Connections up to homotopy and characteristic classes *

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Introduction

The aim of this note is to clarify the relevance of "connections up to homotopy" [4, 5] to the theory of characteristic classes, and to present an application to the characteristic classes of algebroids [3, 6] (and of Poisson manifolds in particular [7]).

We have already remarked [4] that such connections up to homotopy can be used to compute the classical Chern characters. Here we present a slightly different argument for this, and then proceed with the discussion of the flat characteristic classes. In contrast with [4], we do not only recover the classical characteristic classes (of flat vector bundles), but we also obtain new ones. The reason for this is that ($\mathbb{Z}_2$-graded) non-flat vector bundles may have flat connections up to homotopy. As we shell explain here, in this category fall e.g. the characteristic classes of Poisson manifolds [7].

As already mentioned in [4], one of our motivations is to understand the intrinsic characteristic classes for Poisson manifolds (and algebroids) of [6, 7], and the connection with the characteristic classes of representations [3]. Conjecturally, Fernandes' intrinsic characteristic classes [6] are the characteristic classes [3] of the "adjoint representation". The problem is that the adjoint representation is a "representation up to homotopy" only. Applied to algebroids, our construction immediately solves this problem: it extends the characteristic classes of [3] from representations to representations up to homotopy, and shows that the intrinsic characteristic classes [6, 7] are indeed the ones associated to the adjoint representation [5].

Nonlinear connections

Here we recall some well-known properties of connections on vector bundles. Up to a very slight novelty (we allow non-linear connections), this section is standard [10] and serves to fix the notations.

Let M be a manifold, and let $E = E^0 \oplus E^1$ be a super-vector bundle over M. A *non-linear connection* on E is a linear operator

$$\mathcal{X}(M) \times \Gamma E \longrightarrow \Gamma E, \quad (X, s) \mapsto \nabla_X(s) \tag{1}$$

with the property that the operators ∇_X preserve the degree, are local in X, and satisfy

$$\nabla_X(fs) = f\nabla_X(s) + X(f)s$$

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for all $X \in \mathcal{X}(M)$, $s \in \Gamma E$, and $f \in C^{\infty}(M)$. The curvature k_{∇} of ∇ is defined by the usual formula

$$k_{\nabla}(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} : \Gamma E \longrightarrow \Gamma E \ . \tag{2}$$

A *non-linear differential form* on M is an antisymmetric ($\mathbb{R}$-multilinear) map

$$\omega : \underbrace{\mathcal{X}(M) \times \ldots \times \mathcal{X}(M)}_{n} \longrightarrow C^{\infty}(M) \tag{3}$$

which is local in the X_i's. It is easy to see (and it has been already remarked in [4]) that many of the usual operations on differential forms do not use the $C^{\infty}(M)$-linearity, hence they apply to non-linear forms as well. In particular we obtain the algebra $(\mathcal{A}_{\rm nl}(M), d)$ of non-linear forms endowed with De Rham operator. (This defines a contravariant functor from manifolds to dga's.) Considering ΓE-valued operators instead, we obtain a version with coefficients, denoted $\mathcal{A}_{\rm nl}(M;E)$. Note that a non-linear connection ∇ can be viewed as an operator $\mathcal{A}^0_{\rm nl}(M;E) \longrightarrow \mathcal{A}^1_{\rm nl}(M;E)$ which has a unique extension to an operator

$$d_{\nabla} : \mathcal{A}^{*}_{\rm nl}(M;E) \longrightarrow \mathcal{A}^{*+1}_{\rm nl}(M;E)$$

satisfying the Leibniz rule. Explicitly,

$$\begin{aligned} d_{\nabla}(\omega)(X_1, \ldots, X_{n+1}) &= \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots X_{n+1})) \\ &+ \sum_{i=1}^{n+1} (-1)^{i+1} \nabla_{X_i} \omega(X_1, \ldots, \hat{X}_i, \ldots, X_{n+1}). \end{aligned} \tag{4}$$

We now recall the definition of the (non-linear) connection on $\mathrm{End}(E)$ induced by ∇. For any $T \in \Gamma\mathrm{End}(E)$, the operators $[\nabla_X, T]$ acting on $\Gamma(E)$ are $C^{\infty}(M)$-linear, hence define elements $[\nabla_X, T] \in \Gamma\mathrm{End}(E)$. The desired connection is then $\nabla_X(T) = [\nabla_X, T]$. Clearly $k_{\nabla} \in \mathcal{A}^2_{\rm nl}(M; \mathrm{End}(E))$, and one has Bianchi's identity $d_{\nabla}(k_{\nabla}) = 0$.

We will use the algebra $\mathcal{A}_{\rm nl}(M; \mathrm{End}(E))$ and its action on $\mathcal{A}_{\rm nl}(M;E)$. The product structure that we consider here is the one which arises from the natural isomorphisms

$$\mathcal{A}_{\rm nl}(M;E) \cong \mathcal{A}_{\rm nl}(M) \otimes_{C^{\infty}(M)} \Gamma(E)$$

and the usual sign conventions for the tensor products (i.e. $\omega \otimes x \cdot \eta \otimes y = (-1)^{|x||\eta|} \omega\eta \otimes xy$). The usual super-trace on $\mathrm{End}(E)$ induces a super-trace

$$Tr_s : (\mathcal{A}_{\rm nl}(M; \mathrm{End}(E)), d_{\nabla}) \longrightarrow (\mathcal{A}_{\rm nl}(M), d) \tag{5}$$

with the property that $Tr_s d_{\nabla} = d Tr_s$. We conclude (and this is just a non-linear version of the standard construction of Chern characters [10]):

Lemma 1 *If ∇ is a non-linear connection on E, then*

$$ch_p(\nabla) = Tr_s(k^p_{\nabla}) \in \mathcal{A}^{2p}_{nl}(M) \tag{6}$$

are closed non-linear forms on M.

Up to a boundary, these classes are independent of ∇. This is an instance of the Chern-Simons construction that we now recall. Given $k+1$ non-linear connections ∇_i on E $(0 \leq i \leq k)$ we form their affine combination $\nabla^{\mathrm{aff}} = (1-t_1-\ldots-t_k)\nabla_0+t_1\nabla_1+\ldots+t_k\nabla_k$. This is a non-linear connection on the pullback of E to $\Delta^k \times M$, where $\Delta^k = \{(t_1, \ldots, t_k) : t_i \geq 0, \sum t_i \leq 1\}$ is the standard k-simplex. The classical integration along fibers has a non-linear extension

$$\int_{\Delta^k} : \mathcal{A}^*_{\mathrm{nl}}(M \times \Delta^k) \longrightarrow \mathcal{A}^{*-k}_{\mathrm{nl}}(M) \tag{7}$$

given by the explicit formula

$$\left(\int_{\Delta^k} \omega\right)(X_1, \ldots, X_{n-k}) = \int_{\Delta^k} \omega\left(\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_k}, X_1, \ldots, X_{n-k}\right) dt_1 \ldots dt_k \ .$$

We then define

$$cs_p(\nabla_0, \ldots, \nabla_k) = \int_{\Delta^k} ch_p(\nabla^{\mathrm{aff}}) \ . \tag{8}$$

Using a version of Stokes' formula [2] (or integrating by parts repeatedly) we conclude

Lemma 2 *The elements (8) satisfy*

$$dcs_p(\nabla_0, \ldots, \nabla_k) = \sum_{i=0}^{k} (-1)^i cs_p(\nabla_0, \ldots, \widehat{\nabla_i}, \ldots, \nabla_k) \ . \tag{9}$$

Connections up to homotopy and Chern characters

From now on, (E, ∂) is a super-complex of vector bundles over the manifold M,

$$(E, \partial) : \quad E^0 \underset{\partial}{\overset{\partial}{\rightleftarrows}} E^1 \quad . \tag{10}$$

We now consider non-linear connections ∇ on E such that $\nabla_X \partial = \partial \nabla_X$ for all $X \in \mathcal{X}(M)$. We say that ∇ is a (linear) *connection* on (E, ∂) if it also satisfies the identity $\nabla_{fX}(s) = f\nabla_X(s)$ for all $X \in \mathcal{X}(M)$, $f \in C^\infty(M)$, $s \in \Gamma E$. The notion of *connection up to homotopy* [4, 5] on (E, ∂) is obtained by relaxing the $C^\infty(M)$-linearity on X to linearity up to homotopy. In other words we require

$$\nabla_{fX}(s) = f\nabla_X(s) + [H_\nabla(f, X), \partial] \ ,$$

where $H_\nabla(f, X) \in \Gamma\mathrm{End}(E)$ are odd elements which are linear and local in X and f.

We say that two non-linear connections ∇ and ∇' are *equivalent* (or homotopic) if

$$\nabla'_X = \nabla_X + [\theta(X), \partial]$$

for all $X \in \mathcal{X}(M)$, for some $\theta \in \mathcal{A}^1_{\mathrm{nl}}(M; \mathrm{End}(E))$ of even degree. We write $\nabla \sim \nabla'$.

Lemma 3 *A non-linear connection is a connection up to homotopy if and only if it is equivalent to a (linear) connection.*

Proof: Assume that ∇ is a connection up to homotopy. Let U_a be the domain of local coordinates x^k for M, and put

$$\nabla^a_X = \nabla_X + [u^a(X), \partial] \ ,$$

where $u_a \in \mathcal{A}_{\rm nl}(U_a; {\rm End}(E))$ is given by

$$u_a(\sum_k f_k \frac{\partial}{\partial x_k}) = -\sum_k H_\nabla(f_k, \frac{\partial}{\partial x_k}) \ ,$$

for all $f_k \in C^\infty(U_a)$. Note that ∇_X is linear on X. Indeed, for any two smooth functions f, g and $X = g\frac{\partial}{\partial x_k}$ we have

$$\begin{aligned}\nabla^a_{fX} - f\nabla^a_X &= (\nabla_{fX} + [u^a(fX), \partial]) - f(\nabla_X + [u^a(X), \partial]) = \\ &= (\nabla_{fg\frac{\partial}{\partial x_k}} - [H_\nabla(fg, \tfrac{\partial}{\partial x_k}), \partial]) + f(\nabla_{g\frac{\partial}{\partial x_k}} - [H_\nabla(g\tfrac{\partial}{\partial x_k}), \partial]) = \\ &= fg\nabla_{\frac{\partial}{\partial x_k}} - fg\nabla_{\frac{\partial}{\partial x_k}} = 0 \ .\end{aligned}$$

Next we take $\{\nu_a\}$ to be a partition of unity subordinate to an open cover $\{U_a\}$ by such coordinate domains and put $\nabla'_X = \sum_a \nu_a \nabla^a_X$, $u(X) = \sum_a \nu_a u^a(X)$. Then $\nabla' = \nabla + [u, \partial]$ is a connection equivalent to ∇. $\square$

Lemma 4 *If ∇_0 and ∇_1 are equivalent, then $ch_p(\nabla^0) = ch_p(\nabla^1)$.*

Proof: So, let us assume that $\nabla^1 = \nabla^0 + [\theta, \partial]$. A simple computation shows that

$$k_{\nabla_1} = k_{\nabla_0} + [d_\nabla(\theta) + R, \partial] \ , \tag{11}$$

where $R(X, Y) = [\theta(X), [\theta(Y), \partial]]$. We denote by $Z \subset \mathcal{A}_{\rm nl}(M; {\rm End}(E))$ the space of non-linear forms ω with the property that $[\omega, \partial] = 0$, and by $B \subset Z$ the subspace of element of type $[\eta, \partial]$ for some non-linear form η. The formula

$$[\partial, \omega\eta] = [\partial, \omega]\eta + (-1)^{|\omega|}\omega[\partial, \eta]$$

shows that $ZB \subset B$, hence (11) implies that $k^p_{\nabla_1} \equiv k^p_{\nabla_0}$ modulo B. The desired equality follows now from the fact that Tr_s vanishes on B. $\square$

For (linear) connections ∇ on (E, ∂), $ch_p(\nabla)$ are clearly (linear) differential forms on M whose cohomology classes are (up to a constant) the components of the Chern character $Ch(E) = Ch(E^0) - Ch(E^1)$. Hence an immediate consequence of the previous two lemmas is the following [4]

Theorem 1 *If ∇ is a connection up to homotopy on (E, ∂), then $ch_p(\nabla) = Tr_s(k^p_\nabla)$ are closed differential forms on M whose De Rham cohomology classes are (up to a constant) the components of the Chern character $Ch(E)$.*

Flat characteristic classes

As usual, by flatness we mean the vanishing of the curvature forms. Theorem 1 immediately implies

Corollary 1 *If (E, ∂) admits a connection up to homotopy which is flat, then $Ch(E) = 0$.*

As usual, such a vanishing result is at the origin of new "secondary" characteristic classes. Let ∇ be a flat connection up to homotopy. To construct the associated secondary classes we need a metric h on E. We denote by ∂^h be the adjoint of ∂ with respect to h. Using the isomorphism $E^* \cong E$ induced by h (which is anti-linear if E is complex), ∇ induces an adjoint connection ∇^h on (E, ∂^h). Explicitely,

$$L_X h(s,t) = h(\nabla_X(s), t) + h(s, \nabla^h_X(t)) \ .$$

The following describes various possible definitions of the secondary classes, as well as their main properties (note that the role of $i = \sqrt{-1}$ below is to ensure real classes).

Theorem 2 *Let ∇ be a flat connection up to homotopy on (E, ∂), $p \geq 1$.*

(i) For any (linear) connection ∇_0 on (E, ∂) and any metric h,

$$i^{p+1}(cs_p(\nabla, \nabla_0) + cs_p(\nabla_0, \nabla^h_0) + cs_p(\nabla^h_0, \nabla^h)) \in \mathcal{A}^{2p-1}_{\mathrm{nl}}(M) \tag{12}$$

are differential forms on M which are real and closed. The induced cohomology classes do not depend on the choice of h or ∇_0, and are denoted $u_{2p-1}(E, \partial, \nabla) \in H^{2p-1}(M)$.

(ii) For any connection ∇_0 equivalent to ∇, and any metric h,

$$i^{p+1} cs_p(\nabla_0, \nabla^h_0) \in \mathcal{A}^{2p-1}(M) \tag{13}$$

are real and closed, and represent $u_{2p-1}(E, \partial, \nabla)$ in cohomology.

(iii) If ∇ is equivalent to a metric connection (i.e. a connection which is compatible with a metric), then all the classes $u_{2p-1}(E, \partial, \nabla)$ vanish.

(iv) If $\nabla \sim \nabla'$, then $u_{2p-1}(E, \partial, \nabla) = u_{2p-1}(E, \partial, \nabla')$.

(v) If ∇ is a flat connection up to homotopy on both super-complexes (E, ∂) and (E, ∂'), then $u_{2p-1}(E, \partial, \nabla) = u_{2p-1}(E, \partial', \nabla)$.

(vi) Assume that E is real. If p is even then $u_{2p-1}(E, \partial, \nabla) = 0$. If p is odd, then for any connection ∇_0 equivalent to ∇, and any metric connection ∇_m,

$$(-1)^{\frac{p+1}{2}} cs_p(\nabla_0, \nabla_m) \in \mathcal{A}^{2p-1}(M)$$

are closed differential forms whose cohomology classes equal to $\frac{1}{2} u_{2p-1}(E, \partial, \nabla)$.

Note the compatibility with the classical flat characteristic classes, which correspond to the case where E is a graded vector bundle (and $\partial = 0$), or, more classically, just a vector bundle over M. As references for this we point out [8] (for the approach in terms of frame bundles and Lie algebra cohomology), and [1] (for an explicit approach which we follow here). For the proof of the theorem we need the following

Lemma 5 *Given the non-linear connections ∇, ∇_0, ∇_1,*

(i) If ∇_0 and ∇_1 are connections up to homotopy then $cs_p(\nabla_0, \nabla_1)$ are differential forms;

(ii) If $\nabla_0 \sim \nabla_1$, then $cs_p(\nabla_0, \nabla_1) = 0$;

(iii) For any metric h, $ch_p(\nabla^h) = (-1)^p \overline{ch_p(\nabla)}$ and $cs_p(\nabla_0^h, \nabla_1^h) = (-1)^p \overline{cs_p(\nabla_0, \nabla_1)}$.

Proof: (i) follows from the fact that Chern characters of connections up to homotopy are differential forms. For (ii) we use Lemma 4. The affine combination ∇ used in the definition of $cs_p(\nabla_0, \nabla_1)$ is equivalent to the pull-back $\tilde{\nabla}_0$ of ∇_0 to $M \times \Delta^1$ (because $\nabla = \tilde{\nabla}_0 + t[\theta, \partial]$), while $ch_p(\tilde{\nabla}_0)$ is clearly zero. If h is a metric on E, a simple computation shows that $k_{\nabla^h}(X, Y)$ coincides with $-k_{\nabla}(X, Y)^*$ where $*$ denotes the adjoint (with respect to h). Then (iii) follows from $Tr(A^*) = \overline{Tr(A)}$ for any matrix A. $\square$

Proof of Theorem 2: (i) Let us denote by $u(\nabla, \nabla_0, h)$ the forms (12). Since (∇_0, ∇_0^h) is a pair of connections on E, and (∇, ∇_0), (∇^h, ∇_0^h) are pairs of connections up to homotopy on (E, ∂) and (E, ∂^h), respectively, it follows from (i) of Lemma 5 that $u(\nabla, \nabla_0, h)$ are differential forms. From Stokes formula (9) it immediately follows that they are closed. To prove that they are real we use (iii) of the previous Lemma:

$$\begin{aligned} \overline{u(\nabla, \nabla_0, h)} &= (-i)^{p+1}(\overline{cs_p(\nabla, \nabla_0)} + \overline{cs_p(\nabla_0, \nabla_0^h)} + \overline{cs_p(\nabla_0^h, \nabla^h)}) = \\ &+(-i)^{p+1}(-1)^p cs_p(\nabla^h, \nabla_0^h) + cs_p(\nabla_0^h, \nabla_0) + cs_p(\nabla_0, \nabla)) = \\ &= (-i)^{p+1}(-1)^p(-1)u(\nabla, \nabla_0, h) = u(\nabla, \nabla_0, h) \end{aligned}$$

If ∇_1 is another connection, using (9) again, it follows that $u(\nabla, \nabla_0, h) - u(\nabla, \nabla_1, h) = i^{p+1} dv$ where v is the (linear!) differential form

$$v = cs_p(\nabla, \nabla_0, \nabla_1) - cs_p(\nabla^h, \nabla_0^h, \nabla_1^h) + cs_p(\nabla_0, \nabla_0^h, \nabla_1) - cs_p(\nabla_0^h, \nabla_1, \nabla_1^h) \ .$$

(iii) clearly follows from (ii), which in turn follows from (ii) of Lemma 5 and the fact that $\nabla \sim \nabla_0$ implies $\nabla^h \sim \nabla_0^h$. To see that our classes do not depend on h, it suffices to show that given a linear connection ∇ on a vector bundle F, $cs_p(\nabla, \nabla^h)$ is independent of h up to the boundary of a differential form. Let h_0 and h_1 be two metrics. Although the proof below works for general ∇'s, simpler formulas are possible when ∇ is flat. So, let us first assume that (actually we may assume that ∇ is the canonical connection on a trivial vector bundle). From Stokes' formula (9) applied to $(\nabla, \nabla^{h_0}, \nabla^{h_1})$, it suffices to show that $cs_p(\nabla^{h_0}, \nabla^{h_1})$ is a closed form. We choose a family h_t of metrics joining h_0 and h_1. One only has to show that $\frac{\partial}{\partial t} cs_p(\nabla^{h_0}, \nabla^{h_t})$ are closed forms. Writing $h_t(x, y) = h_0(u_t(x), y)$, these Chern-Simons forms are, up to a constant, $Tr(\omega_t^{2p-1})$ where

$$\omega_t = \nabla^{h_t} - \nabla^{h_0} = u_t^{-1} d_{\nabla^{h_0}}(u_t)$$

(here is where we use the flatness of ∇). A simple computation shows that

$$\frac{\partial \omega_t}{\partial t} = d_{\nabla^{h_0}}(v_t) + [\omega_t, v_t] \ ,$$

where $v_t = u_t^{-1} \frac{\partial u_t}{\partial t}$. Since $d_{\nabla^{h_0}}(\omega_t^2) = 0$, this implies

$$\frac{\partial \omega_t}{\partial t} \omega_t^{2p-2} = d_{\nabla^{h_0}}(v_t \omega_t^{2p-2}) + [\omega_t, v_t \omega_t^{2p-2}] \ .$$

Now, by the properties of the trace it follows that

$$\frac{\partial}{\partial t} Tr_s(\omega_t^{2p-1}) = dTr_s(v_t \omega_t^{2p-2})$$

as desired. Assume now that ∇ is not flat. We choose a vector bundle F' together with a connection ∇' compatible with a metric h', such that $\tilde{F} = F \oplus F'$ admits a flat connection ∇_0. We put $\tilde{\nabla} = \nabla \oplus \nabla'$ and, for any metric h on F, we consider the metric $\tilde{h} = h \oplus h'$ on $\tilde{F}$. Clearly $cs_p(\tilde{\nabla}, \tilde{\nabla}^{\tilde{h}}) = cs_p(\nabla, \nabla^h)$. Using also (iii) of Lemma 5 and Stokes' formula, we have:

$$\begin{aligned} cs_p(\nabla, \nabla^h) &= cs_p(\nabla_0, \nabla_0^{\tilde{h}}) - cs_p(\nabla_0, \tilde{\nabla}) + (-1)^p \overline{cs_p(\nabla_0, \tilde{\nabla})} \\ &+ d(cs_p(\nabla_0, \tilde{\nabla}, \tilde{\nabla}^{\tilde{h}}) - cs_p(\nabla_0, \tilde{\nabla}_0, \tilde{\nabla}^{\tilde{h}})). \end{aligned}$$

Hence, by the flat case, $cs_p(\nabla, \nabla^h)$ modulo exact forms does not depend on h.
For (iv) one uses Stokes' formula (9) and (ii) of Lemma 5 to conclude that $cs_p(\nabla', \nabla_0) - cs_p(\nabla, \nabla_0)$ is the differential of the linear form $cs_p(\nabla, \nabla', \nabla_0)$. To prove (v) we only have to show (see (i)) that there exists a linear connection ∇^0 on E which is compatible with both ∂ and ∂'. For this, one defines ∇^0 locally by $\nabla^0_{f\frac{\partial}{\partial x_k}} = f\nabla_{\frac{\partial}{\partial x_k}}$, and then use a partition of unity argument.
We now assume that E is real. From Lemma 5,

$$cs_p(\nabla_m, \nabla_0^h) = (-1)^p cs_p(\nabla_m^h, \nabla_0) = (-1)^{p+1} cs_p(\nabla_0, \nabla_m) \ .$$

Combined with Stokes' formula (9), this implies

$$dcs_p(\nabla_0, \nabla_m, \nabla_0^h) = (1 + (-1)^{p+1}) cs_p(\nabla_0, \nabla_m) - cs_p(\nabla_0, \nabla_0^h) \ ,$$

which proves (vi). □

Note that the construction of the flat characteristic classes presented here actually works for ∇'s which are "flat up to homotopy", i.e. whose curvatures are of type $[-, \partial]$. Moreover, this notion is stable under equivalence, and the flat characteristic classes only depend on the equivalence class of ∇ (cf. (iv) of the Theorem). Note also that, as in [4] (and following [1]), there is a version of our discussion for super-connections [10] up to homotopy. Some of our constructions can then be interpreted in terms of the super-connection $\partial + \nabla$.

If E is regular in the sense that $Ker(\partial)$ and $Im(\partial)$ are vector bundles, then so is the cohomology $H(E, \partial) = Ker(\partial)/Im(\partial)$, and any connection up to homotopy ∇ on (E, ∂) defines a linear connection $H(\nabla)$ on $H(E)$. Moreover, $H(\nabla)$ is flat if ∇ is, and the characteristic classes $u_{2p-1}(E, \partial, \nabla)$ coincide with the classical [1, 8] characteristic classes of the flat vector bundle $H(E, \partial)$. In general, the $u_{2p-1}(E, \partial, \nabla)$'s should be viewed as invariants of $H(E, \partial)$ constructed in such a way that no regularity assumption is required. Let us discuss here an instance of this. We say that E is $\mathbb{Z}$-graded if it comes from a cochain complex

$$0 \longrightarrow E(0) \stackrel{\partial}{\longrightarrow} E(1) \stackrel{\partial}{\longrightarrow} \ldots \stackrel{\partial}{\longrightarrow} E(n) \longrightarrow 0 \ , \tag{14}$$

In other words, it must be of type $E = \oplus_{k=0}^n E(k)$ with the even/odd $\mathbb{Z}_2$-grading, and with $\partial(E(k)) \subset E(k+1)$.

Proposition 1

(i) If (E, ∂) is acyclic, then any two connections up to homotopy on (E, ∂) are equivalent. Moreover, if E is $\mathbb{Z}$-graded, then $u_{2p-1}(E, \partial, \nabla) = 0$.

(ii) If $(E^k, \partial^k, \nabla^k)$ are $\mathbb{Z}$-graded complexes of vector bundles endowed with flat connections up to homotopy which fit into an exact sequence

$$0 \longrightarrow E^0 \xrightarrow{\delta} E^1 \xrightarrow{\delta} \dots \xrightarrow{\delta} E^n \longrightarrow 0 \tag{15}$$

compatible with the structures (i.e. $[\delta, \partial] = [\delta, \nabla] = [\delta, H_\nabla] = 0$), then

$$\sum_{k=0}^{n} (-1)^k u_{2p-1}(E^k, \partial^k, \nabla^k) = 0 \ .$$

Proof: The second part follows from (i) above and (v) of Theorem 2. To see this, we form the super-vector bundle $E = \oplus_k E^k$ (which is $\mathbb{Z}$-graded by the total degree) and the direct sum (non-linear) connection ∇ acting on E. Then ∇ is a connection up to homotopy in both (E, ∂) and $(E, \partial + \delta)$. Clearly $u_{2p-1}(E, \partial, \nabla) = \sum_{k=0}^{n} (-1)^k u_{2p-1}(E^k, \partial^k, \nabla^k)$, while the exactness of (15) implies that $\partial + \delta$ is acyclic. Hence we are left with (i). For the first part we remark that the acyclicity assumption implies that $\partial^* \partial + \partial \partial^*$ is an isomorphism ("Hodge"). Then any operator u which commutes with ∂ can be written as a commutator $[-v, \partial]$ where

$$v = ua, \quad a = -(\partial^* \partial + \partial \partial^*)^{-1} \partial^* \ . \tag{16}$$

This applies in particular to $u = \nabla' - \nabla$ for any two connections up to homotopy on (E, ∂). We now have to prove that $cs_p(\nabla, \nabla^h)$ is zero in cohomology, where ∇ is a linear connection on (E, ∂), and h is a metric. For this we use a result of [1] (Theorem 2.16) which says that $cs_p(A, A^h)$ are closed forms provided $A = A_0 + A_1 + A_2 + \dots$ is a flat super-connection [10] on E with the properties:

(i) A_1 is a connection on E preserving the $\mathbb{Z}$-grading,

(ii) $A_k \in \mathcal{A}^k(M; \mathrm{Hom}(E^*, E^{*+1-k}))$ for $k \neq 1$.

Lemma 6 *Given a (linear) connection ∇ on the acyclic cochain complex (14), there exists a super-connection on E of type*

$$A = \partial + \nabla + A_2 + A_3 + \dots \quad : \mathcal{A}(M; E) \longrightarrow \mathcal{A}(M; E) \ ,$$

which is flat and satisfies (i) and (ii) above.

Let us show that this lemma, combined with the result of [1] mentioned above, prove the desired result. Using Stokes' formula it follows that

$$\begin{aligned} cs_p(\nabla, \nabla^h) \ &= cs(A, A^h) + d(cs_p(\nabla, \nabla^h, A^h) - cs_p(\nabla, A, A^h)) + \\ &+ cs_p(\nabla, A) - cs_p(\nabla^h, A^h) \ , \end{aligned}$$

and we show that $cs_p(\nabla, A) = 0$ (and similarly that $cs_p(\nabla^h, A^h) = 0$). Writing $\theta = A - \nabla$ and using the definition of the Chern-Simons forms, it suffices to prove that

$$Tr_s(((1 - t^2)\nabla^2 + (t - t^2)[\nabla, \theta])^{p-1}\theta) = 0$$

for any t. Since the only endomorphisms of E which count are those preserving the degree, we see that the only term which can contribute is $Tr_s(\nabla^{2(p-2)}[\nabla, \theta]\theta) = Tr_s(\nabla^{2(p-2)}[\nabla, A_2]\partial)$. But $\nabla^{2(p-2)}[\nabla, A_2]\partial$ commutes with ∂ hence its super-trace must vanish (since Tr_s commutes with taking cohomology). $\square$

Proof of Lemma 6: The flatness of A gives us certain equations on the A_k's that we can solve inductively, using the same trick as in (16) above. For instance, the first equation is $[\partial, A_2] + \nabla^2 = 0$. Since $u_1 = \nabla^2$ commutes with ∂, this equation will have the solution $A_2 = u_1 a$ (with a as in (16)). The next equation is $[\partial, A_3] + [A_1, A_2] = 0$. It is not difficult to see that $u_2 = [A_1, A_2]$ commutes with ∂, and we put $A_3 = u_2 a$. Continuing this process, at the n-th level we put $A_{n+1} = u_n a$ where $u_n = [\nabla, A_n] + [A_1, A_{n-2}] + \dots$ as they arise from the coresponding equation. We leave to the reader to show that the u_n's also satisfy the equations

$$u_n = u_{n-1}[\nabla, a] + (\sum_{i+j=n-1} u_i u_j)a^2.$$

Since $[\partial, a] = -1$, ∂ will commute with both $[\nabla, a]$ and a^2, hence also with the u_n's (induction on n). It then follows that A_{n+1} satisfies the desired equation $[\partial, A_{n+1}] = -u_n$. $\square$

Application to algebroids

Recall [9] that an *algebroid* over M consists of a Lie bracket $[\cdot, \cdot]$ defined on the space $\Gamma\mathfrak{g}$ of sections of a vector bundle $\mathfrak{g}$ over M, together with a morphism of vector bundles $\rho : \mathfrak{g} \longrightarrow TM$ (*the anchor* of $\mathfrak{g}$) satisfying $[X, fY] = f[X, Y] + \rho(X)(f) \cdot Y$ for all $X, Y \in \Gamma(\mathfrak{g})$ and $f \in C^\infty(M)$. Important examples are tangent bundles, Lie algebras, foliations, and algebroids associated to Poisson manifolds. It is easy to see (and has already been remarked in many other places [9], [3], [6], etc. etc.) that many of the basic constructions involving vector fields have a straightforward $\mathfrak{g}$-version (just replace $\mathcal{X}(M)$ by $\Gamma(\mathfrak{g})$). Let us briefly point out some of them.

(a) *Cohomology:* the Lie-type formula (4) for the classical De Rham differential makes sense for $X \in \Gamma\mathfrak{g}$ and defines a differential d on the space $C^*(\mathfrak{g}) = \Gamma\Lambda^*\mathfrak{g}^*$, hence a cohomology theory $H^*(\mathfrak{g})$. Particular cases are De Rham cohomology, Lie algebra cohomology, foliated cohomology, and Poisson cohomology.

(b) *Connections and Chern characters:* According to the general philosophy, $\mathfrak{g}$-connections on a vector bundle E over M are linear maps $\Gamma(\mathfrak{g}) \times \Gamma E \longrightarrow \Gamma E$ satisfying the usual identities. Using their curvatures, one obtains $\mathfrak{g}$- Chern classes $Ch^{\mathfrak{g}}(E) \in H^*(\mathfrak{g})$ independent of the connection.

(c) *Representations:* Motivated by the case of Lie algebras, and also by the relation to groupoids (see e.g. [3]), vector bundles E over M together with a flat $\mathfrak{g}$-connection are called representations of $\mathfrak{g}$. This time ∇ should be viewed as an (infinitesimal) action of $\mathfrak{g}$ on E.

(d) *Flat characteristic classes:* The explicit approach to flat characteristic classes (as e.g. in [1], or as in the previous section) has an obvious $\mathfrak{g}$-version. Hence, if E is a representation of $\mathfrak{g}$, then $Ch^{\mathfrak{g}}(E) = 0$, and one obtains the secondary characteristic classes $u_{2p-1}(E) \in H^{2p-1}(\mathfrak{g})$. Maybe less obvious is the fact that one can also extend the Chern-Weil type approach, at the level of frame bundles (as e.g. in [8]). This has been explained in [3], and has certain advantages (e.g. for proving "Morita invariance" of the $u_{2p-1}(E)$'s and for relating them to differentiable cohomology).

(e) *Up to homotopy:* All the constructions and results of the previous sections carry over to algebroids without any problem. As above, a *representation up to homotopy* of $\mathfrak{g}$ is a supercomplex (10) of vector bundles over M, together with a flat $\mathfrak{g}$-connection up to homotopy.

(f) *The adjoint representation:* The main reason for working "up to homotopy" is that the adjoint representation of $\mathfrak{g}$ only makes sense as a representation up to homotopy [5]. Roughly speaking, it is the formal difference $\mathfrak{g} - TM$. The precise definition is:

$$\mathrm{Ad}(\mathfrak{g}) : \quad \mathfrak{g} \underset{\rho}{\overset{0}{\rightleftarrows}} TM \quad , \tag{17}$$

with the flat $\mathfrak{g}$-connection up to homotopy ∇^{ad} given by $\nabla^{ad}_X(Y) = [X, Y]$, $\nabla^{ad}_X(V) = [\rho(X), Y]$ (and the homotopies $H(f, X)(Y) = 0$, $H(f, X)(V) = V(f)X$), for all $X, Y \in \Gamma\mathfrak{g}$, $V \in \mathcal{X}(M)$.

Let us denote by $u^{\mathfrak{g}}_{2p-1}$ the characteristic classes $u_{2p-1}(\mathrm{Ad}(\mathfrak{g}))$ of the adjoint representation. The most useful description from a computational (but not conceptual) point of view is given by (vi) of Theorem 2 (more precisely, its $\mathfrak{g}$-version).

1 Definition We call *basic $\mathfrak{g}$-connection* any $\mathfrak{g}$-connection on $\mathrm{Ad}(\mathfrak{g})$ which is equivalent to the adjoint connection ∇^{ad}.

It is not difficult to see that any such connection is also basic in sense of [6] (and the two notions are equivalent at least in the regular case). Hence we have the following possible description of the $u^{\mathfrak{g}}_{2p-1}$'s, which shows the compatibility with Fernandes' intrinsic characteristic classes [6, 7]:

$$u^{\mathfrak{g}}_{2p-1} = \begin{cases} 0 & \text{if } p = \text{even} \\ \frac{1}{2}(-1)^{\frac{p+1}{2}} cs_p(\nabla_{\mathrm{bas}}, \nabla_{\mathrm{m}}) & \text{if } p = \text{odd} \end{cases} ,$$

where ∇_{bas} is any basic $\mathfrak{g}$-connection, and ∇_{m} is any metric connection on $\mathfrak{g} \oplus TM$. Hence the conclusion of our discussion is the following (which can also be taken as definition of the characteristic classes of [6, 7]).

Theorem 3 *If E is a representation up to homotopy then $Ch^{\mathfrak{g}}(E) = 0$, and the secondary characteristic classes $u_{2p-1}(E) \in H^{2p-1}(\mathfrak{g})$ of representations [4] can be extended to such representations up to homotopy. When applied to the adjoint representation* $\mathrm{Ad}(\mathfrak{g})$*, the resulting classes $u^{\mathfrak{g}}_{2p-1}$ are (up to a constant) the intrinsic characteristic classes of $\mathfrak{g}$ [6].*

More on basic connections: Let us try to shed some light on the notion of basic $\mathfrak{g}$-connection. In our context these are the linear connections which are equivalent to the adjoint connection, while in [6] they appear as a natural extension of Bott's basic connections for foliations. Although not flat in general, they are always flat up to homotopy. The existence of such connections is ensured by Lemma 3 and it was also proven in [6]. There is however a very simple and explicit way to produce them out of ordinary connections on the vector bundle $\mathfrak{g}$.

Proposition 2 *Let ∇ be a connection on the vector bundle $\mathfrak{g}$. Then the formulas*

$$\check{\nabla}^0_X(Y) = [X,Y] + \nabla_{\rho(Y)}(X)$$
$$\check{\nabla}^1_X(V) = [\rho(X),V] + \rho(\nabla_V(X))$$

($X, Y \in \Gamma\mathfrak{g}$, $V \in \Gamma TM$) define a basic $\mathfrak{g}$-connection $\check{\nabla} = (\check{\nabla}^0, \check{\nabla}^1)$.

Proof: We have $\check{\nabla} = \nabla^{ad} + [\theta, \partial]$, where θ is the (non-linear) $\mathrm{End}(\mathrm{Ad}(\mathfrak{g}))$-valued form on $\mathfrak{g}$ given by $\theta(X)(V) = \nabla_V(X)$, $\theta(X)(Y) = 0$. $\Box$

Depending on the special properties of $\mathfrak{g}$, there are various other useful basic connections. This happens for instance when $\mathfrak{g}$ is regular, i.e. when the rank of the anchor ρ is constant. Let us argue that, in this case, the adjoint representation is (up to homotopy) the formal difference $K - \nu$, where K is the kernel of ρ, and ν is the normal bundle $TM/\mathcal{F}$ of the foliation $\mathcal{F} = \rho(\mathfrak{g})$. This time, Bott's formulas [2] trully make sense on ν and K, making them into honest representations of $\mathfrak{g}$:

$$\nabla_X(\bar{Y}) = \overline{[X,Y]}, \quad \forall\ X \in \Gamma\mathfrak{g},\ \bar{Y} \in \Gamma\nu \tag{18}$$
$$\nabla_X(Y) = [X,Y], \quad \forall\ X \in \Gamma\mathfrak{g},\ Y \in \Gamma K\ . \tag{19}$$

Now, choosing splittings $\alpha : \mathcal{F} \longrightarrow \mathfrak{g}$ for ρ, and $\beta : TM \longrightarrow \mathcal{F}$ for the inclusion, we have induced decompositions

$$\mathfrak{g} \cong K \oplus \mathcal{F}, \quad TM \cong \nu \oplus \mathcal{F}\ .$$

As mentioned above, the formal difference $K - \nu$ (view it as a graded complex with K in even degree, ν in odd degree, and zero differential) is a representation of $\mathfrak{g}$. On the other hand, any $\mathcal{F}$-connection ∇ on $\mathcal{F}$ defines a $\mathfrak{g}$-connection on the super-complex

$$D(\mathcal{F}):\ \mathcal{F} \underset{id}{\overset{0}{\rightleftarrows}} \mathcal{F}$$

(and its homotopy class does not depend on ∇). Hence one has an induced $\mathfrak{g}$-connection $\nabla^{\alpha,\beta}$ on $\mathrm{Ad}(\mathfrak{g})$, so that $(\mathrm{Ad}(\mathfrak{g}), \nabla^{\alpha,\beta})$ is isomorphic to $(K - \nu) \oplus D(\mathcal{F})$. Explicitly,

$$\nabla^{\alpha,\beta}_X(Y) = [X, Y - \alpha\rho(Y)] + \alpha\nabla_{\rho(Y)}(\rho X)$$
$$\nabla^{\alpha,\beta}_X(V) = [\rho(X), V] - \beta[\rho(X), V] + \nabla_{\rho(X)}(\beta(V))$$

for all $X, Y \in \Gamma\mathfrak{g}$, $V \in \mathcal{X}(M)$. Note that the second part of the following proposition can also be derived from (iv) of Proposition 1.

Proposition 3 *Assume that $\mathfrak{g}$ is regular. For any $\mathcal{F}$-connection ∇ on $\mathcal{F}$, and any splittings α, β as above, $\nabla^{\alpha,\beta}$ is a basic $\mathfrak{g}$-connection. In particular*

$$u^{\mathfrak{g}}_{2p-1} = u_{2p-1}(K) - u_{2p-1}(\nu)\ ,$$

where K and ν are the representations of $\mathfrak{g}$ defined by Bott's formulas (18), (19).

Proof: We have $\nabla^{\alpha,\beta} = \nabla^{\mathrm{ad}} + [\theta, \partial]$, where θ is the $\mathrm{End}(\mathrm{Ad}(\mathfrak{g}))$-valued non-linear form which is given by

$$\theta(X)(V) = \alpha[\rho(X), \beta(V)] - \alpha\beta[\rho(X), V] - [X, \alpha\beta(V)] + \alpha\nabla_{\rho(X)}\beta(V)$$

for $V \in \Gamma(TM)$, while $\theta(X) = 0$ on $\mathfrak{g}$ (we leave to the reader to check that the previous formula is $C^{\infty}(M)$-linear on V). $\square$.

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