivalent in PA to a term in L, and therefore it suffices to prove the claim for every term in L. We prove the claim by induction on the structure of $t \in L$.

Let us observe that $\phi(p) = p$ for every $p \in SST(Q)$.
We consider all cases generated by the inductive clauses (i),...,(v).

(i) is immediate
(ii) if $c.t(\underline{X}_1,...,\underline{X}_n) \in SST(Q)$ then $t(\underline{X}_1,...,\underline{X}_n) \in SST(Q)$. So $\phi(t(\underline{X}_1,...,\underline{X}_n)) = t(\phi(\underline{X}_1),...,\phi\underline{X}_n))$ and $\phi(c.t(\underline{X}_1,...,\underline{X}_n)) = c.t(\phi(\underline{X}_1),...,\phi(\underline{X}_n))$.
(iii) if $\phi(X_i t(\underline{X}_1,...,\underline{X}_n)) \in SST(Q)$ then $(\underline{X}_i t(\underline{X}_1,...,\underline{X}_n)) = \underline{X}_i t(\underline{X}_1,...,\underline{X}_n) = \phi(\underline{X}_i) = \phi(\underline{X}_i).t(\phi(\underline{X}_1),...,\phi(\underline{X}_n)$.
(iv) if $t_1(\underline{X}_1,...,\underline{X}_n) + t_2(\underline{X}_1,...,\underline{X}_n) \in SST(Q)$ then both summands are in $SST(Q)$ hence $\phi(t_1(\underline{X}_1,...,\underline{X}_n) + t_2(\underline{X}_1,...,\underline{X}_n)) = t_1(\phi(\underline{X}_1),...,\phi(\underline{X}_n)) + t_2(\phi(\underline{X}_1),...,\phi(\underline{X}_n))$.
(v) c is not in $SST(Q)$.

2.3.3 Lemma.

There is no finite subset $\pi(\sigma_1)...\pi(\sigma_K)$ *of* $ST(Q)$ *which generates all of* $ST(Q)$ *within* $A^{\infty}(+,\cdot)$.

Proof.

Suppose otherwise. Choose for each $\pi(\sigma_i)$ a triple $\pi(t_i{}^1)$, $\pi(t_i{}^2)$, $\pi(t_i{}^3)$ such that for appropriate $c_i{}^j \in A$, e_i: $\pi(\sigma_i) = c_i{}^1.\pi(\tau_i{}^1) + c_i{}^2\pi(\tau_i{}^2) + c_i{}^3\pi(\tau_i{}^3)$, then choose for each $\pi(\tau_i{}^j)$ a term $t_i{}^j(X_1,...,X_K)$ such that $\pi(\tau_i{}^j) = t_i{}^j(\pi(\sigma_1),...,\pi(\sigma_K))$. The term $t_i{}^j$ may be chosen such that it contains + and prefix multiplication only because all $\pi(\sigma_i)$ are perpetual and $\pi(\sigma_i).t$ can be replaced by $\pi(\sigma_i)$.

Substituting these identities into e_i one obtains a linear system of equations for the processes $\pi(\sigma_i)$. According to [5] the $\pi(\sigma_i)$ will then be regular which is certainly not the case. Combining lemmas 2.3.1, 2.3.2 and 2.3.3 we obtain the main result of this paper:

Theorem.

Q *cannot be recursively defined in* $A^{\infty}(+,\cdot,\|,\lfloor\!\lfloor)$.

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