Zeta function rigidity

A view from noncommutative geometry

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Zeta functie-rigiditeit

Vanuit het gezichtspunt van de niet-commutatieve meetkunde

(met een samenvatting in het Nederlands)

Proefschrift

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Do not go where the path may lead, go instead where there is no path and leave a trail - *Ralph Waldo Emerson*

Cover illustation:
On the cover one sees a drum emitting the zeta invariants of Theorem 3.3.1. This illustration is inspired by a paper of Mark Kac in 1966 entitled: “Can one hear the shape of a drum?” In that case the answer is no. However, when one listens to the generalized spectrum mentioned in the theorem, one can.
Note that the part of the drum that determines the spectrum you actually hear is not closed, nonetheless it is a nice metaphor.

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Chapter 1

Introduction

Noncommutative geometry

This thesis belongs to the field of noncommutative geometry and in particular to the study of finitely summable spectral triples and the corresponding zeta-function formalism.
In noncommutative geometry, algebraic commutative structure arising from geometric objects is generalized to the noncommutative setting. Usually this is done in such a way that the commutative structure determines the geometric object. The noncommutative generalization is then called a 'noncommutative space', which usually does not refer to a genuine space (for instance a manifold).
To illustrate this, let us start with a well-known duality: Take a compact Hausdorff space (geometric object), then take the continuous complex valued functions on this space considered as a unital commutative C^*-algebra, with as norm the supremum norm and as involution complex conjugation (commutative structure). According to the Gelfand-Naimark theorem, this commutative C^*-algebra encodes the space we started with and also behaves well with respect to maps, that is to say, that the construction is functorial.
For example, the circle S^1 has as C^*-algebra

$$\{f \mid f : S^1 \to \mathbb{C}, \text{continuous}\},$$

and according to the Gelfand-Naimark theorem, only the circle has this particular C^*-algebra.

A noncommutative generalization of this is to consider all unital C^*-algebras as 'noncommutative compact Hausdorff spaces': noncommutative generalization of a topological space.

Spectral triples

Let us now consider smooth Riemannian spin manifolds. These have a lot more structure than the previous example: Apart from the topology there is a manifold structure consisting of an atlas, a metric tensor and finally a spin structure.
The translation of this geometric object to a commutative structure is a priori far from clear.
It turns out that one can encode the whole geometry in a 'spectral triple'. This was proven by Alain Connes in 2008 in [12].
We will construct this triple and then give the general definition of a spectral triple, generalized to the noncommutative setting. Given a smooth compact Riemannian spin manifold X, define a triple (A,H,D) as follows:
Take $A = C(X,\mathbb{C})$, the continuous complex valued functions on the manifold, considered as C^*-algebra, so encoding the topology. For H we take $H = L^2(X,E)$, the space of square integrable sections of its spin bundle E, seen as Hilbert space. Finally for D we take the Dirac operator. Note that A acts on H by multiplication, which is a bounded operator.
Now Alain Connes has given necessary and sufficient conditions for a spectral triple (A,H,D) to correspond to a Riemannian spin manifold and actually gave a way to reconstruct it from the triple (for details see [12]). These triples are commutative, which means that A is commutative.
As depicted before, we want to generalize this concept to the noncommutative setting and eventually this boiled down to the following definition (see for instance [10],[11]):

Definition (Spectral triples)**.** A unital *spectral triple* $\mathscr{S}$ consists of a triple $\mathscr{S} = (A,H,D)$, where A is a unital C^*-algebra, H a Hilbert-space on which A acts faithfully by bounded operators and D is an unbounded, self adjoint operator, densely defined on H, such that $(D-\lambda)^{-1}$ is a compact operator for $\lambda \notin \mathrm{Spec}(D)$, and such that all the commutators $[D,a]$ are bounded operators for a in a dense, involutive subalgebra $A_\infty \subset A$ satisfying $a(\mathrm{Dom}(D)) \subset \mathrm{Dom}(D)$.
A triple (A,H,D) is called p-summable if the trace $\mathrm{tr}((1+D^2)^{-p/2})$ is finite.

We will call D a Dirac operator. In case of spin manifolds, the metric data is stored in D: if p and q are points on the manifolds, then the distance can be expressed in the following way (see [10], Chapter 6):

$$d(p,q) = \sup\{|f(p)-f(q)| \mid \|[D,f]\| \leq 1\},$$

where on the right hand side, f ranges over all $f \in A$ for which the commutator with D is bounded by one. The p-summability in case of spin manifolds means that the dimension of the manifold is at most p.

Let us again look at the circle. Regard it now as Riemannian manifold by giving it a radius $r > 0$. We will construct a spectral triple for it.
Functions on the circle correspond to $2\pi r$-periodic functions on the real line. Denote this algebra of functions by A_r.
Let H_r be the Hilbert-space of square integrable functions on the circle with standard Lebesgue measure:

$$H_r = L^2([-\pi r, \pi r])$$

The algebra A_r acts on H_r by multiplication. The Dirac operator D, densely defined on H_r is given by

$$(Df)(x) = \frac{1}{i}\frac{\mathrm{d}f}{\mathrm{d}x}(x).$$

The eigenfunctions of D are given by

$$\phi_k^r(x) = \frac{1}{\sqrt{2\pi r}} e^{\frac{ikx}{r}}$$

for $k \in \mathbf{Z}$, which form a basis for H_r. The eigenvalues λ are given by $\{k/r \mid k \in \mathbf{Z}\}$ and hence the zeta function as

$$\begin{aligned} \zeta_{S_r^1}(s) &= \operatorname{tr}(|D|^{-s}) \\ &= \sum_{\lambda \neq 0} |\lambda|^{-s} \\ &= 2r^{-s}\zeta(s), \end{aligned}$$

where the latter is the Riemann zeta function.
In case we restrict ourselves to circles, this single zeta function would be a complete invariant, because a circle is as Riemannian manifold determined by its radius.

To mention some more spectral triples: There are spectral triples constructed for Kleinian Schottky groups in [14], for trees and buildings in [21], for certain fractal sets like the Sierpinski gasket in [9] and there are many more.

Zeta functions and rigidity

Let us briefly forget about the spectral triple $\mathscr{S}$ and just consider the Dirac operator. Even more, just consider its spectrum Λ, where we include the multiplicities of the eigenvalues. The non-zero spectrum is in absolute value contained in the following zeta function, which is the meromorphic continuation of

$$\zeta_{\mathscr{S}}(s) = \sum_{\lambda \in \Lambda \setminus \{0\}} |\lambda|^{-s}$$

to $\mathbb{C}$ (depending on the context, sometimes this is defined by $-s$ replaced by $+s$).
Let us ask the following question: Does the spectrum of the Dirac operator determine the spin manifold? And if not, does it capture its isometry class, or at least its homeomorphism class?
The answer to this question is that it determines neither: there exist non-homeomorphic isospectral spin manifolds (see [32], [48], [36]). It does however determine geometric invariants like its volume and dimension.
The next question is then of course: Is there a natural way to remedy this?
This is where we go back to spectral triples. Finitely summable triples give rise to whole families of zeta functions, indexed using the algebra A. Let us start with the first family:
For any $a \in A$, define the zeta-function as (the meromorphic continuation of):

$$\zeta_{\mathscr{S},a}(s) = \mathrm{tr}(a|D|^{-s}).$$

Here $|D|$ is the absolute value of the Dirac operator and the trace is over the Hilbert space H.
We see that we recover the first zeta function as distinguished member of this family at $a = 1$:

$$\zeta_{\mathscr{S},1}(s) = \zeta_{\mathscr{S}}(s)$$

We can define a second family of zeta functions. For $a_1, a_2 \in A$ define:

$$\zeta_{\mathscr{S},a_1,a_2}(s) = \mathrm{tr}(a_1[D,a_2]|D|^{-s}).$$

And further, for any $p \geq 2$ we define for any $a_1, a_2, \dots a_p \in A$ the following family of zeta functions as:

$$\zeta_{\mathscr{S},a_1,a_2,\dots,a_p}(s) = \mathrm{tr}(a_1[D,a_2]\cdots[D,a_p]|D|^{-s}).$$

Now we can state the following general question:
Start with some geometric objects with (or construct if not available / appropriate) spectral triples attached to it. Do there exist some (sub)families of zeta-functions which determine the geometric object?
This question is pursued for instance in [19]. In this paper a spectral triple is assigned to a Riemann surface. The main result is that if two Riemann surfaces X_1, X_2 of genus at least 2 have the same zeta functions of the first class, i.e., there exists a C^*-algebra morphism of the algebras such that the corresponding zeta functions match up, then the Riemann surfaces are equivalent as complex analytic manifolds (up to complex conjugation).
Furthermore, there is some analogy between L-series in number theory and spectral zeta functions as discussed here. To illustrate this, one of the first constructions of isospectral, non-isometric Riemannian manifolds was given by Sunada in [45]. The ideas used there to construct these are inspired by number theory, where Dedekind zeta functions of non-isomorphic number fields can be equal. Now, in [20] the isospectrality phenomenon of number fields is tackled by families of L-series.

Zeta rigidity for graphs

We considered this question for graphs, more specific, for finite, connected unoriented graphs with valencies at least 3 and first Betti number g at least 2, which we will call the genus from now on.
The first problem is that there is no suitable spectral triple for a graph for this construction: In [39] a spectral triple is constructed for graphs using incidence relations on the graph, but these spectral triples are not finitely summable and hence there are no zeta functions (in our sense) to start with.
In this chapter we will construct a completely different spectral triple. It turns out that this spectral triple is finitely summable. Furthermore, we can prove a rigidity theorem for this spectral triple.

For this spectral triple $\mathscr{S}$ there are many isospectral graphs: The spectrum of the Dirac operator itself, contained in $\zeta_{\mathscr{S},1}(s) = \operatorname{tr}(|D|^s)$, does not determine the graph. In fact, it only determines the genus by the function (cf. formula 2.21):

$$\zeta_{\mathscr{S},1}(s) = 1 + (2g)^{3s}(2g-1)\left\{\frac{1-(2g-1)^{3s-1}}{1-(2g-1)^{3s+1}}\right\}$$

We will now briefly outline the construction of the spectral triple $\mathscr{S}_X$.
Given a graph X as before with universal covering tree E_X, represent the fundamental group Γ as a group of isometries of E_X. By Proposition 2.2.3 there exists a Γ-equivariant homeomorphism $\Phi_X : \partial F_g \to \partial X$, where ∂F_g is the boundary of the Cayley graph of Γ. For the triple, take for the algebra $A = C(\partial F_g, \mathbb{C})$, the continuous, complex valued functions on the boundary of the Cayley graph. Note that by the aforementioned Gelfand-Naimark theorem this algebra encodes the topology on ∂F_g, which is the Cantor set.
As a side remark, a natural candidate for the algebra in noncommutative geometry would have been $A \rtimes \Gamma$, but by Connes' result on non-amenable discrete groups (which a free group of rank at least 2 is), this would not give rise to a *finitely* summable spectral triple which we are after (see [10], Theorems 17 and 19, pp. 214-215).
The Hilbert space H is the completion of A with respect to the norm induced by integration of the induced measure from the Patterson-Sullivan measure on ∂X to ∂F_g via Φ_X (see Definition 2.1.9) on which A acts by multiplication. Finally, the Dirac operator D is composed of projection operators depending on the word grading in Γ (see Definition 2.2.9).

The first main result is the following (see Theorem 2.4.6):

Theorem A. *The spectral triple $\mathscr{S}_X$ determines the graph X.*

In this theorem by the word 'determines' we mean the following:
Consider all graphs under consideration and construct all spectral triples, also exhausting the choices. Then the theorem says that if one knows that a spectral triple comes from a graph, then this graph is the *unique* graph giving this spectral triple. In particular it is not a reconstruction theorem, but another way to look at graphs.

It turns out that the construction of the spectral triple depends on the choice of an origin (a vertex in the graph) and on a minimal set of chosen generators for Γ. To deal with these arbitrary choices we have to consider them all.
As it turns out, we only need the first family of zeta functions to determine the graph we started with, the theorem is the following (see Theorem 2.4.1):

Theorem B. *Let X_1, X_2 be finite, connected graphs of first Betti number $g \geq 2$ and valencies ≥ 3. Then:*

(a) $\zeta_1^{X_1} = \zeta_1^{X_2}$ *if and only if* X_1 *and* X_2 *have the same genus.*

(b) *If (a) holds, then* $X_1 \cong X_2$ *if and only if there exists a choice of origins and and minimal sets of generators such that*

$$\zeta_a^{X_1} = \zeta_a^{X_2}$$

for all $a \in A_\infty$.

Note that to be able to formulate (b) the genus must be the same, because A_∞ depends on the genus.
A key ingredient in the proof is that one is able to extract the Patterson-Sullivan measure from the family of zeta functions (see Proposition 2.3.3), which can be used for the reconstruction of the graph.

Zeta rigidity for Riemannian manifolds

The ultimate goal would be to prove some zeta function rigidity for spin manifolds, as this is the original habitat of spectral triples.
This however runs not only into technical trouble, but there is also the problem of which families of zeta functions to take. Hence we started with a somewhat simpler case, namely Riemannian manifolds, basically omitting the spin structure. Consequently there is no Dirac operator, but there is a natural operator to study, namely the Laplace-Beltrami operator, which in case of spin manifolds is the square of the Dirac operator.
For closed Riemannian manifolds there is also the phenomenon of isospectrality, there exist non-isometric/homeomorphic Riemannian manifolds for which the Laplace-Beltrami operator has the same spectrum (including multiplicities), see [36] and later work and also [32], [48].
The spectrum Λ does contain geometric information about the manifold. For this one can use the asymptotic Weyl formulae (see for instance [40]): If X is a closed Riemannian manifold of dimension n, then one can write down the following asymptotic expansion for $t \downarrow 0$:

$$\sum_{\lambda \in \Lambda} e^{-\lambda t} \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} a_k t^k,$$

with

$$a_k = \int_X u_k(x,x)\mathrm{d}_{\mu_X},$$

where the u_k are universal polynomials in the covariant derivatives of the curvature tensor.
Now, if Y is a Riemannian manifold of dimension m, which is isospectral to X, then the behaviour of t implies that $m = n$ and furthermore

$$\int_X u_k \mathrm{d}_{\mu_X} = \int_Y u_k \mathrm{d}_{\mu_Y}.$$

So this is gives infinite constraints for isospectrality. These constraints pose geometric conditions. For instance, $u_0(x,x) = 1$, which implies that X and Y must have the same volume. And for the next one,

$$u_1(x,x) = \frac{1}{6}s(x),$$

with $s(x)$ the total scalar curvature, so that both manifolds must have the same integrated scalar curvature. The meaning of the higher orders is not so clear.

As before, we regard the spectrum as the normal zeta function

$$\zeta_X(s) = \sum_{\lambda\in\Lambda\setminus\{0\}} |\lambda|^{-s}$$

and regard this as a member of a larger family at the unit of the algebra. So we define analogously

$$\zeta_{X,a}(s) = \mathrm{tr}(a|\Delta|^{-s}).$$

Here Δ is the Laplace-Beltrami operator and $a \in C^\infty(X)$, a smooth function on the Riemannian manifold X and the trace is taken in $L^2(X, d\mu_X)$. Furthermore, these functions can also be extended to a meromorphic function on the complex plane. In this case however it turns out that this one family does not suffice, so that we will need a second family as well. For functions $a_1 \in C^\infty(X)$, we define

$$\widetilde{\zeta}_{X,a_1}(s) := \mathrm{tr}(a_1[\Delta_X, a_1]\Delta_X^{-s}).$$

Note that this is the diagonal version of equation (1.1). These zeta functions contain geometric information, expressed in the following relation between the first class of zeta functions and the second (see Lemma 3.2.1) by means of the metric g_X:

$$\widetilde{\zeta}_{X,a_1}(s) = \zeta_{X,g_X(da_1,da_1)}(s)$$

In particular we see that the second class of zeta functions contains information about the metric. What these families mean for a map in terms of being an isometry is the following theorem (see Theorem 3.3.1):

Theorem C. *Let $\varphi : X \to Y$ denote a C^∞-diffeomorphism between closed connected smooth Riemannian manifolds with smooth metric. The following are equivalent:*

(i) *We have that*

   (a) $\zeta_{Y,a_0} = \zeta_{X,\varphi^*(a_0)}$ *for all $a_0 \in C^\infty(Y)$, and*
   (b) $\widetilde{\zeta}_{Y,a_1} = \widetilde{\zeta}_{X,\varphi^*(a_1)}$ *for all $a_1 \in C^\infty(Y)$.*

(ii) *The map φ is an isometry.*

So this answers the isospectrality question for Riemannian manifolds: next to the first family one also needs the diagonal of the second family.

An even stronger theorem can be obtained (see Theorem 3.5.1):

Theorem D. *Let $\varphi : X \to Y$ denote a C^∞-diffeomorphism between closed Riemannian manifolds with smooth metric. Then*

(a) In theorem C, it suffices in condition (b) *to have equality of* one *arbitrary expansion coefficient of the Dirichlet series (for all a_1) for conditions* (a) *and* (b) *to be equivalent to* (ii).

(b) If the spectrum of X or Y is simple, then condition (a) *alone is equivalent to* (ii) *in theorem 3.3.1.*

Actually, by a result of Uhlenbeck in [47], (b) treats the 'generic' situation.

A length category of closed Riemannian manifolds

Using zetas, we will make the category of closed Riemannian manifolds with C^∞-diffeomorphisms into a length category. A length category is the following:

Definition. We call a pair $(\mathscr{C}, \ell)$ a *length category* if $\mathscr{C}$ is a category endowed with a subcategory $\mathscr{D}$, full on objects, such that every morphism of $\mathscr{D}$ is an isomorphism. These are called the isomorphisms from now on.
Furthermore, for every $X, Y \in \mathrm{Ob}(\mathscr{C})$ and every $\varphi \in \mathrm{Hom}(X, Y)$, there is defined a positive real number $\ell(\varphi) \in \mathbf{R}_{\geqslant 0}$, called the *length of* φ such that

(L1) $\ell(\varphi) = 0$ if and only if φ is an isomorphism;

(L2) If $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$ and $\varphi \in \mathrm{Hom}(X, Y), \psi \in \mathrm{Hom}(Y, Z)$, then

$$\ell(\psi \circ \varphi) \leqslant \ell(\varphi) + \ell(\psi).$$

We will define the length of a morphism. The idea is to measure the distance between the zeta functions on the one manifold and the pull back by the map of the zeta function on the other manifold. The definition is as follows:

Definition. Let $\varphi : X \to Y$ denote a C^∞-diffeomorphism of closed Riemannian manifolds of dimension d. For $n \geqslant 1$, define

$$d_n(\varphi, a_0, a_1) := \sup_{d \leqslant s \leqslant d+n} \max\{|\log\left|\frac{\zeta_{X,a_0^*}(s)}{\zeta_{Y,a_0}(s)}\right||, |\log\left|\frac{\widetilde{\zeta}_{X,a_1^*}(s)}{\widetilde{\zeta}_{Y,a_1}(s)}\right||\}.$$

The *length of* φ is defined by

$$\ell(\varphi) := \sup_{\substack{a_0 \in C^\infty(Y, \mathbf{R}_{\geq 0}) - \{0\} \\ a_1 \in C^\infty(Y) - \{\mathbf{R}\}}} \frac{d_1(\varphi, a_0, a_1)}{1 + d_1(\varphi, a_0, a_1)}.$$

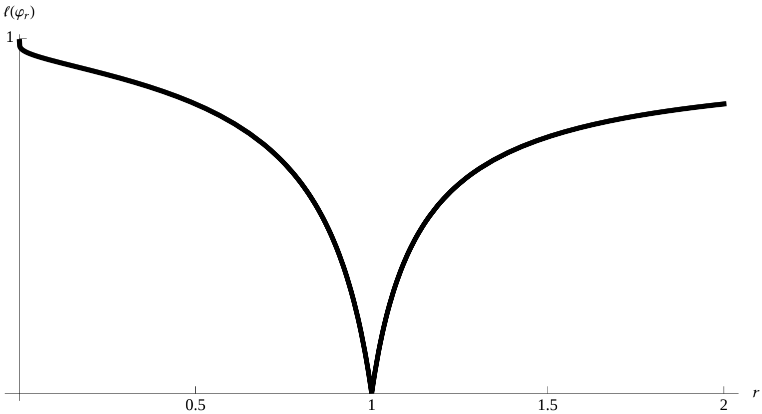
As an example, let us look at the length of the map which rescales a circle:

$$\varphi_{r_1, r_2} : S_{r_1} \to S_{r_2} : \theta \mapsto \theta \quad (\theta \in [0, 2\pi[).$$

The calculation is done in section 4.3.1 and the result is

$$\ell(\varphi_{r_1, r_2}) = \frac{1}{1 + \frac{1}{5|\log(r_1/r_2)|}}.$$

For $r_2 = 1$, as function of $r_1 = r$, this looks like:



So we see that in accordance with the theorem, this map is an isometry if and only if the circles are isometric (i.e., $r = 1$).

Now using this length, we can show that this indeed defines a length category satisfying **(L1)** and **(L2)** with as isomorphisms the isometries (see Proposition 4.2.5 and Proposition 4.2.6 respectively).

In a length category one can sometimes define the distances between objects in the category of the objects modulo isomorphisms as the infimum of all lengths of maps between them. An obstruction to this is that whenever the infimum is zero, there should be an actual isomorphism between the objects - which the infimum a priori does not give. In our case this can be solved: an actual isometry can be found. We will call the obtained distance d_ξ, i.e.,

$$d_\xi(X,Y) := \max\{ \inf_{C^\infty(X \xrightarrow{\varphi} Y)} \ell(\varphi), +\infty\}$$

Objects which are not in the same C^∞-type are at infinite distance and otherwise at finite distance from each other, essentially giving a metric on the space of smooth metrics on a given manifold. Convergence in this metric can be expressed in terms of a geometric notion, namely the distortion of a map.

We prove the following theorem (see Theorem 4.4.3):

Theorem E. *Let $\mathscr{M}$ denote a space of closed Riemannian manifolds up to isometry. Then d_ζ induces the topology of uniform convergence in C^∞-diffeomorphic types on $\mathscr{M}$, i.e., if two such manifolds are not C^∞-diffeomorphic, then the manifolds are at infinite distance, and otherwise, a sequence of manifolds converge if and only if there is a sequence of C^∞-bijections between them whose distortion tends to zero.*

There are various forms of convergence, for instance Lipschitz convergence, uniform convergence, Gromov-Hausdorff convergence and less well-known Kasue-Kumura convergence, d_t- and δ_t-convergence. It is interesting to see where this new convergence fits into the bigger picture (see figure 4.2): our convergence is weaker than Lipschitz convergence, but stronger than Gromov-Hausdorff convergence.

Future directions

As already noted, the ultimate goal would be to assign suitable families of zeta functions to spin manifolds and prove similar results so that these induce a distance between spin manifolds. This concept can then possibly be generalized to more general spectral triples, hopefully inducing interesting lengths / distances in genuinely noncommutative spaces as well.

Chapter 2

Zeta rigidity for graphs

In this chapter we will construct finitely summable spectral triples for graphs and we will prove that the family of corresponding zeta-functions encodes the graph.
In section 2.1 we recall some preliminaries about graphs and their boundaries. In section 2.2 we define the spectral triple associated to such a graph and prove it is one-summable. This construction of the spectral triple is inspired by [19] where a similar construction was applied to Riemann surfaces, which was based on [21] where, among other things, θ-summable spectral triples are constructed for actions on trees (in [19] this was improved to finitely summable spectral triples), which was in fact a refinement of some results of [14]. The chain of ideas started in [8] with a construction of spectral triples for AF C^*-algebras and closely related work in [38].
In section 2.3 we compute the constant terms of its associated family of zeta functions, and in section 2.4 we prove the main theorem 2.4.1, that says that the family of zeta functions associated to the spectral triple determine the isomorphism type of the graph.

This chapter is based on the article "Graphs, spectral triples and Dirac zeta functions" [22].

2.1 Preliminaries

In this section we will recall some facts about graphs, covering spaces, boundaries of these and finally the Patterson-Sullivan measure.

2.1.1. Let X be a finite, connected, unoriented graph with first Betti number $g \geq 2$ (genus) and valencies ≥ 3, the number of edges emerging from a vertex. We will assume this throughout this chapter. Let E_X denote the universal covering tree of X, which is a tree, and let Γ be the fundamental group of X. The following is a well-known result.

Proposition 2.1.1. *Let Γ be the fundamental group of a graph as above. Then Γ is a free group of rank g and acts freely on E_X, with*

$$X \cong E_X/\Gamma \tag{2.1}$$

Remark 2.1.2. Recall that the action of the fundamental group is defined by regarding the fundamental group as group of covering transformations, i.e. the group of homeomorphisms $f : E_X \to E_X$ with $\pi \circ f = \pi$ where π is the covering map $\pi : E_X \to X$.
We will not distinguish between regarding an element of Γ as closed path or as isometry of E_X, it will be clear from the context.

Definition 2.1.3 (Distance on E_X)**.** The tree E_X is a complete, geodesic, metric space for a natural distance function l: Every edge is defined to be a closed interval of fixed length $L \in \mathbf{R}_{>0}$. Since two points in the tree are connected by a unique path, this induces a distance between them, the length of the path.

Definition 2.1.4 (Boundary)**.** Let X be a finite, connected graph with universal covering tree E_X.
The boundary ∂X of E_X consists of all infinite words $w = w_0 w_1 w_2 \dots$ in adjoining, unoriented edges, without backtracking, in the covering tree. Two such words are considered the same if they agree from some point on; more precisely $w = w'$ if there exist N, K such that for all $n \geq N$ we have $w_{n+K} = w'_n$.

2.1.2. Given a vertex $V \in E_X$ and an element $w \in \partial X$, then w can be uniquely represented by a word starting at vertex V. This specific representation of w will be denoted by $[V, w)$.
Given two distinct points $\xi_1, \xi_2 \in \partial X$, we can form the unique geodesic connecting these, which will be denoted by $]\xi_1, \xi_2[$.

Definition 2.1.5 (Distance on the boundary)**.** Let O be a distinguished vertex in the universal covering tree E_X, called the origin. The distance $d_{X,O}$ on ∂X is defined by

$$\begin{array}{rcl} d_{X,O} : \partial X \times \partial X & \to & \mathbf{R}_{\geq 0} \\ d(w,w') & = & 2^{-n}, \end{array} \tag{2.2}$$

with

$$n \quad = \quad \#\{[O,w) \cap [O,w')\} \in \mathbf{N} \cup \{+\infty\}, \tag{2.3}$$

the number of coinciding edges of the two words, with the convention $2^{-\infty} = 0$.

Remark 2.1.6. One can also define a boundary and distance by means of a 'scalar product' (see Chapter 2 of [17]). It is defined as follows: Let $\omega_1, \omega_2 \in \partial X$, now considered as equivalence classes of single ended geodesic rays tending to infinity, where two geodesic rays are equivalent if their Hausdorff distance is finite. Now define a scalar product $\langle .,. \rangle_O$ with respect to a chosen origin O by

$$\langle w, w' \rangle_O = \lim_{t,t' \to \infty} \frac{1}{2}(d(w(t),O) + d(O,w'(t)) - d(w(t),w'(t)).$$

where $w(t)$ and $w'(t)$ are the corresponding rays for w and w'. Now define the corresponding distance function as

$$d(w,w') = \begin{cases} e^{-\langle w,w' \rangle_O} & \text{for } w \neq w', \\ 0 & \text{else} \end{cases}$$

In our case this gives an equivalent metric.

2.1.3. The distance turns ∂X into a metric space which induces a topology on the boundary ∂X.
For fixed $v = [O,v) = v_0 v_1 \ldots \in \partial X$ and $N \in \mathbf{N}_{>0}$, basic open balls around v of radius 2^{-N} are described by

$$\begin{array}{rcl} B(v,2^{-N}) & = & \{w \in \partial X \mid d(w,v) < 2^{-N}\} \\ & = & \{w \in \partial X \mid [O,w) = v_0 v_1 \ldots v_{N+1} Q \text{ where } Q \text{ runs over} \\ & & \text{all infinite connecting strings with no backtracking starting at the} \\ & & \text{endpoint of the path } P = v_0 \ldots v_{N+1}\} \end{array} \tag{2.4}$$

We will denote this open ball by U_P (see figure 2.1).

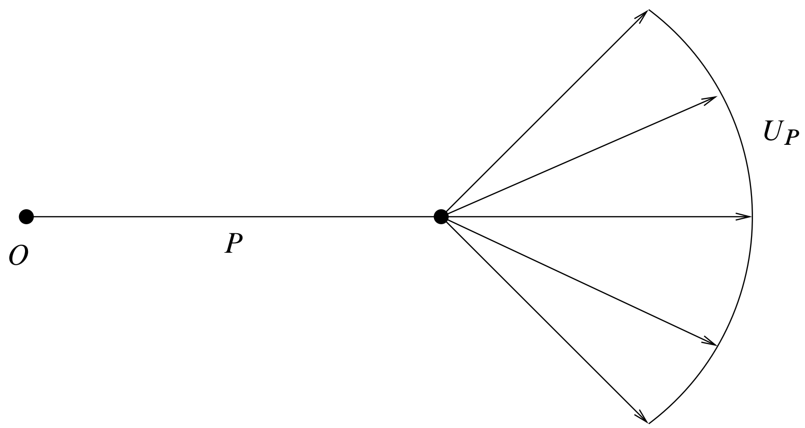


Figure 2.1: The open set U_P.

Proposition 2.1.7. *The boundary is a totally disconnected, compact Hausdorff space and the set U_P as above is clopen (open and closed).*

Proof. For the Hausdorff property: Let $[O,w),[O,w')$ be two distinct elements of ∂X, then there exists an N such that $w_N \neq w'_N$, define $P = w_0 \dots w_N$ and $Q = w'_0 \dots w'_N$, then $w \in U_P$ and $w' \in U_Q$ and $U_P \cap U_Q = \emptyset$, so ∂X is Hausdorff. Now we show that U_P^c is open:

$$U_P^c = \bigcup_{R \neq P, l(R)=l(P)} U_R, \tag{2.5}$$

where l assigns to a word its length as in Definition 2.1.3. This is a union of open sets and hence open.
Finally, suppose that w and w' are in a connected component A of ∂X, then, $U_P \cap A$ and $U_P^c \cap A$ is a separation of A and hence $w = w'$, so ∂X is totally disconnected. Finally, as this space is clearly complete and totally bounded it is compact. □

Definition 2.1.8 (Critical exponent)**.** Let Γ be the fundamental group of X, acting as isometry on E_X. Consider the formal Poincaré series,

$$\sum_{\gamma \in \Gamma} e^{-sd(O,\gamma O)}. \tag{2.6}$$

This series converges absolutely for sufficiently big enough $\mathrm{Re}(s) \in \mathbf{R} \cup \{\infty\}$ (with the convention $e^{-\infty} = 0$, so that by convention this series always converges in ∞).

Define $\delta \in \mathbf{R}_{\geq 0} \cup \{\infty\}$ as the critical exponent, i.e. for $\mathrm{Re}(s) > \delta$ the series converges and for $\mathrm{Re}(s) < \delta$, the series diverges. If $\delta \notin \{0, \infty\}$ and diverges at δ, then Γ is called a *divergence group* with respect to its action on X (see for instance [30]). In our case Γ acts as divergence group, see Remark 2.4.2.

Definition 2.1.9 (Patterson-Sullivan measure)**.** Let δ be the critical exponent of Γ acting on E_X and let D_P be the Dirac measure at the point $P \in E_X$, i.e. $D_P(U) = 1$ if $P \in U$ and 0 else.
Define the Patterson-Sullivan measure as

$$\mu = \mu_{\mathrm{PS},O} = \lim_{s \downarrow \delta} \Big(\frac{\sum_{\gamma \in \Gamma} e^{-sd(O,\gamma O)} D_{\gamma O}}{\sum_{\gamma \in \Gamma} e^{-sd(O,\gamma O)}} \Big). \tag{2.7}$$

This converges weakly to a probability measure on $X \cup \partial X$ with support on the boundary ∂X and hence turns ∂X into a measure space (see [30]).

2.2 The spectral triple

In this section we will define a finitely summable spectral triple in the sense of Connes, a non-commutative analogue of Riemannian manifolds (see the introduction), and we will study its associated zeta function formalism.

Definition 2.2.1. Let $S = \{\gamma_1, \ldots, \gamma_g\}$ be a set of generators of the fundamental group Γ of X. Define F_g as the graph consisting of one vertex with g loops attached to it, corresponding to the generators of the fundamental group. Denote by ∂F_g the boundary of the universal covering tree of this graph (see figure 2.2). This corresponds to the boundary of the *Cayley-graph* of Γ, with respect to the chosen generators of the fundamental group. The Cayley-graph is defined as follows: Every $g \in \Gamma$ corresponds to a vertex and for any pair of the form (g, gs) with $s \in S$ and $g \in \Gamma$ is assigned an edge between them (see for instance [24]). Note that normally the Cayley graph is a directed graph, but here we will not do this.

Definition 2.2.2. A group G is called hyperbolic if G is δ-hyperbolic for some δ, which is defined as follows: Let C be the Cayley graph of the group G, seen as metric space regarding each edge as unit interval in $\mathbb{R}$. If Δ is a geodesic triangle

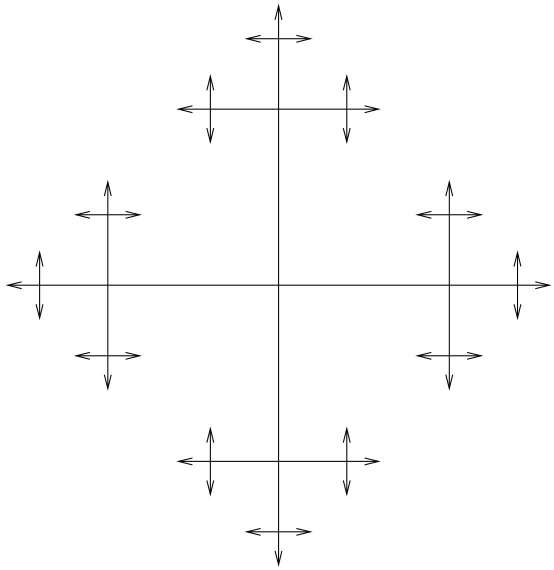


Figure 2.2: The (truncated) covering tree of F_2.

having x, y, z as vertices, then Δ is called δ-slim if

$$[x,y] \subset B_\delta([y,z] \cup [z,x])$$
$$[y,z] \subset B_\delta([z,x] \cup [x,y])$$
$$[z,x] \subset B_\delta([x,y] \cup [y,z])$$

If every geodesic is δ-slim, then G is called δ-hyperbolic.

A free group has as Cayley graph a tree and is hence 0-hyperbolic. We will use Theorem 4.1 of chapter 4 in [17] with $Y = E_X$ and $G = \Gamma$, the fundamental group of X, whose boundary is denoted by ∂F_g.

Proposition 2.2.3. *Let Y be a proper geodesic space and let G be an isometry group of Y, acting properly discontinuous, such that the quotient Y/G is compact. Then G is hyperbolic if and only if Y is. Moreover, if G is hyperbolic, then there is a canonical homeomorphism:*

$$\Phi_Y : \partial G \to \partial Y. \tag{2.8}$$

2.2.1. In this theorem, properly discontinuous means that for any compact set $K \subset Y$ the set

$$\{g \in G \mid g(K) \cap K \neq \emptyset\} \tag{2.9}$$

is finite. In case of the covering space E_X, this is the case because the group acts freely and compact here implies bounded and there are only finitely many group elements mapping a bounded set into itself.
After fixing an origin O, the map Φ_Y is induced by the map

$$\begin{aligned} \phi_Y : G &\rightarrow Y \\ g &\mapsto g(O), \end{aligned} \tag{2.10}$$

which is a quasi-isometry, which means that there are $K \geq 1$ and $C \geq 0$ such that $\frac{1}{K} d(g_1, g_2) - C \leq d(\phi_Y(g_1, g_2)) \leq K d(g_1, g_2) + C$ for all $g_1, g_2 \in G$. By Theorem 2.2 of [17], the map ϕ_Y then induces a map on the boundaries. The map ϕ_Y is G-equivariant with respect to the left action of G on itself (see next Definition 2.2.4) and hence the map Φ_Y is G-equivariant as well.

Definition 2.2.4 (Equivariant maps)**.** Let Y_1, Y_2 be sets. Suppose that the group G_i acts on Y_i and let $\alpha : G_1 \rightarrow G_2$ be a group homomorphism, then a map $f : Y_1 \rightarrow Y_2$ is called *equivariant* (with respect to G_1, G_2, α), if for all $g \in G_1, y \in Y_1$ we have

$$f(g.y) = \alpha(g).f(y)$$

Remark 2.2.5. Note that the map Φ_Y is not determined by the abstract group G, but depends on the representation of G as an isometry group of Y and on a choice of generators when passing to the Cayley graph. Different choices do not give isometric, but quasi-isometric Cayley graphs.

Definition 2.2.6 (Algebra of functions, Hilbert space)**.** Let A_X be the algebra of continuous, $\mathbb{C}$-valued functions on ∂F_g, i.e. $C(\partial F_g, \mathbb{C})$. This algebra contains the subalgebra $A_{X,\infty} = C(\partial F_g, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbb{C}$ of locally constant functions. Let μ be the Patterson-Sullivan measure on ∂X (see Definition 2.1.9), then the measure $\Phi_X^*(\mu)$ (see Proposition 2.2.3) induces an inner product by integration on ∂F_g and hence a norm on A_X. Let H_X be the completion of A_X with respect to this norm.

Remark 2.2.7. Note that after fixing the genus, only H_X depends on the graph X, especially the innerproduct on H_X.
We will use the notation $A = A_X$ and $A_\infty = A_{X,\infty}$ and $H = H_X$ for the algebras.

Proposition 2.2.8. *The subalgebra A_∞ is dense in A.*

Proof. The boundary is a totally disconnected Hausdorff space. The sets U_P form a basis for the topology and hence the result. □

Definition 2.2.9 (Dirac operator)**.** The space A_∞ has a natural filtration $\{A_n\}_{n\geq 0}$ by setting

$$A_n = \mathrm{span}_{\mathbb{C}}\{1_P \mid P \text{ a finite word of length } \leq n\}. \tag{2.11}$$

This filtration is inherited by its completion H. Let P_n be the orthogonal projection operator on A_n, with respect to the inner product on H,

$$P_n : H \to A_n \subset H. \tag{2.12}$$

The Dirac operator,

$$D \quad : \quad H \to H, \tag{2.13}$$

with domain A_∞, is defined by

$$D \quad = \quad \lambda_0 P_0 + \sum_{n\geq 1} \lambda_n Q_n, \tag{2.14}$$

with

$$Q_n \quad = \quad P_n - P_{n-1}, \tag{2.15}$$

$$\lambda_n \quad = \quad (\dim A_n)^3. \tag{2.16}$$

Lemma 2.2.10. *We have:*

a. $\dim A_0 = 1$,

b. $\dim A_1 \ominus A_0 = 2g-1$

For $n \geq 1$ we have

i. $\dim A_n = 2g(2g-1)^{n-1}$,

ii. $\dim A_{n+1} \ominus A_n = (2g)(2g-2)(2g-1)^{n-2}$.

Proof. The space A_0 corresponds to the constant functions which is one-dimensional. Then, from the origin $2g$ edges emerge and every next step there are $2g-1$ choices, giving the results. □

Proposition 2.2.11. *The data (A, H, D) as constructed forms a 1-summable spectral triple.*

Proof. The $*$-operation is complex conjugation, A acts on H by multiplication which is bounded. The operator D consists of a sum of a $\mathbf{R}$-linear sum of projection operators, which are self-adjoint and hence D is self-adjoint.
For the compactness of $(D+\lambda)^{-1}$ for $\lambda \notin \mathrm{Spec}(D)$ note that in uniform operator norm

$$(D+\lambda)^{-1} = \lim_{N\to\infty} (D_N+\lambda)^{-1},$$

with

$$D_N = \lambda_0 P_0 + \sum_{n=1}^{N} \lambda_n Q_n.$$

Hence $(D+\lambda)^{-1}$ is the limit of sequence of finite rank operators $(D_N+\lambda)^{-1}$ and hence compact.
For $a \in A_n$ and $m > n$ we have $P_m(a) = P_{m-1}(a)$. So $P_m|_{A_k} = P_k$ for $k \leq m$ and so $[Q_m, a] = 0$ for $m > n$, in particular $[D, a]$ is a finite linear combination of finite rank operators of the form $[Q_i, a]$ and hence bounded.
For the 1-summability, note that

$$\begin{aligned}
\mathrm{tr}((1+D^2)^{-1/2}) &= \sum_{n=0}^{\infty} (1+\lambda_n^2)^{-1/2} (\dim A_n - \dim A_{n-1}) \\
&\leq \sum_{n=0}^{\infty} (1+\lambda_n^2)^{-1/2} \dim A_n \\
&< \sum_{n=0}^{\infty} (\dim A_n)^{-2} \\
&\leq \sum_{n=0}^{\infty} (n+1)^{-2} \\
&< \infty, \qquad (2.17)
\end{aligned}$$

where we used that $\dim A_n \geq n+1$. □

2.3 Zeta functions

In this section we will look at the zeta-functions induced by the constructed spectral triple.

Definition 2.3.1 (Zeta functions)**.** Given a finitely summable spectral triple. Then for any $a \in A_\infty$ and $\mathrm{Re}(s) \ll 0$ define a zeta function by

$$\zeta_a^X(s) \quad = \quad \mathrm{tr}_H(a|D|^s). \tag{2.18}$$

This defines holomorphic function in a right half plane of $\mathbb{C}$. Note that in the literature often $-s$ is used in the definition of zeta functions.

2.3.1. Let us expand ζ_a^X for later convenience. Let I_m stand for an inductively obtained orthogonal basis for A_m with $A_k \subset A_{k+1}$ for all $k \in \mathbf{N}_{\geq 0}$ (obtained by for instance a Gram-Schmidt orthogonalization procedure). By convention, I_0 corresponds to the constant functions and $I_{-1} = \emptyset$.

$$\begin{aligned}
\zeta_a^X(s) &= \mathrm{tr}_H(a|D|^s) \\
&= \sum_{n\geq 0} \sum_{\Psi \in I_n - I_{n-1}} \langle \Psi | \lambda_0^s a P_0 + \sum_{m\geq 1} \lambda_m^s a Q_m | \Psi \rangle \\
&= \sum_{n\geq 0} \lambda_n^s c_n(a),
\end{aligned} \tag{2.19}$$

with

$$c_n(a) \quad := \quad \sum_{\Psi \in I_n - I_{n-1}} \langle \Psi | a | \Psi \rangle. \tag{2.20}$$

The zeta function at the unit of the algebra does not contain a lot of information about the graph, just the genus, as is expressed in the following proposition:

Proposition 2.3.2. *The zeta function ζ_1^X does not depend on any choices and is equivalent to knowing the genus of X.*

Proof. We will explicitly calculate it. Let $\mathrm{Re}(s) \ll 0$, then

$$
\begin{aligned}
\operatorname{tr}(|D|^s) &= 1+\sum_{n\geq 1}\lambda_n^s(\dim A_n-\dim A_{n-1}) \\
&= 1+(2g)^{3s}(2g-1)+ \\
&\quad \sum_{n\geq 2}((2g)(2g-1)^{n-1})^{3s}\cdot(2g)(2g-1)^{n-2}(2g-2) \\
&= 1+(2g)^{3s}(2g-1)\Big\{\frac{1-(2g-1)^{3s-1}}{1-(2g-1)^{3s+1}}\Big\}. \qquad (2.21)
\end{aligned}
$$

The first order expansion around $s=-\infty$ is

$$1+(2g)^{3s}(2g-1). \qquad (2.22)$$

So the formula determines g and on the other hand g determines ζ_1 by the formula above. □

The zeta-functions contain essentially all the information about the measure, expressed in the following proposition.

Proposition 2.3.3. *Let X_1, X_2 be graphs with the same genus $g\geq 2$. If $\zeta_a^{X_1}=\zeta_a^{X_2}$ for all $a\in A_\infty$, then the induced measures $\Phi^*_{X_i}(\mu_i)$ on ∂F_g are equal (with these specific choices of origin and representation).*
More precise,

$$\lim_{s\to-\infty}\zeta_a^{X_i}(s)=\int_{\partial F_g} a\,\mathrm{d}(\Phi^*_{X_i}(\mu_i)).$$

Proof. The limit on the left handside equals the zeroth term in the expansion. So it suffices to show that the zeroth term, which is $\operatorname{tr}(aP_0)$, is given by the integral on the right.
For this it suffices to show the equality for a basis of the topology, i.e. it suffices to show that $\nu(U_P)=\int_{\partial F_g}1_P\,d\nu$ for all P, here ν is one of the induced measures and P is a finite path starting in O as before.
Fix P and take the canonical orthogonal basis B_P for $A_{|P|}$, i.e.

$$B_P=\{\frac{1_Q}{||1_Q||}\mid \text{with } l(Q)=l(P)\}. \qquad (2.23)$$

The constant term of the zeta function corresponds to $\operatorname{tr}(aP_0)$. First we will trace only over $A_{|P|}$ and get the result. Then we prove that a refinement of the basis

does not change the trace, proving the theorem.
We have

$$\begin{aligned}
\mathrm{tr}_{A_{|P|}}(aP_0) &= \sum_{w\in B_P} \langle w|1_P P_0|w\rangle \\
&= \frac{\int_{\partial F_g} 1_P P_0(1_P)\,\mathrm{d}\nu}{\int_{\partial F_g} 1_P 1_P\,\mathrm{d}\nu} \\
&= P_0(1_P)\frac{\nu(U_P)}{\nu(U_P)} \\
&= P_0(1_P),
\end{aligned} \tag{2.24}$$

here we used that $P_0(1_P)$ is a constant function and by abuse of notation we denoted its value by $P_0(1_P) \in \mathbb{C}$ as well. The projection is characterized by

$$1_P - P_0(1_P) \perp A_0 \tag{2.25}$$

and because $A_0 \cong \mathbb{C}$, this is equivalent with

$$\int_{\partial F_g} (1_P - P_0(1_P))\,\mathrm{d}\nu = 0 \tag{2.26}$$

and so $P_0(1_P) = \nu(U_P)$, proving the first assertion.
Now let $B' \supset B$ be an orthogonal extension of the basis B, in particular we have for $v \in B' \setminus B$, $\int 1_P \cdot v\mathrm{d}\nu = 0$, so

$$\begin{aligned}
\mathrm{tr}_{B'}(1_P P_0) &= \mathrm{tr}_B(1_P P_0) + \mathrm{tr}_{B'\setminus B}(1_P P_0) \\
&= \nu(U_P) + \sum_{v\in B'} \int_{\partial F_g} v \cdot 1_P \cdot P_0(v)\,\mathrm{d}\nu \\
&= \nu(U_P) + \sum_{v\in B'} P_0(v) \int_{\partial F_g} v \cdot 1_P\,\mathrm{d}\nu \\
&= \nu(U_P) + \sum_{v\in B'} P_0(v)\cdot 0 \\
&= \nu(U_P),
\end{aligned} \tag{2.27}$$

proving the proposition. □

Remark 2.3.4. The computation of the corresponding term in [19] is wrong and should be replaced by a calculation similar to the above. This only affects the proofs, not the results, of [19].

2.4 The main theorem

In this section we will prove the main theorem using some concepts which will be introduced as well.

Theorem 2.4.1 (Main theorem)**.** *Let X_1, X_2 be finite, connected graphs of genus $g \geq 2$ and valencies ≥ 3. Then:*

(a) $\zeta_1^{X_1} = \zeta_1^{X_2}$ *if and only if X_1 and X_2 have the same genus.*

(b) *If (a) holds, then $X_1 \cong X_2$ if and only if there exists a choice of origins and and minimal sets of generators such that*

$$\zeta_a^{X_1} = \zeta_a^{X_2}$$

for all $a \in A_\infty$.

Before proving this theorem we will need some definitions and results.

Definition 2.4.2 (Cross-ratio, Möbius)**.** For $\xi_1, \ldots, \xi_4$ four distinct points on the boundary ∂X, define the cross-ratio as:

$$b(\xi_1, \xi_2, \xi_3, \xi_4) := \frac{d(\xi_3, \xi_1)}{d(\xi_3, \xi_2)} : \frac{d(\xi_4, \xi_1)}{d(\xi_4, \xi_2)} \tag{2.28}$$

This can be written as $\exp(L)$ with L (up to sign) the distance between the geodesics $]\xi_1, \xi_3[$ and $]\xi_2, \xi_4[$, see figure 2.3.
A function f preserving the cross-ratio, i.e. such that for all distinct points $\xi_1, \ldots, \xi_4$ we have

$$b(\xi_1, \xi_2, \xi_3, \xi_4) = b(f(\xi_1), f(\xi_2), f(\xi_3), f(\xi_4)),$$

is called *Möbius* (see [30],[16]).

Lemma 2.4.3. *Let $\widetilde{\phi} : \partial X_1 \to \partial X_2$ be an (equivariant) isomorphism which is Möbius, then $\widetilde{\phi}$ induces an (equivariant) isomorphism of trees $F : E_{X_1} \to E_{X_2}$ and hence an isomorphism of graphs $X_1 \cong X_2$.*

Proof. (compare [16])
We define a map F as follows:
Let $x \in V(E_{X_1})$, a vertex in the covering graph. Pick $\xi_1, \xi_2, \xi_3 \in \partial X_1$ such that x

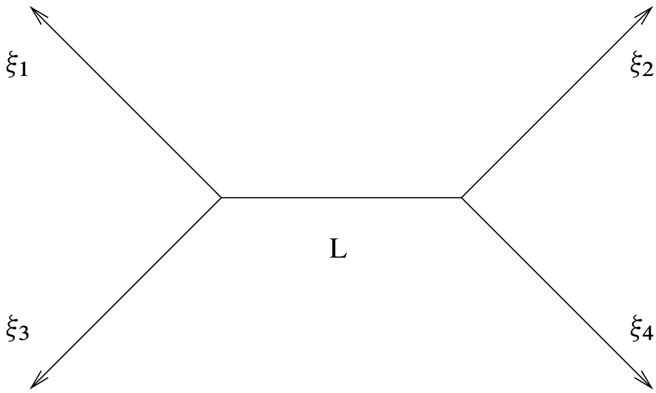


Figure 2.3: The cross-ratio.

is the center of the tripod induced by these (see figure 2.4). Define $F(x) \in V(E_{X_2})$ as the unique center of the tripod of $\widetilde{\phi}(\xi_1), \widetilde{\phi}(\xi_2), \widetilde{\phi}(\xi_3) \in \partial X_2$. To show that this is well-defined, if x is also the center of ξ_1, ξ_2, ξ_4, then this is the same as saying that the geodesics $]\xi_1, \xi_2[$ and $]\xi_3, \xi_4[$ are touching, i.e. $L = 0$, this is preserved because $\widetilde{\phi}$ is Möbius and hence the map is well-defined. The map is an isomorphism on the vertex sets, because the construction can be reversed.
To see that this extends to a map of covering trees, let e be an edge of length L_1, then the initial and terminal vertex of e are mapped to two distinct vertices in $V(E_{X_2})$. Connect these by a path P of length $L = L_1$ (because the map is Möbius it is L_1 again). We have to show that P is an edge again. If not there is a new vertex e' on P, this point is mapped by the inverse to a point on distance on distance $K, K' < L$ from the initial and terminal vertex of e which is impossible, hence F is an isomorphism of graphs.
By well-definedness, if $\widetilde{\phi}$ is equivariant for some group-action, then so is F. □

In the proof of the main theorem we will use the following theorem (see [30], Theorem A). The terminology will be explained right after.

Theorem 2.4.4. *Let Y_1, Y_2 be locally compact complete CAT(-1) metric spaces. Let G_1 and G_2 be discrete groups of isometries of Y_1 and Y_2, having the same critical exponent. Suppose that G_2 is a divergence group. Let $\widetilde{\phi} : \partial Y_1 \to \partial Y_2$ be a Borel map, equivariant for some morphism $G_1 \to G_2$, which is non-singular with*

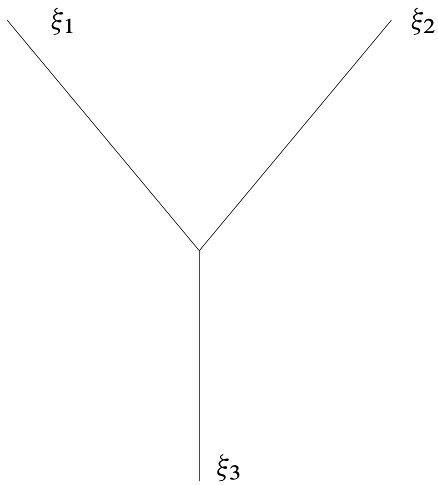


Figure 2.4: The tripod.

respect to the Paterson-Sullivan measures. Then $\widetilde{\phi}$ is Möbius on the limit set of G_1.

2.4.1. Recall that a CAT(k)-space is a geodesic metric space X (such that the distance between points is achieved by the length of a minimizing curve), where triangles are thinner than in spaces of homogeneous constant curvature k.
More precisely, let M_k be the simply connected surface with curvature k. And let $D_k \in [0, \infty)$ be its diameter. Now, let Δ be a geodesic triangle in X, which means that the sides are geodesics in X. Then a comparison triangle Δ' in M_k is a geodesic triangle isometrically mapping the edges $\Delta \to \Delta'$. Now, Δ is said to satisfy the CAT(k)-inequality if points on Δ are at most the same distance apart as the corresponding points in Δ'. Finally, X is called CAT(k) if every geodesic triangle of diameter at most $2D_k$ satisfies the CAT(k) inequality. For instance, a tree is a CAT(k)-space for any k.
Also, recall that 2 measures μ, ν on a measure space Ω are called *singular* with respect to each other, if one can find 2 disjoint sets A, B such that $\Omega = A \cup B$ and μ is 0 on all measurable subsets of A and ν is 0 on all measurable subsets of B.

2.4.2. Let us apply the theorem for $Y_1 = E_{X_1}, Y_2 = E_{X_2}$, the covering trees of the graphs X_1 and X_2 respectively, having the same genus $g \geq 2$ with $G_1 = \Gamma = G_2$, the fundamental groups of X_1 and X_2.
The trees under consideration are locally compact and are CAT(k) metric spaces

for any k (see for instance [5]) and the fundamental group acts by isometries.
To construct an equivariant map $\widetilde{\phi}$, note that by Proposition 2.2.3 we have an equivariant homeomorphism $\Phi_{X_i} : \partial F_g \to \partial X_i$, so $\widetilde{\phi} = \Phi_{X_2} \circ \Phi_{X_1}^{-1} : \partial X_1 \to \partial X_2$ is an equivariant homeomorphism.
The group Γ acts as divergence group on E_{X_i}. This is because the Poincaré series is bounded by the one of the covering tree of F_g for real s whenever it converges (because the group acts free and F_g is a retract of X) and the graph F_g has a finite critical exponent which we will compute now.

$$\begin{aligned} \sum_{\gamma \in F_g} e^{-sd(O,\gamma(O))} &= \sum_{n \geq 1} \#\{\gamma \mid d(O,\gamma(O)) = n\} \cdot e^{-snl} \\ &= \sum_{n \geq 1} (2g)^n e^{-snl} \\ &= \sum_{n \geq 1} (2g \cdot e^{-snl})^n \end{aligned} \tag{2.29}$$

And the latter converges for $2g \cdot e^{-snl} < 1$, giving $\delta = \frac{log(2g)}{l} < \infty$.
Furthermore, $\delta > 0$ because the fundamental group is infinite and diverges at δ. So Γ_i acts as divergence group on X_i.
Note that if we scale the metric of the covering tree by λ, the critical exponent scales by λ^{-1}, so by rescaling we can assume that the critical exponents are the same. Combining this with the previous lemma, we get the following proposition:

Proposition 2.4.5. *Let X_1, X_2 be graphs of genus $g \geq 2$ with covering trees E_{X_1}, E_{X_2}. If $\widetilde{\phi} = \Phi_{X_2} \circ \Phi_{X_i}^{-1} : \partial X_1 \to \partial X_2$ is non-singular with respect to the Patterson-Sullivan measures, then $\widetilde{\phi}$ induces an equivariant isomorphism of $E_{X_1} \to E_{X_2}$ and hence an isomorphism of graphs $X_1 \cong X_2$.*

Proof of the main theorem, Theorem 2.4.1. By Proposition 2.3.2, the zeta functions at the unit are the same if and only if the genus is the same.
Now for the second part consider the following commuting diagram:

$$\begin{tikzcd} (F_g, \Phi_{X_1}^*(\mu_1)) \arrow[r, "\mathrm{id}"] & (F_g, \Phi_{X_2}^*(\mu_2)) \\ (\partial X_1, \mu_1) \arrow[u, "\Phi_{X_1}"] \arrow[r, "\widetilde{\phi}"] & (\partial X_2, \mu_2). \arrow[u, "\Phi_{X_2}"'] \end{tikzcd}$$

Now if for some choice the zeta functions are equal we know that $\Phi^*_{X_1}(\mu_1) = \Phi^*_{X_2}(\mu_2)$ and hence $\mu_2 = \widetilde{\phi}^*(\mu_1)$ and so $\widetilde{\phi}$ is non-singular with respect to the Patterson-Sullivan measures and hence the previous proposition applies to the equivariant map $\widetilde{\phi}$. $\square$

Interesting in its own is that the constructed spectral triple contains all the data of the graph in the sense that if one knows that a spectral triple is coming from a graph by this construction then this graph is the unique graph corresponding to this spectral triple. This is what we mean by 'determines' in the following theorem:

Theorem 2.4.6. *The spectral triple $\mathscr{S}_X$ determines the graph X.*

Proof. The spectral triple determines the zeta functions, which by the main theorem determines the graph. $\square$

Remark 2.4.7. It would be desirable to give a functorial version of the construction. The graphs under consideration form a category in the obvious way, the objects being graphs and the arrows being graph-homomorphisms.
First let us get rid of the arbitrary choices by putting them in a big set:

Definition 2.4.8. The zeta functions constructed depend on a choice of origin and on a representation of Γ as group of isometries of E_X. Write $\zeta_a^{X,O,\alpha}$ to show the dependence in the notation. Here $O \in X$ denotes the (arbitrary chosen) origin and α denotes a representation of Γ as isometry group of X, including a minimal set of generators. Denote by $\mathrm{R}(\Gamma, X)$ the set of all such α's. Let

$$\zeta[X] = \{(\zeta_a^{X,O,\alpha})_{a \in A_\infty} \mid O \in X, \alpha \in \mathrm{R}(\Gamma, X)\} \tag{2.30}$$

denote the set of all indexed rows of zeta functions which can be obtained by varying the origin and α's as described.

Now, Theorem 2.4.1 says that the map

$$(X, O, \alpha) \mapsto \left(\zeta_a^{X,O,\alpha}\right)_{a \in A_\infty}$$

descends to a bijection of isomorphism classes

$$[X] \mapsto \zeta[X],$$

where the isomorphism classes and the image on the right are defined as the images of the isomorphism classes of X. Furthermore, by Theorem 2.4.6 this induces an equivalence relation on spectral triples coming from the construction as well, so this defines a map:

$$\begin{aligned} \{\text{graphs}\}/\{\text{isomorphisms}\} &\rightarrow \{\text{spectral triples}\}/\{\text{induced isomorphisms}\} \\ [X] &\mapsto [\mathscr{S}_X] \end{aligned}$$

At this very moment however, there does not exist a (generally accepted) category of spectral triples (see [4] and the introduction for some ideas using strict equivalence, but not (yet) involving Morita equivalence). The spectral triples constructed here are commutative, which is restrictive, and therefore missing morphisms which are not clearly visible in the commutative setting. For instance, two commutative algebras are Morita equivalent if and only if these are isomorphic as algebras, a statement which is not true in the non-commutative setting (see for instance [35]).

Let us conclude this chapter with a calculation.

Example 2.4.9. Consider the graph consisting of one point and two loops attached to it, the figure eight:

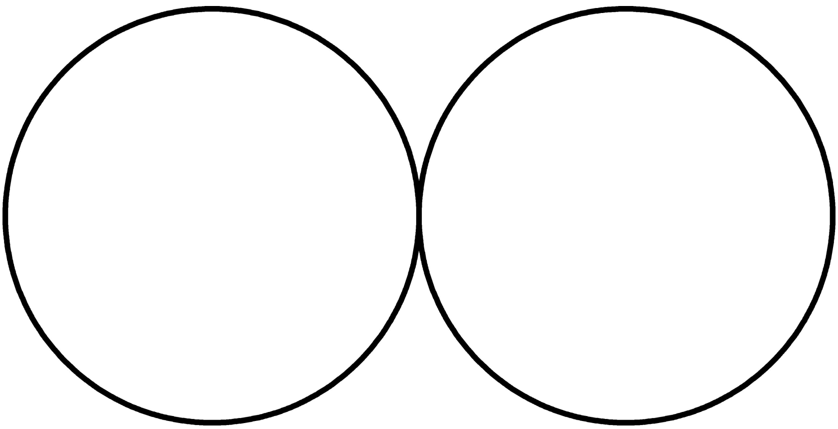


Figure 2.5: The figure eight.

The genus of the graph is two and the universal covering tree is F_2 as is depicted in figure 2.2.
The zeta function follows immediately from formula 2.21 and is given by

$$\zeta_1(s) = 1 + 3 \cdot 4^{3s} \left\{ \frac{1 - 3^{3s-1}}{1 - 3^{3s+1}} \right\}$$

The Patterson-Sullivan is homogeneous because of the symmetry. Hence on basic open subsets U_T as defined below equation 2.4 it is given by

$$\mu_{\mathrm{PS}}(U_T) = \frac{1}{4} \cdot 3^{-N},$$

where $T = T_0 \dots T_N$. If 1_T is the indicator function on U_T as before, then the zeta function is given by

$$\begin{aligned} \zeta_{1_T}(s) &= \mu_{\mathrm{PS}}(U_T) + \frac{3}{4} \cdot \mu_{\mathrm{PS}}(U_T) 4^{3s} + \dots \\ &= \frac{1}{4} \cdot 3^{-N} + \frac{3}{16} \cdot 3^{-N} 4^{3s} + \dots, \end{aligned}$$

which can be obtained by explicitly picking an orthogonal basis for A_0 and $A_1 \ominus A_0$. For instance, a basis for A_0 is given by the constant function 1 and a basis $\{b_1, b_2, b_3\}$ for $A_1 \ominus A_0$ is given by

$$\begin{aligned} b_1 &= \frac{1}{\sqrt{2}}(1_{w_1} - 1_{w_2}) \\ b_2 &= \frac{1}{\sqrt{2}}(1_{w_3} - 1_{w_4}) \\ b_3 &= \frac{1}{2}(1_{w_1} - 1_{w_2} + 1_{w_3} - 1_{w_4}), \end{aligned}$$

where the w_i stand for the letters of the alphabet used in the graph F_2. Now note that every 1_T is nonzero in exactly one of the U_{w_i} to get the result.

Remark that to be able to compute zeta functions for more complicated graphs one has to compute the Patterson-Sullivan measure which is in general a lot harder.

Chapter 3

Zeta rigidity for Riemannian manifolds

In this chapter we assign two families of zeta functions to a closed Riemannian manifold, indexed by functions on the manifold. We study the meaning of equality of families of zeta functions under pullback along a map. This allows us to give a spectral characterization of when a C^∞-diffeomorphism between Riemannian manifolds is an isometry, in terms of equality along pullback.
In section 3.1 we will start with some notation, preliminaries, assumptions and the definition of the families of zeta functions are given.
Next, in section 3.2 the two families are related and suitable residues are computed. In section 3.3 one of the main theorems is stated, namely how one can relate equality of these families to being an isometry.
The final section is dedicated to the computation of these families for flat tori.

This chapter is based on the first part of the article "The spectral length of a map between Riemannian manifolds" [18].

3.1 Notations and preliminaries

We will set up some notation and assumptions. Suppose that (X, g_X) is a closed (which means compact without boundary) smooth, connected manifold of strictly positive dimension, with a Riemannian metric $g = g_X$. We assume throughout this chapter that the metric is smooth. Denote by μ_X the induced measure on X, so that the volume form is given in a coordinate patch by

$$\sqrt{\det(g)}\mathrm{d}x_1 \wedge \ldots \mathrm{d}x_{\dim(X)},$$

Let Δ_X denote the (self-adjoint extension of the) elliptic Laplace-Beltrami operator acting on $L^2(X) = L^2(X, \mu_X)$, with domain the smooth functions to $\mathbb{C}$. Note that this is not the largest domain possible, for instance think of the twice differentiable continuous functions, or the domain in a more distributional sense (i.e. functions f such that the $\Delta(f) \in L^2(X)$ as distribution), or a canonical extension of the domain, such as the Friedrichs extension. As we only use smooth functions for the zeta functions there is no need to consider these larger domains.
In a coordinate patch Δ_X acting on a function f is given by

$$\Delta_X f = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{\dim(X)} \partial_i \left(\sqrt{\det(g)} g^{ij} \partial_j f \right),$$

where $\{g^{ij}\}$ denote the entries of the inverse of the metric. We have defined the Laplace-Beltrami operator with a minus sign, but sometimes it is defined with an opposite sign as well. For convenience write Λ_X for the spectrum of Δ (which is a point spectrum), including multiplicities.

The space $L^2(X)$ has an orthonormal basis of smooth real eigenfunctions for Δ_X (see for instance paragraph 1.3 [40]). Suppose we picked an orthonormal basis of smooth real eigenfunctions for the Laplacian on a Riemannian manifold X. We will use various notations for these. We denote an eigenfunction in the chosen basis, with eigenvalue λ, by $\Psi_{X,\lambda}$ or Ψ_λ suppressing X. Let us also write

$$\Psi_X \vdash \lambda$$

if Ψ_X is an eigenfunction on X in our chosen basis that belongs to the eigenvalue λ.

Let $C^\infty(X) = C^\infty(X, \mathbf{R})$ denote the set of smooth real-valued functions on X. Later on we will also use complex valued functions, this will be clear from the context. Define the zeta-function parametrized by $a_0 \in C^\infty(X)$ as

$$\zeta_{X,g_X;a_0}(s) = \zeta_{X,a_0} := \mathrm{tr}(a_0|\Delta_X|^{-s}) \tag{3.1}$$

where the complex exponent is taken using spectral theory, see for instance [44] and also formula 3.5 for an explicit expression.
In this chapter we will also need the double zeta function parametrized by C^∞-functions a_1, a_2, defined as

$$\zeta_{X,g_X;a_1,a_2}(s) = \zeta_{X,a_1,a_2}(s) = \mathrm{tr}(a_1[\Delta_X, a_2]\Delta_X^{-s}). \tag{3.2}$$

These zeta functions are well defined by [31], see Lemma 3.2.2.

We will be concerned with the diagonal version of this two-variable zeta function, defined by:

$$\widetilde{\zeta}_{X,g_X;a_1}(s) := \zeta_{X,g_X;a_1,a_1}(s) = \mathrm{tr}(a_1[\Delta_X, a_1]\Delta_X^{-s}). \tag{3.3}$$

Some more notation. If a map $\varphi : (X, g_X) \to (Y, g_Y)$ is fixed, we set

$$a^* := \varphi^*(a)$$

for $a \in C^\infty(Y)$.

Finally let

$$\begin{aligned} K_{X,g}(t,x,y) &= K_X(t,x,y) \\ &= \sum_{\substack{\lambda\in\Lambda \\ \text{distinct}}} e^{-\lambda t} \sum_{\Psi \vdash \lambda} \Psi(x)\Psi(y),\ (t > 0) \end{aligned} \tag{3.4}$$

denote the heat kernel of X. Sometimes the X and/or g will be suppressed if no confusion can arise. Seeing the usual zeta function as part of a larger family, we also make the convention to write $\zeta_X = \zeta_{X,1_X}$.

3.2 Residue computations

In this section we will look at the residues of the zeta functions and prove that these are diffeomorphism invariants.

If we pick an orthogonal real basis of eigenfunctions we can expand the zeta functions of the first class as

$$\begin{aligned}\zeta_{X,a_0}(s) &= \sum_{\Psi_X} \langle \Psi_X | a_0 \Delta_X^{-s} | \Psi_X \rangle \\ &= \sum_{\lambda \in \Lambda_X - \{0\}} \lambda^{-s} \sum_{\Psi \vdash \lambda} \int_X a_0 \Psi^2 d\mu_X. \end{aligned} \tag{3.5}$$

It turns out that the two-variable zeta function can be expressed in terms of the one-variable version using the metric, as follows:

Lemma 3.2.1. $\zeta_{X,a_1,a_2} = \zeta_{X,g_X(da_1,da_2)}$.

Proof. Pick a basis of real eigenfunctions and expand as follows:

$$\operatorname{tr}(a_1[\Delta_X, a_2]\Delta_X^{-s}) = \sum_{\Psi_X} \langle \Psi_X | a_1 [\Delta_X, a_2] \Delta_X^{-s} | \Psi_X \rangle \tag{3.6}$$

$$= \sum_{\lambda \neq 0} \lambda^{-s} \sum_{\Psi_X \vdash \lambda} \int_X (\Psi_Y a_1 \Delta_X (a_2 \Psi_Y) - a_1 a_2 \Psi_X \Delta_X \Psi)\, \mathrm{d}\mu_X.$$

$$= \sum_{\lambda \in \Lambda_X \setminus \{0\}} \lambda^{-s} \sum_{\Psi \vdash \lambda} \int_X (g(d(a_1\Psi), d(a_2\Psi)) - g(d(a_1 a_2 \Psi), d\Psi))\, \mathrm{d}\mu_X$$

Now we use the product rule for differentials $d(f_1 f_2) = f_1 df_2 + f_2 df_1$ and the bilinearity of g to get

$$\begin{aligned}\operatorname{tr}(a_1[\Delta_X, a_2]\Delta_X^{-s}) &= \sum_{\lambda \neq 0} \lambda^{-s} \sum_{\Psi_X \vdash \lambda} \int_X \Psi_X^2 g_X(da_1, da_2)\, \mathrm{d}\mu_X \\ &= \operatorname{tr}(g(da_1, da_2)\Delta_X^{-s}), \end{aligned} \tag{3.7}$$

which is the required first class zeta function. □

For the following lemma we will use the following general result from [31], Theorem 2.1 (cf. also [37]):

Lemma 3.2.2. *Let D be any smooth differential operator of order q and Δ a positive second order smooth elliptic partial differential operator on a smooth closed manifold.*
Then the function $\mathrm{tr}(D\Delta^{-s})$ extends to a meromorphic function on $\mathbb{C}$, it has only simple poles, and if order$(D) = q$ and if the dimension of the manifold is n, then these are located within the sequence of points

$$\frac{q+n}{2}, \frac{q-1+n}{2}, \frac{q-2+n}{2}, \ldots$$

□

Lemma 3.2.3. *The series ζ_{X,a_0} and ζ_{X,a_1,a_2} converge for $\mathrm{Re}(s) > \frac{\dim(X)}{2}$ and can be extended to a meromorphic function on $\mathbb{C}$ with at most simple poles at*

$$\frac{1}{2}(\dim(X) - \mathbf{Z}_{\geqslant 0}).$$

Proof. Using the previous lemma, we see that $\mathrm{tr}(D\Delta^{-s})$ has at most simple poles at $\frac{1}{2}(\dim(X) + q - \mathbf{Z}_{\geqslant 0})$. For ζ_{X,a_0}, we have $q = 0$ and the statement follows; for ζ_{X,a_1,a_2}, we have $q = 1$, but from the previous lemma it follows that there is no pole at $\frac{1}{2}(\dim(X) + 1)$, giving the result. □

We are going to look how these zeta behave under pullback of a map. One essential ingredient will be whether the Laplace-Beltrami operators commute with pullback, i.e., if $\varphi : X \to Y$,

$$\varphi^* \Delta_Y = \Delta_X \varphi^*,$$

acting on smooth functions on Y. This condition is studied for instance in [49] and [29]. They both work in a somewhat different setting, so we will present a short proof that this intertwining is equivalent with being an isometry as is expressed in the following proposition.

Proposition 3.2.4. *If $\varphi : X \to Y$ is a C^∞-diffeomorphism between closed Riemannian manifolds, then*

$$\varphi^* \Delta_{Y,h} = \Delta_{X,g} \varphi^*$$

as operators if and only if φ is an isometry.

Proof. Pick Riemann normal coordinates such that $\Delta_X = -g^{ij}\partial_i\partial_j$ when evaluated at a point $p \in X$ and similarly for Δ_Y. Now let $f : Y \to \mathbb{C}$ be a smooth function on which Δ_Y acts. Then comparing the terms in front of $\frac{\partial^2 f}{\partial y_\alpha \partial y_\beta}$ we find that evaluated at p

$$h^{\alpha\beta} \circ \varphi = \sum_{i,j} g^{ij} \frac{\partial \varphi^\alpha}{\partial x_i} \frac{\partial \varphi^\beta}{\partial x_j}. \tag{3.8}$$

This implies that φ is a Riemannian covering map at p and hence everywhere. We assume the map to be bijective and hence the map is an isometry.
Comparing the term in front of $\frac{\partial f}{\partial y_\alpha}$ expresses that φ is harmonic, which an isometry always is (see for instance [49] for details). □

We will introduce some extra notation for convenience. If $\varphi : X \to Y$ is a C^∞-diffeomorphism of Riemannian manifolds, we denote by w_φ the change of the volume element by the map φ (Radon-Nikodym derivative). Locally in a chart it is given by

$$w_\varphi = |\det(J_\varphi)| \sqrt{\det(g_Y)/\det(g_X)}, \tag{3.9}$$

where J_φ is the Jacobian matrix of φ. Now we can state the change of variables formula rather short as

$$\int_Y a_0 \mathrm{d}\mu_Y \quad = \quad \int_X a_0^* w_\varphi \mathrm{d}\mu_X, \tag{3.10}$$

holding for any function $a_0 \in L^2(Y)$.

By noting that for isometries $w_\varphi = 1$, we can also formulate the previous proposition in operator language as:

Lemma 3.2.5. *Suppose that $\varphi : X \to Y$ is a C^∞-diffeomorphism of closed Riemannian manifolds. Let*

$$U = \varphi^* \ : \ L^2(Y) \to L^2(X)$$

denote the induced pullback map. Then φ is a Riemannian isometry if and only if U is a unitary operator that intertwines the Laplace operators:

$$\Delta_X U = U \Delta_Y,$$

holding for all smooth functions on Y. □

Remark 3.2.6. If we do not assume that U arises as actual pullback from a map, then the existence of such a U merely implies that X and Y are isospectral (including multiplicities) induced by sending an eigenfunction on X with some eigenvalue to some eigenfunction on Y with the same eigenvalue. See also the discussion in [51] where isospectral pairs of manifolds are created as well.

There is another way of detecting isometries. The idea is that if we look at the intertwining, we might as well say that for enough f_1, f_2:

$$\int_X f_1 \varphi^*(\Delta_{Y,h} f_2) \mathrm{d}\mu_X = \int_X f_1 \Delta_{X,g} \varphi^*(f_2) \mathrm{d}\mu_X.$$

Now if we use the divergence theorem and apply a change of variables one basically arrives at the following proposition:

Proposition 3.2.7. *If $\varphi : X \to Y$ is a C^∞-diffeomorphism between closed Riemannian manifolds with $w_\varphi = 1$. Then φ is an isometry if and only if for all a_1, a_2 in a dense set of $C^\infty(Y)$*

$$\int_X g_X(da_1^*, da_2^*) \mathrm{d}\mu_X = \int_Y g_Y(da_1, da_2) \mathrm{d}\mu_Y.$$

Proof. First, perform on the right side a change of variables back to X and use $w_\varphi = 1$. Then expand da_i^* using the chain rule and gather the terms in front of $\partial_\mu a_1 \partial_\nu a_2$ together. By comparing these by the corresponding terms on the left and using the fundamental lemma of the calculus of variations for these derivatives one finds again equation 3.8. □

After this little detour we can now prove that the zeta functions are diffeomorphism invariants, which is to say that if all Riemannian structure is preserved by a map, then the zeta functions are equal:

Lemma 3.2.8. *The zeta-functions ζ_{X,a_0} and ζ_{X,a_1,a_2} are 'diffeomorphism invariants', in the sense that if $\varphi : X \to X$ is a C^∞-diffeomorphism, then*

$$\zeta_{X,g_X,a_0} = \zeta_{X,\varphi^*(g_X),\varphi^*(a_0)},$$

for all $a_0 \in C^\infty(Y)$ and

$$\zeta_{X,g_X,a_1,a_2} = \zeta_{X,\varphi^*(g_X),\varphi^*(a_1),\varphi^*(a_2)}.$$

for all $a_1, a_2 \in C^\infty(Y)$.

Proof. Suppose that φ is a smooth Riemannian isometry $(X,\varphi^*(g_X)) \to (X,g_X)$. We have

$$\varphi^* \Delta_{X,g_X} = \Delta_{X,g_X*}\varphi^*$$

for all smooth functions on X and we see that φ^* sends eigenfunctions to eigenfunctions:

$$\begin{aligned} \Delta_{X,g_X*}\varphi^*(\Psi_\lambda) &= \varphi^*(\Delta_{X,g_X}\Psi_\lambda) \\ &= \lambda\varphi^*(\Psi_\lambda) \end{aligned} \tag{3.11}$$

Furthermore, φ preserves integrals, i.e.,

$$\int f \mathrm{d}\mu_{g_X} = \int f^* \mathrm{d}\mu_{g_X^*}.$$

Hence φ^* sends normalized eigenfunctions to normalized eigenfunctions with the same eigenvalue, so that we have:

$$\begin{aligned} \zeta_{X,g_X,a_0}(s) &= \sum_{\lambda\neq 0}\lambda^{-s}\sum_{\Psi\vdash\lambda}\langle\Psi|a_0|\Psi\rangle_{g_X} \\ &= \sum_{\lambda\neq 0}\lambda^{-s}\sum_{\Psi\vdash\lambda}\langle\Psi^*|a_0^*|\Psi^*\rangle_{g_X^*} \\ &= \zeta_{X,g_X^*,a_0^*}(s). \end{aligned}$$

We can use corollary 3.4.3 to give an alternative proof. By using this corollary we find that the heat kernels are the same:

$$K_X(t,x,x) = K_Y(t,\varphi(x),\varphi(x)).$$

From this it follows that the sum of the squares of the eigenfunctions match up under pullback:

$$\sum_{\Psi_{X,g_X^*}\vdash\lambda}\Psi^2_{X,g_X^*}(x) = \sum_{\Psi_{X,g_X}\vdash\lambda}\Psi^2_{X,g_X}(\varphi(x)).$$

Now multiply this expression by a_0^* and integrate over X:

$$\begin{aligned} \int_X a_0^*(x)\sum_{\Psi_{X,g_X^*}\vdash\lambda}\Psi^2_{X,g_X^*}(x)\mathrm{d}\mu_X &= \int_X a_0^*(x)\sum_{\Psi_{X,g_X}\vdash\lambda}\Psi^2_{X,g_X}(\varphi(x))\mathrm{d}\mu_X \\ &= \int_X a_0(y)\sum_{\Psi_{X,g_X}\vdash\lambda}\Psi^2_{X,g_X}(y)\mathrm{d}\mu_Y. \end{aligned}$$

This is an equality of the coefficient of the zeta function at λ^{-s} and hence we are done for the first part.
For the 2-variable version the invariance follows from the previous computation of the one-variable version by using Lemma 3.2.1. □

3.3 Detecting isometries by zeta functions

In this section we will prove the following theorem, expressing how one can detect whether or not a C^∞-diffeomorphism is an isometry by looking at certain families of zeta functions. It is expressed by the following:

Theorem 3.3.1. *Let $\varphi : X \to Y$ denote a C^∞-diffeomorphism between closed connected smooth Riemannian manifolds with smooth metric. The following are equivalent:*

(i) *We have that*

(a) $\zeta_{Y,a_0} = \zeta_{X,\varphi^*(a_0)}$ *for all* $a_0 \in C^\infty(Y)$*, and*
(b) $\widetilde{\zeta}_{Y,a_1} = \widetilde{\zeta}_{X,\varphi^*(a_1)}$ *for all* $a_1 \in C^\infty(Y)$.

(ii) *The map φ is an isometry.*

Remarks 3.3.2.
We will start with some remarks. First observe that in conditions (a) and (b) of the theorem, we only pull back the functions a_i, not the Riemannian structure with corresponding Laplace operator, so the identities in (a) and (b) are in general non-void as in Lemma 3.2.8, because there, for diffeomorphism invariance, we pull back all structure, including the Laplace operator.
Secondly, as already noted in the introduction, the spectrum encoded in $\zeta_X(s)$ is an *incomplete* invariant of a Riemannian manifold: there exist isospectral non-isomorphic manifolds.

Recently Connes described in [13] a complete diffeomorphism invariant of a Riemannian manifold by adding to the spectrum the "relative spectrum" (viz., the relative position of two von Neumann algebras in Hilbert space).

Other work done by Bérard, Besson and Gallot ([2]) gave a faithful embedding of Riemannian manifolds into $\ell^2(\mathbf{Z})$, but by "wave functions", which are not diffeomorphism invariant (see also the end of section 4.4).

The family of zeta functions introduced here is some kind of diffeomorphism invariant when the C^∞-diffeomorphism type of the manifold is fixed, meaning that the algebra of functions $C^\infty(X)$ is given. These algebras of functions are used as "labels" for the zeta functions and cannot be omitted in our construction. We do not know whether the *sets* of functions $\zeta_{X,a_0}, \widetilde{\zeta_{X,a_1}}$ (without an explicit labeling) determine the isometry type of the manifold.

Let us now prove (ii) $\Rightarrow$ (i).

Proof of (ii) $\Rightarrow$ (i) in Theorem 3.3.1. Pull-back by φ induces a unitary transformation U between $L^2(Y)$ and $L^2(X)$ that intertwines the respective Laplace operators on smooth functions on Y. From this intertwining, we find that for every λ, $U\Psi_{Y,\lambda}$ is a normalized eigenfunction of eigenvalue λ. From (3.5), we get that $\zeta_{Y,a_0}(s) = \zeta_{X,a_0^*}(s)$ for all functions $a \in C^\infty(Y)$, and similarly for the two-variable version (cf. proof of Lemma 3.2.8). □

For the other direction of the proof, we will start with a rather short and formal argument by computing suitable residues of the zeta functions. Later on we will also compare expansion coefficients in the region of absolute convergence, rather than the residues. This will allow us to prove some stronger statements expressed for instance in Theorem 3.5.1.

We will use the following lemma (see [23], Lemma 1.3.7 and Thm. 3.3.1(1)):

Lemma 3.3.3. *Let X denote a closed d-dimensional Riemannian manifold, $d > 0$. Then for any $a_0 \in C^\infty(X)$ the function $\Gamma(s)\zeta_{X,a_0}(s)$ has a simple pole at $d/2$ with residue*

$$\mathrm{Res}_{s=\frac{d}{2}}\, \zeta_{X,a_0} = \frac{1}{\Gamma(\frac{d}{2})(4\pi)^{d/2}} \int_X a_0 \,\mathrm{d}\mu_X. \quad \square$$

We can use this to compute the value of the residue at $d/2$ of the second class zeta-functions.

Lemma 3.3.4. *Let X be a closed d-dimensional Riemannian manifold, $d > 0$. For any $a_1, a_2 \in C^\infty(X)$ we have*

$$Res_{s=\frac{d}{2}} \zeta_{X,a_1,a_2} = \frac{1}{\Gamma(\frac{d}{2})(4\pi)^{d/2}} \int_X g_X(da_1, da_2)\,\mathrm{d}\mu_X.$$

Proof. Follows immediately from the previous Lemma and Lemma 3.2.1. □

We will now give the first proof of the other direction. We will give a second one later on.

First proof of Theorem 3.3.1 ((i) $\Rightarrow$ (ii)). We will start by proving the following lemma:

Lemma 3.3.5. *The map φ has $w_\varphi = 1$.*

Proof. It follows from $\zeta_{Y,a_0} = \zeta_{X,a_0^*}$ by taking residues that

$$\mathrm{Res}_{s=\frac{d}{2}} \zeta_{Y,a_0}(s) = \mathrm{Res}_{s=\frac{d}{2}} \zeta_{X,a_0^*}(s).$$

At $a_0 = 1$, we find that X and Y have the same volume. Now using Lemma 3.3.3 the general equality of residues becomes

$$\int_Y a_0 \,\mathrm{d}\mu_Y = \int_X a_0^* \,\mathrm{d}\mu_X. \tag{3.12}$$

Furthermore, the change of variables formula (3.10) implies that

$$\int_X a_0^*(1-w_\varphi)\,\mathrm{d}\mu_X = 0 \quad (\forall a_0^* \in C^\infty(X)).$$

and the fundamental lemma of the calculus of variations gives that we must have

$$w_\varphi = 1 \tag{3.13}$$

identically. □

By using the polarization identity for the quadratic form g, i.e.

$$g(dp+dq,dp+dq) - g(dp-dq,dp-dq) = 4g(dp,dq),$$

we see that

$$\begin{aligned}
4\zeta_{X,a_1^*,a_2^*} &= 4\zeta_{X,g_X(da_1^*,da_2^*)} \\
&= \zeta_{X,g_X(d(a_1^*+a_2^*),d(a_1^*+da_2^*))} - \zeta_{X,g_X(d(a_1^*-a_2^*),d(a_1^*-da_2^*))} \\
&= \widetilde{\zeta}_{X,(a_1+a_2)^*} - \widetilde{\zeta}_{X,(a_1-a_2)^*} \\
&= \widetilde{\zeta}_{Y,(a_1+a_2)} - \widetilde{\zeta}_{Y,(a_1-a_2)} \\
&= 4\zeta_{Y,a_1,a_2}
\end{aligned}$$

for all $a_1, a_2 \in C^\infty(Y)$.

Now we can use Lemma 3.3.4 to get

$$\int_Y g_Y(da_1, da_2)\,\mathrm{d}\mu_Y = \int_X g_X(da_1^*, da_2^*)\,\mathrm{d}\mu_X.$$

By noting that $w_\varphi = 1$ we are done using by Proposition 3.2.7.

We could also have used the operator formalism as in proposition 3.2.5 to finish the proof as follows: After base change, using $w_\varphi = 1$, the previous equality of integrals gives

$$\int_X a_1^* \left((\Delta_Y(a_2))^* - \Delta_X(a_2^*)\right)\mathrm{d}\mu_X = \int_X a_1^* \left(\Delta_Y^* - \Delta_X\right)(a_2^*)\mathrm{d}\mu_X \qquad (3.14)$$

for all $a_1, a_2 \in C^\infty(Y)$. Here, $\Delta_Y^* = U \Delta_Y U^*$ with

$$U = \varphi^* : L^2(Y) \to L^2(X)$$

the pullback, and U^* the push-forward. Since this holds for all a_1^*, we find that

$$\Delta_Y^* = \Delta_X.$$

Also, U is unitary, since $w_\varphi = 1$, so that

$$\begin{aligned} \langle Uf, Ug\rangle_X &= \int_X f^* g^* w_\varphi \mathrm{d}\mu_X \\ &= \int_Y fg\mathrm{d}\mu_Y \\ &= \langle f, g\rangle_Y \end{aligned}$$

for all $f, g \in L^2(Y)$. Hence from Lemma 3.2.5, we find that φ is an isometry. □

Testing uncountably many functions at uncountably many points seems redundant. It indeed is:

Proposition 3.3.6. *We can restrict to countably many s_i and countably many test functions a_i such that the theorem still holds.*

Proof. Since we are dealing with usual Dirichlet series (the spectrum is an increasing sequence of positive real numbers with finite multiplicities), the condition $\zeta_{Y,a_0} = \zeta_{X,a_0^*}$ is for instance satisfied when $\zeta_{Y,a_0}(k) = \zeta_{X,a_0^*}(k)$ for all sufficiently

large integers k, and similarly for the higher order zeta functions (cf. [43], Section 2.2).
By going through the proofs, one sees that one does not need all $a \in C^\infty(Y)$, but that actually a dense subset is sufficient, as these functions are basically used as test functions in integrals.
Furthermore we assume that the manifolds are compact so that we can pick a countable basis. So we are done by picking $\zeta_{X,a_0}(k)$ and $\widetilde{\zeta}_{X,a_1}(k)$ for a_i in a countable basis, and k running through all sufficiently large integers. $\square$

Remark 3.3.7. Note that we can restrict for instance to functions of unit norm (supremum norm, or any other norm). We do not know whether it is possible to pick a *finite* set of functions, depending only on some topological characteristics of the manifold.
Notice that these zeta-invariants presupposes some knowledge of the manifold: one needs to be able to "label" by the functions.

3.4 Matching squared eigenfunctions

In this section, we investigate more closely the meaning of condition (a) in Theorem 3.3.1.

For convenience we use the following notation for the sum of the squares of eigenfunctions belonging to a fixed eigenvalue:

$$\sigma_{X,\lambda} = \sigma_\lambda := \sum_{\Psi_X \vdash \lambda} \Psi_X^2.$$

We will use the following lemma for generalized Dirichlet series ([28], Thm. 6.) for proving a proposition afterwards:

Lemma 3.4.1. *Suppose that the generalized Dirichlet series, $\sum_{n=1}^\infty a_n e^{-\lambda_n s}$, is convergent for $s = 0$, and let E denote the region*

$$\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq \delta > 0, \mathrm{angle}(s) \leq a < \frac{1}{2}\pi\}.$$

Suppose further that $f(s) = 0$ for infinitely many values of s lying inside E. Then $a_n = 0$ for all values of n.

Note that the origin does not play a crucial role, because if we translate s to $s - s_0$ a constant factor e^{s_0} appears in front. The set should then be shifted accordingly.

Proposition 3.4.2. *Suppose that $\varphi : X \to Y$ is a C^∞-diffeomorphism between connected closed Riemannian manifolds. Let $\{\Psi_{X,\lambda}\}$ and $\{\Psi_{Y,\mu}\}$ denote two complete sets of orthonormal real eigenfunctions for Δ_X and Δ_Y, respectively. Condition (a) in Theorem 3.3.1 is equivalent to the statement that the spectra of Δ_X and Δ_Y agree with multiplicities, we have $w_\varphi = 1$, and for any eigenvalue λ we have*

$$\sigma_{X,\lambda} := \sum_{\Psi_X \vdash \lambda} (\Psi_X)^2 = \sum_{\Psi_Y \vdash \lambda} (\Psi_Y^*)^2 =: \sigma_{Y,\lambda}^*.$$

Proof. The assumption is

$$\zeta_{Y,a_0}(s) = \zeta_{X,a_0^*}(s)$$

for all $a_0 \in C^\infty(Y)$. If we evaluate at the unit $a_0 = 1$, it follows that the nonzero spectra of Δ_X and Δ_Y, including multiplicities, agree using the previous lemma after shifting the origin to the right. The volumes agree because $w_\varphi = 1$ and using the residue from Lemma 3.3.3 for $a_0 = 1$ implies that the volume of X and Y agree as well.

The coefficients of the above Dirichlet series, when grouped according to fixed λ as in 3.5, are uniquely determined by it, using the identity theorem for Dirichlet series (lemma 3.4.1). If we spell out the assumption for the individual coefficients in λ^s in the two Dirichlet series, we find that for any $a_0 \in C^\infty(Y)$ we have

$$\int_Y \sum_{\Psi_Y \vdash \lambda} |\Psi_Y|^2 a_0 \mathrm{d}\mu_Y = \int_X \sum_{\Psi_X \vdash \lambda} |\Psi_X|^2 a_0^* \mathrm{d}\mu_X,$$

holding for any $\lambda \neq 0$. We perform a coordinate change in the first integral by using the map $\varphi : X \to Y$. Since we know that $w_\varphi = 1$, we find

$$\int_X \sum_{\Psi_Y \vdash \lambda} |\Psi_Y^*|^2 a_0^* \mathrm{d}\mu_X \quad = \quad \int_X \sum_{\Psi_X \vdash \lambda} |\Psi_X|^2 a_0^* \mathrm{d}\mu_X \tag{3.15}$$

for any $a_0 \in C^\infty(Y)$. Again the fundamental lemma of the calculus of variations gives us

$$\sum_{\Psi_X \vdash \lambda} (\Psi_X)^2 = \sum_{\Psi_Y \vdash \lambda} (\Psi_Y^*)^2, \tag{3.16}$$

for any $\lambda \neq 0$. The eigenvalue $\lambda = 0$ has multiplicity one, since the manifold is connected, and the normalized eigenfunction on Y is equal to $1/\sqrt{\mathrm{vol}(Y)}$, which pulls back to $1/\sqrt{\mathrm{vol}(Y)}$. Since X and Y have the same volume (from equality of their zeta functions at $a_0 = 1$), we find this is equal to $1/\sqrt{\mathrm{vol}(X)}$, the normalized eigenfunction for $\lambda = 0$ on X.

The other direction of the equivalence is obtained by reversing the steps, finishing the proof. □

We can use this proposition to prove a corollary about the diagonal of the heat kernel:

Corollary 3.4.3. *If $\varphi : X \to Y$ is a C^∞-diffeomorphism of closed connected Riemannian manifolds, then the following conditions are equivalent:*

(A) *For all $a_0 \in C^\infty(Y)$, we have that $\zeta_{Y,a_0} = \zeta_{X,\varphi^*(a_0)}$;*

(B) $K_X(t,x,x) = K_Y(t,\varphi(x),\varphi(x))$ *for all $t > 0$ and all $x \in X$, and $w_\varphi = 1$.*

Proof. Recall the following well known expression for the heat kernel (see for instance [3])

$$K_X(t,x,y) = \sum_{\substack{\lambda\in\Lambda \\ \text{distinct}}} e^{-\lambda t} \sum_{\Psi\vdash\lambda} \Psi(x)\Psi(y),\ (t > 0) \tag{3.17}$$

and setting $x = y$, we find

$$K_X(t,x,x) = \sum_{\substack{\lambda\in\Lambda \\ \text{distinct}}} e^{-\lambda t} \sigma_{X,\lambda}(x),\ (t > 0). \tag{3.18}$$

This implies the result by the previous proposition by noting that the multiplicities can be obtained by integrating the heat kernel:

$$\int_X K_X(t,x,x)\mathrm{d}\mu_X = \sum_{\substack{\lambda\in\Lambda \\ \text{distinct}}} n_\lambda e^{-\lambda t},$$

with n_λ the multiplicity of the eigenvalue λ. □

3.5 Improvements and the case of a simple spectrum

In this section we will improve some results of Theorem 3.3.1. Also we will consider the case where Δ_X has a simple spectrum.

The goal is to prove the following.

Theorem 3.5.1. *Let $\varphi : X \to Y$ denote a C^∞-diffeomorphism between closed Riemannian manifolds. Then:*

(a) In theorem 3.3.1, it suffices in condition (b) *to have equality of* one *arbitrary expansion coefficient of the Dirichlet series (for all a_1) for conditions* (a) *and* (b) *to be equivalent to* (ii).

(b) If the spectrum of X or Y is simple, then condition (a) *alone is equivalent to* (ii) *in theorem 3.3.1.*

We will start by taking a closer look at the expansion coefficients of the two-variable zeta functions, under the assumptions of (i) in Theorem 3.3.1. This computation provides an alternative proof of Theorem 3.3.1 and will also be used in proving part of Theorem 3.5.1:

Alternative proof of Theorem 3.3.1. Again looking at $a_0 = 1$, we find that X and Y have the same spectra with multiplicities. We denote this spectrum by Λ, as usual with multiplicities. As we have already seen, the polarization identity for the quadratic form g implies that $\zeta_{Y,a_1,a_2} = \zeta_{X,a_1^*,a_2^*}$. Our starting point is an expression for $\mathrm{tr}(a_1[\Delta, a_2]\Delta_Y^{-s})$:

$$\begin{aligned}&\mathrm{tr}(a_1[\Delta_X, a_2]\Delta_X^{-s}) \\ &= \sum_{\lambda\neq 0} \lambda^{-s} \sum_{\Psi_X \vdash \lambda} \int_X (\Psi_X a_1 \Delta_X(a_2)\Psi_X - 2a_1 g_X(da_2, d\Psi_X)\Psi_X)\, \mathrm{d}\mu_X .\end{aligned} \tag{3.19}$$

Hence that the coefficient for λ^{-s} is given by

$$\int_Y a_1 \left[\sigma_{Y,\lambda}\Delta_Y(a_2) - 2g_Y(da_2, d\sigma_{Y,\lambda})\right] \mathrm{d}\mu_Y .$$

If we equate this to the corresponding coefficient of the other zeta function, and then perform a base change to X where we use that $w_\varphi = 1$ we find

$$\begin{aligned}\int_X a_1^* \left[\sigma_{X,\lambda}\Delta_X(a_2^*) - 2g_X(da_2^*, d\sigma_{X,\lambda})\right] \mathrm{d}\mu_X \\ = \int_X a_1^* \left[\sigma_{Y,\lambda}^*\Delta_Y^*(a_2^*) - 2g_Y(da_2^*, d\sigma_{Y,\lambda}^*)\right] \mathrm{d}\mu_X. \qquad (3.20)\end{aligned}$$

After noting that $\sigma_{Y,\lambda}^* = \sigma_{X,\lambda}$, we use the fundamental lemma of calculus of variations to remove the integral and find the following equation of operators:

$$\begin{aligned}\sigma_\lambda(\Delta_Y^* - \Delta_X) &= 2g_Y^*(d-, d\sigma_\lambda) - 2g_X(d-, d\sigma_\lambda) \\ &= \text{first order operator.} \qquad (3.21)\end{aligned}$$

This means that the leading symbol of $\Delta_Y^* - \Delta_X$ vanishes outside the zero set of σ_λ, which again by Proposition 3.2.5 implies that $g_Y^* = g_X$ on this set. Since for every x there is a λ with $\sigma_\lambda(x) \neq 0$, we find $g_Y^* = g_X$ everywhere. Hence φ is an isometry. □

Now we consider how to improve the theorem in case the spectrum of Δ_X is simple and we will prove Theorem 3.5.1. We will prove that only condition (a) will suffice to conclude that φ is an isometry. We first start by reviewing some consequences of results related to condition (a):
First, condition (a) in Theorem 3.3.1 does not always suffice to imply that φ is an isometry, see Corollary 3.7.2.
Secondly, there exist isospectral, non-isometric compact Riemannian manifolds with simple spectrum (see for instance Zelditch [50], Theorem C), so (for maps with unit Jacobian) condition (a) is not equivalent to isospectrality (which would be condition (a) only for the identity function).
Finally, a result of Uhlenbeck ([46]) says that the condition of having non-simple Laplace spectrum is meager in the space of C^∞-Riemannian metrics on a given manifold X. Thus, Theorem 3.5.1 treats the ‘generic’ situation.

The following lemma will be used to prove the theorem.

Lemma 3.5.2. *Suppose that X is a closed smooth Riemannian manifold. Then the zero set of any nonzero eigenfunction of Δ_X is not dense. If we let $\widetilde{X} \subseteq X$ denote complement of the union of all such zero sets, then $\widetilde{X}$ is dense in X, and the following holds: for any real Δ_X-eigenfunction Φ, and any function $h \in C(X)$ that satisfies $h^2 = \Phi^2$, we have that $h = \pm\Phi$ on every connected component of $\widetilde{X}$.*

Proof. We can write $c\Phi = h$ where c is a function (a priori not necessarily globally constant) that takes values in $\{+1, -1\}$. We can assume that X is connected. We choose for $\widetilde{X}$ the complement of the union of all zero sets of non-zero Δ_X-eigenfunctions on X. By continuity, c is obviously constant on connected components of $\widetilde{X}$. All we have to show is that $\widetilde{X}$ is dense.

We claim that the complement of the zero set of an eigenfunction Φ is an open dense subset of X. Granting this for the moment, since the spectrum is discrete, the intersection $\widetilde{X}$ of all such complements of zero sets is a countable intersection of open dense subsets of X. Since X is compact and hence a complete metric space for the Riemannian metric, the Baire category theorem, saying that every countable intersection of dense open sets is dense, implies that this intersection is itself dense.

The claim will follow if we show that the zero set $\mathscr{Z}$ of Φ is nowhere dense. So suppose on the contrary that $\mathscr{Z}$ is dense in a neighbourhood U of some point $x \in X$. Then $\Phi \equiv 0$ on $\overline{U}$. Since we assume that the metric is smooth, the unique continuation theorem applies to the Laplacian (cf. [1], Rmk. 3, p. 449) and we find $\Phi \equiv 0$ on all of X, because we assumed the manifold to be connected, so that we have a contradiction. □

Now we finish the proof of the theorem.

Proof of Theorem 3.5.1. We start with part (b) of the theorem. Suppose that φ is an isometry between X and Y. Pull-back by φ induces a unitary transformation U between $L^2(Y)$ and $L^2(X)$ that intertwines the respective Laplace operators. From this intertwining, we find that for every λ, $U\Psi_{Y,\lambda}$ is a normalized eigenfunction of eigenvalue λ, hence equal to $\pm\Psi_{X,\lambda}$ by the simplicity assumption on the spectrum. From equation 3.5, we get that $\zeta_{Y,a_0}(s) = \zeta_{X,a_0^*}(s)$ for all functions $a_0 \in C^\infty(Y)$, proving one direction. For the converse direction, we know from the previous section by using the proof of proposition 3.4.2 that the pullback map $U = \varphi^*$ takes on the form

$$\begin{aligned} U : L^2(Y) &\to L^2(X) \\ \Psi_{Y,\lambda}(\lambda \neq 0) &\mapsto \varphi^*(\Psi_{Y,\lambda}) = c_\lambda \Psi_{X,\lambda};\ \ c_\lambda \in \{\pm 1\} \\ \Psi_{Y,0} = \frac{1_Y}{\sqrt{\mathrm{vol}(Y)}} &\mapsto \varphi^*(\Psi_{Y,0}) = \frac{1_X}{\sqrt{\mathrm{vol}(Y)}} = \frac{1_X}{\sqrt{\mathrm{vol}(X)}} = \Psi_{X,0} \end{aligned}$$

The map is unitary and bijective, because it sends a basis of normalized eigenfunctions to a basis of normalized eigenfunctions.

Next we prove that the map U also intertwines the Laplace-Beltrami operators. For this, let $\widetilde{X}_\lambda$ denote the zero set of the eigenfunction $\Psi_{X,\lambda}$. Let $x \in \widetilde{X}_\lambda$. We can find an open neighbourhood $\mathscr{U}_x$ of x on which $c_\lambda = \pm 1$ (defined by $\varphi^*(\Psi_{Y,\lambda}) = c_\lambda \Psi_{X,\lambda}$ as above) is constant. For any $\widetilde{x} \in \mathscr{U}_x$, we find

$$\begin{aligned} \Delta_X U \Psi_{Y,\lambda}(\widetilde{x}) &= \Delta_X (c_\lambda \Psi_{X,\lambda})(\widetilde{x}) \\ &= c_\lambda \lambda \Psi_{X,\lambda}(\widetilde{x}) \\ &= U(\lambda \Psi_{Y,\lambda})(\widetilde{x}) \\ &= U \Delta_Y \Psi_{Y,\lambda}(\widetilde{x}). \end{aligned} \tag{3.22}$$

This is an equality of continuous functions: for the right hand side, this is clear and for the continuity of the left hand side use that the map φ is assumed to be smooth. This equality holds on $\widetilde{X}_\lambda$ and since $\widetilde{X}_\lambda$ is dense in X (as in the previous proof), we find that it holds on X. Now since the eigenfunctions form a basis for $L^2(X)$, we find an equality of operators

$$\Delta_X U = U \Delta_Y.$$

This implies that X and Y are isometric by lemma 3.2.5, and finishes the proof of the second part of Theorem 3.5.1.

To prove part (a) of the theorem, we first observe that the zero set of σ_λ is nowhere dense. This is because σ_λ is a finite linear combination of the positive functions Ψ^2 for $\Psi \vdash \lambda$, so $\sigma_\lambda = 0$ implies $\Psi = 0$ for all $\Psi \vdash \lambda$, so we can use Lemma 3.5.2. Hence in this case, in the proof of Theorem 3.3.1, it suffices to have equation (3.21) for *only one* λ, i.e., equality of *one* coefficient of the Dirichlet series in condition (b) suffices. □

3.6 Further improvements

We can further improve some of the auxiliary results from the previous section.

Lemma 3.6.1. *If in Lemma 3.5.2 $h \in C^\infty(X)$, then $h = \pm\Phi$ with the sign constant everywhere.*

Note that we can write $c\Phi = h$ where c is just a function that takes values in $\{+1, -1\}$ and might a priori wildly vary. We have to prove that c is globally constant.

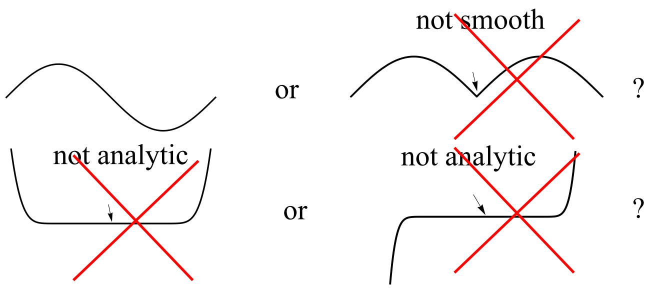


Figure 3.1: An eigenfunction around a nodal set

The gist of the proof is to use the regularity of eigenfunctions at zeros. Think of the prototypical $\sin(x) = c(x)h(x)$ on $[0, 2\pi]$. If $h(x)$ is *not* equal to $\pm\sin(x)$, then we have $h(x) = |\sin(x)|$. But that function is not even C^1 at $x = \pi$. On the other hand, functions like $f(x)^2 = e^{-2/|x|}(x \neq 0)$ and $f(0) = 0$ have different smooth $f(x)$ as square root, but have a zero of infinite order. See Figure 3.1.

Proof. It follows e.g. from the analysis in Caffarelli and Friedman ([7], Example 3 pp. 432–433, compare: [27], chapter 4, proof of Lemma 4.1.1) that for every point x_0 of the manifold X, there exists a small enough neighbourhood U of x_0 that intersects the zero set $\mathscr{Z}$ of Φ in the union of finitely many submanifolds of dimension $\leqslant m-1$. First note that if $x_0 \notin \mathscr{Z}$ there exists an open set $W \ni x_0$ for which $\Phi\,|_W \neq 0$. Then the function

$$\left(\frac{h}{\Phi}\right)|_W = c\,|_W$$

is smooth and hence $c\,|_W$ must be constant.

Now, let $x_0 \in \mathscr{Z}$, then both h and Φ vanish at x_0. Choose any smooth path $i :]-\varepsilon, \varepsilon[\to X$ such that

$$\mathrm{im}(i) \cap \mathscr{Z} = \{x_0\} \text{ with } i(0) = x_0 \text{ and } ||i'(t)|| > 0.$$

Then we get:

$$h(i(t))) = c(i(t)) \cdot \Phi(i(t)) \tag{3.23}$$

Assume that $c(i(t))$ changes sign at 0. Differentiating the equation at $t = 0$ gives:

$$\lim_{t\downarrow 0} h(i(t))' = \lim_{t\downarrow 0} \Phi(i(t))' = -\lim_{t\uparrow 0} \Phi(i(t))'.$$

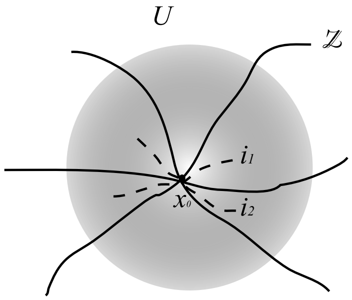


Figure 3.2: Local structure of the zero set $\mathscr{Z}$ in a neighbourhood U of x_0, with independent paths i_1, i_2

Because of smoothness, this implies

$$(\Phi \circ i)'(0) = (h \circ i)'(0) = 0.$$

Any path in $\mathscr{Z}$ is mapped identically to 0 by both h and Φ and hence also has derivative zero. It follows that both h and Φ have all (directional) derivatives in x_0 equal to 0. Indeed, because locally around x_0 the zero set $\mathscr{Z}$ is contained in a finite union of codimension $\geqslant 1$ submanifolds, we can find m paths i as above whose tangent vectors at x_0 span $T_{x_0}X$, and since all directional derivatives along these vectors are zero, so is the total derivative.
By induction it follows that up to any order all derivatives vanish. Hence x_0 is a zero of the eigenfunction Φ of infinite order, which is impossible by Aronszajn's unique continuation theorem (cf. [1], Rmk. 3, p. 449). We conclude that locally c does not change sign at zeros (and anyhow not at nonzeros). We assume X to be connected, so this implies that c is constant. $\square$

We deduce the following corollary about the diagonal of the heat kernel, essentially saying that in case of a simple spectrum, the diagonal determines the Riemannian manifold.

Corollary 3.6.2. *If $\varphi : X \to Y$ is a C^∞-diffeomorphism of closed connected C^∞-Riemannian manifolds and with simple Laplace spectrum, such that the diagonals of the heat kernels match up in the sense that*

$$K_X(t,x,x) = K_Y(t,\varphi(x),\varphi(x))$$

for sufficiently small $t > 0$, then the heat kernels match up in the sense that

$$K_X(t,x,y) = K_Y(t,\varphi(x),\varphi(y))$$

for all $t > 0$.

In particular, if g and g' are two smooth Riemannian structures on a closed connected manifold and with simple Laplace spectrum, then

$$K_g(t,x,x) = K_{g'}(t,x,x)$$

for sufficiently small $t > 0$ implies that

$$K_g(t,x,y) = K_{g'}(t,x,y)$$

for all $t > 0$ and hence $g = g'$.

Proof. Since

$$K_X(t,x,x) = K_Y(t,\varphi(x),\varphi(x))$$

for sufficiently small $t > 0$, and we have simple Laplace spectrum, formula (3.18) implies that the corresponding spectra, and the squares of eigenfunctions match up:

$$(\Psi_{X,\lambda})^2 = (\varphi^*\Psi_{Y,\lambda})^2.$$

The smoothness on both sides implies via the previous lemma that the functions agree up to a global sign. Applying (3.17), we find

$$\begin{aligned} K_X(t,x,y) &= \sum_\lambda e^{-\lambda t}\Psi_{X,\lambda}(x)\Psi_{X,\lambda}(y) \\ &= \sum_\lambda e^{-\lambda t}(\pm 1)^2\varphi^*\Psi_{Y,\lambda}(x)\varphi^*\Psi_{Y,\lambda}(y) \\ &= K_Y(t,\varphi(x),\varphi(y)). \end{aligned}$$

The particular case follows by setting φ to be the identity map. □

3.7 Example: flat tori

In this section we will compute some zeta functions for tori and see what some of the statements become. We also show that there exist non-isometric manifolds for which condition (a) of Theorem 3.3.1 holds.

Let $\mathbf{T} = \mathbf{R}^n/\Lambda$ denote a flat torus, corresponding to a lattice Λ in $\mathbf{R}^n$. Let $\Lambda^\vee$ denote the dual lattice to Λ. The Laplacian is

$$\Delta_{\mathbf{T}} = -\sum_k \partial_k^2,$$

and the spectrum is

$$\{4\pi^2||\lambda^\vee||^2\}_{\lambda^\vee \in \Lambda^\vee}.$$

A basis of orthogonal eigenfunctions of eigenvalue ℓ is computed by considering the periods and is given by

$$\Psi_{\lambda^\vee} := \frac{e^{2\pi i \langle \lambda^\vee, x\rangle}}{\sqrt{\mathrm{vol}(T)}}$$

if $||\lambda^\vee||^2 = \ell$. This is not a real basis as usual in this chapter, but we will make appropriate adaptations. The crucial property is that these functions satisfy

$$|\Psi_{\lambda^\vee}|^2 = \Psi_{\lambda^\vee} \cdot \overline{\Psi_{\lambda^\vee}} = \frac{1}{\mathrm{vol}(T)},$$

making computations doable.

Let us consider the meaning of condition (a) in Theorem 3.3.1 for the torus $\mathbf{T}$. Let $a_0 \in C^\infty(\mathbf{T})$. Then

$$\begin{aligned} \zeta_{\mathbf{T},a_0}(s) &= \sum_{\lambda^\vee \in \Lambda^\vee} \frac{1}{||4\pi^2\lambda^\vee||^{2s}} \cdot \frac{1}{\mathrm{vol}(T)} \int_{\mathbf{T}} a_0 |\Psi_{\lambda^\vee}|^2 \, \mathrm{d}\mu_{\mathbf{R}^n} \\ &= \left(\frac{1}{\mathrm{vol}(T)} \int_{\mathbf{T}} a_0 \, \mathrm{d}\mu_{\mathbf{R}^n} \right) \cdot \zeta_{\mathbf{T}}(s). \end{aligned} \tag{3.24}$$

By noting that the volume is determined by the spectrum we deduce the following proposition:

Proposition 3.7.1. *Let $\varphi : \mathbf{T}_1 \to \mathbf{T}_2$ denote a C^∞-diffeomorphism between two flat tori. Then the following are equivalent:*

(i) *For all $a_0 \in C^\infty(Y)$, we have that $\zeta_{\mathbf{T}_2,a_0} = \zeta_{\mathbf{T}_1,\varphi^*(a_0)}$;*

(ii) $\mathbf{T}_1$ *and* $\mathbf{T}_2$ *are isospectral, and* φ *has Jacobian* $w_\varphi = 1$. □

Corollary 3.7.2. *There exist non-isometric manifolds for which condition (a) of Theorem 3.3.1 holds.*

Proof. Take the following isospectral, non-isometric tori $\mathbf{T}_\pm$ as in [41] and [15]. These are 4 dimensional tori, whose lattice is spanned by the column vectors of the respective matrices G_+ and G_-

$$G_\pm = \frac{1}{2\sqrt{3}} \begin{pmatrix} \pm 3 & -\sqrt{7} & -\sqrt{13} & -\sqrt{19} \\ 1 & \pm 3\sqrt{7} & \sqrt{13} & -\sqrt{19} \\ 1 & -\sqrt{7} & \pm 3\sqrt{13} & \sqrt{19} \\ 1 & \sqrt{7} & \sqrt{13} & \pm 3\sqrt{19} \end{pmatrix}.$$

Consider the linear map $A : \mathbf{R}^4 \to \mathbf{R}^4$ given by

$$A = G_+ G_-^{-1} = \frac{1}{5} \begin{pmatrix} -3 & -2 & -1 & -3 \\ 2 & -2 & 4 & -3 \\ 3 & -3 & -4 & 3 \\ 1 & 4 & 2 & -4 \end{pmatrix}$$

with determinant $\det(A) = 1$. This map factors through to a map $\mathbf{T}_- \to \mathbf{T}_+$ with Jacobian $w_\varphi = \det(A) = 1$. □

Let us finish by considering condition (b) in Theorem 3.3.1 for the torus $\mathbf{T}$. We compute for $a_1, a_2 \in C^\infty(\mathbf{T})$, using Lemma 3.2.1, that

$$\zeta_{\mathbf{T},a_1,a_2}(s) = \left(\frac{1}{\mathrm{vol}(T)} \int_{\mathbf{T}} \nabla(a_1)^\top \nabla(a_2)\, \mathrm{d}\mu_{\mathbf{R}^n} \right) \zeta_{\mathbf{T}}(s) \tag{3.25}$$

where ∇ is the gradient operator. Hence the diagonal is given by

$$\widetilde{\zeta}_{\mathbf{T},a_1}(s) = \left(\frac{1}{\mathrm{vol}(T)} \int_{\mathbf{T}} |\nabla(a_1)|^2 \mathrm{d}\mu_{\mathbf{R}^n} \right) \zeta_{\mathbf{T}}(s).$$

Chapter 4

Lengths and distances

In this chapter we will use the invariant assigned to a Riemannian manifold in the previous chapter, consisting of certain families of zeta functions, to define what we call the spectral length of a map between Riemannian manifolds. We will show that a map of length zero between manifolds is an isometry and that this length induces a distance between Riemannian manifolds up to isometry.
We will discuss this notion in a more abstract framework of general length categories, and equip the category of Riemannian manifolds with C^∞-diffeomorphisms between them with such a length.
In the first section we will define the notion of length category and we will formulate a criterion when this length induces a distance between objects.
In the second section we will define the length of a C^∞-diffeomorphism between Riemannian manifolds and show that this induces a distance.
Then after we have computed some examples in section 4.3 we will study convergence in this category in section 4.4.

This chapter is based on the second part of the article "The spectral length of a map between Riemannian manifolds" [18].

4.1 Length categories

In this section we will introduce length categories. Let us start with the definition:

Definition 4.1.1. We call a pair $(\mathscr{C}, \ell)$ a *length category* if $\mathscr{C}$ is a category endowed with a subcategory $\mathscr{D}$, full on objects, such that every morphism of $\mathscr{D}$ is an isomorphism. These are called the isomorphisms from now on.
Furthermore, for every $X, Y \in \mathrm{Ob}(\mathscr{C})$ and every $\varphi \in \mathrm{Hom}(X, Y)$, there is defined a positive real number $\ell(\varphi) \in \mathbf{R}_{\geqslant 0}$, called the *length of* φ such that

(L1) $\ell(\varphi) = 0$ if and only if φ is an isomorphism;

(L2) If $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$ and $\varphi \in \mathrm{Hom}(X, Y), \psi \in \mathrm{Hom}(Y, Z)$, then

$$\ell(\psi \circ \varphi) \leqslant \ell(\varphi) + \ell(\psi).$$

Remark 4.1.2. In particular, in **(L1)** we do not assume that the isomorphism classes are necessarily the categorical isomorphism classes (i.e, the maps for which there exists an inverse in the category), but we do assume that the isomorphisms form a category and are categorical isomorphisms of $\mathscr{C}$. For instance, think of the category of metric spaces and continuous maps, but with isomorphisms the isometries (instead of the homeomorphisms). Note also that the morphisms of $\mathscr{D}$ can be recovered from the pair $(\mathscr{C}, \ell)$ as those morphisms with length 0.

To illustrate the concept, let us look at some examples. If there is no subcategory specified then implicitly it is understood that the morphism classes of $\mathscr{D}$ are the full isomorphism classes of $\mathscr{C}$.

Examples 4.1.3.
- Any category is a length category in a trivial way, defining the "discrete" length by

$$\ell(X \cong Y) = 0 \text{ and } \ell(\varphi) = 1 \text{ otherwise.}$$

However, categories can carry other, more meaningful lengths.

-Let Grp denote the category of finite commutative groups, and for $\varphi \in \mathrm{Hom}(G, H)$ a homomorphism of groups G and H, define its length as

$$\ell(\varphi) = \max\{\log(|\ker(\varphi)|), \log(|\mathrm{coker}(\varphi)|)\}.$$

This obviously satisfies **(L1)** and also **(L2)** since $|\ker(\psi\circ\varphi)| \le |\ker(\varphi)|\cdot|\ker(\psi)|$ and similarly for the cokernel. Hence (Grp,ℓ) is a length category.

- Let Fred denote the category of vectorspaces with Fredholm maps, i.e. linear maps with finite dimensional kernel and cokernel. Then analogously one can define

$$\ell(\varphi) = \max\{\dim(\ker(\varphi)), \dim(\mathrm{coker}(\varphi))\}.$$

- More generally, an *abelian* category with in some sense "measurable" kernels and cokernels is a length category by a similar construction.

However, non-abelian categories can also be length categories for an interesting length function. In some sense, this is a metric substitute for the non-existence of kernels/cokernels.

- The category of compact metric spaces with bi-Lipschitz homeomorphisms, i.e. compact metric spaces where the morphisms between X and Y are homeomorphisms $X \to Y$ for which there exists a $K \in \mathbf{R}_{\geq 0}$ such that for all $x, y \in X$ we have

$$\frac{1}{K} d_X(x,y) \le d_Y(f(x), f(y)) \le K d_X(x,y).$$

We can turn this into a length category as follows:

Lemma 4.1.4. *The category of compact metric spaces with bi-Lipschitz homeomorphisms and isomorphism classes the isometries form a length category for the length*

$$\ell(\varphi) := \max\{|\log \mathrm{dil}(\varphi)|, \left|\log \mathrm{dil}(\varphi^{-1})\right|\},$$

where $\mathrm{dil}(\varphi)$ *is the dilatation of the map* φ*, defined by*

$$\mathrm{dil}(\varphi) = \sup_{x,y\in X} \frac{d_Y(\varphi(x),\varphi(y))}{d_X(x,y)}. \tag{4.1}$$

Proof. If $\ell(\varphi) = 0$, then we find $\mathrm{dil}(\varphi) = \mathrm{dil}(\varphi^{-1}) = 1$. Hence for all $x, y \in X$, $d_Y(\varphi(x), \varphi(y)) \le d_X(x,y)$ and vice versa. So φ is an isometry.

For the triangle inequality we first write

$$\begin{aligned}
\mathrm{dil}(\psi \circ \varphi) &= \sup_{x,y \in X} \frac{d_Z(\psi \circ \varphi(x), \psi \circ \varphi(y))}{d_X(x,y)} \\
&= \sup_{x,y \in X} \frac{d_Z(\psi \circ \varphi(x), \psi \circ \varphi(y))}{d_Y(\varphi(x), \varphi(y))} \frac{d_Y(\varphi(x), \varphi(y))}{d_X(x,y)} \\
&\leq \sup_{x,y \in X} \frac{d_Z(\psi \circ \varphi(x), \psi \circ \varphi(y))}{d_Y(\varphi(x), \varphi(y))} \sup_{x,y \in X} \frac{d_Y(\varphi(x), \varphi(y))}{d_X(x,y)} \\
&= \mathrm{dil}(\psi)\mathrm{dil}(\phi), \qquad (4.2)
\end{aligned}$$

where in the final line we used that the maps are homeomorphisms.
Note that at least one of $\mathrm{dil}(\psi \circ \varphi)$ or $\mathrm{dil}((\psi \circ \varphi)^{-1})$ is bigger than one. Suppose that $\mathrm{dil}(\psi \circ \varphi) \geq 1$, then

$$\begin{aligned}
|\log(\mathrm{dil}(\psi \circ \varphi))| &\leq \log(\mathrm{dil}(\psi)\mathrm{dil}(\varphi)) \\
&= \log(\mathrm{dil}(\psi)) + \log(\mathrm{dil}(\varphi)) \\
&\leq |\log(\mathrm{dil}(\psi))| + |\log(\mathrm{dil}(\varphi))| \\
&\leq \ell(\varphi) + \ell(\psi) \qquad (4.3)
\end{aligned}$$

□

Lengths in categories sometimes give rise to a metric on the moduli space of objects of the category up to isomorphism, as the following lemma shows:

Lemma 4.1.5. *If $(\mathscr{C}, \ell)$ is a length category and we put*

$$d(X,Y) = \frac{1}{2}\left(\inf_{\varphi \in \mathrm{Hom}(X,Y)} \{\ell(\varphi), +\infty\} + \inf_{\psi \in \mathrm{Hom}(Y,X)} \{\ell(\psi), +\infty\} \right)$$

then d is an extended (i.e., $(\mathbf{R} \cup \{+\infty\})$-valued) metric on the "moduli space" $\mathrm{Ob}(\mathscr{C})/\mathrm{Iso}$ if for $d(X,Y) = 0$, the infimum in the definition of d is attained in $\mathrm{Hom}(X,Y)$. If furthermore $\mathrm{Hom}(X,Y) \neq \emptyset$ for any $X, Y \in \mathrm{Ob}(\mathscr{C})$, d is a (finite) metric.

Proof. First of all, length is well-defined on objects up to isomorphism: if φ is arbitrary and ψ is an isomorphism, then

$$\begin{aligned}
\ell(\varphi \circ \psi) \underset{\textbf{(L2)}}{\leqslant} \ell(\varphi) + \ell(\psi) \underset{\textbf{(L1)}}{=} \ell(\varphi) = \ell(\varphi \circ \psi \circ \psi^{-1}) \\
\underset{\textbf{(L2)}}{\leqslant} \ell(\varphi \circ \psi) + \ell(\psi^{-1}) \underset{\textbf{(L1)}}{=} \ell(\varphi \circ \psi). \qquad (4.4)
\end{aligned}$$

The positivity of d is clear. For the triangle inequality, since d is defined as the symmetrization of the hemimetric (which satisfies the axioms of a metric, except being symmetric and the property that distinct points have a strict positive distance)

$$d'(X,Y) = \inf_{\varphi \in \mathrm{Hom}(X,Y)} \{\ell(\varphi), +\infty\},$$

it suffices to prove the triangle inequality for d'. Let $\varepsilon > 0$. Let $\varphi \in \mathrm{Hom}(X,Y)$ and $\psi \in \mathrm{Hom}(Y,Z)$ be such that

$$\ell(\varphi) \leqslant d'(X,Y) + \varepsilon/2$$

and

$$\ell(\psi) \leqslant d'(Y,Z) + \varepsilon/2$$

(which is possible by the definition of length as an infimum). We have

$$d'(X,Z) = \inf_{\theta \in \mathrm{Hom}(X,Z)} \ell(\theta) \leqslant \ell(\psi \circ \varphi).$$

By axiom **(L2)** we find

$$\begin{aligned} \ell(\psi \circ \varphi) &\leqslant \ell(\psi) + \ell(\varphi) \\ &\leqslant d'(Y,Z) + d'(X,Y) + \varepsilon. \end{aligned}$$

The triangle inequality follows by letting ε tend to zero. Finally, assume

$$d(X,Y) = 0.$$

Since the infimum in the definition is attained, we find a map $\varphi \in \mathrm{Hom}(X,Y)$ of length zero. Then axiom **(L1)** implies that $X \cong Y$.
Finally, if all the Hom-sets are nonempty, then the infima are all finite and hence the metric d is finite. □

4.2 The length of a map between Riemannian manifolds

Let us now consider the category $\mathscr{R}$ of closed smooth Riemannian manifolds, with morphisms the C^∞-diffeomorphisms, and construct a length function in this category using the diffeomorphism invariants of the previous chapter (the isomorphisms will be the isometries). Essentially we construct a metric on the set of smooth metrics on a given smooth manifold. We will measure how far the one- and two-variable zeta functions ζ_{Y,a_0} and $\widetilde{\zeta}_{Y,a_1}$ are apart under pullback by the map φ with a suitable distance on the set of meromorphic functions when varying the test-functions a_0, a_1 over carefully chosen sets.

Definition 4.2.1. Let f and g denote two functions that are analytic (i.e. locally it can be written as a power series expansion) and non-zero in a right half line in $\mathbf{R}$

$$H_\sigma := \{s \in \mathbf{R} \mid s \geqslant \sigma\},$$

where σ is fixed once and for all.
For $n \geqslant 1$, define

$$W_{n,\sigma} = \{s \in \mathbf{R} \mid \sigma \leqslant s \leqslant \sigma + n\},$$

an interval of length n starting at σ. Note that these sets exhaust H_σ.
Now we define a distance between two of those functions on this set as follows:

$$\delta_n(f,g) := \sup_{s \in W_{n,\sigma}} \{ | \log \left| \frac{f(s)}{g(s)} \right| | \},$$

and set

$$d_\sigma(f,g) := \frac{\delta_1(f,g)}{1 + \delta_1(f,g)}.$$

Remark 4.2.2. One might as well take a distance more natural to the set of analytic, nonzero functions on H_σ, namely

$$\widehat{d}_\sigma(f,g) := \sum_{n \geqslant 1} 2^{-n} \frac{\delta_n(f,g)}{1 + \delta_n(f,g)}.$$

Convergence in d_σ and $\widehat{d}_\sigma$ is *not* uniform convergence of general analytic functions without zeros on $W_{1,\sigma}$ and H_σ respectively (because the absolute value signs cause an indeterminacy up to an analytic function with values in the unit circle), but when specialized to our Dirichlet series, this problem disappears.
Note that if a sequence of meromorphic functions $\{f_n\}_{n \geq 1}$ on $\mathbb{C}$ converges absolutely on an open subset of $\mathbf{R}$ to a meromorphic function f on its domain, then by unique continuation, the sequence of meromorphic functions converges pointwise to f on its domain.

With these definitions we can define the length of a C^∞-diffeomorphism of Riemannian manifolds as follows:

Definition 4.2.3. The *length of a C^∞-diffeomorphism* $\varphi : X \to Y$ of Riemannian manifolds of dimension N is defined by

$$\ell(\varphi) := \sup_{\substack{a_0 \in C^\infty(Y,\mathbf{R}_{\geqslant 0})-\{0\} \\ a_1 \in C^\infty(Y)-\mathbf{R}}} \max\{d_N(\zeta_{X,a_0^*},\zeta_{Y,a_0}), d_N(\widetilde{\zeta}_{X,a_1^*},\widetilde{\zeta}_{Y,a_1})\},$$

where $\mathbf{R}$ denotes the set of constant functions. This will give a length category, as we will prove.

Note that since d_σ is bounded by 1, the length of a map also takes values in $[0,1]$.
There is some arbitrariness in the definition of $\ell(\varphi)$: our zeta functions are holomorphic whenever $s > \frac{N}{2}$, so one might also take the suprema over another interval (or another exhaustion in case of $\widehat{d}$).

The following lemma shows that the length is well-defined, i.e., the zeta functions have no zeros on $H_N = \cup_{n\geq 1} W_{n,N}$.

Lemma 4.2.4. *The zeta functions $\zeta_{Y,a_0}, \zeta_{X,a_0^*}$ are nonzero on H_N for all nonzero $a_0 \in C^\infty(Y,\mathbf{R}_{\geqslant 0})$ and the zeta functions $\widetilde{\zeta}_{X,a_1^*}, \widetilde{\zeta}_{Y,a_1}$ are nonzero on H_N for $a_1 \in C^\infty(Y) - \{\mathbf{R}\}$.*

Proof. First note that the eigenvalues are positive. The $a_0 \in C^\infty(Y,\mathbf{R}_{\geqslant 0}) - \{0\}$ are strictly positive on some open subset and positive on the rest of the manifold. The set of eigenfunctions where we trace over form a basis and hence at least one of

$$\sum_{\Psi \vdash \lambda} \int_X a_0 \Psi^2$$

is nonzero. Hence the functions ζ_{X,a_0^*} and ζ_{Y,a_0} are indeed strictly positive on H_N.
For the second pair of zeta functions, note that by lemma 3.2.1

$$\widetilde{\zeta}_{Y,a_1} = \zeta_{Y,g(da_1,da_1)}.$$

The function

$$g(da_1,da_1) : X \to \mathbf{R}$$

is not identically zero because a_1 is nonconstant. Furthermore, g is a metric, so positive on the diagonal. And the same reasoning holds for a_1^*. Hence the previous argument applies. $\square$

The main theorem can be rephrased as follows, which shows that $(\mathscr{R},\ell)$ satisfies axiom **(L1)** of a length category:

Proposition 4.2.5. *If X and Y are closed Riemannian manifolds, then a C^∞-diffeomorphism $\varphi : X \to Y$ has length zero if and only if it is an isometry.*

Proof. If φ has length zero, then we have an equality of absolute values of zeta functions under pullback, at positive functions $a_0 \in C^\infty(Y,\mathbf{R}_{\geqslant 0}) - \{0\}$ and functions $a_1 \in C^\infty(Y) - \{\mathbf{R}\}$.
We can use lemma 3.4.1 to deduce that the two series are everywhere equal. We find that $\zeta_{X,a_0^*} = \zeta_{Y,a_0}$ for $a_0 \in C^\infty(Y,\mathbf{R}_{\geqslant 0})$ and $\widetilde{\zeta}_{X,a_1^*} = \widetilde{\zeta}_{Y,a_1}$ for all $a_1 \in C^\infty(Y)$. Since any $a_0 \in C^\infty(Y)$ is a linear combination of positive functions, we can apply Theorem 3.3.1 to conclude that φ is an isometry.
The converse statement follows directly from the same theorem. □

We now prove that $(\mathscr{R},\ell)$ also satisfies axiom **(L2)** of a length category with isometries as isomorphisms:

Proposition 4.2.6. *If X,Y,Z are closed Riemannian manifolds, and $\varphi : X \to Y$ and $\psi : Y \to Z$ are two C^∞-diffeomorphisms, then*

$$\ell(\psi \circ \varphi) \leqslant \ell(\varphi) + \ell(\psi).$$

Proof. We observe that we have injections of algebras of functions

$$\psi^* : C^\infty(Z,\mathbf{R}_{\geq 0}) \hookrightarrow C^\infty(Y,\mathbf{R}_{\geq 0})$$

and

$$\psi^* : C^\infty(Z) \hookrightarrow C^\infty(Y).$$

Note that the function

$$\mathbf{R}_{\geq 0} \to \mathbf{R} : x \mapsto \frac{x}{1+x}$$

is increasing, hence it suffices to show the inequality when replacing d_N for δ_n.
We have the identity

$$\frac{\zeta_{Z,a_0}}{\zeta_{X,\varphi^*\psi^*(a_0)}} = \frac{\zeta_{Z,a_0}}{\zeta_{Y,\psi^*(a_0)}} \cdot \frac{\zeta_{Y,\psi^* a_0}}{\zeta_{X,\varphi^*\psi^*(a_0)}},$$

which allows us to write

$$
\begin{aligned}
&\sup_{a_0 \in C^\infty(Z,\mathbf{R}_{\geqslant 0})\backslash\{0\}} \delta_{n,N}(\zeta_{X,\varphi^*\psi^*(a_0)}, \zeta_{Z,a_0}) \\
&= \sup_{a_0 \in C^\infty(Z,\mathbf{R}_{\geqslant 0})\backslash\{0\}} \log\left|\frac{\zeta_{Z,a_0}}{\zeta_{X,\varphi^*\psi^*(a_0)}}\right| \\
&= \sup_{a_0 \in C^\infty(Z,\mathbf{R}_{\geqslant 0})\backslash\{0\}} \left(\log\left|\frac{\zeta_{Z,a_0}}{\zeta_{Y,\psi^*(a_0)}}\right| + \log\left|\frac{\zeta_{Y,\psi^* a_0}}{\zeta_{X,\varphi^*\psi^*(a_0)}}\right|\right) \\
&\leq \sup_{a_0 \in C^\infty(Z,\mathbf{R}_{\geqslant 0})\backslash\{0\}} \log\left|\frac{\zeta_{Z,a_0}}{\zeta_{Y,\psi^*(a_0)}}\right| + \sup_{b_0 \in C^\infty(Y,\mathbf{R}_{\geqslant 0})\backslash\{0\}} \log\left|\frac{\zeta_{Y,b_0}}{\zeta_{X,\varphi^* b_0}}\right| \\
&= \sup_{b_0 \in C^\infty(Y,\mathbf{R}_{\geqslant 0})\backslash\{0\}} \delta_{n,N}(\zeta_{Y,\psi^*(a_0)}, \zeta_{Z,a_0}) \\
&\qquad + \sup_{a_0 \in C^\infty(Z,\mathbf{R}_{\geqslant 0})\backslash\{0\}} \delta_{n,N}(\zeta_{X,\phi^*(a_0)}, \zeta_{Y,b_0}),
\end{aligned} \tag{4.5}
$$

which proves the inequality for the first zeta functions.
For the two-variable zeta functions the same lines hold. □

We cannot directly apply Lemma 4.1.5 to conclude that ℓ induces a distance because at this point it is not clear that the infimum is attained, but we will remedy this in Section 4.4.

4.3 Examples

In general it is hard to compute the length of a map, let alone the distance between two Riemannian manifolds. We can however compute the length of the rescaling map of the circle to itself. Afterwards we will look at isospectral tori.

4.3.1 Example: length of rescaling a circle

Let S_r denote the circle of radius r, which we parameterize by an angle $\theta \in [0, 2\pi[$. The metric is $ds^2 = r^2 d\theta, g_{11} = r^2, g^{11} = r^{-2}$, the Laplacian is $-r^{-2}\partial_\theta^2$, with spectrum $\{n^2 r^{-2}\}_{n\in\mathbf{Z}_{>0}}$, with multiplicity two, and eigenspace for n spanned by

$\{\sin(n\theta), \cos(n\theta)\}$.
Let $\zeta(s)$ denote the Riemann zeta function. One sees directly that

$$\zeta_{S_r,a_0} = 2r^{2s+1}\left(\int_0^{2\pi} a_0(\theta)d\theta\right)\zeta(2s)$$

and

$$\zeta_{S_r,a_1,a_2} = 2r^{2s-1}\left(\int_0^{2\pi} a_1(\theta)\partial_\theta^2(a_2)(\theta)d\theta\right)\zeta(2s).$$

Hence

$$\zeta_{S_r,a_1,a_2} = r^2\zeta_{S_r,a_1\partial_\theta^2(a_2)}. \tag{4.6}$$

Let us compute the length of the natural rescaling homeomorphism

$$\varphi_{r_1,r_2} : S_{r_1} \to S_{r_2} : \theta \mapsto \theta \quad (\theta \in [0, 2\pi[).$$

We find

$$\left|\frac{\zeta_{S_{r_1},a_0^*}}{\zeta_{S_{r_2},a_0}}\right| = (r_1/r_2)^{2s+1}$$

and

$$\left|\frac{\zeta_{S_{r_1},a_1^*,a_2^*}}{\zeta_{S_{r_2},a_1,a_2}}\right| = (r_1/r_2)^{2s-1},$$

so we find for the length of φ_{r_1,r_2}:

$$\ell(\varphi_{r_1,r_2}) = \frac{1}{1+\frac{1}{5|\log(r_1/r_2)|}}.$$

Figure 4.1 depicts the $\ell(\varphi_{r,1})$ for $0 \leqslant r \leqslant 2$.

The two-variable zeta function is not involved in this computation (as is to be expected from the fact that the spectrum characterizes a circle).
Actually, the ratio of the one-variable zeta function does not depend on the test function a_0, so we see that the distance between circles is genuinely given by a distance between ordinary spectral zeta functions (of which we know they determine the circle).

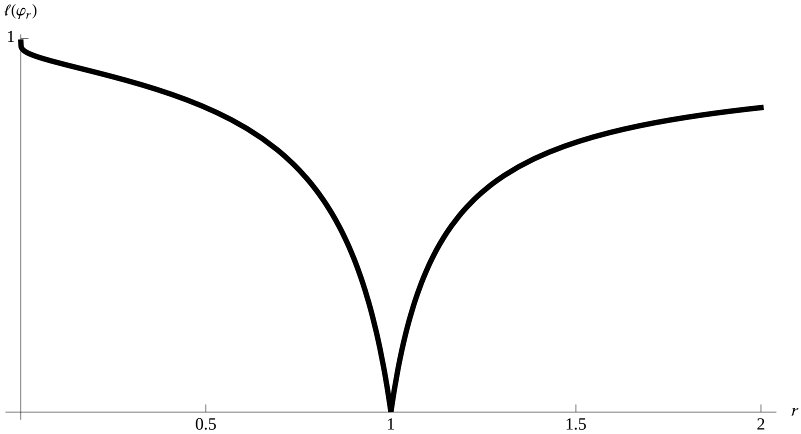


Figure 4.1: Length of the rescaling homeomorphism φ_r between a circle of radius r and a circle of unit radius

4.3.2 Example: length of a linear map between isospectral tori

To show how the construction remedies the isospectrality phenomenon (i.e., the fact that non-isometric Riemannian manifolds can be isospectral), consider the following example:

Let $\mathbf{T}_1$ and $\mathbf{T}_2$ denote two *isospectral* tori. Let $\varphi : \mathbf{T}_1 \to \mathbf{T}_2$ denote a C^∞-diffeomorphism, and assume that φ arises from a linear map A in the universal cover (any map of tori is homotopic to such a linear map with the same action on the homology of the torus, cf. [26], Lemma 1):

$$\begin{tikzcd} \mathbf{R}^n \arrow[r, "A"] \arrow[d, "\pi_1"'] & \mathbf{R}^n \arrow[d, "\pi_2"] \\ \mathbf{T}_1 = \mathbf{R}^n/\Lambda_1 \arrow[r, "\varphi"] & \mathbf{T}_2 = \mathbf{R}^n/\Lambda_2 \end{tikzcd}$$

This is well-defined if $A\Lambda_1 \subseteq \Lambda_2$. If we denote by G_1 and G_2 the generator matrices of the two tori (matrices whose columns are basis vectors of the lattice), the condition is that

$$G_2^{-1} A G_1 \in \mathrm{GL}(n, \mathbf{Z}). \tag{4.7}$$

Taking determinants, we find

$$\begin{aligned} w_\varphi &= |\det(A)| \\ &= |\det(G_1^{-1}G_2)| \\ &= \mathrm{vol}(\mathbf{T}_2)/\,\mathrm{vol}(\mathbf{T}_1). \end{aligned} \tag{4.8}$$

An example of such a map is the "change of basis" $A = G_2G_1^{-1}$. Write $A^\top$ for the transpose of the matrix A.

Since we assume $\mathbf{T}_1$ and $\mathbf{T}_2$ isospectral tori, they have the same (common) spectral zeta function. Hence from formula (3.24) we find that

$$\begin{aligned} \left|\frac{\zeta_{\mathbf{T}_1,a_0^*}}{\zeta_{\mathbf{T}_2,a_0}}\right| &= \left|\frac{\int_{\mathbf{T}_2} a_0 w_{\varphi^{-1}}\,d\mu_{\mathbf{R}^n}}{\int_{\mathbf{T}_2} a_0\,d\mu_{\mathbf{R}^n}}\right| \\ &= |\det(A^{-1})| \\ &= \frac{\mathrm{vol}(\mathbf{T}_1)}{\mathrm{vol}(\mathbf{T}_2)} \\ &= 1. \end{aligned} \tag{4.9}$$

Via formula (3.25), the two-variable zeta functions satisfy

$$\begin{aligned} \sup_{\nabla a_1 \neq 0} \left|\frac{\widetilde{\zeta}_{\mathbf{T}_1,a_1^*}}{\widetilde{\zeta}_{\mathbf{T}_2,a_1}}\right| &= \sup_{\nabla a_1 \neq 0} \frac{\int_{\mathbf{T}_1} |\nabla(a_1^*)|^2\,d\mu_{\mathbf{R}^n}}{\int_{\mathbf{T}_2} |\nabla(a_1)|^2\,d\mu_{\mathbf{R}^n}} \\ &= \sup_{\nabla a_1 \neq 0} \frac{\int_{\mathbf{T}_1} |A\nabla(a_1)|^2\,d\mu_{\mathbf{R}^n}}{\int_{\mathbf{T}_2} |\nabla(a_1)|^2\,d\mu_{\mathbf{R}^n}}. \end{aligned} \tag{4.10}$$

For every $v \in T_x\mathbf{T}_2$, we have

$$|Av|^2 \leqslant ||A||_2|v|^2,$$

where $||A||_2$ is the spectral norm of the matrix A (= the square root of the largest eigenvalue of $AA^\top$). Hence

$$\ell(A) \leqslant \frac{\log||A||_2}{1+\log||A||_2}.$$

One may wonder whether this bound is attained.

Example 4.3.1. The smallest dimension in which there exist non-isometric isospectral tori is four, as was shown by Schiemann ([41]), and an example is given by the two tori in the proof of 3.7.2. For the specific map $A = G_- G_+^{-1}$ between these tori, we have $||A||_2 \approx 3.21537$, and

$$\ell(A) \leqslant 0.538733.$$

4.4 Convergence in the spectral metric

In this section we will prove that the spectral length induces a distance between Riemannian manifolds. We will finish with an overview of various convergences of Riemannian manifolds and how this new distance compares.

Let us start by proving that our lengths behave well with respect to sequences of maps between fixed Riemannian manifolds.

Theorem 4.4.1. *Suppose we are given two Riemannian manifolds (X, g_X) and (Y, g_Y) and a collection of C^∞-diffeomorphisms $\varphi_i : X \to Y$ whose length converges to zero. Then X and Y are isometric.*

Proof. The proof is basically the "convergent" version of the first proof of Theorem 3.3.1.

In the definition of $\ell(\varphi)$, we observe that both zeta functions $\zeta_{Y,a_0}(s)$ and $\zeta_{X,\varphi_i^*(a_0)}(s)$ have their right most pole at $s = d/2$. Both poles are simple (see lemma 3.2.2), hence they cancel in the quotient. Therefore, the quotient function $\zeta_{X,\varphi_i^*(a_0)}(s)/\zeta_{Y,a_0}(s)$ is holomorphic in $\mathrm{Re}(s) \geqslant d/2 - 1/2$. Also, since a_0 is positive, the quotient is a positive real valued function on $s \in H_{\frac{d}{2}-\frac{1}{2}}$. We conclude from $\ell(\varphi) \to 0$ that

$$\zeta_{X,\varphi_i^*(a_0)}(s)/\zeta_{Y,a_0}(s) \to 1 \text{ for } s \in H_{\frac{d}{2}-\frac{1}{2}}.$$

In particular, we have convergence at $s = d/2$, and hence a convergence of residues

$$\begin{aligned} \mathrm{Res}_{s=\frac{d}{2}} \zeta_{X,\varphi_i^*(a_0)}(s) &= \lim_{s \to \frac{d}{2}+} \zeta_{X,\varphi_i^*(a_0)}(s)(s - \frac{d}{2}) \\ &\to \lim_{s \to \frac{d}{2}+} \zeta_{Y,a_0}(s)(s - \frac{d}{2}) = \mathrm{Res}_{s=\frac{d}{2}} \zeta_{Y,a_0}(s). \end{aligned} \quad (4.11)$$

By the computation of these residues in Lemma 3.3.3, we conclude that the jacobians converge to 1:

$$w_{\varphi_i} \to 1.$$

For the two-variable zeta functions, one may reason in a similar way, using that $g(da, da)$ is a totally positive function. From Lemma 3.3.4, we get in a similar way a convergence of metrics

$$\varphi_i^*(g_Y) \to g_X,$$

which is uniformly on X because X is compact.

Recall that the *distortion* of a map $\varphi : X \to Y$ is defined to be

$$\mathrm{dis}(\varphi) := \sup_{x_1, x_2 \in X} |d_Y(\varphi(x_1), \varphi(x_2)) - d_X(x_1, x_2)|.$$

The distance in terms of the metric tensor is

$$d(x_1, x_2) := \inf_{\substack{\gamma \in C^1([0,1], X) \\ \gamma(0) = x_1, \gamma(1) = x_2}} \int_0^1 \sqrt{\sum_{i,j} g^{ij}(\gamma(t))\gamma(t)_i' \gamma(t)_j'}\, dt.$$

By uniform convergence of metric tensors on the manifold X, we can interchange the infimum in the definition of the distance with the limit in metrics to conclude that

$$\mathrm{dis}(\varphi_i) \to 0. \tag{4.12}$$

We can now finish the proof as in [6] (proof of Thm. 7.3.30):
Since X is compact, we can find a dense countable set $S \subset X$, and we can find a subsequence $\{\varphi_i'\}$ of $\{\varphi_i\}$ that converges pointwise in Y at every $x \in S$. This allows us to define a limit map

$$\varphi : S \to Y \ \text{ by } \ \varphi(x) := \lim \varphi_i'(x)$$

for $x \in S$. This limit map is distance-preserving by (4.12), and so can be extended to a distance-preserving bijection from $X \to Y$. Now the Myers-Steenrod theorem ([34], 3.10), which states that a distance-preserving and surjective map is a smooth isometry, implies that φ is a smooth isometry between X and Y. □

Corollary 4.4.2. *The function "zeta-distance"*

$$d_\zeta(X, Y) := \max\{ \inf_{C^\infty(X \xrightarrow{\varphi} Y)} \ell(\varphi), +\infty\}$$

defines an extended metric between isometry classes of Riemannian manifolds.

Proof. It suffices to prove that if $d_\xi(X,Y) = 0$, then X and Y are isometric, and this follows from the previous theorem. □

We discuss the relation between our distance and other existing distances between Riemannian manifolds.

- First there is the *Lipschitz distance*. The distance between spaces X and Y is defined as the infimum of all R for which there is a bijective bi-Lipschitz map with constant $K = e^{-R}$ (see example 4.1.3 for the definition of these maps).

- Secondly, *Gromov-Hausdorff distance* (see [25], [6]).
It is defined as follows: One considers all isometric embeddings of X and Y into a common space Z for every space Z for which this is possible. And for every such pair $f, g : X \to Z$ one assigns the Hausdorff distance d_H between the images, i.e. $d_H(f(X), g(Y))$, where we recall that the Hausdorff distance d_H between two subsets A, B of a metric space Z is defined as

$$d_H(A,B) = \max\left(\sup_{a\in A} \inf_{b\in B} d_Z(a,b), \sup_{b\in B} \inf_{a\in A} d_Z(a,b) \right).$$

The Gromov-Hausdorff distance between two metric spaces X and Y is then defined as the infimum of all these Hausdorff distances between images.

- In Bérard-Besson-Gallot ([2]) two distances called d_t and δ_t are defined. These are defined using both the eigenfunctions of the Laplacian and the Gromov-Hausdorff distance:
Recall that the space ℓ^2 consists of real sequences

$$\overline{a} = \{a_i\}_{i\geq 1}$$

for which $\sum_{i=1}^{\infty} |a_i|^2 < \infty$, which is a metric space by setting

$$d(\overline{a}, \overline{b}) = \sqrt{\sum_{i=1}^{\infty} |a_i - b_i|^2}.$$

If M is an Riemannian manifold of dimension n. Denote by $a = \{\phi_j^a\}$ an orthonormal basis of eigenfunctions of the Laplacian with eigenvalue λ_j. Then one defines maps I_t^a for $t > 0$ as:

$$\begin{aligned} I_t^a &: M \to \ell^2 \\ I_t^a(x) &= \sqrt{\mathrm{vol}(M)}\{e^{-\lambda_j t/2}\phi_j^a(x)\}_{j\geq 1}. \end{aligned} \tag{4.13}$$

Next define

$$\mathscr{B}(M) = \prod_{i=1}^{\infty} \mathscr{B}(E_i),$$

where $\mathscr{B}(E_i)$ is the set of all possible orthonormal bases for the eigenspace E_i corresponding to eigenvalue λ_i. The distance d_t between two Riemannian manifolds X and Y is now defined as follows:

$$\begin{aligned} d_t(M, M') = \max \Bigg\{ & \sup_{a\in\mathscr{B}(X)} \inf_{a'\in\mathscr{B}(Y)} d_H(I_t(a)(X), I_t^{a'}(Y)), \\ & \sup_{a'\in\mathscr{B}(Y)} \inf_{a\in\mathscr{B}(X)} d_H(I_t(a)(X), I_t^{a'}(Y)) \Bigg\}. \end{aligned} \tag{4.14}$$

This measures how far eigenfunctions are apart from each other, where the arbitrary choice of basis is eliminated.
The metric δ_t, which is actually a quasi-metric (i.e., it does not satisfy the symmetry axiom for a metric $d_t(x, y) = d_t(y, x)$ for all x, y), is defined for a fixed Riemannian manifold M with two Riemannian metrics g, h as:

$$d_t((M, g), (M, h)) = \max \left\{ \sup_{a\in\mathscr{B}(M,h)} \inf_{b\in\mathscr{B}(M,g)} d_H(I_t(a)(M), I_t^b(M)) \right\}. \tag{4.15}$$

- Finally, there are spectral distances found by Kasue-Kumura in [33]. They are defined by using heat kernels:
Let $K_X(t, x, y), K_Y(t, x, y)$ denote the heat kernel of X and Y respectively. Modify these by including a volume factor, i.e. define

$$\begin{aligned} p_X(t, x, y) &= \mathrm{vol}(X) K_X(t, x, y) \\ p_Y(t, x, y) &= \mathrm{vol}(Y) K_Y(t, x, y) \end{aligned} \tag{4.16}$$

Then a pair of maps $f : X \to Y, h : Y \to X$ (not necessarily continuous), are called *ϵ-spectral approximations* between X and Y if for all $x, y \in X$ and all $t > 0$:

$$e^{-t+1/t}|p_X(t, x, y) - p_Y(t, f(x), f(y))| < \epsilon$$

and for all $w, z \in Y$

$$e^{-t+1/t}|p_X(t, h(w), h(z)) - p_Y(t, w, z)| < \epsilon.$$

The spectral distance between X and Y is then defined as the infimum of all ϵ such that there is an ϵ-spectral approximation between them.

All of these distances pose computational challenges. In the previous sections, we have hinted at some of the computational aspects of the "zeta-distance" we defined. There are various relations between these distances. Let us start by relating convergence of d_ζ to uniform convergence.

We conclude by comparing our "zeta-distance" d_ζ to the other distances:

Theorem 4.4.3. *Let $\mathscr{M}$ denote a space of closed Riemannian manifolds up to isometry. Then d_ζ induces the topology of uniform convergence in C^∞-diffeomorphic types on $\mathscr{M}$, i.e., if two such manifolds are not C^∞-diffeomorphic, then the manifolds are at infinite distance, and otherwise, a sequence of manifolds converge if and only if there is a sequence of C^∞-bijections between them whose distortion tends to zero.*

Proof. Suppose $(X_i, g_i) \to (X, g)$ converges in d_ζ. This means that there is a sequence of C^∞-diffeomorphisms $\varphi_i : (X_i, g_i) \to (X, g)$ whose length converges to zero. We precompose this with φ_i^{-1}:

$$\begin{tikzcd}
(X_i, g_i) \arrow[r, "\varphi_i"] & (X, g) \\
(X, (\varphi_i^{-1})^*(g_i)) \arrow[u, "\varphi_i^{-1}"] \arrow[ur, "\mathrm{Id}"'] &
\end{tikzcd}$$

Hence we have a sequence of metrics $h_i := (\varphi_i^{-1})^*(g_i)$ for which the length of the identity map converges to 0. Taking residues in the two-variable zeta functions, we find that $h_i \to g$ uniformly. $\square$

Now let us see how this new distance compares with the other distances by looking at the following diagram for the relation between various forms of convergence (the proofs can be found in the denoted references):

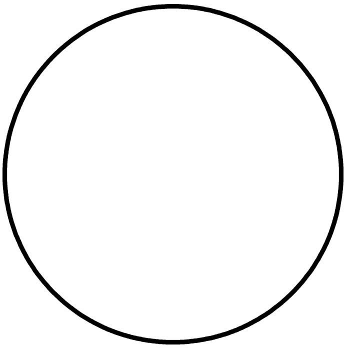
$$
\begin{tikzcd}
 & \text{Lipschitz-conv.} \arrow[dl, Rightarrow, "\text{simple spectrum}"', "{[2]}"] \arrow[d, Rightarrow, "{[2]}"] \arrow[dr, Rightarrow, "{[6]}"] & & \\
d_t\text{-conv.} & \delta_t\text{-conv.} & \text{unif. conv.} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow, "{[6]}"] & \boldsymbol{d_\zeta}\textbf{-conv.} \\
\text{Kasue-Kumura-conv.} \arrow[rr, Rightarrow, "{[33]}"'] & & \text{GH-conv.} &
\end{tikzcd}
$$

Figure 4.2: Some relations between convergence in various distances (in a fixed C^∞-type)

We see that our convergence is implied by Lipschitz convergence and implies Gromov-Hausdorff convergence.

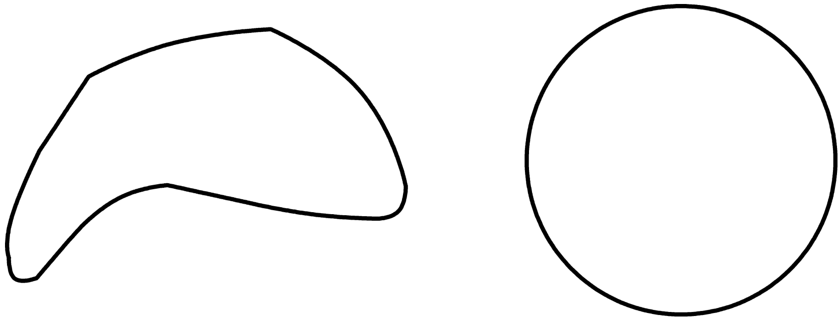
Samenvatting

Dit proefschrift speelt zich binnen de wiskunde af in het vakgebied genaamd 'niet-commutatieve meetkunde'. Dit is in tegenstelling tot de meeste andere vakgebieden binnen de wiskunde een relatief jong vakgebied, het ontstond aan het einde van de 20e eeuw.
Alvorens te vertellen waar dit proefschrift over gaat, zal ik eerst (een deel van) dit vakgebied introduceren aan de hand van een voorbeeld.
Neem daarvoor een cirkel in gedachte.



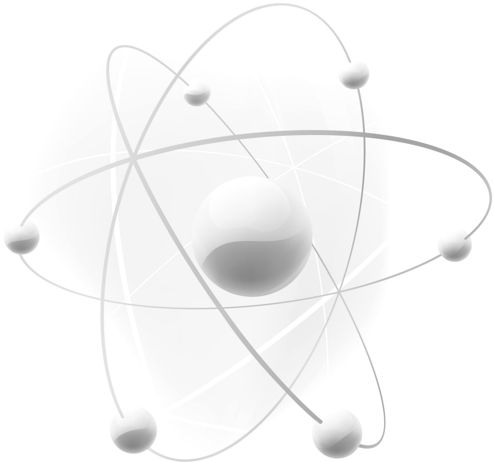
De cirkel.

De cirkel willen we liever algebraisch beschrijven dan als figuur en daarom bekijken we de ruimte van functies hierop, om precies te zijn de ruimte van continue functies. Deze ruimte kan beschreven worden als functies op een interval die in het begin- en eindpunt dezelfde waarde hebben. Verrassend genoeg blijkt deze ruimte van functies enkel van de cirkel afkomstig te kunnen zijn, men kan de cirkel eruit reconstrueren. Echter, enkel topologisch, wat wil zeggen dat je de cirkel kunt reconstrueren *op vervormen na* en is bijvoorbeeld op deze manier niet te onderscheiden van een elastiek.



Een elastiek is topologisch niet te onderscheiden van een cirkel.

De ruimte van functies op een cirkel is commutatief, wat wil zeggen dat je de volgorde van vermenigvuldigen mag omdraaien, net zoals $3 \cdot 4 = 4 \cdot 3$. De niet-commutatieve meetkunde laat, zoals de naam al doet vermoeden, deze eis vallen en de ruimtes die men dan verkrijgt worden 'niet-commutatieve ruimtes' genoemd. Op het eerste gezicht klinkt dit vrij exotisch, maar niks is minder waar. In de wereld van het kleine, waar de quantummechanica de natuurwetten beschrijft, blijken deze niet-commutatieve ruimtes bij uitstek geschikt voor een solide wiskundige beschrijving van de fenomenen die daar optreden.
Laten we weer terug naar de cirkel gaan. Nu met een zogenaamde spinbundel, die ik hier niet verder zal introduceren. Zoals gezegd beschrijven de functies op de cirkel de cirkel *op vervormen na*. Om deze laatste complicatie op te lossen en om ook de spinbundel te beschrijven moest er wat verzonnen worden. Rondom 1995 kreeg Alain Connes het idee van 'spectrale drietallen'. Deze bestaan zoals de naam doet vermoeden uit drie objecten, waaronder de eerder genoemde ruimte van functies op het object. Het idee bleek het juiste te zijn, maar om dat daadwerkelijk te bewijzen bleek verre van eenvoudig, een bewijs werd namelijk pas 13 jaar later in 2008 gegeven.
Het idee van Alain Connes heeft een niet-commutatieve generalisatie en heeft diverse toepassingen. Zo is bijvoorbeeld het 'standaard model' uit de natuurkunde, dat gebruikt wordt in de deeltjes fysica, relatief eenvoudig te verkrijgen uit een spectraal drietal.

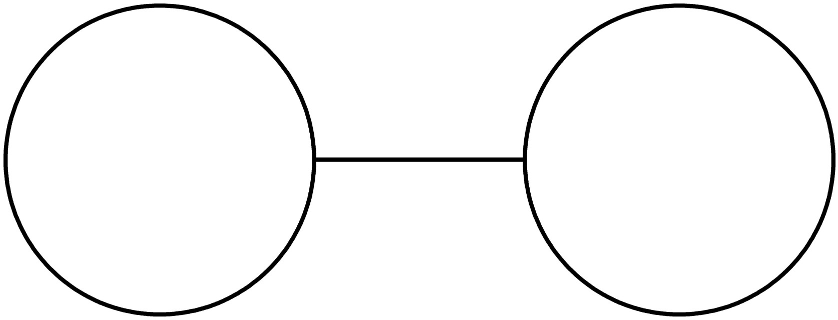


Een atoom uit de deeltjes fysica.

Met spectrale drietallen kun je dus bepaalde objecten beschrijven en daarmee komen we aan bij het onderwerp van dit proefschrift. In het eerste deel van dit proefschrift wordt er een spectraal drietal geconstrueerd voor eindige grafen en de hoofdvraag vraag die daar beantwoord wordt is of dit een 'accurate beschrijving' is van grafen, wat wil zeggen dat er niet toevallig twee grafen bij hetzelfde spectrale drietal kunnen horen. Denkt u bijvoorbeeld aan de graaf wat verderop.

De vraag is a priori geen eenvoudige opgave, hoe toon je zoiets aan? Dat is waar spectrale drietallen van pas komen. Aan bepaalde spectrale drietallen kunnen namelijk oneindig veel invarianten toegekend worden, de zogenoemde *zeta functies*. De meer conventionele wiskundige beschrijving geeft meestal aanleiding tot slechts één zeta functie en daar treedt vaak het eerder genoemde probleem op, meerdere grafen hebben dezelfde zeta functie. Dit wordt het fenomeen van *isospectraliteit* genoemd. In het geval van grafen zit in die ene functie het geslacht, wat wil zeggen, het aantal onafhankelijke gesloten paden in de graaf. Voor de cirkel, nu even beschouwd als graaf, is het geslacht bijvoorbeeld 1 en bij het gegeven voorbeeld is dat 2.

Een belangrijk resultaat uit dit proefschrift is dat deze oneindig veel zeta functies,



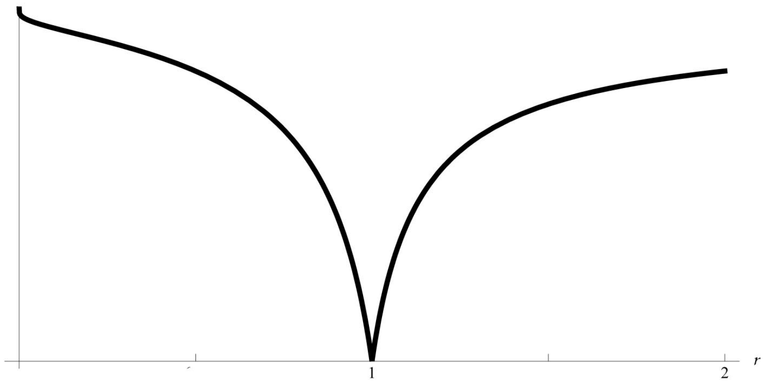
Een voorbeeld van een graaf.

die uit het geconstrueerde spectrale drietal verkregen worden, de graaf helemaal bepalen. Dit noemen we *zeta-rigiditeit*. Een consequentie hiervan is dat het oorspronkelijke spectrale drietal de graaf ook vastlegt, waarmee we de originele vraag beantwoorden.
Met onder andere dit positieve resultaat in het achterhoofd zijn we vervolgens gaan kijken naar Riemannse varieteiten. U kunt hiervoor weer de cirkel in gedachte nemen, maar het is illustratiever één van de trommels op de voorkant van dit proefschrift in gedachte te nemen.

Eerst wat geschiedenis. In 1966 vroeg wiskundige Mark Kac zich af of je door enkel te luisteren naar het geluid dat een trommel voortbrengt je de vorm van de trommel kunt bepalen. Of bestaan er soms trommels van verschillende vorm die toch dezelfde klanken kunnen produceren, d.w.z., hetzelfde spectrum hebben? Al vrij snel wist John Milnor de vraag van Mark Kac te beantwoorden, zulke trommels bestaan! Milnor gaf in eerste instantie enkele voorbeelden in 16 dimensies, welke dus nog geen trommels waren, maar wat jaren later werden er ook verschillende twee dimensionale voorbeelden gevonden. We zien hier dus hetzelfde fenomeen als eerder optreden, namelijk dat van *isospectraliteit*!
Met de machinerie van spectrale drietallen nog vers in het geheugen hebben we dit fenomeen op een zelfde manier weten te beschrijven. Net zoals bij grafen is het spectrum wederom bevat in een enkele zeta functie die wederom lid is van een familie van zeta functies, of eigenlijk twee families van zeta functies, namelijk ζ_a en $\widetilde{\zeta}_a$ voor iedere functie a op het object. In dit proefschrift hebben we bewezen dat deze invarianten de trommel vastleggen, of in het algemeen een Riemannse varieteit. Op de voorkant van dit proefschrift ziet u daarom geen noten de trommel uitkomen, maar de zojuist genoemde invarianten. Het credo is dus, je moet niet

Enkele trommels.

enkel naar het klassieke spectrum luisteren, maar naar het volledige spectrum van ζ_a en $\widetilde{\zeta_a}$ die opduiken uit ideeën uit de niet-commutatieve meetkunde. Het volledige spectrum, in het geval van een trommel, kun je niet écht horen met het oor, maar het is een goede analogie.
Om de eerder genoemde resultaten te formaliseren hebben we in het laatste hoofdstuk een afstand tussen objecten gedefinieerd. Om terug naar het voorbeeld te gaan. U kunt zich voorstellen dat sommige trommels qua klanken erg op elkaar lijken, we zeggen dan dat deze trommels een korte afstand tot elkaar hebben en trommels die qua spectrum minder op elkaar lijken staan verder uiteen.
In de volgende grafiek ziet u ter illustratie de afstand tussen een cirkel van straal r een cirkel van straal 1 uitgezet. U ziet dat de afstand precies 0 is als de circkels gelijk zijn, i.e, $r = 1$, en van daaruit toeneemt in zowel de positieve als de negatieve richting. Deze afstand maakt het mogelijk om meetkunde te bedrijven in veranderende objecten, denk bijvoorbeeld aan een cirkel die steeds groter wordt, of een trommel die vervormt in een andere trommel. Mogelijkerwijs hebben deze resultaten daarom toepassing in de natuurkunde. Seriu heeft al in 1996 een artikel geschreven (zie [42]) waarin geprobeerd wordt de dynamiek van de kosmos te be-



De afstand tussen een cirkel van straal r en een cirkel van straal 1.

schrijven met behulp van spectrale invarianten, maar niet met een complete familie zoals in dit proefschrift. Als gevolg heeft zijn aanpak het probleem van isospectraliteit waardoor de evolutie van de kosmos niet eenduidig vastgelegd wordt en zelfs niet continue overgangen toelaat, wat op z'n minst als onorthodox beschouwd kan worden. Mogelijkerwijs kunnen de geconstrueerde zeta invarianten en / of de gevonden afstand uitkomst bieden in een beschrijving van de kosmos door spectrale invarianten.

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Curriculum Vitae

Jan Willem de Jong was born on the 5th of June in 1982 in Ede. After spending his first years in Dieren he obtained his high school degree in 2000 at the Christelijk College Nassau Veluwe in Harderwijk while living in Nijkerk.
Afther that he started the 'twin program' physics and mathematics at Utrecht University and obtained both propedeuses in 2001. His master's degree mathematics was obtained cum laude in 2005 under the supervision of Prof. dr. E.J.N. Looijenga. In 2007, now under the supervision of Dr. S.J.G. Vandoren, he obtained his master's degree physics cum laude as well.
In that same year Jan Willem began his PhD studies in Utrecht together with Prof. dr. G.L.M. Cornelissen which resulted in this thesis. During his research he taught several undergraduate courses and visited seminars in the Netherlands and also abroad, this included Trieste, Nashville, Columbus Ohio, Bonn and Granada.

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