*A Geometrical Approach to*
*Two-Dimensional*

# Conformal Field Theory

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# A Geometrical Approach to Two-Dimensional Conformal Field Theory

# A Geometrical Approach to Two-Dimensional Conformal Field Theory

*Een Meetkundige Benadering van Tweedimensionale Conforme Veldentheorie*

(met een samenvatting in het Nederlands)

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*Aan mijn moeder*

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# 1
# General Introduction

A great enigma in theoretical physics is how to reconcile two very successful theories of Nature, that both describe physical phenomena in excellent agreement with experiment, albeit it at completely different ranges of energy and distance. On the one hand, at the microscopic scales in reach of present day accelerators, quantum field theory—the theory of elementary particles and the forces that govern their interactions—has been shaped into almost perfection with the highly acclaimed Standard Model. Many of its theoretical predictions have been verified, some with an accuracy that is unprecedented in the history of physics. On the other hand, the macroscopic structures in the universe, the stars, the galaxies, the universe itself, obey without noticeable deviations the laws of Einstein's theory of gravitation—General Relativity. However, these two theories, although both internally consistent and founded on principles of great beauty, have resisted all attempts to be unified in a consistent fashion into one fundamental theory of Nature. Nevertheless, there are many interesting areas where the two theories have to meet, most notably in the very early stages of the evolution of the universe, but also in the vicinity of black holes and all other places where particles interact with strong gravitational fields. This problem, how to reconcile gravity with the principles of quantum physics, is known as *quantum gravity*. It has haunted physicists for many years, and has led them to propose various theoretical models, many of which have failed by lack of consistency before meeting with experiment. However, one such proposal is still alive, and that is superstring theory, or simply string theory [90]. Although this thesis does not concern string theory *per se*, its subject—the study of two-dimensional conformal field theory—owes much of its impetus and motivation to string theory. Conformal field theory might be regarded as the study of the classical solutions of string theory, and is a main ingredient in any perturbative string calculation. So, let us digress a bit on the presumed qualities of string theory as a theory of quantum gravity.

The essential problems in quantum gravity are caused by the fact that the strength of the gravitational interaction, as measured by Newton's constant $G$, is not dimensionless. Since $G$ acts as the coupling constant in a perturbative quantization of the gravitational field, there is a length scale, the Planck length [129]

$$L_{Planck} = (\hbar G/c^3)^{\frac{1}{2}} \sim 10^{-33}\,\mathrm{cm}, \tag{1.1}$$

with a corresponding energy range, $(\hbar c^5/G)^{\frac{1}{2}} \sim 10^{19}\,\mathrm{GeV}$, where the quantum fluctuations of the gravitational field become dominant. At these scales the higher order quantum corrections are as important as the lower order contributions, and the whole concept of a perturbation theory becomes meaningless. This is usually understood to signal a catastrophe: the complete breakdown of the smooth structure of space-time at Planckian scales. Space-time changes from an inert spectator to an active participant in the interactions of quantum particles. This is the cause of serious infinities that render any standard field theory approach useless.

String theory tries to tackle these problems in a rather bold way. It assumes that fundamental particles are not point-like but excitations of an extended, one-dimensional object—the string. Among these excitations are the gravitons, the quanta of the gravitational field. The size of a string is roughly of the Planck length. So, although for any conceivable realistic experiment it can be considered to be point-like, string theory embodies a very natural cut-off at distances of relevance to quantum gravity. The infinities that torment point particle theories are softened by the intrinsic 'fuzziness' of the string.

Although we will not be able to experimentally verify the presumed 'stringy' structure of elementary particles, string theory does give very strict predictions on the particle spectrum. In fact, a string can be regarded as an infinite tower of particles, whose masses are equally spaced with intervals of the Planck mass. The particles with non-zero masses would be extremely heavy ($\sim 10^{-5}$ g), so only the massless modes should correspond to directly observable particles in our low energy world—among others the graviton, gauge bosons and fermionic matter fields. Since the string is supersymmetric, at least in flat space-time, the massless modes will correspond to a supergravity theory, a supersymmetric form of General Relativity.

However, the structure of string theory is also very rigid. The conditions on the possible space-times in which string theory can be consistently formulated are very restrictive. In particular, the dimension should always be 10, which is certainly not the observed 'macroscopic' value. But, this is not such a severe constraint as it might seem at first sight. Strings can move on objects that resist an interpretation as space-time at all, and we could easily visualize a splitting of the degrees of freedom into a four-dimensional space-time and an internal space, which accounts for the missing six dimensions, extending the ideas of Kaluza-Klein theories. In fact, in this way many interesting models can be obtained that produce more or less realistic, grand unified gauge groups like $E_6$.

As a theory of quantum gravity, string theory does in principle not need an external gravitational field in order to describe the metric on the space-time on which its excitations propagate; the graviton is one of the modes of the string and the curved background should be generated dynamically. This fundamental principle is unfortunately still very poorly understood. One of its manifestations appears if one tries to describe a string in a curved background metric $G_{\mu\nu}(x)$ [26,27,111,97,142,66]. In the Polyakov [131] approach to (bosonic) string theory, the quantum mechanical description of a propagating string corresponds to a two-dimensional quantum field $x^\mu(\sigma,\tau)$, coupled to two-dimensional gravity with a metric $g_{ab}$, and with an action

$$S = \frac{1}{2\pi\alpha'} \int d\sigma\, d\tau \sqrt{g} g^{ab} G_{\mu\nu}(x) \partial_a x^\mu \partial_b x^\nu. \tag{1.2}$$

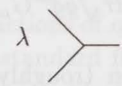
Here $\alpha'$ is the inverse string tension (roughly related to the Planck length as $\alpha' \sim L^2_{Planck}$). This action is a straightforward generalization of the analogous action of a massless point particle $x^\mu(\tau)$ moving in a curved background. One easily verifies that it is invariant under (local) scalings of the $2d$ metric $g_{ab} \to \rho\, g_{ab}$. In general this scale invariance is spoiled in the quantization process, since it cannot be preserved in a regularization scheme. But, the string equation of motion, or perhaps more correctly a consistent quantization scheme in which all the unphysical modes decouple, requires that the two-dimensional non-linear sigma model is also scale invariant on the quantum level. This implies that the $\beta$-function of the 'coupling constant' $G_{\mu\nu}(x)$ should vanish. A perturbative calculation in $\alpha'$ shows that consequently the metric has to satisfy Einstein's field equation in vacuo [72]

$$R_{\mu\nu} = 0, \tag{1.3}$$

up to corrections of the order of $\alpha'$. So the field equations of General Relativity are recovered from string theory in the limit $\alpha' \to 0$, where the target space-time metric $G_{\mu\nu}(x)$ is slowly varying with respect of the characteristic size of the string and the string can be considered as essentially point-like.

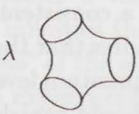
One can conclude that a consistent quantum mechanical description of a propagating string is equivalent to a two-dimensional scale invariant quantum field theory. In fact, starting from this field theory one can formulate a complete string perturbation theory, and thus one considers the sigma model to be a 'string vacuum.' Scale invariant theories are also known as conformal field theories (CFT's) [16], and will be the central theme of this thesis. The fundamental importance of conformal field theory to string theory is due to the following observation. One can take a giant leap in abstraction and consider in principle *any* conformal field theory to be a perturbative vacuum of string theory. Of course, a general CFT does not have to possess an interpretation as a non-linear $\sigma$-model, although in some cases there may be a hidden geometric interpretation as was shown for instance in the spectacular analysis of [82]. From this point of view the investigation and possibly the classification of conformal field theories is of great interest to string theory. It amounts to a determination of all perturbative string vacua.

Given a particular CFT the spectrum and also the interactions of the string excitations can be directly inferred. This should be contrasted with the introduction of interactions in point particle theory, which are always to some extent *ad hoc.* Although there are guiding principles like general covariance, gauge invariance, and other basic symmetries that one wishes to respect, the interactions are hardly ever uniquely determined. One naive way to understand this is that a possible interaction between point particles, like a three-point interaction (with coupling constant $\lambda$)




(1.4)

is always of a singular nature, and the rules that govern the interactions have to be imposed by hand. How different in string theory. A string that propagates in time sweeps out a two-dimensional surface, the world-sheet, and interactions are very naturally given by the 'pant' or 3-string vertex




(1.5)

with $\lambda$ the string coupling constant. All $n$-point vertices can be derived from this geometric object by 'sewing.' This geometrical structure of string theory will be further pursued in this thesis.

With these interaction rules the radiative corrections can be calculated in a rather straightforward fashion. The perturbation expansion is very simple and geometric. For a fixed number of loops, there is only one Feynman diagram to consider: a $g$-loop amplitude corresponds to a surface with $g$ handles, and the Schwinger parameters range over a finite dimensional space, the moduli space of genus $g$ Riemann surfaces. One of the convenient qualities of superstring theory is the alleged finiteness of this perturbation expansion. Although the general proof is not yet completed, there are very strong results that point to that direction [7,156].

We already mentioned that strings provide a natural cut-off of the order of $L_{Planck}$. It does not make sense to speak about distances smaller than the Planck scale, since a string is the only object with which one can possibly probe the structure of space-time. This gives us a kind of 'string uncertainty principle,' that forbids the determination of positions with absolute precision. In fact, in string theory manifolds loose their structure at scales of $L_{Planck}$, and in this regime bizarre quantum equivalences will equate string theory on one manifold to string theory on another. This is very neatly demonstrated in the following toy model.

Consider a string propagating in a flat toroidal space-time, that is, a box with periodic boundary conditions. Let us denote the size of this toy universe by $R$, in natural units. As we will see in the next chapter, this model can be exactly solved for all values of $R$, and has the following remarkable property. It is invariant under the duality symmetry [107]

$$R \to 1/R, \tag{1.6}$$

that interchanges large and small scales. So, although we can solve our model for scales much smaller than the Planck scale, we are effectively describing a large universe of inverse radius $1/R$. This can intuitively be understood as follows. One way to see that the size of the box is large, is through the existence of very low energy modes with momentum $p \sim 1/R$. However, in string propagation we also have so-called winding modes that describe configurations where the string wraps around the torus. For large $R$ these modes will be very massive, since it is energetically unfavorable to make such an extended object. However, for small values of $R$ the energy will be very small, and the winding states are completely similar to the momentum states at large $R$. In particular, the limit $R \to 0$, which would produce a singularity in field theory, is in string theory perfectly regular, and corresponds by virtue of (1.6) to the case of flat space ($R = \infty$).

Two-dimensional scale invariant field theories were first introduced in statistical mechanics in the description of systems with a second order phase transition [101, 130]. At the critical point, in the continuum limit, these systems can be described by scale invariant field theories [31]. In this context, the classification of CFT's corresponds to the knowledge of all universality classes of $2d$ critical phenomena. Many of these models, with as most famous example undoubtedly the critical Ising model, are formulated as lattice models. At criticality, the renormalization group washes away the lattice structure, and a continuous field theory emerges. This approach provides links with ideas in quantum gravity, that propose a fundamental lattice description of space-time at the Planck scale.

Apart from their application to string theory and the study of two-dimensional critical phenomena, conformal field theories are also of interest from a more academic point of view. They provide examples of highly nontrivial, non-perturbative, interacting field theories, that in many cases can be solved exactly. In this way they shed light on the (geometrical) structures behind quantum field theory in general, and act as a playground for many new ideas. Their solvability is a consequence of the infinite dimensional symmetries that can occur in two dimensions. These exceptionally large symmetry algebras are currently actively investigated in the mathematical literature. In fact, there exist various deep relations between string theory and what is sometimes mockingly called 'high energy mathematics,' an agglomerate of diverse mathematical fields that are all related in one way or another to two-dimensional quantum field theory, like affine algebras [100,88], the Monster group [38,70,54], knot theory [98,162], quantum groups [57] *etc.*

It is perhaps appropriate to close this introductory chapter with the question: what are the fundamental principles of string theory? One of the great progresses that has been made in the last few years, and to which this thesis is a modest contribution, is a growing plethora of possible CFT's that can be used as perturbative vacua, and a better understanding of their structure. Of course, this raises immediately the question of non-perturbative effects, and the nature of the true ground state. Moreover, this is hardly the way one would have liked to have discovered General Relativity. Suppose we were unaware of manifolds, metrics, Riemannian geometry, but only had access to the Feynman rules describing the interactions of some spin 2 field around various classical solutions of Einstein's equations. We would certainly be in great need of finding fundamental guiding principles, like general covariance and the correspondence principle. It can only be hoped that equally beautiful, unifying concepts lie at the heart of string theory.

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This thesis is organized in the following way. In Chapter 2 we will give a brief introduction to conformal field theory along the lines of standard quantum field theory, without any claims to originality. We introduce the important concepts of the stress-energy tensor, the Virasoro algebra, and primary fields. The general principles are demonstrated by fermionic and bosonic free field theories. This also allows us to discuss some general aspects of moduli spaces of CFT's. In particular, we describe in some detail the space of inequivalent toroidal compactifications, giving examples of the quantum equivalences that we already mentioned. In Chapter 3 we will reconsider general quantum field theory from a more geometrical point of view, along the lines of the so-called operator formalism. Crucial to this approach will be the consideration of topology changing amplitudes. After a simple application to $2d$ topological theories, we proceed to give our second introduction to CFT, stressing the geometry behind it.

In Chapter 4 the so-called rational conformal field theories are our object of study. These special CFT's have extended symmetries with only a finite number of representations. If an interpretation as non-linear sigma model exists, this extra symmetry can be seen as a kind of resonance effect due to the commensurability of the size of the string and the target space-time. The structure of rational CFT's is extremely rigid, and one of our results will be that the operator content of these models is—up to some discrete choices—completely determined by the symmetry algebra. The study of rational models is in its rigidity very analogous to finite group theory. In Chapter 5 this analogy is further pursued and substantiated. We will show how one can construct from general grounds rational conformal field theories from finite groups. These models are abstract versions of non-linear

$\sigma$-models describing string propagation on 'orbifolds.' An orbifold is a singular manifold obtained as the quotient of a smooth manifold by a discrete group.

In Chapter 6 our considerations will be of a somewhat complementary nature. We will investigate models with central charge $c = 1$ by deformation techniques. The central charge is a fundamental parameter in any conformal invariant model, and the value $c = 1$ is of considerable interest, since it forms in many ways a threshold value. For $c < 1$ a complete classification of all unitary models has been obtained, but $c > 1$ is still very much *terra incognita*. Our results give a partial classification for the intermediate case of $c = 1$ models. The formulation of these $c = 1$ CFT's on surfaces of arbitrary topology is central in Chapter 7. Here we will provide many explicit results that provide illustrations for our more abstract discussions of higher genus quantities in Chapters 3 and 4. Unfortunately, our calculations will become at this point rather technical, since we have to make extensive use of the mathematics of Riemann surfaces and their coverings.

Finally, in Chapter 8 we leave the two-dimensional point of view that we have been so loyal to up to then, and ascend to three dimensions where we meet topological gauge theories. These so-called Chern-Simons theories encode in a very economic way much of the structure of two-dimensional (rational) conformal field theories, and this direction is generally seen to be very promising. We will show in particular how many of our results of Chapter 5 have a natural interpretation in three dimensions.

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The following material of this thesis has appeared in separate publications: Chapter 2 [51], Chapter 4 [48,46], Chapter 5 [47], Chapter 6 [49,50,47], Chapter 7 [49], Chapter 8 [52].

# 2

# Introduction to Conformal Field Theory

This will be a brief chapter on a vast subject. The study of two-dimensional conformal field theory, originating simulaneously in string theory and the theory of critical phenomena, has been a very active field the last few years. Its applications have found their way to numerous fields—ranging from spin chains to number theory. The literature on conformal field theory is immense, see *e.g.* the reprint collection [96], and our introduction is but a simple variation on a theme that might by now sound very familiar to some readers. Besides the excellent original papers [16,76,75,30] and reviews [71,73,31], that make an introduction rather unnecessary, recently many good introductory lecture notes to conformal field theory have appeared, among others by Ginsparg [85] and Cardy [33]. It is mainly the approach of the latter that we will follow here, albeit at a less leisure pace. We will make up for this succinctness, and give another, more geometric introduction to conformal field theory in Chapter 3.

## 2.1. Conformal Invariance in Two Dimensions

Let us consider a two-dimensional Euclidean quantum field theory, described by a path-integral

$$Z = \int \mathcal{D}\phi \, e^{-S[\phi]}, \tag{2.1}$$

with the action $S[\phi]$ a functional of a set of local fields $\phi(x)$. We will assume space-time to be flat, with metric $g_{ab} = \delta_{ab}$. Under infinitesimal coordinate transformations $x^a \to x^a + \xi^a(x)$ the action will change by

$$\delta S = -\int \frac{d^2x}{2\pi} \, T_{ab} \partial^a \xi^b, \tag{2.2}$$

with $T_{ab}$ the symmetric stress-energy tensor. Translation invariance implies the conservation law $\partial^a T_{ab} = 0$. If one further assumes that the model is scale invariant, *i.e.* invariant under rescalings $x^a \to \lambda \, x^a$, the stress-energy tensor is also traceless

$$T_a{}^a = 0. \tag{2.3}$$

This property automatically implies invariance under the extended class of coordinate transformations that satisfy

$$\partial^a \xi^b + \partial^b \xi^a = \partial_c \xi^c \delta^{ab}, \tag{2.4}$$

the so-called *conformal* transformations. These are exactly the transformations that transform the metric with a local scale factor $g_{ab} \to \rho(x) g_{ab}$. Locally, a conformal transformation acts as a combination of a translation, a rotation, and a dilatation. So, all angles will be preserved.

At this point it is unavoidable to follow standard routine and introduce complex coordinates

$$z = x^1 + ix^2, \qquad \overline{z} = x^1 - ix^2. \tag{2.5}$$

Note that $z$ and $\overline{z}$ are related by complex conjugation $z^* = \overline{z}$. General functions will be denoted as $f(z, \overline{z})$, analytic functions as $f(z)$. After analytic continuation to a Minkovskian metric, $z$ and $\overline{z}$ become independent, real light-cone coordinates, and conformal transformations leave the light-cone structure invariant. In our (Euclidean) complex notation the vector fields $\xi = \xi_z$, $\overline{\xi} = \xi_{\overline{z}}$ (with a reality condition $\xi^* = \overline{\xi}$) satisfy

$$\partial_{\overline{z}} \xi = 0, \qquad \partial_z \overline{\xi} = 0. \tag{2.6}$$

These are exactly the Cauchy-Riemann equations, which implies that in complex coordinates conformal transformations correspond to analytic transformations $z \to z + \xi(z)$.

If we add a point at infinity, we can identify the compactified plane with the Riemann sphere $S^2$. It is an important observation that not all conformal transformations can be seen as global diffeomorphisms (conformal isometries) of $S^2$. In general the vector field $\xi(z)$ will have poles at some value of $z$, possibly at infinity. If we make a Laurent expansion $\xi(z) = \sum_n \xi_n z^{-n+1}$, we see that solutions regular at the origin satisfy $\xi_n = 0$ for $n > 1$. To investigate the behavior at infinity, one makes a coordinate transformation $z \to w = 1/z$, under which the vector field $\xi(z)$ transforms as $\xi(z) \to \frac{\partial z}{\partial w} \xi(w) = -z^2 \xi(1/z)$. So, regularity at infinity implies that also $\xi_n = 0$ for $n < -1$. This leaves us with the global vector fields

$$\xi(z) = \xi_1 + \xi_0 z + \xi_{-1} z^2. \tag{2.7}$$

These infinitesimal transformations exponentiate to the global conformal group $SL(2, \mathbf{C})$,

$$z \to w(z) = \frac{az + b}{cz + d}. \tag{2.8}$$

Stated otherwise, fractional linear transformations are the only one-to-one holomorphic maps of the Riemann sphere. The group $SL(2, \mathbf{C}) \cong SO(3, 1)$ is the two-dimensional analogue of the $d$-dimensional Euclidean conformal group $SO(d+1, 1)$.

Global conformal transformations are also known as Möbius transformations. The other, meromorphic solutions of (2.6) should be regarded as *local* analytic reparametrizations.

This is a good point to introduce the concept of a scaling or quasi-primary field. This is a field $\phi(z,\overline{z})$ that transforms under the Möbius transformations (2.8) as

$$\phi(z,\overline{z}) \to \left(\frac{\partial w}{\partial z}\right)^{h}\left(\frac{\partial \overline{w}}{\partial \overline{z}}\right)^{\overline{h}} \phi(w,\overline{w}). \tag{2.9}$$

This relation defines the (real) conformal weights or dimensions $(h,\overline{h})$. Note, that under $z \to \lambda\, z$ (with complex $\lambda$) this transformation property reads $\phi \to \lambda^{h}\overline{\lambda}^{\overline{h}}\phi$. So the scaling dimension is given by $x = h+\overline{h}$, and the spin by $s = h-\overline{h}$. By $SL(2,\mathbf{C})$ invariance the two and three-point functions of quasi-primary fields are completely determined up to normalization. With $z_{ij} = z_i - z_j$, and $h_{ijk} = h_i + h_j - h_k$ we have

$$\left\langle \phi_1(z_1,\overline{z}_1)\phi_2(z_2,\overline{z}_2)\right\rangle \propto \begin{cases} (z_{12})^{-2h_1}(\overline{z}_{12})^{-2\overline{h}_1}, & \textit{if } h_1 = h_2,\, \overline{h}_1 = \overline{h}_2, \\ 0 & \textit{otherwise.} \end{cases} \tag{2.10}$$

$$\left\langle \phi_1(z_1,\overline{z}_1)\phi_2(z_2,\overline{z}_2)\phi_3(z_3,\overline{z}_3)\right\rangle \propto z_{12}^{-h_{123}}\, z_{23}^{-h_{231}}\, z_{31}^{-h_{312}} \times c.c. \tag{2.11}$$

In complex coordinates the (symmetric) stress-energy tensor $T_{ab}$ has components $T \equiv T_{zz}$, $\overline{T} \equiv T_{\overline{z}\overline{z}}$ and $T_{z\overline{z}} = \frac{1}{4}T_a{}^a = 0$. The conservation law $\partial^a T_{ab} = 0$ gives the equations

$$\partial_{\overline{z}} T = 0, \qquad \partial_z \overline{T} = 0. \tag{2.12}$$

So, $T$ is a holomorphic field, a fact we will stress by writing $T(z)$. Since we have $(h,\overline{h}) = (2,0)$, its two-point function is of the form

$$\left\langle T(z)T(w)\right\rangle = \frac{c/2}{(z-w)^4}. \tag{2.13}$$

This will be our definition of the *central charge c*. Note that the normalization of $T$ is fixed by the relation (2.2). The central charge can in principle be any real number, though positive for unitary theories. In some sense $c$ measures the degrees of freedom in the model; as we will see, $c = n + \frac{1}{2}m$ for $n$ free bosons and $m$ (real) fermions.

The central charge is related to the trace anomaly in curved spaces. For an arbitrary metric $g_{ab}$ the trace $T_{z\overline{z}}$ does not necessarily vanish. In fact, its expectation value on a curved worldsheet is given by

$$\langle T_{z\overline{z}}\rangle = -\frac{c}{48}R(g), \tag{2.14}$$

with $R(g)$ the two-dimensional curvature. Note that general covariance dictates the form of the anomaly, it is only the constant of proportionality that we have to establish. This can be done perturbatively [71,33]. Consider a deformation $\delta g_{ab}$ of the flat metric that is *not* a conformal transformation. This is equivalent to deforming the action by a term that reads in complex coordinates

$$\delta S = -\int \frac{d^2z}{2\pi}\, \delta g^{zz} T + c.c. \tag{2.15}$$

If one calculates $\langle T_{zz}\rangle$ perturbatively, with the aid of the two-point function (2.13), and then uses the conservation law $\partial_{\bar{z}} T_{zz} + \partial_z T_{z\bar{z}} = 0$, the final result reads

$$\langle T_{z\bar{z}}\rangle = -\frac{c}{48}\partial_z^2 \delta g^{zz} + \ldots \tag{2.16}$$

where the ellipsis denotes terms that make the expression covariant. This result is indeed a first order approximation to (2.14).

### 2.1.1. *The Hamilton Picture*

Let us now turn to the Hamiltonian formalism, and consider quantization on the cylinder $S^1 \times \mathbf{R}$, with a (periodic) space coordinate $\sigma$ and Euclidean time coordinate $\tau$, that can be combined into one complex coordinate $w = \sigma + i\tau$. In fact, the transformation $z = e^{iw}$ relates the cylinder to the punctured plane, and we will keep on using the coordinate $z$ of Eq. (2.5). We will assume the existence of a Hilbert space $\mathcal{H}$ with a vacuum $|0\rangle$, and the usual relation between correlation functions and time-ordered expectation values of operators

$$\Big\langle \prod_i \phi_i(z_i, \bar{z}_i)\Big\rangle = \langle 0| T \prod_i \phi_i(z_i, \bar{z}_i)|0\rangle. \tag{2.17}$$

Conformal transformations will be unitarily represented on $\mathcal{H}$. The conserved charges $Q_\xi$ that generate the transformation $z \to z + \xi(z)$ are given by

$$Q_\xi = \oint \frac{dz}{2\pi i}\, \xi(z) T(z) - \oint \frac{d\bar{z}}{2\pi i}\, \bar{\xi}(\bar{z}) \overline{T}(\bar{z}) \tag{2.18}$$

Now consider the components $L_n$, that generate $\delta z = z^{n+1}$,

$$T(z) = \sum_{n\in\mathbf{Z}} L_n z^{-n-2}, \qquad L_n = \oint \frac{dz}{2\pi i} T(z) z^{n+1}. \tag{2.19}$$

The modes $L_0, L_\pm$ and $\overline{L}_0, \overline{L}_\pm$ generate the global transformations $SL(2, \mathbf{C})$. In particular the Hamiltonian equals $L_0 + \overline{L}_0$, and the momentum operator $L_0 - \overline{L}_0$. What are the commutation relations of the $L_n$'s? First, classically we have $L_n =$

$-z^{n+1}\partial_z$ with algebra $[L_m, L_n] = (m-n)L_{m+n}$. Quantum mechanically we should allow for a possible central extension, that will make the representation projective. The only possible extension, that vanishes for the global conformal group, is the famous Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n}. \tag{2.20}$$

Of course, a similar algebra holds for the modes $\overline{L}_n$ of $\overline{T}$. In order to show that this $c$ is the same central charge as defined by the two-point function (2.13), one considers an arbitrary correlation function

$$\langle 0|\phi_1(z_1, \overline{z}_1) \cdots \phi_n(z_n, \overline{z}_n) T(z)|0\rangle. \tag{2.21}$$

There is no reason why this expression should be singular at either $z = 0$ or $z = \infty$. Comparison with the mode expansion (2.19) shows that consequently the vacuum is annihilated by the modes $L_n$, $L_n|0\rangle = 0$, if $n \geq -1$. Similarly, for the out vacuum $\langle 0|L_n = 0$, if $n \leq 1$. We see in particular that $|0\rangle$ is $SL(2, \mathrm{C})$ invariant. If we now calculate the two-point function $\langle TT\rangle$ and compare with (2.13), we easily establish that $c$ in (2.20) is indeed the central charge. The commutation relations (2.20) imply the following form of the singular contribution to the operator product expansion (OPE)

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}. \tag{2.22}$$

The Hilbert space $\mathcal{H}$ will decompose into irreducible representations $[\phi_\alpha] \otimes [\overline{\phi_{\overline{\alpha}}}]$ of the holomorphic and anti-holomorphic Virasoro algebras generated by $T$ and $\overline{T}$

$$\mathcal{H} = \bigoplus_{(\alpha,\overline{\alpha})} [\phi_\alpha] \otimes [\overline{\phi_{\overline{\alpha}}}]. \tag{2.23}$$

One representation, the basic representation $[\phi_0]$, consists of all states of the form $L_{-n_1} \cdots L_{-n_k}|0\rangle$. This should be compared to a representation of, say, $SO(3)$, with $L_{\pm 1}$ the raising and lowering operators. The other representations $[\phi_\alpha]$ are constructed from highest weight states

$$L_0|h_\alpha\rangle = h_\alpha|h_\alpha\rangle, \qquad L_n|h_\alpha\rangle = 0 \quad \textit{if } n > 0. \tag{2.24}$$

The states $|h_\alpha\rangle$ are created by the so-called primary fields at $\tau = -\infty$, *i.e.* $z = 0$,

$$|h_\alpha\rangle = \lim_{z \to 0} \phi_\alpha(z, \overline{z})|0\rangle \tag{2.25}$$

Primary fields of conformal weights $h, \overline{h}$ satisfy (2.9) for *all* (local) conformal transformations. The infinitesimal form of this transformation is generated by the charges $Q_\xi$:

$$\delta\phi(w,\overline{w}) = \left[Q_\xi, \phi(w,\overline{w})\right] = \left(\xi(w)\partial_w + h\partial_w\xi(w)\right)\phi(w,\overline{w}) + c.c. \tag{2.26}$$

Since this can be re-expressed in terms of a contour integral as

$$\delta\phi(w,\overline{w}) = \oint \frac{dz}{2\pi i}\, \xi(z)T(z)\phi(w,\overline{w}) + c.c., \tag{2.27}$$

we can read off the singular term in the OPE of $T$ and $\phi$

$$T(z)\phi(w,\overline{w}) \sim \frac{h\,\phi(w,\overline{w})}{(z-w)^2} + \frac{\partial_w\phi(w,\overline{w})}{z-w}. \tag{2.28}$$

The so-called descendant states $L_{-n_1}\ldots L_{-n_k}|h_\alpha\rangle$ correspond to non-primary operators, in a similar way as in (2.25). In particular the state $L_{-2}|0\rangle = T(0)|0\rangle$ gives the stress-energy tensor $T(z)$. Although quasi-primary, $T$ is not a primary field, as we see from (2.22). Under conformal transformations it transforms as,

$$T(z) \rightarrow \left(\frac{\partial w}{\partial z}\right)^2 T(w) + \frac{c}{12}S(w,z). \tag{2.29}$$

with $S(w,z)$ the Schwarzian derivative $(\partial^3 w/\partial w) - \frac{3}{2}\left(\partial^2 w/\partial w\right)^2$, that vanishes for the global $SL(2,\mathbf{C})$ transformations. Note in particular that the stress-energy tensors on the plane and cylinder are related by

$$T_{cyl}(w)(dw)^2 = \left(T_{plane}(z) - \frac{c}{24}\right)(dz)^2. \tag{2.30}$$

An important result, that we will explain in more detail in the next chapter, is that a CFT is completely determined by the central charge $c$, the spectrum of primary fields $\phi_\alpha$ of weight $(h_\alpha, \overline{h}_\alpha)$, and the operator product coefficients $c_{\alpha\beta}{}^\gamma$ of the primary fields, defined by the OPE

$$\phi_\alpha(z,\overline{z})\phi_\beta(w,\overline{w}) \sim \sum_\gamma c_{\alpha\beta}{}^\gamma (z-w)^{-h_{\alpha\beta\gamma}}(\overline{z}-\overline{w})^{-\overline{h}_{\alpha\beta\gamma}}\phi_k(w,\overline{w}) + \cdots \tag{2.31}$$

We can introduce an integer $N_{\alpha\beta}{}^\gamma = 0, 1$ depending on whether the coefficient $c_{\alpha\beta}^\gamma$ vanishes or not. The formal relations

$$[\phi_\alpha]\cdot[\phi_\beta] = \sum_\gamma N_{\alpha\beta}^\gamma[\phi_\gamma], \tag{2.32}$$

between the representations $[\phi_\alpha]$ encode the interaction rules of the primary fields, and are known as the fusion rules.

## 2.2. Free Field Theories

Without any doubt the simplest class of conformal field theories are free field theories. In this section we will very briefly discuss free fermions and bosons in two dimensions. We will meet these theories at various other places in this thesis. It will also allow us to establish some further notations and conventions.

### 2.2.1. Free Fermions

The action for a free real Majorana fermion $(\psi, \overline{\psi})$ in two dimensions reads

$$S = \int \frac{d^2z}{2\pi} \left( \psi \overline{\partial} \psi + \overline{\psi} \partial \overline{\psi} \right). \tag{2.33}$$

This action is manifestly invariant under conformal transformations $z \to w(z)$. The equation of motion reads $\overline{\partial}\psi = \partial\overline{\psi} = 0$. The operator $\psi(z)$ is a meromorphic field of conformal weights $(\frac{1}{2}, 0)$, and can be decomposed in its Laurent modes as

$$\psi(z) = \sum_{n \in \mathbf{Z}+\epsilon} \psi_n z^{-n-\frac{1}{2}}, \tag{2.34}$$

with anti-commutation relations $\{\psi_n, \psi_m\} = \delta_{n+m}$. So a two-dimensional fermion is simply an infinite set of fermionic oscillators. We can have either $\epsilon = 0$ or $\epsilon = \frac{1}{2}$ in (2.34). The two choices correspond to the two possible spin structures

$$\psi(e^{2\pi i} z) = \pm\, \psi(z) \tag{2.35}$$

on the punctured plane, or equivalently to periodic respectively anti-periodic boundary conditions on the cylinder. The two different Hilbert spaces that can be constructed are known as the Neveu-Schwarz sector $\mathcal{H}_{NS}$ $(\epsilon = 0)$ and the Ramond sector $\mathcal{H}_R$ $(\epsilon = \frac{1}{2})$. The stress-energy tensor is given by $T = \frac{1}{2}\psi\partial\psi$, and the central charge is easily verified to be $c = \frac{1}{2}$. In the Ramond sector translational invariance is lost. The field $\psi(z)$ is expanded in half-integer powers of $z$ and we have a branch cut from $z = 0$ to $z = \infty$. The propagator reads (see *e.g.* [85])

$$\big\langle \psi(z)\psi(w) \big\rangle = \langle \sigma | \psi(z)\psi(w) | \sigma \rangle = \tfrac{1}{2} \frac{1}{z-w} \left( \sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}} \right). \tag{2.36}$$

From this expression we see that the Ramond ground state $|\sigma\rangle$ is not $SL(2, \mathbf{C})$ invariant. In fact, it can be inferred (see *e.g.* [85]) that the ground state is created out of the vacuum by a weight $(\frac{1}{16}, \frac{1}{16})$ spin operator $\sigma(z, \overline{z})$. The spin field changes the boundary condition of the fermion from periodic into anti-periodic, and corresponds to the order parameter of the Ising model. It is furthermore doubly degenerated, because of the zero-mode $\psi_0$.

Strictly speaking a free fermion is not a conformal field theory, since it contains half-integer spin operators. Correlation functions are only well-defined after a choice of spin structure on a punctured surface. However, a bosonic model, with only integer spin operators, can be obtained by the so-called spin projection [77]. One introduces a $\mathbf{Z}_2$ grading $(-1)^F$, with $\psi$ odd, $|0\rangle$ and $\sigma$ even, and projects onto the even states

$$\mathcal{H} = Inv\,(\mathcal{H}_{NS} \oplus \mathcal{H}_R)\,. \tag{2.37}$$

The primary fields in the bosonic model, that corresponds to the critical Ising model, are $1, \sigma$, and $\epsilon = \psi\overline{\psi}$. In the case of several fermions we have the option to assign them independent spin structures or not. In this way different bosonic theories can be constructed. In particular two real Majorana fermions $\psi_1, \psi_2$ with equal spin structures can be combined into a single complex Dirac fermion $\psi = \psi_1 + i\psi_2$.

### 2.2.2. Gaussian Models

In contrast to a free fermion, a free bosonic field $\phi(\sigma, \tau)$ in two dimensions represents a *noncompact* conformal field theory, *i.e.* a theory with a continuous energy spectrum. This is due to the center of mass mode

$$\phi(\tau) = \int d\sigma\, \phi(\sigma, \tau), \tag{2.38}$$

that corresponds to a free point particle (moving on a line) and that gives rise to a continuous spectrum with arbitrary momenta. It will be much more convenient to first consider compact models, where the bosonic field is a periodic variable. In that case $\phi(\tau)$ represents a particle on a circle, which makes the allowed momenta discrete. The non-periodic case can be retrieved in the limit where the period becomes infinite. Periodic bosonic free field theories are also known as *gaussian* or *toroidal* models.

The $d$-dimensional toroidal model is described by $d$ real massless scalar fields $\phi^\mu(z, \overline{z})$ $(\mu = 1, \ldots, d)$ defined modulo $2\pi\Lambda$, where $\Lambda$ is a $d$-dimensional lattice. That is, we have an identification

$$\phi^\mu \equiv \phi^\mu + 2\pi w^\mu, \qquad \textit{for all } w^\mu \in \Lambda. \tag{2.39}$$

It can be considered as a non-linear $\sigma$-model describing string propagation on the multi-dimensional torus $\mathbf{R}^d/2\pi\Lambda$. The most general action is given by the gaussian one, together with a constant antisymmetric background field $B_{\mu\nu}$ for $d > 1$

$$S = \int \frac{d^2z}{2\pi} \left( G_{\mu\nu} \partial\phi^\mu \overline{\partial}\phi^\nu + i B_{\mu\nu} \partial\phi^\mu \overline{\partial}\phi^\nu \right). \tag{2.40}$$

By a redefinition of the field $\phi$ we can always put the metric $G_{\mu\nu}$ equal to the identity $\delta_{\mu\nu}$, and we will drop it in the subsequent expressions. The second term in the action is a topological term that only gives a contribution when we consider nontrivial worldsheet topology. Although it can be written as a total derivative

$$\frac{i}{2\pi}\int d^2z\ \partial_a\Big(B_{\mu\nu}\phi^\mu\partial_b\phi^\nu\epsilon^{ab}\Big), \tag{2.41}$$

the integral does not vanish, since in general the field $\phi$ is not single-valued on arbitrary topology. If the lattice vectors $v_i, w_i \in \Lambda$ denote the winding numbers of the $\phi$-field around the cycles $A_i, B_i$ of a canonical homology basis (see *fig.* 11 in Chapter 7 for its definition), this term contributes $4\pi i\sum_i(v_i\cdot B\cdot w_i)$. Since the action enters the path-integral as the exponential $e^{-S}$, the $B$-field behaves as a periodic variable, just like the $\theta$-angle in QCD. If $\Lambda^*$ denotes the dual lattice of $\Lambda$, then $B$ and $B+\delta B$ lead to identical theories if $\delta B$ satisfies

$$\tfrac{1}{2}\delta B\cdot\Lambda\subset\Lambda^*. \tag{2.42}$$

Or, less abstractly, if $e_\alpha$ is an orthonormal basis of $\Lambda$ the matrix elements $\frac{1}{2}B_{\mu\nu}e_a^\mu e_b^\nu$ are defined modulo integers.

The model (2.40) clearly represents a conformal field theory, since the action is invariant under analytic transformations. The stress-energy tensor is given by $T(z)=-\frac{1}{2}\partial\phi^\mu\partial\phi^\mu$, and the central charge equals the dimension of the torus, $c=d$. Its spectrum is conveniently discussed in terms of the spin one currents $\partial\phi^\mu(z)$ and $\overline{\partial\phi}^\mu(\bar z)$. Because of the equation of motion ($\partial\bar\partial\phi^\mu=0$) these are meromorphic fields, with operator products

$$\partial\phi^\mu(z)\cdot\partial\phi^\nu(w)\sim\frac{\delta^{\mu\nu}}{(z-w)^2}. \tag{2.43}$$

The modes $\partial\phi^\mu(z)=\sum_n\alpha_n^\mu z^{-n-1}$ satisfy the commutation relations of harmonic oscillators $[\alpha_n^\mu,\alpha_m^\nu]=n\delta^{\mu\nu}\delta_{n+m}$. The Hilbert space is created by the operators $\alpha_n^\mu,\bar\alpha_n^\mu$ $(n<0)$ out of the eigenstates $|p^\mu,\bar p^\mu\rangle$ of the zero-modes

$$\alpha_0^\mu=\oint\frac{dz}{2\pi i}\,\partial\phi^\mu(z),\qquad \bar\alpha_0^\mu=\oint\frac{d\bar z}{2\pi i}\,\overline{\partial\phi}^\mu(\bar z). \tag{2.44}$$

If we use the operator identity

$$\phi^\mu(z,\bar z)=\phi^\mu(z)+\bar\phi^\mu(\bar z), \tag{2.45}$$

then the states $|p^\mu,\bar p^\mu\rangle$ are created from the vacuum $|0\rangle$ by the primary fields,

$$V_{p\bar p}(z,\bar z)=\exp\Big(ip\cdot\phi(z)+i\bar p\cdot\bar\phi(\bar z)\Big). \tag{2.46}$$

Using the OPE

$$\partial\phi^\mu(z)\cdot e^{ip\cdot\phi(w)} \sim \frac{ip^\mu}{z-w}e^{ip\cdot\phi(w)}, \tag{2.47}$$

one finds that these vertex operators $V_{p\overline{p}}$ have dimension $h=\frac{1}{2}p^2$. Locality forces the momenta $(p,\overline{p})$ to be elements of the $2d$-dimensional lattice $\Gamma^{d,d}$ given by [123,125]

$$(p,\overline{p}) = \big(k+\tfrac{1}{2}(1+B)\cdot w, k-\tfrac{1}{2}(1-B)\cdot w\big) \tag{2.48}$$

with the winding numbers $w\in\Lambda$ and the momenta $k\in\Lambda^*$.

As is well-known the lattice $\Gamma^{d,d}$ is even and self-dual when one adopts a metric of signature $(d,d)$. Even, self-dual Lorentzian lattices are unique up to $SO(d,d)$ transformations; they are given by Lorentz transforms of the lattice

$$\underbrace{\Gamma^{1,1}\oplus\cdots\oplus\Gamma^{1,1}}_{d \text{ times}} \tag{2.49}$$

with

$$\Gamma^{1,1} = \tfrac{1}{2}\sqrt{2}\,\mathbf{Z}\begin{pmatrix}1\\1\end{pmatrix}\oplus\mathbf{Z}\begin{pmatrix}1\\-1\end{pmatrix}. \tag{2.50}$$

In fact, due to the particular properties of these lattices certain surprising identifications exist between toroidal models that at first sight seem to be completely unrelated. However, in order to give a proper description of these phenomena, we will first turn to a general description of moduli spaces of CFT's. These considerations will be somewhat more technical and serve mainly as a preparation for calculations in Chapter 6, so the reader might wish to skip the following section.

## 2.3. Moduli Spaces of CFT's

We have seen that one can construct toroidal models for any lattice $\Lambda$ and background field $B_{\mu\nu}$. In this way one obtains multi-parameter families of conformal models. This is a genuine feature of conformal field theory; in general one can consider smooth moduli spaces of CFT's [166,14]. Infinitesimal deformations are defined to be small variations in the data that determine a CFT, that is, the central charge $c$, the conformal weights $(h_\alpha,\overline{h}_\alpha)$ and the multiplicities of the primary fields $\phi_\alpha$, and their operator product coefficients $c_{\alpha\beta\gamma}$.

Deformations that preserve the infinite conformal symmetry and the value of the central charge are generated in first order by marginal operators $\psi_i$, *i.e.* primary fields with conformal weight (1,1) [103]. In a path-integral formulation of the theory these perturbations can be represented by an additional term in the action

$$\delta S = \sum_i \delta g_i \int \frac{d^2z}{2\pi}\,\psi_i(z,\overline{z}). \tag{2.51}$$

This deformation only respects the conformal invariance if $\psi_i(z,\bar{z})$ transforms as a density, *i.e.* a weight (1,1) primary field. Equivalently, the correlation function of any product of operators $\mathcal{O}$ is modified to

$$\frac{\delta}{\delta g_i}\langle\mathcal{O}\rangle = \int \frac{d^2z}{2\pi}\langle\psi_i(z,\bar{z})\mathcal{O}\rangle. \tag{2.52}$$

In string theory this corresponds to a condensate of on-shell modes described by the background field $\delta g_i$. We will call those weight (1,1) operators $\psi_i(z,\bar{z})$ for which (2.52) can be integrated to a finite perturbation *integrable* marginal operators. Locally, their coupling constants $g_i$ can serve as a coordinate system for the space of CFT's in the neighbourhood of the unperturbed theory. One can visualize the motion in CFT-space generated by these marginal operators as a flow of weights and operator product coefficients of the primary fields. The change in the conformal weights can be derived from the variation of the two-point function. With (2.52) we find [103]

$$\frac{\delta}{\delta g_i}\big\langle\phi_\alpha(z,\bar{z})\phi_\alpha(w,\bar{w})\big\rangle = 2c_{i\alpha\alpha}(z-w)^{-2h_\alpha}(\bar{z}-\bar{w})^{-2\bar{h}_\alpha}\log|z-w|^2. \tag{2.53}$$

We now see that the weights $(h_\alpha,\bar{h}_\alpha)$ are shifted by

$$\delta h_\alpha = \delta\bar{h}_\alpha = -\sum_i c_{i\alpha\alpha}\delta g_i. \tag{2.54}$$

Note that in the case that there are several operators with the same weight one has to diagonalize the matrix $c_{i\alpha\beta}$ to find the variation of the weights. In a similar way one can derive expressions for higher order terms (see *e.g.* [32] for an interesting discussion), and variational formulas for the operator product coefficients $c_{\alpha\beta\gamma}$.

Following [103] we can now formulate the conditions a marginal operator $\psi_i$ has to satisfy in order to be integrable to first order in $g_i$. First of all, it is evident that its weight must not be changed by the perturbation generated by $\psi_i$ itself. So from (2.54) we see that at least $c_{iii}$ must vanish. An additional constraint arises if there are more weight (1,1) primary operators $\psi_j$. An easy application of degenerate perturbation theory shows that if $c_{iij}\neq 0$ for some of these $\psi_j$ then in the perturbed CFT $\psi_i$ becomes a linear combination of primary fields with different weights and its marginality is destroyed by the perturbation. So, in summary, we have the following necessary conditions for an integrable marginal operator $\psi_i$

(*i*) $h_i = \bar{h}_i = 1$.

(*ii*) $c_{iij} = 0$ for any primary field $\psi_j$ of weight $(1,1)$.

Note that because of condition (*ii*) linear combinations of integrable marginal operators need in general not be integrable.

In a local neighborhood of a generic conformal field theory the space of CFT's connected to this theory has the structure of a smooth manifold, which can be parametrized by the couplings $g_i$ of the marginal operators $\psi_i$ that satisfy the relation

$$\langle \psi_i \psi_j \psi_k \rangle_g = 0. \tag{2.55}$$

In fact, a metric $G_{ij}(g)$ (the Zamolodchikov metric [166]) can be defined by

$$\Big\langle \psi_i(z,\overline{z}) \psi_j(w,\overline{w}) \Big\rangle_g = \frac{G_{ij}(g)}{|z-w|^4}. \tag{2.56}$$

If we investigate a particular CFT this metric can of course always be chosen to be orthonormal.

It can happen that at some value of the $g_i$ the number of integrable marginal operators in the theory jumps through some accidental degeneracy to a larger value. The points where this happens are called *multi-critical*. In general, the extra integrable marginal operators signal the presence of new independent directions in which the theory can be deformed. In this case the multi-critical point lies at the intersection locus of two or more submanifolds of CFT space. However, it often happens that in such a point the conformal field theory has an enhanced symmetry. This symmetry group can reduce by its action on the marginal operators the number of inequivalent deformations of the theory. The coupling constants $g_i$ then give a redundant coordinatization of the CFT space, which can have a singularity at the multi-critical point. More precisely, if a symmetry group $\Gamma$ acts on the $n$ integrable marginal operators $\psi_i$, the moduli space will locally be an orbifold* $\mathbf{R}^n/\Gamma$.

### 2.3.1. *Toroidal Compactifications*

As an application we will now consider more carefully the geometry of the moduli space $\mathcal{T}^d$ of $d$-dimensional toroidal compactifications, that we discussed in section 2.2.2. For generic values of the parameters $\Lambda, B_{\mu\nu}$ the $d^2$ marginal operators $\partial\phi^\mu \overline{\partial\phi}^\nu$ are the only dimension $(1,1)$ fields, and one can easily verify that they are indeed integrable. In fact, their coupling constants correspond to a constant background metric $g_{\mu\nu}$ and antisymmetric tensor field $b_{\mu\nu}$ that can be absorbed by a redefinition of $\Lambda$ and $B_{\mu\nu}$. The action of these marginal deformations on the weights $(h, \overline{h})$ of the vertex operators $V_{p\overline{p}}$ can be calculated using the OPE (2.47). This gives the following flow for the marginal deformation as induced by $\partial\phi^\mu \overline{\partial\phi}^\nu$

$$\delta h = \delta \overline{h} = -p^\mu \, \overline{p}^\nu. \tag{2.57}$$

*The concept of an orbifold will be extensively discussed in Chapter 5.

So we can identify the marginal deformations with linear transformations in the momenta $p^\mu, \overline{p}^\mu$ with generators

$$p^\mu \frac{\partial}{\partial \overline{p}^\nu} + \overline{p}^\nu \frac{\partial}{\partial p^\mu}. \tag{2.58}$$

The expressions on the right-hand side clearly represent tangent vectors to the coset space

$$X = \frac{SO(d,d)}{SO(d) \times SO(d)}. \tag{2.59}$$

Indeed this noncompact homogeneous space is the universal covering space of the moduli space $\mathcal{T}^d$, a fact due to the uniqueness of the even, self-dual Lorentzian lattice $\Gamma^{d,d}$ [123]. Any lattice is given by a $SO(d,d)$ Lorentz transformation of the lattice (2.49). Since the conformal weights $(h, \overline{h}) = (\frac{1}{2}p^2, \frac{1}{2}\overline{p}^2)$ and the interaction rules are invariant under independent $SO(d)$ rotations of the left-moving and the right-moving momenta, we have to take the quotient with $SO(d) \times SO(d)$. The invariance under these transformations has no direct physical interpretation and must be considered to be a typical quantum effect. Only for the diagonal $SO(d)$ this invariance can be understood as a reparametrization of the torus $\mathbf{R}^d/2\pi\Lambda$ [86]. A local coordinatization of the moduli space is furnished by the background fields $g_{\mu\nu}$, $b_{\mu\nu}$ [125].

The actual moduli space $\mathcal{T}^d$ is obtained by taking the quotient of the covering space $X$ by the discrete group $O(d,d;\mathbf{Z})$, which are basis transformations of the Lorentzian lattice $\Gamma^{d,d}$. The group $SO(d,d)$ acts transitively on the moduli space, and the elements $g \in O(d,d\,;\mathbf{Z})$ can be seen as the coset classes of $SO(d,d)$ transformations that are automorphisms of the full momentum lattice $\Gamma^{d,d}$ and that accordingly leave the spectrum invariant. The elements $g$ will however permute the separate weights and the corresponding operators. Note that this implies the possibility of *nontrivial spectral flow* along closed paths in moduli space. Points that are invariant under some subgroup of $O(d,d\,;\mathbf{Z})$ correspond to orbifold singularities. At these points the space $\mathcal{T}^d$ looses its manifold structure.

Let us give some examples of elements in the discrete group $O(d,d\,;\mathbf{Z})$ [123,86]. An evident class of elements are $SL(d,\mathbf{Z})$ transformations acting on a basis of the lattice $\Lambda$. These are automorphisms of $\Lambda$ (and $\Lambda^*$) and have a nontrivial action on the individual momentum states. Fixed points correspond to tori with extra automorphisms. A second example is the background field $B_{\mu\nu}$. We have seen that the $B$ field behaves as a periodic variable and consequently we have generators $B \to B + \delta B$ (*cf* Eq. (2.42)) of infinite order. A final example is given by the *duality* transformation (of order two) that we alluded to in Chapter 1. For vanishing antisymmetric background field it interchanges the lattice $\Lambda$ and its dual lattice $\Lambda^*$. For non-zero $B$ the transformation is modified to

$$\Lambda \to 2(1 - B^2)^{-\frac{1}{2}}\,\Lambda^*, \qquad B \to -B. \tag{2.60}$$

This transformation interchanges the winding and momentum states and as such large and small compactification scales. The self-dual points are necessarily of order one and thus in physical units of the order of the Planck scale. Note that in this sense we cannot consider the compactified space to be essentially smaller than the Planck scale, since it will always be equivalent to a space of much larger scales. This is from a string point of view a very natural situation. In string theory we cannot ascribe any physical meaning to space-times smaller than the string itself.

### 2.3.2. Two-Dimensional Tori

As a more concrete example we can consider the toroidal compactifications in $d = 2$ [51]. They are determined by 4 parameters: a $2d$ lattice $\Lambda$ and an antisymmetric tensor $B_{\mu\nu} = B\,\varepsilon_{\mu\nu}$, or equivalently by background fields $g_{\mu\nu}$ and $b_{\mu\nu}$ with respect to some *a priori* given toroidal compactification. It will prove to be very convenient to trade these 4 real parameters for 2 complex ones. A two-dimensional (flat) torus can be parametrized in terms of the area $g$ and the modular parameter $\tau$ in the upperhalf plane $H = \{z \in \mathbf{C} \mid \mathrm{Im}\, z > 0\}$, defined as $\tau = \omega_1/\omega_2$ with $\omega_1$ and $\omega_2$ two basis vectors of the lattice $\Lambda \subset \mathbf{C} \simeq \mathbf{R}^2$. The variable $\tau$ can be seen to parametrize the complex structure of the two-torus. The area $g$ and the background field $B$ can be combined into a second complex variable

$$\sigma = \tfrac{1}{2}(B + i)g \in H. \tag{2.61}$$

Let us first discuss the dependance on the parameter $\tau$. Modular $PSL(2, \mathbf{Z})$ transformations are automorphisms of the lattice $\Lambda$ and act on $\tau$ by fractional linear transformations

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}, \qquad a, b, c, d \in \mathbf{Z},\ ad - bc = 1. \tag{2.62}$$

The modular group $PSL(2, \mathbf{Z})$ is generated by

$$T\ :\ \tau \longrightarrow \tau + 1, \tag{2.63a}$$

$$S\ :\ \tau \longrightarrow -1/\tau. \tag{2.63b}$$

The space $H/PSL(2, \mathbf{Z})$ can be represented by the fundamental domain $\mathcal{F}$, with $|z| > 1, |\mathrm{Re}\, z| < \frac{1}{2}$ as depicted in *fig.* 1. Note that $\mathcal{F}$ is topologically a sphere, but contains three orbifold singularities, at $\tau = i$, $\tau = \omega \equiv e^{2\pi i/3}$, and $\tau = i\infty$. These are respectively the fixed points of the transformations $S$, $ST$ and $T$ (of order 2, 3 and $\infty$).

On the other hand the periodicity of $B$ (2.42) and duality transformation (2.60) can be expressed in terms of the second complex parameter $\sigma$ as

$$T'\ :\ \sigma \longrightarrow \sigma + 1, \tag{2.64a}$$

$$S'\ :\ \sigma \longrightarrow -1/\sigma. \tag{2.64b}$$

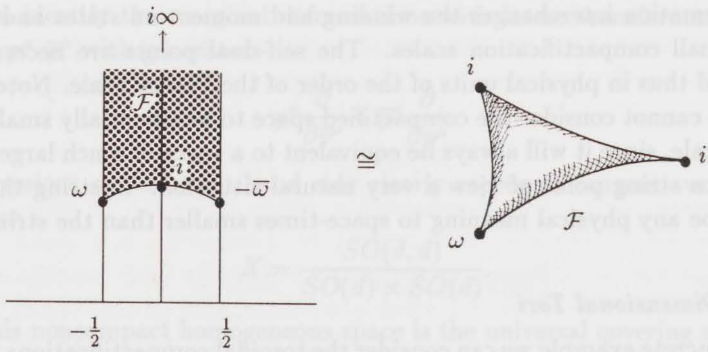



FIGURE 1: The fundamental domain $\mathcal{F} = H/PSL(2,\mathbf{Z})$.

The identical form of the above transformations of $\sigma$ and $\tau$ is not a coincidence; the model is invariant under interchange of $\sigma$ and $\tau$

$$U\,:\,(\sigma,\tau)\longrightarrow(\tau,\sigma). \tag{2.65}$$

This invariance is quite bizarre in terms of a description of background fields but becomes clear if one writes down the 4-dimensional Lorentzian momentum lattice

$$\Gamma^{2,2} = i\,(2\,\mathrm{Im}\,\sigma\,\mathrm{Im}\,\tau)^{-\frac{1}{2}}\,\mathbf{Z}\begin{pmatrix}1\\1\end{pmatrix}\oplus\mathbf{Z}\begin{pmatrix}\overline{\sigma}\\\sigma\end{pmatrix}\oplus\mathbf{Z}\begin{pmatrix}\tau\\\tau\end{pmatrix}\oplus\mathbf{Z}\begin{pmatrix}\overline{\sigma}\tau\\\sigma\tau\end{pmatrix}. \tag{2.66}$$

The group $O(2,2\,;\mathbf{Z})$ of discrete automorphisms is freely generated by $S$, $ST$, $U$ and the parity transformation $V$

$$V\,:\,(\sigma,\tau)\longrightarrow(-\overline{\sigma},-\overline{\tau}). \tag{2.67}$$

This is due to the identity $O(2,2\,;\mathbf{Z})/\mathbf{Z}_2 = SL(2,\mathbf{Z})\times SL(2,\mathbf{Z})$. So, the moduli space of toroidal compactifications can be represented by the product of two copies of $\mathcal{F}$ with the symmetries $U$ and $V$ divided out. The group $SO(2,2)$ is isomorphic to $SL(2,R)\times SL(2,R)/\mathbf{Z}_2$ and acts by separate linear fractional transformations on the parameters $(\sigma,\tau)$. For some application of these observations see [144].

# 3

# Geometry and Quantum Field Theory

Throughout the history of science, physics and geometry have been intimately connected. It has become more and more clear in the last decade—perhaps first through the study of Yang-Mills theories, instantons, and anomalies, and more recently through two-dimensional conformal field theory and Witten's work on topological field theories in three and four dimensions [162,161]—that quantum field theories are also strongly related to geometry. Not only have all kinds of field theories served as a very useful tool in studying mathematical questions in low dimensional topology, giving rise to a variety of new invariants as in the work of Donaldson and Floer [9,56,65]. But, somewhat surprisingly, geometry also seems to be at the heart of many interesting quantum field theories.

It is a fundamental principle, that a quantum field theory can be defined on manifolds of arbitrary topology, possibly given the existence of some extra geometrical objects such as an orientation or a spin structure. This property is a reflection of the essential locality of a field theory. In a path-integral formulation based on some Lagrangian density this is rather evident, since a density can be integrated over any oriented manifold. But, also in a Hamiltonian framework one can consider canonical quantization on a space-time of the form $\Sigma \times \mathbf{R}$, where $\Sigma$ is a space-like manifold with arbitrary topology. The resulting Hilbert space $\mathcal{H}_\Sigma$ will depend crucially on the geometry of $\Sigma$, as we will see in particular in Chapter 8. Of course, in the case of the quantum field theories that we usually meet in particle physics, one is more than happy to limit one's understanding to the structure of the theory in a trivial, flat background, but *a priori* nothing forbids us to consider nontrivial geometries. Any reasonable approach to quantum gravity will at some point have to discuss quantum field theory in curved backgrounds of possibly nontrivial topology.

## 3.1. A Functorial Approach to Quantum Field Theory

We will now consider a first draft of an axiomatic definition of quantum field theory fitted to our geometric needs, following the functorial approach of Segal [139], see also [10]. In this formulation one defines a $d+1$ dimensional quantum field theory to be (among many other things) a 'functor' $\Phi$ from the category of closed $d$-manifolds into the category of vector spaces [139]. Before we will explain this somewhat enigmatic definition, let us make one additional remark. In the subsequent we will use the term 'manifold' somewhat loosely; the correct terminology for the objects of our category would be $*$-manifolds. Here $*$ can be any extra, fixed structure on the manifold, *e.g.* an orientation. Two manifolds are considered equivalent if there exists a diffeomorphism that preserves the structure. It is important to stress that this structure does not constitute an additional degree of freedom. On the contrary, it should be regarded as an *a priori* given background, on equal footing with the topology of the manifold. The most familiar example of such a structure occurs in ordinary quantum field theories such as QED, where we will always need a *metric* in order to define the theory. We will refer to field theories with the epithets *topological*, *spin*, or *conformal*, if the extra structure is respectively an *orientation*, a *spin structure* or a *complex structure*.

The axiomatic definition of a quantum field theory comprises two important ingredients. The first one we already alluded to. We will assume that to any closed $d$-dimensional space-like manifold $\Sigma$ we can associate a Hilbert space $\mathcal{H}_\Sigma$. This Hilbert space should be considered as the space of quantum states obtained by quantizing the theory on the space-time $\Sigma \times \mathbf{R}$. Since we assume that $\Sigma$ is closed and compact, the definition of $\mathcal{H}_\Sigma$ should be unambiguous.

The second ingredient is motivated by the concept of a transition amplitude. Recall that an evolution operator of time $t$ is a linear unitary map

$$\Phi_t : \mathcal{H}_\Sigma \to \mathcal{H}_\Sigma, \tag{3.1}$$

describing the time evolution of states in $\mathcal{H}_\Sigma$. It can be associated to a 'slice' of space-time $\Sigma \times I$, with $I$ a time-interval of length $t$. Furthermore, two such operators $\Phi_t$ and $\Phi_{t'}$ can be composed to obtain the evolution operator of time $t + t'$

$$\Phi_{t+t'} = \Phi_t \circ \Phi_{t'}. \tag{3.2}$$

Note that we can relate the operator $\Phi_t$ to the manifold $\Sigma \times I$ with an 'incoming' and 'outgoing' boundary component $\Sigma$. We will extend this correspondence and will assume that in all generality any $d+1$ dimensional manifold $M$ with two boundaries $\Sigma$ and $\Sigma'$ gives rise to a linear map

$$\Phi_M : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}. \tag{3.3}$$

Here we have located the Hilbert space $\mathcal{H}_\Sigma$ at the boundary $\Sigma$, although the 'space-time' $M$ is not of the form $\Sigma \times \mathbf{R}$. Our justification is that quantization only needs a *local* arrow of time. So we can restrict to a collar $\Sigma \times I \subset M$ around the boundary, in order to construct the Hilbert space $\mathcal{H}_\Sigma$. In general the map $\Phi_M$ will describe a tunneling amplitude from one space topology to another, possibly through a

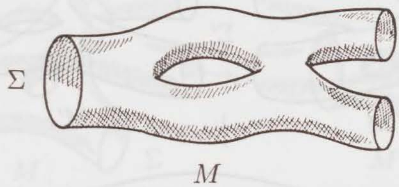



FIGURE 2: The path-integral on a manifold $M$ with boundary components $\Sigma$ and $\Sigma'$ will give rise to a (topology changing) transition amplitude $\Phi_M : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}$.

process of joining and splitting of space-like manifolds, as in *fig.* 2. Of course, the in- and outgoing boundaries can consist of several disconnected components.

The existence of the $\Phi_M$'s can be physically motivated as follows. If $\phi$ is a local set of fundamental fields in the theory, a state in $\mathcal{H}_\Sigma$ will be a wave function $\Psi(\phi)$ on the space of field configurations $\phi(x)$ on $\Sigma$. The path-integral on $M$ with fixed values $\phi, \phi'$ at the boundaries $\Sigma, \Sigma'$ equals the kernel $K_M(\phi', \phi)$ of the evolution operator $\Phi_M$. The transition amplitude reads

$$\Psi_{out}(\phi') = \int \mathcal{D}\phi\, K_M(\phi', \phi)\Psi_{in}(\phi). \tag{3.4}$$

Just as we could compose the transition amplitudes $\Phi_t$ (as in (3.2)), we will demand a similar composition rule for the $\Phi_M$'s. That is, given a $d+1$ dimensional manifold $M$ that interpolates from $\Sigma$ to $\Sigma'$, and a similar manifold $M'$ that interpolates from $\Sigma'$ to $\Sigma''$, we can consider the manifold $M' \circ M$, that is obtained by gluing $M, M'$ at their common boundary (as illustrated in *fig.* 3) and that describes tunneling from $\Sigma$ to $\Sigma''$. The amplitude $\Phi_{M' \circ M}$ should respect this relation:

$$\Phi_{M' \circ M} = \Phi_{M'} \circ \Phi_M. \tag{3.5}$$

This essentially corresponds to the superposition principle of quantum mechanics

$$K_{M' \circ M}(\phi'', \phi) = \int \mathcal{D}\phi'\, K_{M'}(\phi'', \phi')K_M(\phi', \phi). \tag{3.6}$$

Note that we sum over all intermediate states, but keep the intermediate topology $\Sigma'$ fixed. We can read Eq. (3.5) also backwards: if we slice a space-time into two halves, the corresponding amplitude will be the composition of the two partial amplitudes. In particular it will not matter how we have chosen to cut the manifold in smaller parts. (This fundamental principle is known as *duality*.)

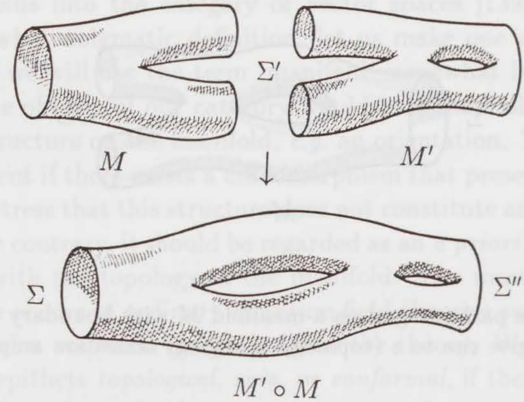



FIGURE 3: If two manifolds $M$ and $M'$ can be composed to form a manifold $M' \circ M$, the transition amplitudes satisfy $\Phi_{M' \circ M} = \Phi_{M'} \circ \Phi_M$.

The structure we just sketched can be formulated more elegantly, albeit somewhat more abstract, in the language of categories. A category (see *e.g.* [121] for a 'pedestrian' introduction) contains besides a set of *objects* $x$, a set of *arrows* or morphisms $f : x \to y$. These should be regarded as abstract quantities that satisfy an associative composition law. That is, given two arrows $f : x \to y$ and $g : y \to z$, we can form the composite arrow $g \circ f : x \to z$, such that $(h \circ g) \circ f = h \circ (g \circ f)$. A category further presumes for each object $x$ an identity arrow $1_x : x \to x$. An elementary example, that the reader should keep in mind, is the category of vector spaces, where the arrows correspond to linear maps. A functor is a map from one category to another, that respects all these relations.

In the category of $d$-dimensional closed manifolds, the arrows are cobordisms. An arrow $M : \Sigma \to \Sigma'$ is a $d+1$ dimensional manifold $M$ that interpolates from $\Sigma$ to $\Sigma'$, *i.e.* a tunneling amplitude as we considered in *fig.* 2. More precisely, $M$ is a manifold that satisfies $\partial M = \Sigma \cup (-\Sigma')$, and whose $*$-structure reduces to the respective structures on $\Sigma$ and $-\Sigma'$ at its boundary. Here '$\cup$' denotes the disjoint sum, and $-\Sigma$ is the manifold $\Sigma$ with inverse structure, *e.g.* reversed orientation. It is defined by $\partial(\Sigma \times I) = \Sigma \cup (-\Sigma)$, where the structure on $\Sigma \times I$ along the time-component is trivial. (For instance, if we presume a metric $g_{00} = 0$.) Note that we have distinguished the boundary components $\Sigma$ and $\Sigma'$ of $M$ by labeling them respectively as incoming and outgoing. The composition of arrows in this category corresponds to gluing the manifolds.

Our previous considerations can now be neatly summarized by stating that a field theory gives rise to a functor $\Phi$, that maps the objects and arrows of the

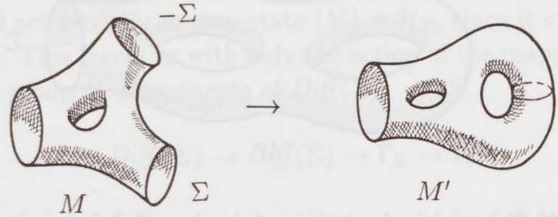



FIGURE 4: If a manifold $M'$ is obtained by sewing two ends of a manifold $M$, the amplitudes satisfy $\Phi_{M'} = \mathrm{Tr}\ \Phi_M$.

category of $d$-manifolds to the objects and arrows in the category of vector spaces, preserving their relations. That is, $\Phi$ associates to each $d$-manifold $\Sigma$ a vector space $\mathcal{H}_\Sigma$, and to each $d+1$ manifold $M : \Sigma \to \Sigma'$ an evolution operator $\Phi_M : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}$. It should preserve the associate composition law, which gives equation (3.5).

The other axioms are mainly concerned with situations where the boundary consists of several disjoint components.

(*i*) In the degenerated case, that $\Sigma$ consists of the empty set, the corresponding Hilbert space is defined to be one-dimensional

$$\mathcal{H}_\emptyset = \mathbf{C}. \tag{3.7}$$

This definition is appropriate in the light of the following axiom.

(*ii*) If $\Sigma$ is the disjoint union of several manifolds, the corresponding Hilbert space is the tensor product:

$$\mathcal{H}_{\Sigma \cup \Sigma'} = \mathcal{H}_\Sigma \otimes \mathcal{H}_{\Sigma'}. \tag{3.8}$$

A physically evident condition, since the spaces $\Sigma$ and $\Sigma'$ are causally disconnected.

(*iii*) The 'collapsing' axiom: if $M$ has two boundary components $-\Sigma$ and $\Sigma$, labeled respectively as outgoing and incoming, we can glue the two boundaries together to form a new manifold $M'$, see *fig.* 4. $\Phi$ should also respect this 'trace' operation. So, if $v_i$ and $v^i$ are conjugate bases in $\mathcal{H}_\Sigma$ and $\mathcal{H}^*_\Sigma$, we have

$$\Phi_{M'} = \mathrm{Tr}\ \Phi_M = \sum_i \Phi_M(v_i, v^i). \tag{3.9}$$

(*iv*) An (optional) reality condition: In general we have only a canonical isomorphism

$$\mathcal{H}_{-\Sigma} \cong \mathcal{H}^*_\Sigma, \tag{3.10}$$

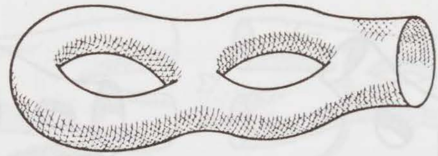



FIGURE 5: If $M$ has a single outgoing boundary it defines a state $|M\rangle \in \mathcal{H}_\Sigma$.

but if $\mathcal{H}_\Sigma$ also carries a hermitian structure $\mathcal{H}_{-\Sigma}$ is isomorphic to $\overline{\mathcal{H}_\Sigma}$ (the complex conjugate space) and one has

$$\Phi_{-M} = \overline{\Phi}_M. \tag{3.11}$$

($v$) A natural action of the permutation group, possibly graded in the case of fermionic theories, if several boundary components are isomorphic.

This concludes our axiomatic considerations. We close this abstract exposé with some related remarks.

First, we observe that a closed $d+1$ manifold $M$, *i.e.* a manifold with an empty boundary, can be seen as an arrow $\emptyset \to \emptyset$, and will give rise to a morphism $\Phi_M : \mathbf{C} \to \mathbf{C}$. This implies that we can associate to $M$ a number $Z(M)$, the *partition function*.

Secondly, if the manifold $M$ has only a single, outgoing boundary $\Sigma$, the transition amplitude $\Phi_M$ is a map $\mathbf{C} \to \mathcal{H}_\Sigma$ and defines a special state (*fig.* 5), that we denote as

$$|M\rangle \in \mathcal{H}_\Sigma. \tag{3.12}$$

This can also be understood in a path-integral language. The path-integral on a space-time with boundary $\Sigma$ is a functional of the fundamental field variables $\phi(x)$ at the boundary, and defines a wave function $\Psi_M(\phi)$. If we split a closed manifold $M$ into two parts $M_1, M_2$ by cutting it at $\Sigma$, the partition function can be recovered as

$$Z(M) = \langle M_1 | M_2 \rangle = \int \mathcal{D}\phi \; \Psi^*_{M_1}(\phi)\, \Psi_{M_2}(\phi). \tag{3.13}$$

Of course, we can also glue $M_1, M_2$ together after applying a diffeomorphism on one of the boundaries $\Sigma$. This procedure is known in topology as *surgery*, and is a very effective way of obtaining a new manifold $M'$. The action of a diffeomorphism $\gamma : \Sigma \to \Sigma$ will leave a state $|M\rangle \in \mathcal{H}_\Sigma$ invariant if $\gamma$ can be extended to a map $\gamma' : M \to M$. So for a *topological* theory—that presumes no more structure than the topology of $M$ and an orientation—the identity component of the diffeomorphism group $Diff(\Sigma)$ will act trivially on any state $|M\rangle \in \mathcal{H}_\Sigma$, since it can be deformed over a collar $\Sigma \times I$. This leaves us with only the action of the mapping class group $\Gamma_\Sigma$, defined as the group of components of $Diff(\Sigma)$

$$1 \to Diff_0(\Sigma) \to Diff(\Sigma) \to \Gamma_\Sigma \to 1, \tag{3.14}$$

that is, the group of global diffeomorphisms that cannot be deformed to the identity. If $U(\gamma)$ is the unitary representation of $\gamma \in \Gamma_\Sigma$ in $\mathcal{H}_\Sigma$, we can write the partition function of the manifold $M'$ obtained by surgery as

$$Z(M') = \langle M_2|U(\gamma)|M_1\rangle. \tag{3.15}$$

Finally, we point out that the axiom (3.9) allows us, at least in principle, to calculate the dimensions of the Hilbert spaces $\mathcal{H}_\Sigma$ by gluing the two ends of the cylinder $\Sigma \times I$ together to form the closed manifold $\Sigma \times S^1$:

$$\dim \mathcal{H}_\Sigma = \mathrm{Tr}_{\mathcal{H}_\Sigma} 1 = Z(\Sigma \times S^1). \tag{3.16}$$

Of course, this formula only makes sense for finite dimensions. We will see examples of such finite theories in Chapter 8.

## 3.2. A Toy Model in Two Dimensions

The simplest realization of the above axioms is $0+1$ dimensional quantum field theory, better known as non-relativistic quantum mechanics. According to the functorial point of view, its definition entails a Hilbert space $\mathcal{H}$ associated to a point, and a set of evolution operators $\Phi_t$ for each line segment of length $t$. The sewing axiom gives us the semigroup law

$$\Phi_t\,\Phi_{t'} = \Phi_{t+t'}, \tag{3.17}$$

so we can write $\Phi_t = \exp(itH)$ with $H$ the Hamiltonian. As a second, less elementary illustration, that emphasizes the relation with geometry, we will consider in this section two-dimensional *topological* field theories. Although these theories have no direct physical application and serve here only as a toy model, we will meet much of their geometric structure, albeit in a somewhat different form, when we discuss conformal field theories in the following section and in Chapter 4.

In two dimensions life simplifies enormously. First, up to diffeomorphisms there is only a single one-dimensional closed manifold: the circle $S^1$, with two possible orientations. So we only have to consider the Hilbert space $\mathcal{H} \equiv \mathcal{H}_{S^1}$ and $\mathcal{H}^* = \mathcal{H}_{-S^1}$. In order to facilitate our discussion we will assume that this space is finite dimensional. The full structure of a topological theory now requires the definition of the morphisms $\Phi_M$, where $M$ is any surface, possibly with boundaries. However, these morphisms cannot be independently defined, but should respect the relations that follow from the composition rule (3.5). In fact, for these two-dimensional theories *all* $\Phi_M$'s are uniquely determined by considering just three special manifolds: the sphere with one, two, and three holes, or punctures.

According to the general formalism the once-punctured sphere, with the boundary labeled as 'out-going', determines a map $\mathbf{C} \to \mathcal{H}$, *i.e.* a special state in the Hilbert space $\mathcal{H}$ that we will denote as $|0\rangle$

$$|0\rangle = \tag{3.18}$$



The state $|0\rangle$ has every right to be called the *vacuum* of the theory, although we remind the reader that in a topological theory the Hamiltonian vanishes and energy is not defined; we lack the concept of time which requires a metric. However, there is a natural operator interpretation of the states in the Hilbert space $\mathcal{H}$ which makes the identification with a vacuum state clear. If we consider a surface $M$ with $n$ punctures $P_1, \ldots, P_n$ and with states $|\phi_1\rangle, \ldots, |\phi_n\rangle$ inserted at these punctures, the morphism $\Phi_M : \mathcal{H}^{\otimes n} \to \mathbf{C}$ can be seen as a topological correlation function on the *closed* manifold $\overline{M}$, obtained by removing the punctures,

$$\Phi_M|\phi_1, \ldots, \phi_n\rangle = \left\langle \phi_1(P_1) \cdots \phi_n(P_n) \right\rangle_{\overline{M}} \tag{3.19}$$

Here we have identified the states $|\phi_i\rangle \in \mathcal{H}$ with operators $\phi_i$ on the surface. We will use the two notations interchangeably. Because of the topological nature of the theory the correlator does not depend on the positions of the punctures. It is now geometrically evident that the state $|0\rangle$ corresponds to the identity operator; inserting $|0\rangle$ at a puncture is equivalent to filling the puncture, because of (3.18). In particular, we have

$$\Phi_M|0, \ldots, 0\rangle = Z(\overline{M}). \tag{3.20}$$

In a similar way the twice-punctured sphere corresponds to a bilinear form $g : \mathcal{H} \otimes \mathcal{H} \to \mathbf{C}$, if we choose to label both boundaries as 'incoming'. With a basis $\phi_i$ in $\mathcal{H}$, we can calculate the matrix elements $g_{ij}$ as

$$g_{ij} = \langle \phi_i \phi_j \rangle = \tag{3.21}$$

where the correlation function is understood to be on the sphere $S^2$. We can use this metric $g_{ij}$ to convert outgoing states, that are elements of the dual space $\mathcal{H}^*$ with dual basis $\phi^i$, into incoming states by lowering indices. The matrix $g^{ij}$ is defined by similarly considering the twice-punctured sphere with outgoing states

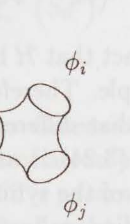
$$g^{ij} = \langle \phi^i \phi^j \rangle = \tag{3.22}$$




By the composition axiom (3.5) it satisfies $g_{ik}g^{kj} = \delta_i{}^j$, so $g_{ij}$ is indeed invertible and defines an inner product on $\mathcal{H}$.

Our claim is now that the map $\Phi_Y$, with $Y$ the sphere with *three* punctures (the 'pant' or trinion, that describes the space-time history of the merging of two strings into one string), will enable us to calculate the morphism associated to any surface. This is easily seen to be true if we recall that a two-dimensional manifold, which is always topologically equivalent to a sphere with $g$ handles and $n$ holes, can be represented by the union of $2g-2+n$ copies of $Y$. So let us define the coefficients $c_{ij}{}^k$ by the three-point function

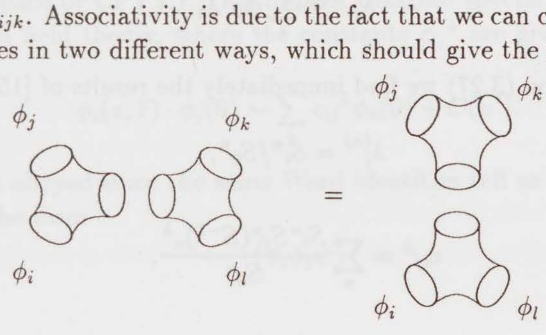
$$c_{ij}{}^k = \left\langle \phi_i \phi_j \phi^k \right\rangle = \tag{3.23}$$




We will now proceed to show that this map $\Phi_Y : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ makes the vector space $\mathcal{H}$ into a commutative, associative algebra with unity. Here the product on $\mathcal{H}$ is defined by the structure coefficients $c_{ij}{}^k$

$$\phi_j \times \phi_j = \sum_k c_{ij}{}^k \phi_k. \tag{3.24}$$

Commutativity is a straightforward consequence of the permutation symmetry of the quantities $c_{ijk}$. Associativity is due to the fact that we can construct the sphere with 4 punctures in two different ways, which should give the same answer




(3.25)

As we already mentioned, this fundamental principle that a morphism $\Phi_M$ is independent of the way in which it can be constructed from elementary amplitudes is known as *duality*. It implies the relation

$$\sum_n c_{ij}{}^n c_{nkl} = \sum_n c_{jk}{}^n c_{nil}. \tag{3.26}$$

As we already explained, the unit element of the algebra (3.24) corresponds to the vacuum $\phi_0 = |0\rangle$. We have the identity

$$c_{0j}{}^k = \delta_j{}^k, \tag{3.27}$$

which implies $\phi_0 \times \phi_i = \phi_i$. It is easily seen, that no new conditions are obtained by imposing the duality condition for higher $n$-point functions, and this completes our one-to-one correspondence between $2d$ topological field theories and commutative, associative algebras with unity and inner product. Note that the algebra (3.24) also furnishes the operator interpretation of the states that we already mentioned. For correlation functions on the sphere this implies the representation

$$\left\langle \phi_{i_1} \cdots \phi_{i_n} \right\rangle_{S^2} = \langle 0|\phi_{i_1} \cdots \phi_{i_n}|0\rangle. \tag{3.28}$$

It is a very important fact that $\mathcal{H}$ has a natural inner product, since it implies that the algebra is semi-simple. Therefore we can follow the line of arguments of [153] (as applied in a somewhat different context) and write the algebra $\mathcal{H}$ in a basis of idempotents. Since (3.24) is associative and commutative, we have a regular representation in terms of the symmetric, commuting matrices $(\phi_i)_j{}^k = c_{ij}{}^k$. They can be simultaneously diagonalized by a matrix $S_i{}^j$ that is orthogonal with respect to the metric $g_{ij}$. Thus we find the relation

$$c_{ij}{}^k = \sum_n S_j{}^n \lambda_i^{(n)} (S^{-1})_n{}^k. \tag{3.29}$$

The eigenvalues and characters $\lambda_i^{(n)}$ are the irreducible, one-dimensional representations of the algebra (3.24)

$$\lambda_i^{(n)} \lambda_j^{(n)} = \sum_k c_{ij}{}^k \lambda_k^{(n)}. \tag{3.30}$$

If we use the relation (3.27) we find immediately the results of [153]

$$\lambda_i^{(n)} = S_i{}^n / S_0{}^n, \tag{3.31}$$

and

$$c_{ij}{}^k = \sum_n \frac{S_i{}^n S_j{}^n (S^{-1})_n{}^k}{S_0{}^n}. \tag{3.32}$$

If we now introduce the new basis $\tilde{\phi}_i = (S^{-1})_i{}^j \phi_j$, we easily verify that they are idempotents

$$\tilde{\phi}_i \times \tilde{\phi}_j = \frac{\delta_{ij}}{S_0{}^i} \tilde{\phi}_i. \tag{3.33}$$

For closed surfaces $M$ of genus $g$ the partition function is now very easily calculated [153]. We can decompose $M$ into $2g-2$ thrice-punctured spheres, which in the above basis $w_i$ in $\mathcal{H}$ each give rise to a delta-function with multiplicative constant $1/S_0{}^k$. So we immediately find the result

$$Z(M) = \sum_k \left(S_0{}^k\right)^{2(1-g)} = \sum_k \left(S_0{}^k\right)^{\chi(M)}, \tag{3.34}$$

with $\chi(M)$ the Euler characteristic of $M$. Note in particular that

$$Z(S^1 \times S^1) = \dim \mathcal{H} \tag{3.35}$$

in accordance with (3.16). Similarly we find for the '$n$-point functions' on $M$

$$\left\langle \phi_{i_1} \cdots \phi_{i_n} \right\rangle_M = \sum_k S_{i_1}{}^k \cdots S_{i_n}{}^k \left(S_0{}^k\right)^{2(1-g)-n} \tag{3.36}$$

This completes the 'solution' of this toy model.

We would like to mention here that these two-dimensional topological theories arise in a natural way if we take a certain limit of a conformal field theory. Suppose, following [74], that we have a family of CFT's parametrized by a continuous variable $\alpha' > 0$, and consider the limit $\alpha' \to 0$. Let $\{\phi_i\}$ be the set of primary fields whose conformal weights are of order $\alpha'$. The conformal Ward identities tell us that in the limit $\alpha' \to 0$ the correlation functions

$$\left\langle \phi_{i_1}(z_1, \overline{z}_1) \cdots \phi_{i_n}(z_n, \overline{z}_n) \right\rangle_M \tag{3.37}$$

will no longer depend on the positions $z_k$. (In a regular CFT the identity is the only field that satisfies this constraint, so we see that in general this limit brings us out of the space of CFT's.) This implies that the special fields $\phi_i$ constitute a $2d$ topological field theory, where the constants $c_{ij}{}^k$ are given by the operator products

$$\phi_i(z, \overline{z}) \cdot \phi_j(0) \sim \sum_k c_{ij}{}^k \phi_k(0) + \mathcal{O}(\alpha'). \tag{3.38}$$

Associativity is obeyed since the same Ward identities tell us that to order $\alpha'$ all correlators of the form

$$\left\langle \phi_i \phi_j \psi \right\rangle \tag{3.39}$$

vanish, where $\psi$ is a field whose conformal dimensions do *not* vanish in the limit $\alpha' \to 0$. In the particular case that the CFT is a nonlinear $\sigma$-model on a target manifold $X$ (with action (1.2)) the commutative, associative algebra $\mathcal{H}$ will be the algebra of functions on $X$ with some suitable normalization requirement, and the coefficients $c_{ijk}$ are given by overlap functions [74].

## 3.3. The Geometry of CFT

We will now proceed to include more structure to our 'skeleton' theory and 'dress' the topological surface of the previous section with a complex structure. This will lead us naturally to conformal field theories. The geometrical formulation of conformal field theory found its origin in the seminal work of Friedan and Shenker [78]. It received further impetus from the so-called operator formalism of string theory and CFT (see [95,3,150,2] and in particular [151]), an approach dating back to the 'early string days' that uses techniques developed in the study of integrable hierarchies [40,140]. Recently, it has been put on more rigorous, axiomatic grounds by Segal [139], see also the review in [80].

Just as in a topological theory, the objects one considers in CFT are two-dimensional manifolds, possibly with boundaries. However, a conformal field theory presumes more data; all surfaces come with a complex structure, or equivalently a conformal class of metrics. That is, our category consists of Riemann surfaces [61,43]. Recall that a complex structure is a linear map $J$, defined in the tangent space at each point of the surface, that satisfies $J^2 = -1$ and the integrability condition $\nabla J = 0$. The complex structure $J$ allows us to split the complexified tangent space in holomorphic and anti-holomorphic vectors, with eigenvalues $i$ and $-i$, and tells us what is (locally) an analytic coordinate on the surface. In two dimensions any metric $g_{ab}$ defines a complex structure through the relation

$$J_a{}^b = \sqrt{g}\, \epsilon_{ac} g^{cb}, \tag{3.40}$$

with $\epsilon_{ab}$ the Levi-Civita symbol and $\sqrt{g} = \det g_{ab}$ (not to be confused with the genus $g$). Furthermore, all $J$'s are obtained in this way. As we see, $J$ is invariant under local rescaling (Weyl transformations) of the metric $g_{ab} \to \rho(x) g_{ab}$, and so is only determined by the so-called conformal class of the metric. Locally we can always choose coordinates $x$ such that $g_{\mu\nu}(x) = \rho(x)\delta_{\mu\nu}$, and in these coordinates the complex structure reduces to that of the complex plane $\mathbf{C}$, with an analytic coordinate $z = x_1 + ix_2$. Any Riemann surface can be seen as a local set of patches of $\mathbf{C}$ glued together with conformal reparametrizations, *i.e.* holomorphic transition functions. Two complex structures are regarded as identical if they are related by a diffeomorphism.

One way to parametrize the possible complex structures $J$ on a genus $g$ surface $\Sigma$ is to start with the space of all metrics modulo diffeomorphisms, and choose a unique representative in each conformal class. This can be conveniently done by requiring the curvature $R$ to be constant. Taking into account the Gauss-Bonnet theorem

$$\int_\Sigma \frac{d^2z}{2\pi}\sqrt{g}\,R = 2-2g, \tag{3.41}$$

$R$ can be normalized to be $1, 0, -1$ for genus $g = 0, 1, \geq 2$. (For genus one we also have to normalize the area $\int \sqrt{g} = 1$.) The moduli space $\mathcal{M}_g$ of compex structures on $\Sigma$ is however not a smooth manifold. This is due to the fact that we have to identify (constant curvature) metrics by the diffeomorphism group $\mathit{Diff}(\Sigma)$, which does not act freely. We can do this in two steps, by considering first the diffeomorphisms $\mathit{Diff}_0(\Sigma)$ connected to the identity, and subsequently the mapping class group $\Gamma_g$ (see (3.14)). The first step is a smooth operation, and produces the so-called Teichmüller space $\mathcal{T}_g$—a smooth, complex manifold of dimension $0, 1, 3g-3$ for $g = 0, 1, \geq 2$. However, the second operation, that expresses the moduli space $\mathcal{M}_g$ as

$$\mathcal{M}_g = \mathcal{T}_g/\Gamma_g, \tag{3.42}$$

can have fixed points, that correspond to surfaces with extra automorphisms.

A simple example is the torus $g = 1$. As we already discussed in Chapter 2, a two-dimensional torus with a flat metric and fixed area can be represented as a $\mathbf{C}/\Lambda$, with $\Lambda$ the lattice $\mathbf{Z} \oplus \tau\mathbf{Z}$, and $\tau$ an element of the upperhalf plane $H = \mathcal{T}_1$. That is, we have the identifications $z \sim z+1 \sim z+\tau$. However, we also have to make identifications under the modular group $\Gamma_1 = PSL(2,\mathbf{Z})$. So the moduli space $\mathcal{M}_1$ equals $H/PSL(2,\mathbf{Z})$ and can be represented by the fundamental domain $\mathcal{F}$ of *fig.* 1. The fixed points correspond to $\tau = i, \omega$, or $i\infty$ where the torus has automorphism group $\mathbf{Z}_4, \mathbf{Z}_6$, or $\mathbf{Z}$. (There is always the $\mathbf{Z}_2$ symmetry $z \to -z$.) The point $\tau = i\infty$ corresponds to a singular surface, where the torus is pinched to an infinitely thin tube. This boundary of $\mathcal{M}_g$ is called in general the compactification divisor.

For Riemann surfaces with boundaries, we will also assume a local coordinatization of the boundary. This is conveniently done as follows [150]. Any Riemann surface $\Sigma$ with boundaries can be obtained by excising disks out of a closed surface $\overline{\Sigma}$. For a surface with $n$ holes one chooses $n$ points $Q_1, \ldots, Q_n$ on $\overline{\Sigma}$ together with a local analytical coordinate* $z_i$ around $Q_i$. The surface $\Sigma$ can now be obtained from $\overline{\Sigma}$ by removing the disks $|z_i| > 1$. The corresponding map from the unit circle in the $z_i$-plane to $\Sigma$ gives the required coordinatization of the boundaries. This representation of the parametrized boundary is unique, since any diffeomorphism of $S^1$ can be extended to a holomorphic map of a neighborhood of the unit

*Following a long history, the point $Q_i$ is usually taken to correspond to the point $z = \infty$.

circle. A Riemann surface with genus $g$ and $n$ punctures or marked points has a moduli space $\mathcal{M}_{g,n}$ of inequivalent complex structures. For $g > 1$ the dimension is $\dim \mathcal{M}_{g,n} = 3g - 3 + n$.

The geometric structure of conformal field theories resembles very much the structure we sketched in the previous section for topological theories. Only we now have to distinguish surfaces with similar topology but different complex structures. Since we have chosen to represent parametrized holes by local coordinates around punctures the sewing procedure of two Riemann surfaces is easily formulated. The two local analytic coordinates $z, z'$ at the two punctures are simply related by $zz' = 1$ [151].

Of course, just as in the case of our topological toy model, we can now proceed to show that a CFT is completely determined by its one-, two- and three-point functions. In particular, the unit disk $D = \{|z| \leq 1\}$ corresponds with the vacuum $|0\rangle$. In this approach we can also make a direct identification between operators and states. The unit disk with an operator $\phi$ inserted at $z = 0$ corresponds to the state $|\phi\rangle$ at the boundary of the disk

$$|\phi\rangle = \phi(0)|0\rangle \tag{3.43}$$

The two-point function represents a cylinder or annulus $A$, and the CFT associates to this an operator $\Phi_A : \mathcal{H} \to \mathcal{H}$. Since two cylinders can be sewn together to form a new cylinder, the space of all cylinders naturally has the structure of a semi-group. One would like to derive the Virasoro algebra by looking at infinitesimal transformations, just as in the case (3.17) in quantum mechanics. Intuitively such a relation is evident, since we can induce in this way arbitrary (local) analytical transformations of the coordinate $z$ around the puncture. The mathematical rigorous derivation can be found in [139]. The important result is that the conformal symmetry can be derived from *a priori* grounds from the gluing axioms.

An annulus $A$ can be uniquely represented as $0 < e^{-t} \leq |z| \leq 1$, with $t$ a real, positive parameter. With the induced parametrizations the amplitude $\Phi_A$ reads

$$\Phi_A = e^{-tH}, \tag{3.44}$$

with $H$ the Hamiltonian on the cylinder (see (2.30)

$$H = L_0 + \overline{L}_0 - \frac{c}{12}. \tag{3.45}$$

If we glue the two ends together after a constant rotation $z \to e^{i\theta} z$, we obtain a torus $z \sim q\, z$, with $q = e^{2\pi i \tau} = e^{-t + i\theta}$. The partition function of the torus reads

$$Z(S^1 \times S^1) = \mathrm{Tr}\, q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}} \tag{3.46}$$

Since the partition function should not depend on the particular way the torus is constructed, it should be modular invariant under $PSL(2,\mathbf{Z})$ transformations

$$\tau \to \frac{a\tau+b}{c\tau+d}, \tag{3.47}$$

generated by

$$T:\tau \longrightarrow \tau+1, \tag{3.48}$$
$$S:\tau \longrightarrow -1/\tau. \tag{3.49}$$

The implications of the condition of modular invariance are quite strong, as we will see in the following chapters.

We would like to close this chapter with one subtlety that might confuse the reader. When considered on a plane or cylinder, a conformal field theory depends only on the complex structure, not on the full metric, since we have

$$\frac{1}{\sqrt{g}}\frac{\delta}{\delta g_{z\bar{z}}}\log Z = \langle T_{z\bar{z}}\rangle, \tag{3.50}$$

and the trace of the stress-energy tensor $T_{z\bar{z}}$ vanishes. However, because of the trace anomaly (2.14) this relation is no longer true on arbitrary topology. If we integrate the trace anomaly we find the behavior under Weyl transformations

$$Z(e^{\lambda}g) = e^{-cS_L(\lambda,g)}Z(g), \tag{3.51}$$

with $S_L$ the Liouville action.

$$S_L(\lambda,g) = \frac{1}{24\pi}\int d^2z\,\sqrt{g}\left(g^{ab}\partial_a\lambda\partial_b\lambda - R(g)\lambda\right). \tag{3.52}$$

However, we do see that the dependence on the Weyl factor is completely determined by the central charge, and does not depend on the specific details of the model. By taking suitable quotients of quantities we can always eliminate this dependence.

# 4

# Rational Conformal Field Theory

It has always been a guiding principle in physics to search for the maximal symmetry underlying a problem. In a quantum field theory a symmetry group will organize the operator content in multiplets, and will require all interactions to be given by invariant couplings. It has become more and more clear that in the exceptional case of two-dimensional conformal invariant systems the symmetry groups, or more properly symmetry algebras, can be so extensive that one is left with only a *finite* number of representations. This reduces quantum field theory in a sense to finite quantum mechanics. As we will see in this chapter, in this class of field theories, known as *Rational Conformal Field Theories* (RCFT's), the symmetry algebra almost uniquely determines the model. The only degree of freedom is a discrete choice of left-right pairing. For a RCFT algebraic equations can be deduced for many of the physical parameters. These algebraic relations and their solutions have been the object of many recent studies, see *e.g.* [153,152,119,48,120,121,133,114,148], and it seems that the classification of RCFT's is a much more feasible task than the classification of general CFT's. The importance of RCFT has first been stressed by Friedan and Shenker, in [78] and in unpublished work.

It was a surprising result of the work of Belavin *et al.* [16] that in the range $c < 1$ special conformal field theories can be constructed, the so-called *minimal models*, with only a *finite* number of irreducible representations of the Virasoro algebra and a closed operator algebra. The constraint of unitarity further restricts these models to the famous unitary discrete series [76], with central charges

$$c = 1 - \frac{6}{m(m+1)}, \qquad (m \geq 3), \tag{4.1}$$

and special *rational* conformal weights

$$h_{r,s} = \frac{[r(m+1) - sm]^2 - 1}{4m(m+1)}, \qquad (0 < r < m,\ 0 < s < r), \tag{4.2}$$

or a subset of these [81,28]. The simplest example of a minimal model is the two-dimensional, critical Ising model, that appears at $m = 3$, with central charge $c = \frac{1}{2}$ and conformal dimensions $h = 0, \frac{1}{16}, \frac{1}{2}$. The corresponding primary fields can be identified with respectively the identity operator, the spin operator $\sigma(z, \overline{z})$ and the energy operator $\epsilon(z,\bar{z})$. We stress here that the Ising model is the *only* unitary model that forms a representation of the $c=\frac{1}{2}$ Virasoro algebra.

The amplitudes of minimal CFT's have an analytic structure, that is a reflection of the particular representation content. An unnormalized multi-point function of primary fields on an arbitrary Riemann surface $\Sigma$, with complex moduli $m_a$, is of the form*

$$\left\langle \phi_{i_1}(z_1,\bar{z}_1)\cdots\phi_{i_n}(z_n,\bar{z}_n)\right\rangle_\Sigma = \sum_I \mathcal{F}_I(z_i;m_a)\overline{\mathcal{F}}_I(\bar{z}_i;\overline{m}_a), \tag{4.3}$$

where the summation index $I$ ranges over a *finite* set. The quantities $\mathcal{F}_I(z_i,m_a)$ are called the holomorphic blocks, and are— in general multi-valued—meromorphic functions of the moduli $z_i, m_a$ of the punctured surface.

Equation (4.3) will be our starting point in this chapter. Conformal invariant models whose correlation functions satisfy this finite factorization condition will be referred to as rational, although we will show in the next sections that it is sufficient to demand this property for the genus one vacuum amplitude and the three-point function on the sphere. The term 'rational' is appropriate here, since it can be proved that these models will always have rational central charge and rational conformal weights [5]. The importance of the complex structure of the moduli space of (punctured) Riemann surfaces, which is a crucial ingredient in (4.3), for the formulation of string and conformal field theory was probably first fully realized in [15].

Let us make one final general remark. In the following sections our approach to RCFT will always start from the presumed existence of a physical model that satisfies certain conditions. This is somewhat against the spirit of RCFT, which due to its rigid structure lends itself *par excellence* for a more axiomatic approach, as developed in *e.g.* [121].

## 4.1. An Example — The Four-Point Function of Ising Spin Fields

The factorization property (4.3) of the minimal models was first investigated for correlation functions on the sphere. Here it is a direct consequence of the special

*Equation (4.3) deserves some comments. In a coordinate invariant regularization it is not quite true. More precisely, the amplitude is of the form $e^\alpha \sum_I \mathcal{F}_I\overline{\mathcal{F}}_I$, where $\alpha$ is a function that is neither holomorphic nor independent of the scale factor of the metric, but that depends only on the central charge $c$ and the weights $h_i$ and not on the precise details of the model. In fact, $\alpha$ transforms under Weyl transformation with the Liouville action (3.51). In the subsequent discussions we will always assume a normalization such that the factor $e^\alpha$ drops from our expressions. An alternative approach, in which (4.3) is correct as it stands, is to use a Weyl invariant regularization scheme to define all correlation functions. However, in this way general reparametrization invariance is lost.

conformal Ward identities that follow from the existence of null states. These Ward identities determine (holomorphic) partial differential equations for the correlators. The differential equations allow only for a finite number of linear independent solutions, which are the holomorphic blocks $\mathcal{F}_I$. We will explain this with a simple example—the four-point function of the spin field $\sigma(z,\bar{z})$ in the Ising model [16].

Although the correlator depends *a priori* on the four positions $z_1,\ldots,z_4$, global conformal invariance tells us that there is effectively only one independent parameter, the anharmonic ratio $z = z_{12}z_{34}/z_{13}z_{24}$, with $z_{ij} = z_i - z_j$. One way to see this, is to use the global conformal group $SL(2,\mathbf{C})$ and move the three positions $z_1, z_3, z_4$ to, say, $0, 1$, and $\infty$. In this way the correlator becomes a function of only the remaining position $z = z_2$. So, we can restrict our attention to the reduced quantity

$$A(z,\bar{z}) = \lim_{z_\infty \to \infty} |z_\infty|^{\frac{1}{4}} \left\langle \sigma(z_\infty)\sigma(1)\sigma(z,\bar{z})\sigma(0)\right\rangle. \tag{4.4}$$

Here the prefactor compensates for the pole of the propagator for the field $\sigma(z_\infty)$ at infinity. The representation of the Virasoro algebra with $c = \frac{1}{2}, h = \frac{1}{16}$ has a null-state at level 2

$$\left(L_{-2} - \tfrac{4}{3}L_{-1}^2\right)|\tfrac{1}{16}\rangle = 0. \tag{4.5}$$

When we express the operators $L_n$ in contour integrals of $T(z)$, and use the operator products of $T$ and $\sigma$, this implies the following differential equation for $A(z,\bar{z})$ [16]

$$\left(z(1-z)\partial_z^2 + (\tfrac{1}{2} - z)\partial_z + \tfrac{1}{16}\right)\left[|z(1-z)|^{\frac{1}{4}} A(z,\bar{z})\right] = 0, \tag{4.6}$$

together with its anti-holomorphic partner. This second order equation has two independent solutions, for which we choose (with some hindsight)

$$\mathcal{F}_\pm(z) = (z(1-z))^{-\frac{1}{8}}\sqrt{\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1-z}}. \tag{4.7}$$

Similarly the $\overline{\mathcal{F}}_\pm(\bar{z})$'s are the linear independent solutions of the complex conjugate of (4.6). We observe that the functions $\mathcal{F}_\pm(z)$ (the holomorphic blocks) are multi-valued functions on the thrice-punctured Riemann sphere $S^2/\{0,1,\infty\}$. For instance, if we move the point $z$ clock-wise around the point $z = 1$, the blocks transform as

$$\mathcal{F}_\pm(z) \to e^{\frac{i\pi}{4}}\mathcal{F}_\mp(z). \tag{4.8}$$

The full correlation function is of the form

$$A(z,\bar{z}) = \sum_{I,\bar{I}=\pm} c_{I\bar{I}}\,\mathcal{F}_I(z)\overline{\mathcal{F}}_{\bar{I}}(\bar{z}), \tag{4.9}$$

where the constant matrix $c_{I\bar{I}}$ is determined by crossing invariance. That is, the correlation function should be a real, single-valued quantity of the positions $z_i$. It is not difficult to see that the combination that is both local and produces the right two-point functions in the factorization limit is $c_{I\bar{I}} = \delta_{I\bar{I}}$.

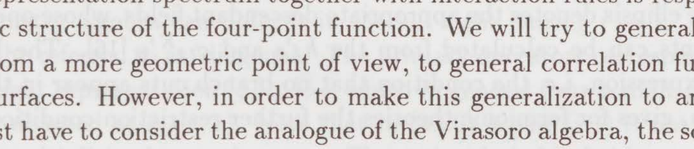
To give a correct interpretation of the holomorphic blocks $\mathcal{F}_I$, we have to take a closer look at the interactions in the Ising model. Recall that the possible interactions are encaptured in the fusion rules, which read in this case

$$\epsilon \times \epsilon = 1, \qquad \sigma \times \sigma = 1 + \epsilon. \tag{4.10}$$

We see that the OPE of two spin fields produces fields in the conformal family of either the identity or the energy operator. Now consider the four point function in the limit where we split the operators into two pairs, and move the pair $(z_3, z_4)$ to infinity. In terms of the reduced quantity $A(z, \bar{z})$ this means taking the limit $z \to 0$, since $|z| \sim r^{-2}$, with $r$ the distance between the two pairs. What do we expect to see? Since we are effectively describing a two-point function, we expect propagation of certain intermediate states. In this limit the holomorphic blocks behave as

$$\mathcal{F}_+ \sim z^{-\frac{1}{8}}, \qquad \mathcal{F}_- \sim z^{-\frac{1}{8}+\frac{1}{2}}. \tag{4.11}$$

We can immediately recognize the two intermediate states: the identity operator with $h = 0$ (plus descendants) in $\mathcal{F}_+$ and the energy operator $\epsilon$ with $h = \frac{1}{2}$ in $\mathcal{F}_-$. Graphically this can be represented in a self-evident notation as




(4.12)

So the finite representation spectrum together with interaction rules is responsible for the analytic structure of the four-point function. We will try to generalize this observation, from a more geometric point of view, to general correlation functions on Riemann surfaces. However, in order to make this generalization to arbitrary RCFT's we first have to consider the analogue of the Virasoro algebra, the so-called chiral algebras.

## 4.2. Chiral Algebras

A special class of primary fields in any conformal field theory is the set of *chiral* or *holomorphic* fields $W_I$ that satisfy $\bar{h}_I = 0$. The vanishing of the anti-holomorphic conformal dimension $\bar{h}_I$ implies that the state $\bar{L}_{-1}|h_I, \bar{h}_I\rangle$ has zero norm and thus vanishes, as can be seen by using the commutator $[\overline{L}_1, \overline{L}_{-1}] = 2\overline{L}_0$. Since the operator $\overline{L}_{-1}$ corresponds to the anti-holomorphic derivative $\partial_{\bar{z}}$, this in turn gives rise to operator relation

$$\partial_{\bar{z}} W_I = 0. \tag{4.13}$$

So all correlation functions of $W_I$ are meromorphic functions of $z$, and we can write $W_I(z)$. Note that $h_I$ equals the spin $s_I = h_I - \overline{h}_I$, so we necessarily have $h_I \in \mathbf{Z}$ for a bosonic field. For fermionic fields $h_I$ can be half-integer, but in that case correlation functions of $W_I(z)$ are only well-defined after a choice of spin structure on the punctured surface.

The Ward identities with respect to the anti-holomorphic coordinate tell us that the operator product of two chiral fields can only produce chiral fields. To see this, consider a three-point function on the sphere

$$\big\langle \phi(w_1, \overline{w}_1) W_I(w_2) W_J(w_3) \big\rangle, \tag{4.14}$$

and an insertion of a contour integral of the anti-holomorphic stress-energy tensor $\overline{T}(\overline{z})$ that does not encircle any of the three operators, and thereby vanishes. By deforming the contour we obtain the relation

$$\begin{aligned} 0 &= \oint \frac{d\overline{z}}{2\pi i} \big\langle \overline{T}(\overline{z}) \phi(w_1, \overline{w}_1) W_I(w_2) W_J(w_3) \big\rangle \\ &= \sum_i \frac{\partial}{\partial \overline{w}_i} \big\langle \phi(w_1, \overline{w}_1) W_I(w_2) W_J(w_3) \big\rangle, \end{aligned} \tag{4.15}$$

and we find that also $\partial_{\bar{z}} \phi = 0$. So, a closed operator algebra can be defined on the set of chiral fields through their OPE

$$W_I(z) \cdot W_J(w) \sim \sum_K c_{IJ}{}^K (z - w)^{-h_{IJK}} W_K(w) + \cdots. \tag{4.16}$$

Here the ellipsis denotes the appropriate descendant fields, whose operator product coefficients can be calculated from the $h_I$'s and $c_{IJ}{}^K$'s [16]. The locality of the above expression, *i.e.* the condition that no branch cuts appear in the three-point function, gives for fermionic theories the further restriction condition that $h_{IJK}$ is integer.

We will denote the collection of the chiral fields $W_I(z)$ and their descendants as the *chiral algebra* $\mathcal{A}$ of the conformal field theory. Of course, we can similarly define the anti-chiral algebra $\overline{\mathcal{A}}$. In the following we will restrict us to conformal field theories whose chiral and anti-chiral symmetry algebra consist of only integer spin generators and both contain the Virasoro algebra at the same value $c$ of the central charge. For convenience we will also take $\mathcal{A}$ and $\overline{\mathcal{A}}$ to be isomorphic—a restriction that is quite reasonable in the context of critical phenomena, but excludes for example heterotic string models. Chiral fields are conserved currents of arbitrary spin, and $\mathcal{A} \times \overline{\mathcal{A}}$ can be regarded as the maximal symmetry algebra of the model. Note that, since $W_I(z)$ is a chiral field, we can introduce its mode decomposition

$$W_I(z) = \sum_n W^I_n z^{-n-h}, \qquad W^I_n = \oint \frac{dz}{2\pi i} W_I(z) z^{n+h-1}, \tag{4.17}$$

and (4.16) can be made into a commutator algebra

$$\left[W^I_n, W^J_m\right] = \sum_K c_{IJ}{}^K \binom{n+h_I-1}{h_{IJK}-1} W^K_{n+m} + \cdots \tag{4.18}$$

### 4.2.1. Representation Theory

We can now consider the action of the algebra $\mathcal{A}$ on the other, non-chiral primary fields $\phi_\alpha(z,\overline{z})$ of the theory, through the operator product coefficients $c_{I\alpha}{}^\beta$

$$W_I(z) \cdot \phi_\alpha(w,\overline{w}) \sim \sum_\beta c_{I\alpha}{}^\beta (z-w)^{-h_{I\alpha\beta}} \phi_\beta(w,\overline{w}) + \cdots. \tag{4.19}$$

In this way the representations $[\phi_\alpha]$ of the Virasoro algebra form orbits under the action of $\mathcal{A}$, and we can combine one orbit into one irreducible 'highest weight' representation $[\phi_i]$ of the chiral algebra $\mathcal{A}$. The orbit $[\phi_0]$ of the identity corresponds to chiral algebra itself. Thus, the existence of the chiral algebra $\mathcal{A}$ implies that the Hilbert space $\mathcal{H}$ will have the structure

$$\mathcal{H} = \bigoplus_{(i,\overline{\imath})} [\phi_i] \otimes [\overline{\phi}_{\overline{\imath}}]. \tag{4.20}$$

In this way the field content is organized in representations of the full symmetry algebra $\mathcal{A} \times \overline{\mathcal{A}}$ of the model. We will argue that the indices $i$ and $\overline{\imath}$ range over the same set of labels and are paired in a one-to-one fashion. The only way models with the same symmetry algebra can differ is in the left-right pairing $i \to \overline{\imath}$. A model with $\overline{\imath} = i$ will be referred to as 'diagonal.' The primary operators corresponding to the highest weight representation $[\phi_i] \otimes [\overline{\phi}_{\overline{\imath}}]$, which in general form a multiplet, will be collectively denoted as $\phi_{i\overline{\imath}}(z,\overline{z})$. Note that in (4.20) a pair $(i,\overline{\imath})$ is not allowed to occur with multiplicity larger than one. The rationale behind this assumption is that there will always be some quantum number that distinguishes between two non-identical operators. However, we might have to include a discrete symmetry group in order to make this distinction. Charge conjugation will be denoted as $C: i \to \hat{\imath}$.

The one-loop partition function on the torus $z \sim q\,z$ is given by a trace over the Hilbert space $\mathcal{H}$, and decomposes as

$$Z(q,\overline{q}) = \mathrm{Tr}_{\mathcal{H}}\left(q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}}\right) = \sum_{(i,\overline{\imath})} \chi_i(q) \chi_{\overline{\imath}}(\overline{q}), \tag{4.21}$$

with the one-loop characters $\chi_i(q)$ defined by

$$\chi_i(q) = \mathrm{Tr}_{[\phi_i]}(q^{L_0 - \frac{c}{24}}). \tag{4.22}$$

Of course, it is well possible that the characters of two different representations $[\phi_i]$ have the same $q$-expansion.*

If we impose the condition of finite factorization (4.3) on the genus one partition function, we immediately see that for a *rational* conformal field theory, the number of irreducible representations of $\mathcal{A}$ that occur in (4.20) is *finite*.

### 4.2.2. *Chiral Algebras of Compact Groups*

What are possible chiral algebras that occur as symmetry algebras of conformal models? In the simplest case, that appears in the minimal models, the only *primary* holomorphic field is the identity. In that case the chiral algebra consists of just the descendants of the identity, *i.e.* the stress-energy tensor $T(z)$ and polynomials in $T(z)$ and its (multiple) derivatives: the universal enveloping algebra of the Virasoro algebra.

Another very important class of chiral algebras are the algebras $\mathcal{A}(G)$ associated to a compact Lie group $G$. These algebras are the maximal symmetry algebras of the so-called Wess-Zumino-Witten (WZW) models [158]—sigma models that describe string propagation on the group manifold $G$. The WZW model is formulated in terms of a $G$-valued field $g(z,\bar{z})$. Its action is discussed in section 8.3.1, and requires the choice of an integer parameter $k$, the so-called 'level.'

It is well-known that these models have an extended symmetry, because we have spin one currents $J(z) = -\frac{1}{2}k\, \partial_z g\, g^{-1}$, in the adjoint representation of $G$, that generate the Kac-Moody algebra associated to $Lie(G)$ of level $k$ [109,83]. With a set of generators $T^a$ of $Lie(G)$ satisfying $[T^a, T^b] = if^{abc}T^c$ and $\mathrm{Tr}\, T^aT^b = \frac{1}{2}\delta^{ab}$, the components $J^a(z)$ have OPE's

$$J^a(z)J^b(w) \sim \frac{\frac{1}{2}k\delta^{ab}}{(z-w)^2} + \frac{if^{abc}J^c(w)}{z-w}. \tag{4.23}$$

The stress-energy tensor is of the Sugawara form, and generates a Virasoro algebra of central charge

$$c = \frac{k \dim G}{k+h}, \tag{4.24}$$

with $h$ the dual Coxeter number of $G$.

*An example is furnished by the 3-state Potts model ($m = 5$ in (4.1)) which has characters of weight $h = \frac{1}{15}$ and $h = \frac{2}{3}$ that are both twice degenerated. This degeneracy is resolved by a discrete $\mathbf{Z}_3$ symmetry [62].

For simply connected $G$ the chiral algebra $\mathcal{A}(G)$ equals the universal enveloping algebra of the affine Kac-Moody algebra (and as a vector space the basic representation $[\phi_0]$.) The other representations of $\mathcal{A}(G)$ are given by the integrable representations $[\phi_i]$ of the affine algebra [83]. The primary fields $\phi_{i\bar{i}}(z,\bar{z})$ transform in the representation $R_i \otimes R_{\bar{i}}$ of the global symmetry group $G \times G$, with highest weights $(\mu_i, \mu_{\bar{i}})$. The conformal dimensions are given

$$h_i = \frac{\langle \mu_i, \mu_i + 2\rho \rangle}{2(k+h)}, \tag{4.25}$$

with $\rho$ half the sum of the positive roots.

In order to discuss the chiral algebra for non-simply connected $G$, we have to make a small digression. There is an alternative way to obtain the chiral algebra and that is via holomorphic quantization [132,64,122]. Consider the space of free loops $LG$, *i.e.* maps $S^1 \to G$. This is the configuration space of the group manifold model in a Hamiltonian approach. The space of based loops $LG/G$ consists of maps $g(\sigma)$ that satisfy $g(0) = 1$. There exists a natural holomorphic line bundle $\mathcal{L}$ over $LG/G$, that can be constructed from the Wess-Zumino term. By geometric quantization we recover $\mathcal{A}(G)$ as the space of holomorphic sections of $\mathcal{L}$

$$\mathcal{A}(G) = Hol(\mathcal{L}). \tag{4.26}$$

This line bundle and its space of holomorphic sections can still be defined for arbitrary compact groups. However, if $\pi_1(G) \neq 0$, $LG$ will have several components, since $\pi_0(LG) = \pi_1(G)$. The sections over the identity component will give the current algebra of $Lie(G)$ at level $k$. The holomorphic sections over the other components of $LG/G$, that correspond to the elements $z_a \neq 1$ of $\pi_1(G)$, will give rise to special representations $[\phi_a]$ of the current algebra. These representation can be seen as creation operators of vortices of the field $g(z,\bar{z})$ of winding number $z_a$. The full chiral algebra is given by direct sum of the $[\phi_a]$. Of course, the conformal weight of $\phi_a$ should be integer, and this implies a quantization condition on the level $k$. We will return to this point in section 8.3.4.

A group manifold model on a non-connected group does not really make much sense. However, there is a sensible definition of the chiral algebra $\mathcal{A}(G)$ [122]. If $G$ has more than one connected component, we have an exact sequence

$$1 \to G_0 \to G \to \Gamma \to 1, \tag{4.27}$$

with $G_0$ the identity component and $\Gamma$ the discrete group of components. We can write $G = G_0 \rtimes \Gamma$. The chiral algebra $\mathcal{A}(G)$ consists of the $\Gamma$-invariant sections of the line bundle $\mathcal{L}$ over $LG_0/G_0$. This is the symmetry algebra of a $\Gamma$ orbifold model of the $G_0$ WZW model, as will be discussed in Chapter 5 and 8.

Another important class of chiral algebras that we should certainly mention here are the so-called Casimir algebras [13,138] that are associated to coset models [87]. For a subgroup $H \subset G$ these algebras are roughly defined to be the set of all operators in $\mathcal{A}(G)$ that commute with $\mathcal{A}(H)$.

## 4.3. Interactions and Fusion Algebras

Let us now proceed to include interactions. As we have discussed in section 3.3 all information about the interactions in a CFT is contained in the three-point functions. A CFT assigns to a three-punctured sphere the three-point vertex $|V\rangle \in \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$. It is defined by

$$\langle \phi_1, \phi_2, \phi_3 | V \rangle = \langle \phi_1 | \phi_2(1) | \phi_3 \rangle. \tag{4.28}$$

Here we used the fact that the three-punctured sphere has no moduli, to fix the three punctures at $0, 1, \infty$. The three-point functions of descendants (with respect to the Virasoro algebra) are completely determined from the corresponding primary correlator. Therefore the vertex $|V\rangle$ is specified by its matrix elements with Virasoro primary states. Using the form (4.20) of the Hilbert space $\mathcal{H}$, we can decompose $|V\rangle$ as follows:

$$|V\rangle = \sum_{i,j,k} \sum_{\alpha} |V_{ijk}^{\alpha}\rangle \otimes |\overline{V_{\bar{i}\bar{j}\bar{k}}^{\alpha}}\rangle, \tag{4.29}$$

with the chiral vertices

$$|V_{ijk}^{\alpha}\rangle \in [\phi_i] \otimes [\phi_j] \otimes [\phi_k]. \tag{4.30}$$

These chiral vertices have to satisfy the Ward identities with respect to $\mathcal{A}$. These can be given quite generally as follows. Given a chiral field $W(z) \in \mathcal{A}$ of integer spin $h$ and a differential $\rho(z)$ of weight $1-h$, holomorphic on $S^2/\{0,1,\infty\}$ (but possibly with poles at $z = 0, 1, \infty$) we will have the relation

$$\sum_{w=0,1,\infty} \oint_w \frac{dz}{2\pi i} W(z)\rho(z)|V_{ijk}\rangle = 0. \tag{4.31}$$

This equation can be understood by deforming the contour just as in (4.15). It relates the action of $\mathcal{A}$ on the three representations at the punctures. In fact, the three-point vertices $|V_{ijk}^{\alpha}\rangle$ define a natural tensor product structure for the representations $[\phi_i]$. One should note, that the naive tensor product $[\phi_i] \otimes [\phi_j]$ is not appropriate for an operator algebra with central extensions. For example, if we have a field $W$ that satisfies a relation of the form $W \cdot W \sim c \cdot 1$ (like the stress-energy tensor), the representation $W' = W \otimes 1 + 1 \otimes W$, which would be the naive implementation of the action of $W$ on the tensor product, would reproduce an algebra with central charge $c' = 2c$.

So the chiral vertices should be considered as the invariant couplings or Clebsch-Gordan coefficients of $\mathcal{A}$. These couplings span a vector space, that is finite dimensional for a rational theory by application of condition (4.3) to the three-point vertex. This finite dimension will be denoted as $N_{ijk}$. Note that by the permutation properties of the vertex $|V\rangle$ the symbol $N_{ijk}$ is totally symmetric. Since the sectors $[\phi_i]$ are representations of $\mathcal{A}$ we furthermore have $N_{ij0} = C_{ij}$ with $C$ the charge-conjugation matrix. The representation algebra of $\mathcal{A}$ is known as the *fusion algebra*. It is defined as the associative, commutative algebra [153]

$$\phi_i \times \phi_j = \sum_k N_{ij}{}^k \phi_k. \tag{4.32}$$

Indices are raised and lowered by the conjugation matrix $C_{ij} = \delta_{ij}$.

The existence of more than one independent chiral coupling is a general feature of RCFT's that is not present in the minimal models. There the conformal Ward identities relate the couplings of all descendants to the primary correlator, which either vanishes or not. This gives us $N_{ijk} = 0$ or 1. For affine models more independent couplings can exist, because the primary fields form multiplets of the finite dimensional Lie group $G$. In fact, for these models the fusion algebra will always be a truncation of the representation algebra of $G$, since interactions should be $G$-invariant. However, in general it is the coupling between primary and descendant fields that is responsible for $N_{ijk} > 1$, as we will demonstrate with a simple example [46,25,120].

### 4.3.1. $SU(2)$ Affine Models

As is well-known, the $SU(2)$ level $k$ Kac-Moody algebra allows for $k+1$ integrable representations $[\phi_j]$ $(2j = 0, 1, \ldots, k)$ with $SU(2)$ isospin $j$ [83]. The weights of the corresponding primary fields are given by $h_j = \frac{j(j+1)}{k+2}$. The fusion algebra is a truncation of the $SU(2)$ representation algebra

$$\phi_j \times \phi_{j'} = \sum_{j''=|j-j'|}^{\frac{1}{2}k - |\frac{1}{2}k - j - j'|} \phi_{j''}. \tag{4.33}$$

This truncation is a result of the existence of null-states.

The investigation of the modular invariant partition functions that can be constructed with the corresponding one-loop characters has yielded the famous $ADE$ classification of Cappelli, Itzykson, and Zuber [28]. All modular invariant combinations can be labeled by the Dynkin diagrams of the simply-laced Lie algebras

$A_n, D_n, E_6, E_7, E_8$. The $A_n$ series corresponds to the diagonal sum over all representations. The $D_{even}$, $E_6$, and $E_8$ models are diagonal combinations with an extended chiral algebra, which in the latter two cases has been identified with the $C_2$ and $G_2$ level one Kac-Moody algebra [23]. The $D_{odd}$ series and $E_7$ are non-diagonal combinations and will be discussed in section 4.6.

The models in the $D$ series have even level $k$ and correspond to $SO(3)$ group manifolds models. For the $D_{even}$ models, with $k = 0 \pmod 4$, the field $W \equiv \phi_{\frac{k}{2}}$ is seen to have an integer dimension $h = \frac{k}{4}$. Accordingly, we can consider an extension $\mathcal{A}$ of the $SU(2)$ chiral algebra with this extra chiral field $W(z)$. This extended algebra is the chiral algebra $\mathcal{A}(SO(3))$ [122]. Using (4.33) it is seen to satisfy $W \times W = 1$, so the extension indeed closes. Furthermore, we have

$$W \times \phi_j = \phi_{\frac{k}{2}-j}, \tag{4.34}$$

which implies that the extended representations $[\tilde{\phi}_j]$ will be given by

$$[\tilde{\phi}_j] = [\phi_j] \oplus [\phi_{\frac{k}{2}-j}]. \tag{4.35}$$

Only integer spins $j$ can occur here, because the locality of relation (4.34) demands $h_j = h_{\frac{k}{2}-j} \pmod 1$.

We can assign a $\mathbf{Z}_2$ charge to the affine representations $[\phi_j]$ which is even if $j < \frac{k}{4}$ and odd for $j > \frac{k}{4}$. (The representation with $j = \frac{k}{4}$ doubles in the $D_n$ models, and its interactions should be treated separately.) This implies a corresponding grading of all $SU(2)$ primary fields, where the odd elements will be descendants with respect to the extended algebra. In particular, we have a grading of $\mathcal{A}$ itself, with $W$ odd. This grading is preserved in the action of $\mathcal{A}$ on the representations. Consequently, the interaction vertex $|V\rangle$ will split into an even and odd part

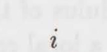
$$|V\rangle = |V\rangle^{+} + |V\rangle^{-}. \tag{4.36}$$

The Ward identities corresponding to the extended symmetry will always interrelate correlators of an equal charge, *i.e.* within a given vertex $|V\rangle^{\pm}$. This implies that *a priori* the $N_{ijk}$ for the extended algebra can equal $0, 1$, or $2$, since there are at most two independent couplings. For a coupling $\tilde{\phi}_{j_1} \times \tilde{\phi}_{j_2} = 2\,\tilde{\phi}_{j_3}$, we need both $\phi_{j_1} \times \phi_{j_2} = \phi_{j_3}$ and $\phi_{j_1} \times \phi_{j_2} = \phi_{\frac{k}{2}-j_3}$. It is not difficult to find examples of this using the fusion rules (4.33). For example, for $k = 12$ one finds $\tilde{\phi}_2 \times \tilde{\phi}_2 = 2\,\tilde{\phi}_2 + \cdots$ [25].

## 4.4. Holomorphic Blocks and Duality Transformations

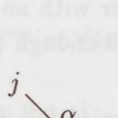
We will now proceed to show that the presence of a finite number of representations $\phi_i$ with a finite number of independent couplings $N_{ijk}$ accounts for the finite factorization of the multi-loop amplitudes. Let us consider an $n$-point amplitude on genus $g$ Riemann surface $\Sigma$, as in (4.3). Let $m$ be a $3g-3+n$ dimensional, complex coordinate on the moduli space $\mathcal{M}_{g,n}$. The Riemann surface $\Sigma$ can be obtained by sewing together $2g-2+n$ trinions or pants. That is to say, we can describe it by a collection of three-punctured spheres $Y_i \cong S^2 \backslash \{0,1,\infty\}$ and a set of holomorphic transition functions $f_{ij}: U_i \rightarrow U_j$, where $U_i$ is a neighborhood of one of the punctures on $Y_i$. This decomposition in trinions can be represented by marking the Riemann surface with non-intersecting, primitive homotopy cycles, indicating the 'sewing stitches.'

The amplitude can be obtained by similarly sewing three-point vertices $|V\rangle$ [146]. Here the sewing consists of a summation over all intermediate states. If we restrict all these summations to a particular representation $[\phi_i] \otimes [\overline{\phi}_{\overline{i}}]$ and choose a definite chiral vertex $|V^{\alpha}_{ijk}\rangle$ for each coupling, the summations over left- and right-movers decouple. Since left-movers couple holomorphically to the moduli $m$ through $T(z)$, we obtain a factorized amplitude $\mathcal{F}_I(m)\overline{\mathcal{F}}_I(\overline{m})$, with $\mathcal{F}_I(m)$ the holomorphic block. We denote by $V_\Sigma$ the vector space of $\mathcal{F}_I$'s. The map $\Sigma \rightarrow V_\Sigma$ is called a *modular functor* [139]. The individual blocks can be labeled by a $\varphi^3$-diagram, as in *fig.* 6, with propagators



(4.37)

and vertices



(4.38)

where we denoted the type of coupling by the index $\alpha = 1, \ldots, N_{ijk}$. For complex representations the propagators carry an orientation. We note however that the correspondence with $\varphi^3$-diagrams is not unambiguous. We must also choose a set of dual cycles that fix the possible phase ambiguities due to rotations of the boundaries of the trinions before gluing. This can be considered as defining a local direction of time in the intermediate channels. For example the genus one characters $\chi_i(\tau)$ are defined by cutting the torus along the $A$-cycle. The $B$-cycle defines the time flow, and fixes the phase ambiguity $\chi_i(\tau) \rightarrow \chi_i(\tau) e^{n 2\pi i (h_i - c/24)}$.

This construction is particularly clear in the vicinity of the compactification divisor, where we can consider the singular limit of the punctured surface with all intermediate channels pinched. Here we can represent the Riemann surface as a thickened version of the $\varphi^3$-diagram. That is, a set of $2g-2+n$ thrice-punctured

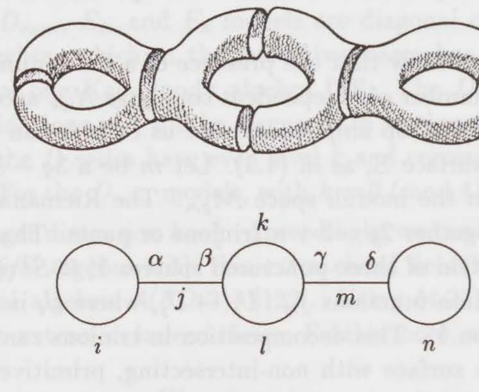



FIGURE 6: The holomorphic blocks can be labeled by $\varphi^3$-diagrams that indicate how the Riemann surface is decomposed into pants. The propagators are labeled by the representations $i, j, \ldots$, and the vertices by the invariant couplings $\alpha, \beta, \ldots$.

spheres connected with $3g-3+n$ infinitely thin tubes. The holomorphic transition functions relate local coordinates $z, z'$ on two spheres by the 'plumbing fixture' $zz' = q$. Here $q$ is the complex modulus of the connecting tube, $-\log|q|$ can be regarded as its length. The $q$'s form a local coordinate on the moduli space $\mathcal{M}_{g,n}$. The chiral vertices are sewn together with an insertion of $q^{L_0}$. This perturbative expression for $\mathcal{F}_I$ converges for small enough $q$. Analytical continuation defines it over all of $\mathcal{M}_{g,n}$.

However, the moduli space $\mathcal{M}_{g,n}$ is not a smooth manifold. In general the holomorphic blocks are only single-valued, analytic functions on its universal analytical covering, the Teichmüller space $T_{g,n}$. This implies that on the actual moduli space they are sections of a possibly nontrivial vector bundle, the Friedan-Shenker bundle $V_{g,n}$ [78]. The fibre of this bundle at a Riemann surface $\Sigma \in \mathcal{M}_{g,n}$ is the vector space $V_\Sigma$. Naively, the definition of the sections $\mathcal{F}_I(m)$ is locally unambiguous, and suffers only from monodromy ambiguities. This would imply that the bundle $V_{g,n}$ is flat, *i.e.* determined by a (unitary) representation of the mapping class group. However, as argued in [78] for genus $g > 1$ this vector bundle is only projectively flat, *i.e.* up to a phase, due to the conformal anomaly.

So each set of holomorphic blocks $\mathcal{F}_I$, as constructed by gluing together chiral 3-point functions, defines a basis in the vector bundle $V_{g,n}$. The basis $\mathcal{F}_I$ is completely determined by the cycles that cut the punctured surface into trinions and a set of dual cycles. The full amplitude $A$ (in a suitable normalization of the $\mathcal{F}_I$'s) is given by

$$A(m, \overline{m}) = \sum_I \mathcal{F}_I(m)\overline{\mathcal{F}}_{\overline{I}}(\overline{m}). \tag{4.39}$$

Of course there is no unique marking of the surface. Different choices of intermediate channels furnish different bases in the vector space $V_\Sigma$, corresponding to factorization limits in different 'corners' of moduli space. The completeness of each basis, corresponding to a $\varphi^3$-diagram of a particular type, is a consequence of the consistency of the model, and is of fundamental importance in all further considerations. It is known as the *postulate of duality.* That there is always a *linear* transformation that relates two choices $\mathcal{F}_I$ and $\mathcal{F}'_I$, can be seen from the duality condition

$$\sum_I \mathcal{F}_I(m)\overline{\mathcal{F}}_{\overline{I}}(\overline{m}) = \sum_I \mathcal{F}'_I(m)\overline{\mathcal{F}}'_{\overline{I}}(\overline{m}). \tag{4.40}$$

By a simple argument [121] this implies immediately that the two sets of holomorphic blocks are necessarily related by a linear, unitary transformation

$$\mathcal{F}'_I(m) = A_{IJ}\mathcal{F}_J(m). \tag{4.41}$$

The transformation matrices $A_{IJ}$ are called duality transformations. Moore and Seiberg have made a careful analysis of the duality 'groupoid,' and have given a complete set of generators and relations [121]. Modular and monodromy transformations are particular examples of duality transformations.

## 4.5. RCFT at Genus One

Much of the structure of a conformal field theory can be analyzed by considering its properties on a genus one Riemann surface. As first shown by Cardy [30] the one-loop partition function gives us the operator spectrum of the theory. It is a quite remarkable result due to E. Verlinde [153] that also the interactions of the model follow from genus one considerations.

Since the fusion algebra (4.32) is an associative, commutative algebra with an inner product, it will be diagonalized by some matrix, just as in our discussion in section 3.2. It was conjectured in [153], and consequently proved in [119,121] (see also [48]), that the matrix that diagonalizes the fusion algebra (4.32) is given by the modular transformation $S : \tau \to -1/\tau$ that acts on the characters $\chi_i(\tau)$. This relation implies that the irreducible one-dimensional representations of the fusion algebra

$$\lambda_i^{(n)}\lambda_j^{(n)} = \sum_k N_{ij}{}^k\lambda_k^{(n)}, \tag{4.42}$$

are given by (*cf* (3.31))

$$\lambda_i^{(j)} = \frac{S_{ij}}{S_{0j}}. \tag{4.43}$$

Furthermore, the dimensions $N_{ijk}$ can be expressed as (*cf* (3.32))

$$N_{ijk} = \sum_n \frac{S_{in} S_{jn} S_{nk}}{S_{0n}}. \tag{4.44}$$

One can use the result (4.44) to prove that the matrix $S$ is *symmetric* and *unitary*, irrespective of whether the CFT itself is unitary [48]. The first property follows from the permutation symmetry of the $N_{ijk}$, and is in fact equivalent to it. We can now use in addition the invariance of the fusion algebra under conjugation $C : \phi_i \to \phi_{\bar{i}}$, which implies $\lambda_{\bar{i}}^{(j)} = \lambda_i^{*(j)}$. Then from (4.43) we find $CS = S^*$, which together with $S^2 = C$ gives unitarity.

For unitary CFT's, which have positive conformal weights, the quantities $\lambda_i^{(0)}$ have a special interpretation. They indicate the relative dimension $[\phi_i : \phi_0]$ of the representation $[\phi_i]$ compared to identity representation $[1] = [\phi_0]$. Hereto we use the fact that in the limit $q \to 1$ the character $\chi_i(q)$ will be an unweighted trace over the representation $[\phi_i]$, and formally counts the number of states in $[\phi_i]$. The modular transformation $S$ allows us to exchange this limit by $q \to 0$. More precisely we have

$$[\phi_i : \phi_0] = \lim_{q \to 1} \frac{\chi_i(q)}{\chi_0(q)} = \lim_{q \to 0} \frac{\sum_j S_{ij} \chi_j(q)}{\sum_j S_{0j} \chi_j(q)} = \frac{S_{i0}}{S_{00}} = \lambda_i^{(0)}. \tag{4.45}$$

From this we conclude that $\lambda_i^{(0)} > 0$. The relative dimensions, which are not necessarily integer, satisfy the fusion algebra, and hence we even have $\lambda_i^{(0)} \geq 1$, where equality holds iff $\phi_i \times \phi_{\bar{i}} = 1$.

The proof of the relations (4.43-4.44) is based on the construction of a set of operators $\phi_i(C)$ for each cycle $C$ of the torus $\Sigma$, that act on the vector space of characters $V_\Sigma$. The definition of the operator $\phi_i(C)$ is as follows [153]. One inserts the identity operator inside the trace over $[\phi_j]$ at some point on the torus, and rewrites it as the OPE of an operator in the representation $[\phi_i]$ and its conjugate. Next one moves this operator around the cycle $C$, and when it is returned to its original position, one takes again the OPE with the conjugate operator to reobtain the identity. This manipulation defines a linear operator on the space of characters, and is, upto normalization, equal to $\phi_i(C)$. By applying the operation $\phi_i(B)$ to the character $\chi_0$ of the identity, one finds that the representation $[\phi_0]$ changes into $[\phi_i]$. We use this fact to fix the normalization of the operators $\phi_i(C)$ by the condition

$$\phi_i(B)\, \chi_0 = \chi_i. \tag{4.46}$$

The key equation is that, with this normalization, the operator $\phi_i(B)$ acts on the other characters as

$$\phi_i(B)\,\chi_j = N_{ij}{}^k \chi_k. \tag{4.47}$$

Intuitively an equation of this form can be expected, but the integrality of the coefficients is not obvious from the definition of $\phi_i(B)$. The rigorous proof involves a consideration of the three-point function on the torus*. Here a consistency condition obeyed by the duality groupoid, called the pentagon identity [119,121], leads to the above relation. Granted equation (4.47), the proof of relation (4.43) is immediate. First we observe that, when we move an operator with representation $[\phi_i]$ around the $A$-cycle, the representation $[\phi_j]$, over which the trace is taken, does not change. Consequently the characters $\chi_j$ are eigenstates of the operators $\phi_i(A)$

$$\phi_i(A)\,\chi_j = \lambda_i^{(j)} \chi_j. \tag{4.48}$$

Since the modular transformation $S$ interchanges the $A$ and $B$-cycle, the operator $\phi_i(B)$ is, by conjugation with the matrix $S$, mapped to $\phi_i(A)$. By comparing (4.47) and (4.48) we immediately conclude that the modular transformation $S$ diagonalizes the coefficients $N_{ijk}$. Finally, the condition (4.46) enables us to determine the eigenvalues $\lambda_i^{(j)}$, and express everything in terms of the entries of the matrix $S$. The resulting expressions are given in (4.44).

## 4.6. Non-diagonal Invariants and Algebra Automorphisms

The fusion algebra is clearly related to the selection rules for the operator algebra of the primary fields $\phi_{i\bar{\imath}}(z,\bar{z})$. However, these are not really one and the same thing, since the fusion algebra, as we define it, concerns only one chiral half of the CFT. In fact, one can associate two fusion algebras with a CFT, one for the left and one for the right representations; the primary fields $\phi_{i\bar{\imath}}$ feel both of them. The coefficients $N_{ijk}$ and $N_{\bar{\imath}\bar{\jmath}\bar{k}}$ give us information about the left and right chiral blocks, which have to be combined in a consistent way to build the correlation functions. We therefore expect some relation between $N_{ijk}$, $N_{\bar{\imath}\bar{\jmath}\bar{k}}$ and the allowed representations $(i,\bar{\imath})$ of the primary fields.

First, let us consider what modular invariance tells us. By writing the partition function as $Z = \overline{\chi}_i \Pi_{ij} \chi_j$, we define a matrix $\Pi_{ij}$, with non-negative, integer entries. Full modular invariance of the partition function gives two conditions. First, the spins $h_i - h_{\bar{\imath}}$ are required to be integer if $\Pi_{i\bar{\imath}} \neq 0$. Furthermore, invariance under $\tau \to -1/\tau$ tells us that $S^\dagger \Pi S = \Pi$. Since $S$ is unitary, this is equivalent to

$$S\,\Pi = \Pi\,S. \tag{4.49}$$

*Recently other proofs have been given by Witten [162] and Cardy [34].

Note that for every chiral algebra there always exists a diagonal modular invariant with $\Pi_{ij} = \delta_{ij}$. However, in this section we are mainly interested in the non-diagonal invariants, because most of the things we are about to discuss are more or less trivial in the diagonal case.

For the following it is important that we consider the maximal chiral symmetry algebra of the model. In other words, we assume that all chiral operators are contained in the identity representation $[\phi_0] = [1]$. This can be stated as

$$\Pi_{i0} = \Pi_{0i} = \delta_{i0}. \tag{4.50}$$

We now like to show that (4.49) and (4.50) imply that each representation $[\phi_i]$ occurs precisely once and is uniquely paired with a representation $[\overline{\phi}_{\bar{\imath}}]$ of the right algebra. More precisely, we will prove that the integral matrix $\Pi$ defines a one-to-one mapping, *i.e.* a permutation, of the representations

$$\Pi : \phi_i \to \phi_{\bar{\imath}}, \tag{4.51}$$

and furthermore that this mapping gives an automorphism of the fusion algebra. Our proof works only for unitary CFT's, but we believe that these facts also hold for non-unitary models.

We first consider the relative dimensions defined in (4.45). Combining the relations (4.49) and (4.50) gives

$$\lambda_i^{(0)} = \Pi_{ij} \lambda_j^{(0)} \tag{4.52}$$

Now we make use of the observation that for unitary CFT's all relative dimensions satisfy $\lambda_i^{(0)} \geq 1$. Since $\Pi$ has non-negative integer entries, equation (4.52) clearly implies that for each representation $i$ there is one and precisely one representation $\bar{\imath}$ such that $\Pi_{ij} = \delta_{\bar{\imath}j}$, and hence $\Pi$ is indeed of the form (4.51).

Next we consider the interplay between this mapping and the left and right fusion algebras. Using the condition (4.49) together with (4.52) one easily shows that for the one-dimensional representations one has: $\lambda_{\bar{\imath}}^{(j)} = \lambda_i^{(\bar{\jmath})}$. This in turn is sufficient to prove that the integer fusion coefficients satisfy

$$N_{\bar{\imath}\bar{\jmath}\bar{k}} = N_{ijk}, \tag{4.53}$$

which shows that (4.51) is an automorphism of the fusion algebra.

Another immediate consequence of (4.53) is that for any correlation function of the primary fields $\phi_{i\bar{\imath}}$ the numbers of left and right chiral blocks are equal. This suggests that also in this more general case the correlator defines a one-to one pairing between the chiral blocks, such that corresponding blocks transform in isomorphic representations of the modular group. In particular, by considering the three-point function of the primary fields we may conclude that the selection rules for the operator product algebra of the primary fields $\phi_{i\bar{\imath}}(z,\bar{z})$ are determined by only one of the labels $i$ or $\bar{\imath}$. Consequently, these selection rules have the same form as the fusion algebra. For non-diagonal CFT's this a nontrivial fact, which follows from the modular invariance of the one-loop partition function, and is some cases even equivalent to it.

#### 4.6.1. *Non-diagonal $SU(2)$ Models*

To illustrate the relation between automorphisms of the fusion algebra and non-diagonal modular invariant partition functions we return to the $SU(2)$ WZW-models, but now consider the non-diagonal combinations $D_{odd}, E_7$.

The $D_{2n+1}$ combinations occur at level $k = 4n - 2$. In the partition function the same characters occur as in the corresponding $A_n$ form, only the pairing is not diagonal. More precisely, in these models the left and right representation $[\phi_j]$ and $[\overline{\phi}_{\bar{\jmath}}]$ are paired as

$$\begin{aligned} \bar{\jmath} &= \tfrac{1}{2}k - j, && \text{if } j \in \mathbf{Z} + \tfrac{1}{2}, \\ \bar{\jmath} &= j, && \text{if } j \in \mathbf{Z}. \end{aligned} \tag{4.54}$$

The fusion rules are of course identical to those of the $A_n$ series (4.33), since they only depend on the chiral algebra, not on the left-right pairing.

It is not difficult to verify that the transformation (4.54) constitutes an automorphism of the fusion rules (4.33), and to verify that there are no other automorphisms. Note that this is true for all even $k$. The fact that the $D_{2n+1}$ invariants only occur at $k = 2 \pmod 4$ follows from the condition $h_j = h_{\bar{\jmath}} \pmod 1$.

A more peculiar example is the single exceptional, non-diagonal modular invariant $E_7$. The corresponding diagonal invariant is the one labeled by $D_{10}$. If we follow conventions and denote the spin $j$ characters by $\chi_{2j+1}$, the two partition functions are given by

$$\begin{aligned} Z_{D_{10}} &= |\chi_1 + \chi_{17}|^2 + |\chi_3 + \chi_{15}|^2 + |\chi_5 + \chi_{13}|^2 \\ &\quad + |\chi_7 + \chi_{11}|^2 + 2|\chi_9|^2, \end{aligned} \tag{4.55}$$

$$\begin{aligned} Z_{E_7} &= |\chi_1 + \chi_{17}|^2 + |\chi_5 + \chi_{13}|^2 + |\chi_7 + \chi_{11}|^2 \\ &\quad + |\chi_9|^2 + [(\chi_3 + \chi_{15})\overline{\chi}_9 + c.c.]\,. \end{aligned} \tag{4.56}$$

We see that in both models the character $\chi_9$ occurs twice, and accordingly two different fields with $j = 4$ have to be distinguished. We shall label the characters as $\chi_9^+$ and $\chi_9^-$. Both the $D_{10}$ and $E_7$ combinations have the same chiral algebra, the $SO(3)$ $k = 16$ algebra. The characters are given by $\chi_1 + \chi_{17}$, $\chi_3 + \chi_{15}$, $\chi_5 + \chi_{13}$, $\chi_7 + \chi_{11}$, $\chi_9^+$, and $\chi_9^-$. The corresponding operators will be denoted as $1$, $\phi_1$, $\phi_2$, $\phi_3$, $\phi_4^+$, and $\phi_4^-$. The non-diagonal $E_7$ pairing is given by $\phi_1 \leftrightarrow \phi_4^+$.

Since both models have the same chiral algebra, the fusion rules will be identical. We will derive them from the modular behavior of the characters. The matrix $S$ is most easily obtained by folding up the $SU(2)$ $k=16$ matrix and requiring $S^2=1$. The result reads upto normalization, in a basis $(\phi_j, \phi_4^+, \phi_4^-)$

$$S = \begin{pmatrix} 2s_{jj'} & (-1)^j & (-1)^j \\ (-1)^{j'} & 2 & -1 \\ (-1)^{j'} & -1 & 2 \end{pmatrix}, \tag{4.57}$$

with $s_{jj'} = \sin((2j+1)(2j'+1)\pi/18)$. From this we get the following set of fusion rules for these representations

$$\begin{aligned}
\phi_1 \times \phi_1 &= 1 + \phi_1 + \phi_2, & \phi_4^+ \times \phi_4^+ &= 1 + \phi_2 + \phi_4^+, \\
\phi_1 \times \phi_2 &= \phi_1 + \phi_2 + \phi_3, & \phi_2 \times \phi_4^+ &= \phi_2 + \phi_3 + \phi_4^+, \\
\phi_1 \times \phi_3 &= \phi_2 + \phi_3 + \phi_4^+ + \phi_4^-, & \phi_3 \times \phi_4^+ &= \phi_1 + \phi_2 + \phi_3 + \phi_4^-, \\
\phi_1 \times \phi_4^+ &= \phi_3 + \phi_4^+, & \phi_4^+ \times \phi_4^- &= \phi_1 + \phi_3,
\end{aligned}$$
$$\begin{aligned}
\phi_2 \times \phi_2 &= 1 + \phi_1 + \phi_2 + \phi_3 + \phi_4^+ + \phi_4^-, \\
\phi_3 \times \phi_3 &= 1 + \phi_1 + 2\,\phi_2 + 2\,\phi_3 + \phi_4^+ + \phi_4^-, \\
\phi_2 \times \phi_3 &= \phi_1 + \phi_2 + 2\,\phi_3 + \phi_4^+ + \phi_4^-.
\end{aligned} \tag{4.58}$$

It is readily verified that the $E_7$ pairing $\phi_1 \leftrightarrow \phi_4^+$ is indeed a symmetry of these fusion rules. It is also not hard to convince oneself that this automorphism does not generalize to the whole $D_{even}$ series, and that $D_{10}$ is in fact the only possibility.

# 5

# Orbifolds and Finite Groups

Orbifolds are singular manifolds, obtained by identifying points on a smooth manifold under the action of a discrete group. The original motivation of studying orbifolds in string theory was to obtain simple models of string compactification, which are more or less realistic [55]. The simplest orbifold models are constructed out of tori with certain isometries. One considers the string propagation on the quotient of the torus by some subgroup of its symmetry group to obtain a new theory, which is only slightly more complicated to analyze than the toroidal model one started with. In general, the operation of taking the quotient of a space by its symmetry leads to singular points if the symmetry operation has fixed points. These singularities would render field theories on such spaces inconsistent. However string theory manages to avoid this problem by the introduction of string states which are closed only up to the action of the group element, *i.e.* by enlarging the Hilbert space to include twisted string states, as we will explain in much more detail in the next section.

The whole idea of orbifolds can be applied to general conformal theories with discrete symmetries. Given any conformal invariant model $\mathcal{C}$ with a particular symmetry group $G$, one could in principle try to construct a model $\mathcal{C}/G$ where one imposes an equivalence relation modulo $G$. Again the main new feature of the theory $\mathcal{C}/G$ is the introduction of twisted Hilbert space sectors. We shall continue to call such conformal theories *orbifold models,* although an interpretation in terms of sigma models on orbit spaces is in general not possible.

Due to a lack of geometrical understanding of the twisted states for a general conformal theory, the investigation of abstract orbifold theories has had only a limited amount of progress. Some of the properties of general orbifolds, such as the partition functions for twisted sectors can only be deduced by the requirement of modular invariance. This is in sharp contrast to toroidal orbifolds where we can *a priori* determine the partition function of the twisted sectors by geometric arguments and that always turns out to be consistent with modular invariance.

In this chapter we will investigate the structure of general orbifold CFT's, with a strong emphasis on rational theories, from the viewpoint of the general principles that we discussed in Chapter 4. One motivation is the compelling analogy we found between the structure of RCFT and finite group theory, as summarized in table 1, see also [121]. (See the appendix for a very brief review of some elements of group theory.) In both cases we have a finite number of representations with a closed

| RCFT | Group Theory |
|---|---|
| chiral algebra $\mathcal{A}$ | finite group $G$ |
| representations $\phi_i$ | representations $R_\alpha$ |
| fusion algebra: | representation ring: |
| $\phi_i \times \phi_j = \sum_k N_{ij}{}^k \phi_k$ | $R_\alpha \otimes R_\beta = \bigoplus_\gamma N_{\alpha\beta}{}^\gamma R_\gamma$ |
| characters $\lambda_i^{(n)} = S_{in}/S_{0n}$ | characters $\rho_\alpha(g) = \mathrm{Tr}\ R_\alpha(g)$ |
| $\lambda_i^{(n)}\lambda_j^{(n)} = \sum_k N_{ij}{}^k \lambda_k^{(n)}$ | $\rho_\alpha(g)\rho_\beta(g) = \sum_\gamma N_{\alpha\beta}{}^\gamma \rho_\gamma(g)$ |
| relative dimensions $\lambda_i^{(0)} \geq 1$ | dimensions $d_\alpha = \rho_\alpha(1) \in \mathbf{N}$ |

TABLE 1: A suggestive analogy between rational conformal field theory and the theory of finite groups.

representation algebra. Our aim in this chapter will be to substantiate this analogy by showing how on general grounds one can associate RCFT's to finite groups. In particular we will study the operator algebra and the role of the modular group. The abstract setting that we shall follow permits for quite general orbifolds, and not just the toroidal ones. It is pleasant to find that the basic structures of orbifolds are for a large part dictated by the group structure and depend relatively little on what the underlying conformal field theory is. This is particularly true for the case of orbifolds constructed from so-called holomorphic conformal theories, *i.e.* theories for which all operators are contained in the chiral algebra. We will show that for this class of orbifolds the modular geometry can be described entirely in terms of the finite group, up to certain phases. The reason that even in these holomorphic theories more structure appears than can be obtained from the finite group lies in the fact that we are interested in the *chiral* group action. So one has to split the Hilbert space into chiral blocks, and there is some additional information about the underlying CFT in the form of how this left-right splitting is accomplished.

The organization of this chapter is as follows. We start in section 5.1 with a brief review of the role of orbifolds in conformal field theory. In section 5.2 we derive, using general arguments, the operator content and the possible form of interactions in orbifold theories. Here we will mainly discuss orbifolds of holomorphic conformal field theories. These arguments will be substantiated in section 5.3 where the formulation of orbifolds on the torus is discussed. Using the modular properties of these theories we will derive the fusion rules and conformal dimensions of the operators. We also discuss briefly higher genus surfaces and discrete torsion and give a concrete example. The results are applied and illustrated for abelian groups in section 5.4. In section 5.5 we consider the much more difficult case of non-holomorphic models. We will make a start in characterizing this general class of orbifolds, but our results are not yet complete. Finally, in an appendix we have collected some useful identities obtained in the theory of finite groups.

## 5.1. Introduction to Orbifolds

Consider a *point particle* moving in some space-time $M_0$. Suppose that we have correctly quantized this problem and have constructed a Hilbert space $\mathcal{H}_0$ of quantum states. If the space-time $M_0$ has a discrete symmetry group $G$, we can consider the related problem of a particle moving on the quotient space $M = M_0/G$. If the group $G$ acts free on $M_0$, $M$ will be again a smooth manifold. But, if $G$ has fixed points, the space $M$ will no longer be a manifold but will possess singularities. Singular manifolds that are obtained from smooth manifolds by the action of a group $G$, *i.e.* whose points correspond to *orbits* of $G$, are generally referred to as *orbifolds*. The singularities occur whenever the order of the orbit or if one wishes the order of the isotropy group jumps. The canonical example of an orbifold is the cone $\mathbf{R}^2/\mathbf{Z}_2$, where the action of the $\mathbf{Z}_2$ group is a reflection in the origin of the plane $\mathbf{R}^2$. Clearly the cone has a singularity in the origin, where the orbit contains one point instead of the two points $x$ and $-x$.

For a point particle the quantum theory on the orbifold $M$ is easily deduced from the theory describing propagation on $M_0$. The symmetry group $G$ will be unitarily represented in the Hilbert space $\mathcal{H}_0$, and the spectrum $\mathcal{H}$ of the theory on $M$ is given by the $G$-invariant states in $\mathcal{H}_0$,

$$\mathcal{H} = Inv_G(\mathcal{H}_0). \tag{5.1}$$

In a naive implementation of the symmetry group $G$, the wave functions in $\mathcal{H}$ satisfy $\psi(g \cdot x) = \psi(x)$. But, if the particle carries internal quantum numbers, the symmetry group $G$ can combine space-time transformations with internal transformations $U_g$. In that case the wave functions will satisfy a condition of the form $\psi(g \cdot x) = U_g \psi(x)$. A simple illustration is a fermion on a circle $S^1 = \mathbf{R}/\mathbf{Z}$, which can be *anti*-periodic: $\psi(x+1) = -\psi(x)$.

### 5.1.1. *From Point Particles to Strings*

The simple picture we sketched above changes rather remarkably when we consider the related problem of a *string* moving on the target manifold $M$. The essential

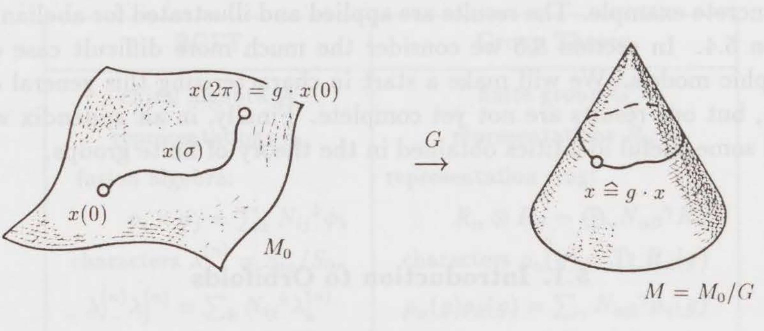



FIGURE 7: Propagation of a closed string on the orbifold $M = M_0/G$ includes configurations that should be regarded as open strings on the smooth manifold $M_0$.

modification is the existence of new states in the model describing string propagation on $M$, that are not present in the original theory describing propagation on $M_0$. If $x$ is a coordinate on $M_0$ and $\sigma \in [0, 2\pi]$ a coordinate on $S^1$, the string configurations $S^1 \to M_0$ satisfy

$$x(\sigma + 2\pi) = x(\sigma). \tag{5.2}$$

However, on the space-time $M = M_0/G$ the points $x$ and $g \cdot x$ are identified. Accordingly, we must allow configurations that obey the weaker condition

$$x(\sigma + 2\pi) = g \cdot x(\sigma) \tag{5.3}$$

for some $g \in G$. These configurations, which are called twisted, can be regarded as *open* string configurations on $M_0$, that become *closed* when we make the identification under $G$, as depicted in *fig.* 7. So, apart from the invariant states $\mathcal{H} = Inv(\mathcal{H}_0)$ in the original, untwisted Hilbert space $\mathcal{H}_0$, we also have to include states that satisfy the boundary condition (5.3). That is, we must associate to each group element $g \in G$ a space of twisted states $\mathcal{H}_g$. Note that for non-abelian groups this prescription is ambiguous, since the element $g$ is not uniquely determined. In fact, if we could just as well have chosen the coordinate $y = h \cdot x$ with some $h \in G$. When expressed in $y$, the boundary condition (5.3) reads

$$y(\sigma + 2\pi) = hgh^{-1} \cdot y(\sigma). \tag{5.4}$$

So $g$ is only unique up to conjugation, and the correct statement is that one has to introduce a twisted sector for each conjugation class of $G$. Note that in the twisted sectors $\mathcal{H}_g$ we still have an action of a subgroup of $G$, the stabilizer subgroup

$$N_g = \{h \in G \mid [g,h] = ghg^{-1}h^{-1} = 1\} \tag{5.5}$$

of all elements that commute with $g$. So, just as we had to project on $G$-invariant states in the untwisted sector $\mathcal{H}_0$, we will have to project on $N_g$-invariant states in the twisted sector $\mathcal{H}_g$, and the full spectrum of the theory describing string propagation on $M$ is given by

$$\mathcal{H} = \bigoplus_{g\in C} Inv_{N_g}(\mathcal{H}_g). \tag{5.6}$$

Here $C$ is any set of representatives of the conjugacy classes $C_A$ of $G$. We would like to stress the difference between this expression and equation (5.1) obtained in the analogous problem for a point particle.

### *5.1.2. A Path-Integral Derivation*

It is very illuminating to derive (5.6) in the path-integral language. The general idea is that we can transform the path-integral describing the sigma model on $M_0$ into a model on $M$ by allowing the fundamental field $x$ to be multi-valued. That is, around a closed loop, parametrized by a coordinate $\sigma$, $x(\sigma)$ can satisfy boundary conditions of the type (5.3). If we calculate the partition function on a general Riemann surface $\Sigma$, this implies that the configurations of the field $x$ are no longer functions on $\Sigma$, but sections of a nontrivial $G$-bundle over the surface. Furthermore, we will have to sum over all $G$-bundles, since all possible multi-valued configurations can occur. These bundles are described by giving the monodromies for all nontrivial cycles on $\Sigma$, *i.e.* for each element $\gamma$ of the fundamental group $\pi_1(\Sigma)$ of the surface we should specify an element $g_\gamma \in G$. This should be a homomorphisms of groups; so, if two cycles $\gamma, \gamma'$ starting and ending at the same point are composed to give a cycle $\gamma'' = \gamma' \circ \gamma$ we will require $g_{\gamma''} = g_{\gamma'} g_\gamma$.

From this point of view it is interesting to consider the genus one partition function $Z$. Since we have a periodic time variable, this corresponds in the Hamiltonian formalism to the trace of the evolution operator, $Z = \mathrm{Tr}\ e^{-tH}$. The configurations on the torus $S^1 \times S^1$ with coordinates $\sigma, \sigma'$ (mod $2\pi$) are now given by a field $x(\sigma,\sigma')$ satisfying

$$\begin{aligned} x(\sigma+2\pi,\sigma') &= g\cdot x(\sigma,\sigma'), \\ x(\sigma,\sigma'+2\pi) &= h\cdot x(\sigma,\sigma'), \end{aligned} \tag{5.7}$$

for two group elements $g, h \in G$. Note that by consistency $g$ and $h$ should commute, since one can argue from these equations that $x(\sigma+2\pi,\sigma'+2\pi)$ equals both $gh\cdot x(\sigma,\sigma')$ and $hg\cdot x(\sigma,\sigma')$.

The path-integral over configurations satisfying (5.7) will be denoted as $Z(g,h)$, with $Z(1,1)$ the original partition function of the $\sigma$-model on $M_0$. The full partition function includes a summation over all commuting pairs $g,h$, and reads

$$Z = \frac{1}{|G|} \sum_{\substack{g,h \in G \\ gh=hg}} Z(g,h). \tag{5.8}$$

The normalization factor is explained by making contact with equation (5.6). We can rewrite the partition sum as

$$Z = \sum_{g \in C} \frac{1}{|N_g|} \sum_{h \in N_g} Z(g,h). \tag{5.9}$$

Here we used the $G$ equivariance $x \to c \cdot x$ of the coordinate $x$, which implies conjugation invariance of the quantities $Z(g,h)$

$$Z(cgc^{-1}, chc^{-1}) = Z(g,h). \tag{5.10}$$

Furthermore, we made use of the relation $|N_g| = |G|/|C_g|$, with $C_g$ the conjugacy class of $g$. The twisted partition function $Z(g,h)$ corresponds in the Hamiltonian picture to a trace of the group element $h$ in the twisted sector $\mathcal{H}_g$. The operators

$$P^g = \frac{1}{|N_g|} \sum_{h \in N_g} h \tag{5.11}$$

project onto the $N_g$-invariant states of $\mathcal{H}_g$. So (5.8) can indeed be rewritten as

$$Z = \sum_{g \in C} \mathrm{Tr}_{\mathcal{H}_g} P^g e^{-tH} = \mathrm{Tr}_{\mathcal{H}} e^{-tH}, \tag{5.12}$$

in accordance with the Hamiltonian picture (5.6). Here $e^{-tH}$ is the two-dimensional evolution operator on the torus with modulus $\tau = it/2\pi$, and Hamiltonian

$$H = L_0 + \overline{L}_0 - \frac{c}{12}. \tag{5.13}$$

### *5.1.3. Discrete Torsion*

Part of our prescription to derive the spectrum of the orbifold model was a projection on $N_g$-invariant states in the twisted sectors $\mathcal{H}_g$. Here we tacitly assumed that there is a unique implementation of the stabilizer group $N_g$. This is however not true. One has to face a fundamental ambiguity that is present in any implementation of a symmetry in a quantum system.

Consider a representation $R(g)$ of some group $G$ on a Hilbert space $\mathcal{H}$. It could just as well be implemented as $\epsilon(g)R(g)$ with $\epsilon(g) \in U(1)$ a one-dimensional representation of $G$. If $\mathcal{H}$ splits up into multiplets $\mathcal{H}_\alpha$ transforming in irreducible representations $R_\alpha$, this corresponds to a relabeling of the sectors, since $\epsilon(g)R_\alpha(g) = R_{\alpha'}(g)$ is again an irreducible representation. It is clear that such a relabeling matters when we want to project on the invariant states. One could argue that this indeterminacy could be settled by demanding the vacuum to be invariant. However, this is only a valid argument in the untwisted sector $\mathcal{H}_0$, since in the twisted Hilbert spaces $\mathcal{H}_g$ there is in general not a unique state with lowest energy. So we have the freedom to choose in each twisted sector $\mathcal{H}_g$ a phase $\epsilon_g(h)$, which is a one-dimensional representation of $N_g$. These phases, that we will denote somewhat more symmetrically as $\epsilon(g,h)$, satisfy for $h_1, h_2 \in N_g$

$$\epsilon(g,h_1)\,\epsilon(g,h_2) = \epsilon(g,h_1h_2). \tag{5.14}$$

In the path-integral description this would imply a modified definition of the one-loop partition function

$$Z = \frac{1}{|G|}\sum_{\substack{g,h\in G\\ gh=hg}} Z(g,h)\,\epsilon(g,h). \tag{5.15}$$

Vafa coined the word *discrete torsion* for the quantity $\epsilon(g,h)$ [149], and has derived the following extra conditions from modular invariance:

$$\epsilon(h,g) = \epsilon(g^{-1},h), \qquad \epsilon(g,g) = 1. \tag{5.16}$$

As is explained in [149] the solutions to these equations are classified by the cohomology group $H^2(BG,U(1))$ (see also the discussion in [52, section 6.7]).

### 5.1.4. Interactions

Although we can include arbitrary interactions for a point particle, interactions in string theory are naturally dictated by two-dimensional geometry. We have a natural three-point coupling given by the manifold $Y$, the sphere with three disks removed. Since the Hilbert space of the two-dimensional sigma model splits up in different twisted sectors, we should investigate the possible superselection rules.

There are indeed some general rules that govern orbifold interactions, see also [92,53]. Fix three conjugacy classes $C_A$, $C_B$ and $C_C$, and consider an interaction between two states in $\mathcal{H}_A$ and $\mathcal{H}_B$ that gives rise to a state in $\mathcal{H}_C$. If we choose $g_1 \in C_A$ and $g_2 \in C_B$ to represent the twisted boundary conditions, then it is clear that the fused state is twisted by an element $g_3$ in the class of $g_1g_2$, see *fig.* 8. This gives a selection rule on the possible classes $C_C$. (Note that for non-abelian

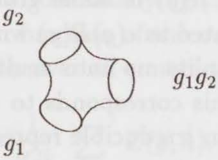



FIGURE 8: The interaction of two states twisted by group elements $g_1$ and $g_2$ give rise to a state twisted by $g_1g_2$.

groups $g_1$ and $g_2$ need not commute. But this does not cause an inconsistency, since $g_1g_2$ and $g_2g_1$ are equivalent under conjugation by $g_1^{-1}$.) Given an allowed triplet of conjugacy classes, how many independent couplings exist? Although the three twists $g_1$, $g_2$, $g_3$ are only defined up to conjugation, it is not difficult to see that the three-point interaction vertex allows only a simultaneous conjugation

$$(g_1, g_2, g_3) \to (cg_1c^{-1}, cg_2c^{-1}, cg_3c^{-1}). \tag{5.17}$$

Therefore different interaction channels between three classes can occur, corresponding to the inequivalent triplets of representatives of the three conjugacy classes*. To be more precise, consider the set of possible interactions

$$I = \{(g_1, g_2, g_3) \in C_A \times C_B \times C_C \mid g_1g_2 = g_3\}. \tag{5.18}$$

The group $G$ acts on these triplets by simultaneous conjugation of all three elements and under this action we can decompose $I$ into orbits $I^{(i)}$. The fact that the three-point coupling is only invariant under simultaneous conjugation implies that each orbit $I^{(i)}$ corresponds to a different channel between three external states in the sectors $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}_C$. If we denote by $N^{ABC}$ the number of orbits $I^{(i)}$ that $I$ splits into, we can define the class algebra

$$\phi^A \times \phi^B = \sum_C N^{ABC} \phi^C. \tag{5.19}$$

It is not difficult to verify that this algebra is associative and commutative. It encodes in a simple way the selection rules in the orbifold model.

*As an example with more than one way to couple three given conjugacy classes, consider the tetrahedron group $T$, or equivalently the group $A_4$ of even permutations of four elements. This group has two conjugacy classes of order three elements. There are two inequivalent couplings possible between two elements in the same conjugacy class with elements of order three, and one in the other conjugacy class with elements of order three.

*5.1.5. String Motivation*

Let us close this introductory section with some further observations on the original motivation of studying orbifolds in string theory. A special class of orbifolds are quotients of tori. Since toroidal compactifications are exactly solvable, the spectrum and interactions of these special orbifold models can also be exactly determined [55,53,92]. This is of considerable interest, since some highly nontrivial manifolds can degenerate to quotients of tori in a suitable limit. A famous example of such a manifold is the $K3$ manifold, which is the unique four-dimensional compact manifold of $SU(2)$ holonomy (*i.e.* the unique compact gravitational instanton in 4 dimensions). The moduli space of string theories on $K3$ is parametrized by the possible metrics and anti-symmetric background fields and is 80-dimensional. For special values of the moduli describing the metric, $K3$ degenerates to the quotient space $T^4/\mathbf{Z}_2$, with $T^4$ a four-dimensional hypertorus and $\mathbf{Z}_2$ the transformation $x \to -x$. This is an orbifold with 16 conical singularities. A smooth version of $K3$ can be obtained by 'resolving' the singularities, for example by replacing the cones by Eguchi-Hanson instantons [58]. In CFT language this resolution corresponds to perturbing the model by special marginal operators that control the size of the 'blown-up' fixed points. In principle, such a perturbation theory is expected to have a finite radius of convergence, and can be used to describe string theory on $K3$ in a neighborhood of the orbifold point.

An example that is of more relevance to string theory is a six-dimensional analogue of $K3$, a so-called Calabi-Yau manifold of $SU(3)$ holonomy, which has become known as the $Z$ manifold [27,55]. It is constructed as follows. Let $T^2$ be the two torus with $\mathbf{Z}_3$ isometry. That is $T^2 = \mathbf{R}^2/\Lambda$ with $\Lambda$ a triangular lattice. The $Z$ manifold is obtained as the orbifold $T^2 \times T^2 \times T^2$ modulo the diagonal $\mathbf{Z}_3$ action. The possible phenomenological implication of this and other orbifold models was first investigated in [55,53], and followed in numerous other papers.

## 5.2. Rational Orbifold Models

We will now take a giant leap in abstraction and generalize the setting of the equivariance problem from sigma models to arbitrary (rational) conformal field theories. So we consider a RCFT $\mathcal{C}$ that allows for the action of a finite group $G$ and investigate the operator algebra of the new orbifold model $\mathcal{C}/G$ obtained by dividing out the symmetry $G$. In order to get a consistent model, $G$ has to be a subgroup of the group of endomorphisms of the operator algebra, and respect the left-right decomposition (4.20) of the Hilbert space. Furthermore $G$ should commute with the Virasoro algebra. We further make the restriction to a left-right symmetric action of $G$. It is clear that the structure of the resulting orbifold theory strongly depends on the way the group of $G$ intertwines with the fusion algebra of the original model. In general the group $G$ will act on the different representations $[\phi_i]$ of the chiral algebra $\mathcal{A}$ by some permutation group, which commutes with the fusion algebra coefficient $N_{ijk}$. In other words, if $g \in G$ acts on the set of indices labeling the Hilbert subsectors by $i \rightarrow i'$, then

$$N_{ijk} = N_{i'j'k'}. \tag{5.20}$$

The cases in which the group $G$ acts nontrivially on the operator algebra will be referred to as the 'outer' case. The word outer is suitable for describing the action of $G$ on conformal theories which permutes the operators, because the chiral algebra, which is constructed out of states in the identity block, acts only within each chiral block. This means that in such cases the action of the group cannot be represented by the operators inside the chiral algebra we started with. The case where all elements of $G$ act trivially on the fusion algebra is called 'inner.'*

The determination of the full representation content of a general orbifold theory can be quite complicated, especially because in general the group $G$ consists of both inner and outer automorphisms. Roughly speaking, one expects that the outer action of the group results in the identification of representations, while the inner action gives rise to splitting of representations and the emergence of twisted sectors. Our aim is to make this intuitive picture more precise. We will mainly concentrate on the effect of the inner action. To simplify the discussion we will discuss the case where the representation theory of chiral algebra $\mathcal{A}$ of the original model is trivial, *i.e.* when no other irreducible representations occur, except the basic representation $\mathcal{A}$ itself. Such a model will be called a holomorphic CFT†. We return to more generic orbifold CFT's in section 5.5 and in Chapter 6 where we discuss special models with central charge $c=1$.

### *5.2.1. Holomorphic Theories: Untwisted Sector*

The Hilbert space of a holomorphic conformal theory is equal to the tensor product $\mathcal{A} \otimes \overline{\mathcal{A}}$ of the left and right chiral algebra. Consequently, the one-loop partition function factorizes

$$Z(\tau, \overline{\tau}) = \chi(\tau)\overline{\chi}(\overline{\tau}). \tag{5.21}$$

Famous examples of holomorphic theories are the level one $E_8$ and $Spin(32)/\mathbf{Z}_2$ group manifold models, or more generally models based on even self-dual lattices, and certain quotients of them, *e.g.* the Moonshine module [70,54]. If $G$ is any

*This does not necessarily mean that in the inner case the automorphism can always be represented by elements in the chiral algebra. An example of this kind is provided by $E_8 \times E_8$ (both at level one) and the automorphism which exchanges the two $E_8$'s.

†Although usually the term holomorphic is reserved for a CFT $\mathcal{C}$ with only holomorphic operators [54], here we will use it for theories of the form $\mathcal{C} \times \overline{\mathcal{C}}$.

discrete symmetry of such a model, we can consider the corresponding $G$ orbifold, which we will call a holomorphic orbifold. We will give the reader a simple example of a holomorphic orbifold to keep in mind in the subsequent, rather abstract considerations.

Consider the level $k=1$ $E_8$ group manifold model. The Lie group $E_8$ has a cyclic subgroup of order two under which it breaks as

$$E_8 \to E_7 \times SU(2). \tag{5.22}$$

At level one both $E_7$ and $SU(2)$ have two representations, and we will denote their characters as $\chi_i^{E_7}, \chi_i^{SU(2)}$ with $i=0,1$. The single $k=1$ $E_8$ character can be written as

$$\chi = \chi_0^{E_7}\chi_0^{SU(2)} + \chi_1^{E_7}\chi_1^{SU(2)}. \tag{5.23}$$

The orbifold model $E_8/\mathbf{Z}_2$ will be simply the tensor product of the $E_7$ and $SU(2)$ models, and has consequently four characters

$$\chi_i^{E_7}\chi_j^{SU(2)} \quad (i,j=0,1), \tag{5.24}$$

with conformal weights $0, \frac{1}{4}, \frac{3}{4}, 1$. The fusion algebra is the tensor product of the $k=1$ $E_7$ and $SU(2)$ fusion algebras, which are both $\mathbf{Z}_2$. Our aim will be to derived these fusion rules and the conformal dimensions from general considerations, just using data from the finite group.

We now consider a general discrete group $G$ whose elements $g$ act as endomorphisms on the chiral algebra $\mathcal{A}$ and leave the Virasoro algebra invariant. Under the action of $G$ the chiral algebra $\mathcal{A}$ decomposes into subsectors $\mathcal{A}_\alpha$ containing the operators that transform in the irreducible representations $R_\alpha$ of $G$

$$\mathcal{A} = \bigoplus_\alpha \mathcal{A}_\alpha. \tag{5.25}$$

Our aim is to construct RCFT's described by the subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ which is invariant under the $G$ action. We will also write $\mathcal{A}_0 = \mathcal{A}/G$. These models are the $G$ orbifold models associated with the original model with algebra $\mathcal{A}$. We will try to analyze these orbifold theories using the relation between $\mathcal{A}$ and $\mathcal{A}_0$. Indeed many of the properties of the $\mathcal{A}_0$ models only become clear once we realize that they can be obtained by this orbifold construction.

The Hilbert space sectors $\mathcal{H}_\alpha$, that correspond to the operators $\mathcal{A}_\alpha$, form highest weight representations of $\mathcal{A}_0$. They are however in general not irreducible. This is due to the fact that $G$ acts within $\mathcal{H}_\alpha$ and commutes with the action of $\mathcal{A}_0$. Accordingly we have a decomposition

$$\mathcal{H}_\alpha = [\phi_\alpha] \otimes R_\alpha, \tag{5.26}$$

where we want to identify $[\phi_\alpha]$ with an irreducible $\mathcal{A}_0$ representation. This can be considered a coset construction, since we identify orbits of states under $G$. Furthermore, we immediately see that $[\phi_\alpha]$ occurs with multiplicity $d_\alpha = \dim R_\alpha$. Let us stress here that $G$ does *not* act in the orbifold model although we label the operators with a representation index of $G$.

What can we say about the possible couplings of the representations $[\phi_\alpha]$ occurring in (5.26). The starting point is that the couplings in $\mathcal{A}$ are uniquely fixed, and, since the elements of $G$ are endomorphisms of $\mathcal{A}$, they are $G$-invariant. These couplings are encoded in the (unique) chiral vertex $|V_\mathcal{A}\rangle \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$. Since we have the decomposition $\mathcal{A} = \bigoplus_\alpha [\phi_\alpha] \otimes R_\alpha$ we can write

$$|V_\mathcal{A}\rangle = \sum_{\alpha,\beta,\gamma} \sum_a c^{(a)}_{\alpha\beta\gamma} |V^{(a)}_{\alpha\beta\gamma}\rangle. \tag{5.27}$$

This defines the chiral vertex operators of the $\mathcal{A}_0$ model

$$|V^{(a)}_{\alpha\beta\gamma}\rangle \in [\phi_\alpha] \otimes [\phi_\beta] \otimes [\phi_\gamma]. \tag{5.28}$$

The $G$ invariance of $|V_\mathcal{A}\rangle$ implies that the coefficients

$$c^{(a)}_{\alpha\beta\gamma} \in R_\alpha \otimes R_\beta \otimes R_\gamma \tag{5.29}$$

are invariant tensors of $G$, *i.e.* Clebsch-Gordan coefficients. Thus we see that for each independent $G$-invariant tensor $c^{(a)}_{\alpha\beta\gamma}$ we have a corresponding $\mathcal{A}_0$-invariant chiral coupling $|V^{(a)}_{\alpha\beta\gamma}\rangle$. Consequently the integers $N_{\alpha\beta}{}^\gamma$ occurring in the fusion algebra of the representations $[\phi_\alpha]$

$$\phi_\alpha \times \phi_\beta = \sum_\gamma N_{\alpha\beta}{}^\gamma \phi_\gamma \tag{5.30}$$

equal the number of independent Clebsch-Gordan coefficients of $G$. Hence we conclude that the fusion algebra (5.30) is identical to the representation algebra of $G$

$$R_\alpha \otimes R_\beta = \bigoplus_\gamma N_{\alpha\beta}{}^\gamma R_\gamma. \tag{5.31}$$

We can further extend the above arguments of the thrice-punctured sphere to any number of punctures. In this case one finds that the holomorphic blocks that appear in the decomposition of the correlator of a set fields $\phi_\alpha$ are in one-to-one correspondence with the invariant tensors

$$Inv_G(\bigotimes_\alpha R_\alpha). \tag{5.32}$$

This gives a partial explanation of the analogy between group theory and rational conformal field theory. However, this cannot be the complete picture, because the analogy of table 1 is not really one-to-one. The main discrepancy is that the fusion algebra of a RCFT is diagonalized by a matrix $S$ that satisfies $S^2 = 1$. The representation algebra a finite group is also diagonalized by a matrix $S$ (see Eq. (A.8) in the appendix), but this is a map from the space of irreducible representations to the space of conjugacy classes. These spaces are not *canonically* isomorphic, so it does not make sense to consider $S^2$. In order to obtain a full analogy we also have to include twisted representations.

### 5.2.2. *Inclusion of the Twisted Sectors*

One should ask whether the $[\phi_\alpha]$'s are the only possible representations of $\mathcal{A}_0$. It is clear that in general this will not be the case, because we have reduced the algebra and as a consequence there will be extra operators that are local with respect to it. Concretely, since $\mathcal{A}_0$ is $G$-invariant, we also have to consider fields that are local with respect to $\mathcal{A}$ up to the action of an element $g \in G$. These extra fields should be considered as twist fields and can be organized in 'twisted' representations of $\mathcal{A}_0$. Further, since we are considering the case for which in the original model only the trivial representation of $\mathcal{A}$ occurs, it is natural to assume that these twisted representations are the only other representations of $\mathcal{A}_0$ besides the $[\phi_\alpha]$. A more rigorous derivation of this operator content uses the invariance under modular transformations. We will return to that point of view in section 5.3.

We now turn to the couplings of the twisted sectors. Twist fields can be introduced for any element $g \in G$ and have the property that the operators in $\mathcal{A}$ have nontrivial monodromy around the twist fields given by the action of $g$. This implies in particular that they have local operator products with the chiral algebra $\mathcal{A}_0$. We denote the set of twisted states associated with $g$ as $\mathcal{H}_g$. For a non-abelian group the spaces $\mathcal{H}_g$ only depend on the conjugacy class $C_A$ of $g$. In an analogous way to (5.25), we can decompose $\mathcal{H}_g$ into the irreducible representations of $\mathcal{A}_0$, using the action of $N_g$

$$\mathcal{H}_g = \bigoplus_\alpha [\phi^g_\alpha] \otimes R^g_\alpha. \tag{5.33}$$

This defines the $\mathcal{A}_0$ modules $[\phi^g_\alpha] = [\phi^A_\alpha]$ labeled by the conjugacy class $C_A$ of $g$ and the irreducible representation $\alpha$ of $N_g$.

Let us however make the following comments. First of all, since we are dealing with chiral objects, it is possible that the *chiral* action of the stabilizer subgroups in the twisted sectors is projective. So we should in principle allow for a nontrivial $U(1)$ cocycle $c_g(h_1, h_2)$, which is common to all representations $R^g_\alpha$ in a given twisted sector. Although most of the following analysis will not depend on the presence of these cocycles, we will for convenience assume that the representations $R^g_\alpha$ are non-projective.

A related point is the following. As in our discussion of discrete torsion in section 5.1.3 the labeling of the above operators is not unique, since we can redefine the action of $N_g$ by tensoring with a one-dimensional representation $\epsilon_g(h)$. We will regard the transformation

$$R^g_\alpha \longrightarrow \epsilon_g \otimes R^g_\alpha \tag{5.34}$$

as a gauge degree of freedom in our description.

What are the possible couplings between these twisted $A_0$ representations $[\phi^A_\alpha]$, *i.e.* what is the fusion algebra

$$\phi^A_\alpha \times \phi^B_\beta = \sum_{\gamma,C} N^{ABC}_{\alpha\beta\gamma}\, \phi^C_\gamma. \tag{5.35}$$

Let us first concentrate on the index labeling the classes of $G$. Given the discussion in section 5.1.4, we know that it is covered by the class algebra

$$\phi^A \times \phi^B = \sum_C N^{ABC}\, \phi^C. \tag{5.36}$$

This class algebra is of course not yet the complete orbifold fusion algebra (5.35), since we still have to include the representations of the stabilizer subgroups $N_{g_i}$. Again, similar as in the untwisted sector, we have to determine the selection rules that follow from the group action on the couplings. It is clear that the three-point coupling for a given triplet $(g_1, g_2, g_1g_2) \in I^{(i)}$ cannot be invariant under the full action of the three $N_{g_i}$'s but only under that of the common stabilizer $N^{(i)} = N_{g_1} \cap N_{g_2}$. One would now like to repeat the arguments given for the non-twisted sector to construct a correspondence between the possible invariant couplings and the Clebsch-Gordan coefficients of $N^{(i)}$. However, to do this we will have to identify the action of a subgroup of $G$ in different Hilbert spaces. As explained above, this action is only well-defined up to a one-dimensional representation. In principle it is possible that when comparing the different representations of $N^{(i)}$ in each of the sectors $(g_1, g_2, g_1g_2)$ we may have to redefine our representation labelings. So, without further information about the details of the orbifold model, the only restriction we can make is that the couplings in $\mathcal{H}_{g_1} \otimes \mathcal{H}_{g_2} \otimes \mathcal{H}_{g_1g_2}$ correspond to the $N^{(i)}$-invariant tensors

$$Inv_{N^{(i)}}(R^{g_1}_\alpha \otimes R^{g_2}_\beta \otimes R^{g_1g_2}_\gamma \otimes \epsilon^{(i)}), \tag{5.37}$$

where $\epsilon^{(i)}$ is an *a priori* arbitrary one-dimensional representation of $N^{(i)}$. Let us denote the number of inequivalent $N^{(i)}$-invariant tensors by $N^{(i)}_{\alpha\beta\gamma}$. The coefficients $N^{ABC}_{\alpha\beta\gamma}$ of the fusion algebra, are now given by

$$N^{ABC}_{\alpha\beta\gamma} = \sum_{i=1}^{N^{ABC}} N^{(i)}_{\alpha\beta\gamma}, \tag{5.38}$$

where the sum is over all orbits $I^{(i)}$ of the set $I$ defined in (5.18). By construction, the number on the right-hand side is the total number of inequivalent couplings in $[\phi^A_\alpha]\otimes[\phi^B_\beta]\otimes[\phi^B_\gamma]$.

This concludes our discussion of the interactions of the general orbifolds of holomorphic conformal theories. It will be clear that the reasoning given in this section has been more intuitive than rigorous. In the next section, however, we will provide a more solid base to the arguments presented here, by deriving the same results using the modular transformation of the one-loop characters.

## 5.3. Orbifold Models at Genus One

As we have seen in Chapter 4 much of the structure of a RCFT can be investigated by considering the modular properties of the model on a one-loop Riemann surface. Not only the representation content can be determined [30], but quite unexpectedly also the fusion algebra can be expressed in terms of the representation of the modular group [153]. In this section we will use these powerful results to study the operator algebra of the orbifold CFT's, and to find confirmation of the general arguments presented in the previous section.

### 5.3.1. *Partition Functions and Characters*

Our starting point will be the path-integral representation of an orbifold partition function $Z(\tau,\bar{\tau})$ on a torus with modular parameter $\tau$.

$$Z=\frac{1}{|G|}\sum_{\substack{g,h\in G\\ [g,h]=1}} Z(g,h)\,\epsilon(g,h). \tag{5.39}$$

We will set $\epsilon=1$ in the subsequent discussions, but we will return to its interpretation in section 5.3.5. For the case of a holomorphic theory we can now make contact with the description given in section 5.2. We have complete factorization of the Hilbert space in holomorphic and anti-holomorphic parts $\mathcal{H}_0\otimes\overline{\mathcal{H}}_0$. This generalizes to the twisted Hilbert spaces; they are also of the form $\mathcal{H}_g\otimes\overline{\mathcal{H}}_g$. Furthermore the action of the stabilizers $N_g$ decomposes in a left and right action. Accordingly we can define the *holomorphic* blocks

$$h\underset{g}{\square}=\mathrm{Tr}_{\mathcal{H}_g}\, h\, q^{L_0-c/24}, \tag{5.40}$$

and similar expressions for the anti-holomorphic blocks. The partition function $Z(g,h)$ is the product of the holomorphic part times the anti-holomorphic part.

For a left-right symmetric theory this would be

$$Z(g,h) = \Big| h \underset{g}{\square} \Big|^2. \tag{5.41}$$

The holomorphic space $\mathcal{H}_g$ decomposes into sectors that transform in an irreducible representation $R^g_\alpha$ of the stabilizer $N_g$

$$\mathcal{H}_g = \bigoplus_\alpha [\phi^g_\alpha] \otimes R^g_\alpha. \tag{5.42}$$

Similar to the projection operators $P^g$ (5.11) that project on the invariant states of $\mathcal{H}_g$, we can introduce projection operators $P^g_\alpha$ that project on the irreducible representation $R^g_\alpha$

$$P^g_\alpha = \frac{1}{|N_g|} \sum_{h \in N_g} \rho^g_\alpha(1) \rho^g_\alpha(h^{-1}) h. \tag{5.43}$$

With the aid of these operators the one-loop characters $\chi^g_\alpha(q)$ of the module $[\phi^g_\alpha]$ are calculated as follows

$$\begin{aligned} \chi^g_\alpha(q) &= \mathrm{Tr}_{[\phi^g_\alpha]} q^{L_0 - c/24} = (d^g_\alpha)^{-1} \mathrm{Tr}_{\mathcal{H}_g} P^g_\alpha \, q^{L_0 - c/24} \\ &= \frac{1}{|N_g|} \sum_{h \in N_g} \rho^g_\alpha(h^{-1}) \;\; h \underset{g}{\square} \\ &= \frac{1}{|G|} \sum_{\substack{h \in G, g \in C_A \\ [h,g]=1}} \rho^g_\alpha(h^{-1}) \;\; h \underset{g}{\square}. \end{aligned} \tag{5.44}$$

It is clear that these characters $\chi^g_\alpha(q)$ have a $q$-expansion with positive integer coefficients. The above relation between the characters and the holomorphic blocks can be inverted to give

$$h \underset{g}{\square} = \sum_\alpha \rho^g_\alpha(h) \, \chi^g_\alpha(q). \tag{5.45}$$

Of course the characters depend only on the class $C_A$ of $g$. Using this relation the full partition function can now be written as

$$Z(q, \bar{q}) = \sum_{\alpha, A} |\chi^A_\alpha(q)|^2. \tag{5.46}$$

We note that the ambiguity (5.34) in the labeling of the operators corresponds in terms of the chiral blocks to a redefinition

$$h \underset{g}{\square} \longrightarrow \epsilon_g(h) \, h \underset{g}{\square}, \tag{5.47}$$

with $\epsilon_g$ a one-dimensional representation of $N_g$. A transformation of type (5.47) is always accompanied by a similar transformation with $\epsilon_g^*$ acting on the anti-holomorphic blocks. In this way the modular invariance of the partition function is preserved.

### 5.3.2. Modular Transformations

For the non-chiral blocks $Z(g,h)$ in genus one, we have a simple transformation rule under the modular group. Recall that modular transformations act on the homology basis $(A,B)$ as $SL(2,\mathbf{Z})$ transformations. This gives a straightforward action on the boundary conditions used to define $Z(g,h)$:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z}), \qquad M : Z(g,h) \to Z(g^a h^b, g^c h^d). \tag{5.48}$$

For the chiral blocks however, which are the holomorphic square roots of the twisted partition function $Z(g,h)$, this transformation rule is somewhat modified, since in general a phase has to be included

$$M : h\,\underset{g}{\square} \to \epsilon_M(g|h)\; g^c h^d \underset{g^a h^b}{\square}, \tag{5.49}$$

where $\epsilon_M$ is a phase factor independent of $\tau$. Since the anti-chiral blocks transform with the opposite phases, it is evident that in the partition function the two phases will cancel*. In section 5.4 we will show that for generic models these phases indeed do occur. They will play an important part in the subsequent analysis, and we will treat them in detail.

As is well-known, the modular group is generated by the transformation $S$ and $T$, defined on the cycles as:

$$\begin{aligned} S : (A,B) &\to (B,-A), & (5.50)\\ T : (A,B) &\to (A, A+B), & (5.51) \end{aligned}$$

with the relations $S^2 = (ST)^3 = C$, where the charge conjugation reverses the orientation of the cycles. We will first consider $S$, which interchanges the 'space' and 'time' directions. Among other things, $S$ will give us the fusion rules of the model. Let us now define the phases $\sigma(g|h)$ by

$$S : \; h\,\underset{g}{\square} \to \sigma(g|h)\;\; g^{-1} \underset{h}{\square}\,. \tag{5.52}$$

*In asymmetric orbifolds [124] this cancellation is not automatic and has to be imposed by the so-called level-matching conditions.

We will derive the following conditions on $\sigma(g|h)$

$$\sigma(h|g) = \sigma(g|h), \tag{5.53}$$

$$\delta_1\delta_2\sigma(h_1, h_2|g_1, g_2) = 1. \tag{5.54}$$

In the last condition we introduced the coboundary operators $\delta_1$, $\delta_2$. They act as the coboundary operator $\delta$ of group cohomology (see appendix A) on the two arguments of $\sigma$:

$$\delta_1\sigma(h_1, h_2|g) \;=\; \sigma(h_1|g)\,\sigma(h_2|g)\,\sigma^{-1}(h_1h_2|g), \tag{5.55}$$

$$\delta_2\sigma(h|g_1, g_2) \;=\; \sigma(h|g_1)\,\sigma(h|g_2)\,\sigma^{-1}(h|g_1g_2). \tag{5.56}$$

Note that $\delta_1\delta_2 = \delta_2\delta_1$. The condition $\delta_1\sigma = 1$ implies that $\sigma(h|g)$, when considered as a function of $h$, is a one-dimensional representation of $N_g$. Consequently $\delta_1\delta_2\sigma = 1$ has the following interpretation: $\delta_2\sigma(h|g_1, g_2)$ is a 2-coboundary on $N_h$ and a representation of $N_{g_1} \cap N_{g_2}$. We will motivate this condition on $\sigma$ when we discuss the fusion algebra.

The requirement of symmetry (5.53) can be proved as follows. First we observe that the chiral blocks are functions of $\tau$. This implies that the transformation $S$ is implemented as the $PSL(2, \mathbf{C})$ transformation $\tau \to -1/\tau$, where it satisfies $S^2 = 1$. Furthermore, if we take $\tau$ pure imaginary, the one-loop characters $\chi^g_\alpha(\tau)$ are real. Consequently we have in that case the identity

$$h\,\underset{g}{\square} = \Big( h^{-1}\,\underset{g}{\square} \Big)^*. \tag{5.57}$$

This is a direct result of $\rho^A_\alpha(h^{-1}) = \rho^A_\alpha(h)^*$ and relation (5.45). If we now use this relation and $S^2 = 1$ we obtain the result $\sigma(g|h) = \sigma(h|g)$.

A further important property of $\sigma(g|h)$ is its relation with the charge conjugation operation. Charge conjugation $\mathcal{C}$ clearly leaves the characters invariant, but can reshuffle the indices if several characters have the same $q$-expansion. This implies that in general we can have

$$\mathcal{C}:\; h\,\underset{g}{\square} \;\to c_g(h)\; h^{-1}\,\underset{g^{-1}}{\square}. \tag{5.58}$$

Here $c_g(h)$ is a one-dimensional representation of $N_g$. By $CPT$ invariance we have $S^2 = \mathcal{C}$, so that

$$\sigma(g|h)\,\sigma(h|g^{-1}) = c_g(h), \tag{5.59}$$

which implies

$$c_g(h) = c_{g^{-1}}(h). \tag{5.60}$$

This guarantees the relation $S^4 = 1$. If $g$ and $g^{-1}$ are not conjugate we can remove the phase $c_g$ by a gauge transformation (5.47). Finally, note that all conditions we imposed on the phase factors are invariant under the transformation (5.47), since it implies

$$\sigma(g|h) \to \epsilon_g(h)\epsilon_h(g)\sigma(g|h), \tag{5.61}$$

which can be readily seen not to interfere with the above conditions.

The action of $S$ in the basis of one-loop characters $\chi_\alpha^A$ (instead of chiral blocks) is given by the following elegant expression

$$S_{\alpha\beta}^{AB} = \frac{1}{|G|} \sum_{\substack{h\in C_B, g\in C_A \\ [h,g]=1}} \rho_\alpha^g(h^{-1})\rho_\beta^h(g^{-1})\sigma(g|h). \tag{5.62}$$

From general arguments we know that $S$ should be symmetric and unitary. The first property is ensured by condition (5.53), while unitarity follows from the fact that $S^* = S^{-1}$.

We will now use the modular transformation $T$ to derive the conformal weights of the twist fields in terms of the $U(1)$ cocycle $\sigma$. It is clear that, similar as for the transformation $S$, we have to include an (*a priori* arbitrary) phase factor $\tau(g|h)$ in the transformation rules of $T$

$$T : h\underset{g}{\square} \to e^{-2\pi i c/24}\tau(g|h)\; hg\underset{g}{\square}. \tag{5.63}$$

Here we extracted the factor $e^{-2\pi i c/24}$ for convenience. Since $T$ should be diagonal in the basis $\chi_\alpha^A$, we see that $\tau(g|h) = \tau_g$. Furthermore $\tau_g$ only depends on the conjugacy class $C_A$ of $g$. Note that

$$\tau_g = e^{2\pi i h_g}, \tag{5.64}$$

where $h_g$ is the ground state energy of $\mathcal{H}_g$. We now apply the fundamental relation $(ST)^3 = \mathcal{C}$ on a chiral block. This gives us

$$\tau_g\tau_h\tau_{gh}\sigma(g|hg)\sigma(h|gh)\sigma(g|h)^{-1} = 1. \tag{5.65}$$

Choosing $h = 1$ we find a relation between $\tau_g$, *i.e.* the conformal weights, and $\sigma$

$$\tau_g^2 = \sigma(g|g)^{-1}. \tag{5.66}$$

Substitution in (5.65) gives a condition that can be written very succinctly as

$$\delta_1\delta_2\sigma(g,h|g,h) = 1, \tag{5.67}$$

which is indeed a special case of the condition (5.54). Consequently the relation $(ST)^3 = C$ does not give any further relations on $\sigma$.

It is now not difficult to determine the weights of the other operators. Since $g$ obviously is an element of the center of $N_g$, we can use Schur's lemma to show that in any irreducible representation $R^g_\alpha$ of $N_g$ is represented as a phase factor $\epsilon^g_\alpha$ (satisfying $(\epsilon^g_\alpha)^n = 1$ with $n$ the order of $g$) times the identity

$$R^g_\alpha(g) = \epsilon^g_\alpha\, 1. \tag{5.68}$$

From this we find that $T$ acts on the character of the chiral sector $[\phi^g_\alpha]$ as

$$\begin{aligned} T : \chi^g_\alpha \ \rightarrow\ & e^{-2\pi i c/24}\tau_g \frac{1}{|N_g|}\sum_{h\in N_g} \rho^g_\alpha(h^{-1}) \ \ hg\,\underset{g}{\Box} & (5.69)\\ =\ & e^{-2\pi i c/24}\tau_g \frac{1}{|N_g|}\sum_{k\in N_g} \rho^g_\alpha(gk^{-1}) \ \ k\,\underset{g}{\Box} & (5.70)\\ =\ & e^{-2\pi i c/24}\tau_g \epsilon^g_\alpha \chi^g_\alpha & (5.71)\end{aligned}$$

Hence we obtain the important result that all operators in $[\phi^g_\alpha]$ have conformal weight given by

$$e^{2\pi i h_{g,\alpha}} = \tau_g \epsilon^g_\alpha. \tag{5.72}$$

### *5.3.3. The Fusion Algebra*

It is a fundamental result that the fusion algebra of a RCFT can be derived from the transformation properties of the modular group [153]. More explicitly we have the relation

$$N_{ijk} = \sum_n \frac{S_{in}S_{jn}S_{kn}}{S_{0n}}. \tag{5.73}$$

We will now use this result to establish the fusion algebra in the orbifold models

$$\phi^A_\alpha \times \phi^B_\beta = \sum_{\gamma, C} N^{ABC}_{\alpha\beta\gamma}\ \phi^C_\gamma. \tag{5.74}$$

Since we have explicitly calculated $S$, (5.73) can be easily applied. In order to simplify the expression we need the following result in group theory [89]. Suppose we are given a function $f(g_1, g_2, g_3)$ that only depends on the classes of its arguments. Then the following identity holds

$$\sum_{g_i\in G} \frac{1}{|G|}\sum_\alpha \frac{1}{d_\alpha}\rho_\alpha(g_1)\rho_\alpha(g_2)\rho_\alpha(g_3^{-1}) f(g_1, g_2, g_3) = \sum_{\substack{g_i\in G\\ g_1g_2=g_3}} f(g_1, g_2, g_3). \tag{5.75}$$

With this result and after some algebra $N_{\alpha\beta\gamma}^{ABC}$ can be rewritten as

$$N_{\alpha\beta\gamma}^{ABC} = \frac{1}{|G|} \sum_{\substack{g_1 \in C_A, g_2 \in C_B, g_3 \in C_C, h \in G \\ [h,g_1]=[h,g_2]=[h,g_3]=1 \\ g_1 g_2 = g_3}} \rho_\alpha^{g_1}(h) \rho_\beta^{g_2}(h) \rho_\gamma^{g_3}(h^{-1})\, \delta_2\sigma(h|g_1,g_2) \tag{5.76}$$

$$= \sum_i \frac{1}{|N^{(i)}|} \sum_{h \in N^{(i)}} \rho_\alpha^{g_1}(h) \rho_\beta^{g_2}(h) \rho_{\gamma^*}^{g_1 g_2}(h)\, \delta_2\sigma(h|g_1,g_2). \tag{5.77}$$

In the second line we can indeed recognize the sum over the different interaction channels, as argued in section 5.2 (see equation (5.38)). Furthermore we can now calculate the phase $\epsilon^{(i)}$. It is given by

$$\epsilon^{(i)}(h) = \delta_2\sigma(h|g_1,g_2), \tag{5.78}$$

where the label $i$ corresponds to the interaction channel $(g_1, g_2, g_1 g_2)$. This relation finally gives our motivation for the condition (5.54) on $\sigma$, since it implies that $\delta_2\sigma$ should be a representation of $N^{(i)}$, *i.e.* $\delta_1\delta_2\sigma = 1$.

The determination of the possible operator algebras of a holomorphic orbifold model is now reduced to a cohomology problem. The fusion algebras are labeled by the solutions of $\delta_1\delta_2\sigma = 1$ modulo the pure gauge solutions $\sigma(g|h) = \epsilon_g(h)\epsilon_h(g)$. These solutions are classified by the cohomology group $H^3(BG, U(1))$. But, we will only be in a good position to clarify this result when we have discussed *three-dimensional* theories in Chapter 8.

The above result for the interaction rules and the phase $\sigma$ can also be given the following interpretation. We can redefine the action $R_g(h)$ of $N_g$ in the sector $\mathcal{H}_g$ by

$$R_g(h) \to R'_g(h) = \sigma(h|g) R_g(h). \tag{5.79}$$

The new representation $R'_g$ has a trivial 2-cocycle

$$R'_g(h_1) R'_g(h_2) = c(h_1, h_2|g) R'_g(h_1 h_2), \tag{5.80}$$

with

$$c(h_1, h_2|g) = \delta_1\sigma(h_1, h_2|g). \tag{5.81}$$

With the use of these representations $R'_g$ the fusion rules can now be interpreted straightforwardly as the couplings invariant under the mutual stabilizer $N^{(i)}$, *i.e.* without the phase ambiguity $\epsilon^{(i)}$. How can we understand the relation $\delta_1\delta_2\sigma = 1$ from this point of view? Let us introduce operators $\phi_g$ that create sectors twisted by $g$ in the non-local model. Now consider the operator $\phi_{g_1}\phi_{g_2}\phi_{(g_1 g_2)^{-1}}$. We claim that this operator is an element of the chiral algebra, and should accordingly transform in a true representation. This implies the following condition on the cocycles

$$\delta_2 c = \delta_2\delta_1\sigma = 1. \tag{5.82}$$

We close this section with the remark that the above fusion algebra (for the case $\sigma = 1$) also appears in the mathematical literature, in particular in the work of Lusztig [112], where it is developed in the study of Lie groups over a finite field. The appropriate objects in that context are $G$ equivariant complex vector bundles over the group $G$. Since $G$ is a finite set, an equivariant vector bundle over $G$ is simply a collection of finite vector spaces $V_g$ with a representation of $G$ on

$$V = \bigoplus_{g \in G} V_g \tag{5.83}$$

such that $g \cdot V_h = V_{ghg^{-1}}$. The set of all these equivariant bundles with the obvious notion of addition is called the Grothendieck group $K_G(G)$. The irreducible vector bundles, *i.e.* bundles $V$ that cannot be written as $V_1 \oplus V_2$, are labeled by a conjugacy class $C_A$ and an irreducible representation of the stabilizer subgroup $N_A$. These bundles satisfy $V_g \neq 0$ if and only if $g \in C_A$ and carry the irreducible representation $R^g_\alpha$ of $N_g$. There is also a definition of multiplication (which is not the tensor product) which makes $K_G(G)$ into a semisimple commutative algebra. The definition is [112]

$$(V \cdot V')_g = \bigoplus_{\substack{g_1, g_2 \in G \\ g_1 g_2 = g}} \left( V_{g_1} \otimes V'_{g_2} \right). \tag{5.84}$$

It is not difficult to see that we can identify the Grothendieck algebra $K_G(G)$ with our fusion algebra for the case $\sigma = 1$. The matrix $S$ also features in the work of Lusztig where it is called a non-abelian Fourier transformation.

### 5.3.4. *Higher Genus Riemann Surfaces*

Just as any other RCFT, the partition function of a (holomorphic) orbifold model on a genus $g$ Riemann surface decomposes in generalized characters, that are labeled by the representations $\phi^g_\alpha$ and their interactions. However, in this case we have another natural basis of the bundle $V_g$ of holomorphic blocks, viz. the twisted sectors. Given a canonical homology basis $A_i, B_i$, $i = 1, \ldots, g$ (see Chapter 7 for its definition) we can have twists $g_i, h_i \in G$ along all these cycles, where the group elements are again defined up to simultaneous conjugation. They have to satisfy [69]

$$\prod_i [g_i, h_i] = 1. \tag{5.85}$$

This is the generalization of the one-loop constraint $[g, h] = 1$. It expresses that the set $g_i, h_i$ defines a homomorphism from the fundamental group of the Riemann surface to the group $G$. As such it defines a $G$-bundle over the surface. Two such homomorphisms give equivalent bundles if they are related by a global conjugation of a group element. The partition function can be expressed as the weighted sum over all possible twists

$$Z = \frac{1}{|G|^g} \sum_{g_i, h_i} Z(g_i, h_i), \tag{5.86}$$

where the $g_i, h_i$ satisfy (5.85). The chiral twisted sectors are defined by taking a holomorphic square root out of the $Z(g_i, h_i)$'s. That it is possible to take this square root can be seen by considering the theory on a covering of the Riemann surface. The partition function can be expressed as a sum of the modulus squared of the chiral twisted sectors, or a similar expression using the holomorphic blocks. This implies that there will be a linear basis transformation that expresses the twisted sectors into the generalized characters.

As explained in [153] one can calculate the number of generalized characters on a Riemann surface of genus $g$ using the matrix elements $S_{0i}$,

$$\dim V_g = \sum_i (S_{0i})^{2(1-g)}. \tag{5.87}$$

If we apply this formula to the case of an orbifold model of a holomorphic theory, we also obtain the number of independent twisted sectors. Mathematically this corresponds to the number of inequivalent $G$-bundles over the Riemann surface. To our knowledge an explicit expression for this quantity was not yet known. Substituting (5.62) in (5.87) we find

$$\dim V_g = \sum_{A,\alpha} \left[ \frac{N_A}{d_\alpha^A} \right]^{2(g-1)} \tag{5.88}$$

This is indeed always an integer, as it is a fundamental result in the theory of finite groups that the dimension of an irreducible representation always divides the dimension of the group [89]. One can further check that the dimension is indeed one on the sphere with the aid of the relation (see the appendix)

$$\sum_\alpha (d_\alpha^A)^2 = |N_A|, \qquad \sum_A |N_A|^{-1} = 1. \tag{5.89}$$

For abelian groups the above equation correctly reproduces

$$\dim V_g = |G|^{2g}. \tag{5.90}$$

Note that the dimensions of the Friedan-Shenker bundles $V_g$ do not depend on the particular $\sigma$ used to define the fusion algebra. However, if some of the twisted representations are projective, this will be the case. We return to this point in Chapter 8.

### 5.3.5. Discrete Torsion Revisited

Up to now we have been only concerned with the chiral structure. We considered the algebra $\mathcal{A}/G$, the representations and their modular properties. Now we want to investigate the combination of the holomorphic and anti-holomorphic characters into a modular invariant partition function. Although in principle we can choose different left and right chiral algebra, we will restrict ourselves here to the symmetric case.

Different modular invariant partition functions are classified by constructing all possible permutations $\Pi$ of the space of one-loop characters that commute with the modular group. As shown in section 4.6 this implies that $\Pi$ also constitutes an automorphism of the fusion algebra. On the other hand, in section 5.1.3 we have seen that different modular invariant partition functions for orbifold models are obtained by the inclusion of discrete torsion $\epsilon(g,h)$ in the expression for the partition function. We will now show that both approaches are equivalent.

Let us recall the conditions that the phase $\epsilon(g,h)$ has to satisfy

$$\delta_1\epsilon(h_1,h_2,g)=1, \quad \epsilon(h,g)=\epsilon(g^{-1},h), \quad \epsilon(g,g)=1. \tag{5.91}$$

The relation $\delta_1\epsilon=1$ implies that $\epsilon(h,g)$ is a one-dimensional representation of $N_h$. The symmetry property $\epsilon(h,g)=\epsilon(g^{-1},h)$ makes it also a representation of $N_g$. This implies that we can define an action of $\epsilon(g,h)$ on the one-loop characters by permuting the representations of $N_g$, completely similar to the action (5.34),

$$\Pi : R^g_\alpha(h)\rightarrow \epsilon(g,h)R^g_\alpha(h). \tag{5.92}$$

However, in contrast with (5.34) it acts here only on the holomorphic characters, while leaving the anti-holomorphic ones invariant, and accordingly the partition function will *not* be invariant. In fact, by construction the permutation of the characters commutes with the fusion rules. We now have to show that $\Pi$ commutes with the modular transformations: $\Pi S = S\Pi$ and $\Pi T = T\Pi$. Both are easy results of the conditions imposed on $\epsilon$.

### 5.3.6. An Example: $D_3$

As an example of a non-abelian group we will work out the details for the group $D_3\cong S_3$. It is generated by the elements $\tau$ and $\theta$ with defining relations $\theta^3=\tau^2=(\theta\tau)^2=1$. The conjugacy classes are $C_1=\{1\}$, $C_\tau=\{\tau,\tau\theta,\tau\theta^2\}$ and $C_\theta=\{\theta,\theta^2\}$. The stabilizer subgroups are given by $N_1=D_3$, $N_\tau=\mathbf{Z}_2$ and $N_\theta=\mathbf{Z}_3$. There are three irreducible representations of $D_3$: two one-dimensional representations defined by $R(\tau)=\pm1$ and $R(\theta)=1$, and a two-dimensional representation obtained by the embedding $D_3\subset O(2)$,

$$R(\tau)=\begin{pmatrix}0&1\\1&0\end{pmatrix} \qquad R(\theta)=\begin{pmatrix}0&-1\\1&-1\end{pmatrix} \tag{5.93}$$

| | $1$ | $1_-$ | $\theta_\nu$ | $\tau_+$ | $\tau_-$ |
|---|---|---|---|---|---|
| $1$ | 1 | 1 | 2 | 3 | 3 |
| $1_-$ | 1 | 1 | 2 | $-3$ | $-3$ |
| $\theta_\mu$ | 2 | 2 | $\epsilon_{\mu\nu}$ | 0 | 0 |
| $\tau_+$ | 3 | $-3$ | 0 | 3 | $-3$ |
| $\tau_-$ | 3 | $-3$ | 0 | $-3$ | 3 |

TABLE 2: The matrix $S$, up to an overall factor 1/6, for the $D_3$ orbifold with $\sigma = 1$. Here $\epsilon_{\mu\nu} = 4$ if $\mu = \nu$ and $-2$ otherwise ($\mu, \nu = 0, \ldots, 3$).

This representation is easily verified to be irreducible. We will denote these representations as $1$, $1_-$ and $\theta_0$ respectively. This notation will be explained in a moment. The stabilizer $N_\theta = \mathbf{Z}_3$ has three representations: $\theta_i$ ($i = 1, 2, 3$), and $N_\tau = \mathbf{Z}_2$ has two: $\tau_+, \tau_-$.

So altogether we have 8 operators in the $D_3$ orbifold. If we set $\sigma = 1$, the matrix $S$ is given as in table 2. From this table we read off the fusion rules

$$1_- \times 1_- = 1, \qquad \theta_\mu \times \theta_\mu = 1 + 1_- + \theta_\mu, \qquad \theta_\mu \times \theta_\nu = \sum_{\lambda \neq \mu, \nu} \theta_\lambda, \tag{5.94}$$

$$\tau_+ \times \tau_+ = \tau_- \times \tau_- = 1 + \sum_\mu \theta_\mu, \qquad \tau_+ \times \tau_- = 1_- + \sum_\mu \theta_\mu, \tag{5.95}$$

The remarkable symmetry in the $\theta_\mu$ can be understood, if we recall that for solvable groups, we can construct orbifolds by modding out by a sequence of abelian groups. If we first consider an orbifold obtained by modding out by the normal subgroup $\mathbf{Z}_3 \triangleleft D_3$, which possesses 9 operators, and then modding out by a $\mathbf{Z}_2$ ( $D_3/\mathbf{Z}_3 \cong \mathbf{Z}_2$). The group $\mathbf{Z}_2$ acts by exchanging all the operators, except the identity, giving rise to the 4 operators $\theta_\mu$ (as will be discussed in the next section).

An explicit realization of a $D_3$ orbifold with $\sigma = 1$ can be constructed using the $E_8$ $k = 1$ model. The appropriate subgroup is $D_3 \subset SU(3)$, with $SU(3) \times E_6 \subset E_8$. As an amusing exercise one can calculate the dimensions of the higher genus vector bundles, *e.g.* for a genus 2 surface one finds $\dim V_2 = 116$, a result that can be checked by hand.

The general form of the interactions is found by solving the cohomology problem. We include the phases $\sigma(g|h)$ and solve the condition $\delta_1 \delta_2 \sigma = 1$ modulo solutions of the form (5.47). In this case we find as the only independent variables

| | $1$ | $1_-$ | $\theta_0$ | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\tau_+$ | $\tau_-$ |
|---|---|---|---|---|---|---|---|---|
| $1$ | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 |
| $1_-$ | 1 | 1 | 2 | 2 | 2 | 2 | $-3$ | $-3$ |
| $\theta_0$ | 2 | 2 | 4 | $-2$ | $-2$ | $-2$ | 0 | 0 |
| $\theta_1$ | 2 | 2 | $-2$ | $2\alpha_0$ | $2\alpha_1$ | $2\alpha_2$ | 0 | 0 |
| $\theta_2$ | 2 | 2 | $-2$ | $2\alpha_1$ | $2\alpha_2$ | $2\alpha_0$ | 0 | 0 |
| $\theta_3$ | 2 | 2 | $-2$ | $2\alpha_2$ | $2\alpha_0$ | $2\alpha_1$ | 0 | 0 |
| $\tau_+$ | 3 | $-3$ | 0 | 0 | 0 | 0 | $3\beta$ | $-3\beta$ |
| $\tau_-$ | 3 | $-3$ | 0 | 0 | 0 | 0 | $-3\beta$ | $3\beta$ |

TABLE 3: The general form of the matrix $S$ for the $D_3$ orbifold with $\alpha_i = \sigma\omega^i + \overline{\sigma}\overline{\omega}^i$ ($\omega = e^{2\pi i/3}$).

$\sigma = \sigma(\theta|\theta)$ and $\beta = \sigma(\tau|\tau)$. Inserting this into $\delta_1\delta_2\sigma = 1$ we obtain $\sigma^9 = \beta^2 = 1$, and so there are 6 gauge inequivalent solutions. This gives the $S$-matrix of table 3. We still have $S = S^T$ and $S^2 = 1$. The fusion rules are modified only by:

$$\theta_i \times \theta_i = 1 + 1_- + N_{ii}{}^k\theta_k, \qquad \theta_i \times \theta_j = \theta_0 + N_{ij}{}^k\theta_k, \tag{5.96}$$

where the integer coefficients $N_{ij}{}^k$ are defined through $\alpha_i\alpha_i = 2 + N_{ii}{}^k\alpha_k$ and $\alpha_i\alpha_j = -1 + N_{ij}{}^k\alpha_k$. This can be considered to be the general fusion algebra of an holomorphic $D_3$ orbifold.

## 5.4. Abelian Groups and Toroidal Models

Many of the above, abstract results become more transparent when we consider an abelian group $G$. In that case the twisted sectors are labeled by the elements of $G$, and contain all irreducible representations. So the total number of primary operators is $|G|^2$. Furthermore the class algebra and the representation algebra are simply isomorphic to $G$. Without the inclusion of the phases $\sigma$ these two algebras would completely decouple. This would result in a fusion algebra given by (the group algebra) of $G \times G$. However, the occurrence of the phases is a generic phenomenon, as can be shown by considering the toroidal compactifications. These examples will also help to clarify the relation between the phases in the modular transformations and the resulting modification of the fusion algebra.

A well-known class of holomorphic theories is given by certain special toroidal compactifications (see [123], and *e.g.* [54]). These models are described in terms of a free scalar field $\phi^\mu$ $(\mu = 1, \ldots, d)$ compactified on a torus $\mathbf{R}^d/2\pi\Lambda$, where $\Lambda$ is a Euclidean even, self-dual lattice, *i.e.* for all $p, p' \in \Lambda$ we have $p^2 \in 2\,\mathbf{Z}$ and $p \cdot p' \in \mathbf{Z}$. (This requires $d = 0 \pmod 8$.) With the inclusion of an appropriate constant antisymmetric background field in the action the partition function is easily evaluated and seen to factorize as

$$Z(q,\overline{q}) = \Big|\frac{1}{\eta(q)^d}\sum_{p\in\Lambda} q^{\frac{1}{2}p^2}\Big|^2. \tag{5.97}$$

We can now construct the following class of abelian orbifolds from these models. We choose a lattice $\Lambda' \supset \Lambda$ and mod out by the transformations

$$\phi \to \phi + 2\pi\alpha, \qquad \alpha \in \Lambda'/\Lambda. \tag{5.98}$$

Here the relevant group $G$ is isomorphic to $\Lambda'/\Lambda$ considered as an additively written abelian group. The chiral action of the shift $\alpha$ is easily written down. It only depends on the chiral momentum $p \in \Lambda$ and is given by

$$\alpha : |p\rangle \to e^{2\pi i p\cdot\alpha}|p\rangle. \tag{5.99}$$

Furthermore, it is not difficult to construct the twisted Hilbert spaces $\mathcal{H}_\alpha$. They consist of momentum states $|p+\alpha\rangle$, $p \in \Lambda$. However, the action of $\beta \in G$ on $\mathcal{H}_\alpha$ is seen to be ambiguous. It depends on the choice of the representatives $\alpha, \beta \in \Lambda'$. This will be reflected in a phase ambiguity in the definition of the chiral blocks. Part of that ambiguity is resolved once we demand that the action of $G$ is linearly, *i.e.* non-projectively, represented. If we furthermore choose a set of representatives in $\Lambda'$, for example by taking $\alpha^2$ minimal (although this can be ambiguous), we obtain a definite expression for the chiral blocks:

$$\beta\underset{\alpha}{\square} = \frac{1}{\eta^d}\sum_{p\in\Lambda} e^{2\pi i\left[\frac{1}{2}(p+\alpha)^2\tau + p\cdot\beta\right]}. \tag{5.100}$$

The above definition does not depend on the choice $\beta \in \Lambda'$

$$(\beta + k)\underset{\alpha}{\square} = \beta\underset{\alpha}{\square}, \quad \forall\, k \in \Lambda. \tag{5.101}$$

This is however no longer true for $\alpha$, since

$$\beta\underset{\alpha+k}{\square} = e^{-2\pi i\beta\cdot k}\,\beta\underset{\alpha}{\square}. \tag{5.102}$$

That is to say, we have been forced to break the symmetry in the cycles $A$ and $B$ by defining time to flow along the $B$-cycle. That is of course in complete accordance with the Hamiltonian point of view. We note that the ambiguity (5.102) is of the form (5.47). It corresponds to multiplication by a one-dimensional representation of $G$. This definition of the chiral blocks gives rise to a modified action of the modular group. Indeed it is easy to verify that

$$S : \beta \underset{\alpha}{\square} \to e^{-2\pi i \alpha\cdot\beta} - \alpha \underset{\beta}{\square}. \tag{5.103}$$

So we find for the phase $\sigma$

$$\sigma(\alpha,\beta) = e^{-2\pi i \alpha\cdot\beta}. \tag{5.104}$$

It is not difficult to check that it satisfies all the conditions we imposed. Further note that $\tau(\alpha)^2 = \sigma(\alpha,\alpha) = e^{2\pi i \alpha^2}$ gives the correct weight $h = \frac{1}{2}\alpha^2$.

Let us now calculate the characters of the orbifold model. The representations of $G$ can be labeled by elements $\mu \in \Lambda/\Lambda'^*$, with $\Lambda'^*$ the dual lattice of $\Lambda'$. This results in the following expression for the characters

$$\chi^\alpha_\mu = \frac{1}{|G|} \sum_{\beta\in\Lambda'/\Lambda} e^{-2\pi i \mu\cdot\beta}\, \beta \underset{\alpha}{\square}, \tag{5.105}$$

which can be rewritten as

$$\chi^\alpha_\mu(q) = \frac{1}{\eta(q)^d} \sum_{p\in\Lambda'^*} q^{\frac{1}{2}(p+\alpha+\mu)^2}. \tag{5.106}$$

An evident result, since we have in fact constructed—in a complicated way—the toroidal model based on the lattice $\Lambda'$. It is now also evident what the fusion algebra is. It is given by $\Lambda'/\Lambda'^*$ which is in general not equal to $\Lambda'/\Lambda \times \Lambda'/\Lambda$. The projective action of $\beta \in G$ on the twisted sectors $\mathcal{H}_\alpha$ that accounts correctly for these fusion rules is according to (5.79) given by

$$\beta|p\rangle \to e^{2\pi i p\cdot\beta}|p\rangle, \qquad p \in \Lambda + \alpha. \tag{5.107}$$

This is indeed the action of a chiral shift $\phi \to \phi + 2\pi\beta$.

After this example it is not too difficult to treat the general abelian case. For simplicity we restrict ourselves here to $G = \mathbf{Z}_N$. Motivated by the above example we write $\mathbf{Z}_N$ additively. With an appropriate gauge transformation the phase $\sigma$ can always be given in the following form

$$\sigma(\alpha,\beta) = e^{2\pi i \alpha\beta k/N^2}, \tag{5.108}$$

with $\alpha,\beta \in \mathbf{Z}_N \equiv \mathbf{Z}/N\mathbf{Z}$. The conditions (5.53),(5.54) and (5.59) restrict $k \in \mathbf{Z}$. With the inclusion of this expression for $\sigma(\alpha,\beta)$ the fusion algebra equals

$\mathbf{Z}_{pN} \times \mathbf{Z}_{N/p}$ with $p = N/(k, N)$. For the case that $N$ is even, there is a further restriction on $\sigma$ which forces $k$ to be even. This is because when we look at the element $g$ of order two in $\mathbf{Z}_N$, since $g^{-1} = g$, when applying $S^2$ to the one-loop character corresponding to twistings by $(g, g)$ in the $(A, B)$ directions, we should get no extra phases. This implies that $\sigma(g|g)^2 = 1$, which in turn implies that $k$ is even. To compare with the toroidal case, consider modding $\Lambda$ by a shift vector $v$, with $Nv \in \Lambda$. If $(Nv)^2 = k \pmod N$, then it is easy to convince oneself that the operator algebra is $\mathbf{Z}_{pN} \times \mathbf{Z}_{N/p}$ with $p = N/(k, N)$. It is clear that because $\Lambda$ is an even lattice, when $N$ is even $k$ is also even, in accordance with the general arguments just discussed (for $N$ odd, since $k$ is only defined modulo addition by $N$ we can take $k$ to be even or odd).

## 5.5. Some Remarks on General Rational Orbifolds

In this section we will consider rational conformal field theories with more than one chiral block. We suppose that the theory has a symmetry $G$, and we wish to mod out by some action of $G$. It turns out that the resulting operator algebra is far more difficult to analyze than the orbifolds built out of holomorphic theories, and we will make only some general remarks about them. Examples of such orbifolds which display the general features discussed in this section will be presented in the next chapter (in the context of $c = 1$ models).

The fundamental difficulty in analyzing the orbifolds constructed from general RCFT's is that their modular properties do not seem to be completely dictated by the group structure and the operator algebra we started with. For example, we will have to consider chiral blocks of the form

$$\mathrm{Tr}_{[\phi_i]}\, g\, q^{L_0}, \tag{5.109}$$

where $[\phi_i]$ is a chiral sector of the theory we started with. Under the modular transformation $S$, this gives a set of new characters. How do we organize these new characters? In the holomorphic case we had the group structure to guide us in the organization, but now with the operator algebra of the initial theory mixed in, one has to find an organizing principle for the characters and this does not seem clear.

What we will do is limit ourselves just to the counting of the operators in generic cases for the case of solvable groups. We will first do this for the case where $G = \mathbf{Z}_k$. Of course we can continue the modding out by a sequence of $\mathbf{Z}_N$'s. Thus we will be able to do the counting for any solvable group. Non-abelian groups of this kind will be encountered in the next chapter.

Let $g$ be the generator of $\mathbf{Z}_k$, where $g^k = 1$. To begin with we will assume that $k$ is a prime. $g$ acts on the operator algebra by either permuting $k$ operators cyclically, or by acting trivially on some operators (this is so because we have chosen $k$ to be prime). Suppose the operator algebra we start with has $N$ chiral sectors. Let $n$ be the number of operators left fixed under the action of $g$, and $m$ be the number of groups of $k$ operators which are cyclically permuted by the action of $g$, then we have

$$N = n + mk. \tag{5.110}$$

We claim that the total number of operators $\tilde{N}$ that we will obtain by modding by $G$ is

$$\tilde{N} = nk^2 + m. \tag{5.111}$$

To see this consider the untwisted sector first. This corresponds to considering linear combination of characters of the form

$$\mathrm{Tr}_{[\phi_0]} g^l q^{L_0}. \tag{5.112}$$

For the sectors which are fixed under the action of the modular group this results in $k$ different characters for each sector*. However, for the $mk$ sectors which are permuted by this action the above trace vanishes and we get nothing new. In fact, since the sectors which are mapped to each other must have the same character (as is required if the $g$ action which permutes them is a symmetry of the conformal theory), for each group of $k$ sectors which are permuted among each other we are left with only one character. In other words, under the action of $g$ the $k$ sectors are identified and should be counted as one. So from the untwisted sector we obtain $nk + m$ operators. Now consider the sectors twisted by $g$. These sectors are obtained by considering the modular transformation $S$ on the characters in (5.112). As we discussed there are $n$ independent non-vanishing characters in (5.112). So we obtain $n$ independent sectors each twisted by $g$. Each of these sectors has to be decomposed into representations of $\mathbf{Z}_k$ which are obtained by taking linear combination of the modular transformations $T^l$ acting on them for $l$ running from 1 to $k$. So in the first twisted sector we obtain $nk$ new chiral sectors. This story repeats in each sector and we finally obtain $(k-1)nk$ chiral sectors from the twisted sectors. Including the contribution of the untwisted sector we see that we have $\tilde{N} = nk^2 + m$ operators, as was to be shown.

If $k$ is not a prime, the operators form orbits of size $l_i$ under the action of $g$, where $l_i$ divides $k$, and we have

$$N = \sum_i l_i. \tag{5.113}$$

*Here we are assuming a generic case, where there is no reason for (5.112) to vanish for the sectors which are fixed by the action of $g$. There are examples of conformal theories where extra symmetries force this to be zero for some sectors.

By a simple modification of the argument presented above, it is easy to see that the total number of operators we obtain is

$$\tilde{N} = \sum_i (k/l_i)^2. \tag{5.114}$$

This concludes our discussion for counting the number of operators in generic cases for solvable groups.

Even though we have not determined the operator algebra for the non-holomorphic case, there are some selection rules which are obvious. One is that group multiplication law dictates certain selection rules. Also the representation tensor products will also indicate a selection rule (modulo the inclusion of tensor products with a one dimensional representations) as discussed for holomorphic theories. The ideas in this section will be illustrated in the light of $c = 1$ conformal theories in the next chapter.

## Appendix A — Finite Group Theory

In this appendix we list some useful identities obtained in the theory of finite groups. To any finite group $G$ with elements $g_i$ we can associate a set $Irr(G)$ of irreducible representations $R_\alpha$ of dimension $d_\alpha$. The character of a representation is defined as the trace of the representation

$$\rho(g) = \mathrm{Tr}\, R(g). \tag{A.1}$$

Useful identities are $\rho(g^{-1}) = \rho(g)^*$ and $\rho(1) = \dim R$. The characters are invariant under similarity transformations $R(g) \to SR(g)S^{-1}$. This implies that they are only functions of the conjugacy classes $C_A$ and we can write $\rho(C_A)$. The collection of conjugacy classes will be denoted by $Cl(G)$. There are as many irreducible representations as there are (conjugacy) classes, although in general there is no natural mapping $R_\alpha \to C_A$. The fact that the $|Irr(G)| = |Cl(G)|$ can be proved by considering the group algebra.

The group algebra $A(G)$ (over $\mathbf{C}$) is the $|G|$-dimensional vector space with basis elements $g_i \in G$, i.e. it consists of elements $\alpha = \sum_i \alpha_i g_i$, $a_i \in \mathbf{C}$. Multiplication is defined as $\alpha \cdot \beta = \sum_{i,j} \alpha_i \beta_j g_i g_j$. The irreducible representations of $A(G)$ (as an algebra) are given by the elements of $Irr(G)$. By the left or right action of $G$, $A(G)$ itself can be regarded as a representation, the so-called regular representation.

The center $Z(G)$ of $A(G)$ is a subalgebra spanned by the elements

$$z_A = \sum_{g \in C_A} g. \tag{A.2}$$

Now we can show using Schur's lemma that $R_\alpha(z)$ is a multiple of the identity, since it commutes with all $R_\alpha(g)$'s. Taking a trace we find the factor of proportionality

$$R_\alpha(z_A) = \frac{\rho_\alpha(C_A)}{\rho_\alpha(1)} 1. \tag{A.3}$$

Further considerations one can show that this mapping is onto, so that $Cl(G)$ can be considered as the dual space to $Irr(G)$

The regular representation $R$ decomposes in $d_\alpha$ copies of the irreducible representation $R_\alpha$

$$R = \bigoplus_\alpha d_\alpha R_\alpha. \tag{A.4}$$

Taking traces of this identity we obtain the relation

$$|G| = \sum_\alpha d_\alpha^2. \tag{A.5}$$

The characters of the irreducible representations satisfy some very useful orthogonality relations

$$\frac{1}{|G|} \sum_{g\in G} \rho_\alpha(gh)\rho_\beta(g^{-1}) \;=\; \delta_{\alpha\beta}\frac{\rho_\alpha(h)}{\rho_\alpha(1)}, \tag{A.6}$$

$$\frac{1}{|G|} \sum_{\alpha} \rho_\alpha(g)\rho_\alpha(h^{-1}) \;=\; \frac{1}{|C_g|}\delta(g,h). \tag{A.7}$$

Here the $\delta$-function is defined by $\delta(g,h) = 1$ if $g$ and $h$ are conjugate, and 0 otherwise. Both relations express the unitarity of the matrix

$$S_\alpha{}^A = \left[\frac{|C_A|}{|G|}\right]^{\frac{1}{2}} \rho_\alpha(C_A). \tag{A.8}$$

The irreducible representations define an algebra

$$R_\alpha \otimes R_\beta = \sum_\gamma N_{\alpha\beta}{}^\gamma R_\gamma. \tag{A.9}$$

This is an associative, commutative algebra and accordingly has only one-dimensional representations. They are given by the characters $\rho_\alpha(C_A)$ as labeled by the classes. In particular there is one representation $\rho_\alpha(1) = d_\alpha \in \mathbf{Z}_{>0}$. Accordingly we have

$$N_{\alpha\beta\gamma} = \frac{1}{|G|} \sum_{g\in G} \rho_\alpha(g)\rho_\beta(g)\rho_\gamma(g). \tag{A.10}$$

# 6

# Conformal Field Theory at $c = 1$

In this chapter we will discuss conformal field theories with central charge $c = 1$. The value $c = 1$ is in many respects a threshold value. The conformal field theories with $c$ smaller than one have been successfully classified, using the severe restrictions imposed by the representation theory of the Virasoro algebra with $c < 1$ [16], unitarity [76], and modular invariance [81,28]. Unfortunately, the picture for $c \geq 1$ is less complete. The only constraint that follows from unitarity is positivity of the conformal weights and, because the number of primary fields will always be infinite [30], the condition of modular invariance is much more difficult to analyze. The possible extension of the classification program to arbitrary $c$-values essentially comprises two different strategies, of a complementary nature.

First, one can presume that the model has an extended symmetry algebra, and try to repeat the classification program in complete analogy with the Virasoro algebra. In this case one can hope to find a similar exhaustive result for models with this extended symmetry and central charges $c < c_{crit}$, with a critical $c$-value $c_{crit} > 1$. Indeed, studies of particular chiral algebras, such as affine algebras and superconformal algebras, for which the analysis can be closely matched to the Virasoro case, have led to exactly these kind of results [28,29,106]. For affine algebras based on a Lie group $G$ the critical value is $c_{crit} = \dim G$, and for $N = 1, 2$ superconformal algebras $c_{crit} = \frac{3}{2}, 3$.

However, as we have shown in Chapter 4 this approach will give us—for a fixed value of the central charge—only a discrete, finite spectrum of models, determined by the possible left-right pairings. It fails to encapture one of the remarkable new features of conformal field theory that appears for $c \geq 1$: the possibility of a *continuum* of inequivalent models with a given $c$-value. We have seen examples of this in section 2.3 where we studied moduli spaces of toroidal compactifications. The existence of continuous deformations is due to the presence of marginal operators in the spectrum, *i.e.* primary fields with conformal dimensions $h = \overline{h} = 1$ that can be regarded as the infinitesimal generators of smooth deformations. The study of these marginal deformations gives a perturbative tool to investigate a local neighborhood of a conformal model in CFT-space. The relation to the discrete approach of the previous paragraph, is that the models with extended symmetry correspond to special points in the moduli space. Perturbing the model with a marginal operator will in general destroy the extended symmetry, leaving only the generic conformal invariance.

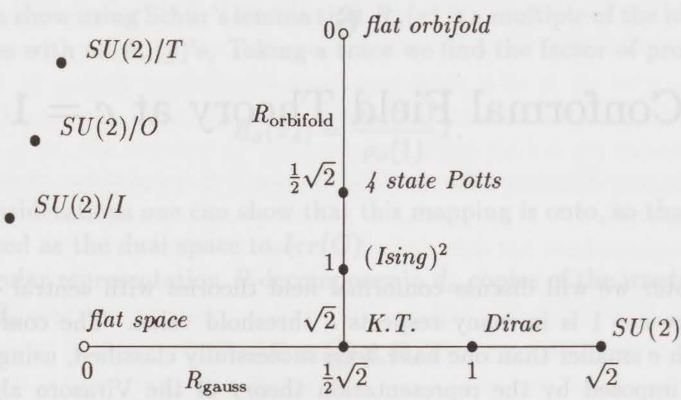



FIGURE 9: The moduli space of known $c = 1$ CFT's. It consists of two continuous half-lines, that meet in a multi-critical point, and three isolated models. The different indicated models have special extended symmetries and are explained in the text.

This manifold structure of the moduli space of CFT's is very nicely illustrated for the value $c=1$. In the following sections we will argue that the space of $c=1$ conformal field theories looks like *fig.* 9. Apart from three isolated models, there exist two one-parameter families of theories—respectively the gaussian models, describing a free bosonic field $\phi$ with period $R$, and their orbifold relatives obtained through the identification $\phi \to -\phi$. The parameter space is in both cases a half-line because of the duality equivalence $R \to 2/R$. At a multi-critical point these two sets intersect. Here the moduli space is no longer a smooth manifold; there are two independent integrable marginal operators. The derivation and interpretation of *fig.* 9 will be our main goal in this chapter.

### 6.1. Operator Content and Multi-Critical Points at $c = 1$

The fundamental model at $c=1$ is the free bosonic field $\phi(z, \bar{z})$. We will always assume $\phi$ to be periodic, $\phi \sim \phi + 2\pi R$, with $R$ the compactification radius. The action is the gaussian one, with coupling constant $g$

$$S = \frac{g}{2\pi} \int d^2z \, \partial\phi\bar{\partial}\phi. \tag{6.1}$$

We observe that only the quantity $gR^2$ is physically relevant. So we will make use of the freedom to rescale $\phi$ and put $g = 1$, leaving $R$ to parametrize the different models. The non-periodic case will be treated as the limit $R \to \infty$.

The easiest way to determine the operator content is to compute the one-loop partition function on the torus $z \sim q\,z$,

$$Z_R(q,\overline{q}) = \mathrm{Tr}\left(q^{L_0-\frac{1}{24}}\overline{q}^{\overline{L}_0-\frac{1}{24}}\right), \tag{6.2}$$

and to decompose it into characters of irreducible representations of the $c = 1$ Virasoro algebra. The partition function of a compactified scalar field is well-known [44], see also our discussion in section 2.2.2,

$$Z_R(q,\overline{q}) = \frac{1}{|\eta(q)|^2} \sum_{(p,\overline{p})\in\Gamma_R} q^{\frac{1}{2}p^2}\overline{q}^{\frac{1}{2}\overline{p}^2}, \tag{6.3}$$

with $\eta(q)$ the Dedekind eta-function

$$\eta(q) = q^{\frac{1}{24}} \prod_{n\geq 1}(1-q^n),$$

and $\Gamma_R$ the momentum lattice

$$\Gamma_R = \frac{1}{R}\mathbf{Z}\begin{pmatrix}1\\1\end{pmatrix} \oplus \frac{R}{2}\mathbf{Z}\begin{pmatrix}1\\-1\end{pmatrix}. \tag{6.5}$$

The lattice $\Gamma_R$ is obtained by a Lorentz boost of the lattice $\Gamma^{1,1}$ of equation (2.50). For the special limit $R \to \infty$ the summation over the momenta $p, \overline{p}$ becomes an integral (with a restriction $p = \overline{p}$), and gives

$$Z_\infty(q,\overline{q}) = (\mathrm{Im}\,\tau)^{-\frac{1}{2}}|\eta(q)|^{-2}, \tag{6.6}$$

with $q = e^{2\pi i\tau}$.

The Virasoro algebra at $c = 1$ does not act freely but possesses null states at the weights $h = \frac{1}{4}n^2$, with $n \in \mathbf{Z}$ [99]. Accordingly, the Virasoro characters are given by the following expressions

$$\chi_h^{(Vir)}(q) = \begin{cases} \dfrac{1}{\eta(q)}\left[q^{\frac{1}{4}n^2} - q^{\frac{1}{4}(n+2)^2}\right], & \text{if } h = \frac{1}{4}n^2,\ n \in \mathbf{Z}, \\[2ex] \dfrac{1}{\eta(q)}q^h, & \text{otherwise.} \end{cases} \tag{6.7}$$

Because of these null states, it is more convenient to discuss the spectrum first in terms of the generic symmetry algebra of the gaussian model: the $U(1) \times U(1)$

current algebra, which is freely generated by the spin one fields $J(z) = i\partial\phi(z)$ and $\overline{J}(z) = i\overline{\partial}\overline{\phi}(z)$. The characters of the $U(1)$ current algebra of charge $p$ are simply given by

$$\chi_p(q) = \frac{1}{\eta(q)} q^{\frac{1}{2}p^2}, \tag{6.8}$$

and the primary fields with respect to the current algebra are the vertex operators

$$V_{p\overline{p}}(z, \overline{z}) = \exp\left(ip\phi(z) + i\overline{p}\overline{\phi}(\overline{z})\right), \tag{6.9}$$

with $p, \overline{p} \in \Gamma_R$. We observe that the vertex operators are not correctly normalized when regarded as Virasoro primary fields, since they are not self-conjugate. The combinations with the correct properties under charge conjugation are

$$\begin{aligned} V_{nm}^{+} &= \sqrt{2}\cos\left(p\phi + \overline{p}\overline{\phi}\right), \\ V_{nm}^{-} &= \sqrt{2}\sin\left(p\phi + \overline{p}\overline{\phi}\right), \end{aligned} \tag{6.10}$$

with the momenta $(p, \overline{p})$ related to the integers $(n, m)$ by

$$p = \frac{n}{R} + \tfrac{1}{2}mR, \qquad \overline{p} = \frac{n}{R} - \tfrac{1}{2}mR. \tag{6.11}$$

The vertex operator $V_{nm}^{\pm}$ has conformal weights $h_{nm} = \frac{1}{2}p^2, \overline{h}_{nm} = \frac{1}{2}\overline{p}^2$, and so in particular spin $s_{nm} = nm$. The special spin zero operators $V_{n0}$ and $V_{0m}$ are usually called *electric* respectively *magnetic*. Electric operators can be represented in the path-integral by insertions of $e^{in\phi/R}$ and represent electric charges coupled to the Coulomb field $\phi$. (Recall that we have a propagator $\langle\phi\phi\rangle \sim \log|z - w|^2$.) The magnetic operators create vortices in the field $\phi$, *i.e.* configurations where $\phi$ has a non-vanishing period around the insertion of the magnetic operator, as can be seen from the OPE

$$\phi(z, \overline{z}) \cdot V_{0m}(0) \sim -\tfrac{1}{2}imR\log(z/\overline{z})V_{0m}(0). \tag{6.12}$$

According to (6.7) we have the following branching of $U(1)$ characters into Virasoro characters

$$\frac{1}{\eta(q)} q^{\frac{1}{2}p^2} = \begin{cases} \displaystyle\sum_{n=0}^{\infty} \chi^{(Vir)}_{(n+\frac{1}{2}\sqrt{2}p)^2}, & \text{if } p \in \frac{1}{\sqrt{2}}\mathbf{Z}, \\ \chi^{(Vir)}_{\frac{1}{2}p^2}, & \text{otherwise.} \end{cases} \tag{6.13}$$

We see that for the special momenta $p \in \frac{1}{\sqrt{2}}\mathbf{Z}$ extra primary fields appear in the decomposition of the charge $p$ sector, besides the vertex operators; fields of the form $F(\partial, \phi)V_{nm}^{\pm}$, with $F$ a polynomial in $\partial\phi$ and its derivatives. In particular, in the charge zero sector ($p = \overline{p} = 0$) we have an infinite set of primary chiral fields $J_{n^2}(z)$ of spin $h = n^2$, with $n \in \mathbf{Z}$. These fields can be expressed in normal ordered Schur polynomials in the current $J(z)$ and its multiple derivatives [157]. For example, $J_1(z) = J(z)$ and the spin 4 field $J_4(z)$ is given by

$$J_4 = J^4 - 2J\,\partial^2 J + \tfrac{3}{2}(\partial J)^2, \tag{6.14}$$

where normal ordering is understood.

For all values of $R$ we have the primary (1,1) tensor $\partial\phi\overline{\partial\phi}(z,\overline{z})$, which can easily be seen to satisfy the conditions of an integrable marginal operator. In fact, an additional term in the action of the form

$$\delta S = \frac{\delta g}{2\pi} \int d^2 z\, \partial\phi\overline{\partial\phi}(z,\overline{z}) \tag{6.15}$$

can be absorbed by a redefiniton of $\phi$ changing its compactification radius by $\delta R^2 = R^2 \delta g$. Indeed, according to (2.54) the resulting shift in the weights $h_{nm}$ of the vertex operators $V_{nm}$ is [103]

$$\frac{\delta}{\delta g} h_{nm} = -\frac{n^2}{R^2} + \frac{m^2 R^2}{4} = \tfrac{1}{2} R \frac{\delta}{\delta R} h_{nm}, \tag{6.16}$$

where we used the operator product relation

$$\partial\phi\overline{\partial\phi}(z,\overline{z}) \cdot V_{nm}(0) \sim \left( \frac{n^2}{R^2} - \frac{m^2 R^2}{4} \right) \frac{1}{|z|^2} V_{nm}(0). \tag{6.17}$$

As we have seen in section 2.3 there is a redundancy in the parameter $R$. The moduli space of one-dimensional toroidal models is given by the half-line $\mathbf{R}/\mathbf{Z}_2$, where the $\mathbf{Z}_2$ is generated by the electric-magnetic duality transformation

$$R \to 2/R. \tag{6.18}$$

Under this duality electric and magnetic vertex operators are interchanged, and the marginal operator changes sign:

$$V_{nm} \to V_{mn}, \qquad \partial\phi\overline{\partial\phi} \to -\partial\phi\overline{\partial\phi}. \tag{6.19}$$

We will parametrize the models by $R \leq \sqrt{2}$.

### 6.1.1. Multi-Critical Points

Let us now look for multi-critical points on the line of gaussian models. For $R$ or $1/R$ equal to a multiple of $\sqrt{2}$ extra marginal operators appear in the form of vertex operators. However, because of the operator product relation (6.17) we find that the condition (*ii*) of integrability of section 2.3 can only be met if both electric and magnetic weight (1,1) vertex operators are present. This condition leaves as only possible multi-critical points $R = \sqrt{2}$ and $R = \frac{1}{2}\sqrt{2}$.

The model at $R = \sqrt{2}$ is the fixed point of the duality transformation (6.18). It has an enhanced symmetry group $SU(2) \times SU(2)$. The associated conserved chiral currents

$$\begin{aligned} J_1 &= \cos\sqrt{2}\phi, \\ J_2 &= \sin\sqrt{2}\phi, \\ J_3 &= i\tfrac{1}{2}\sqrt{2}\partial\phi, \end{aligned} \tag{6.20}$$

generate the level one Kac-Moody algebra $A_1^{(1)}$

$$J_i(z)J_j(w) \sim \frac{\frac{1}{2}\delta_{ij}}{(z-w)^2} - \frac{i\epsilon_{ijk}J_k(w)}{z-w}. \tag{6.21}$$

As is well-known, this particular gaussian model can be identified with the $k=1$ $SU(2)$ Wess-Zumino-Witten model. Each primary field belongs to a multiplet that transforms in the isospin $(j, \bar{j})$ representation of $SU(2) \times SU(2)$. The conformal weights of these fields are $(h, \bar{h}) = (j^2, \bar{j}^2)$. Modular invariance of the one-loop partition function implies the condition $j + \bar{j} \in \mathbf{Z}$ [83]. This implies that the (diagonal) center of $SU(2)$ acts trivially on the spectrum of the model.

In total there are 9 marginal operators $J_i(z)J_j(\bar{z})$. Imposing the conditions of integrability and using the OPE (6.21) one finds that all linear combinations of the form

$$\Big(\sum_{i=1}^{3} \alpha_i J_i\Big)\Big(\sum_{i=1}^{3} \overline{\alpha}_i \overline{J}_i\Big), \tag{6.22}$$

with arbitrary $\alpha_i, \overline{\alpha}_i$, generate a continuous deformation of the $SU(2)$ model. But, since all these operators are related by the global symmetry group to $J_3\overline{J}_3 = \frac{1}{2}\partial\phi\overline{\partial\phi}$, every marginal perturbation is equivalent to a modification of the compactification radius $R$. Note that the electric-magnetic duality can also be understood as a consequence of the $SU(2) \times SU(2)$ symmetry, since $\pm\partial\phi\overline{\partial\phi}$ are equivalent marginal operators at $R = \sqrt{2}$.

Our second candidate multi-critical point is at $R = \frac{1}{2}\sqrt{2}$ which corresponds to the continuum limit of the $XY$-model at the Kosterlitz-Thouless point [103,104]. Let us analyze this model using its relation with the $SU(2)$ model. Quite generally one implements the transformation $R \to \frac{1}{2}R$ on the spectrum by projecting onto even momentum states and adding extra sectors with half-integer winding numbers. These sectors have ground states created out of the vacuum by the magnetic operators $V^{\pm}_{0\frac{1}{2}}$. In the case $R = \sqrt{2}$ we can be more explicit about this. Let us introduce the projection operators $\Pi_k$

$$\Pi_k = \tfrac{1}{2}(1+\theta_k\overline{\theta}_k), \qquad \theta_k = \exp\left[\tfrac{1}{2}\oint dz\, J_k(z)\right]. \tag{6.23}$$

The operators $\theta_k$, which satisfy

$$\theta_k^2 = 1, \quad \theta_1\theta_2 = \theta_3 \quad (+\ \mathit{cyclic}), \tag{6.24}$$

generate the following transformations on the *chiral* field $\phi(z)$ (mod $\sqrt{2}\pi$)

$$\begin{aligned} \theta_1\,\phi(z)\,\theta_1 &= -\phi(z), \\ \theta_2\,\phi(z)\,\theta_2 &= -\phi(z)+\sqrt{2}\pi, \\ \theta_3\,\phi(z)\theta_3 &= \phi(z)+\sqrt{2}\pi. \end{aligned} \tag{6.25}$$

The appropriate projection operator in our context is $\Pi_3$, since it evidently reduces the compactification scale to $R=\frac{1}{2}\sqrt{2}$. The sectors with integer winding number are created by the weight $(\frac{1}{16},\frac{1}{16})$ vertex operators $\sqrt{2}\cos\frac{1}{4}\sqrt{2}(\phi-\overline{\phi})$ and $\sqrt{2}\sin\frac{1}{4}\sqrt{2}(\phi-\overline{\phi})$. The projection $\Pi_3$ leaves a total of 5 independent marginal operators, out of which the following (to first order) integrable combinations can be formed

$$J_3\overline{J}_3, \quad \Big(\sum_{i=1}^{2}\alpha_i J_i\Big)\Big(\sum_{i=1}^{2}\overline{\alpha}_i\overline{J}_i\Big), \tag{6.26}$$

All the elements of the second set are related by the $U(1)\times U(1)$ symmetry generated by the two chiral currents $J_3$ and $\overline{J}_3$. These marginal deformations, which we will show in a moment to be integrable to *all* orders, are inequivalent to the one induced by $J_3\overline{J}_3$. This implies that at this particular point $R=\frac{1}{2}\sqrt{2}$ a new continuous deformation of the theory exists that changes the nature of the compactification! The meaning of this new direction becomes clear when we use the obvious invariance under the permutation of the three projections $\Pi_i$. In particular, we can repeat the above construction with $\Pi_1$ as projection operator instead of $\Pi_3$. As seen from the action of $\theta_1$ on $\phi$, this construction results in an identification of $\phi$ with $-\phi$. So in fact we have constructed a $\mathbf{Z}_2$ orbifold compactification with $R=\sqrt{2}$. This radius is further varied by the action of the marginal operator. Thus the new marginal direction is seen to correspond to the line of $\mathbf{Z}_2$ orbifold models. This equivalence of toroidal and orbifold compactifications has been uncovered in [55], see also [124].

### 6.1.2. Spectrum of the $\mathbf{Z}_2$ Orbifold Models

The spectrum of operators in the orbifold models* consists of a twisted and untwisted part (see [53] and also the review in [85]). The untwisted states are found

*Note that the $Z_2$ orbifold corresponds to the continuum limit of the critical Askin-Teller model [103,104], which describes two Ising models (with spins $\sigma_1$ and $\sigma_2$) coupled by an interaction term of the form $(\sigma_1\sigma_2)^2$.

by projecting the spectrum of the corresponding gaussian model onto even states under $\phi \to -\phi$ using the projection operator $\Pi_1$. Of the vertex operators only the operators $V^{+}_{nm}$ are left. In addition we have the twisted sectors created out of the vacuum by the twist fields $\sigma_1$ and $\sigma_2$ with conformal weights $(\frac{1}{16}, \frac{1}{16})$. These $\sigma_i$, and their partners $\tau_i$ defined by

$$\partial\phi\overline{\partial\phi}(z, \overline{z}) \cdot \sigma_i(w, \overline{w}) \sim \frac{1}{|z - w|}\tau_i(w, \overline{w}), \tag{6.27}$$

are the only relevant operators in the twisted sectors. The weights of the $\tau_i$ are $(\frac{9}{16}, \frac{9}{16})$. In fact, the weights of *all* twisted operators are independent of the radius $R$. One way to see this is to note that the OPE of the marginal operator $\partial\phi\overline{\partial\phi}$ with any twisted field $\psi$ only contains operators with weights which differ from that of $\psi$ by a half-integer, since both $\partial\phi$ and $\overline{\partial\phi}$ acquire a branch cut starting at $\psi$. This forces the three-point function $\langle \partial\phi\overline{\partial\phi}\, \psi\, \psi \rangle$ to be zero, and consequently $\partial\phi\overline{\partial\phi}$ does not change the weight of the field $\psi$.

As discussed in Chapter 5, the general form of the one-loop partition function of an orbifold consists of a summation over twisted sectors. For a $\mathbf{Z}_2$ group we have the representation

$$Z^{orb}_R = \frac{1}{2}\left( {}_{+}\underset{+}{\square} + {}_{-}\underset{+}{\square} + {}_{+}\underset{-}{\square} + {}_{-}\underset{-}{\square} \right), \tag{6.28}$$

where $\pm$ indicates the two possible boundary conditions along the $A$- and $B$-cycle of the torus. The first block equals the gaussian partition function $Z_R$ of Eq. (6.3). The other three twisted blocks are permuted under modular transformations. If we rewrite the above relation as

$$Z^{orb}_R = \tfrac{1}{2} Z_R + Z_{twist}, \tag{6.29}$$

the scale independence of the twisted sector implies that the part $Z_{twist}$ can be determined at the multi-critical point $R = \sqrt{2}$ using the equivalence with the $R = \frac{1}{2}\sqrt{2}$ gaussian model. Equating the two expressions we find

$$Z_{twist} = Z(\tfrac{1}{2}\sqrt{2}) - \tfrac{1}{2}Z(\sqrt{2}). \tag{6.30}$$

This is in accordance with the results of [134].

There also exists an electric-magnetic duality for the line of orbifold models. As for the untwisted part of the spectrum this transformation acts similarly as in the gaussian model. The action of the duality on the twisted part can most easily be understood at the self-dual point $R = \sqrt{2}$. Here the twist fields $\sigma_1$ and $\sigma_2$ are equivalent to the half-integer magnetic operators $V^{\pm}_{0\frac{1}{2}}$ of the $SU(2)$ model. In this correspondence duality translates into shifting

$$\phi(z) \to \phi(z), \qquad \overline{\phi}(\overline{z}) \to \overline{\phi}(\overline{z}) + \tfrac{1}{2}\sqrt{2}\pi, \tag{6.31}$$

since that changes the sign of the marginal operator $J_1\overline{J}_1 = \cos\sqrt{2}\phi\cos\sqrt{2}\overline{\phi}$. The effect of this shift is easily calculated, and the resulting duality transformation is seen to act on the twist fields as

$$\sigma_1 \to \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_2), \qquad \sigma_2 \to \frac{1}{\sqrt{2}}(\sigma_1 - \sigma_2). \tag{6.32}$$

The two twist fields $\sigma_1$ and $\sigma_2$ and their respective sectors correspond to the two conjugacy classes (or equivalently fixed points) of the $\mathbf{Z}_2$ twist:

$$\phi \to -\phi \pmod{4\pi R}, \quad \phi \to -\phi + 2\pi R \pmod{4\pi R}. \tag{6.33}$$

From this we conclude that the product of two operators in the same twisted sector produces operators which create an even winding number, whereas the product of two operators in different twist sectors creates odd winding numbers. In combination with electric-magnetic duality this implies the following form of the operator product relations

$$\begin{aligned}
[\sigma_1]\cdot[\sigma_1] &\sim \sum_{n,m} C_{2n,2m}[V^+_{2n,2m}] + \sum_{n,m} C_{2n+1,2m}[V^+_{2n+1,2m}],\\
[\sigma_2]\cdot[\sigma_2] &\sim \sum_{n,m} C_{2n,2m}[V^+_{2n,2m}] - \sum_{n,m} C_{2n+1,2m}[V^+_{2n+1,2m}],\\
[\sigma_1]\cdot[\sigma_2] &\sim \sum_{n,m} C_{2n,2m+1}[V^+_{2n,2m+1}].
\end{aligned} \tag{6.34}$$

The numerical coefficients $C_{n,m}$ are given in [167,53]

$$C^2_{n,m} = \begin{cases} 1 & \textit{if } n = m = 0,\\ 2\cdot 16^{-h_{nm}-\overline{h}_{nm}} & \textit{otherwise.}\end{cases} \tag{6.35}$$

The global symmetry group of the $\mathbf{Z}_2$ orbifold is the dihedral group $D_4$ (the symmetry group of the square) generated by

$$\begin{aligned}
(\sigma_1,\sigma_2,V_{nm}) &\to (-\sigma_1,\sigma_2,(-)^m V_{nm}),\\
(\sigma_1,\sigma_2,V_{nm}) &\to (\sigma_2,\sigma_1,(-)^n V_{nm}).
\end{aligned} \tag{6.36}$$

The invariance under these transformations follows from (6.34). Of course, the duality transformations (6.19), (6.32) should be read modulo $D_4$.

The twisted sector does not contain any weight $(1,1)$ conformal fields, so we can copy the arguments applied to the gaussian model to show that besides the point $R = \sqrt{2}$ the only other multi-critical point is $R = \frac{1}{2}\sqrt{2}$. This model is known to correspond to the continuum limit of the 4-state Potts model [126,44]. It contains the following relevant and marginal operators of weight $(h,\overline{h})$

$$3\times(\tfrac{1}{16},\tfrac{1}{16}),\ 1\times(\tfrac{1}{4},\tfrac{1}{4}),\ 3\times(\tfrac{9}{16},\tfrac{9}{16}),\ 3\times(1,1). \tag{6.37}$$

The frequent occurrence of the multiplicity 3 can be understood from the fact that this model can be obtained from the $SU(2)$ model by twisting with all three $\theta_i$'s of (6.23). That is, it can be obtained as a $D_2 = \mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold of the $SU(2)$ model. The integrable marginal operators which are not projected out are $J_i\overline{J}_i$ ($i = 1, 2, 3$). It is clear from this construction that the $SU(2)$ invariance is broken to a discrete symmetry that permutes the three operators. So no new deformations can be found. In this picture we have, besides the projected $SU(2)$ spectrum, three extra sectors generated by the twist operators $\sigma_1$, $\sigma_2$ and $\sigma_3 \equiv \sqrt{2}\cos\frac{1}{4}\sqrt{2}(\phi - \overline{\phi})$ with OPE

$$\sigma_1(z,\overline{z})\sigma_2(w,\overline{w}) \sim \frac{2^{\frac{1}{4}}}{|z-w|^{\frac{1}{4}}}\sigma_3(w,\overline{w}), \qquad (+\ cyclic). \tag{6.38}$$

These three twist operators correspond to the spins $\sigma_1, \sigma_2$, and $\sigma_1\sigma_2$ of the Ashkin-Teller model. The full symmetry group is $S_4$ (the permutation group of 4 elements) which acts on the $\sigma_i$'s as

$$(\sigma_1, \sigma_2, \sigma_3) \to \left((-)^p\sigma_{i_1}, (-)^q\sigma_{i_2}, (-)^{p+q}\sigma_{i_3}\right), \tag{6.39}$$

with $(1,2,3) \to (i_1, i_2, i_3)$ a permutation in $S_3$ and $p, q = 0, 1$. Indeed, $S_4$ is the symmetry group of the 4-state Potts model†. Before the projections, each twisted sector is an irreducible representation of the (twisted) $A_1^{(1)}$ algebra. The fact that all three representations are isomorphic is known in the mathematical literature as triality of $A_1^{(1)}$ [70].

The orbifold theory at $R = 1$ corresponds to the decoupling point of the Ashkin-Teller model, where the model reduces to two independent Ising systems [164,59]. In the orbifold model we can indeed identify in the spectrum two sets of the Ising operators $1^{(i)}, \sigma^{(i)}, \epsilon^{(i)}$ ($i = 1, 2$). With $\sigma^{(1)} \equiv \sigma^{(1)} \otimes 1^{(2)}$ *etc.*, we have

$$\begin{aligned}
T^{(1)} &= -\tfrac{1}{4}(\partial\phi)^2 + \tfrac{1}{2}\cos 2\phi, & T^{(2)} &= -\tfrac{1}{4}(\partial\phi)^2 - \tfrac{1}{2}\cos 2\phi,\\
\overline{T}^{(1)} &= -\tfrac{1}{4}(\overline{\partial\phi})^2 + \tfrac{1}{2}\cos 2\overline{\phi}, & \overline{T}^{(2)} &= -\tfrac{1}{4}(\overline{\partial\phi})^2 - \tfrac{1}{2}\cos 2\overline{\phi},\\
\sigma^{(1)} &= \sigma_1, & \sigma^{(2)} &= \sigma_2,\\
\epsilon^{(1)} &= -2\sin\phi\sin\overline{\phi}, & \epsilon^{(2)} &= -2\cos\phi\cos\overline{\phi},
\end{aligned}$$
$$\begin{aligned}
\sigma^{(1)}\otimes\sigma^{(2)} &= \sqrt{2}\cos\tfrac{1}{2}(\phi - \overline{\phi}),\\
\epsilon^{(1)}\otimes\epsilon^{(2)} &= -\partial\phi\overline{\partial\phi},\\
\sigma^{(1)}\otimes\epsilon^{(2)} &= \tau_1,\\
\epsilon^{(1)}\otimes\sigma^{(2)} &= \tau_2.
\end{aligned} \tag{6.40}$$

†Note that $S_4 \cong O$ with $O$ the octahedral group, the symmetry group of the cube. The representation of $S_4$ on the $\sigma_i$'s is the three-dimensional representation of $O \subset SO(3)$.

One easily checks with (6.34) that the two sets of operators have the correct operator product relations. All other primary fields of the orbifold model are descendants of these operators with respect to the two Virasoro algebras generated by $T^{(1)}(z)$ and $T^{(2)}(z)$. The above identifications can be used to calculate Ising correlation functions within the $c=1$ orbifold model.

This completes the first part of our analysis leading to *fig.* 9. We would like to close this section by pointing out a very appealing symmetric relation between the two lines of gaussian and orbifold models [85]. The former are described in terms of the fundamental field $\phi$, and the orbifolds are obtained by the twist $\phi \to -\phi$, that eliminates the current $\partial\phi$. This could give the impression that the gaussian model is in some sense more fundamental. However, the orbifolds can also be described in terms of the spin fields $\sigma_i$ of the Ashkin-Teller model. In this picture the gaussian models are obtained by removing these spin fields with the twist $\sigma_i \to -\sigma_i$.

## 6.2. Completeness?

We have proved in the last section that no new $c = 1$ models other than the $\mathbf{Z}_2$ orbifolds can be obtained by continuous deformations of the gaussian models. This certainly does not rule out the possibility that other, *disconnected* parts of the $c=1$ moduli space might exist. In fact, three isolated models have been found by Ginsparg [84], see also [93]. As we have seen in Chapter 5 a very productive way to generate new models is the orbifold construction. The generic gaussian model only allows a simple twist $\phi \to -\phi$, but in the $SU(2)$ model we can take the quotient by any discrete subgroup of $SU(2)$. The one-loop partition functions of these models have been calculated, in the context of lattice models, by Pasquier [127] and where correctly interpreted as orbifold partition functions in [84].

Recall that the finite subgroups $G$ of $SO(3)$ are in one-to-one correspondence with the simply-laced Lie algebras and allow a $ADE$ type of classification [39,145], see *fig.* 10. We have two series, the cyclic groups $\mathbf{Z}_N$ and the dihedral groups $D_N$, and three exceptional non-abelian groups $T$, $O$, and $I$—the symmetry groups of respectively the tetrahedron, octahedron, and icosahedron (of order 12, 24, and 60). In our case we actually need the lifts of these groups $G$ into $SU(2)$, the double cover of $SO(3)$, which are known as the binary polyhedral groups. However, since the center of $SU(2)$ acts trivially on the Hilbert space of the model*, this is a somewhat academic point. (Compare however the discussion of the *chiral* action of $G$ in section 6.4.) The cyclic and dihedral models can easily be seen to give rise to particular gaussian and $\mathbf{Z}_2$ orbifold models, and so are already included in our picture, but the three exceptional groups lead to new models. They do not

*Recall that diagonal quantum number $j + \bar{\jmath}$ is always integer.

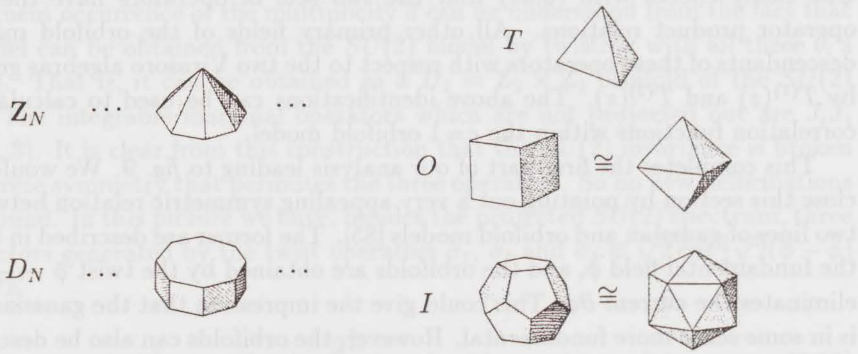



FIGURE 10: The finite subgroups of the three-dimensional rotation group have an $ADE$ classification: two infinite series of cyclic ($\mathbf{Z}_N$) and dihedral ($D_N$) groups, and three exceptional groups ($T, O, I$) associated to the five Platonic solids.

allow for marginal deformations, since the only weight (1,1) field is the abolute $G$-invariant $\sum_i J_i \overline{J}_i$, which is clearly not integrable.

It turns out that the one-loop partition functions of all the $SU(2)/G$ models can be expressed as linear combinations of the gaussian partition functions $Z_R$, with radii $R = \sqrt{2}/N$. We will denote these partition functions as $Z_N$. The fact that a partition function can be written as weighted sums of gaussian ones should not come as a complete surprise. We have already met this phenomenon for the $\mathbf{Z}_2$ orbifolds. Combining (6.29–6.30) $Z_R^{orb}$ can be expressed as

$$Z_R^{orb} = \tfrac{1}{2} Z_R + Z_2 - \tfrac{1}{2} Z_1. \tag{6.41}$$

The corresponding partition functions for the $SU(2)/G$ models are given by [127, 84]

$$\begin{aligned} \mathbf{Z}_N &: \quad Z = Z_N, \\ D_N &: \quad Z = \tfrac{1}{2} Z_N + Z_2 - \tfrac{1}{2} Z_1, \\ T &: \quad Z = Z_3 + \tfrac{1}{2} Z_2 - \tfrac{1}{2} Z_1, \\ O &: \quad Z = \tfrac{1}{2} Z_4 + \tfrac{1}{2} Z_3 + \tfrac{1}{2} Z_2 - \tfrac{1}{2} Z_1, \\ I &: \quad Z = \tfrac{1}{2} Z_5 + \tfrac{1}{2} Z_3 + \tfrac{1}{2} Z_2 - \tfrac{1}{2} Z_1. \end{aligned} \tag{6.42}$$

It is not difficult to show that the above list exhausts all $c = 1$ models whose partition function can be written as a linear sum of $Z_R$'s [51]

$$Z = \sum_R a_R Z_R. \tag{6.43}$$

Here the $a_R$ are *a priori* arbitrary constants, although $\sum_R a_R = 1$, since the identity should occur with multiplicity one. In order to prove this claim we have to distinguish two situations. First, we will consider the case that the chiral algebra contains a spin one current $J(z)$. The condition that all multiplicities are non-negative leads now to two possibilities: we either have a gaussian model with $Z = Z_R$ or we have a partition function of the form

$$Z = \tfrac{1}{2}Z_{R_1} + \tfrac{1}{2}Z_{R_2}, \tag{6.44}$$

for arbitrary $R_1, R_2$. Note that at first sight the division by two does not seem problematic, since all states of dimension $h = \frac{1}{2}p^2$ (mod 1) with $p \neq 0$ occur in the gaussian partition function $Z_R$ with multiplicity two. However, at second sight, this fact allows us to rule out possibility (6.44). Since a $U(1)$ subalgebra generated by the current $J(z)$ is present, we can decompose $Z$ into the $U(1)$ characters $q^{\frac{1}{2}p^2}/\eta(q)$ with $p$ the global $U(1)$ charge, and $h = \frac{1}{2}p^2$. (Here we used the fact that the stress-energy tensor is necessarily given by the Sugawara form $T(z) = \frac{1}{2}J(z)^2$.) Now the partition function (6.44) has the property that the charges $p, \overline{p}$ are not summed over a lattice, since $\Gamma_{R_1}$ and $\Gamma_{R_2}$ are not compatible if $R_1 \neq R_2$. This already points out a problem since $p$ is additively conserved. But, furthermore, it implies that $Z$ contains characters with weight $h = \frac{1}{2}p^2, p \neq 0$, and with multiplicity one. The corresponding operators should have charge $p$ and should also be self-conjugate, which conflicts with the fact that conjugate fields carry the opposite charge $-p$. This proves the impossibility of (6.44).

So, in order to find other models than gaussians we have to exclude spin one fields. Since the multiplicity of the spin one field equals 3 for $Z_1$ and 1 for all other $Z_R$'s, this implies that $Z_1$ should always be included in our sum and that its coefficient should be $-\frac{1}{2}$. The coefficients in (6.43) are further restricted by applying the positivity condition for the multiplicity of chiral fields. Here one needs the fact that the multiplicity $m_j$ of the weight $h = j^2$ chiral field in the partition function $Z_N$ is given by

$$m_j = 1 + 2\left[j/N\right], \tag{6.45}$$

with $[-]$ the *entier* function. It is an easy exercise to verify with the aid of table 4 that the possible combinations are already restricted to the above cases by requiring the multiplicities to be non-negative integers for $j \leq 5$.

We note that any partition function in (6.42) can be written as

$$Z = \tfrac{1}{2}Z_{n_1} + \tfrac{1}{2}Z_{n_2} + \tfrac{1}{2}Z_{n_3} - \tfrac{1}{2}Z_1, \tag{6.46}$$

where $n_1, n_2, n_3$ are three positive integers satisfying

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1 + \frac{2}{N} > 1. \tag{6.47}$$

| $j$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $Z_1$ | 1 | 3 | 5 | 7 | 9 | 11 |
| $Z_2$ | 1 | 1 | 3 | 3 | 5 | 5 |
| $Z_3$ | 1 | 1 | 1 | 3 | 3 | 3 |
| $Z_4$ | 1 | 1 | 1 | 1 | 3 | 3 |
| $Z_5$ | 1 | 1 | 1 | 1 | 1 | 3 |
| $Z_R$ | 1 | 1 | 1 | 1 | 1 | 1 |

TABLE 4: The multiplicities of the weight $h = j^2$ chiral fields in the gaussian partition functions $Z_N$ ($N = 1, \ldots, 5$) and the other $Z_R$'s.

This last condition is exactly the equation that classifies all discrete $SO(3)$ subgroups, with $N$ the order of the group, or the simply-laced Lie algebras, with $n_i$ the number of nodes on each branch of the Dynkin diagram and $N$ the dual Coxeter number of the corresponding affine algebra. The numbers $n_i$ have an interpretation in tilings of the sphere with spherical triangles. Such tilings are only possible if the three angles $\theta_i$ are given by $\theta_i = \pi/n_i$. When a regular solid is projected on the sphere, the vertices and the centers of the faces and edges of the polyhedron correspond to the vertices of the triangular tiling [39]. The inequality (6.47) can be understood in the context of $c=1$ conformal field theory by calculating the multiplicity of the spin $j$ field in (6.46) in the limit $j \to \infty$, and requiring it to be positive.

Starting from the above classification of models with partition functions given by linear combinations of $Z_R$'s and with the aid of the results of Serre and Stark [143] on modular forms of weight one-half, the completeness of *fig.* 9 can be proved for all rational CFT's [108]. So CFT's that are not included in our list have to be irrational. Although it seems very unlikely that such models exist, a rigorous completeness proof for $c=1$, similar to the case $c < 1$, has not yet been given.

## 6.3. Rational Gaussian Models

We have seen that the $c=1$ theories can serve as an instructive set of examples for the general phenomena of marginal operators and multi-critical points, *i.e.* deformation techniques in CFT. However, also from the complementary, rational point of view they are very interesting. It is a remarkable fact that no conformal field theory with $c \geq 1$ (where we do not necessarily impose the condition of rationality) is known whose chiral algebra consists only of the Virasoro algebra, *i.e.* there always seem to appear chiral scaling operators. This is in particular the case for $c=1$, and in this section and the following we will discuss the different chiral algebras that occur at $c=1$, their representation theory and fusion algebras.

The chiral algebra of the gaussian model for generic radii $R$ equals the universal enveloping algebra of the $U(1)$ current algebra. The primary holomorphic fields of this algebra are the spin $h=n^2$ fields $J_{n^2}(z)$. However, for rational values of $R^2$ holomorphic vertex operators appears in the spectrum. These models are known as the rational gaussian models, or rational tori [118,50]. In fact, it is not difficult to see that at $\frac{1}{2}R^2 = p/p'$, with $p, p'$ two relative prime integers, the chiral vertex operators of smallest integer dimension are given by

$$V_\pm(z) = e^{\pm iQ\phi(z)}, \qquad Q = \sqrt{2pp'}, \tag{6.48}$$

and have conformal weight $h = pp'$. Closure of the chiral algebra tells one to include all vertex operators with momenta multiple of $Q$. The chiral algebra we obtain in this way will be denoted as $\mathcal{A}_N$, with $N = 2pp'$. $\mathcal{A}_N$ can be regarded as the level $k = N$ $U(1)$ chiral algebra. Note that $N$ is always an *even* integer. (The models for odd $N$ still make sense as fermionic models. In particular the case $N = 1$ corresponds to a Dirac fermion with $V_\pm = \psi, \psi^*$. We return to this point in section 8.4.) The representations of this algebra will consist of vertex operators $e^{iq\phi}$ that are local with respect to $V_\pm(z)$, *i.e.* with $qQ$ integer. Furthermore, the action of $V_\pm(z)$ will induce an identification $q \sim q + Q$. So we conclude that this chiral algebra possesses only a *finite* number of representations $[\phi_k]$, with chiral vertex operators

$$\phi_k = e^{ik\phi/\sqrt{N}}, \qquad k \in \mathbf{Z}_N. \tag{6.49}$$

Since the momentum $q = k/\sqrt{N}$ is conserved, the fusion algebra (4.32) is given in this case by the cyclic algebra $\mathbf{Z}_N$

$$\phi_k \times \phi_{k'} = \phi_{k+k'}. \tag{6.50}$$

Rational tori can be considered as the most simple examples of rational CFT's. The one-loop characters read (with $q = e^{2\pi i\tau}$)

$$\chi_k(\tau) = \frac{1}{\eta(q)} \sum_{m\in\mathbf{Z}} q^{(k+mN)^2/2N} = \frac{1}{\eta(q)} \vartheta\left[{}^{k/N}_{\ 0}\right](0|N\tau). \tag{6.51}$$

Here we introduced the Jacobi $\vartheta$-function

$$\vartheta\left[{}^{\alpha}_{\beta}\right](z|\tau) = \sum_{n\in\mathbf{Z}+\alpha} e^{i\pi n^2\tau + 2\pi i n(z+\beta)}. \tag{6.52}$$

The behavior of the characters under the modular transformation $S : \tau \rightarrow -1/\tau$ is given by finite Fourier transformation

$$S : \chi_k \rightarrow \frac{1}{\sqrt{N}} \sum_{k' \in \mathbf{Z}_N} e^{-2\pi i k k'/N} \chi_{k'}. \tag{6.53}$$

The partiton function (6.3) can be decomposed in terms of the $\vartheta$-functions as*

$$Z_R(\tau, \overline{\tau}) = \sum_{\substack{\alpha \in \mathbf{Z}/p,\, \beta \in \mathbf{Z}/p' \\ \gamma \in \mathbf{Z}/2}} \vartheta\Big[{\textstyle\frac{1}{2}\alpha+\frac{1}{2}\beta+\gamma \atop 0}\Big](0|N\tau)\, \overline{\vartheta\Big[{\textstyle\frac{1}{2}\alpha-\frac{1}{2}\beta+\gamma \atop 0}\Big](0|N\tau)}, \tag{6.54}$$

which reads in terms of the characters $\chi_k(\tau)$

$$Z_R(\tau, \overline{\tau}) = \sum_{k \in \mathbf{Z}_N} \chi_k(\tau) \overline{\chi_{\overline{k}}(\tau)}, \tag{6.55}$$

with $k$ and $\overline{k}$ related by $k - \overline{k} = 0 \pmod{2p}$, $k + \overline{k} = 0 \pmod{2p'}$. We see that the map $k \rightarrow \overline{k}$ is one-to-one and defines an automorphism of $\mathbf{Z}_N$, in accordance with the results of section 4.6.

A particular example is the case $N = 2$, which equals the $SU(2)$ level $k = 1$ affine algebra. It possesses only two representations: the identity $[1] = [\phi_0]$ and the spin $\frac{1}{2}$ representation $[\phi] = [\phi_1]$ with fusion algebra $\phi \times \phi = 1$. The characters can be decomposed into Virasoro characters as

$$\begin{aligned} \chi_0(\tau) &= \frac{1}{\eta(q)} \vartheta\Big[{0 \atop 0}\Big](0|2\tau) = \sum_{j=0}^{\infty} (2j+1) \chi^{(Vir)}_{j^2}(\tau), \\ \chi_1(\tau) &= \frac{1}{\eta(q)} \vartheta\Big[{\frac{1}{2} \atop 0}\Big](0|2\tau) = \sum_{j=0}^{\infty} (2j+2) \chi^{(Vir)}_{(j+\frac{1}{2})^2}(\tau), \end{aligned} \tag{6.56}$$

and we clearly see the $SU(2)$ multiplet structure appear, with all integer and half-integer representations occurring in respectively $[1]$ and $[\phi]$.

In fact, this model gives a nice way to see the necessity of the existence of the primary fields $J_{n^2}$ in the $U(1)$ current algebra. The vertex operator $e^{in\sqrt{2}\phi}$ is evidently a primary field with $SU(2)$ quantum numbers $(j, m) = (n, n)$ and conformal weight $h = n^2$. It is the highest weight of a $2n+1$ dimensional multiplet, which contains in particular a primary field of charge $m = 0$. This field in the zero charge subsector is necessarily a polynomial in the current and its derivatives and should be identified with the field $J_{n^2}$.

*We will use the notations $\mathbf{Z}_p = \{0, 1, \ldots, p-1\}$ and $\mathbf{Z}/p = \frac{1}{p}\mathbf{Z}_p = \{0, \frac{1}{p}, \ldots, \frac{1-p}{p}\}$.

## 6.4. Rational Orbifolds at $c = 1$

The partition functions of the orbifold models at $c = 1$, as we have presented them in (6.41–6.42), are not suitable for an analysis of the chiral algebra and its representations. In fact, these expressions completely obscure the chiral structure. We cannot read off the characters, since the partition functions are not written in the canonical form $Z(\tau,\bar\tau) = \sum \chi_i(\tau)\chi_{\bar\imath}(\bar\tau)$. In order to derive the characters we will exploit the fact that the partition functions $Z_N$ can be written as

$$Z_N(\tau,\bar\tau) = \sum_{\substack{\alpha\in\mathbf{Z}/2N \\ \beta\in\mathbf{Z}/N}} \frac{1}{N}\left|\tilde\vartheta\left[{\alpha \atop \beta}\right](\tau)\right|^2. \tag{6.57}$$

where we introduced a generalized theta function

$$\tilde\vartheta\left[{\alpha \atop \beta}\right](\tau) = \frac{1}{\eta(\tau)}\vartheta\left[{\alpha \atop \beta}\right](0|2\tau) = \frac{1}{\eta(\tau)}\sum_{m\in\mathbf{Z}} q^{(m+\alpha)^2} e^{2\pi i m\beta}. \tag{6.58}$$

The $\tilde\vartheta$'s have an interpretation as twisted sectors of the $SU(2)$ model—an interpretation we will discuss in detail in a moment. For future reference we observe that these functions enjoy the following transformation properties under the modular transformation $S$

$$S : \tilde\vartheta\left[{\alpha \atop \beta}\right] \to \frac{e^{-2\pi i\alpha\beta}}{\sqrt{2}}\left(\tilde\vartheta\left[{\beta/2 \atop -2\alpha}\right] + e^{-2\pi i\alpha}\tilde\vartheta\left[{(\beta+1)/2 \atop -2\alpha}\right]\right). \tag{6.59}$$

Our strategy will be to decompose the partition function into holomorphic blocks that are linear sums of the $\tilde\vartheta$'s. These will be the conjectured generalized characters of the relevant extended chiral algebra. The behavior under modular transformations will give us the fusion rules. We will give the explicit calculations in the next sections.

### 6.4.1. $\mathbf{Z}_2$ *Orbifolds*

We start our analysis by considering the $\mathbf{Z}_2$ orbifold model, *i.e.* a scalar field $\phi$ defined mod $2\pi R$ and identified by $\iota : \phi \to -\phi$. The partition function is given by (6.41). We will choose $R^2$ to be rational, so that our original algebra equals $\mathcal{A}_N$. The operator content of the algebra $\mathcal{A}_N/\mathbf{Z}_2$ is easy to describe. It consists simply of all the elements of $\mathcal{A}_N$ invariant under the involution $\iota$. If we use the fact that $\iota$ acts on the spin $n^2$ currents as $J_{n^2}(z) \to (-1)^n J_{n^2}(z)$, then it is evident that $\mathcal{A}_N/\mathbf{Z}_2$ contains in particular all fields $J_{n^2}$ with $n$ even. This subalgebra is generated by $J_4$. It is not difficult to verify that the total chiral algebra is generated by the fields

$$T,\ J_4,\ \cos\sqrt{N}\phi, \tag{6.60}$$

of respectively spin 2, 4, and $\frac{1}{2}N$.

In order to determine the representations of this algebra we try to decompose the partition function into holomorphic and anti-holomorphic blocks. After substituting (6.57) in (6.41) this results in $\frac{1}{2}N+7$ operators. The representations with their characters $\chi(\tau)$ and conformal weights $h$ are given below. (Here $\chi_k$ denote the $\mathcal{A}_N$ characters (6.51).)

$$
\begin{array}{llll}
\mathit{1} & : \quad \chi = \frac{1}{2}\chi_0 + \frac{1}{2}\tilde{\vartheta}\begin{bmatrix}0\\1/2\end{bmatrix}, & & h=0,\\
J & : \quad \chi = \frac{1}{2}\chi_0 - \frac{1}{2}\tilde{\vartheta}\begin{bmatrix}0\\1/2\end{bmatrix}, & & h=1,\\
\Phi^i & : \quad \chi = \frac{1}{2}\chi_{\frac{1}{2}N}, & (i=1,2), & h=\frac{1}{8}N,\\
\phi_k & : \quad \chi = \chi_k, & (k=1,\ldots,\frac{1}{2}N-1), & h=k^2/2N,\\
\sigma_i & : \quad \chi = \frac{1}{2}\tilde{\vartheta}\begin{bmatrix}1/4\\0\end{bmatrix} + \frac{1}{2}\tilde{\vartheta}\begin{bmatrix}1/4\\1/2\end{bmatrix} & (i=1,2), & h=\frac{1}{16},\\
\tau_i & : \quad \chi = \frac{1}{2}\tilde{\vartheta}\begin{bmatrix}1/4\\0\end{bmatrix} - \frac{1}{2}\tilde{\vartheta}\begin{bmatrix}1/4\\1/2\end{bmatrix} & (i=1,2), & h=\frac{9}{16}.
\end{array}
$$

The left-right pairing is given by $\phi_k \to \phi_{\bar{k}}$, as in (6.55), and diagonal for all other fields.

We can give the following interpretation of this operator content. Our original model consisted of the representations $[\phi_k]$ $(k \in \mathbf{Z}_N)$. The $\mathbf{Z}_2$ transformation $\iota$ acts on these representations as

$$\iota : [\phi_k] \to [\phi_{N-k}]. \tag{6.61}$$

So it acts as an inner automorphism for the representations $[\phi_0]$ and $[\phi_{\frac{1}{2}N}]$. These are the analogues of the fixed points that occur in the different orbifold models featuring in string compactification models. According to the discussion in section 5.5 we expect the following operators. The identity will split in an invariant part $[\mathit{1}]$ and a non-invariant part $[J]$, whose primary field is the current $J=i\partial\phi$. The same will happen for $[\phi_{\frac{1}{2}N}]$ and we will denote these two representations as $[\Phi^i]$ $(i=1,2)$. Furthermore, we will have 2 twisted sectors, each giving rise to 2 operators, corresponding to the trivial and the nontrivial representation of $\mathbf{Z}_2$. This accounts for the fields $[\sigma_i]$ and $[\tau_i]$. The remaining representations $[\phi_k]$, on which $\iota$ acts as an outer automorphism, are pairwise combined into an invariant operator, which corresponds to the vertex operator $\cos\frac{k}{\sqrt{N}}\phi$.

Now that we have identified the spectrum, we want to discuss the possible interactions. With the formulas given in the previous section one can easily derive the action of $S$ on the above characters. In this reconstruction of the modular transformation one should bear in mind the constraints of symmetry, unitarity and a closed operator product. It turns out that in order to satisfy all these demands we must distinguish between the cases $N=2$ (mod 4) and $N=0$ (mod

| | $\mathit{1}$ | $J$ | $\Phi^j$ | $\phi_{k'}$ | $\sigma_j$ | $\tau_j$ |
|---|---|---|---|---|---|---|
| $\mathit{1}$ | 1 | 1 | 1 | 2 | $\sqrt{\frac{N}{2}}$ | $\sqrt{\frac{N}{2}}$ |
| $J$ | 1 | 1 | 1 | 2 | $-\sqrt{\frac{N}{2}}$ | $-\sqrt{\frac{N}{2}}$ |
| $\Phi^i$ | 1 | 1 | 1 | $2(-1)^{k'}$ | $\sigma_{ij}\sqrt{\frac{N}{2}}$ | $\sigma_{ij}\sqrt{\frac{N}{2}}$ |
| $\phi_k$ | 2 | 2 | $2(-1)^k$ | $4\cos\pi\frac{kk'}{N}$ | 0 | 0 |
| $\sigma_i$ | $\sqrt{\frac{N}{2}}$ | $-\sqrt{\frac{N}{2}}$ | $\sigma_{ij}\sqrt{\frac{N}{2}}$ | 0 | $\delta_{ij}\sqrt{N}$ | $-\delta_{ij}\sqrt{N}$ |
| $\tau_i$ | $\sqrt{\frac{N}{2}}$ | $-\sqrt{\frac{N}{2}}$ | $\sigma_{ij}\sqrt{\frac{N}{2}}$ | 0 | $-\delta_{ij}\sqrt{N}$ | $\delta_{ij}\sqrt{N}$ |

TABLE 5: The matrix elements $S_{ij}$ that encode the modular transformations of the characters of the rational $\mathbf{Z}_2$ orbifold model for $N=0 \pmod 4$, with an overall factor $\frac{1}{\sqrt{4N}}$ and $\sigma_{ij}=2\delta_{ij}-1$.

4). In the latter case the two representations $[\Phi^1]$ and $[\Phi^2]$ are self-conjugate, while in the former case they are each others conjugate. For the case $N=0 \pmod 4$ the matrix $S$ is given in table 5.

The corresponding fusion algebra can be directly deduced. We give only the relevant relations. Other relations follow from associativity. First the elements $\mathit{1}$, $J$, $\Phi^1$, $\Phi^2$ generate a $\mathbf{Z}_2\times\mathbf{Z}_2$ subalgebra

$$J\times J=\mathit{1},\qquad \Phi^i\times\Phi^i=\mathit{1},\qquad \Phi^1\times\Phi^2=J. \tag{6.62}$$

The vertex operators $\phi_k$ have a fusion algebra consistent with their interpretation as $\cos\frac{k}{\sqrt{N}}\phi$

$$\begin{aligned} \phi_k\times\phi_{k'} &= \phi_{k+k'}+\phi_{k-k'}, \qquad (k'\neq k, \tfrac{1}{2}N-k),\\ \phi_k\times\phi_k &= \mathit{1}+J+\phi_{2k},\\ \phi_{\frac{1}{2}N-k}\times\phi_k &= \phi_{2k}+\Phi^1+\Phi^2,\\ J\times\phi_k &= \phi_k. \end{aligned} \tag{6.63}$$

The twist fields generate all vertex operators through the relations

$$\sigma_i\times\sigma_i=\mathit{1}+\Phi^i+\sum_{k\text{ even}}\phi_k,\qquad \sigma_1\times\sigma_2=\sum_{k\text{ odd}}\phi_k. \tag{6.64}$$

The operator product structure of the $\tau_i$ can be easily deduced using $J\times\sigma_i=\tau_i$.

In the case $N=2 \pmod 4$ we find the matrix elements of $S$ of table 6. As

| | $\mathit{1}$ | $J$ | $\Phi^j$ | $\phi_{k'}$ | $\sigma_j$ | $\tau_j$ |
|---|---|---|---|---|---|---|
| $\mathit{1}$ | 1 | 1 | 1 | 2 | $\sqrt{\frac{N}{2}}$ | $\sqrt{\frac{N}{2}}$ |
| $J$ | 1 | 1 | 1 | 2 | $-\sqrt{\frac{N}{2}}$ | $-\sqrt{\frac{N}{2}}$ |
| $\Phi^i$ | 1 | 1 | $-1$ | $2(-1)^{k'}$ | $i\sigma_{ij}\sqrt{\frac{N}{2}}$ | $i\sigma_{ij}\sqrt{\frac{N}{2}}$ |
| $\phi_k$ | 2 | 2 | $2(-1)^k$ | $4\cos 2\pi\frac{kk'}{N}$ | 0 | 0 |
| $\sigma_i$ | $\sqrt{\frac{N}{2}}$ | $-\sqrt{\frac{N}{2}}$ | $i\sigma_{ij}\sqrt{\frac{N}{2}}$ | 0 | $e^{\pi i\sigma_{ij}/4}\sqrt{N}$ | $-e^{\pi i\sigma_{ij}/4}\sqrt{N}$ |
| $\tau_i$ | $\sqrt{\frac{N}{2}}$ | $-\sqrt{\frac{N}{2}}$ | $i\sigma_{ij}\sqrt{\frac{N}{2}}$ | 0 | $-e^{\pi i\sigma_{ij}/4}\sqrt{N}$ | $e^{\pi i\sigma_{ij}/4}\sqrt{N}$ |

TABLE 6: The matrix elements $S_{ij}$ as in table 5 for $N = 2 \pmod 4$.

mentioned before, this results in a somewhat different fusion algebra. The operator algebra of $\mathit{1}, J, \Phi^i$ now equals $\mathbf{Z}_4$

$$J \times J = \mathit{1}, \quad \Phi^1 \times \Phi^2 = \mathit{1}, \quad \Phi^i \times \Phi^i = J. \tag{6.65}$$

and for the twist fields one obtains

$$\sigma_i \times \sigma_i = \Phi^i + \sum_{k \text{ odd}} \phi_k, \qquad \sigma_1 \times \sigma_2 = \mathit{1} + \sum_{k \text{ even}} \phi_k. \tag{6.66}$$

The fusion algebra of the vertex operators $\phi_k$ is left unchanged.

It is interesting to consider the case $N = \infty$. In that case we obtain a symmetry algebra that is present in all the $\mathbf{Z}_2$ orbifold models, independent of the radius of compactification, and equals the maximal algebra for irrational $R^2$. It consists of the fields $J_{n^2}(z)$ with $n$ even, and is generated by the field $J_4(z)$. Its representations are

$$\mathit{1},\, J = i\partial\phi,\, \cos p\phi \ (p > 0),\, \sigma_i,\, \tau_i. \tag{6.67}$$

However, the fusion algebra is not well-defined, since we cannot determine the conjugation properties of the twist fields. A possible solution to this problem would be to keep only one pair $(\sigma, \tau)$ as irreducible representations, which then should be included with multiplicity 2. Furthermore, this is a case of a model that is not even quasi-rational [120], since the fusion of two twist fields will produce all vertex operators

$$\sigma \times \sigma = \mathit{1} + \sum_{p>0} \cos p\phi. \tag{6.68}$$

This concludes our analysis of the $\mathbf{Z}_2$ case. We will turn now to the more complicated groups that occur in the $SU(2)$ models.

### 6.4.2. The $SU(2)$ Orbifolds

The $SU(2)$ WZW model has two blocks $[1]$ and $[\phi]$, with $\mathbf{Z}_2$ operator algebra $\phi \times \phi = 1$. Since the sector $[\phi]$ contains half-integer spin $SU(2)$ representations, the action of a finite subgroup $G \subset SO(3)$ within $[\phi]$ will be projective. This causes no problems. In the general case where $G$ acts as an inner automorphism in more than one representation $[\phi_i]$, the chiral action of $G$ can be projective in all representations except for the identity sector, where the vacuum state should be left invariant. Furthermore, the cocycles that occur should respect the fusion algebra, *i.e.* if we have a cocycle $c_i(g,h)$ in the action on the representation $[\phi_i]$, then

$$c_i(g,h)\, c_j(g,h) = c_k(g,h) \qquad \textit{if } N_{ijk} \neq 0. \tag{6.69}$$

So indeed for the algebra $\phi^2 = 1$ a cocycle $c(g,h) = \pm 1$ is allowed.

In this respect it is convenient to consider the lift $\widehat{G} \subset SU(2)$, defined by

$$1 \to \mathbf{Z}_2 \to \widehat{G} \xrightarrow{\pi} G \to 1 \tag{6.70}$$

where $\pi$ is the projection $SU(2) \to SO(3)$. Elements of $\widehat{G}$ will be denoted as $\hat{g}$, with $\pi(\hat{g}) = g \in G$ and $\pi^{-1}(g) = \{\hat{g}, z\hat{g}\}$. Here $z$ denotes the generator of the center $\mathbf{Z}_2$ of $SU(2)$. As we have discussed, it is a matter of taste whether we want to consider the $SU(2)$ orbifolds as $G$ or $\widehat{G}$ orbifolds. If we adopt the latter point of view, we have the extra condition that the generator of the center $z \in \widehat{G}$ acts as $+1$ and $-1$ in the representations $[1]$ and $[\phi]$ respectively.

Let us now discuss the operator content of these $SU(2)/G$ orbifolds. We will first treat the untwisted sector. Both the representations $[1]$ and $[\phi]$ are evidently fixed points under the action of any subgroup $G$. Accordingly they will split up in smaller blocks that are labelled by the representations. The description is most conveniently done in terms of the representations $R_\alpha$ of $\widehat{G}$. The identity sector $[1]$ will give rise to blocks indexed by the irreducible representations $R_\alpha^+$ of $\widehat{G}$ that satisfy $R_\alpha^+(z) = 1$, *i.e.* the (linear) representations of $G$. Similarly the blocks produced by spinor $[\phi]$ correspond to representations that obey $R_\alpha^-(z) = -1$, and are accordingly genuine projective representations of $G$. Note that consequently there is a one-to-one correspondence between the untwisted operators in the $SU(2)/G$ model and the nodes in the corresponding extended Dynkin diagram. Furthermore the fusion algebra of these untwisted operators just equals the representation ring of $\widehat{G}$.

The characters of these operators are also easily written down. The operators coming from the identity sector $[1]$, giving rise to the characters

$$\chi_\alpha^+ = \frac{1}{|G|} \sum_{h \in G} \rho_\alpha^+(h)\; h \underset{1}{\square}\,, \tag{6.71}$$

This is completely analogous to the chiral case discussed in Chapter 5. The characters of the operators that descend from the $[\phi]$ sector read in an evident notation

$$\chi_{\bar{\alpha}} = \frac{1}{|G|} \sum_{h \in G} \rho_{\bar{\alpha}}(h)\, h \underset{\phi}{\square}\, . \tag{6.72}$$

We can actually calculate these blocks by bosonization [84]. The action of any element $h \in G$ can be represented as a shift $\beta \in \frac{1}{N}\mathbf{Z}_N$. This gives us the following expressions for the chiral blocks

$$h \underset{1}{\square} = \tilde{\vartheta}\begin{bmatrix}0\\ \beta\end{bmatrix}, \qquad h \underset{\phi}{\square} = \tilde{\vartheta}\begin{bmatrix}1/2\\ \beta\end{bmatrix}. \tag{6.73}$$

We now have to include the twist fields. A general prescription for the counting of twist fields and the determination of the correct fusion rules in an orbifold model based on an arbitrary RCFT is still lacking. However, in this case we have explicit partition functions available, and we can actually calculate the characters. The results of this calculation, the details of which we will give in the next section for some examples, can be neatly summarized as follows. The number of twist fields is given by $\sum_i n_i(n_i - 1)$, where the $n_i$ are the integers of (6.47). Thus we obtain for the $SU(2)/G$ orbifolds the following numbers of representations

$$\mathbf{Z}_N \,:\, 2N^2, \quad D_N \,:\, N^2 + 7, \quad T \,:\, 21, \quad O \,:\, 28, \quad I \,:\, 37. \tag{6.74}$$

In the case of solvable groups we can also determine the correct number of twist operators using the explicit action of the different abelian group that feature in the decomposition series. The only non-solvable group in this case is the icosahedral group $I \cong A_5$. The total number of operators we obtain in this way using the composition series

$$\mathbf{Z}_N \triangleleft D_N, \qquad \mathbf{Z}_2 \triangleleft D_2 \triangleleft T \triangleleft O, \tag{6.75}$$

is consistent with above counting, as we will see in the next section.

We can give the following heuristic argument for the counting of the twist fields and the construction of their characters. The $SU(2)$ $k = 1$ model can be viewed as a projection of a single block, nonlocal theory, where the operator $\phi$ is adjoined to the chiral algebra. The projection is onto elements even under the action of the center $z$. This creates the separate representations $[1]$ and $[\phi]$. We now want to construct the orbifold model, by first modding out the single block model by the group $\widehat{G}$ and projecting on even states afterwards. According to this argument the twist sectors are labelled by the classes of $\widehat{G}$. However, not all representations of the stabilizer occur, since we again have to project on those sectors that are even under the $\mathbf{Z}_2$ grading. This gives as the total number of operators, half the number in the holomorphic $\widehat{G}$ orbifold. Note that in this picture the sector twisted by $z$ corresponds to $[\phi]$. With $N_{\hat{g}}$ the projection of the stabilizer of $\hat{g}$ into $SO(3)$, the expressions for the corresponding characters now read

$$\chi_\alpha^{\hat{g}} = \frac{1}{|N_{\hat{g}}|} \sum_{h \in N_{\hat{g}}} \rho_\alpha^{\hat{g}}(h) \; h \underset{\hat{g}}{\square} \tag{6.76}$$

We can again calculate these chiral blocks by bosonization, since we can always represent two commuting elements $\hat{g}$ and $h$ as shifts $\alpha \in \frac{1}{2N}\mathbf{Z}_{2N}, \beta \in \frac{1}{N}\mathbf{Z}_N$. Accordingly we can represent the block as

$$h \underset{\hat{g}}{\square} = \tilde{\vartheta}\begin{bmatrix}\alpha \\ \beta\end{bmatrix}. \tag{6.77}$$

This representation also accounts for some features of the fusion algebra of the twisted operators. We will see an overall selection rule due to the class algebra of $\widehat{G}$. Furthermore, the representations will decompose similarly as in the holomorphic case, if we take into account the phases $\sigma$ that can be read off from the modular properties of the above characters.

### 6.4.3. Some Examples

We will now present the results of some more detailed calculations for specific examples, that confirm the prescription we gave in the previous section. As a first example let us consider the group $G = D_2$. The model $SU(2)/D_2$ is known to correspond to the 4-state Potts model at criticality. The group $D_2$ acts on the $SU(2)$ currents $J_a(z)$ by

$$r_b : \; J_a \longrightarrow \epsilon_{ab} J_a, \tag{6.78}$$

where the symbol $\epsilon_{ab}$ is defined to be 1 if $a = b$, and $-1$ otherwise. The 4-state Potts model also corresponds to the rational $\mathbf{Z}_2$ orbifold model with $N=4$. So we have already the prescription of section 6.4.1 to calculate the characters and fusion rules. We will rewrite the results in such a way that they can be seen to confirm the general prescriptions we gave for $SU(2)$ orbifolds. The partition functions can be decomposed in the following characters

$$\begin{array}{lcll}
\mathit{1} & : & \chi = \frac{1}{4}\tilde{\vartheta}\begin{bmatrix}0 \\ 0\end{bmatrix} + \frac{3}{4}\tilde{\vartheta}\begin{bmatrix}0 \\ 1/2\end{bmatrix}, & h = 0, \\
J_a & : & \chi = \frac{1}{4}\tilde{\vartheta}\begin{bmatrix}0 \\ 0\end{bmatrix} - \frac{1}{4}\tilde{\vartheta}\begin{bmatrix}0 \\ 1/2\end{bmatrix}, & h = 1, \\
\phi & : & \chi = \frac{1}{2}\tilde{\vartheta}\begin{bmatrix}1/2 \\ 0\end{bmatrix}, & h = \frac{1}{4}, \\
\sigma_a & : & \chi = \frac{1}{2}\tilde{\vartheta}\begin{bmatrix}1/4 \\ 0\end{bmatrix} + \frac{1}{2}\tilde{\vartheta}\begin{bmatrix}1/4 \\ 1/2\end{bmatrix}, & h = \frac{1}{16}, \\
\tau_a & : & \chi = \frac{1}{2}\tilde{\vartheta}\begin{bmatrix}1/4 \\ 0\end{bmatrix} - \frac{1}{2}\tilde{\vartheta}\begin{bmatrix}1/4 \\ 1/2\end{bmatrix}, & h = \frac{9}{16}.
\end{array}$$

From the modular properties of the blocks $\tilde{\vartheta}\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ we can calculate the behavior of the characters under the modular transformation $S$. Up to an overall factor $\frac{1}{\sqrt{32}}$ we find the matrix $S$

| | $1$ | $J_b$ | $\phi$ | $\sigma_b$ | $\tau_b$ |
|---|---|---|---|---|---|
| $1$ | 1 | 1 | 2 | 2 | 2 |
| $J_a$ | 1 | 1 | 2 | $2\epsilon_{ab}$ | $2\epsilon_{ab}$ |
| $\phi$ | 2 | 2 | $-4$ | 0 | 0 |
| $\sigma_a$ | 2 | $2\epsilon_{ab}$ | 0 | $\delta_{ab}\sqrt{8}$ | $-\delta_{ab}\sqrt{8}$ |
| $\tau_a$ | 2 | $2\epsilon_{ab}$ | 0 | $-\delta_{ab}\sqrt{8}$ | $\delta_{ab}\sqrt{8}$ |

By applying (6.59) we can read off what the fusion algebra of the 4-state Potts model is. We find

$$\begin{aligned} J_a \times J_a &= 1, & \sigma_a \times \sigma_a &= 1 + \phi + J_a, \\ J_a \times J_b &= J_c \quad (a \neq b \neq c), & \sigma_a \times \sigma_b &= \sigma_c + \tau_c, \\ \phi \times \phi &= 1 + \sum_c J_c, & \sigma_a \times \tau_a &= \phi + \sum_{c \neq a} J_c. \end{aligned} \tag{6.79}$$

In order to see that the above structure indeed confirms our general argument, we consider the representation and class algebra of the lift of $D_2$ into $SU(2)$: $\widehat{D_2} \cong Q$, the quaternionic group. If we denote the 5 irreducible representations of $Q$ as $1, J_a, \phi$, then the representation algebra of $Q$ is indeed equal to the corresponding subalgebra of the above fusion rules. Furthermore $Q$ has 5 conjugacy classes $1, \phi, \sigma_a$. The projections of the stabilizers of the elements $\sigma_a$ into $D_2$ are isomorphic to $\mathbf{Z}_2$. This accounts for the pairs of twist fields $\sigma_a, \tau_a$. The class algebra is given by

$$\phi^2 = 1, \quad \sigma_1\sigma_2 = \sigma_3, \quad \sigma_a^2 = 1 + \phi. \tag{6.80}$$

This algebra can indeed be recognized as an overall selection rule in the fusion algebra.

As a less trivial example consider the tetrahedral group $T \cong A_4$. By explicit computation we found the following characters:

$$\begin{aligned} 1 \quad &: \quad \chi = \tfrac{1}{12}\tilde{\vartheta}\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] + \tfrac{1}{4}\tilde{\vartheta}\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right] + \tfrac{2}{3}\tilde{\vartheta}\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right], & h &= 0, \\ J \quad &: \quad \chi = \tfrac{1}{4}\tilde{\vartheta}\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] - \tfrac{1}{4}\tilde{\vartheta}\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right], & h &= 1, \\ 1_1, 1_2 \quad &: \quad \chi = \tfrac{1}{12}\tilde{\vartheta}\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] + \tfrac{1}{4}\tilde{\vartheta}\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right] - \tfrac{1}{3}\tilde{\vartheta}\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right], & h &= 4, \\ \phi_0 \quad &: \quad \chi = \tfrac{1}{6}\tilde{\vartheta}\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right] - \tfrac{2}{3}\omega\,\tilde{\vartheta}\left[\begin{smallmatrix}1/2\\1/3\end{smallmatrix}\right], & h &= \tfrac{1}{4}, \end{aligned}$$

$$
\begin{array}{llll}
\phi_1, \phi_2 & : & \chi = \frac{1}{6}\tilde{\vartheta}\left[{1/2 \atop 0}\right] + \frac{1}{3}\omega\,\tilde{\vartheta}\left[{1/2 \atop 1/3}\right], & h = \frac{9}{4}, \\
\sigma & : & \chi = \frac{1}{2}\tilde{\vartheta}\left[{1/4 \atop 0}\right] + \frac{1}{2}\tilde{\vartheta}\left[{1/4 \atop 1/2}\right], & h = \frac{1}{16}, \\
\tau & : & \chi = \frac{1}{2}\tilde{\vartheta}\left[{1/4 \atop 0}\right] - \frac{1}{2}\tilde{\vartheta}\left[{1/4 \atop 1/2}\right], & h = \frac{9}{16}, \\
\omega_i^{\pm} & : & \chi = \frac{1}{3}\tilde{\vartheta}\left[{1/3 \atop 0}\right] + \omega^i\tilde{\vartheta}\left[{1/3 \atop 1/3}\right] + \overline{\omega}^i\tilde{\vartheta}\left[{1/3 \atop 2/3}\right], & h = \frac{1}{9}, \frac{4}{9}, \frac{16}{9}, \\
\theta_i^{\pm} & : & \chi = \frac{1}{3}\tilde{\vartheta}\left[{1/6 \atop 0}\right] + \omega^i\tilde{\vartheta}\left[{1/6 \atop 1/3}\right] + \overline{\omega}^i\tilde{\vartheta}\left[{1/6 \atop 2/3}\right], & h = \frac{1}{36}, \frac{25}{36}, \frac{49}{36}.
\end{array}
$$

with $i = 1, 2, 3$ and $\omega = e^{2\pi i/3}$. From the corresponding $S$-matrix we found the following fusion rules (all indices mod 3). First, the untwisted sectors give rise to the representation ring of the binary tetrahedral group $\widehat{T} \cong SL(2,3)$

$$
1_i \times 1_j = 1_{i+j}, \quad \phi_i \times \phi_j = 1_{i+j} + J, \quad J \times J = 2\,J + \sum_k 1_k. \tag{6.81}
$$

Note that the last equation gives an explicit example of a $N_{ijk} > 1$ with non-degenerate ground states, as can be explicitly checked using the above character formulas. The fusion rules of the twisted operators are

$$
\begin{aligned}
\sigma \times \sigma = \tau \times \tau &= \sum_k 1_k + J + \sum_k \phi_k + 2\,\sigma + 2\,\tau, \\
\sigma \times \tau &= 2\,J + \sum_k \phi_k + 2\,\sigma + 2\,\tau,
\end{aligned} \tag{6.82}
$$

$$
\begin{aligned}
\omega_i^+ \times \omega_j^+ &= \theta_i^+ \times \theta_j^+ = \omega_{-i-j}^- + \sum_k \theta_k^-, \quad \omega_i^+ \times \omega_j^- = \theta_i^+ \times \theta_j^- = J + 1_{i-j} + \sigma + \tau, \\
\omega_i^+ \times \theta_j^+ &= \theta_i^+ \times \omega_j^+ = \theta_{-i-j}^- + \sum_k \omega_k^-, \quad \omega_i^+ \times \theta_j^- = \theta_i^+ \times \omega_j^- = \sum_{k \neq i-j} \phi_k + \sigma + \tau.
\end{aligned}
$$

This can also be better understood if we consider the classes of $SL(2,3)$. The 7 conjugacy classes are $1, \phi, \sigma, \omega^{\pm}, \theta^{\pm}$ with elements of orders respectively 1, 2, 4, 3, 6. The projections of the stabilizer subgroups are $\mathbf{Z}_2$ for $\sigma$, and $\mathbf{Z}_3$ for the elements $\omega^{\pm}, \theta^{\pm}$. The class algebra can again be recognized as an overall selection rule in the fusion rules.

We can also consider this tetrahedron model as an abelian $\mathbf{Z}_3$ orbifold of the $D_2$ model that we discussed before. The $\mathbf{Z}_3$ evidently permutes the operators $J_a$, $\sigma_a$ and $\tau_a$, which give rise to a single $J$, $\sigma$ and $\tau$ in the $T$ model. The fields $1$ and $\phi$ are fixed points, and accordingly triple to form the representations $1_i$, $\phi_i$. Since we have two fixed points under an order three action, we expect $2 \times 6$ twist fields, which correspond to the $\omega_i^{\pm}$ and $\theta_i^{\pm}$. This gives indeed the total of 21 operators. If we mod out by another $\mathbf{Z}_2$ we obtain the 28 operators of the octahedral model. This analysis for the solvable groups can without much problems be extended to arbitrary level $k$.

# 7

# $c = 1$ CFT on Riemann Surfaces

It is a fundamental property of quantum field theory that we can consider partition and correlation functions on manifolds of arbitrary geometry. Since conformal field theories couple trivially to the scale factor of the metric—that is, only through the conformal anomaly, which is completely fixed by the central charge $c$ and does not depend on the precise details of the theory—the only relevant geometric data are the topology of the surface and its complex structure. That is, the relevant class of objects are Riemann surfaces. The determination of higher genus quantities is an interesting problem for several reasons. First, in string theory correlation functions on surfaces of genus $g > 0$ represent the radiative corrections to the tree amplitudes. A general $n$-particle amplitude $A_n$ with particles described by two-dimensional vertex operators $V_i(z, \overline{z})$ is of the form

$$A_n = \sum_{g=0}^{\infty} \lambda^{2g-2} \int_{\mathcal{M}_{g,n}} \mu(m, \overline{m}) \left\langle V_1(z_1, \overline{z}_1) \cdots V_n(z_n, \overline{z}_n) \right\rangle, \tag{7.1}$$

with $\lambda$ the string coupling constant. The integral is over the moduli space $\mathcal{M}_{g,n}$ of $n$-punctured Riemann surfaces of genus $g$, and includes the positions $z_i$ of the vertex operators. The dependence on the ghosts, which are an essential ingredient in any string theory, is incorporated in the measure $\mu$ on $\mathcal{M}_{g,n}$. The ghost contribution reduces the central charge to $c = 0$ and represents a natural density on moduli space. From this perturbative point of view, the higher genus correlation functions are a necessary ingredient in any multi-loop string calculation.

The study of critical phenomena gives a second motivation. Finite geometries—in particular the torus—are interesting, because they are easier to realize in computer simulations. In this way analytic results can be checked with numerical calculations. Last but not least, nontrivial topology is of considerable interest to the general study of two-dimensional conformal field theory. In principle the partition functions on higher genus surfaces encode through their dependence on the moduli parameters all the information of the conformal field theory. In particular, correlation functions can be produced by factorization, that is, by pinching a handle. We will see examples of this in sections 7.3 and 7.4.

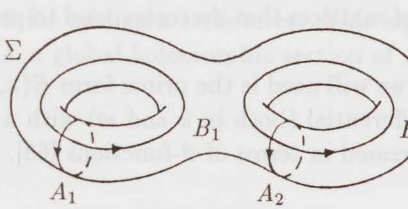



FIGURE 11: A genus two Riemann surface $\Sigma$ with a canonical homology basis $A_i, B_i$.

## 7.1. Partition Functions of Free Field Theories

Before we discuss the formulation of fermionic and bosonic free field theories on arbitrary topology, let us first add a few words to the geometry of Riemann surfaces in general [61,63,91,43]. A genus $g$ surface $\Sigma$ has a $2g$-dimensional first homology group

$$H_1(\Sigma, \mathbf{Z}) = \mathbf{Z}^{2g}. \tag{7.2}$$

We will presume that all surfaces come equipped with a 'marking,' *i.e.* with a basis of canonical homology cycles $A_i$, $B_i$ $(i = 1, \ldots, g)$ normalized with respect to the intersection product as in *fig.* 11

$$\#(A_i, A_j) = \#(B_i, B_j) = 0, \qquad \#(A_i, B_j) = -\#(B_j, A_i) = \delta_{ij}. \tag{7.3}$$

The intersection of two cycles is just the number of times the cycles cross, counted with appropriate signs depending on the two possible orientations of the crossing.

One-forms are naturally dual to cycles, over which they can be integrated. The Riemann-Roch theorem tells us that on a genus $g$ surface there are $g$ holomorphic and anti-holomorphic one-forms. We will denote them by as $\omega_i(z), \overline{\omega}_i(\overline{z})$ $(i = 1, \ldots, g)$. They are the global holomorphic sections of the canonical line bundle $K$. The integrals $\int^z \omega_i$ are holomorphic functions, but multi-valued. (The only single-valued holomorphic function is the constant function.) The period matrix $\tau_{ij}$ is defined by the periods of the one-forms along the $B$-cycles, in a basis that is normalized with respect to the $A$-cycles

$$\oint_{A_i} \omega_j = \delta_{ij}, \qquad \oint_{B_i} \omega_j = \tau_{ij}. \tag{7.4}$$

It is a symmetric $g \times g$ matrix with a positive-definite imaginary part [63], and it can be regarded as a coordinate on the moduli space of Riemann surfaces. Note however, that for $g > 2$ this coordinatization is not one-to-one. Since $\frac{1}{2}g(g+1) > 3g - 3$, not all symmetric matrices with $\operatorname{Im} \tau > 0$ are period matrices of Riemann surfaces. The subspace of matrices that do correspond to period matrices is known as the Schottky locus.

Another object that we will need is the prime form $E(z, w)$. This is the unique holomorphic spin $-\frac{1}{2}$ differential (both in $z$ and $w$) with a single, first order zero at $z = w$. It can be expressed in terms of $\vartheta$-functions [63].

*7.1.1. Fermions*

For fermionic and bosonic free theories the partition functions and correlators on arbitrary surfaces have been thoroughly studied [4,1,155], and we will be very short in these matters. Since a fermion can be either periodic or anti-periodic around a closed, non-contractible loop, fermionic theories are only well-defined after a choice of spin structure [141]

$$\left[{\alpha \atop \beta}\right] = \left[{\alpha_1 \cdots \alpha_g \atop \beta_1 \cdots \beta_g}\right], \tag{7.5}$$

with $\alpha_i, \beta_i = 0, \frac{1}{2}$. Here $e^{2\pi i \alpha_i}, e^{2\pi i \beta_i}$ are the monodromies of a fermion $\psi(z)$ along the cycles $A_i, B_i$ with respect to a reference spin structure. More precisely, fermion correlation functions are sections of a spin bundle $\mathcal{L}$, whose square equals the canonical line bundle $K$. The difference between two possible spin bundles $\mathcal{L}$ and $\mathcal{L}'$ is given by a spin structure $\left[{\alpha \atop \beta}\right]$. A spin structure is called even or odd depending on whether $4\,\alpha \cdot \beta$ is even or odd. Under modular transformations all spin structures of a given parity are transformed into each other.

For a given spin bundle $\mathcal{L}$ the partition function of a (Dirac) fermion is simply given by the modulus squared of the determinant of the $\overline{\partial}_{\frac{1}{2}}$ operator ($\overline{\partial}_j$ denotes the Cauchy operator $\partial/\partial\overline{z}$ acting on spin $j$ differentials)

$$Z = \int \mathcal{D}\psi \mathcal{D}\overline{\psi}\; e^{-\frac{1}{2\pi}\int(\psi\overline{\partial}\psi + \overline{\psi}\partial\overline{\psi})} = |\det \overline{\partial}_{\frac{1}{2}}|^2. \tag{7.6}$$

Bosonization [155] allows one to re-express this result in terms of the scalar determinant

$$\det \overline{\partial}_{\frac{1}{2}} = (\det \overline{\partial}_0)^{-\frac{1}{2}} \vartheta\left[{\alpha \atop \beta}\right](0|\tau). \tag{7.7}$$

Here the genus $g$ $\vartheta$-function is defined analogous to the $g = 1$ expression (6.58)

$$\vartheta\left[{\alpha \atop \beta}\right](z|\tau) = \sum_{n \in \mathbf{Z}^g + \alpha} e^{i\pi(n\cdot\tau\cdot n) + 2\pi i n\cdot(z+\beta)}. \tag{7.8}$$

It encaptures the spin structure dependence of (7.6). Explicit expressions for the chiral determinants can be found in [155]. The bosonic spin model is obtained by a summation over all spin structures

$$Z^{spin} = \sum_{\alpha,\beta \in (\mathbf{Z}/2)^g} |\det \overline{\partial}_0|^{-1} \left|\vartheta\left[{\alpha \atop \beta}\right](0|\tau)\right|^2. \tag{7.9}$$

Note that $\vartheta\big[{\alpha \atop \beta}\big](0|\tau) = 0$ for odd spin structures. This signals the existence of a fermionic zero-mode, *i.e.* a global holomorphic section of $\mathcal{L}$.

### 7.1.2. Bosons

The computation of the partition function of the free *bosonic* field $\phi$, compactified at radius $R$, has also become quite standard [4,154]. It is given by the product of a factor $Z_0^{qu}$ representing the quantum fluctuations of $\phi$ times a classical part $Z^{cl}(R)$ given by the partition sum over the classical solutions in the different winding sectors. These multi-valued instanton-like configurations are given by integrals of the holomorphic and anti-holomorphic one-forms $\omega_i, \overline{\omega}_i$ on $\Sigma$. Their classical action can be expressed in terms of the period matrix $\tau_{ij}$. Referring for the details of the calculation to the literature, we proceed to give the result for the partition function

$$Z(R) = Z_0^{qu} Z^{cl}(R), \qquad Z_0^{qu} = |\det \overline{\partial}_0|^{-1}, \tag{7.10a}$$

$$Z^{cl}(R) = \sum_{p,\overline{p} \in \Gamma_R^g} \exp\left[i\pi\left(p\cdot\tau\cdot p - \overline{p}\cdot\overline{\tau}\cdot\overline{p}\right)\right]. \tag{7.10b}$$

The quantities $p_i$, $\overline{p}_i$ $(i = 1, \ldots, g)$ have a natural interpretation as the momenta of the zero-mode of the scalar field running through the cycles $B_i$. They are eigenvalues of the loop operators [49,154] around the $A$-cycles

$$\oint_{A_i} \frac{dz}{2\pi i}\, \partial\phi(z), \qquad \oint_{A_i} \frac{d\overline{z}}{2\pi i}\, \overline{\partial\phi(\overline{z})}. \tag{7.11}$$

For rational values of the compactification radius $R^2$ the momentum lattice $\Gamma_R$ can be built up from a finite number of square sublattices. As a consequence the classical part of the partition function can be expressed as a finite sum of $\vartheta$-functions. This is completely analogous to the one-loop case (6.54). If $\frac{1}{2}R^2 = p/p'$ with $(p,p') = 1$ the result reads

$$Z^{cl}(R) = \sum_{\alpha,\beta,\gamma} \vartheta\Big[{\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+\gamma \atop 0}\Big](0|2pp'\tau)\, \overline{\vartheta\Big[{\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta+\gamma \atop 0}\Big](0|2pp'\tau)} \tag{7.12a}$$

$$= 2^{-g} \sum_{\alpha,\beta,\gamma} e^{4\pi i \beta\cdot\gamma}\, \vartheta\Big[{\alpha+\beta \atop \gamma}\Big]\big(0|\tfrac{1}{2}pp'\tau\big)\, \overline{\vartheta\Big[{\alpha-\beta \atop \gamma}\Big]\big(0|\tfrac{1}{2}pp'\tau\big)}, \tag{7.12b}$$

where the summation is over $\alpha \in (\mathbf{Z}/p)^g$, $\beta \in (\mathbf{Z}/p')^g$, and $\gamma \in (\mathbf{Z}/2)^g$. Note the manifest duality $(p,p') \to (p',p)$ that corresponds to the duality relation (6.18). Cases of particular relevance to us in the following discussions will be the representations

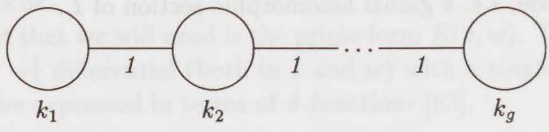



FIGURE 12: The generalized characters for the rational gaussian model with $\mathbf{Z}_N$ fusion rules have a simple interpretation: they are fixed by giving the representation $\phi_{k_i}$ that propagates in the $i^{th}$ loop.

$$Z^{cl}(\sqrt{2}) = \sum_{\alpha\in(\mathbf{Z}/2)^g} \left|\vartheta\left[{\alpha \atop 0}\right](0|2\tau)\right|^2, \tag{7.13a}$$

$$Z^{cl}(1) = 2^{-g} \sum_{\alpha,\beta\in(\mathbf{Z}/2)^g} \left|\vartheta\left[{\alpha \atop \beta}\right](0|\tau)\right|^2, \tag{7.13b}$$

$$Z^{cl}(\tfrac{1}{2}\sqrt{2}) = 2^{-g} \sum_{\alpha,\beta,\gamma\in(\mathbf{Z}/2)^g} \left|\vartheta\left[{\alpha+\frac{1}{2}\gamma \atop \beta}\right](0|2\tau)\right|^2. \tag{7.13c}$$

Let us add a few words to the two expressions (7.12a) and (7.12b). The first one corresponds to a decomposition

$$Z(\tau,\overline{\tau}) = \sum_{k\in\mathbf{Z}_N^g} \chi_k(\tau)\chi_{\overline{k}}(\overline{\tau}) \tag{7.14}$$

in terms of the generalized $g$-loop characters $\chi_k(\tau)$ of the rational gaussian model with chiral algebra $\mathcal{A}_N$, $N = 2pp'$. Given the simple $\mathbf{Z}_N$ fusion rules of these rational tori, the holomorphic blocks are completely determined by fixing the representations $k_1, \dots, k_g \in \mathbf{Z}_N$ in the channels dual to the $A$-cycles (see *fig.* 12). So, for the chiral algebra $\mathcal{A}_N$ the genus $g$ characters are a natural generalization of (6.51)

$$\chi_k(\tau) = (\det \overline{\partial}_0)^{-\frac{1}{2}} \vartheta\left[{k/N \atop 0}\right](0|N\tau). \tag{7.15}$$

As we discussed in section 6.3 we can also consider the $\mathcal{A}_N$ algebras for odd $N$. This produces *fermionic* models, whose spin projections correspond to the *bosonic* $\mathcal{A}_{4N}$ model. This is exactly the representation (7.12b). In particular, for $N=1$ we recover the partition function (7.9) of the spin model of a free Dirac fermion.

## 7.2. The $\mathbf{Z}_2$ Orbifold Model

As an illustrative example of the higher genus formulation of an interacting field theory we will discuss in this section the $c = 1$ $\mathbf{Z}_2$ orbifold models of Chapter 6. Since these theories are obtained from free theories by a simple operation, their structure is quite tractable and we can exactly calculate the partition and correlation functions on arbitrary Riemann surfaces.

### *7.2.1. The Partition Function*

The partition function of a $\mathbf{Z}_2$ orbifold model is represented by a functional integral over a free field $\phi(z,\bar{z})$ that can be *double-valued* on $\Sigma$. These field configurations naturally fall into $2^{2g}$ distinct topological sectors corresponding to the monodromies of $\phi$ around the elements of $H_1(\Sigma)$. The partition function is the normalized sum over all sectors (*cf* Eq. (6.28))

$$Z^{orb}(R) = 2^{-g} \sum_{\epsilon,\delta\in(\mathbf{Z}/2)^g} Z_{\epsilon,\delta}(R), \tag{7.16}$$

where the 'twist structure'

$$\begin{bmatrix}\epsilon\\ \delta\end{bmatrix} = \begin{bmatrix}\epsilon_1\cdots\epsilon_g\\ \delta_1\cdots\delta_g\end{bmatrix}, \tag{7.17}$$

labels the contribution of the field configurations having a branch cut along the cycle $2(\delta\cdot A + \epsilon\cdot B)$. The untwisted part is simply the gaussian partition function $Z_{0,0}(R) = Z(R)$ and is by itself modular invariant. The other $Z_{\epsilon,\delta}$ with $\begin{bmatrix}\epsilon\\ \delta\end{bmatrix} \neq \begin{bmatrix}0\\ 0\end{bmatrix}$ are all permuted by the modular group, which implies that only the sum over all twisted sectors is modular invariant.

We will calculate the contribution of the twisted sectors using the theory of unramified coverings of Riemann surfaces [63] following the general strategy as advocated in [92,53]. (For a related calculation, see [115,17]). Let us concentrate on one of the terms $Z_{\epsilon,\delta}$. We can always choose a new homology basis such that the branch cut is along $A_g$, *i.e.*

$$\begin{bmatrix}\epsilon\\ \delta\end{bmatrix} = \begin{bmatrix}0\cdots 0\\ 0\cdots \frac{1}{2}\end{bmatrix}. \tag{7.18}$$

This branch cut defines an unramified double cover $\pi : \hat{\Sigma} \to \Sigma$ (as depicted in *fig.* 13 for the $g = 2$ case) The surface $\hat{\Sigma}$ is obtained by taking two copies of the surface $\Sigma$, cutting each open along the cycle $A_g$ and then pasting the two copies together. This defines a natural complex structure on $\hat{\Sigma}$, and we will denote by $\hat{z}$ the local analytic coordinate on $\hat{\Sigma}$. By the Riemann-Hurwitz theorem it is a compact Riemann surface of genus $2g - 1$ and admits an involutive conformal automorphism $\iota : \hat{\Sigma} \to \hat{\Sigma}$ satisfying $\pi \circ \iota = \pi$, which is simply sheet interchange.

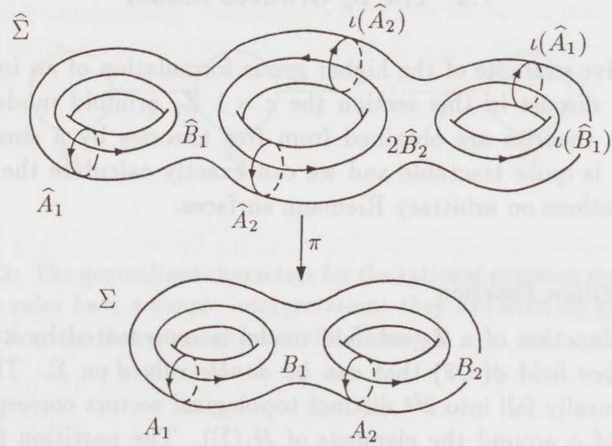



FIGURE 13: The double cover $\widehat{\Sigma}$ of a genus two Riemann surface $\Sigma$, both with a canonical homology basis. The branched cycle is $A_2$.

Once we have chosen a basis for $H_1(\Sigma, \mathbf{Z})$ and specified the twist characteristic $\left[\begin{smallmatrix}\epsilon\\ \delta\end{smallmatrix}\right]$ the double cover $\widehat{\Sigma}$ is uniquely determined. A convenient choice for the homology basis on $\widehat{\Sigma}$ is one that projects onto the homology basis on $\Sigma$. That is, $H_1(\widehat{\Sigma}, \mathbf{Z})$ is generated by $\widehat{A}_1, \widehat{B}_1, \ldots, \widehat{A}_{g-1}, \widehat{B}_{g-1}$, their images under $\iota$, and $\widehat{A}_g, 2\widehat{B}_g$, with $\pi(\widehat{A}_1) = A_1$ *etc.* as depicted in *fig.* 13. Note that $\iota(\widehat{A}_g) = \widehat{A}_g$ and $\iota(2\widehat{B}_g) = 2\widehat{B}_g$. We should mention here that such a homology basis is not uniquely fixed by these conditions. Different choices are related by modular transformations on $\widehat{\Sigma}$ which projected onto $\Sigma$ leave the homology basis fixed. We return to this point at the end of this section.

In order to calculate the classical part of the partition function we observe that $\partial\phi(z)$ will be a double-valued, holomorphic one-form on $\Sigma$. Its lift to $\widehat{\Sigma}$ can therefore be expanded in the so-called Prym differentials. The Prym differentials $\nu_i = \nu_i(\hat{z})d\hat{z}$ $(i = 1, \ldots, g-1)$ are the holomorphic one-forms on $\widehat{\Sigma}$, which are odd under the involution $\nu_i(\iota(\hat{z})) = -\nu_i(\hat{z})$. They are normalized with respect to the $\widehat{A}$-cycles

$$\oint_{\widehat{A}_i} \nu_j \;=\; -\oint_{\iota(\widehat{A}_i)} \nu_j \;=\; \delta_{ij}, \tag{7.19a}$$

$$\oint_{\widehat{B}_i} \nu_j \;=\; -\oint_{\iota(\widehat{B}_i)} \nu_j \;=\; \Pi_{ij}, \qquad (i,j = 1, \ldots, g-1). \tag{7.19b}$$

The matrix $\Pi_{ij}$ is the period matrix of the Prym differentials [63]. Of course, $\Pi$ depends on the twist characteristics—a fact we will sometimes stress by writing $\Pi_{\epsilon,\delta}$.

The Prym differentials have no periods around $\widehat{A}_g$ and $2\widehat{B}_g$. In terms of the underlying surface $\Sigma$, the Prym differentials are double-valued holomorphic one-forms which are anti-periodic around $B_g$. The complex torus $\mathbf{C}^{g-1}/\left(\mathbf{Z}^{g-1}+\Pi\mathbf{Z}^{g-1}\right)$ is called the Prym variety and is isomorphic to the set of degree zero divisor classes on $\widehat{\Sigma}$ odd under $\iota$. The isomorphism is given by the Abel map

$$Q-\iota(Q)\rightarrow\tfrac{1}{2}\int_{\iota(Q)}^{Q}\nu_i. \tag{7.20}$$

Divisors on the surface $\widehat{\Sigma}$ have (up to a minus sign) a unique image in the Prym variety.

The contribution of the twisted sector to the orbifold partition function is now given by the functional integral over fields $\widehat{\phi}$ defined on $\widehat{\Sigma}$ which are odd under $\iota:\widehat{\phi}(\iota(\hat{z}))=-\widehat{\phi}(\hat{z})\pmod{2\pi R}$. Furthermore, because of the doubled area of $\widehat{\Sigma}$, the action of $\widehat{\phi}$ has to be rescaled with a factor of $\frac{1}{2}$. The computation of the soliton contribution to the partition function is analogous to the untwisted case. We split the scalar field $\widehat{\phi}$ into a classical part and a quantum part $\widehat{\phi}=\widehat{\phi}^{qu}+\widehat{\phi}^{cl}$. The classical part is now given by the integral of the Prym differentials

$$\widehat{\phi}^{cl}=-\tfrac{1}{2}i\pi R(m-\overline{\Pi}n)\cdot(\operatorname{Im}\Pi)^{-1}\cdot\int_{\iota(\hat{z})}^{\hat{z}}\nu+c.c. \tag{7.21}$$

Its action can be expressed in terms of the period matrix $\Pi$

$$S[\widehat{\phi}^{cl}]=\tfrac{1}{2}\pi R^2(n-\overline{\Pi}m)\cdot(\operatorname{Im}\Pi)^{-1}\cdot(n-\Pi m). \tag{7.22}$$

After a Poisson resummation of the soliton sum the expression for the contribution $Z_{\epsilon,\delta}$ to the partition function reads

$$Z_{\epsilon,\delta}(R)=Z^{qu}_{\epsilon,\delta}\sum_{p,\overline{p}\in\Gamma_R^{g-1}}\exp\left[i\pi\left(p\cdot\Pi\cdot p-\overline{p}\cdot\overline{\Pi}\cdot\overline{p}\right)\right]. \tag{7.23}$$

For rational $R^2$ this expression can of course also be rewritten in terms of $\vartheta$-functions defined on the Prym variety using (7.12a–b) with $\tau$ replaced by $\Pi$.

The quantum contribution to the partition function seems much more difficult to determine. However, since it is independent of the compactification radius, and with the discussion of Chapter 6 in mind, we can choose the multi-critical value $R=\sqrt{2}$, where we can equate the partition function of the orbifold to that of the $R=\frac{1}{2}\sqrt{2}$ gaussian model. Combining Eqs (7.13a) and (7.13c) we find for the twisted part the identity

$$2^{-g}\sum_{\epsilon,\delta,\gamma}Z^{qu}_{\epsilon,\delta}\left|\vartheta\left[\begin{smallmatrix}\gamma\\0\end{smallmatrix}\right](0|2\Pi_{\epsilon,\delta})\right|^2=2^{-g}\sum_{\epsilon,\delta,\gamma}Z^{qu}_0\left|\vartheta\left[\begin{smallmatrix}\frac{1}{2}\epsilon+\gamma\\\delta\end{smallmatrix}\right](0|2\tau)\right|^2. \tag{7.24}$$

Here on both sides the summation runs over $(\epsilon,\delta)\neq(0,0)$. In order to interpret the separate terms in this equation, let us conceive an operator formalism in which the partition function is obtained by summing in each loop over all states in the Hilbert space. The characteristics $\epsilon$, $\delta$, and $\gamma$ then indicate which sector of the spectrum contributes to the different loops. Both theories correspond to a twisted $SU(2)$ model. So, in order to identify these sectors, let us first compare (7.24) with the untwisted $SU(2)$ partition function

$$Z(\sqrt{2}) = \sum_{\gamma\in(\mathbf{Z}/2)^g} Z_0^{qu}\left|\vartheta\begin{bmatrix}\gamma\\0\end{bmatrix}(0|2\tau)\right|^2. \tag{7.25}$$

Here $\gamma_i=0,\frac{1}{2}$ can be seen to distinguish the integer respectively half-integer spin representations of $SU(2)$ in the $i^{th}$ loop. Indeed, for $g=1$ the two terms are given by the $k=1$ $A_1^{(1)}$ characters $|\chi_0(\tau)|^2$ and $|\chi_1(\tau)|^2$ (6.56). It is evident that $\gamma$ has the same function for the untwisted loops in equation (7.24). The interpretation of $\epsilon$ and $\delta$ is now also clear. On the right-hand side the sum over $\delta$ corresponds to a projection on even momentum states, while $\epsilon$ labels the integer resp. half-integer winding number sectors. On the left-hand side $\epsilon$ and $\delta$ have a comparable task. The untwisted and twisted sectors are indexed by $\epsilon=0,\frac{1}{2}$ whereas the summation over $\delta$ projects onto $\mathbf{Z}_2$ even states.

This can be neatly demonstrated in the genus one case. Here we can explicitly evaluate the twisted contributions to the orbifold partition function using an oscillator representation

$$\begin{aligned} Z_{0,\frac{1}{2}}(\sqrt{2}) &= -\underset{+}{\square} = \mathrm{Tr}_0\left[\Theta\, q^{L_0-\frac{1}{24}}\bar{q}^{\bar{L}_0-\frac{1}{24}}\right] = \left|q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1+q^n)\right|^{-2} \\ &= |\eta(q)|^{-2}\left|\vartheta\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}(0|2\tau)\right|^2, \end{aligned} \tag{7.26a}$$

$$\begin{aligned} Z_{\frac{1}{2},0}(\sqrt{2}) &= +\underset{-}{\square} = \mathrm{Tr}_{\frac{1}{2}}\left[q^{L_0-\frac{1}{24}}\bar{q}^{\bar{L}_0-\frac{1}{24}}\right] = 2\left|q^{-\frac{1}{48}}\prod_{n=1}^{\infty}(1-q^{n+\frac{1}{2}})\right|^{-2} \\ &= 2\,|\eta(q)|^{-2}\left|\vartheta\begin{bmatrix}\frac{1}{4}\\0\end{bmatrix}(0|2\tau)\right|^2, \end{aligned} \tag{7.26b}$$

$$\begin{aligned} Z_{\frac{1}{2},\frac{1}{2}}(\sqrt{2}) &= -\underset{-}{\square} = \mathrm{Tr}_{\frac{1}{2}}\left[\Theta\, q^{L_0-\frac{1}{24}}\bar{q}^{\bar{L}_0-\frac{1}{24}}\right] = 2\left|q^{-\frac{1}{48}}\prod_{n=1}^{\infty}(1+q^{n+\frac{1}{2}})\right|^{-2} \\ &= 2\,|\eta(q)|^{-2}\left|\vartheta\begin{bmatrix}\frac{1}{4}\\\frac{1}{2}\end{bmatrix}(0|2\tau)\right|^2. \end{aligned} \tag{7.26c}$$

Here $\Theta$ is the twist operator, $\Theta^\dagger\phi\,\Theta=-\phi$, and the suffices $0,\frac{1}{2}$ attached to the trace denote the restriction to the untwisted resp. twisted Hilbert space sector. So we see that for $g=1$ the separate terms in (7.24) can indeed be equated.

Generalizing this observation to higher genus surfaces we can now solve for the twisted partition function $Z^{qu}_{\epsilon,\delta}$

$$Z^{qu}_{\epsilon,\delta} = \left| c\begin{bmatrix}\epsilon\\ \delta\end{bmatrix}\right|^{-2} Z^{qu}_0, \tag{7.27}$$

where $c\begin{bmatrix}\epsilon\\ \delta\end{bmatrix}$ is defined as the ratio of the classical contributions. For our canonical twist $\begin{bmatrix}\epsilon\\ \delta\end{bmatrix} = \begin{bmatrix}0\cdots 0\\ 0\cdots \frac{1}{2}\end{bmatrix}$ we have

$$c\begin{bmatrix}0\cdots 0\\ 0\cdots \frac{1}{2}\end{bmatrix} = \frac{\vartheta\begin{bmatrix}\gamma\\ 0\end{bmatrix}(0|2\Pi)}{\vartheta\begin{bmatrix}\gamma & 0\\ 0 & \frac{1}{2}\end{bmatrix}(0|2\tau)}. \tag{7.28}$$

As we have just argued, $c$ is independent of the characteristic

$$\begin{bmatrix}\gamma\\ 0\end{bmatrix} = \begin{bmatrix}\gamma_1\cdots\gamma_{g-1}\\ 0\cdots 0\end{bmatrix},$$

a fact that is indeed known in the mathematical literature as one of the Schottky relations [63]. The other $c\begin{bmatrix}\epsilon\\ \delta\end{bmatrix}$ are related to (7.28) by a modular transformation. Finally, combining equations (7.16), (7.23), and (7.27) we arrive at the following general result for the orbifold partition function on a genus $g$ Riemann surface

$$Z^{orb}(R) = 2^{-g} Z(R) + Z_{twist}(R), \tag{7.29a}$$

$$Z_{twist}(R) = Z^{qu}_0 \sum_{\epsilon,\delta} \left| c\begin{bmatrix}\epsilon\\ \delta\end{bmatrix}\right|^{-2} \sum_{p,\overline{p}\in\Gamma_R^{g-1}} \exp\left[ i\pi\left(p\cdot \Pi_{\epsilon,\delta}\cdot p - \overline{p}\cdot \overline{\Pi}_{\epsilon,\delta}\cdot \overline{p}\right)\right]. \tag{7.29b}$$

From this result one can now extract all correlation functions of twist fields, vertex operators, *etc.* by factorization at the compactification divisor and projecting onto the relevant sector, indexed by the twist structure $(\epsilon,\delta)$ and loop momenta $(p,\overline{p})$. This will be worked out in detail in sections 7.3 and 7.4.

### 7.2.2. *Dirac versus Majorana Fermions*

Let us make some comments on the case $R=1$. As mentioned in section 6.1 the model can be described as two decoupled Ising systems, *i.e.* two Majorana fermions. This should be compared with the corresponding $R=1$ gaussian model which is equivalent to the spin model of a complex Dirac fermion $\psi = e^{i\phi}$. On a higher genus Riemann surface the construction of a spin model implies a summation over all spin structures. In the orbifold model one introduces extra twisted sectors. In terms of the complex Dirac fermion $\psi = \psi_1 + i\psi_2$ the twist $\phi \to -\phi$ introduces a difference between the spin structures of its real and imaginary component. So the $R=1$ orbifold model is given by the spin model of two *un*correlated

Majorana fermions [59]. Indeed using $\vartheta$-function doubling formulas we can rewrite the Schottky relation (7.28) as

$$c^2\left[\begin{smallmatrix}0\dots0\\0\dots\frac{1}{2}\end{smallmatrix}\right] = \frac{\left[\vartheta\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right](0|\Pi)\right]^2}{\vartheta\left[\begin{smallmatrix}\alpha&0\\\beta&0\end{smallmatrix}\right](0|\tau)\vartheta\left[\begin{smallmatrix}\alpha&0\\\beta&\frac{1}{2}\end{smallmatrix}\right](0|\tau)}. \tag{7.30}$$

Hence the twisted contribution to the $R = 1$ orbifold partition function can be re-expressed as

$$\begin{aligned} Z_{twist}(1) &= Z_0^{qu}2^{-2g}\sum_{\epsilon,\delta}\left|c\left[\begin{smallmatrix}\epsilon\\\delta\end{smallmatrix}\right]\right|^{-2}\sum_{\alpha,\beta}\left|\vartheta\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right](0|\Pi)\right|^2 \\ &= Z_0^{qu}2^{-2g}\sum_{\substack{\alpha,\beta,\mu,\nu\\(\alpha,\beta)\neq(\mu,\nu)}}\left|\vartheta\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right](0|\tau)\vartheta\left[\begin{smallmatrix}\mu\\\nu\end{smallmatrix}\right](0|\tau)\right|^2. \end{aligned} \tag{7.31}$$

Adding the untwisted part we obtain with (7.29a-b)

$$Z^{orb}(1) = (Z_{Ising})^2 \tag{7.32}$$

with $Z_{Ising}$ the Ising partition function [78]

$$Z_{Ising} = (Z_0^{qu})^{\frac{1}{2}}\,2^{-g}\sum_{\alpha,\beta}\left|\vartheta\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right](0|\tau)\right|. \tag{7.33}$$

This confirms the identification with the Ashkin-Teller model at the decoupling point for arbitrary genus.

### *7.2.3. The Torelli Group*

We close this section with a discussion of the action of the the mapping class (or modular) group $\Gamma_g$ [19], *i.e.* the group of all disconnected diffeomorphisms of $\Sigma$. The mapping class group is generated by Dehn twists around cycles. The action of such a twist $D_C$ can be represented by cutting the Riemann surface along the cycle $C$ and gluing it together after rotating one of the boundaries over $2\pi$. The effect of this transformation on the elements $\gamma$ of the homology group $H_1(\Sigma)$ is

$$D_C : \gamma \rightarrow \gamma + \#(\gamma, C)\,C. \tag{7.34}$$

Note that this transformation does not depend on the orientation of either $C$ or $\gamma$. Dehn twists of the form (7.34) are called positive, the inverse twists $D_C^{-1}$ are called negative. The mapping class group leaves the intersection product invariant, hence it acts on the homology basis $A_i, B_i$ by $Sp(2g, \mathbf{Z})$ transformations. Moreover, all elements of the symplectic group $Sp(2g, \mathbf{Z})$ correspond to modular transformations.

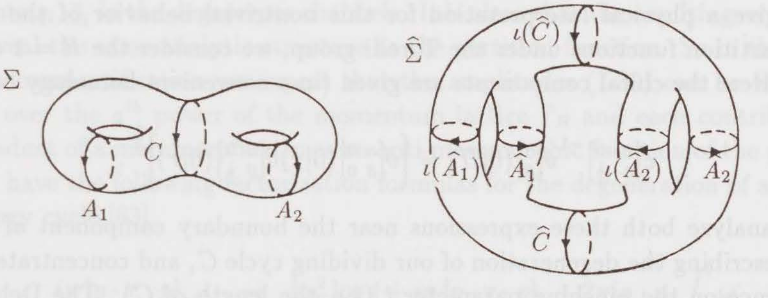



FIGURE 14: A genus 2 surface $\Sigma$ with branch cut along $A_1 + A_2$ and its double cover $\widehat{\Sigma}$. The lifts $\widehat{C}$ and $\iota(\widehat{C})$ of the zero-homology cycle $C$ are homologically nontrivial.

The subgroup of $\Gamma_g$ that leaves the homology fixed is known as the *Torelli* group. It is generated by the Dehn twists around the homologically trivial cycles on $\Sigma$. So, in summary, we have the following exact sequence

$$1 \rightarrow \mathit{Torelli} \rightarrow \Gamma_g \rightarrow Sp(2g, \mathbf{Z}) \rightarrow 1. \tag{7.35}$$

We have seen that for rational $R^2$ both the gaussian and the orbifold partition function can be written as a finite sum of products of a holomorphic times an anti-holomorphic function on moduli space. This total sum is invariant under the mapping class group, but the individual analytic parts are not. For the gaussian models these are—apart from an overall factor $(\det \overline{\partial}_0)^{-\frac{1}{2}}$—all functions of the period matrix $\tau$, on which $\Gamma_g$ acts by the $Sp(2g, \mathbf{Z})$ transformations

$$\tau \rightarrow (A\tau + B)(C\tau + D)^{-1}. \tag{7.36}$$

Hence all gaussian models are insensitive to the Torelli group. The chiral orbifold partition functions, on the other hand, are functions of the period matrix $\Pi$ of the Prym differentials and in general they do feel the Torelli group. To explain this let us consider a zero homology cycle $C$ on $\Sigma$ and a twist structure with two branched cycles, one on each side of $C$ (see *fig.* 14). For this case the lifts $\widehat{C}$ and $\iota(\widehat{C})$ to the double cover $\widehat{\Sigma}$ are homologically nontrivial. The positive Dehn twist $D_C$ on $\Sigma$ lifts to the composite transformation $D_{\widehat{C}} \circ D_{\iota(\widehat{C})}$ on $\widehat{\Sigma}$, which has a nontrivial action on the homology of the double cover. More specifically, it transforms the elements of $H_1(\widehat{\Sigma}, \mathbf{Z})$ odd under the involution $\iota$, whereas the even elements are inert. Consequently the period matrix $\Pi$ will not be invariant under the Dehn twist $D_C$. The Torelli group projects onto a subgroup of the symplectic group $Sp(2g-2, \mathbf{Z})$ that acts on $\Pi$ as in (7.36). For example, in the case considered in *fig.* 14 the transformation reads $\Pi \rightarrow \Pi + 4$.

To give a physical interpretation for this nontrivial behavior of the twisted chiral partition functions under the Torelli group, we consider the $R=1$ orbifold model. Here the chiral components are given (in a convenient homology basis) by

$$c\begin{bmatrix} 0\cdots 0 \\ 0\cdots \frac{1}{2}\end{bmatrix}^{-1} \vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0|\Pi) = \left[\vartheta\begin{bmatrix}\alpha & 0\\ \beta & 0\end{bmatrix}(0|\tau)\vartheta\begin{bmatrix}\alpha & 0\\ \beta & \frac{1}{2}\end{bmatrix}(0|\tau)\right]^{\frac{1}{2}}. \tag{7.37}$$

Let us analyze both these expressions near the boundary component of moduli space describing the degeneration of our dividing cycle $C$, and concentrate on the dependence on the pinching parameter $t$ (*i.e.* the length of $C$). The Dehn twist $D_C$ acts on $t$ as

$$D_C : t \to e^{2\pi i} t. \tag{7.38}$$

If we choose the (even) spin structure on the left-hand side of (7.37) such that it splits into two odd spin structures for $t \to 0$, then the same is true for one of the spin structures on the right-hand side. It is quite easy to see that for this situation the chiral partition function behaves to first order as $t^{\frac{1}{2}}$ and hence is not invariant under (7.38). The fermionic explanation is that the partition function of one of the chiral Majorana fermions factorizes on the one-point functions of two $\psi$ fields absorbing the zero-mode on each side. The power of $t$ equals the conformal dimension of $\psi$ which is $\frac{1}{2}$. In the orbifold picture (7.37) is seen to factorize on the one-point function of the corresponding chiral vertex operator $V = \cos\phi$. The key observation here is that $\phi$ charge is not conserved around twisted cycles. This argumentation makes it clear that also for other radii $R$ the chiral orbifold partition function are not invariant under the Torelli group, as long as the corresponding chiral theories contain vertex operators with non-integer conformal weights. Of course, the total partition function (7.29a-b) is invariant, since the full theory only contains operators with integer spin $h-\overline{h}$.

## 7.3. Vertex Operators and Factorization

In this section we compute the correlation functions of vertex operators in both the gaussian and the orbifold model on a genus $g$ Riemann surface $\Sigma$. In these calculations we will employ the factorization expansion of the partition function to determine the analytic structure of the correlators. The results of this section will be used for the construction of the operator formalism on Riemann surfaces, as described in section 7.5. The $n$-point functions of vertex operators

$$A(z_i,\overline{z}_i;q_i,\overline{q}_i) = \left\langle V_{q_1\overline{q}_1}(z_1,\overline{z}_1)\ldots V_{q_n\overline{q}_n}(z_n,\overline{z}_n)\right\rangle \tag{7.39}$$

can be obtained by considering the partition function near the compactification divisor, where an appropriate number of handles is pinched, and projecting on loop momenta $q_i$, $\overline{q}_i$ in the degenerate channels. It is clear that the resulting expressions will reveal the same analytic structure as the partition function. So in the case of a torus compactification we expect that the amplitudes (7.39) can be written as a sum over the $g^{th}$ power of the momentum lattice $\Gamma_R$ and each contribution is the product of a meromorphic times an anti-meromorphic function of the positions $z_i$. We have the following factorization formulas for the degeneration of a nonzero homology cycle [63]

$$i\pi\left[p\cdot\tau\cdot p\right]_{g+1} \longrightarrow \tfrac{1}{2}p_0^2\log t + i\pi\left[p\cdot\tau\cdot p\right]_g + 2\pi i p\cdot p_0\int_a^b \omega \\ -p_0^2\log E(a,b) + \mathcal{O}(t), \tag{7.40}$$

$$\omega_0(z) \longrightarrow \partial_z \log\left[\frac{E(a,z)}{E(b,z)}\right] + \mathcal{O}(t), \tag{7.41}$$

while all other quantities factorize trivially, to first order. By repeatedly applying these formulas one deduces the following expression for the unnormalized $n$-point function in terms of the prime form $E(z,w)$ and the holomorphic one-forms $\omega_i(z)$ $(i=1,\ldots,g)$ of the Riemann surface $\Sigma$:

$$A_0(z_i,\overline{z}_i;q_i,\overline{q}_i) = Z_0^{qu}\sum_{p,\overline{p}\in\Gamma_R^g} A_0^p(z_i;q_i)\overline{A_0^p(z_i;q_i)},$$

$$A_0^p(z_i;q_i) = \prod_{i<j}E(z_i,z_j)^{q_iq_j}\exp\left[i\pi(p\cdot\tau\cdot p)+2\pi i\left(p\cdot\sum_i q_i\int^{z_i}\omega\right)\right] \tag{7.42}$$

In this expression complex conjugation also takes $p\rightarrow\overline{p}$ and $q\rightarrow\overline{q}$. The momenta $q_i$, $\overline{q}_i$ are forced by crossing symmetry of the amplitude to be elements of the lattice $\Gamma_R$. Factorization automatically gives momentum conservation:

$$\sum_i(q_i,\overline{q}_i) = (0,0). \tag{7.43}$$

The expressions for all other correlation functions follow from (7.42) by taking suitable operator products, *i.e.* limits of the positions $z_i$. The result (7.42) can also be obtained by a generalization of the path-integral method of [155]. Of course the expression (7.39) cannot be directly inserted into the path-integral, since it is not a function of the integration variable $\phi(z,\overline{z})$. Instead, for each vertex operator in (7.39) we insert $\exp\left[i\pi(q_i+\overline{q}_i)\phi(z_i,\overline{z}_i)\right]$ and integrate over field configurations with winding number $q_i-\overline{q}_i$ around the point $z_i$. This is achieved by adding to the classical solitons the extra piece

$$-\sum_i \tfrac{1}{2}i(q_i-\overline{q}_i)\left[\log E(z_i,z_j) - \tfrac{1}{2}\pi\int_{z_i}^{z}\omega\cdot(\mathrm{Im}\,\tau)^{-1}\cdot\int_{z_i}^{z}\omega\right] + c.c. \tag{7.44}$$

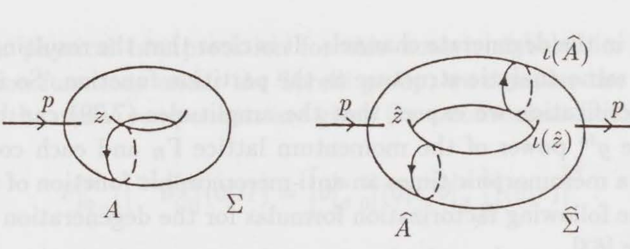



FIGURE 15: The insertion of a vertex operator on the surface $\Sigma$ corresponds on the double cover $\widehat{\Sigma}$ to the insertion of a conjugate pair of operators.

A straightforward application of Wick's theorem and the Poisson resummation formula then yields (7.42).

### *7.3.1. Vertex Operators in the Orbifold Model*

Let us now turn to the correlators of vertex operators in the $\mathbf{Z}_2$ orbifold model. On the sphere or complex plane these are the same as in the gaussian model. For higher genus, however, some new features appear due to the fact that we can have twisted states in the loops. In particular, momentum conservation is lost, since the chiral currents $\partial\phi$ and $\overline{\partial}\phi$ are no longer allowed operators in the orbifold model. However, we still have a selection rule as a result of the $D_4$ symmetry (6.36)

$$\sum_i (q_i, \overline{q}_i) \in 2\,\Gamma_R. \tag{7.45}$$

The analytic structure of the amplitudes again follows from factorization. So, just like the partition function, they will be a sum of two terms which are by itself single-valued and modular invariant. The first equals $2^{-g}$ times the corresponding untwisted expression, while the latter, which we will denote as $A_t$, contains the contribution of the twisted intermediate states. If we choose a particular twisted cycle and project onto definite loop momenta in the untwisted loops, the amplitudes will factorize into an analytic times an anti-analytic part. Because there is no explicit factorization formula available for the period matrix of the Prym differentials, we will compute $A_t$ via the path-integral method described above. To carry out this calculation, it will again be convenient to consider the double cover $\widehat{\Sigma}$ of the multi-loop surface $\Sigma$. This enables us to express the amplitudes in terms of the prime form $\widehat{E}(\hat{z}, \hat{w})$ on $\widehat{\Sigma}$ and the Prym differentials $\nu_i$. Note that we have to insert vertex operators on both sheets of the double cover with opposite momenta as depicted in *fig.* 15. This automatically guarantees momentum conservation on $\widehat{\Sigma}$.

The propagator of an uncompactified double-valued scalar field twisted around the cycle $B_g$ is given by

$$\left\langle \phi(z,\overline{z})\phi(w,\overline{w})\right\rangle = \log F_t(z,w), \tag{7.46}$$

where

$$F_t(z,w) = |E_t(z,w)|^2 \exp\left[2\pi \mathrm{Im}\int_{\iota(\hat{z})}^{\hat{z}} \nu \cdot (\mathrm{Im}\,\Pi)^{-1}\cdot \mathrm{Im}\int_{\iota(\hat{z})}^{\hat{z}} \nu\right], \tag{7.47}$$

with

$$E_t(z,w) = \frac{\widehat{E}(\hat{z},\hat{w})}{\widehat{E}(\hat{z},\iota(\hat{w}))} e^{-\frac{1}{2}i\pi\tau_{gg}}, \tag{7.48}$$

is single-valued around all cycles except for the twisted cycle $B_g$, around which it transforms into $1/F_t$. The above result can be proved by a similar method as described in [155, section 5]. The expression for the amplitude (7.39) for $R=0$ (or $\infty$) now follows directly from Wick's theorem. The modification at finite $R$ due to the soliton contributions is computed completely analogous to the gaussian case. Again skipping the details of the calculation we proceed with the final expression for the twisted contribution to the $n$-point function (7.39)

$$A_t(z_i,\overline{z}_i;q_i,\overline{q}_i) = Z_0^{qu}\sum_{\epsilon,\delta}\sum_{p,\overline{p}\in\Gamma_R^{g-1}} A^p_{\epsilon,\delta}(z_i;q_i)\overline{A^p_{\epsilon,\delta}(z_i;q_i)}, \tag{7.49}$$

where the meromorphic contributions are given by

$$\begin{aligned} A^p_{\epsilon,\delta}(z_i;q_i) &= c\begin{bmatrix}\epsilon\\ \delta\end{bmatrix}^{-1}\prod_{i<j}E(z_i,z_j)^{q_iq_j}\prod_i \varepsilon(z_i)^{\frac{1}{2}q_i^2} \\ &\quad\times \exp\left[i\pi(p\cdot\Pi\cdot p) + 2\pi i\left(p\cdot\sum_i \tfrac{1}{2}q_i\int_{\iota(\hat{z})}^{\hat{z}}\nu\right)\right]. \end{aligned} \tag{7.50}$$

Here the presence of the factors

$$\varepsilon(z) = \widehat{E}(\hat{z},\iota(\hat{z}))^{-1}e^{-\frac{1}{2}i\pi\tau_{gg}} \tag{7.51}$$

is due to the self-energy of the vertex operators. Of course, all quantities on the right-hand side of (7.50) are understood to be those related to the twist $\epsilon,\delta$. The above expression for the amplitude $A$ is not yet a single-valued function of the positions of all vertex operators; only the combinations $V^+_{nm}(z,\overline{z})$ (see (6.10)) have single-valued correlators.

We have shown that vertex operators of the form $V^+_{2n,2m}$ indeed have vacuum expectation values in the orbifold theory. As we have argued this implies directly that the twisted chiral partition functions of the $\mathbf{Z}_2$ orbifold theories are sensitive to the Torelli group. We are now in a position to make this relation more explicit.

So let us consider again a Dehn twist around a homologically trivial cycle $C$ (as in *fig.* 14), and the behavior of the partition function near the boundary of moduli space corresponding to the degeneration of $C$. Here we can compare the expression (7.29a-b) for the partition function with the factorization expansion

$$Z_{g_1+g_2} \to Z_{g_1} Z_{g_2} \Big[1 + \sum_{n,m \in 2\mathbf{Z}} t^{h_{nm}} \bar{t}^{\bar{h}_{nm}} \left\langle V_{nm}^+ \right\rangle_{g_1} \left\langle V_{nm}^+ \right\rangle_{g_2} + (\textit{descendants})\Big], \qquad (7.52)$$

where $g_1$ and $g_2$ are the genera of the two parts of $\Sigma$ separated by $C$, and $\langle V_{nm}^+ \rangle_{g_i}$ denotes the 1-point function of the vertex operator $V_{nm}^+$ on each part. The chiral partition functions on $\Sigma$ with twist characteristics which are divided by $C$ into two nontrivial parts, will receive nonvanishing contributions of these 1-point functions. Equating the separate momentum contributions on both sides of (7.52) and isolating the part quadratic in the loop momenta we read off that, if $C$ degenerates, the $\Pi$ matrix of such a twist characteristic factorizes as

$$\begin{aligned} i\pi[p \cdot \Pi \cdot p]_{g_1+g_2} \quad \to \quad & 2p_0^2 \log t + i\pi[p \cdot \Pi \cdot p]_{g_1} + i\pi[p \cdot \Pi \cdot p]_{g_2} \\ & + 2\pi i p_1 \cdot p_0 \int_{\iota(\hat{Q}_1)}^{\hat{Q}_1} \nu_1 + 2\pi i p_2 \cdot p_0 \int_{\iota(\hat{Q}_2)}^{\hat{Q}_2} \nu_2 \\ & + 2p_0^2 \log\left[\varepsilon(Q_1)\varepsilon(Q_2)\right] \; + \; \mathcal{O}(t). \end{aligned} \qquad (7.53)$$

The first term on the right-hand side signals the momentum $2p_0$ running through the tube enclosed by $C$ and is clearly not invariant under $t \to e^{2\pi i}t$. This factorization behavior of the $\Pi$ matrix should be contrasted with that of the period matrix $\tau$

$$i\pi[p \cdot \tau \cdot p]_{g_1+g_2} \to \; 2p_0^2 \log t + i\pi[p \cdot \tau \cdot p]_{g_1} + i\pi[p \cdot \tau \cdot p]_{g_2} \; + \; \mathcal{O}(t). \qquad (7.54)$$

where $\mathcal{O}(t)$ is a single-valued function of $t$.

### 7.3.2. *Some Strange Identities*

For the specific models in the $c = 1$ spectrum many relevant correlation functions can now be calculated using (7.49)–(7.51). As an example let us determine in the $R = 1$ orbifold model the two-point function of the magnetic vertex operator $\sqrt{2} \cos \frac{1}{2}(\phi - \overline{\phi})$. According to (2.36) this operator equals the composite operator $\sigma^{(1)} \otimes \sigma^{(2)}$ in the doubled Ising system, so the result should be equal to the square of the spin-spin correlator of the Ising model [59,44]. Let us see how this comes about. By lack of momentum conservation we have two different twisted contributions to this correlation function, with holomorphic components

$$\left\langle e^{\frac{1}{2}i\phi(z)} e^{-\frac{1}{2}i\phi(w)} \right\rangle \; = \; c\begin{bmatrix} 0 \cdots 0 \\ 0 \cdots \frac{1}{2} \end{bmatrix}^{-1} E_t(z,w)^{-\frac{1}{4}} \epsilon(z)^{\frac{1}{8}} \epsilon(w)^{\frac{1}{8}}$$

$$\times \vartheta\Big[{}^{\alpha}_{\beta}\Big]\left(\tfrac{1}{4}\int_{\iota(\hat{z})}^{\hat{z}} \nu - \tfrac{1}{4}\int_{\iota(\hat{w})}^{\hat{w}} \nu \Big| \Pi\right), \tag{7.55a}$$

$$\left\langle e^{\frac{1}{2}i\phi(z)} e^{\frac{1}{2}i\phi(w)}\right\rangle = c\Big[{}^{0\cdots 0}_{0\cdots \frac{1}{2}}\Big]^{-1} E_t(z,w)^{\frac{1}{4}}\epsilon(z)^{\frac{1}{8}}\epsilon(w)^{\frac{1}{8}}$$

$$\times \vartheta\Big[{}^{\alpha}_{\beta}\Big]\left(\tfrac{1}{4}\int_{\iota(\hat{z})}^{\hat{z}} \nu + \tfrac{1}{4}\int_{\iota(\hat{w})}^{\hat{w}} \nu \Big| \Pi\right). \tag{7.55b}$$

After some manipulations using a generalized version of the Schottky relation [63] the right-hand sides can be rewritten as

$$\left\langle e^{\frac{1}{2}i\phi(z)} e^{-\frac{1}{2}i\phi(w)}\right\rangle = E_t(z,w)^{-\frac{1}{4}}\left[\vartheta\Big[{}^{\alpha\ 0}_{\beta\ 0}\Big]\left(\tfrac{1}{2}\int_w^z \omega \Big| \tau\right)\right.$$

$$\left.\times\, \vartheta\Big[{}^{\alpha\ 0}_{\beta\ \frac{1}{2}}\Big]\left(\tfrac{1}{2}\int_w^z \omega \Big| \tau\right)\right]^{\frac{1}{2}}, \tag{7.56a}$$

$$\left\langle e^{\frac{1}{2}i\phi(z)} e^{\frac{1}{2}i\phi(w)}\right\rangle = E_t(z,w)^{-\frac{1}{4}}\left[\vartheta\Big[{}^{\alpha\ \frac{1}{2}}_{\beta\ 0}\Big]\left(\tfrac{1}{2}\int_w^z \omega \Big| \tau\right)\right.$$

$$\left.\times\, \vartheta\Big[{}^{\alpha\ \frac{1}{2}}_{\beta\ \frac{1}{2}}\Big]\left(\tfrac{1}{2}\int_w^z \omega \Big| \tau\right)\right]^{\frac{1}{2}}. \tag{7.56b}$$

In terms of the two Majorana fermions we see that the first term gives the contribution of the $(even)\times(even)$ or $(odd)\times(odd)$ spin structures (depending on whether $\big[{}^{\alpha}_{\beta}\big]$ in (7.55a) is even or odd), whereas the second corresponds to $(odd)\times(even)$. This identification is in agreement with the leading behavior of (7.55a) for $z \to w$. Performing the summation over all sectors and taking the square root, we obtain for the unnormalized spin 2-point function

$$\left\langle \sigma(z,\overline{z})\sigma(w,\overline{w})\right\rangle = 2^{-\frac{1}{2}g}(Z_0^{qu})^{\frac{1}{2}}|E_t(z,w)|^{-\frac{1}{4}}\sum_{\alpha,\beta}\left|\vartheta\Big[{}^{\alpha}_{\beta}\Big]\left(\tfrac{1}{2}\int_w^z \omega\Big|\tau\right)\right|. \tag{7.57}$$

Of course, this expression can also be obtained by direct factorization of the Ising partition function (7.33). The result (7.57) agrees with that of [18,45] for $g = 1$. Finally, we notice that the equivalence of the $R = \sqrt{2}$ and $R = 1$ orbifold theory with the $R = \frac{1}{2}\sqrt{2}$ gaussian model resp. squared Ising model may be exploited to derive various nontrivial identities relating geometrical objects on the multi-loop surface $\Sigma$ to objects defined on its double cover. To give a simple example: the equality of the two vacuum expectation values

$$\left\langle \tfrac{1}{2}\sqrt{2}i\partial\phi(z)\right\rangle_{gauss,R=\frac{1}{2}\sqrt{2}} = \left\langle \cos\sqrt{2}\phi(z)\right\rangle_{orb,R=\sqrt{2}} \tag{7.58}$$

implies, when projected on the relevant subsectors, the identity [63, p. 83]

$$\sum_j i\partial_j\vartheta\Big[{}^{\alpha\frac{1}{2}}_{\beta\frac{1}{2}}\Big](0|2\tau)\omega_j(z) = c\Big[{}^{0\cdots 0}_{0\cdots\frac{1}{2}}\Big]^{-1}\epsilon(z)\vartheta\Big[{}^{\alpha}_{0}\Big]\left(\int_{\iota(\hat{z})}^{\hat{z}} \nu\Big|2\Pi\right). \tag{7.59}$$

Pursuing this line much further would lead us deep into the mathematics of unramified coverings of Riemann surfaces, which is of course not our intent. We would like to emphasize, however, that many existing mathematical identities can be of great help in the analysis of the correspondences between the various physical models.

## 7.4. Correlation Functions of Twist Operators

As a further application of the methods developed in the previous sections we will now proceed to calculate correlation functions of orbifold twist operators

$$\Big\langle \sigma_1(z_1,\overline{z}_1)\ldots\sigma_1(z_m,\overline{z}_m)\sigma_2(z_{m+1},\overline{z}_{m+1})\ldots\sigma_2(z_n,\overline{z}_n)\Big\rangle. \tag{7.60}$$

To evaluate this function we will again make use of the theory of double covers of Riemann surfaces. We will describe the calculation for arbitrary genus, and then turn to the sphere for more explicit results. A somewhat different approach is given by [115] and for $g=0$ in [53,18,168]. This latter approach has been generalized to arbitrary genus in [6].

Twist field correlation functions are produced by factorization of the partition function when a non-zero homology cycle is pinched. The different constituents $Z_{\epsilon,\delta}$ of the genus $g+1$ partition function factorize in lowest order of the pinching parameter $t$ either on the genus $g$ partition function or on the unnormalized twist field two-point function depending on whether or not the pinched loop is twisted. Schematically

$$Z_{g+1} \to Z_g + \ldots + |t|^{\frac{1}{8}} \sum_{i=1}^{2} \langle \sigma_i(z_1)\sigma_i(z_2)\rangle_g + \ldots \tag{7.61}$$

In general we see that pinching $n$ twisted loops of a genus $g+n$ surface leaves a genus $g$ Riemann surface $\Sigma$ with $2n$ twist field insertions. In this factorization process the genus of the double cover changes from $2g+2n-1$ to $2g+n-1$. The resulting cover $\widehat{\Sigma}$ of $\Sigma$ is ramified over $2n$ branch points, *i.e.* fixed points of the defining involution $\iota$, at the positions of the twist fields. Of course, besides the twists around the operators $\sigma(z_k,\overline{z}_k)$ we should also account for possible twists along the nontrivial cycles of $\Sigma$ indexed by the characteristics $\epsilon_i,\delta_i$ $(i=1,\ldots,g)$. Modular invariance at genus $g+n$ translates into modular invariance at genus $g$ and crossing symmetry of the $\sigma$'s [78,167]. Indeed, by transporting a twist operator around an untwisted loop one converts it into a twisted loop, and vice versa. So any twist characteristic $\left[\begin{smallmatrix}\epsilon\\ \delta\end{smallmatrix}\right]$ can be mapped to $\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right]$ by the crossing transformation

$$\int^{z_k}\omega \to \int^{z_k}\omega + 2\epsilon\cdot\tau + 2\delta. \tag{7.62}$$

As a homology basis of the cover $\widehat{\Sigma}$ we choose the cycles $\widehat{A}_i$, $\widehat{B}_i$, $\iota(\widehat{A}_i)$, $\iota(\widehat{B}_i)$ $(i = 1, \ldots, g)$ and the $2(n-1)$ extra cycles $\widehat{A}_\lambda$, $\widehat{B}_\lambda$ $(\lambda = 1, \ldots, n-1)$. The latter correspond on the underlying surface $\Sigma$ to loops encircling an even number of twist fields.

The number of Prym differentials does not change by the factorization; it stays $g+n-1$. We will normalize them with respect to the $\widehat{A}_i$ and $\widehat{A}_\lambda$-cycles of $\widehat{\Sigma}$. All the Prym differentials $\nu_i$, $\nu_\lambda$ have square root singularities at the branch points $z_k$. It will be convenient to define their period matrix $\Pi$ around the cycles $\widehat{B}_i$ and $\frac{1}{2}\widehat{B}_\lambda$. This normalization is consistent with factorization.

The essential idea now is to equate any multi-point correlation function with twist field insertions to the corresponding quantity without the twist fields calculated on the ramified double cover $\widehat{\Sigma}$ with the field $\widehat{\phi}$ odd under $\iota$. The dependence on the positions of the twist operators $\sigma(z_k)$ is incorporated in the definition of $\widehat{\Sigma}$. Note however, that because of the conformal anomaly all these quantities depend on the coordinatization we choose. The prescription is to use the coordinate $z$ of the Riemann surface $\Sigma$. The conformal dimension $(\frac{1}{16}, \frac{1}{16})$ of the twist operators is due to the fact that $z$ is a singular coordinate at the positions of the twist fields. The field $\widehat{\phi}$ cannot have arbitrary winding numbers along the cycles $\widehat{A}_\lambda$, $\widehat{B}_\lambda$. Around any cycle $C$ that projected on $\Sigma$ encircles an even number of $\sigma$'s, the operator product relations (6.34) imply the monodromy condition [53]

$$\oint_C \frac{dz}{2\pi R}\, \partial\widehat{\phi}(z) + c.c. = Q \pmod 2, \tag{7.63}$$

where $Q$ is a $\mathbf{Z}_2$ charge defined as

$$Q = \#\sigma_1 = \#\sigma_2 \pmod 2. \tag{7.64}$$

We define $\mathbf{Z}_2$ charges $a_\lambda$, $b_\lambda$ for the cycles $\widehat{A}_\lambda$, $\widehat{B}_\lambda$ as in (7.63).

Each quantity can now be written as a sum over $\epsilon_i$ and $\delta_i$ numbering the possible twists around the cycles $A_i$, $B_i$, and loop momenta $(p_i, p_\lambda; \overline{p}_i, \overline{p}_\lambda)$. The appropriate momentum lattice is fixed by the configuration of twist fields $\sigma_1$ and $\sigma_2$. To investigate this relation let us analyze the pure twist correlation function

$$A(z_k, \overline{z}_k) = \Big\langle \prod_{k=1}^{2n_1} \sigma_1(z_k, \overline{z}_k) \prod_{k=1}^{2n_2} \sigma_2(z_k, \overline{z}_k) \Big\rangle, \tag{7.65}$$

*i.e.* the partition function $\widehat{Z}$ on $\widehat{\Sigma}$. (Note that the discrete symmetry group $D_4$ forces correlation functions of an odd number of $\sigma_1$'s or $\sigma_2$'s to vanish.) This partition function is again given by a summation over all twisted sectors. Each term is the product of a quantum contribution $\widehat{Z}^{qu}$ times a sum over winding sectors. The soliton sum $\widehat{Z}^{cl}_{a,b}$ is restricted by the monodromy conditions labeled by $a_\lambda, b_\lambda$. The calculation is quite analogous to the one described in section 7.2.

The classical solutions $\widehat{\phi}^{cl}$ are labeled by $m_i, n_i \in \mathbf{Z}$, $m_\lambda \in \mathbf{Z} + \frac{1}{2}b_\lambda$, $n_\lambda \in 2\mathbf{Z} + a_\lambda$, and have action

$$S[\widehat{\phi}^{cl}] = \tfrac{1}{2}\pi R^2 (n - \overline{\Pi}) \cdot (\mathrm{Im}\,\Pi)^{-1} \cdot (n - \Pi m). \tag{7.66}$$

The soliton summation yields [115,17]

$$\widehat{Z}^{cl}_{a,b}(R) = \sum_{p,\overline{p}} (-1)^{b\cdot n} \exp\left[ i\pi \left( p \cdot \Pi \cdot p - \overline{p} \cdot \overline{\Pi} \cdot \overline{p} \right) \right], \tag{7.67}$$

where

$$p_i, \overline{p}_i \in \Gamma_R \qquad , i = 1, \ldots, g, \tag{7.68}$$

$$\begin{aligned} p_\lambda &= \frac{n}{R} + (m + \tfrac{1}{2}a_\lambda)R, \\ \overline{p}_\lambda &= \frac{n}{R} - (m + \tfrac{1}{2}a_\lambda)R, \end{aligned} \qquad n, m \in \mathbf{Z}, \quad \lambda = 1, \ldots, n-1. \tag{7.69}$$

As for the quantum part we again make use of the multi-critical point $R = \sqrt{2}$. Namely, as we have seen, in this point the twist fields $\sigma_1$ and $\sigma_2$ are equivalent to the magnetic vertex operators $\sqrt{2}\cos\frac{1}{4}\sqrt{2}(\phi - \overline{\phi})$ and $\sqrt{2}\sin\frac{1}{4}\sqrt{2}(\phi - \overline{\phi})$ of the $R = \frac{1}{2}\sqrt{2}$ torus model, whose correlation functions are simply a sum of generalized Koba-Nielsen amplitudes (7.42),

$$\begin{aligned} A(z_k, \overline{z}_k) \;=\; & 2^{-g-n+1} Z_0^{qu} \sum_{\gamma,\delta,\epsilon} \sum_{(x|y)} (-1)^{Q(x)} \left| \frac{\prod_{i<j} E(x_i, x_j) E(y_i, y_j)}{\prod_{i,j} E(x_i, y_i)} \right|^{\frac{1}{4}} \\ & \times \left| \vartheta\!\left[\begin{smallmatrix} \gamma + \frac{1}{2}\epsilon \\ \delta \end{smallmatrix}\right] \left( \sum_{i=1}^{n} \int_{y_i}^{x_i} \omega \middle| 2\tau \right) \right|^2 . \end{aligned} \tag{7.70}$$

Here we sum over all $\frac{1}{2}\binom{2n}{n}$ partitions $(x|y)$ of the branch points $z_k$ into two subsets $\{x_k\}$ and $\{y_k\}$ of $n$ elements. The charge $Q(x)$ is the total $\mathbf{Z}_2$ charge in the subset $\{\sigma(x_k)\}$. Note that we regard $(x|y)$ and $(y|x)$ as equivalent. On the other hand we can express the partition function on $\widehat{\Sigma}$ in terms of $\vartheta$-functions as explained in section 7.2,

$$\widehat{Z}_{a,b}(\sqrt{2}) = 2^{-g-n+1} \sum_{\epsilon,\delta} \widehat{Z}^{qu}_{\epsilon,\delta} \sum_{\mu,\nu,\gamma} (-1)^{2\gamma\cdot b + b\cdot a} \left| \vartheta\!\left[\begin{smallmatrix} \gamma & \mu \\ 0 & \nu \end{smallmatrix}\right] (0 | 2\Pi_{\epsilon,\delta}) \right|^2 . \tag{7.71}$$

Here the summation is over the half-integer characteristics

$$\gamma, \epsilon, \delta \in (\mathbf{Z}/2)^g, \quad \mu, \nu \in (\mathbf{Z}/2)^{n-1}. \tag{7.72}$$

Remarkably, it turns out that again one can identify the separate holomorphic terms in (7.70) and (7.71). For $\gamma, \delta, \epsilon$ we can repeat the argument of section

7.2. As for the characteristics $\mu, \nu$ we should note that not all even spin structures contribute in (7.71). In fact, there are only $\frac{1}{2}\binom{2n}{n}$ non-singular even spin structures, *i.e.* with non-vanishing $\vartheta$-function, which exactly correspond to the partitions $(x|y)$ of the branch points. This is explained in all detail in [63]. This allows us to solve

$$\widehat{Z}^{qu}_{\epsilon,\delta} = \left|c\left[{\epsilon \atop \delta}\right]\right|^{-2} Z^{qu}_0$$

as the ratio of the two corresponding contributions in (7.70) and (7.71). The result is the modulus squared of a holomorphic function of the positions of the branch points. So for example for the two-point function we find

$$\left\langle \sigma(z,\overline{z})\sigma(w,\overline{w})\right\rangle = 2^{-g} Z^{qu}_0 \sum_{\epsilon,\delta} \left|c\left[{\epsilon \atop \delta}\right]\right|^{-2} \widehat{Z}^{cl}(R, \Pi_{\epsilon,\delta}), \tag{7.73}$$

where

$$c\left[{\epsilon \atop \delta}\right]^{-1} = E(z,w)^{-\frac{1}{8}} \frac{\vartheta\left[{\gamma+\frac{1}{2}\epsilon \atop \delta}\right]\left(\frac{1}{2}\int_z^w \omega | 2\tau\right)}{\vartheta\left[{\gamma \atop 0}\right](0|2\Pi_{\epsilon,\delta})}, \tag{7.74}$$

which is independent of the characteristic $\gamma$. For $R=1$ this expression is consistent with the result (7.57), since we can rewrite (7.74) as

$$c\left[{\epsilon \atop \delta}\right]^{-1} = E(z,w)^{-\frac{1}{8}} \frac{\left[\vartheta\left[{\alpha+\epsilon \atop \beta+\delta}\right]\left(\frac{1}{2}\int_z^w \omega|2\tau\right)\vartheta\left[{\alpha \atop \beta}\right](0|\tau)\right]^{\frac{1}{2}}}{\vartheta\left[{\alpha \atop \beta}\right](0|\Pi_{\epsilon,\delta})}, \tag{7.75}$$

for arbirary $\alpha$, $\beta$.

For higher $n$-point functions the resulting expressions have not the same structure as (7.73). An unpleasant feature is that they are not manifestly independent of the choice of the spin structure $\left[{\mu \atop \nu}\right]$ or partition $(x|y)$. It would be very interesting to find such a manifest independent formulation. The computation of correlation functions including vertex operators is completely analogous to the one described in the previous section, provided one uses the branched double cover $\widehat{\Sigma}$ and the momentum lattice (7.69).

### 7.4.1. Correlation Functions on the Sphere

Now let us specialize to the case of twist operator correlation functions on the sphere. Here we can explicitly write down the double cover. It is the hyperelliptic Riemann surface defined by the polynomial equation in $\mathbf{C}^2$

$$w^2 = \prod_{i=1}^{2n}(z - z_i)^2. \tag{7.76}$$

This is a double cover of the Riemann sphere $S^2$, with branch points $z_i \in S^2$. The involution $\iota$ is simply $(w,z) \to (-w,z)$. The period matrix $\tau$ of $\widehat{\Sigma}$ equals $2\Pi$. For the case $g=0$ the equations (7.70) and (7.71) simplify for $R=\sqrt{2}$ considerably

$$A(z_k,\overline{z}_k) = 2^{-n-1} \sum_{(x|y)} (-1)^{Q(x)} \left| \frac{\prod_{i<j}(x_i-x_j)(y_i-y_j)}{\prod_{i,j}(x_i-y_j)} \right|^{\frac{1}{4}}, \tag{7.77}$$

$$\widehat{Z}_{a,b}(\sqrt{2}) = 2^{-n-1} \widehat{Z}^{qu} \sum_{\mu,\nu} (-1)^{2\gamma\cdot b + b\cdot a} \left| \vartheta\!\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right](0|\tau) \right|^2. \tag{7.78}$$

The correspondence between the non-singular even spin structures $\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]$ and the partitions $(x|y)$ is expressed by the Thomae identity valid for hyperelliptic $\vartheta$-functions [63]

$$\left[\vartheta\!\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right](0|\tau)\right]^8 = (\det M)^{-4} \prod_{i<j}(z_i-z_j) \frac{\prod_{i<j}(x_i-x_j)(y_i-y_j)}{\prod_{i,j}(x_i-y_j)}. \tag{7.79}$$

The matrix $M$ is related to the canonical one-forms $\omega_i$ on $\widehat{\Sigma}$ by

$$\omega_i(z) = \sum_{i=1}^{n-1} M_{ij} \frac{z^{j-1}dz}{y}. \tag{7.80}$$

Thus we find as a final answer for the twist correlator

$$A(z_k,\overline{z}_k) = |\det M| \prod_{i<j} |z_i-z_j|^{-\frac{1}{4}} \sum_{p,\overline{p}} (-1)^{b\cdot n} \exp\left[\tfrac{1}{2} i\pi \left(p\cdot\tau\cdot p - \overline{p}\cdot\overline{\tau}\cdot\overline{p}\right)\right], \tag{7.81}$$

where the momentum summation is as in (7.69) with $g=0$. This reproduces the results obtained in [53,18,168].

Some comments on (7.81) are due here. First we see that $R \to 1/R$ leaves the correlation function invariant. (Note however that this symmetry is only valid for genus zero.) So, in particular the twist field $n$-point function at $R=\frac{1}{2}\sqrt{2}$ (the 4-state Potts model) is also given by the Koba-Nielsen amplitude (7.77). This is in accordance with the permutational symmetry $S_3$ between the two twist operators $\sigma_1$ and $\sigma_2$, and the magnetic vertex operator $\sigma_3 = \sqrt{2}\cos\frac{1}{4}\sqrt{2}(\phi-\overline{\phi})$ at this special point.

At first sight this invariance under $R \to 1/R$ seems hard to reconcile with the duality $R \to 2/R$. We must realize however that the latter transformation also acts on the twist fields. This can serve as a check on our transformation rule

(2.27). Let us consider the simple case $n_2 = 0$

$$
\begin{aligned}
\Big\langle \prod_{k=1}^{2n} \sigma_1(z_k,\overline{z}_k)\Big\rangle_R &= \Big\langle \prod_{k=1}^{2n} \sigma_1(z_k,\overline{z}_k)\Big\rangle_{1/R} \\
&= \widehat{Z}_{0,0}(1/R) = 2^{1-n} \sum_{a,b\in \mathbf{Z}_2} \widehat{Z}_{a,b}(2/R) \\
&= 2^{-n} \Big\langle \prod_{k=1}^{2n} \big(\sigma_1(z_k,\overline{z}_k)+\sigma_2(z_k,\overline{z}_k)\big)\Big\rangle_{2/R}.
\end{aligned} \tag{7.82}
$$

Here we used that the sum over all monodromy conditions is equivalent to the transformation $2/R \to 1/R$. Thus, the transformation rule (2.27) is confirmed.

## 7.5. Operator Formalism on Riemann Surfaces

As we have seen in Chapter 3 a two-dimensional conformal field theory will associate to each Riemann surface $\Sigma$ with a single connected, parametrized boundary a state $|\Sigma\rangle$ in the Hilbert space $\mathcal{H}$. Why are we interested in the construction of such a state $|\Sigma\rangle$? The fundamental property of $|\Sigma\rangle$ is that for any local set of operators $\mathcal{O}_i(z,\overline{z})$ we have

$$
\Big\langle \prod_i \mathcal{O}_i(z_i,\overline{z}_i)\Big\rangle_\Sigma = \langle 0|\prod_i \mathcal{O}_i(z_i,\overline{z}_i)\,|\Sigma\rangle, \tag{7.83}
$$

where $\langle\cdots\rangle_\Sigma$ denotes the unnormalized expectation value on the surface $\Sigma$. This is illustrated in *fig.* 16. In particular the partition function on the surface is given by $Z(\Sigma) = \langle 0|\Sigma\rangle$. So an explicit construction of $|\Sigma\rangle$ allows us to calculate correlation functions on a Riemann surface with nontrivial topology using the operator representation in the Hilbert space $\mathcal{H}$. In that respect (7.83) is the generalization of the well-known operator representation (2.17) of the correlation functions on the sphere or cylinder where $|\Sigma\rangle = |0\rangle$, and the study of higher genus quantities in this representation of has become known as the operator formalism on Riemann surfaces [95,150,3,160,2,49]. Note that in this formalism the only reference to the global geometry of the Riemann surface is contained in the boundary conditions imposed by the state $|\Sigma\rangle$. Furthermore the condition (7.83) determines $|\Sigma\rangle$ uniquely as we will show in some examples in the subsequent sections.

### *7.5.1. Ward Identities*

The structure of the chiral algebra $\mathcal{A}$ of the conformal field theory gives rise to important constraints on $|\Sigma\rangle$. Let $W(z)$ be a chiral primary field of (possibly

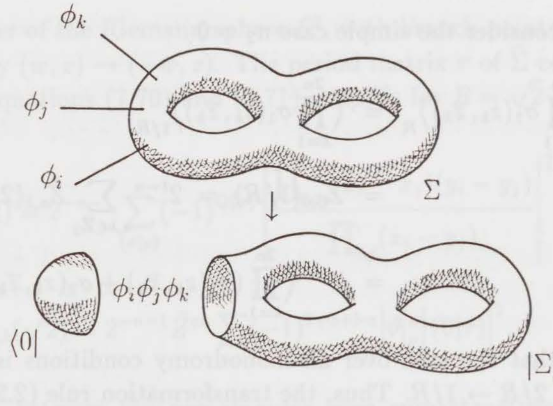



FIGURE 16: The operator formalism allows us to calculate correlation functions on an arbitrary Riemann surface in a similar fashion as on the cylinder. The ingoing vacuum $|0\rangle$ is replaced by the state $|\Sigma\rangle$.

half-integer) dimension $h$, and $\xi(z)$ a $c$-number holomorphic differential of degree $1-h$. The correlation functions of the field $W(z)$ are sections of the holomorphic line bundle $\mathcal{L} = K^h$, with $K$ the canonical line bundle over $\Sigma$. We now have the fundamental identity*

$$\oint \frac{dz}{2\pi i}\xi(z)W(z)\,|\Sigma\rangle = 0, \tag{7.84}$$

whenever $\xi \in H^0(\Sigma - Q, K \otimes \mathcal{L}^{-1})$, *i.e.* $\xi(z)$ is a section of the bundle $K \otimes \mathcal{L}^{-1}$ that extends holomorphically to $\Sigma - Q$. Equation (7.84) can be best understood within a correlation function. Consider the correlator

$$\Big\langle \prod_i \mathcal{O}_i(z_i, \bar{z}_i) \oint \frac{dz}{2\pi i}\,\xi(z)W(z)\Big\rangle, \tag{7.85}$$

where the contour is taken to encircle the operator insertions at $z_i$. Since both $\xi(z)$ and $W(z)$ are analytical on the part of $\Sigma$ outside the contour, the above integral can be deformed and 'pulled off' the surface without leaving any residues.

A similar analysis can be performed for descendant fields in particular the stress-energy tensor $T(z)$, although we should be more careful since $T(z)$ is not a conformal tensor. We can adjoin to vector fields $\xi(z)$, that are holomorphic in the local coordinate patch except for possible poles in $Q$, the Virasoro generators

$$L[\xi] = \oint \frac{dz}{2\pi i}\,\xi(z)T(z). \tag{7.86}$$

*Here and in the subsequent equations all contours will be understood to encircle the point $Q$ ($z = \infty$) unless otherwise stated.

These are generalizations of the Fourier modes $L_n$ on the sphere. Due to operator product (2.22) they satisfy the commutation relations

$$\big[L[\xi], L[\eta]\big] = L[\xi\partial\eta - \eta\partial\xi] + \frac{c}{12}\oint \frac{dz}{2\pi i}\,\xi(z)\partial^3\eta(z), \tag{7.87}$$

a natural generalization of the Virasoro algebra (2.20). The different components of $T(z)$ can now be classified in terms of the dual objects $\xi(z)$. If $\xi(z)$ extends holomorphically to $\Sigma - Q$ then in view of the above one concludes that $L[\xi]$ annihilates the state $|\Sigma\rangle$:

$$L[\xi]\,|\Sigma\rangle = 0, \quad \textit{if} \ \ \xi \in H^0(\Sigma - Q, K^{-1}), \tag{7.88}$$

just as we have $L_n|0\rangle = 0$ for $n \geq -1$ on the sphere. However, there are some subtleties here, since $T(z)$ is not a conformal tensor. In fact $T(z)$ is a projective connection [91], *i.e.* it transforms with the Schwarzian derivative (2.29). The problems associated with this are avoided by using a coordinate covering $\{U_\alpha, z_\alpha\}$ on $\Sigma$ with transition functions $z_\alpha \circ z_\beta^{-1} \in SL(2, \mathbf{C})$, since for these functions the Schwarzian derivative vanishes. The equivalence class of such coverings is called a projective structure and always exists by the uniformization theorem. So if we choose the local coordinate $z$ compatible with the projective structure on $\Sigma$ then (7.88) is correct. The Virasoro operators $L_n$ $(n \leq 1)$ that correspond to vector fields non-singular at $Q$ generate analytic coordinate transformations in the local patch and will in general modify the property (7.88). When $\xi(z)$ extends neither to $Q$ nor to $\Sigma - Q$ the corresponding components of $T(z)$ change the $3g-3$ moduli parameters $m_\alpha$ of the surface $\Sigma$. More precisely, we can choose $3g-3$ elements $\xi_\alpha$ dual to the quadratic differentials such that for the partition function $Z(\Sigma)$ on $\Sigma$ we have

$$\frac{\partial}{\partial m_\alpha} Z(\Sigma) = \langle 0|\, L[\xi_\alpha]\,|\Sigma\rangle. \tag{7.89}$$

The operator formulation of conformal field theory is the natural language to describe the factorization expansion of the partition function and other quantities at the boundary of moduli space. For example, in the case of the degeneration of a dividing cycle the behavior of the partition function as a function of the pinching parameter $t$ is exactly described by [78]

$$Z_{g_1+g_2}(t,\bar{t}) = \langle \Sigma_1 | t^{L_0} \bar{t}^{\bar{L}_0} | \Sigma_2 \rangle, \tag{7.90}$$

where $|\Sigma_1\rangle$ and $|\Sigma_2\rangle$ describe the conformal field theory on the left resp. right half of the surface $\Sigma$. In a similar way the pinching and creation of handles can be dealt with. One chooses two points on $\Sigma$ and attributes a density matrix $\rho_\Sigma$ to this twice-punctured Riemann surface. The partition function of the surface with the extra handle connecting the two points is now calculated as

$$Z_{g_1+1}(t,\bar{t}) = \mathrm{Tr}\Big(\rho_\Sigma t^{L_0} \bar{t}^{\bar{L}_0}\Big). \tag{7.91}$$

*7.5.2. The Gaussian Model*

Let us first turn to the construction of the state $|\Sigma\rangle$ for the compactified conformal scalar field theories, see also [95,150,3,2]. In the operator language $\phi(z,\overline{z})$ can be expanded as:

$$\phi(z,\overline{z}) = \phi(z) + \overline{\phi}(\overline{z}),$$

$$\phi(z) = \tfrac{1}{2}q + a_0 \log z + \sum_{n=1}^{\infty}\left[z^{-n}\frac{a_n}{\sqrt{n}} + z^n \frac{a_n^\dagger}{\sqrt{n}}\right], \tag{7.92}$$

with commutation relations $[a_0, q] = [a_n, a_n^\dagger] = 1$. The Hilbert space consists of all states of the form

$$|\Phi\rangle = \sum_{p_0,\overline{p}_0 \in \Gamma_R} \Phi^{p_0,\overline{p}_0}[\phi_+;\overline{\phi}_+]e^{ip_0 q - i\overline{p}_0\overline{q}}\,|0\rangle, \tag{7.93}$$

where

$$\phi_+(z) = \sum_{n=1}^{\infty} z^n \frac{a_n^\dagger}{\sqrt{n}} \tag{7.94}$$

is the creation part of $\phi(z)$, that is, $\phi_+(z)$ is regular at $z \to 0$. The functionals $\Phi$ are obtained from the state $|\Phi\rangle$ by taking the inner product with coherent states

$$\begin{aligned}\Phi^{p_0,\overline{p}_0}[\lambda;\overline{\lambda}] &= \langle 0|\exp\Big[ip_0 q - i\overline{p}_0\overline{q} + \oint \frac{dz}{2\pi i}\,\lambda(z) i\partial\phi(z) \\ &\quad + \overline{\oint \frac{dz}{2\pi i}\,\lambda(z) i\partial\phi(z)}\Big]|\Phi\rangle,\end{aligned} \tag{7.95}$$

where $\lambda(z)$, as $\phi_+(z)$, has an expansion in only positive Fourier modes. The action of the creation and annihilation operators $a_n^\dagger$ and $a_n$ on these coherent states, and accordingly on $\Phi$, is given by

$$a_n = \frac{1}{i\sqrt{n}}\frac{\partial}{\partial\lambda_n}, \qquad a_n^\dagger = i\sqrt{n}\lambda_n, \tag{7.96}$$

with $\lambda_n$ is the $n^{th}$ Laurent coefficients of $\lambda(z)$.

Combining the equations (7.83) and (7.95), it is clear that with a straightforward application of the path-integral method described in section 7.3, we can immediately obtain the answer for our state $|\Sigma\rangle$ for the gaussian model. It has momentum $p_0 = \overline{p}_0 = 0$, because of $\phi$ charge conservation. Projected onto a given set of loop momenta $p, \overline{p} \in \Gamma_R^g$, it factorizes into a left-moving times a right-moving state

$$|\Sigma\rangle = Z_0^{qu} \sum_{p,\overline{p}\in\Gamma_R^g} A^p[\phi_+]\overline{A^p[\phi_+]}\,|0\rangle,$$

where

$$A^p[\lambda] = \exp\left[\oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\,\tfrac{1}{2}\lambda(z)\lambda(w)\partial_z\partial_w \log E(z,w)\right] \times \exp\left[i\pi\,(p\cdot\tau\cdot p) + \left(p\cdot\oint \lambda(z)\omega(z)\right)\right]. \tag{7.97}$$

This formula may be considered as a generating functional of all correlation functions on the surface $\Sigma$. In particular, by acting with vertex operators $V(\lambda, \partial/\partial\lambda)$ onto (7.97) one may reobtain expression (7.42).

For rational values of $R^2$ the momentum summation in (7.97) can be replaced by a finite sum of $\vartheta$-functions, using (7.12a-b). For example, for the $R=1$ model the chiral components of $|\Sigma\rangle$ are given by the famous $\tau$-function of the KP-hierarchy [40]

$$\tau_{KP}[\lambda] = \exp\left[\oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\,\tfrac{1}{2}\lambda(z)\lambda(w)\partial_z\partial_w \log E(z,w)\right] \times \vartheta\!\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]\left(\oint \frac{dz}{2\pi i}\lambda(z)\omega(z)|\tau\right). \tag{7.98}$$

This $\tau$-function describes a vacuum state of a chiral Dirac fermion. The space of such fermionic vacuum states, which can be obtained from the standard vacuum by a Bogoliubov transformation, is called the universal Grassmannian manifold [40,140].

### 7.5.3. *An Algebraic Construction*

In the previous section we constructed $|\Sigma\rangle$ starting from the knowledge of all correlation functions on $\Sigma$ as derived from the path-integral. In this subsection we give an alternative derivation of the result (7.97) in terms of (generalized) Bogoliubov transformations without using the path-integral. This more algebraic approach will clarify a great deal of the structure of the state $|\Sigma\rangle$.

We introduce the chiral creation/annihilation operators

$$a[f] = \oint \frac{dz}{2\pi i} f(z)\partial\phi(z), \tag{7.99}$$

with commutation relations

$$\big[a[f], a[g]\big] = \oint \frac{dz}{2\pi i} f(z)\partial g(z). \tag{7.100}$$

The key observation that will enable us to reconstruct the state $|\Sigma\rangle$ is that $\partial\phi(z)$ is a chiral primary field and hence the state representing $\Sigma$ is annihilated by the operators $a[f_-]$ precisely for those $f_-(z)$ that can be extended to a holomorphic function on $\Sigma - Q$, *i.e.* $f_- \in H^0(\Sigma - Q)$. It is clear from (7.100) that these modified annihilation operators $a[f_-]$ mutually commute. A basis $\{f_-^{(n)}\}$ for $H^0(\Sigma - Q)$ can be chosen of the form

$$f_-^{(n)}(z) = z^n + (\textit{regular at } Q). \tag{7.101}$$

However, by the Weierstrass gap theorem [61], not all $n > 0$ occur: there are $g$ values of $n$ (between 1 and $2g$) missing. For generic positions of the point $Q$ these values are $n = 1, \ldots, g$. The Weierstrass gap forms an obstruction to view the state $|\Sigma\rangle$ as a genuine Bogoliubov transform of the standard vacuum $|0\rangle$. As we will see many nontrivial features are due to the presence of this gap.

The state $|\Sigma\rangle$ is not uniquely determined by the condition that it is annihilated by all the elements of the set $\{a[f_-]; f_- \in H^0(\Sigma - Q)\}$. In fact there is an infinite dimensional space $\mathcal{H}_{vac}$ of such vacuum states. To further analyze this space $\mathcal{H}_{vac}$ we need additional operators that commute with all the annihilation operators $a[f]$. An obvious way to find such operators is to extend the set $\{a[f]; f \in H^0(\Sigma - Q)\}$ to a complete set of annihilation and creation operators. Because of the Weierstrass gap theorem such a complete set will contain $2g$ additional modes $a[g]$ satisfying

$$\big[[a[f_-], a[g]\big] = 0, \qquad \text{for all } f_- \in H^0(\Sigma - Q). \tag{7.102}$$

The idea is now to use a commuting subset of these operators to decompose the space $\mathcal{H}_{vac}$ into eigenspaces. In order to have an interpretation the corresponding eigenvalues we need to know a bit more about the space of functions $g(z)$ satisfying (7.102). First we observe that this space is of course only defined modulo elements of $H^0(\Sigma - Q)$. So we are dealing with a cohomology problem. Furthermore, we see from (7.102) that $\partial g(z)$ must be an element of $H^0(\Sigma - Q, K)$, which implies that $g(z)$ is extendible to a multi-valued holomorphic function with constant shifts around the nontrivial cycles of the surface. The space of such functions modulo $H^0(\Sigma - Q)$ is naturally dual to $H_1(\Sigma, \mathbf{C})$, the space of cycles on $\Sigma$, and hence is indeed $2g$-dimensional. The duality is expressed by the map

$$(C, g) \to \oint_C \partial g(z), \tag{7.103}$$

where $C$ is a cycle on the surface. We can now use the intersection product to adjoin to any cycle $C$ a function $g_C(z)$ by

$$\oint_D \partial g_C(z) = \#(D, C), \qquad \textit{for all } D \in H_1(\Sigma, \mathbf{Z}). \tag{7.104}$$

This relation defines a one-to-one map between the functions $g(z)$ (mod $H^0(\Sigma - Q)$), such that $a[g]$ commutes with all annihilation operators, and the nontrivial cycles on the surface. A reformulation of the condition (7.104) is given by the requirement that for any 1-form $\Omega(z)$ holomorphic on $\Sigma - Q$

$$\oint g_C(z)\Omega(z) = \oint_C \Omega(z). \tag{7.105}$$

Commutation relations among the $a[g_C]$ are translated into intersection products of the corresponding cycles

$$\big[a[g_C], a[g_D]\big] = \oint \frac{dz}{2\pi i} g_C(z)\partial g_D(z) = \frac{1}{2\pi i}\#(D,C). \tag{7.106}$$

We can refine the correspondence between cycles $C \in H_1(\Sigma, \mathbf{C})$ and modes $g_C$ to $C \in H_1(\Sigma, \mathbf{Z})$ by demanding $\exp(2\pi i g_C)$ to be extendible to a nowhere vanishing, single-valued holomorphic function on $\Sigma - Q$. A natural basis for $H_1(\Sigma, \mathbf{Z})$ is given by a canonical set of homology cycles $A_i, B_i$ satisfying (7.3). The representatives $a[f_i]$ and $a[g_i]$ of these cycles then satisfy canonical commutation relations

$$\big[a[f_i], a[f_j]\big] = \big[a[g_i], a[g_j]\big] = 0, \qquad \big[a[f_i], a[g_j]\big] = \frac{i}{2\pi}\delta_{ij}. \tag{7.107}$$

We can now define the chiral state $|\Sigma; p\rangle$ which is annihilated by the operators $a[f_-]$ and is a common eigenstate of the $a[g_i]$

$$\begin{aligned} a[f_-]\,|\Sigma;p\rangle &= 0, \qquad f_- \in H^0(\Sigma - Q), \\ a[g_i]\,|\Sigma;p\rangle &= \oint_{A_i} \frac{dz}{2\pi i}\,\partial\phi(z)|\Sigma;p\rangle = p_i|\Sigma;p\rangle. \end{aligned} \tag{7.108}$$

The eigenvalues $p_i$ can be interpreted as the loop momenta flowing through the cycles $A_i$. The dependence on the momenta is fixed by the action of the conjugated operators

$$a[f_i]\,|\Sigma;p\rangle = \oint_{B_i} \frac{dz}{2\pi i}\,\partial\phi(z)|\Sigma;p\rangle = \frac{1}{2\pi i}\frac{\partial}{\partial p_i}|\Sigma;p\rangle. \tag{7.109}$$

The final modular invariant state $|\Sigma\rangle$ is given by

$$|\Sigma\rangle = \sum_{p,\bar{p}\in\Gamma_R} |\Sigma;p\rangle \otimes \overline{|\Sigma;p\rangle}. \tag{7.110}$$

We like to stress that up to this point we did not use the explicit form of $|\Sigma; p\rangle$ in terms of the oscillators $a_n^\dagger$. We will now show that the properties (7.108) and (7.109) are sufficient to determine this state up to normalization. To this end we first observe that the one-to-one correspondence between the modes $f_i, g_i$ and the homology basis $A_i$, $B_i$ can be formulated equivalently by the following local conditions

$$\begin{aligned} &\oint dz\ f_i(z)\partial_z\partial_w \log E(z,w) = 0, &&\oint dz\ f_i(z)\omega(z) = \delta_{ij}, \\ &\oint dz\ g_i(z)\partial_z\partial_w \log E(z,w) = 2\pi i\, \omega_i(w), &&\oint dz\ g_i(z)\omega(z) = \tau_{ij}, \end{aligned} \tag{7.111}$$

where the contour encircles both $Q$ and $w$. These equations follow directly from the fact that the space of 1-forms holomorphic on $\Sigma - Q$ is spanned by $\omega_i(z)$ and the forms

$$\omega^{(n)}(z) = \partial_z\partial_w^n \log E(Q,z) \quad (n \geq 1). \tag{7.112}$$

By the same observation it is straightforward to verify that the state $|\Sigma;p\rangle$ satisfies

$$\partial\phi(z)|\Sigma;p\rangle = \left[\oint \frac{dz}{2\pi i}\,\phi_+(w)\partial_z\partial_w \log E(z,w) + 2\pi i \sum_i p_i\omega_i(z)\right]|\Sigma;p\rangle. \tag{7.113}$$

The right-hand side, which is manifestly holomorphic on all of $\Sigma - Q$, may serve as a definition of the operator $\partial\phi(z)$ outside the coordinate patch. This equation can be read as a compact way of writing the generalized Bogoliubov transformation relating the state $|\Sigma;p\rangle$ to the standard vacuum $|0\rangle$. Now using (7.111) we find

$$\frac{\partial}{\partial p_i}|\Sigma;p\rangle = \left[2\pi i(\tau\cdot p)_i + \oint \phi_+(z)\omega_i(z)\right]|\Sigma;p\rangle. \tag{7.114}$$

Equations (7.113)–(7.114) can be integrated to the following result for the chiral vacuum state $|\Sigma;p\rangle$

$$|\Sigma;p\rangle = C\,A^p[\phi_+]\,|0\rangle, \tag{7.115}$$

where $A^p[\lambda]$ is given in (7.97). The normalization constant $C$ is determined by using the fact that the modular dependence of $|\Sigma;p\rangle$ is given by the action of the stress-energy tensor (7.89). Using the variational formulas of [155], this yields $C = (\det\partial_0)^{-\frac{1}{2}}$. This concludes our algebraic derivation of the result (7.97).

### 7.5.4. *Operator Formulation of Orbifold Models*

For the twisted scalar field the state $|\Sigma\rangle$ has a slightly more complicated structure. It is given by a sum of $2^{2g}$ states $|\Sigma\rangle_{\epsilon,\delta}$ representing the different twist structures. All states $|\Sigma\rangle_{\epsilon,\delta}$ lie in the untwisted sector of the Hilbert space. The contribution $|\Sigma\rangle_{0,0}$ simply equals $2^{-g}$ times the gaussian result (7.97). For the other characteristics there is no momentum conservation, so the corresponding states will have components with non-zero momentum $p_0$. Again using (7.95) and the expressions

(7.49)–(7.51) for the correlator of the vertex operators we obtain after a straightforward calculation

$$|\Sigma\rangle_{\epsilon,\delta} = 2^{-q} Z_0^{qu} \sum_{\substack{p_0,\bar{p}_0 \in 2\Gamma_R \\ p,\bar{p} \in \Gamma_R^{g-1}}} A_{\epsilon,\delta}^{p_0,p}[\phi_+] \overline{A_{\epsilon,\delta}^{p_0,p}[\phi_+]} \, |0\rangle, \tag{7.116}$$

with

$$\begin{aligned} A_{\epsilon,\delta}^{p_0,p}[\lambda] \;&=\; c\begin{bmatrix}\epsilon\\ \delta\end{bmatrix}^{-1} \exp\left[\oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i}\, \tfrac{1}{2}\lambda(z)\lambda(w)\partial_z\partial_w \log E(z,w)\right] \\ &\times \exp\left[ i\pi(p\cdot \mathit{\Pi}\cdot p) + 2\pi i\Big(p\cdot \tfrac{1}{2}p_0 \int_{\iota(\hat{Q})}^{\hat{Q}} \nu\Big) + \tfrac{1}{2}p_0^2 \log\varepsilon(Q) \right. \\ &\left. + \oint \frac{dz}{2\pi i}\, \lambda(z)\Big(2\pi i p\cdot \nu(z) + p_0 \partial_z \log E_t(z,Q)\Big)\right], \end{aligned} \tag{7.117}$$

where $\mathit{\Pi}$, $\nu_i(z)$, $\varepsilon(z)$, and $E_t(z,w)$ are defined with respect to the twist characteristics $\epsilon, \delta$. Note that $|\Sigma\rangle_{\epsilon,\delta}$ is even under $\phi \to -\phi$, as it should be.

One of the striking features of equation (7.117) is the symmetric role of the state momentum $p_0$ and the loop momenta $p_i$ ($i = 1, \ldots, g-1$). An intuitive explanation for this is that on the double cover $\widehat{\Sigma}$ we have in fact constructed a density matrix, *i.e.* an element in the tensor product of two Hilbert spaces, one at $\hat{Q}$ and one at $\iota(\hat{Q})$, whose quantum numbers are related by $\phi \to -\phi$. By taking the trace of this density matrix as in (7.91) one creates in a way an extra handle of $\widehat{\Sigma}$ of which $p_0$ is the loop momentum. Comparing with (7.41) one can indeed recognize in (7.117) the factorization expansion of the partition function on the resulting surface of a scalar field which is odd under the involution $\iota$.

### 7.5.5. *Twist Fields*

We end this section with an extension of the operator formulation on Riemann surfaces to twist operators. Since twist fields are intertwining operators between the two sectors of the Hilbert space, their action on the state $|\Sigma\rangle$ is, although perfectly well-defined, not very easily expressed in the formalism as developed up to now. As we have seen, the presence of twist fields is responsible for a summation over the additional loop momenta $p_\lambda, \bar{p}_\lambda$. So they can be regarded as topological objects comparable to extra handles on the Riemann surface. Accordingly we will attribute a state $|\Sigma;\sigma^N\rangle$ in the Hilbert space of the orbifold theory to the punctured Riemann surface $\Sigma - Q$ with $N$ twist field insertions. By definition we have

$$|\Sigma;\sigma^N\rangle = \prod_{i=1}^{N} \sigma(z_i,\bar{z}_i)\,|\Sigma\rangle. \tag{7.118}$$

This state is an element of the twisted or untwisted sector depending on whether $N$ is odd or even. The untwisted case is completely analogous to (7.117) with the appropriate double cover $\widehat{\Sigma}$ and momentum lattice (7.69). As for the twisted case, we now expand $\phi(z)$ in half-integer modes and write an arbitrary element in the twisted Hilbert space as

$$|\Phi\rangle = \sum_{i=1}^{2} \Phi^i[\phi_+, \overline{\phi}_+]\, \sigma_i\, |0\rangle, \tag{7.119}$$

where $\phi_+(z)$ is the creation part of $\phi(z)$. (Note that the twisted states do not carry any definite momentum). The functionals $\Phi$ are related to the state $|\Phi\rangle$ by

$$\Phi^i[\lambda, \overline{\lambda}] = \langle 0|\, \sigma_i \exp\left[\oint \frac{dz}{2\pi i}\, \lambda(z) i\partial\phi(z) + c.c.\right] |\Phi\rangle. \tag{7.120}$$

This expression is well-defined, since both $\lambda(z)$ and $\phi(z)$ are expanded in half-integer powers of $z$. Using the twisted chiral propagator,

$$E_t(z,w) = \frac{\widehat{E}(\hat{z}, \hat{w})}{\widehat{E}(\hat{z}, \iota(\hat{w}))}, \tag{7.121}$$

we find for $|\Sigma; \sigma^N\rangle$, $N$ odd

$$|\Sigma; \sigma^N\rangle = Z_0^{qu} \sum_{\substack{i=1,2 \\ \epsilon,\delta \in (\mathbf{Z}/2)^g}} \sum_{p,\overline{p} \in \Gamma_R^{g-1}} A^{i,p}_{\epsilon,\delta}[\phi_+] \overline{A^{i,p}_{\epsilon,\delta}[\phi_+]}\, \sigma_i\, |0\rangle,$$

$$\begin{aligned} A^{i,p}_{\epsilon,\delta}[\lambda] \;&=\; c\big[{}^{\epsilon}_{\delta}\big]^{-1} \exp\left[\tfrac{1}{2} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i}\, \lambda(z)\lambda(w) \partial_z \partial_w \log E_t(z,w)\right] \\ &\times \exp\left[i\pi(p \cdot \Pi \cdot p) + \left(p \cdot \oint \lambda(z)\nu(z)\right)\right], \end{aligned} \tag{7.122}$$

where all quantities on the right-hand side are those on the branched cover defined by the positions of the $N+1$ twist fields $\sigma(z_i, \overline{z}_i)$ and $\sigma(Q)$, and the twist structure $\epsilon, \delta$. The momenta are summed over (5.9). An interesting special case is $R = 1, N = 1$ where $|\Sigma; \sigma^N\rangle$ is a finite sum with chiral components proportional to

$$\begin{aligned} \tau_{BKP}[\lambda] \;&=\; \exp\left[\oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i}\, \tfrac{1}{2} \lambda(z)\lambda(w) \partial_z \partial_w \log E_t(z,w)\right] \\ &\times \vartheta\big[{}^{\alpha}_{\beta}\big] \left(\oint \frac{dz}{2\pi i}\, \lambda(z)\omega(z) | \Pi\right). \end{aligned} \tag{7.123}$$

As discussed extensively in [40,41], this function solves a hierarchy of differential equations known as the BKP hierarchy and is called the BKP $\tau$-function. It is furthermore shown in [40,41] that it is related to the KP $\tau$-function (see Eq. (7.98)) by

$$\tau_{BKP}[\lambda(z)]^2 = const \cdot \tau_{KP}[\hat{\lambda}(\hat{z})], \tag{7.124}$$

where the KP $\tau$-function is defined on $\widehat{\Sigma}$. Here $\hat{\lambda}$ is the lift of $\lambda$ to $\widehat{\Sigma}$ and satisfies $\hat{\lambda}(\iota(\hat{z})) = -\hat{\lambda}(\hat{z})$. (Note that locally $z \sim \hat{z}^2$.) At first sight this relation seems very mysterious. However, it has a fairly simple physical explanation as follows. The chiral state corresponding to the BKP $\tau$-function (7.123) describes a chiral Dirac fermion $\psi_D(z)$ on $\Sigma$ with twisted boundary conditions: if $\psi_D(z)$ is moved around one of the twist fields or twisted cycles, it is transformed (modulo a sign) into $\overline{\psi}_D(z)$. When we lift this situation to $\widehat{\Sigma}$ we describe the theory of a 'Majorana fermion' $\psi_M(\hat{z})$ living on $\widehat{\Sigma}$, satisfying the reality condition

$$\psi_M^*(\hat{z}) = \psi_M(\iota(\hat{z})). \tag{7.125}$$

The BKP $\tau$-function is obtained from this fermion theory through

$$\tau_{BKP}[\lambda(z)] = \left\langle \exp\left( \tfrac{1}{2} \oint \frac{d\hat{z}}{2\pi i}\, \lambda(\hat{z}) \psi_M(\iota(\hat{z})) \psi_M(\hat{z}) \right) \right\rangle. \tag{7.126}$$

Thus (7.123) has an interpretation as a vacuum state in a real fermion Hilbert space—although with modified commutation relations due to the unusual reality condition (7.125)—and is as such an element of the orthogonal Grassmannian manifold [41]. Note that the construction makes essential use of the fact that $\widehat{\Sigma}$ has an involutive automorphism $\iota$. The KP $\tau$-function, on the other hand, describes a chiral fermion $\psi_D(\hat{z})$ on $\widehat{\Sigma}$ and can be expressed as the expectation value

$$\tau_{KP}[\hat{\lambda}(\hat{z})] = \left\langle \exp\left( \tfrac{1}{2} \oint \frac{d\hat{z}}{2\pi i}\, \hat{\lambda}(\hat{z}) \psi_D^*(\hat{z}) \psi_D(\hat{z}) \right) \right\rangle. \tag{7.127}$$

Out of this Dirac fermion $\psi_D(\hat{z})$ we can construct two Majorana fermions of the type (7.125) by taking the combinations

$$\begin{aligned} \psi_M^{(1)}(\hat{z}) &= \frac{1}{\sqrt{2}} \left( \psi_D(\hat{z}) + \psi_D^*(\iota(\hat{z})) \right), \\ \psi_M^{(2)}(\hat{z}) &= \frac{1}{i\sqrt{2}} \left( \psi_D(\hat{z}) - \psi_D^*(\iota(\hat{z})) \right). \end{aligned} \tag{7.128}$$

the relation (7.124) between (7.126) and (7.127) is now readily verified.

# 8

# Topological Gauge Theories in Three Dimensions

In a remarkable work Witten [162] has shown that much of the structure of two-dimensional rational conformal field theory has an elegant interpretation in the framework of *three*-dimensional topological field theories. In particular there exists a rather direct relation between Wess-Zumino-Witten models based on a compact Lie group $G$, and Chern-Simons gauge theories with gauge group $G$. The correspondence relates the generalized characters of the partition function of the conformal model on a Riemann surface $\Sigma$ to states in the Hilbert space of the topological theory quantized on the $3d$ space-time $\Sigma \times R$. The general covariance in three dimensions puts certain puzzling properties of two-dimensional quantities in the right perspective. In particular it explains why certain braid group representations, that appear as the monodromies of holomorphic blocks of correlators in RCFT's, can be used to construct knot (or more general link) invariants. The use of higher dimensional theories in order to explain phenomena in a lower dimension is not completely unfamiliar in quantum field theory. Gauge anomalies in $n$ dimensions are closely related to effective actions in $n+1$ dimensions, and to chiral anomalies and the Index Theorem in $n+2$ dimensions. It could be argued that Witten's work has taken this connection to its extreme.

With considerable hindsight we can give some arguments why such a connection with three-dimensional topological theories should exist at all. As we have seen in Chapter 3, it is a general feature of a topological theory that it associates to each possible boundary—in three dimensions to each surface $\Sigma$—a Hilbert space of states $\mathcal{H}_\Sigma$. Furthermore, the space $\mathcal{H}_\Sigma$ carries a natural representation of the group of global diffeomorphisms of $\Sigma$. The map $\Sigma \to \mathcal{H}_\Sigma$ bears a close analogy to the modular functor $\Sigma \to V_\Sigma$, that assigns to a surface a vector space of holomorphic blocks. We can pursue this analogy further if we observe that the dimensions of $V_\Sigma$ and $\mathcal{H}_\Sigma$ are calculated in a similar fashion. As we have demonstrated the number of holomorphic blocks can be determined from the spectrum $\phi_i$ and the fusion rules $N_{ijk}$. This was essentially due to the fact that any surface can be decomposed into 'pants' $Y$. On the other hand, according to Eq. (3.16) the dimension of $\mathcal{H}_\Sigma$ can be calculated as the partition function of the closed 3-manifold $\Sigma \times S^1$,

$$\dim \mathcal{H}_\Sigma = Z(\Sigma \times S^1). \tag{8.1}$$

This three-manifold decomposes similarly into copies of $Y \times S^1$. The manifold $Y \times S^1$ is not closed; its boundary consists of three copies of the 2-torus $S^1 \times S^1$. If $\mathcal{H}$ is the Hilbert space associated to this torus, the amplitude $\Phi_{Y\times S^1}$ can be regarded as a map

$$\Phi_{Y\times S^1} : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}. \tag{8.2}$$

So the states in $\mathcal{H}$ are analogous to the representations $\phi_i$, and the amplitude $\Phi_{Y\times S^1}$ to the fusion algebra. In fact, this analogy between RCFT and topological field theories can be made into an exact one-to-one correspondence [163,110].

Moore and Seiberg [122] have made the observation that all presently known rational conformal field theories can be obtained from group manifolds models, by coset and orbifold constructions. They then demonstrated that cosets and orbifolds can still be associated with three-dimensional gauge theories. This let them to conjecture that *all* RCFT's are related to compact Lie groups. Whatever the status of that conjecture, it certainly shows that one important step in the direction of a possible classification of rational theories is the construction of three-dimensional topological gauge theories. Thereto we have to consider arbitrary compact gauge groups $G$, not necessarily connected or simply connected. (As we will see in a moment, compactness is equivalent to rationality, since it implies that the Hilbert spaces $\mathcal{H}_\Sigma$ are finite dimensional.) One of our main results in this chapter will be that these topological theories are classified by the cohomology group $H^4(BG, \mathbf{Z})$ of the classifying space $BG$. These concepts are explained in some detail in the appendix.

This chapter is organized as follows. In section 8.1 we give a very short introduction to Chern-Simons theories. After this introduction we will discuss the construction of topological actions for general compact gauge groups in section 8.2. In section 8.3 we will address the relation of these three-dimensional topological theories to two-dimensional WZW conformal field theories and generalizations of them. We will give an explanation of why for non-simply connected groups, chiral algebras only exist at certain particular values of the level $k$, a fact that we already pointed out in section 4.2.2. The quantization condition on $k$ can be explained entirely by topological considerations in four dimensions. Section 8.4 contains an extension of the construction of topological gauge theories to the category of spin manifolds. These topological 'spin' theories will require a definite choice of spin structure on the manifold in order to be well-defined. They are related in two dimensions to what one might call $\mathbf{Z}_2$ graded chiral algebras, or chiral superalgebras. Superconformal field theories are examples of theories with interesting chiral superalgebras, and it seems more natural to think about superconformal field theories as theories with chiral superalgebras than to regard them as theories with chiral algebras in which there just happens to be a primary field of dimension 3/2 with certain interesting properties. Finally, in section 8.5 we return to a theme that has run throughout this thesis—field theories associated to finite groups. Here we will consider three-dimensional topological gauge theories with a finite gauge group $G$, and show that these theories can be very neatly represented in a form similar to lattice gauge theory. These theories have some claims to being the most simple quantum field theories, being completely finite, and topological in nature. They also provide an elementary but enlightening illustration of the functorial description of quantum field theory as described in Chapter 3. We will furthermore establish in some detail the connection between our $3d$ results and those obtained in the analysis of two-dimensional (holomorphic) orbifold models in Chapter 5. In an appendix we have briefly reviewed some essentials in the theory of group cohomology that we will make use of in the following sections.

## 8.1. Introduction to Chern-Simons Theory

A topological theory is, by definition, independent of the choice of metric on spacetime. One way to achieve this 'general covariance' is to consider the metric as a dynamical field, and integrate over all metrics in the path-integral. In this way some theories of quantum gravity can be considered to be topological. Alternatively, one can begin with an action that does not need a metric at all, and hope that the quantum theory, in particular the measure of the path-integral, will not break this classical general covariance. Chern-Simons gauge theories are of the latter type.

One of the key ingredients in formulating three-dimensional topological gauge theories is the action functional. Thus, let $M$ be an oriented three-manifold, $G$ a compact gauge group, Tr an invariant quadratic form on the Lie algebra of $G$, and $A$ a gauge field on a $G$-bundle $E$. If $E$ is trivial, the connection $A$ can be regarded as a Lie algebra valued one-form, and we can define the Chern-Simons functional by the well-known formula [36]

$$S(A)=\frac{k}{8\pi^2}\int_M \mathrm{Tr}\left(A\wedge dA+\tfrac{2}{3}A\wedge A\wedge A\right). \tag{8.3}$$

Since we can write the action entirely in terms of differential forms, the independence of the metric is clear. Throughout this chapter we will use a somewhat unconventional normalization of the action in which the path-integral reads

$$Z(M)=\int \mathcal{D}A\, e^{2\pi i S(A)}. \tag{8.4}$$

It is an important result in Chern-Simons theory that the measure in (8.4), which *a priori* does need a metric, respects the general covariance [162]. The parameter $k$ in (8.3) must be an integer, since under gauge transformations

$$A\rightarrow A^g=g^{-1}Ag+g^{-1}dg \tag{8.5}$$

the action on a *closed* manifold $M$ transforms as

$$S(A^g) = S(A) - kN(g). \tag{8.6}$$

The integer $N(g)$ measures the winding number of the map $g : M \to G$

$$N(g) = \frac{1}{24\pi^2} \int_M \mathrm{Tr}\left(g^{-1}dg\right)^3. \tag{8.7}$$

Although the action is not inert under large gauge transformations (with $N(g) \neq 0$) the weight $e^{2\pi i S}$ in the path-integral is invariant if $k$ is an integer. The Chern-Simons action has been introduced as an additional term in ordinary $3d$ Yang-Mills theories to induce a topological mass term [42]. Here the complete action is given by $S(A)$, and the dynamics simplifies considerably.

An important property of (8.3) is that the equations of motion imply the vanishing of the curvature $F = dA + A^2$,

$$F = 0. \tag{8.8}$$

What is the interpretation of this equation? We learn that the classical solutions of the theory consist of gauge fields $A$ on $M$ with vanishing field strength, the so-called *flat connections*. Furthermore, gauge invariance tells us that the configurations $A$ and $A^g$ should be considered equivalent. How do we characterize a flat connection modulo gauge transformations? Since $F = 0$, $A$ will always be pure gauge, $A = g^{-1}dg$, in a local, contractible patch. So, locally we can gauge $A$ to zero and the solution is trivial. The only obstructions to write $A$ *globally* as pure gauge are the monodromies around the nontrivial closed curves of $M$. That is, to every element $C$ of the fundamental group $\pi_1(M)$ of the 3-manifold $M$ we can associate the group element

$$g_C = P \exp \oint_C A \in G, \tag{8.9}$$

where $P$ indicates a path-ordered exponential. The map $\pi_1(M) \to G$ should be a group homomorphism. Note that these objects depend on the begin point (= end point) of the closed curve. If we calculate the holonomy around the cycles, but start (and end) at another point, this amounts to conjugation of the $g_C$'s

$$g_C \to h\, g_C\, h^{-1}. \tag{8.10}$$

So, only the conjugacy classes are invariant quantities. Summarizing, the configuration space of classical solutions is given by the moduli space of flat $G$-bundles on $M$, $\mathrm{Hom}(\pi_1(M), G)/G$, where '$/G$' denotes modulo conjugation.

What are the physical objects in the Chern-Simons theory? If we want to construct a physical, gauge invariant correlation function, we are seriously handicapped by the field equations $F = 0$. So, on-shell, local expressions in the curvature always vanish. Furthermore, in a general covariant theory correlation functions have no coordinate dependence, since coordinates have no physical content.

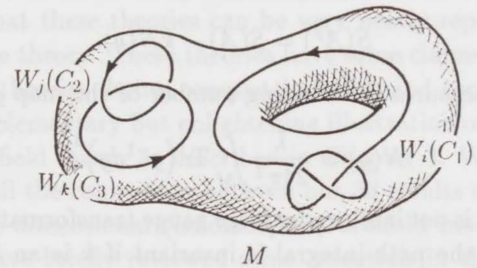



FIGURE 17: In *3d* topological gauge theories the interesting quantities are expectation values of, possibly knotted, Wilson loops $W_i(C)$.

One class of objects that do satisfy all conditions of physical operators are the expectation values of Wilson loops,

$$W_i(C) = \mathrm{Tr}_{R_i} P \exp \oint_C A, \tag{8.11}$$

where $R_i$ is a representation of $G$. In a classical solution of the field equations, these Wilson loops correspond to the characters of the monodromies of the gauge field. In the quantum theory the interesting quantities are expectation values

$$\left\langle W_{i_1}(C_1) \cdots W_{i_n}(C_n) \right\rangle. \tag{8.12}$$

The loops might be knotted as in *fig.* 17, and it is exactly this relation with knot invariants, in particular the Jones invariant [98], that has been the main motivation to consider Chern-Simons theories [162].

### 8.1.1. *Quantization*

The quantization of the theory proceeds as follows. One chooses the space-time $M$ to be of the special form $\Sigma \times \mathbf{R}$, and adopts the temporal gauge $A_0 = 0$. In this gauge the action reduces to the bilinear form

$$S = \frac{k}{8\pi^2} \int_M \mathrm{Tr}\, A_i \partial_t A_j \epsilon^{ij}. \tag{8.13}$$

If we write $A_i = A_i^a T^a$, with generators $T^a$ satisfying $\mathrm{Tr}\, T^a T^b = -\frac{1}{2}\delta^{ab}$, this gives the Poisson brackets

$$\left\{ A_i^a(x), A_j^b(y) \right\} = -\frac{4\pi}{k} \delta^{ab} \epsilon_{ij} \delta(x-y). \tag{8.14}$$

So $A_i\epsilon^{ij}$ is the conjugate momentum of $A_j$, or—in light-cone coordinates—$A_{\bar z}$ the conjugate momentum of $A_z$. We also have to impose the equation of motion associated to $A_0$. This gives the Gauss law constraint $F_{ij}=0$. So the (reduced) phase space of the theory consists of gauge fields $A_j$ on $\Sigma$ with vanishing curvature, *i.e.* flat connections on $\Sigma$. These are again fully characterized by the monodromies around the nontrivial cycles in $\pi_1(\Sigma)$, and the phase space $\mathcal{V}_\Sigma$ is given by

$$\mathcal{V}_\Sigma = \mathrm{Hom}(\pi_1(\Sigma), G)/G. \tag{8.15}$$

If the Lie group $G$ is compact, the phase space is clearly also compact. In quantum mechanics one obtains by a heuristic argument one quantum state per volume $\hbar$. This indicates that here the Hilbert space $\mathcal{H}_\Sigma$ will be finite dimensional. Compact phase spaces cannot be quantized canonically. Since there are no natural momenta—momenta are by definition noncompact—one has to resort to geometrical quantization. We will not pursue these very interesting problems (see [122,60], and also *e.g.* [22] for a different approach), but we will restrict our discussion to the group $U(1)$ which can be treated with completely elementary methods.

For the abelian group $U(1)$ the analysis is much more simple. If $A_i, B_i$ $(i = 1,\ldots,g)$ is a canonical homology basis on the genus $g$ Riemann surface $\Sigma$ (as in *fig.* 11), the flat connections are parametrized by the holonomies $e^{ix_i}, e^{ip_i}\in U(1)$, with

$$x_i = \oint_{A_i} A, \qquad p_i = \oint_{B_i} A. \tag{8.16}$$

The Poisson brackets (8.14) imply the canonical commutation relations

$$[x_i, p_j] = -\frac{2\pi i}{k}\delta_{ij}. \tag{8.17}$$

So the field theory reduces to a set of $g$ uncoupled point particles! In fact, this problem is even more elementary than a free particle, since there are only a finite number of independent states. Both $x_i$ and $p_i$ are periodic modulo $2\pi$. The periodicity of the coordinates $x_i$ requires the momenta $p_i$ to be discrete, quantized in units of $2\pi/k$. But, the momenta are also periodic and this restricts the eigenvalues of $p_i$ to the finite set $\{2\pi n/k\}$ with $n\in\mathbf{Z}_k$. Consequently the Hilbert space $\mathcal{H}_\Sigma$ is $k^g$-dimensional. (Note that quantization only makes sense for integer $k$. We will see in section 8.4 that in the correct normalization of (8.3) $k$ is in fact always an even integer .) So, we exactly reproduce the spectrum of holomorphic blocks of the $\mathbf{Z}_k$ rational gaussian model, which can be regarded as the level $k$ $U(1)$ group manifold model. This analogy can be continued much further [22]. For instance, the Wilson loops

$$\phi_q(C) = \exp iq\oint_C A \tag{8.18}$$

correspond exactly to the loop operators [153,154] that we discussed in Chapter 4, and that act on the space of generalized characters. We will not pursue the interesting correspondence between Chern-Simons theories and WZW models much further here, although we will return to the relation of the classical actions of the two theories in section 8.3.

## 8.2. Topological Actions and Group Cohomology

We will now turn to the correct formulation of Chern-Simons theory for arbitrary compact Lie groups. If $G$ is a connected, simply connected compact Lie group, then a $G$ bundle on a three-manifold is necessarily trivial, and the definition (8.3) of the action is adequate. For more general Lie groups (such as those studied in [122]) nontrivial bundles over $M$ may exist and we will include in the path-integral also a summation over all possible bundles $E$. The inclusion of nontrivial bundles actually tells us that we are considering the gauge group $G$ and not a connected, simply connected group whose Lie algebra equals $Lie(G)$.

If the bundle $E$ is not trivial, the formula (8.3) for the action $S$ does not make sense, since a connection on a nontrivial bundle cannot be represented by a Lie algebra valued one-form as in that formula. A more general definition can be obtained as follows. Any three-manifold $M$ can be realized as the boundary of a four-manifold $B$. If it is possible to choose $B$ so that $E$ extends over $B$ then (upon picking an extension of $A$ over $B$) we can define the Chern-Simons functional by the formula

$$S(A) = \frac{k}{8\pi^2} \int_B \mathrm{Tr}\,(F \wedge F)\,. \tag{8.19}$$

A standard argument shows that if $k$ is an integer, $S(A)$ is independent, modulo 1, of the choice of $B$ and of the extensions of $E$ and $A$. Equation (8.19) reduces to (8.3) when (8.3) makes sense, and so does represent a more general definition of the Chern-Simons functional.

Depending on $\pi_1(M)$ and $G$, there may exist nontrivial flat connections on $M$. The action $S(A)$ for a flat connection $A$ is in general not zero, but is an interesting invariant of the representation of the fundamental group of $M$ determined by the flat connection $A$. However, (8.19) implies the important fact that

$$S(A) = 0 \tag{8.20}$$

for a flat connection $A$ which extends *as a flat connection* over some bounding four-manifold $B$. In other words, if $B$ and $E$ and the extension of $A$ can be chosen so that $F = 0$ on $B$, then obviously $S(A) = 0$.

In general it will be impossible to find a four-manifold $B$, with boundary $M$, over which $E$ can be extended, and therefore (8.19) is still not a completely general definition of the topological action. One of our goals in this paper is to give a completely general definition (for an arbitrary compact group $G$, not necessarily connected or simply connected). To understand a bit better the nature of the problem, note that if $A$ and $A'$ are two different connections on the same bundle $E$, then (8.19) can always be used to define the difference $S(A) - S(A')$. In fact, the four-manifold $B = M \times I$ has boundary $M \cup (-M)$. (Here $-M$ denotes $M$ with opposite orientation). Since $B$ retracts onto $M$, the bundle $E$ has (up to homotopy) a unique extension, which we will also call $E$, over $B$, and it is possible to find a connection $A''$ on $B$ that interpolates between $A$ on $M \times \{0\}$ and $A'$ on $M \times \{1\}$. So a special case of (8.19) is

$$S(A) - S(A') = \frac{k}{8\pi^2} \int_B \mathrm{Tr}\,(F \wedge F)\,, \tag{8.21}$$

where $F$ is the curvature of $A''$; in fact, by standard arguments the right hand side of (8.21) depends modulo 1 only on $A$ and $A'$ and not on the choice of $A''$. Since (8.21) defines the difference $S(A) - S(A')$ for any two connections $A$ and $A'$ on $E$, what remains to be fixed is just an integration constant that depends on $M$ and $E$ but not on the particular choice of a connection $A$. To define a topological quantum field theory, one needs a way to fix these integration constants for all possible three-manifolds $M$ and $G$-bundles $E$, in a way compatible with basic physical requirements of unitarity and factorization.

To give an orientation to this problem (and an example which is quite typical of our interests), consider an example which is of the opposite type from the connected, simply connected groups for which (8.3) serves as an adequate definition of the topological action. Let us consider the case in which $G$ is a finite group. Every principal $G$-bundle has a unique, flat connection, and corresponds to a homomorphism $\lambda : \pi_1(M) \to G$. Since the connections are unique, the integration constants that we previously isolated by using (8.21) are in this case from the beginning all that there is to discuss.

Since we want to be able to consider transition amplitudes between initial and final states (defined on Riemann surfaces), we consider three-manifolds $M$ whose boundaries are not necessarily empty. A 'topological action' $S$ for the gauge group $G$ would be a rule which to every pair $(M, \lambda)$ (with $M$ being a three-manifold and $\lambda$ a homomorphism of $\pi_1(M)$ to $G$) assigns a value $S(\lambda)$ in $\mathbf{R}/\mathbf{Z}$ subject to the following:

($i$) Two actions $S$ and $S'$ should be considered equivalent if they differ by a functional that only depends on the restriction of $\lambda$ to the boundary of $M$—since in that case the difference between the transition amplitudes $e^{2\pi i S}$ and $e^{2\pi i S'}$ can be absorbed in a redefinition of the external state wave functions.

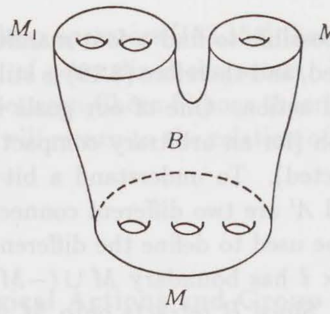



FIGURE 18: The 4-manifold $B$ represents the 'space-time history' of the factorization of the 3-manifold $M$ into $M_1$ and $M_2$.

*(ii)* If $M$ has no boundary, and it is possible to find a four-manifold $B$ such that $\partial B = M$ (that is, the boundary of $B$ is $M$) and such that $\lambda$ extends to a homomorphism $\lambda_B : \pi_1(B) \to G$, then we require $S(\lambda) = 0$. As we have seen in (8.20), this requirement holds for arbitrary $G$, not just finite groups.

Physically, this requirement amounts to a requirement of factorization. This point may require some discussion. In fact, if $M$ is the connected sum of three-manifolds $M_1$ and $M_2$, one could find a four-manifold $B$ of boundary $M \cup (-M_1) \cup (-M_2)$. If one picks $B$ to represent a space-time history of $M$ splitting into $M_1 \cup M_2$ (as in *fig.* 18), then every $G$ connection $\lambda$ on $M$ extends over $B$, and (8.20) implies

$$e^{i2\pi S(M)} = e^{i2\pi S(M_1)} \cdot e^{i2\pi S(M_2)}, \tag{8.22}$$

which is the statement of factorization.

The problem of classifying action functionals $S(\lambda)$ subject to *(i)*, *(ii)* is a standard problem and the answer is as follows. Such action functionals are in one to one correspondence with elements of the cohomology group $H^3(BG, \mathbf{R}/\mathbf{Z})$, where $BG$ is the classifying space of the group $G$. These concepts are explained to some extent in the appendix.

This answer can be reexpressed in the following way. Looking at the long exact sequence in cohomology derived from the exact sequence of groups

$$0 \to \mathbf{Z} \to \mathbf{R} \to \mathbf{R}/\mathbf{Z} \to 0, \tag{8.23}$$

and using the fact that for a finite group, the cohomology with coefficients $\mathbf{R}$ vanishes, we find that $H^k(BG, \mathbf{R}/\mathbf{Z}) \cong H^{k+1}(BG, \mathbf{Z})$. In particular, $H^3(BG, \mathbf{R}/\mathbf{Z}) \cong H^4(BG, \mathbf{Z})$. Therefore, we can consider the topological actions for finite groups to be classified by $H^4(BG, \mathbf{Z})$.

This way of looking at things is fruitful for the following reason. Let us go back to the case in which $G$ is a connected, simply connected group and the simple definition (8.3) of the topological action is adequate. For a group of this type, the topological actions are classified by the integer $k$ that appears in (8.3) or (8.19). On the other hand, it is also so for connected, simply connected $G$ that $H^4(BG, \mathbf{Z}) \cong \mathbf{Z}$. What is more, the generator of $H^4(BG, \mathbf{Z})$ corresponds exactly to the characteristic class $\frac{1}{8\pi^2}\mathrm{Tr}(F \wedge F)$ that appears in (8.19). Thus, we can consider the topological actions for connected, simply connected groups to be classified by $H^4(BG, \mathbf{Z})$.

Thus, a common answer arises for the two opposite kinds of groups—the connected, simply connected ones in which the classification of the components of the space of connections is trivial and the finite groups in which this classification is the whole story (since there is only one connection on any given principal bundle). This strongly suggests that the same result will hold for gauge groups intermediate between these extreme kinds. We will show that this is so—that for an arbitrary compact Lie group $G$ one can construct a topological action corresponding to any element of $H^4(BG, \mathbf{Z})$.

One reason that this is natural is that general Lie groups can be built in simple ways from the types considered above. In fact, any Lie group $G$ appears in an exact sequence

$$1 \to G_0 \to G \to \Gamma \to 1, \tag{8.24}$$

where $G_0$ is the component of the identity and $\Gamma$ is the group of components. If $G$ is compact, then $\Gamma$ is a finite group, one of the two types that we have considered. $G$ also appears in the fundamental exact sequence

$$1 \to \pi_1(G) \to \widetilde{G} \to G \to 1, \tag{8.25}$$

where $\widetilde{G}$ is the simply connected universal cover of $G$. Combining these two exact sequences, any compact $G$ with finite fundamental group is built from a connected, simply connected group and some finite groups—the two extreme cases that we have just considered.

### 8.2.1. *Definition of the Topological Action*

As we have already discussed, the form

$$S(A) = \frac{k}{8\pi^2} \int_B \mathrm{Tr}\,(F \wedge F) \quad (\mathrm{mod}\ 1), \tag{8.26}$$

of the Chern-Simons action, with $B$ a four-manifold that bounds the three-manifold $M$ with $G$-bundle $E$ is for arbitrary compact groups problematic. If the bundle $E$ is not topologically a product $G \times M$, the above representation needs to be modified, since it will in general not be possible to extend the bundle $E$ to a similar bundle over the bounding 4-manifold $B$. To deal with this problem, we can be somewhat more general and allow $B$ to be a smooth singular 4-chain, since a differential form can be integrated over any such chain. Since we are looking for a 4-chain $B$ with a bundle $E'$ that restricts to $E$ at the boundary $M$, we are actually trying to find a 4-chain in the classifying space $BG$ that bounds the image $\gamma(M)$ of $M$ under the classifying map $\gamma$. The restriction of the universal bundle to this 4-chain would give us the bundle $E'$. The obstruction to the existence of such a 4-chain is exactly measured by the image $\gamma_*[M]$ in the cohomology* group $H_3(BG, \mathbf{Z})$. We note that if the bundle $E$ has an extension over $B$, the connection can always be extended using a partition of unity. For connected, simply connected Lie groups $H_3(BG, \mathbf{Z})$ vanishes, and (8.26) can serve as a general definition of the action (8.3) also for bundles with a nontrivial topology. However, for general compact $G$ we have to take this possible obstruction into proper account.

As we mentioned in the previous section, the third homology group, and in fact all odd homology of $BG$, consists only of torsion. This implies that for each bundle $E$ over $M$ there always exist a positive integer $n$ such that

$$n \cdot \gamma_*[M] = 0. \tag{8.27}$$

Stated otherwise, $E$ can be extended to a bundle $E'$ over a 4-chain $B$, whose boundary consists of $n$ copies of $M$, such that the restriction of $E'$ on all boundary components is isomorphic to $E$. We shall call such a bundle $E$ of order $n$. Of course, it is always possible to choose the connection such that $A'$ also reduces to $A$ at $\partial B$. So we have no problem in defining the action modulo $1/n$ as

$$n \cdot S = \frac{k}{8\pi^2} \int_B \mathrm{Tr}\, F \wedge F \quad (\mathrm{mod}\ 1). \tag{8.28}$$

This makes it clear that our task is to resolve an $n$-fold ambiguity consisting of the ability to add a multiple of $1/n$ to the definition of $S$. We must resolve this ambiguity, for all possible three-manifolds and bundles, in a fashion compatible with factorization and unitarity.

So far, the basic object that we have used is the differential form

$$\Omega(F) = \frac{k}{8\pi^2} \mathrm{Tr}\, F \wedge F,$$

which represents an element $\Omega$ of the De Rham cohomology group $H^4(BG, \mathbf{R})$. This differential form has integral periods, so it is in the image of the natural

*By permitting $B$ to be a general 4-chain, we reduce the problem to homology and avoid having to consider the bordism theory of $BG$. If we require $B$ to be a smooth 4-manifold, the obstruction to the existence of $B$ with a compatible bundle lies in the bordism group $\Omega_3(BG, \mathbf{Z})$ [147]. Bordism groups are as generalizations of homology groups. The homology and bordism groups of $BG$ only differ in their torsion. In fact, if $BG$ has no odd torsion one can prove $\Omega_3(BG, \mathbf{Z}) = H_3(BG, \mathbf{Z})$, see also [37].

map $\rho : H^4(BG, \mathbf{Z}) \to H^4(BG, \mathbf{R})$. Thus, there exists a cohomology class $\overline{\omega} \in H^4(BG, \mathbf{Z})$ such that $\rho(\overline{\omega}) = \Omega$. However, the choice of $\overline{\omega}$ may not be unique. It is unique only modulo a torsion element in $H^4(BG, \mathbf{Z})$. We will now show that the choice of a particular $\overline{\omega}$ such that $\rho(\overline{\omega}) = \Omega$ gives a way to resolve the ambiguity in the definition of the action in (8.28). This should not come as a surprise, since the torsion part of $H^4$ is related to the torsion in $H_3$ through the universal coefficient theorem.

Let $\omega$ be any integer-valued cocycle representing the cohomology class $\overline{\omega}$. Then, we define the topological action for a connection on a bundle of order $n$ to be [35]

$$S = \frac{1}{n}\left\{\int_B \Omega(F) - \langle\gamma^*\omega, B\rangle\right\} \quad (\text{mod } 1), \tag{8.29}$$

with $\gamma$ the classifying map $B \to BG$. (Note that $\langle\gamma^*\omega, B\rangle$ is an integer for all chains $B$.) We can now perform some consistency checks on this definition. First on *closed* 4-manifolds we have

$$\int_B \Omega(F) = \langle\gamma^*\omega, B\rangle, \tag{8.30}$$

so that (8.29) is manifestly independent of the bounding manifold $B$ and the way we have continued the bundle and the connection on $B$. It is not difficult to verify that our definition is also invariant under homotopy transformations of the classifying map $\gamma$. Also, the action depends only on the cohomology class $\overline{\omega}$ and not on the particular cocycle chosen to represent it, since under shifts $\omega \to \omega + \delta\epsilon$, with $\epsilon$ an integer cochain, the action changes by

$$\delta S = -\frac{1}{n}\langle\gamma^*\delta\epsilon, B\rangle = -\langle\gamma^*\epsilon, M\rangle = 0 \quad (\text{mod } 1). \tag{8.31}$$

We note that the phase choice made in (8.29) is very sensitive to the torsion information in $\omega$. If we transform $\omega \to \omega + \omega'$, where $\omega'$ is an $n$-torsion element, the action will pick up a $\mathbf{Z}_n$ phase. This is in particular relevant if $\Omega(F) = 0$, *i.e.* 'level' $k = 0$, as is always the case for finite $G$. Then the class $\omega$ is torsion and determines a 3-cocycle $\alpha \in H^3(BG, \mathbf{R}/\mathbf{Z})$ through the isomorphism

$$\operatorname{Tor} H^4(BG, \mathbf{Z}) \cong H^3(BG, \mathbf{R}/\mathbf{Z}). \tag{8.32}$$

In that case we can rewrite the action, which is now independent of the connection, as

$$S = \langle\gamma^*\alpha, [M]\rangle. \tag{8.33}$$

For a discussion of the definition of $S$ on manifolds with boundary see [52].

## 8.3. Correspondence with CFT

Part of the interest of three-dimensional Chern-Simons theories comes from their relation [162] to two-dimensional current algebra theories. In this section, we will discuss those aspects of this relation that are illuminated by the topological considerations of the last section. In particular, we wish to gain a better understanding of subtleties in this correspondence that arise [122] for groups that are not simply connected.

### *8.3.1. The Wess-Zumino Action*

To begin with, we recall [158] that conformally invariant sigma models in two dimensions with target space a group manifold require the introduction of the so-called Wess-Zumino term. Let us recall how this is defined. We are given a Riemann surface $\Sigma$ and a map $g : \Sigma \to G$, $G$ being some compact Lie group of interest. We wish to define the Wess-Zumino term $S(g)$. To begin with, if $G$ is simply connected, the map $g$ is homotopic to a trivial map, and extends to $g : W \to G$, where $W$ is a three-manifold with $\partial W = \Sigma$. Just as in the formulation (8.19) of the Chern-Simons action, in this situation the Wess-Zumino term has a convenient definition

$$S(g) = \frac{k}{24\pi^2} \int_W \mathrm{Tr}\left(g^{-1}dg\right)^3, \tag{8.34}$$

where for reasons explained in [158], $k$ must be an integer. The key object in (8.34) is the differential form $\Phi = \frac{k}{24\pi^2}\mathrm{Tr}\,(g^{-1}dg)^3$ on the group manifold $G$. This form defines an element of $H^3(G, \mathbf{R})$, and since it has integral periods it lies in the image of the natural map $\rho : H^3(G, \mathbf{Z}) \to H^3(G, \mathbf{R})$.

If $G$ is not simply connected, the maps $\Sigma \to G$ come in distinct homotopy classes $\mathcal{U}_i$. It may happen, in general, that for suitable $i$, the definition (8.34) does not make sense for $g \in \mathcal{U}_i$, since a three-manifold $W$ and an extension of $g$ over $W$ may not exist. The obstruction lies in $H_2(G, \mathbf{Z})$. (An example of a semi-simple Lie group with $H_2(G, \mathbf{Z}) \neq 0$ would be $SO(3) \times SO(3)$, or more generally the groups $Spin(4n)/D_2$ as discussed in [64].) Even if $W$ does not exist, if we are given two maps $g$ and $g'$ both in the same homotopy class $\mathcal{U}_i$, the difference $S(g) - S(g')$ can be defined as in (8.34),

$$S(g) - S(g') = \frac{k}{24\pi^2} \int_W \mathrm{Tr}\left(\hat{g}^{-1}d\hat{g}\right)^3, \tag{8.35}$$

where now $W = \Sigma \times I$, and $\hat{g} : W \to G$ is any map that agrees with $g$ on $\Sigma \times \{0\}$ and with $g'$ on $\Sigma \times \{1\}$. Just as in our study of the Chern-Simons term, (8.35) defines the Wess-Zumino term except for an integration constant in each topological sector $\mathcal{U}_i$. What remains is to fix these integration constants, for all possible $\Sigma$ and all $\mathcal{U}_i$, in a way that is compatible with factorization.

If $G$ is semi-simple, $H^2(G,\mathbf{R}) = 0$ and the obstruction to definition (8.34) is the torsion class $g_*[\Sigma] \in H_2(G,\mathbf{Z})$. If this class has order $n$, then $S(g)$ can be defined as

$$S(g) = \frac{1}{n}\left\{\int_W \Phi - \langle g^*\phi, W\rangle\right\} \pmod 1, \tag{8.36}$$

with $\partial W = n \cdot \Sigma$ and $\phi$ an integer class in $H^3(G,\mathbf{Z})$ such that $\rho(\phi) = \Phi$. So the torsion information in $H^3(G,\mathbf{Z})$ (that gives rise to different 'periodic vacua' [64]) suffices to fix the phase ambiguity in the definition of the Wess-Zumino term, completely analogous to our discussion in section 8.2.1 of the Chern-Simons action. Thus, for semi-simple $G$, the Wess-Zumino terms—and therefore, according to [158], the conformally invariant sigma models on group manifolds—are classified by $H^3(G,\mathbf{Z})$.

*8.3.2. The Natural Map $H^4(BG) \to H^3(G)$*

We know now that in general, Chern-Simons theories in three dimensions are classified by $H^4(BG,\mathbf{Z})$, and Wess-Zumino terms in two dimensions (and hence conformally invariant sigma models) are classified by $H^3(G,\mathbf{Z})$. A correspondence between them must therefore involve a natural map from $H^4(BG,\mathbf{Z})$ to $H^3(G,\mathbf{Z})$. Let us first discuss in geometrical terms the map that proves to be relevant. The universal bundle

$$G \to EG \to BG$$

gives rise to a map $\tau : H^k(BG,F) \to H^{k-1}(G,F)$, with $F$ any group of coefficients, as follows [36]. Since $EG$ is a contractible space, any cocycle representing an element $\omega \in H^k(BG,F)$ becomes exact when lifted to $EG$. So we have a relation of the form

$$\pi^*\omega = \delta\beta. \tag{8.37}$$

We now define $\tau(\omega)$ as the restriction of $\beta$ to the fibre $G$. Since the restriction of $\pi^*\omega$ vanishes, the cochain $\tau(\omega)$ is closed and it is easily verified that the cohomology class of $\tau(\omega)$ does not depend on the choice made in the above definition. The *inverse* of the map $\tau$ is a well-known tool in the study of characteristic classes and cohomology of Lie groups and is known as transgression [20].

We now want to show that the map $\tau$ is actually the correspondence between Chern-Simons actions and Wess-Zumino terms that arises in connecting three-dimensional quantum field theory with two-dimensional quantum field theory. As has been shown in the concluding section of [162], the chiral algebras of two-dimensional current algebra can be obtained from three dimensions by quantizing the three-dimensional Chern-Simons theory on the three-manifold $M = D \times \mathbf{R}$, with $D$ a disk.

In fact, [122,60], the two-dimensional WZW action can be explicitly derived from the three-dimensional Chern-Simons action by first integrating over the 'time' component $A_0$ of the gauge field in the functional integral. The portion of the action (8.3) that depends on $A_0$ is

$$S_0 = \frac{k}{4\pi^2} \int_M \mathrm{Tr}\,(A_0 \cdot F_{12}), \tag{8.38}$$

where $F_{12}$ is the spatial component of the curvature, tangent to $D$. The functional integral over $A_0$ therefore gives a delta function setting $F_{12}$ to zero, and so we are left with a connection on $M$ whose components tangent to $D$ are pure gauge, *i.e.* $A = g^{-1}dg$ for a map $g : M \to G$ ($g$ is unique up to a transformation $g \to ug$ where $u$ depends only on 'time'). Since $M$ is contractible, any bundle $E$ over $M$ is necessarily trivial, and we can evaluate the topological action by choosing a global section and pulling the Chern-Simons form to $M$

$$S = \int_M Q(g^{-1}dg) \ + \textit{exact}. \tag{8.39}$$

The exact terms that we will ignore here just correspond to local terms in the two-dimensional action. The important contribution is the first term which corresponds to the Wess-Zumino term. With $\theta$ the Maurer-Cartan form on $G$, *i.e.* the restriction of the connection to the fibre, we can rewrite the first term as

$$\int_M g^* Q(\theta), \tag{8.40}$$

where $Q(\theta)$ is a closed differential form on $G$ whose class is integer. That is, the Chern-Simons form defines an integer cohomology class in $H^3(G, \mathbf{R})$.

The transformation just found from an element of $H^4(BG, \mathbf{R})$ with integral periods used to define (8.3) to the element of $H^3(B, \mathbf{R})$ with integral periods that appears in (8.39) is precisely the map $\tau$ written out in terms of differential forms. One should go on to show that even when one takes torsion into account, the map from three-dimensional theories classified by $H^4(BG, \mathbf{Z})$ to two-dimensional theories classified by $H^3(G, \mathbf{Z})$ is the inverse transgression map. However, we will not tackle this here.

Now, the crucial map $\tau$ from $H^4(BG, \mathbf{Z})$ to $H^3(G, \mathbf{Z})$ is not necessarily onto. (The special classes in $H^*(G)$ that are images of the map $\tau$ are usually referred to as *universally transgressive* [20].) This fact implies in particular that not all group manifold models 'descend' from a three-dimensional Chern-Simons theory. In fact, we will see that only those group manifold models that allow a description in terms of a so-called extended chiral algebra will be generated by three-dimensional gauge theories.

*8.3.3. $SU(2)$ versus $SO(3)$*

As an example we first consider the cases $G = SU(2)$ and $G = SO(3)$, with the relation

$$1 \to \mathbf{Z}_2 \to SU(2) \xrightarrow{\pi} SO(3) \to 1. \tag{8.41}$$

For both groups we have $H^4(BG) = H^3(G) = \mathbf{Z}$. (Here and in the subsequent all cohomology groups are understood to be with integer coefficients unless otherwise stated.) The generators for $H^4(BG)$ are respectively the second Chern class $c_2$ for $SU(2)$, and the first Pontryagin class $p_1$ for $SO(3)$. We will denote the respective generators of the cohomology groups $H^3(G)$ by $\alpha$ for $SU(2)$ and $\beta$ for $SO(3)$, with the important relation

$$\pi^*(\beta) = 2\,\alpha. \tag{8.42}$$

This factor of two is simply due to the fact that the volume of $SO(3)$ is half the volume of $SU(2)$. This corresponds to the familiar fact that, if we normalize the Wess-Zumino term with respect to the group $SU(2)$, the corresponding term for $SO(3)$ can only exist for even $k$ [83].

It is well-known that all classes of $H^*(SU(2))$ are transgressive [20], so $\tau(c_2) = \alpha$ and this implies a one-to-one correspondence between the $SU(2)$ WZW models and Chern-Simons theories, which are both characterized by their level $k \in \mathbf{Z}$. This will however not be the case for $SO(3)$. In fact, we will see that only the models based on *even* elements of $H^3(SO(3))$, *i.e.* level $k$ divisible by four, correspond to three-dimensional topological theories. Note that it has been observed [122] that exactly for these values the chiral algebra for $SO(3)$ exists, since the chiral vertex operators that are associated with the nontrivial loops in $SO(3)$ have conformal dimensions $k/4$, and these dimensions should be integer.

This restriction to $k = 0 \pmod 4$ has a completely topological explanation. Let us recall that, although every $SU(2)$ bundle naturally gives rise to an $SO(3)$ bundle, the opposite is not true. Not every $SO(3)$ bundle can be extended to an $SU(2)$ bundle. In fact, this can only happen for certain specific values of the characteristic classes. To determine these values we have to compare the 'instanton charges' in the four-dimensional $SU(2)$ and $SO(3)$ gauge theories. In a normalization where $SU(2)$ instantons have integer charge the $SO(3)$ instantons can have fractional charges. The fact that a non-simply connected group can have fractional instantons is a well-known phenomenon, *e.g.* on the hypertorus $T^4$ one can construct $SU(n)/\mathbf{Z}_n$ instantons with charge $1/n$ [94,11]. We will actually show that the minimal charge of an $SO(3)$ instanton is $\frac{1}{4}$, and this naturally quantizes $k$ in units of four. Equivalently, if an $SO(3)$ bundle $E$ on a four-manifold extends to an $SU(2)$ bundle, the first Pontryagin class $p_1(E)$ always has to be divisible by four. That is, under the map $B\pi : BSU(2) \to BSO(3)$ as induced by the exact sequence (8.41), we have (see *e.g.* [68])

$$B\pi^*(p_1) = 4\,c_2. \tag{8.43}$$

This can be seen as follows, though perhaps in a slightly abstract way. After lifting to a suitable flag space, any $SU(2)$ vector bundle $V$ (that is, any rank two complex vector bundle of structure group $SU(2)$) splits as a sum of line bundles $V = L \oplus L^{-1}$. Now recall that $p_1(E)$ can also be defined as the second Chern class $c_2(W)$ of the complexified three-dimensional vector bundle $W$ in the adjoint representation of $SO(3)$. In this case we find $W = L^2 \oplus L^0 \oplus L^{-2}$, so that $p_1(E) = c_2(W) = 4\,c_2(V)$.

A concrete example of an $SO(3)$ bundle that has instanton charge $\frac{1}{4}$ can be constructed on $CP^2$. Our normalization will be as follows. Let $\lambda_a$ denote the generators of the Lie algebra of $SO(3)$, satisfying $[\lambda_a, \lambda_b] = i\epsilon_{abc}\lambda_c$. A general curvature can be expressed as $F = \sum_a F^a \lambda_a$ and the instanton number reads

$$q = \frac{1}{16\pi^2}\int \sum_a F^a \wedge F^a. \tag{8.44}$$

For the basic instanton over the 4-sphere $q = 1$. Now consider the fundamental line bundle $L$ over $CP^2$. Its curvature $F'$ satisfies

$$\int_{CP^2} F' \wedge F' = 4\pi^2. \tag{8.45}$$

We can now make $L$ into an $SO(3)$ bundle using the embedding $U(1) \subset SO(3)$, which maps $e^{i\theta} \to e^{i\theta\lambda_3}$. This gives $F = F'\lambda_3$ and in this case the contribution in (8.44) is $4\pi^2$ for $a = 3$, and zero otherwise, so $q = \frac{1}{4}$, as promised.

The existence of $SO(3)$ bundles of instanton number $1/4$ means that in $SO(3)$ Chern-Simons gauge theory, the level $k$ must be divisible by four (in units in which an arbitrary integer is allowed for $SU(2)$). This result was first established in [122].

We can now easily establish that the $SO(3)$ Chern-Simons theories lead to group manifold models corresponding to even elements of $H^3(SO(3))$. Since we have a commuting diagram

$$\begin{array}{ccc} H^4(BSO(3)) & \xrightarrow{B\pi^*} & H^4(BSU(2)) \\ \big\downarrow \tau & & \big\downarrow \tau \\ H^4(SO(3)) & \xrightarrow{\pi^*} & H^4(SU(2)) \end{array} \tag{8.46}$$

the equations (8.42) and (8.43) immediately imply the relation

$$\tau(p_1) = 2\,\beta. \tag{8.47}$$

That $\tau(p_1)$ is necessarily even can also be proved (and generalized to arbitrary $SO(n)$) using the fact that the class $p_1$ satisfies $p_1 = w_2 \cup w_2 \pmod 2$. This gives $\tau(p_1) = 0 \pmod 2$, since for any coefficient field $F$ the inverse transgression $\tau : H^k(BG, F) \to H^{k-1}(G, F)$ satisfies $\tau(u \cup u) = 0$ [36].

### 8.3.4. Non-Simply Connected Groups

Let us now consider the somewhat more general situation where we have an exact sequence
$$1 \to Z \to \widetilde{G} \xrightarrow{\pi} G \to 1,$$
with $\widetilde{G}$ a connected, simply connected, simple group, and $Z$ a cyclic subgroup of the center of $\widetilde{G}$. In that case all relevant cohomology groups are still isomorphic to $\mathbf{Z}$. The relation between the generators of $H^3(\widetilde{G})$ and $H^3(G)$ has been carefully investigated in [64]: the constant of proportionality is either one or two. As to the cohomology of the classifying space, let $\tilde{\omega}$ and $\omega$ denote the generators of respectively $H^4(B\widetilde{G})$ and $H^4(BG)$. In all generality we have a relation
$$B\pi^*(\omega) = N\tilde{\omega}, \tag{8.48}$$
where we wish to determine the integer $N$. The interpretation will be again that in four dimensions $G$ instantons can have instanton charge $1/N$ when compared to $\widetilde{G}$ instantons, and that the Chern-Simons theory is only well-defined for $k$ divisible by $N$. The calculation of $N$ is as follows. Let $T \subset \widetilde{G}$ be the maximal torus of $\widetilde{G}$ with rank $r$, and let $\Lambda$ be the weight lattice of $\widetilde{G}$ as generated by the fundamental weights $\mu_i$. The inclusion $T \subset \widetilde{G}$ gives a natural map $H^*(B\widetilde{G}) \to H^*(BT)$. Now $H^*(BT)$ is generated by the 2-cocycles $x_i$, the first Chern classes in the decomposition $T = U(1)^r$. The $x_i$ are the images of the fundamental weights $\mu_i$ under transgression in the universal bundle $ET$, *i.e.* under the isomorphism $H^1(T) \cong H^2(BT)$. The image of the generator $\tilde{\omega}$ of $H^4(B\widetilde{G})$ in $H^4(BT)$ is given by the Weyl group invariant combination
$$\sum_{i,j} \tfrac{1}{2} A_{ij} x_i \cup x_j, \tag{8.49}$$
with $A_{ij}$ the Cartan matrix.

Now let the sublattice $\Lambda' \subset \Lambda$ be the weight lattice of $G = \widetilde{G}/Z$ with generators $v_i$. The corresponding elements of $H^2(BT)$ we will denote by $y_i$. The $y_i$ are linear combinations of the $x_i$ with integer coefficients. Since $\tilde{\omega}$ is again the smallest Weyl invariant integer combination of the $y_i$ (recall that $G/T$ is torsion free), the relation (8.48) between $\tilde{\omega}$ and $\omega$ is simply determined by comparing the images of $\tilde{\omega}$ and $\omega$ in $H^*(BT)$. This gives the following result for $N$. Every element $z_a$ of the center $Z$ corresponds to a fundamental weight $\mu_a$, and $N$ is defined as the smallest integer that satisfies for all $a$
$$\tfrac{1}{2} N \langle \mu_a, \mu_a \rangle = 0 \pmod 1. \tag{8.50}$$
This result confirms the relation found in [122] (see also section 4.2.2) where it was established that the conformal dimensions $h_a$ of the vertex operators that create

vortices associated to the fundamental group $\pi_1(G) = Z$, and that extend the chiral algebra of $\widetilde{G}$ to the chiral algebra of $G$ are given by

$$h_a = \frac{\langle k\mu_a, k\mu_a + 2\rho\rangle}{2(k+h)}, \tag{8.51}$$

with $h$ the dual Coxeter number and $\rho$ half the sum of positive roots of $\widetilde{G}$. The conformal dimensions $h_a$ should be integer and this reproduces the condition $k = 0$ (mod $N$) using the relation $2\langle\rho, \mu_a\rangle = h\langle\mu_a, \mu_a\rangle$ [64].

An interesting example is $G = SU(n)/\mathbf{Z}_n$. According to [64] the WZW models based on $G$ exist at level $k \in 2\mathbf{Z}$ or $k \in \mathbf{Z}$ depending on whether $n$ is even or odd respectively. But according to (8.50) the quantization of $k$ for the corresponding Chern-Simons theories is in multiples of $N$, with $N = 2n$ for even $n$ and $N = n$ for odd $n$. So we see that the map $\tau : H^4(BG) \to H^3(G)$ is simply multiplication by $n$.

## 8.4. Topological Spin Theories

Up to now all topological theories were defined on oriented 3-manifolds, possibly with boundary. In general we can consider manifolds with extra structure, and in this section we want to discuss topological theories defined on *spin manifolds*. We recall that a spin manifold $M$ is an oriented manifold with a choice of spin structure. A spin structure on an oriented manifold exists if the second Stiefel-Whitney class $w_2(T)$ of the tangent bundle $T$ of $M$ vanishes. For three-dimensional manifolds, this is always so. (But an oriented three-dimensional manifold may admit more than one spin structure if there is two-torsion in $H^1(M, \mathbf{Z})$.) We will refer to topological theories which require choices of spin structure as 'topological spin theories' or simply 'spin theories' for short. These theories will have the fundamental property that the definition of partition functions and transition amplitudes associated with $M$ require a choice of spin structure on $M$.

Just as ordinary topological theories in three dimensions lead to ordinary chiral algebras in two dimensions, spin theories lead to what one might call $\mathbf{Z}_2$ graded chiral algebras or chiral superalgebras. A chiral superalgebra consists of a collection of holomorphic fields $A_i(z)$ of integer or half-integer dimension $h_i$ which are closed under operator products,

$$A_i(z)\,A_j(w) \sim \sum_k c_{ij}{}^k (z-w)^{h_k - h_i - h_j} A_k(w) \tag{8.52}$$

(and with $c_{ij}{}^k = 0$ unless $h_k - h_i - h_j$ is an integer) and obeying certain other axioms that are just analogous to the axioms for bosonic chiral algebras. If the

$A_i$ are all of integer dimension, this reduces to the notion of an ordinary (bosonic) chiral algebra. The superconformal algebra in two dimensions should be regarded as a $\mathbf{Z}_2$ graded chiral algebra. But there are many other theories that are not superconformal but can be conveniently regarded as theories with $\mathbf{Z}_2$ graded chiral algebras.

The general axioms of quantum field theory tell us that a topological spin theory will associate to each two-dimensional closed surface $\Sigma$ with a particular spin structure $\alpha$ a Hilbert space $\mathcal{H}_{\Sigma,\alpha}$. We would like to identify this Hilbert space as the space of holomorphic blocks of a $\mathbf{Z}_2$ graded conformal field theory on $\Sigma$. Elementary examples are of course free fermion theories, where the chiral superalgebra is freely generated by the spin $\frac{1}{2}$ currents $\psi_i(z)$. These theories possess for a given spin structure only a single holomorphic block whose dependence on the spin structure is given by a theta-function $\vartheta[\alpha](0|\tau)$.

We will not treat here the general theory of 'spin' Chern-Simons theories with arbitrary compact gauge group $G$, but restrict ourselves to two examples. Consider first the group $U(1)$, and let $u$ be the generator of $H^4(BU(1),\mathbf{Z})$. (Here $u = c_1^2$, with $c_1$ the first Chern class.) Each class $k \cdot u$ defines a topological action, and consequently there are topological $U(1)$ theories in three dimensions with an arbitrary integer level $k$. But if we are given a three-manifold with a spin structure, the level need not be an integer; it can be half-integer.

The reason for this is the following. Recall that $H_3(BU(1),\mathbf{Z})$ vanishes, so that the action of the $U(1)$ theory on a 3-manifold $M$ can always be defined as

$$S = \frac{k}{4\pi^2}\int_B F \wedge F \pmod 1, \tag{8.53}$$

with $B$ a four-manifold that bounds $M$. The curvature form $\frac{1}{2\pi}F$ represents the first Chern class $c_1(L)$ of some complex line bundle $L$ over $M$. This formula for the action is well-defined since the integral

$$q = \frac{1}{4\pi^2}\int_B F \wedge F, \tag{8.54}$$

is an integer on any *closed* 4-manifold $B$. But if $B$ is a *spin* manifold this integer is always *even*. The reason for this is the following. Equation (8.54) can be interpreted in terms of the intersection pairing in $H^2(B,\mathbf{Z})$. In fact, the right hand side of (8.54) is a De Rham representation of $\langle c_1(L) \cup c_1(L), [B]\rangle$. But on a four-dimensional spin manifold, the intersection pairing in $H^2(B,\mathbf{Z})$ is even, so (8.54) is even. This statement can be given a rather elementary, geometrical proof. Alternatively, one purely analytic way to prove that $c_1(L)^2$ is even on a four dimensional spin manifold is to note that the index theorem for the Dirac operator $D_L$ on a four-manifold $B$ twisted by the line bundle $L$ gives

$$\operatorname{Index} D_L = \tfrac{1}{12}p_1(T) + \tfrac{1}{2}c_1(L)^2, \tag{8.55}$$

($T$ is the tangent bundle of $B$.) Taking $L$ to be trivial and requiring Index $D_L$ to be an integer, we learn that $\frac{1}{12}p_1(T)$ is an integer. (In fact, it can be shown to be even). Generalizing to arbitrary $L$ and requiring that the index should still be an integer, we learn that $\frac{1}{2}c_1(L)^2 \in \mathbf{Z}$, so that $c_1(L)^2$ and thus (8.54) is even.

Because of this, the definition (8.53) of the action still makes sense modulo 1 for half-integer level $k$ if $M$ is a spin manifold. Note that we tacitly assumed that the spin bordism group $\Omega_3^{spin}(BU(1))$ vanishes, so that both the line bundle and the spin structure of $M$ can always be extended to $B$. This fact is proved by a spectral sequence argument, using the fact that $\Omega_n^{spin}(\text{point}) = 0$ for $n = 1, 2, 3$ [147], and that $H_*(BU(1))$ is torsion free. Thus, we may conclude that there is a topological spin theory with $U(1)$ gauge group and half-integer $k$. These theories should correspond to $\mathbf{Z}_2$ graded chiral algebras in two dimensions. Indeed in our normalization the chiral vertex operators that appear in the two-dimensional $U(1)$ chiral algebra have weight $k$. Since we quantize the theory on a Riemann surface with a fixed spin structure, we do not require an implementation of the full modular group, but only of the subgroup which leaves a given spin structure fixed.

Note that $k$ as we define it is half as big as the usual $k$ in most discussions of the abelian theory. Thus, to compare our discussion to other treatments one must make a redefinition $k \to 2k$. (However, comparison to [122] needs a redefinition $k \to 4k$.) So the 'half-integers' become integers, and it is usually said that $k$ must be even in order to define a topological $U(1)$ Chern-Simons theory in three dimensions or in order to be able to define the $U(1)$ chiral algebra in two dimensions (with $\mathbf{Z}_k$ fusion rules). Note that in this normalization the $k = 1$ theory represents the theory of a free Dirac fermion. In general the bosonic subalgebra of the level $k$ theory equals the chiral algebra at $4k$, so the spin projection of the Dirac fermion occurs at $k = 4$, as is well-known to be true.

A second example of a spin theory is $SO(3)$ Chern-Simons theory. It is likewise true that on a spin 4-manifold the first Pontryagin number of an $SO(3)$ bundle $E$ is always even*. Therefore, in the spin category, the level $k$ (normalized with respect to $SU(2)$) can be half as big as in the bosonic category; that is, $k$ can be any even number, not necessarily a multiple of four. So our claim is that for $k = 2 \pmod 4$, the $SO(3)$ affine models do have a chiral superalgebra and have a diagonal partition function if formulated on Riemann surfaces with spin structure. This corresponds to the results in [122]. Indeed, if we calculate the conformal weight $h$ of the chiral field that extends the $SU(2)$ current algebra to $SO(3)$ we find $h = k/4$, which is half-integer for $k = 2 \pmod 4$. The extended characters $\tilde{\chi}_j$ (with *integer* spin $j$) in the Neveu-Schwarz sector are of the form $\tilde{\chi}_j = \chi_j + \chi_{\frac{k}{2}-j}$, where the $\chi_j$'s represents the $SU(2)$ characters. If we calculate the partition function of the corresponding

*In fact this is true for all $SO(n)$. It follows from a fact that we used earlier, namely that $p_1(E) = w_2(E)^2 \bmod 2$. As a result, the first Pontryagin number of $E$, which is $\langle p_1(E), [M] \rangle$, is equal modulo two to $\langle w_2(E) \cup w_2(E), [M] \rangle$, and this vanishes because the intersection form on $H^2(M)$ is even for spin manifolds.

bosonic model that is obtained by the summation over spin structures, we find the familiar expressions for the $SO(3)$ partition functions. (See also the discussion in [136].) An elementary example is the case $k = 2$ which can be described by three free Majorana fermions. The fermionic model has a single character $(\vartheta[\alpha]/\eta)^{\frac{3}{2}}$, and the chiral algebra is generated by the three fermionic currents $\psi_i(z)$.

## 8.5. Finite Gauge Groups

We will now turn to the very special case of a finite gauge group $G$. Our main result will be that the structure of topological gauge theories with finite gauge group will correspond to the two-dimensional holomorphic orbifold models that we considered in Chapter 5. Holomorphic orbifolds can for instance be obtained by taking the quotient of the level one $E_8$ WZW model with any finite subgroup $G$ of $E_8$. According to [122] the modular geometry of these orbifold CFT's will be reproduced by Chern-Simons theories whose gauge group is the semi-direct product of $E_8$ and $G$. However, here the group $E_8$ is essentially used to reproduce a trivial theory in two dimensions, at least for closed surfaces. This can be accomplished much more economically by simply omitting the $E_8$ gauge theory, and this leads us naturally to consider Chern-Simons theories with finite gauge group $G$. In our opinion these theories are also of some intrinsic interest, since they are very simple examples of topological 'quantum field theories.' That is, they provide an elementary illustration of the approach to quantum field theory along the lines of category theory as described in Chapter 3.

### *8.5.1. Topological Gauge Theories with Finite Gauge Group*

In gauge theories with finite gauge groups there are no gauge fields, and the only degree of freedom will be the topology of the principal $G$-bundle $E$ over the manifold $M$. For a discrete group all $G$-bundles are of course necessarily flat, and the topology can only be detected in the possible holonomy around homotopically nontrivial closed curves. Accordingly, $G$-bundles are completely determined by homomorphisms of the fundamental group $\pi_1(M)$ of the 3-manifold $M$ into the group $G$, up to conjugation. We will denote both this homomorphism and the corresponding homotopy class of the classifying map $M \to BG$ as $\gamma$. In accordance with the general discussion in section 8.2.1 we choose a class $\alpha \in H^3(BG, U(1)) \cong H^4(BG, \mathbf{Z})$ as topological action. (Note that in this section we will identify $\mathbf{R}/\mathbf{Z} \cong U(1)$ with the unit circle in $\mathbf{C}$, and write the cohomology groups accordingly multiplicatively, which might confuse the reader.)

The partition function for a closed 3-manifold $M$ will be defined as the sum over all possible $G$-bundles over $M$, weighted with the action $W = e^{2\pi i S}$

$$Z(M) = \frac{1}{|G|} \sum_{\gamma \in \mathrm{Hom}(\pi_1(M), G)} W(\gamma), \tag{8.56}$$

with

$$W(\gamma) = \langle \gamma^* \alpha, [M] \rangle. \tag{8.57}$$

In (8.56), $\pi_1(M)$ is defined relative to some choice of base point. We notice that the path-integral is reduced to a finite sum. The weights $W(\gamma)$ are manifestly invariant under diffeomorphisms of $M$. Since all bundles over the 3-sphere are trivial, we have in particular

$$Z(S^3) = \frac{1}{|G|}. \tag{8.58}$$

Note that although the isomorphism class of $E$ depends on $\gamma$ only up to conjugacy, we sum over all of $\mathrm{Hom}(\pi_1(M), G)$. This prescription is required for the property*

$$Z(M) \cdot Z(S^3) = Z(M_1) \cdot Z(M_2), \tag{8.59}$$

where $M$ is the *connected* sum of the two manifolds $M_1, M_2$. This relation follows immediately from two facts: (*i*) the fundamental group of $M$ equals the free product $\pi_1(M_1) * \pi_1(M_2)$, and (*ii*) if $\gamma = (\gamma_1, \gamma_2) \in \mathrm{Hom}(\pi_1(M), G) \cong [M, BG]$,

$$\langle \alpha, \gamma(M) \rangle = \langle \alpha, \gamma_1(M_1) \rangle \cdot \langle \alpha, \gamma_2(M_2) \rangle, \tag{8.60}$$

since we can construct a 4-manifold $B$ (the 'world-sheet' swept out during the factorization process $M \to M_1 + M_2$, as in *fig.* 18) that interpolates from $M$ to $M_1 \cup M_2$. Evaluating $\delta\alpha = 1$ on the image of this manifold $B$ into $BG$ gives the required property. The normalization of the partition sum (8.56) is such that

$$Z(S^2 \times S^1) = 1. \tag{8.61}$$

Here we used that $\pi_1(S^2 \times S^1) = \mathbf{Z}$ and $W(\gamma) = 1$, since any bundle over $S^2 \times S^1$ can be continued over the bounding 4-manifold $B^3 \times S^1$, with $B^3$ the 3-ball. Stated otherwise, in the light of (3.16), the Hilbert space $\mathcal{H}_{S^2}$ turns out to be one-dimensional.

### *8.5.2. Hilbert Spaces and Interactions*

An interesting class of objects in any (compact) topological field theory are the dimensions of the Hilbert spaces $\mathcal{H}_\Sigma$ obtained by quantizing the theory on a space-time $\Sigma \times R$ with $\Sigma$ a Riemann surface of genus $g$. At first sight 'quantization' seems

*More generally this definition is necessarily to define a functor as discussed in Chapter 3 that respects the gluing axioms.

quite elementary in this case, since the classical degrees of freedom are discrete and finite. The phase space is simply the moduli space $\mathcal{V}_\Sigma$ of $G$-bundles over $\Sigma$

$$\mathcal{V}_\Sigma = \mathrm{Hom}(\pi_1(\Sigma), G)/G. \tag{8.62}$$

A representation $\gamma$ of the fundamental group of the Riemann surface consists of elements $(g_i, h_i)$ $(i = 1, \ldots, g)$ satisfying $\prod_i [g_i, h_i] = 1$, and a $G$-bundle is determined by $\gamma$ up to conjugation. Although $\mathcal{V}_\Sigma$ is a finite set of points, and naively every point contributes one quantum state, in general we only have an inequality

$$\dim \mathcal{H}_\Sigma \leq |\mathcal{V}_\Sigma|. \tag{8.63}$$

Let us explain why this is true. The dimensions of the Hilbert spaces can be determined in principle—and here also in practice — by calculating the partition functions $Z(\Sigma \times S^1)$. In the case of a trivial cocycle, $\alpha = 1$, the action $W$ will always be one, and the definition of the partition function is just the normalized sum over all representations of the fundamental group of the three-manifold. Representations of $\pi_1(\Sigma \times S^1)$ are given by representations $\gamma = (g_i, h_i)$ of the fundamental group $\pi_1(\Sigma)$ of the Riemann surface, together with an element $k$, the holonomy associated to the factor $S^1$, in the common stabilizer subgroup $N_\gamma$ of the holonomies $g_i, h_i$. For a fixed $\gamma$ the prefactor in the partition sum is $|N_\gamma|/|G|$. Since $|G|/|N_\gamma|$ equals the order of the orbit of the representation $\gamma$ under conjugation, the partition sum yields exactly $Z(\Sigma \times S^1) = |\mathcal{V}_\Sigma|$, as expected.

However, if $\alpha \neq 1$ there can be $k \in N_\gamma$ such that the action $W(\gamma, k)$ is not equal to one. In fact, the action will always be a one-dimensional representation of $N_\gamma$, *i.e.*

$$W(\gamma, k_1)\, W(\gamma, k_2) = W(\gamma, k_1 k_2). \tag{8.64}$$

This relation can be proved by constructing a 4-manifold that has as its boundary three copies of $\Sigma \times S^1$, and which allows a representation of its fundamental group that reduces at the boundary to the representations appearing in (8.64). We can take for this 4-manifold $\Sigma \times Y$, with $Y$ the two sphere with three holes (the trinion or 'pair of pants'). With a suitable orientation the monodromies around the three holes are respectively $k_1, k_2$, and $k_1 k_2$. The summation over all $k \in N_\gamma$ in the partition sum will now give zero if the representation (8.64) is nontrivial. So in that case the bundle over $\Sigma$ described by $\gamma$ does not contribute a quantum state, and the dimension of the Hilbert space is smaller than expected. This effect has been noted in [122] and can be regarded as a global anomaly.

Now for arbitrary genus and general cocycle $\alpha$ the explicit calculation of the dimensions of the Hilbert spaces for arbitrary genus might be a complicated calculation. However, we can make a shortcut. As we explained in the introduction to this chapter it suffices to know the Hilbert space of the 2-torus and the map

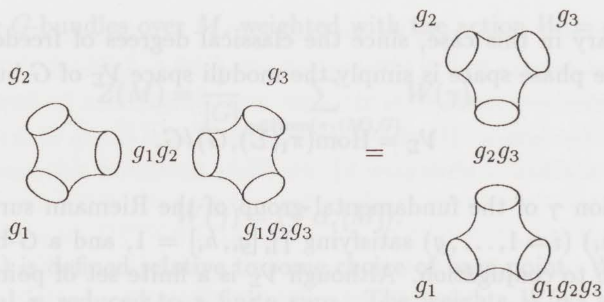



FIGURE 19: The sphere with four holes can be composed in two distinct ways from two copies of the sphere with three holes, as indicated in these diagrams. The group elements correspond to monodromies around the punctures. As explained in the text this duality relation proves the cocycle condition for the quantities $c_h(g_1, g_2)$ associated to a single pant.

$\Phi_{Y\times S^1}$. Let us first evaluate the action for the manifold $Y \times S^1$. Since the manifold has a boundary the action is only well-defined if we choose some fixed classifying maps at the boundaries. So, we will assume that for each homomorphism of the fundamental group of a Riemann surface into $G$ we have been given some fixed classifying map. (We will belabor this point in the next section.) An element of $\gamma \in \mathrm{Hom}(\pi_1(Y \times S^1), G)$ is given by elements $g_i$ $(i = 1, 2, 3)$ satisfying $g_1 \cdot g_2 \cdot g_3 = 1$, that represent the monodromies around the three punctures of $Y$, and an element $h$ that commutes with the $g_i$ and that corresponds to the generator of the factor $S^1$. So the independent variables are $g_1, g_2$, and $h$. Of course, the bundle over $Y \times S^1$ is only properly defined in terms of the elements $g_1, g_2, h$ once we picked a base point and specified the cycles along which the holonomies are determined. Let us denote the action as

$$W(\gamma) = c_h(g_1, g_2). \tag{8.65}$$

Our claim is that the object $c_h$ is an (algebraic) 2-cocycle of the stabilizer group $N_h \subset G$, the subgroup of all elements in $G$ that commute with $h$. That is, we have the relation

$$c_h(g_1, g_2)\, c_h(g_1 g_2, g_3) = c_h(g_1, g_2 g_3)\, c_h(g_2, g_3), \tag{8.66}$$

for *any* three elements $g_1, g_2, g_3 \in N_h$. This statement has a very natural geometric proof. Consider the sphere with four holes. It can be obtained in two different ways from two copies of $Y$, as is represented diagrammatically in *fig.* 19. This is similarly true after taking the direct product with $S^1$. Since the action cannot depend on the way we have chosen to construct the manifold, the above relation follows immediately.

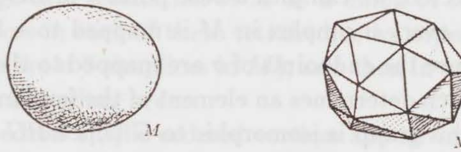



FIGURE 20: A smooth manifold $M$ and a triangulation $M'$.

So we see that on *a priori* grounds the Chern-Simons theory associates group cocycles $c_h$ to each stabilizer subgroup $N_h$ of $G$. However, to actually calculate $c_h$ in terms of the 3-cocycle $\alpha$, we have to resort to a different approach.

*8.5.3. A Lattice Gauge Theory Realization*

We would now like to explain why the abstract description of the topological action can, in the case of a finite gauge group, be reduced to a concrete description somewhat reminiscent of lattice gauge theory.

Recall that a lattice gauge theory, formulated on a lattice with vertices $V_i$, links $L_{ij}$, *etc.*, associates to each link $L_{ij}$, oriented from $V_j$ to $V_i$, a gauge field $g_{ij} \in G$. A gauge transformation is simply a set of elements $h_i \in G$, and the transformation acts on the gauge field as

$$g_{ij} \rightarrow h_i \cdot g_{ij} \cdot h_j^{-1}. \tag{8.67}$$

The total curvature $f_{ijk}$ for a 2-simplex is given by the holonomy

$$f_{ijk} = g_{ij} \cdot g_{jk} \cdot g_{ki}, \tag{8.68}$$

and is only well-defined modulo conjugation. The action is some local functional of the gauge fields. These lattice theories are particularly well suited for finite groups, where no obvious continuum theory exists. We would like to define here something close to a lattice Chern-Simons theory.

We have seen that for a finite gauge group $G$, the integer cohomology group $H^4(BG, \mathbf{Z})$ is isomorphic to $H^3(BG, U(1))$, and the topological action can be specified by giving an element $\alpha \in H^3(BG, U(1))$. Given an oriented three-manifold $M$ without boundary (the orientation is always assumed in what follows), and a map $\gamma : M \rightarrow BG$, the topological action is the pairing $\langle \gamma^*\alpha, [M] \rangle$. We will now discuss how this can be evaluated. We may as well assume that $M$ is connected. In addition, we suppose that we are given a triangulation of $M$ (see *fig.* 20); given such a triangulation, we will exhibit a recipe for computing the topological action.

Since $BG$ is connected, we can pick a base point $*$ in $BG$, and deform the map $\gamma : M \to BG$ so that every 0-simplex in $M$ is mapped to $*$ by $\gamma$. Now let $\sigma$ be a one simplex in $M$. Since the endpoints of $\sigma$ are mapped to the base point $*$, $\gamma(\sigma)$ is a path from $*$ to $*$ which determines an element of the fundamental group $\pi_1(BG)$. On the other hand, this group is isomorphic to $G$ (since $BG$ is the quotient of the contractible space $EG$ by the free action of $G$). Thus, to every one-simplex $\sigma$ in $M$, the map $\gamma$ determines a group element $g_\sigma \in G$. The assignment of group elements to one-simplices is reminiscent of lattice gauge theory. In this situation, however, the lattice field strength vanishes: if the three one-simplices $\sigma_1, \sigma_2$, and $\sigma_3$ bound a two-simplex, then the product $g_{\sigma_1} \cdot g_{\sigma_2} \cdot g_{\sigma_3}$ vanishes (since it represents an element of $\pi_1(BG)$, namely $\gamma(\sigma_1 \cup \sigma_2 \cup \sigma_3)$, which must vanish since $\sigma_1 \cup \sigma_2 \cup \sigma_3$ bounds a two-simplex or disc). This product is precisely the field strength in the sense of lattice gauge theory. Thus, in this lattice gauge theory model, one is limited to flat connections.

Now, if we really want to establish an analogy with lattice gauge theory, the topological action $\langle \gamma^* \alpha, [M] \rangle$ should depend only on the 'gauge field,' that is, on the $g_\sigma$, and not on other details of $\gamma$. In fact, if $\gamma' : M \to BG$ is some other map that determines the same $g_\sigma$'s as those determined by $\gamma$, then $\gamma$ and $\gamma'$ are homotopic to each other. To see this, one constructs a homotopy from $\gamma'$ to $\gamma$ on the $k$ skeleton of $M$, inductively in $k$. For $k = 1$, the existence of a homotopy from $\gamma$ to $\gamma'$ is precisely the statement that they determine the same $g_\sigma$'s. Once the homotopy from $\gamma$ to $\gamma'$ is established on the $k$ skeleton, the obstruction to extending it over the $k+1$ skeleton lies in $\pi_{k+1}(BG)$ (or more precisely in $H^{k+1}(M, \pi_{k+1}(BG))$), and vanishes since for $G$ a finite group, the homotopy groups $\pi_n(BG)$, for $n > 1$, all vanish.

Given that $\gamma$ and $\gamma'$ are homotopic, the cocycle condition on $\alpha \in H^3(BG, U(1))$ implies that for $M$ a manifold without boundary, the topological action is the same for $\gamma$ as for $\gamma'$. Thus, if $M$ has no boundary, the topological action depends only on the 'gauge field' $g_\sigma$.

### *8.5.4. Manifolds with Boundary — Gauge Theory Action*

It remains to understand the case in which $M$ has a boundary. If $M$ has a boundary, the topological action cannot be defined as $W = \langle \gamma^* \alpha, [M] \rangle$, because the fundamental class $[M]$ does not exist for a manifold with boundary. Also, we want a somewhat different formulation that will be concrete and closer to lattice gauge theory.

Given a three-simplex $T$ and a map $\gamma : T \to BG$, the cocycle $\alpha \in H^3(BG, U(1))$, by definition, assigns an element $W(T) \in U(1)$ to this data. From this point of view, a three-simplex $T$ is not just a tetrahedron; it is a tetrahedron with an ordering of the edges as $0, 1, 2, 3$. Roughly, we would like to regard $M$ as a union of three-simplices $M = \cup_i T_i$ and define the topological action as $\prod_i W(T_i)$. A chosen triangulation of $M$ gives a realization of $M$ as a union of tetrahedra. To give an ordering of the vertices in each of these tetrahedra, we order the vertices in $M$ as $1, 2, 3, \ldots, n$ (if there are $n$ 0-simplices in $M$), and then in each tetrahedron $T_i$, we order the vertices in ascending order.

In a given tetrahedron $T_i$, the ordered vertices appear in either a right-handed arrangement or a left-handed arrangement; this determines an orientation of $T_i$ which either agrees or disagrees with the orientation induced from that on $M$. Let us define an integer $\epsilon_i$ that is 1 or $-1$ depending on whether these orientations agree. Then if $M$ has no boundary, the fundamental class of $M$ can be defined as

$$[M] = \sum_i \epsilon_i T_i. \tag{8.69}$$

It follows from the definition of singular cohomology groups that the topological action, which we earlier defined as $W = \langle \gamma^* \alpha, [M] \rangle$, can equivalently be defined as the product over all individual simplices as

$$W = \prod_i W(T_i)^{\epsilon_i}. \tag{8.70}$$

This formula makes sense and is valid whether or not $M$ has a nonempty boundary.

In (8.70) we write the topological action as a product of terms that only depend on the maps to $BG$ of the individual tetrahedra $T_i$. This goes in the direction of a lattice gauge theory description, but we have not achieved such a description yet, since in general the $W(T_i)$'s do not depend only on the $g_\sigma$, as we wish, but on all of the details of the map $\gamma$.

To overcome this problem, we proceed as follows. For each choice of the $g_\sigma$'s (with vanishing curvature), we will describe how to pick a particular map $\gamma$ from $M$ to $BG$. By considering only these $\gamma$'s we will ensure that the $W(T_i)$'s depend only on the $g_\sigma$. The recipe for associating a particular map $\gamma$ with every collection of $g_\sigma$'s is very simple. As in a previous argument, we consider the $k$-skeleton of $M$ and work by induction in $k$. For every homotopy class of paths from $*$ to $*$ in $BG$, that is, for every element $g \in G$, we pick a particular path $u_g$, and we agree to use only these paths. This ensures that the map $\gamma$ on the 1-skeleton is uniquely determined by the $g_\sigma$'s. When we consider extending $\gamma$ over the 2-skeleton, we see that the map to $BG$ of a two-simplex $\Delta_2$ in $M$ is given by a triple $u_{g_1}, u_{g_2}$, and $u_{g_3}$ (with $g_1 g_2 g_3 = 1$). For each such triple we pick a particular map $v : \Delta_2 \to M$. Similarly, when it comes to the 3-skeleton, for each three-simplex $\Delta_3$, the map of its boundary to $BG$ consists of a certain collection of $v$'s, and for each such collection, we pick a particular map $w : \Delta_3 \to BG$. This completes the story for $M$ of dimension three, but otherwise the induction would obviously continue indefinitely.

At this point we have what we want: (8.70) is a formula for the topological action that depends only on the gauge theory data, and is similar to a lattice

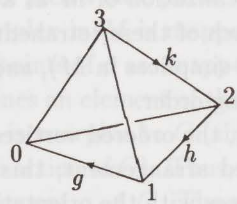



FIGURE 21: A 3-simplex $T$ with gauge fields $g, h, k$ has action $W(T) = \alpha(g,h,k)$.

gauge theory action in that the total action is a product of local terms, one for each three-simplex. The basic object is the value of the action $W(T)$ associated to a 3-simplex $T$. Once we have identified the vertices of the tetrahedron this action is a function of the three independent gauge fields $g$, $h$ and $k$ on the links, that we can choose as in *fig.* 21. We would now like to show that the action

$$W(T) = \alpha(g,h,k), \tag{8.71}$$

is a group cocycle in the algebraic sense, *i.e.* we would like to prove

$$\alpha(g,h,k)\,\alpha(g,hk,l)\,\alpha(h,k,l) = \alpha(gh,k,l)\,\alpha(g,h,kl). \tag{8.72}$$

This relation follows quite easily if we consider a 4-simplex with independent gauge fields $g, h, k, l$. Its boundary consists of 5 tetrahedra, and the above equality just expresses the general fact that the action of a boundary vanishes. Note that under a 'gauge' transformation $\alpha \to \alpha\,\delta\beta$, we have the transformation property

$$\alpha(g,h,k) \;\longrightarrow\; \alpha(g,h,k)\,\frac{\beta(g,hk)\,\beta(h,k)}{\beta(g,h)\,\beta(gh,k)} \tag{8.73}$$

Let us summarize our lattice construction. For a given 3-manifold $M$, possibly with boundary, we choose an arbitrary triangulation. (The definition (8.57) of the topological action makes it clear that the choice of the triangulation does not matter, though this is not completely obvious in the lattice construction.) We assign gauge fields to the links of the lattice, with the restriction that the curvature vanishes for all 2-simplices that occur in the triangulation. We will sum over all gauge field configurations, modulo gauge transformations that leave one point fixed. This leaves overall conjugation of the gauge fields as a physical degree of freedom. Since flat connections have only nontrivial holonomy around non-contractible loops, our gauge field configurations are labeled by homomorphisms of the fundamental group $\pi_1(M)$ into the gauge group $G$. With an arbitrary choice of ordering of the vertices, we associate to three-simplex $T_i$ the action $W(T_i)$ as in (8.71). The total action is simply the product over all elementary simplices (8.70).

We can now explicitly check some properties of the definition, which are clear on *a priori* grounds.

First, for a closed manifold, the value of $W$ does not depend on the choice of a cocycle used to represent $\alpha \in H^3(BG, U(1))$. Under a transformation $\alpha \to \alpha\,\delta\beta$ we will pick up terms that are defined on the 2-simplices. These are summed over twice, once in each orientation, and cancel. For example, the simplex depicted in *fig.* 21 would transform with (among other terms) a term $\beta(g,h)$ associated to the 2-simplex labeled by $0,1,2$. However, since the manifold is closed, there will be a neighboring 3-simplex of opposite orientation that will contribute $\beta(g,h)^{-1}$. So both terms cancel.

It is further not difficult to show that this expression is also invariant under further refinement of the lattice. It is sufficient to consider the barycentric subdivision of a 3-simplex $T$, since every two triangulations have a common subdivision in three dimensions. The barycentric subdivision will replace $T$ by 4 new simplices. It is again exactly due to the cocycle condition (8.72) that the sum of the actions of these 4 simplices equals $W(T)$.

Another important property is gauge invariance on closed manifolds. This is due to the fact that a gauge transformation $h_i \in G$ on a vertex $V_i$ of a simplex $T$ changes the weight $W(T)$ by terms that only depend on the gauge fields on the 2-simplices containing $V_i$. If every plaquette belongs to two 3-simplices, as is the case for a closed manifold, the terms cancel two by two. More precisely, a gauge transformation $c \in G$ acting on the vertex $V_0$ of the simplex of *fig.* 21, will transform the action $W(T)$ as

$$W(T) \to \frac{\alpha(c,g,h)\,\alpha(c,gh,k)}{\alpha(c,g,hk)} W(T). \tag{8.74}$$

Each of the three factors will be cancelled by the neighboring simplices.

We would like to close this section with one related remark. Depending on the divisibility of the order of the group it may be possible to choose the gauge

$$\alpha(g,g^{-1},h) = \alpha(g,h,h^{-1}) = 1. \tag{8.75}$$

In this gauge the above prescription becomes considerably simpler, since now the action $W(T)$ is invariant under chance of labeling of the vertices. One also has the convenient reality condition

$$\alpha(g,h,k)^{-1} = \alpha(k^{-1},h^{-1},g^{-1}). \tag{8.76}$$

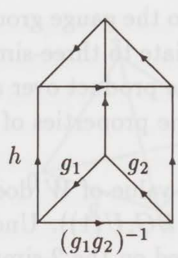



FIGURE 22: The manifold $Y \times S^1$, the direct product of the 2-sphere with three holes and the circle. The top and bottom and all vertices should be identified. The group elements indicate a homomorphism $\pi_1(Y \times S^1) \to G$.

### 8.5.5. *The Partition Function of the 3-Torus*

We can now compute several interesting quantities using triangulations. We will first reconsider the manifold $Y \times S^1$. We have seen that the Chern-Simons theory associates to each group element $h \in G$ a 2-cocycle $c_h$ of the stabilizer group $N_h$, and we would now like to express the cocycles $c_h$ in terms of the fundamental 3-cocycle $\alpha$. Once we have realized that a sphere with three holes can be represented by a 2-simplex with its three vertices identified, it is not difficult to imagine that $Y \times S^1$ can be represented as in *fig.* 22. Three simplices suffice to triangulate $Y \times S^1$, and the corresponding action is given by

$$c_h(g_1, g_2) = \frac{\alpha(h, g_1, g_2)\, \alpha(g_1, g_2, h)}{\alpha(g_1, h, g_2)}. \tag{8.77}$$

It can now be explicitly checked, using repeatedly the cocycle condition $\delta\alpha = 1$, that $c_h$ is indeed a 2-cocycle of the stabilizer subgroup $N_h$. Note that under (8.73) $c_h$ transforms as $c_h \to c_h\, \delta\beta_h$, with

$$\beta_h(g) = \beta(g, h)\, \beta(h, g)^{-1}. \tag{8.78}$$

Let us now move on to the partition function of the 3-torus $S^1 \times S^1 \times S^1$. It can be conveniently triangulated with 6 simplices. This can be easily seen when we represent the 3-torus as a cube with periodic boundaries as in *fig.* 23. If $g, h, k$ are the three commuting gauge fields on the edges of the cube, the partition function can be evaluated to give

$$Z(S^1 \times S^1 \times S^1) = \frac{1}{|G|} \sum_{\substack{g,h,k \in G \\ [g,h]=[h,k]=[k,g]=1}} W(g, h, k), \tag{8.79}$$

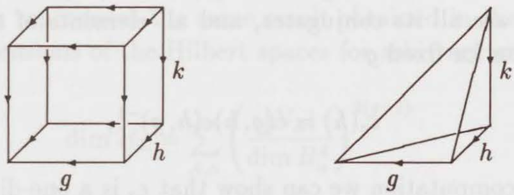



FIGURE 23: The 3-torus and one of the six 3-simplices that can be used to triangulate it.

with the action given by

$$W(g,h,k) = \frac{\alpha(g,h,k)\,\alpha(h,k,g)\,\alpha(k,g,h)}{\alpha(g,k,h)\,\alpha(h,g,k)\,\alpha(k,h,g)}. \tag{8.80}$$

We can now explicitly check some general properties of $W$. First we observe that it is indeed invariant under transformations $\alpha \to \alpha\,\delta\beta$, in particular $W = 1$ whenever $\alpha$ is cohomologically trivial. Furthermore, it can be verified that the above expression is inert under the mapping class group of the 3-torus $SL(3,\mathbf{Z})$.

The action can also be simply rewritten in terms of the 2-cocycles $c_g$. Since the three torus can be constructed out of two copies of the manifold $Y \times S^1$ we have

$$W(g,h,k) = c_g(h,k)\,c_g(k,h)^{-1}. \tag{8.81}$$

This is especially easily visualized with the aid of the triangulated manifolds of *fig.* 22 and *fig.* 23. We will use this observation to evaluate and interpret the partition function of the 3-torus.

For any finite group $G$ let the positive integer $r(G)$ denote the number of non-isomorphic irreducible representations. It is a familiar fact that $r(G)$ also equals the number of conjugacy classes. Similarly we can define for any 2-cocycle $c \in H^2(G,U(1))$ the number $r(G;c)$ of irreducible projective representations $R(g)$ that satisfy

$$R(g)R(h) = c(g,h)R(gh). \tag{8.82}$$

It can be shown that $r(G;c)$ is the rank of the center of the twisted group algebra and that it equals the number of so-called '$c$-regular' conjugacy classes [105], which implies

$$r(G;c) \leq r(G). \tag{8.83}$$

An element $g \in G$ is called $c$-regular if $c(g,h) = c(h,g)$ for all $h \in N_g$. If $g$ is $c$-regular then so are all its conjugates, and all elements of the form $g^n h^m$ with $h \in N_g$. If we write for fixed $g$

$$\epsilon_g(h) = c(g,h)c(h,g)^{-1}, \tag{8.84}$$

then by a simple computation we can show that $\epsilon_g$ is a one-dimensional representation of the stabilizer of $g$. An element $g$ is $c$-regular iff $\epsilon_g = 1$. This implies the following expression for $r(G;c)$

$$r(G;c) = \frac{1}{|G|} \sum_{\substack{g,h\in G \\ [g,h]=1}} c(g,h)c(h,g)^{-1}. \tag{8.85}$$

Here we used again the property that the summation $\sum_{k\in K} \epsilon(k)$ vanishes for any nontrivial one-dimensional representation $\epsilon(k)$ of a group $K$. Comparing with the expressions (8.79) and (8.81) for the partition function of the 3-torus we obtain

$$Z(S^1 \times S^1 \times S^1) = \sum_{g\in C} r(N_g; c_g), \tag{8.86}$$

where $C$ is a set of representatives of the conjugacy classes $C_A$ of $G$. So in particular we find that the partition function is an integer, in accordance with its interpretation as the dimension of the Hilbert space associated to the 2-torus $S^1 \times S^1$.

In this calculation of the Hilbert space for genus one, we recognize the general phenomenon that not all $G$-bundles give rise to quantum states in the theory. According to the result (8.86), only those bundles contribute for which the pair $(g,h)$ satisfies the condition that $h$ is $c_g$-regular (or vice versa, the condition is symmetric). So we can take the following basis in the Hilbert space. Let $R^g_\alpha$ be the irreducible, projective modules of the stabilizer group $N_g$ with cocycle $c_g$. Since the stabilizer subgroups $N_g$ are isomorphic for all $g$ in a conjugacy class $C_A$, we can denote these groups as $N_A$ and their representations by $R^A_\alpha$. The basis elements $v^A_\alpha$ can now be defined by

$$v^A_\alpha(g,h) = \begin{cases} \rho^g_\alpha(h) = \mathrm{Tr}\ R^g_\alpha(h) & \text{if } g \in C_A, \\ 0 & \text{otherwise.} \end{cases} \tag{8.87}$$

Note that the 'wave functions' $v^A_\alpha : \mathcal{V}_\Sigma \to \mathbf{C}$ indeed satisfy $v^A_\alpha(g,h) = 0$, if $h \in N_g$ is *not* a $c_g$-regular class. It can be verified that this basis is orthonormal. (Recall that the manifold $\Sigma \times I$ furnishes a natural inner product on the Hilbert space $\mathcal{H}_\Sigma$.)

We will now proceed to show that the analysis of this three-dimensional topological gauge theory, reproduces the same result obtained in the two-dimensional analysis for the dimensions of the Hilbert spaces for arbitrary genus, namely

$$\dim \mathcal{H}_\Sigma = \sum_{A,\alpha} \left( \frac{|N_A|}{\dim R_\alpha^A} \right)^{2(g-1)} . \tag{8.88}$$

The calculation is not difficult. We only have to check that, when expressed in the basis $v_\alpha^A$ in $\mathcal{H}_{\Sigma_1}$, the morphism $\Phi_{Y\times S^1}$ reproduces the fusion rules (5.77) that we derived in Chapter 5 for the holomorphic orbifold models. This is indeed true, since

$$\Phi_{Y\times S^1}(v_\alpha^A, v_\beta^B, v_\gamma^C) = \frac{1}{|G|} \sum_{\substack{g_1\in C_A, g_2\in C_B, g_3\in C_C, h\in G \\ g_1g_2g_3=1,\ [g_i,h]=1}} \rho_\alpha^{g_1}(h)\rho_\beta^{g_2}(h)\rho_\gamma^{g_3}(h)\, c_h(g_1,g_2), \tag{8.89}$$

which is completely identical to equation (5.77).

As special case occurs when all the cocycles $c_g$ are trivial. This is in particular true for abelian $G$, and implies that there exist phases $\epsilon_g(h)$ (defined up to a 1-cocycle) such that

$$c_g(h,k) = \epsilon_g(h)\epsilon_g(k)\epsilon_g(hk)^{-1}. \tag{8.90}$$

It is easy to check that the phases $\epsilon_g(h)$ satisfy

$$\epsilon_{g^{-1}}(h) = \epsilon_g(h^{-1}) = \epsilon_g(h)^{-1}. \tag{8.91}$$

Let us now introduce the quantities

$$\sigma(g|h) = \epsilon_g(h)\epsilon_h(g). \tag{8.92}$$

These objects are manifestly invariant under the transformation (8.73), and so are determined only by the cohomology class $\alpha \in H^3(BG, U(1))$ and are invariants of the theory. They equal the phases (5.52) that were used to describe the modular transformation properties of holomorphic orbifold models in Chapter 5. We now see that the equations we derived for $\sigma(g|h)$ are solved by giving a cohomology class $\alpha \in H^3(BG, U(1))$. For example we can compare our results with the one obtained in section 5.3.6 for the group $G = D_3 \cong S_3$, where the group of possible phases $\sigma(g|h)$ was calculated to be $\mathbf{Z}_3 \times \mathbf{Z}_2$. This result agrees with the three-dimensional calculation, since $H^3(BS_3, U(1)) = \mathbf{Z}_6 \cong \mathbf{Z}_3 \times \mathbf{Z}_2$.

## Appendix B — Group Cohomology

In order to be more or less self-contained, we will review in this appendix some essential ingredients of homology and algebraic topology that we will need in this chapter. We give a very brief review of the singular homology and cohomology theory of topological spaces, in particular of classifying spaces of compact Lie groups, and their relation to characteristic classes. A much more thorough treatment of the material in this section can of course be found in the mathematical literature, for instance in [20,21,117,113,24]; for an introduction to integer homology, aimed at physicists, that stresses the importance of torsion, see [67].

### *B.1. Singular Cohomology Theory*

We will first recall the definition of singular homology with integer coefficients. For any topological space $T$ we can introduce the groups of singular chains $C_k(T)$. A singular $k$-chain is essentially a map of a collection of $k$-dimensional simplices into the space $T$. The group operation is simply addition with integer coefficients. One further defines certain subgroups $B_k(T)$ and $Z_k(T)$ of $C_k(T)$. The 'boundaries' $B_k(T)$ and the 'cycles' $Z_k(T)$ consist respectively of chains $C$ that satisfy $C = \partial B$ and $\partial C = 0$, with $\partial$ the boundary operator. The homology groups are defined as the quotients $H_k(T, \mathbf{Z}) = Z_k(T)/B_k(T)$. Completely similarly, one can introduce the space of (integer) cochains

$$C^k(T, \mathbf{Z}) = \mathrm{Hom}(C_k(T), \mathbf{Z}), \tag{B.1}$$

and with the aid of the coboundary operator $\delta$, coboundaries, cocycles and cohomology groups $H^k(T, \mathbf{Z})$. Here the coboundary operator is defined by

$$\langle \delta\alpha, C\rangle = (-1)^k \langle \alpha, \partial C\rangle, \tag{B.2}$$

with $\langle \cdot, \cdot \rangle$ the pairing $C_k(T) \otimes C^k(T) \to \mathbf{Z}$. The cohomology groups $H^k(T, F)$ can be defined with coefficients in any abelian group $F$, by replacing $\mathbf{Z}$ by $F$ in the definition (B.1) of the cochains. In particular, with real coefficients we have $H^k(T, \mathbf{R}) = H^k(T, \mathbf{Z}) \otimes \mathbf{R}$. Due to the fundamental theorem of De Rham, these real cocycles can be represented by closed differential forms. We further recall that in the case that $F$ is a divisible group, so in particular for $F = \mathbf{R}$, we have another very simple definition of the cohomology groups:

$$H^k(T, F) = \mathrm{Hom}(H_k(T), F), \tag{B.3}$$

*i.e.* $\alpha \in H^k(T, F)$ is a homomorphism $Z_k(T) \to F$ that vanishes on boundaries.

An element of finite order of an abelian group is called a torsion element. The homology and cohomology groups of a topological space (with arbitrary coefficients) are abelian groups. The universal coefficient theorem gives an isomorphism (but not a completely natural one) between the torsion of $H_{k-1}(T,\mathbf{Z})$ and that of $H^k(T,\mathbf{Z})$. Torsion elements in $H^k(T,\mathbf{Z})$ cannot be represented in the usual fashion by differential forms, since torsion classes are elements of the kernel of the map $\rho : H^k(T,\mathbf{Z}) \to H^k(T,\mathbf{R})$. An Abelian group $A$ has a torsion subgroup Tor $A$, but there is no natural map from $A$ to Tor $A$. Given $\alpha \in H^k(T,\mathbf{Z})$, there is no natural way to identify a torsion part of $\alpha$ unless $\alpha$ is itself a torsion element of $H^k(T,\mathbf{Z})$. On the other hand, if one wishes to study $\alpha$ modulo torsion, this is naturally done by studying the image $\rho(\alpha)$ of $\alpha$ in $H^k(T,\mathbf{R})$.

### *B.2. Group Cohomology and Classifying Spaces*

In order to define the cohomology of a topological group $G$, we first have to introduce the concept of a *classifying space.* A classifying space $BG$ is the base space of a principal $G$-bundle $EG$, the so-called universal bundle, which has the following fundamental property: Any principal $G$-bundle $E$ over a manifold $M$ allows a bundle map into the universal bundle, and any two such morphisms are smoothly homotopic. We will write

$$\gamma : M \longrightarrow BG \tag{B.4}$$

for the induced map of the base manifolds, the so-called classifying map. The topology of the bundle $E$ is completely determined by the homotopy class of the classifying map $\gamma$. That is, the different components of the space $Map(M,BG)$ correspond to the different bundles $E$ over $M$. It can be shown that up to homotopy $BG$ is uniquely determined by requiring $EG$ to be contractible. That is, any contractible space with a free action of $G$ is a realization of $EG$. In general the classifying space $BG$ of a compact group is an infinite dimensional space as the simple examples $B\mathbf{Z}_2 = RP^\infty$, $BU(1) = CP^\infty$, and $BSU(2) = HP^\infty$ show. We notice that for our class of Lie groups, $BG$ will be a fibre bundle over $B\Gamma$ with fibre $BG_0$.

The *group cohomology* of a group—as opposed to its cohomology as topological space—can now be defined as the cohomology of the associated classifying space $BG$. Of course, the group cohomology and the ordinary cohomology of $G$ are intimately related, and one relation between them will be important in section (4). The elements in $H^*(BG,\mathbf{Z})$ are also called *universal characteristic classes*, since under the pullback $\gamma^*$ they give rise to cohomology classes in $H^*(M,\mathbf{Z})$ that depend only on the topology of the bundle $E$.

For a compact Lie group we have the very useful property, due to Borel, that with real coefficients all odd cohomology vanishes:

$$H^{odd}(BG,\mathbf{R}) = 0. \tag{B.5}$$

So the odd cohomology (and homology) consists completely of torsion. For *finite* groups an even stronger result holds: *all* cohomology is finite: $H^*(BG,\mathbf{R}) = 0$. With the use of the exact sequence $0 \to \mathbf{Z} \to \mathbf{R} \to \mathbf{R}/\mathbf{Z} \to 0$ this implies for finite $G$ the isomorphism

$$H^k(BG,\mathbf{Z}) \cong H^{k-1}(BG,\mathbf{R}/\mathbf{Z}). \tag{B.6}$$

For the even, real cohomology an important isomorphism exists due to Weil:

$$H^*(BG,\mathbf{R}) \cong I(G). \tag{B.7}$$

Here $I(G)$ is the ring of polynomials on $Lie(G)$ which are invariant under the adjoint action of $G$. The isomorphism is established using the Chern-Weil homomorphism that maps a polynomial $P \in I(G)$ to the class $[P(F)]$, where $F$ is the curvature of an arbitrary connection in the universal bundle. $P(F)$ is a closed differential form of degree $2k$ if the polynomial $P$ is of degree $k$. It is a fundamental result that the image $[P(F)]$ in $H^*(BG,\mathbf{R})$ is independent of the choice of connection. In this chapter we are mainly interested in the case $k=2$, where $P$ is an invariant quadratic form on $Lie(G)$, which we denote as Tr.

The group cohomology of the unitary groups $U(n)$ is perhaps the most familiar example. It contains no torsion, and is given by the polynomial ring in the Chern classes $c_k$ of degree $2k$

$$H^*(BU(n),\mathbf{Z}) = Pol[c_1,\ldots,c_n]. \tag{B.8}$$

As an example of a finite group, we can consider the cyclic group $\mathbf{Z}_n$. Again the cohomology ring is finitely generated. There is a single generator $x$ of order $n$ and degree 2, so that

$$H^{odd}(B\mathbf{Z}_n,\mathbf{Z}) = 0, \qquad H^{even}(B\mathbf{Z}_n,\mathbf{Z}) = \mathbf{Z}_n. \tag{B.9}$$

Finally, we recall that for discrete groups the cohomology groups $H^k(BG,F)$ have an algebraic description, that is perhaps more familiar to the reader. Cochains are represented as functions $\alpha : G^k \to F$, and, if we write the abelian group $F$ multiplicatively, the coboundary operator is defined as

$$\begin{aligned}\delta\alpha(g_1,\ldots,g_{k+1}) &= \alpha(g_1,\ldots,g_k)^{(-1)^{k+1}}\alpha(g_2,\ldots,g_{k+1}) \\ &\quad \times \prod_{i=1}^{k} \alpha(g_1,\ldots,g_ig_{i+1},\ldots,g_{k+1})^{(-1)^i}.\end{aligned} \tag{B.10}$$

These cochains can be assumed to be normalized, *i.e.* $\alpha(g_1,\ldots,g_k) = 1$ if $g_i = 1$ for some $i$. The equivalence between algebraic cocycles and simplicial cocycles of $BG$ is proved using Milnor's construction of $BG$ [116]. We give an elementary derivation of this result in section (8.5.4) where we treat Chern-Simons theory for finite groups.

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# Samenvatting

De beschrijving van de elementaire natuurkrachten in de huidige theoretische fysica rust op twee fundamenten: de quantumveldentheorie en de algemene relativiteitstheorie. Beide theorieën vonden hun oorsprong in het begin van deze eeuw en zijn sindsdien tot een grote mate van perfectie gebracht. Ze beschrijven en verklaren een groot aantal experimenteel waargenomen verschijnselen. De quantumveldentheorie vertelt ons hoe op microscopisch niveau de interakties plaatsvinden tussen de elementaire deeltjes, zoals die in deeltjesversnellers worden geproduceerd. Einsteins algemene relativiteitstheorie daarentegen beschrijft de gravitationele krachten, die zich vooral manifesteren tussen de macroscopische objecten in het heelal, zoals de sterren en sterrenstelsels. Het is niet alleen een grote uitdaging deze beide theorieën onder één noemer te brengen, maar ook een absolute noodzakelijkheid voor een beter begrip van vele interessante fysische vraagstukken. Overal waar zich zeer sterke gravitationele velden voordoen, zoals bijvoorbeeld in de beginfase van de evolutie van het heelal, maar ook nabij zwarte gaten, zal het gravitatieveld op een quantummechanisch verantwoorde wijze beschreven moeten worden. Een groot probleem hierbij is dat de energieschaal waarbij zulke processen plaatsvinden zo onwaarschijnlijk hoog is (een factor $10^{16}$ hoger dan op dit moment in versnellers gerealiseerd kan worden) dat een direkt experiment volslagen uitgesloten is. We zijn slechts aangewezen op gedachtenexperimenten, en op de interne consistentie van ons theoretisch model.

Tot nu toe zijn er slechts weinig concrete voorstellen voor zo'n theorie van quantumgravitatie. Stringtheorie is er één van, ook al werd het oorspronkelijk gezien als een model voor de sterke wisselwerking. Pas na een aantal jaren realiseerde men zich dat stringtheorie ook opgevat kan worden als een unificerende theorie van alle natuurkrachten, een theorie die in het bijzonder een consistente beschrijving geeft van gravitonen, de quanta van het zwaartekrachtsveld. Stringtheorie veronderstelt dat deeltjes niet puntvormig maar ééndimensionaal zijn, zoals een elastiekje. De string kent verschillende excitaties die zouden moeten corresponderen met de waargenomen en nog waar te nemen elementaire deeltjes. De afmetingen van een string zijn infinitesimaal klein (ongeveer $10^{-33}$ cm), zodat slechts op afstanden waar de quantumeigenschappen van de zwaartekracht belangrijk worden, het verschil tussen een string- en een conventionele veldentheorie zichtbaar wordt. Dit houdt echter in het bijzonder in dat we de veronderstelde 'string'-struktuur van elementaire deeltjes nooit rechtstreeks experimenteel zullen kunnen vaststellen.

Een string die zich voortbeweegt door een (mogelijk gekromde) ruimte beschrijft een tweedimensionaal oppervlak. Aangezien strings interakties kunnen aangaan door zich in tweeën te splitsen en zich weer samen te voegen, zal een interaktieproces tussen een aantal strings corresponderen met een oppervlak van een willekeurige topologie. Een quantummechanische beschrijving van deze processen blijkt overeen te komen met een tweedimensionale veldentheorie op dit oppervlak. Om deze beschrijving consistent te maken, moet deze veldentheorie ook schaalinvariant zijn. Schaalinvariante veldentheorieën worden ook wel conforme veldentheorieën genoemd en zijn het belangrijkste punt van onderzoek in dit proefschrift. Deze theorieën zijn het eerst onderzocht in het kader van tweedimensionale kritische verschijnselen die een tweede orde fase-overgang beschrijven. Precies op het kritieke punt verdwijnen alle natuurlijke lengtematen uit het systeem en wordt schaalinvariantie gerealiseerd. Er zijn een groot aantal conforme veldentheorieën bekend, en de bijzondere struktuur van stringtheorie maakt het mogelijk al deze theorieën op te vatten als grondtoestanden van de string. Het is echter van belang dat niet al deze modellen een directe meetkundige interpretatie hebben in termen van een string die zich door een bepaalde ruimte voortbeweegt.

In dit proefschrift wordt de struktuur van conforme veldentheorieën onderzocht vanuit verschillende (geometrische) gezichtspunten. Hoewel dat in dit proefschrift niet echt aan bod komt, is het doel dat ons daarbij voor ogen staat een beter begrip van stringtheorie te verkrijgen, ook al moeten we ons altijd realiseren dat deze inzichten van storingstheoretische aard zullen zijn. De hoofdstukken 2, 3 en in mindere mate 4 hebben een zekere inleidende funktie. In hoofdstuk 2 worden de grondbeginselen van conforme veldentheorie kort uiteengezet en toegelicht met behulp van fermionische en bosonische vrije veldentheorieën. In hoofdstuk 3 wordt de meetkundige struktuur van meer algemene veldentheorieën onderzocht vanuit een abstract, mathematisch gezichtspunt. Deze benadering is de laatste jaren zeer vruchtbaar gebleken wat betreft toepassingen in de wiskunde, en geeft ons ook een nieuwe kijk op wat een quantumveldentheorie is.

Indien een veldentheorie een bepaalde symmetriegroep bezit, kunnen we het spektrum beschrijven in termen van representaties, en de interakties door invariante koppelingen. In twee dimensies kan het voorkomen dat er maar een eindig aantal representaties zijn, en dan wordt de struktuur een stuk doorzichtiger. We spreken in dat geval van *rationale* theorieën. Deze modellen zijn zeer rigide, en voor een groot aantal grootheden kunnen exacte algebraïsche vergelijkingen bepaald worden. Er is recentelijk veel vooruitgang geboekt in de classificatie van deze speciale conforme veldentheorieën. In hoofdstuk 4 beschrijven we het algemene formalisme voor rationale modellen. Onze belangrijkste conclusie is dat de symmetrie algebra de operatorinhoud geheel vastlegt, op enkele discrete keuzes na. In vele opzichten lijken deze rationale modellen op eindige groepen. Dat dit meer is dan zomaar een analogie, laten we zien in hoofdstuk 5. Uitgaande van eindige groepen construeren we conforme veldentheorieën op zogenaamde 'orbifolds.' De interaktieregels en conforme dimensies kunnen geheel op grond van de struktuur

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van de eindige groep bepaald worden.

Voor iedere conforme veldentheorie is een grootheid te definiëren, de centrale lading $c$, die in zekere zin de dimensie van de ruimte-tijd beschrijft waarin de string zich voortbeweegt, ook al hoeft $c$ in het algemeen geen geheel getal te zijn. Modellen met $0 < c < 1$ zijn volledig geklassificeerd uitgaande van de eisen van unitariteit en modulaire invariantie. Dit ligt echter geheel anders voor $c \geq 1$, waar weinig resultaten bekend zijn. In hoofdstuk 6 doen we een kleine stap in de goede richting en classificeren we een groot gedeelte van de $c = 1$ modellen met behulp van zogenaamde marginale deformaties. De storingstheorie van stringcompactificaties gebaseerd op deze modellen wordt beschreven in hoofdstuk 7. Hiertoe moeten we de formulering op algemene oppervlakken bekijken. Dit is ook van belang voor de meer abstrakte, meetkundige benadering van conforme veldentheorie, die als een rode draad door dit proefschrift loopt. In helaas nogal technische berekeningen bepalen we de correlatiefunkties van operatoren op oppervlakken van willekeurig genus. Naast de padintegraalmethode behandelen we daartoe ook het zogenaamde operatorformalisme. In deze elegante benadering wordt de geometrie van een willekeurig oppervlak als het ware gesimuleerd door geschikte randcondities op het platte vlak te kiezen.

De representatietheorie van rationale conforme veldentheorieën kunnen we ook terugvinden in driedimensionale topologische ijktheorieën, en dit vormt het onderwerp van het laatste hoofdstuk. In het kader van een algemene classificatie is het van belang deze zogenaamde Chern-Simons theorieën te kunnen formuleren voor arbitraire compacte Lie groepen, niet noodzakelijkerwijs (enkelvoudig) samenhangend. Dit probleem wordt in hoofdstuk 8 opgelost. In het bijzondere geval van een eindige groep kan het resultaat geformuleerd worden in termen van een roosterijktheorie, en maken we contact met de resultaten van hoofdstuk 5.

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# Curriculum Vitae

De auteur werd op 24 januari 1960 te Ridderkerk geboren, en doorliep van 1972 tot 1978 het Erasmiaans Gymnasium te Rotterdam. Hierna werd de studie natuurkunde in Utrecht aangevangen. Na het candidaatsexamen (1982, cumlaude) werd deze studie voor twee jaar onderbroken voor een studie schilderen aan de Rietvelt Academie te Amsterdam. Na in 1984 weer op het 'rechte pad' te zijn teruggekeerd werd in 1986 het doctoraal theoretische fysica (cum laude) behaald, waarna de auteur het hier afgeronde promotieonderzoek begon aan het Instituut voor Theoretische Fysica. Na deze promotie wacht een postdoctorale positie aan Princeton University.