

# Connections up to homotopy and characteristic classes <sup>\*</sup>

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## Introduction

The aim of this note is to clarify the relevance of “connections up to homotopy” [4, 5] to the theory of characteristic classes, and to present an application to the characteristic classes of algebroids [3, 6] (and of Poisson manifolds in particular [7]).

We have already remarked [4] that such connections up to homotopy can be used to compute the classical Chern characters. Here we present a slightly different argument for this, and then proceed with the discussion of the flat characteristic classes. In contrast with [4], we do not only recover the classical characteristic classes (of flat vector bundles), but we also obtain new ones. The reason for this is that ( $\mathbb{Z}_2$ -graded) non-flat vector bundles may have flat connections up to homotopy. As we shall explain here, in this category fall e.g. the characteristic classes of Poisson manifolds [7].

As already mentioned in [4], one of our motivations is to understand the intrinsic characteristic classes for Poisson manifolds (and algebroids) of [6, 7], and the connection with the characteristic classes of representations [3]. Conjecturally, Fernandes’ intrinsic characteristic classes [6] are the characteristic classes [3] of the “adjoint representation”. The problem is that the adjoint representation is a “representation up to homotopy” only. Applied to algebroids, our construction immediately solves this problem: it extends the characteristic classes of [3] from representations to representations up to homotopy, and shows that the intrinsic characteristic classes [6, 7] are indeed the ones associated to the adjoint representation [5].

## Nonlinear connections

Here we recall some well-known properties of connections on vector bundles. Up to a very slight novelty (we allow non-linear connections), this section is standard [10] and serves to fix the notations.

Let  $M$  be a manifold, and let  $E = E^0 \oplus E^1$  be a super-vector bundle over  $M$ . A *non-linear connection* on  $E$  is a linear operator

$$\mathcal{X}(M) \times \Gamma E \longrightarrow \Gamma E, \quad (X, s) \mapsto \nabla_X(s) \tag{1}$$

with the property that the operators  $\nabla_X$  preserve the degree, are local in  $X$ , and satisfy

$$\nabla_X(fs) = f\nabla_X(s) + X(f)s$$

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for all  $X \in \mathcal{X}(M)$ ,  $s \in \Gamma E$ , and  $f \in C^\infty(M)$ . The curvature  $k_\nabla$  of  $\nabla$  is defined by the usual formula

$$k_\nabla(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]} : \Gamma E \longrightarrow \Gamma E . \quad (2)$$

A *non-linear differential form* on  $M$  is an antisymmetric ( $\mathbb{R}$ -multilinear) map

$$\omega : \underbrace{\mathcal{X}(M) \times \dots \times \mathcal{X}(M)}_n \longrightarrow C^\infty(M) \quad (3)$$

which is local in the  $X_i$ 's. It is easy to see (and it has been already remarked in [4]) that many of the usual operations on differential forms do not use the  $C^\infty(M)$ -linearity, hence they apply to non-linear forms as well. In particular we obtain the algebra  $(\mathcal{A}_{\text{nl}}(M), d)$  of non-linear forms endowed with De Rham operator. (This defines a contravariant functor from manifolds to dga's.) Considering  $\Gamma E$ -valued operators instead, we obtain a version with coefficients, denoted  $\mathcal{A}_{\text{nl}}(M; E)$ . Note that a non-linear connection  $\nabla$  can be viewed as an operator  $\mathcal{A}_{\text{nl}}^0(M; E) \longrightarrow \mathcal{A}_{\text{nl}}^1(M; E)$  which has a unique extension to an operator

$$d_\nabla : \mathcal{A}_{\text{nl}}^*(M; E) \longrightarrow \mathcal{A}_{\text{nl}}^{*+1}(M; E)$$

satisfying the Leibniz rule. Explicitly,

$$\begin{aligned} d_\nabla(\omega)(X_1, \dots, X_{n+1}) &= \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{n+1}) \\ &+ \sum_{i=1}^{n+1} (-1)^{i+1} \nabla_{X_i} \omega(X_1, \dots, \hat{X}_i, \dots, X_{n+1}). \end{aligned} \quad (4)$$

We now recall the definition of the (non-linear) connection on  $\text{End}(E)$  induced by  $\nabla$ . For any  $T \in \Gamma \text{End}(E)$ , the operators  $[\nabla_X, T]$  acting on  $\Gamma(E)$  are  $C^\infty(M)$ -linear, hence define elements  $[\nabla_X, T] \in \Gamma \text{End}(E)$ . The desired connection is then  $\nabla_X(T) = [\nabla_X, T]$ . Clearly  $k_\nabla \in \mathcal{A}_{\text{nl}}^2(M; \text{End}(E))$ , and one has Bianchi's identity  $d_\nabla(k_\nabla) = 0$ .

We will use the algebra  $\mathcal{A}_{\text{nl}}(M; \text{End}(E))$  and its action on  $\mathcal{A}_{\text{nl}}(M; E)$ . The product structure that we consider here is the one which arises from the natural isomorphisms

$$\mathcal{A}_{\text{nl}}(M; E) \cong \mathcal{A}_{\text{nl}}(M) \otimes_{C^\infty(M)} \Gamma(E)$$

and the usual sign conventions for the tensor products (i.e.  $\omega \otimes x \cdot \eta \otimes y = (-1)^{|x||\eta|} \omega \eta \otimes xy$ ). The usual super-trace on  $\text{End}(E)$  induces a super-trace

$$Tr_s : (\mathcal{A}_{\text{nl}}(M; \text{End}(E)), d_\nabla) \longrightarrow (\mathcal{A}_{\text{nl}}(M), d) \quad (5)$$

with the property that  $Tr_s d_\nabla = d Tr_s$ . We conclude (and this is just a non-linear version of the standard construction of Chern characters [10]):

**Lemma 1** *If  $\nabla$  is a non-linear connection on  $E$ , then*

$$ch_p(\nabla) = Tr_s(k_\nabla^p) \in \mathcal{A}_{\text{nl}}^{2p}(M) \quad (6)$$

*are closed non-linear forms on  $M$ .*

Up to a boundary, these classes are independent of  $\nabla$ . This is an instance of the Chern-Simons construction that we now recall. Given  $k+1$  non-linear connections  $\nabla_i$  on  $E$  ( $0 \leq i \leq k$ ) we form their affine combination  $\nabla^{\text{aff}} = (1-t_1-\dots-t_k)\nabla_0+t_1\nabla_1+\dots+t_k\nabla_k$ . This is a non-linear connection on the pullback of  $E$  to  $\Delta^k \times M$ , where  $\Delta^k = \{(t_1, \dots, t_k) : t_i \geq 0, \sum t_i \leq 1\}$  is the standard  $k$ -simplex. The classical integration along fibers has a non-linear extension

$$\int_{\Delta^k} : \mathcal{A}_{\text{nl}}^*(M \times \Delta^k) \longrightarrow \mathcal{A}_{\text{nl}}^{*-k}(M) \quad (7)$$

given by the explicit formula

$$\left( \int_{\Delta^k} \omega \right) (X_1, \dots, X_{n-k}) = \int_{\Delta^k} \omega \left( \frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_k}, X_1, \dots, X_{n-k} \right) dt_1 \dots dt_k .$$

We then define

$$cs_p(\nabla_0, \dots, \nabla_k) = \int_{\Delta^k} ch_p(\nabla^{\text{aff}}) . \quad (8)$$

Using a version of Stokes' formula [2] (or integrating by parts repeatedly) we conclude

**Lemma 2** *The elements (8) satisfy*

$$dcs_p(\nabla_0, \dots, \nabla_k) = \sum_{i=0}^k (-1)^i cs_p(\nabla_0, \dots, \widehat{\nabla}_i, \dots, \nabla_k) . \quad (9)$$

## Connections up to homotopy and Chern characters

From now on,  $(E, \partial)$  is a super-complex of vector bundles over the manifold  $M$ ,

$$(E, \partial) : E^0 \begin{array}{c} \xrightarrow{\partial} \\ \xleftarrow{\partial} \end{array} E^1 \quad . \quad (10)$$

We now consider non-linear connections  $\nabla$  on  $E$  such that  $\nabla_X \partial = \partial \nabla_X$  for all  $X \in \mathcal{X}(M)$ . We say that  $\nabla$  is a (linear) *connection* on  $(E, \partial)$  if it also satisfies the identity  $\nabla_{fX}(s) = f\nabla_X(s)$  for all  $X \in \mathcal{X}(M)$ ,  $f \in C^\infty(M)$ ,  $s \in \Gamma E$ . The notion of *connection up to homotopy* [4, 5] on  $(E, \partial)$  is obtained by relaxing the  $C^\infty(M)$ -linearity on  $X$  to linearity up to homotopy. In other words we require

$$\nabla_{fX}(s) = f\nabla_X(s) + [H_\nabla(f, X), \partial] ,$$

where  $H_\nabla(f, X) \in \Gamma \text{End}(E)$  are odd elements which are linear and local in  $X$  and  $f$ .

We say that two non-linear connections  $\nabla$  and  $\nabla'$  are *equivalent* (or homotopic) if

$$\nabla'_X = \nabla_X + [\theta(X), \partial]$$

for all  $X \in \mathcal{X}(M)$ , for some  $\theta \in \mathcal{A}_{\text{nl}}^1(M; \text{End}(E))$  of even degree. We write  $\nabla \sim \nabla'$ .

**Lemma 3** *A non-linear connection is a connection up to homotopy if and only if it is equivalent to a (linear) connection.*

*Proof:* Assume that  $\nabla$  is a connection up to homotopy. Let  $U_a$  be the domain of local coordinates  $x^k$  for  $M$ , and put

$$\nabla_X^a = \nabla_X + [u^a(X), \partial],$$

where  $u_a \in \mathcal{A}_{\text{nl}}(U_a; \text{End}(E))$  is given by

$$u_a\left(\sum_k f_k \frac{\partial}{\partial x_k}\right) = -\sum_k H_{\nabla}(f_k, \frac{\partial}{\partial x_k}),$$

for all  $f_k \in C^\infty(U_a)$ . Note that  $\nabla_X$  is linear on  $X$ . Indeed, for any two smooth functions  $f, g$  and  $X = g \frac{\partial}{\partial x_k}$  we have

$$\begin{aligned} \nabla_{fX}^a - f\nabla_X^a &= (\nabla_{fX} + [u^a(fX), \partial]) - f(\nabla_X + [u^a(X), \partial]) = \\ &= (\nabla_{fg \frac{\partial}{\partial x_k}} - [H_{\nabla}(fg, \frac{\partial}{\partial x_k}), \partial]) + f(\nabla_{g \frac{\partial}{\partial x_k}} - [H_{\nabla}(g \frac{\partial}{\partial x_k}), \partial]) = \\ &= fg \nabla_{\frac{\partial}{\partial x_k}} - fg \nabla_{\frac{\partial}{\partial x_k}} = 0. \end{aligned}$$

Next we take  $\{\nu_a\}$  to be a partition of unity subordinate to an open cover  $\{U_a\}$  by such coordinate domains and put  $\nabla'_X = \sum_a \nu_a \nabla_X^a$ ,  $u(X) = \sum_a \nu_a u^a(X)$ . Then  $\nabla' = \nabla + [u, \partial]$  is a connection equivalent to  $\nabla$ .  $\square$

**Lemma 4** *If  $\nabla_0$  and  $\nabla_1$  are equivalent, then  $ch_p(\nabla^0) = ch_p(\nabla^1)$ .*

*Proof:* So, let us assume that  $\nabla^1 = \nabla^0 + [\theta, \partial]$ . A simple computation shows that

$$k_{\nabla_1} = k_{\nabla_0} + [d_{\nabla}(\theta) + R, \partial], \quad (11)$$

where  $R(X, Y) = [\theta(X), [\theta(Y), \partial]]$ . We denote by  $Z \subset \mathcal{A}_{\text{nl}}(M; \text{End}(E))$  the space of non-linear forms  $\omega$  with the property that  $[\omega, \partial] = 0$ , and by  $B \subset Z$  the subspace of element of type  $[\eta, \partial]$  for some non-linear form  $\eta$ . The formula

$$[\partial, \omega\eta] = [\partial, \omega]\eta + (-1)^{|\omega|} \omega[\partial, \eta]$$

shows that  $ZB \subset B$ , hence (11) implies that  $k_{\nabla_1}^p \equiv k_{\nabla_0}^p$  modulo  $B$ . The desired equality follows now from the fact that  $Tr_s$  vanishes on  $B$ .  $\square$

For (linear) connections  $\nabla$  on  $(E, \partial)$ ,  $ch_p(\nabla)$  are clearly (linear) differential forms on  $M$  whose cohomology classes are (up to a constant) the components of the Chern character  $Ch(E) = Ch(E^0) - Ch(E^1)$ . Hence an immediate consequence of the previous two lemmas is the following [4]

**Theorem 1** *If  $\nabla$  is a connection up to homotopy on  $(E, \partial)$ , then  $ch_p(\nabla) = Tr_s(k_{\nabla}^p)$  are closed differential forms on  $M$  whose De Rham cohomology classes are (up to a constant) the components of the Chern character  $Ch(E)$ .*

## Flat characteristic classes

As usual, by flatness we mean the vanishing of the curvature forms. Theorem 1 immediately implies

**Corollary 1** *If  $(E, \partial)$  admits a connection up to homotopy which is flat, then  $Ch(E) = 0$ .*

As usual, such a vanishing result is at the origin of new “secondary” characteristic classes. Let  $\nabla$  be a flat connection up to homotopy. To construct the associated secondary classes we need a metric  $h$  on  $E$ . We denote by  $\partial^h$  be the adjoint of  $\partial$  with respect to  $h$ . Using the isomorphism  $E^* \cong E$  induced by  $h$  (which is anti-linear if  $E$  is complex),  $\nabla$  induces an adjoint connection  $\nabla^h$  on  $(E, \partial^h)$ . Explicitly,

$$L_X h(s, t) = h(\nabla_X(s), t) + h(s, \nabla_X^h(t)) .$$

The following describes various possible definitions of the secondary classes, as well as their main properties (note that the role of  $i = \sqrt{-1}$  below is to ensure real classes).

**Theorem 2** *Let  $\nabla$  be a flat connection up to homotopy on  $(E, \partial)$ ,  $p \geq 1$ .*

(i) *For any (linear) connection  $\nabla_0$  on  $(E, \partial)$  and any metric  $h$ ,*

$$i^{p+1}(cs_p(\nabla, \nabla_0) + cs_p(\nabla_0, \nabla_0^h) + cs_p(\nabla_0^h, \nabla^h)) \in \mathcal{A}_{\text{nl}}^{2p-1}(M) \quad (12)$$

*are differential forms on  $M$  which are real and closed. The induced cohomology classes do not depend on the choice of  $h$  or  $\nabla_0$ , and are denoted  $u_{2p-1}(E, \partial, \nabla) \in H^{2p-1}(M)$ .*

(ii) *For any connection  $\nabla_0$  equivalent to  $\nabla$ , and any metric  $h$ ,*

$$i^{p+1}cs_p(\nabla_0, \nabla_0^h) \in \mathcal{A}^{2p-1}(M) \quad (13)$$

*are real and closed, and represent  $u_{2p-1}(E, \partial, \nabla)$  in cohomology.*

(iii) *If  $\nabla$  is equivalent to a metric connection (i.e. a connection which is compatible with a metric), then all the classes  $u_{2p-1}(E, \partial, \nabla)$  vanish.*

(iv) *If  $\nabla \sim \nabla'$ , then  $u_{2p-1}(E, \partial, \nabla) = u_{2p-1}(E, \partial, \nabla')$ .*

(v) *If  $\nabla$  is a flat connection up to homotopy on both super-complexes  $(E, \partial)$  and  $(E, \partial')$ , then  $u_{2p-1}(E, \partial, \nabla) = u_{2p-1}(E, \partial', \nabla)$ .*

(vi) *Assume that  $E$  is real. If  $p$  is even then  $u_{2p-1}(E, \partial, \nabla) = 0$ . If  $p$  is odd, then for any connection  $\nabla_0$  equivalent to  $\nabla$ , and any metric connection  $\nabla_m$ ,*

$$(-1)^{\frac{p+1}{2}} cs_p(\nabla_0, \nabla_m) \in \mathcal{A}^{2p-1}(M)$$

*are closed differential forms whose cohomology classes equal to  $\frac{1}{2}u_{2p-1}(E, \partial, \nabla)$ .*

Note the compatibility with the classical flat characteristic classes, which correspond to the case where  $E$  is a graded vector bundle (and  $\partial = 0$ ), or, more classically, just a vector bundle over  $M$ . As references for this we point out [8] (for the approach in terms of frame bundles and Lie algebra cohomology), and [1] (for an explicit approach which we follow here). For the proof of the theorem we need the following

**Lemma 5** Given the non-linear connections  $\nabla, \nabla_0, \nabla_1$ ,

(i) If  $\nabla_0$  and  $\nabla_1$  are connections up to homotopy then  $cs_p(\nabla_0, \nabla_1)$  are differential forms;

(ii) If  $\nabla_0 \sim \nabla_1$ , then  $cs_p(\nabla_0, \nabla_1) = 0$ ;

(iii) For any metric  $h$ ,  $ch_p(\nabla^h) = (-1)^p \overline{ch_p(\nabla)}$  and  $cs_p(\nabla_0^h, \nabla_1^h) = (-1)^p \overline{cs_p(\nabla_0, \nabla_1)}$ .

*Proof:* (i) follows from the fact that Chern characters of connections up to homotopy are differential forms. For (ii) we use Lemma 4. The affine combination  $\nabla$  used in the definition of  $cs_p(\nabla_0, \nabla_1)$  is equivalent to the pull-back  $\tilde{\nabla}_0$  of  $\nabla_0$  to  $M \times \Delta^1$  (because  $\nabla = \tilde{\nabla}_0 + t[\theta, \partial]$ ), while  $ch_p(\tilde{\nabla}_0)$  is clearly zero. If  $h$  is a metric on  $E$ , a simple computation shows that  $k_{\nabla^h}(X, Y)$  coincides with  $-k_{\nabla}(X, Y)^*$  where  $*$  denotes the adjoint (with respect to  $h$ ). Then (iii) follows from  $Tr(A^*) = \overline{Tr(A)}$  for any matrix  $A$ .  $\square$

*Proof of Theorem 2:* (i) Let us denote by  $u(\nabla, \nabla_0, h)$  the forms (12). Since  $(\nabla_0, \nabla_0^h)$  is a pair of connections on  $E$ , and  $(\nabla, \nabla_0), (\nabla^h, \nabla_0^h)$  are pairs of connections up to homotopy on  $(E, \partial)$  and  $(E, \partial^h)$ , respectively, it follows from (i) of Lemma 5 that  $u(\nabla, \nabla_0, h)$  are differential forms. From Stokes formula (9) it immediately follows that they are closed. To prove that they are real we use (iii) of the previous Lemma:

$$\begin{aligned} \overline{u(\nabla, \nabla_0, h)} &= (-i)^{p+1} \overline{(cs_p(\nabla, \nabla_0) + cs_p(\nabla_0, \nabla_0^h) + cs_p(\nabla_0^h, \nabla^h))} = \\ &+ (-i)^{p+1} (-1)^p cs_p(\nabla^h, \nabla_0^h) + cs_p(\nabla_0^h, \nabla_0) + cs_p(\nabla_0, \nabla) = \\ &= (-i)^{p+1} (-1)^p (-1) u(\nabla, \nabla_0, h) = u(\nabla, \nabla_0, h) \end{aligned}$$

If  $\nabla_1$  is another connection, using (9) again, it follows that  $u(\nabla, \nabla_0, h) - u(\nabla, \nabla_1, h) = i^{p+1} dv$  where  $v$  is the (linear!) differential form

$$v = cs_p(\nabla, \nabla_0, \nabla_1) - cs_p(\nabla^h, \nabla_0^h, \nabla_1^h) + cs_p(\nabla_0, \nabla_0^h, \nabla_1) - cs_p(\nabla_0^h, \nabla_1, \nabla_1^h).$$

(iii) clearly follows from (ii), which in turn follows from (ii) of Lemma 5 and the fact that  $\nabla \sim \nabla_0$  implies  $\nabla^h \sim \nabla_0^h$ . To see that our classes do not depend on  $h$ , it suffices to show that given a linear connection  $\nabla$  on a vector bundle  $F$ ,  $cs_p(\nabla, \nabla^h)$  is independent of  $h$  up to the boundary of a differential form. Let  $h_0$  and  $h_1$  be two metrics. Although the proof below works for general  $\nabla$ 's, simpler formulas are possible when  $\nabla$  is flat. So, let us first assume that (actually we may assume that  $\nabla$  is the canonical connection on a trivial vector bundle). From Stokes' formula (9) applied to  $(\nabla, \nabla^{h_0}, \nabla^{h_1})$ , it suffices to show that  $cs_p(\nabla^{h_0}, \nabla^{h_1})$  is a closed form. We choose a family  $h_t$  of metrics joining  $h_0$  and  $h_1$ . One only has to show that  $\frac{\partial}{\partial t} cs_p(\nabla^{h_0}, \nabla^{h_t})$  are closed forms. Writing  $h_t(x, y) = h_0(u_t(x), y)$ , these Chern-Simons forms are, up to a constant,  $Tr(\omega_t^{2p-1})$  where

$$\omega_t = \nabla^{h_t} - \nabla^{h_0} = u_t^{-1} d_{\nabla^{h_0}}(u_t)$$

(here is where we use the flatness of  $\nabla$ ). A simple computation shows that

$$\frac{\partial \omega_t}{\partial t} = d_{\nabla^{h_0}}(v_t) + [\omega_t, v_t],$$

where  $v_t = u_t^{-1} \frac{\partial u_t}{\partial t}$ . Since  $d_{\nabla^{h_0}}(\omega_t^2) = 0$ , this implies

$$\frac{\partial \omega_t}{\partial t} \omega_t^{2p-2} = d_{\nabla^{h_0}}(v_t \omega_t^{2p-2}) + [\omega_t, v_t \omega_t^{2p-2}].$$

Now, by the properties of the trace it follows that

$$\frac{\partial}{\partial t} Tr_s(\omega_t^{2p-1}) = dTr_s(v_t \omega_t^{2p-2})$$

as desired. Assume now that  $\nabla$  is not flat. We choose a vector bundle  $F'$  together with a connection  $\nabla'$  compatible with a metric  $h'$ , such that  $\tilde{F} = F \oplus F'$  admits a flat connection  $\nabla_0$ . We put  $\tilde{\nabla} = \nabla \oplus \nabla'$  and, for any metric  $h$  on  $F$ , we consider the metric  $\tilde{h} = h \oplus h'$  on  $\tilde{F}$ . Clearly  $cs_p(\tilde{\nabla}, \tilde{h}) = cs_p(\nabla, \nabla^h)$ . Using also (iii) of Lemma 5 and Stokes' formula, we have:

$$\begin{aligned} cs_p(\nabla, \nabla^h) &= cs_p(\nabla_0, \nabla_0^{\tilde{h}}) - cs_p(\nabla_0, \tilde{\nabla}) + (-1)^p \overline{cs_p(\nabla_0, \tilde{\nabla})} \\ &\quad + d(cs_p(\nabla_0, \tilde{\nabla}, \tilde{h}) - cs_p(\nabla_0, \tilde{\nabla}_0, \tilde{h})). \end{aligned}$$

Hence, by the flat case,  $cs_p(\nabla, \nabla^h)$  modulo exact forms does not depend on  $h$ .

For (iv) one uses Stokes' formula (9) and (ii) of Lemma 5 to conclude that  $cs_p(\nabla', \nabla_0) - cs_p(\nabla, \nabla_0)$  is the differential of the linear form  $cs_p(\nabla, \nabla', \nabla_0)$ . To prove (v) we only have to show (see (i)) that there exists a linear connection  $\nabla^0$  on  $E$  which is compatible with both  $\partial$  and  $\partial'$ . For this, one defines  $\nabla^0$  locally by  $f \frac{\partial}{\partial x_k} = f \nabla \frac{\partial}{\partial x_k}$ , and then use a partition of unity argument.

We now assume that  $E$  is real. From Lemma 5,

$$cs_p(\nabla_m, \nabla_0^h) = (-1)^p cs_p(\nabla_m^h, \nabla_0) = (-1)^{p+1} cs_p(\nabla_0, \nabla_m).$$

Combined with Stokes' formula (9), this implies

$$dcs_p(\nabla_0, \nabla_m, \nabla_0^h) = (1 + (-1)^{p+1}) cs_p(\nabla_0, \nabla_m) - cs_p(\nabla_0, \nabla_0^h),$$

which proves (vi).  $\square$

Note that the construction of the flat characteristic classes presented here actually works for  $\nabla$ 's which are "flat up to homotopy", i.e. whose curvatures are of type  $[-, \partial]$ . Moreover, this notion is stable under equivalence, and the flat characteristic classes only depend on the equivalence class of  $\nabla$  (cf. (iv) of the Theorem). Note also that, as in [4] (and following [1]), there is a version of our discussion for super-connections [10] up to homotopy. Some of our constructions can then be interpreted in terms of the super-connection  $\partial + \nabla$ .

If  $E$  is regular in the sense that  $Ker(\partial)$  and  $Im(\partial)$  are vector bundles, then so is the cohomology  $H(E, \partial) = Ker(\partial)/Im(\partial)$ , and any connection up to homotopy  $\nabla$  on  $(E, \partial)$  defines a linear connection  $H(\nabla)$  on  $H(E)$ . Moreover,  $H(\nabla)$  is flat if  $\nabla$  is, and the characteristic classes  $u_{2p-1}(E, \partial, \nabla)$  coincide with the classical [1, 8] characteristic classes of the flat vector bundle  $H(E, \partial)$ . In general, the  $u_{2p-1}(E, \partial, \nabla)$ 's should be viewed as invariants of  $H(E, \partial)$  constructed in such a way that no regularity assumption is required. Let us discuss here an instance of this. We say that  $E$  is  $\mathbb{Z}$ -graded if it comes from a cochain complex

$$0 \longrightarrow E(0) \xrightarrow{\partial} E(1) \xrightarrow{\partial} \dots \xrightarrow{\partial} E(n) \longrightarrow 0, \quad (14)$$

In other words, it must be of type  $E = \bigoplus_{k=0}^n E(k)$  with the even/odd  $\mathbb{Z}_2$ -grading, and with  $\partial(E(k)) \subset E(k+1)$ .

### Proposition 1

- (i) If  $(E, \partial)$  is acyclic, then any two connections up to homotopy on  $(E, \partial)$  are equivalent. Moreover, if  $E$  is  $\mathbb{Z}$ -graded, then  $u_{2p-1}(E, \partial, \nabla) = 0$ .
- (ii) If  $(E^k, \partial^k, \nabla^k)$  are  $\mathbb{Z}$ -graded complexes of vector bundles endowed with flat connections up to homotopy which fit into an exact sequence

$$0 \longrightarrow E^0 \xrightarrow{\delta} E^1 \xrightarrow{\delta} \dots \xrightarrow{\delta} E^n \longrightarrow 0 \quad (15)$$

compatible with the structures (i.e.  $[\delta, \partial] = [\delta, \nabla] = [\delta, H_\nabla] = 0$ ), then

$$\sum_{k=0}^n (-1)^k u_{2p-1}(E^k, \partial^k, \nabla^k) = 0 .$$

*Proof:* The second part follows from (i) above and (v) of Theorem 2. To see this, we form the super-vector bundle  $E = \bigoplus_k E^k$  (which is  $\mathbb{Z}$ -graded by the total degree) and the direct sum (non-linear) connection  $\nabla$  acting on  $E$ . Then  $\nabla$  is a connection up to homotopy in both  $(E, \partial)$  and  $(E, \partial + \delta)$ . Clearly  $u_{2p-1}(E, \partial, \nabla) = \sum_{k=0}^n (-1)^k u_{2p-1}(E^k, \partial^k, \nabla^k)$ , while the exactness of (15) implies that  $\partial + \delta$  is acyclic. Hence we are left with (i). For the first part we remark that the acyclicity assumption implies that  $\partial^* \partial + \partial \partial^*$  is an isomorphism (“Hodge”). Then any operator  $u$  which commutes with  $\partial$  can be written as a commutator  $[-v, \partial]$  where

$$v = ua, \quad a = -(\partial^* \partial + \partial \partial^*)^{-1} \partial^* . \quad (16)$$

This applies in particular to  $u = \nabla' - \nabla$  for any two connections up to homotopy on  $(E, \partial)$ . We now have to prove that  $cs_p(\nabla, \nabla^h)$  is zero in cohomology, where  $\nabla$  is a linear connection on  $(E, \partial)$ , and  $h$  is a metric. For this we use a result of [1] (Theorem 2.16) which says that  $cs_p(A, A^h)$  are closed forms provided  $A = A_0 + A_1 + A_2 + \dots$  is a flat super-connection [10] on  $E$  with the properties:

- (i)  $A_1$  is a connection on  $E$  preserving the  $\mathbb{Z}$ -grading,
- (ii)  $A_k \in \mathcal{A}^k(M; \text{Hom}(E^*, E^{*+1-k}))$  for  $k \neq 1$ .

**Lemma 6** *Given a (linear) connection  $\nabla$  on the acyclic cochain complex (14), there exists a super-connection on  $E$  of type*

$$A = \partial + \nabla + A_2 + A_3 + \dots : \mathcal{A}(M; E) \longrightarrow \mathcal{A}(M; E) ,$$

which is flat and satisfies (i) and (ii) above.

Let us show that this lemma, combined with the result of [1] mentioned above, prove the desired result. Using Stokes' formula it follows that

$$\begin{aligned} cs_p(\nabla, \nabla^h) &= cs(A, A^h) + d(cs_p(\nabla, \nabla^h, A^h) - cs_p(\nabla, A, A^h)) + \\ &\quad + cs_p(\nabla, A) - cs_p(\nabla^h, A^h) , \end{aligned}$$

and we show that  $cs_p(\nabla, A) = 0$  (and similarly that  $cs_p(\nabla^h, A^h) = 0$ ). Writing  $\theta = A - \nabla$  and using the definition of the Chern-Simons forms, it suffices to prove that

$$Tr_s(((1 - t^2)\nabla^2 + (t - t^2)[\nabla, \theta])^{p-1}\theta) = 0$$

for any  $t$ . Since the only endomorphisms of  $E$  which count are those preserving the degree, we see that the only term which can contribute is  $Tr_s(\nabla^{2(p-2)}[\nabla, \theta]\theta) = Tr_s(\nabla^{2(p-2)}[\nabla, A_2]\partial)$ . But  $\nabla^{2(p-2)}[\nabla, A_2]\partial$  commutes with  $\partial$  hence its super-trace must vanish (since  $Tr_s$  commutes with taking cohomology).  $\square$

*Proof of Lemma 6:* The flatness of  $A$  gives us certain equations on the  $A_k$ 's that we can solve inductively, using the same trick as in (16) above. For instance, the first equation is  $[\partial, A_2] + \nabla^2 = 0$ . Since  $u_1 = \nabla^2$  commutes with  $\partial$ , this equation will have the solution  $A_2 = u_1 a$  (with  $a$  as in (16)). The next equation is  $[\partial, A_3] + [A_1, A_2] = 0$ . It is not difficult to see that  $u_2 = [A_1, A_2]$  commutes with  $\partial$ , and we put  $A_3 = u_2 a$ . Continuing this process, at the  $n$ -th level we put  $A_{n+1} = u_n a$  where  $u_n = [\nabla, A_n] + [A_1, A_{n-2}] + \dots$  as they arise from the corresponding equation. We leave to the reader to show that the  $u_n$ 's also satisfy the equations

$$u_n = u_{n-1}[\nabla, a] + \left( \sum_{i+j=n-1} u_i u_j \right) a^2.$$

Since  $[\partial, a] = -1$ ,  $\partial$  will commute with both  $[\nabla, a]$  and  $a^2$ , hence also with the  $u_n$ 's (induction on  $n$ ). It then follows that  $A_{n+1}$  satisfies the desired equation  $[\partial, A_{n+1}] = -u_n$ .  $\square$

## Application to algebroids

Recall [9] that an *algebroid* over  $M$  consists of a Lie bracket  $[\cdot, \cdot]$  defined on the space  $\Gamma \mathfrak{g}$  of sections of a vector bundle  $\mathfrak{g}$  over  $M$ , together with a morphism of vector bundles  $\rho : \mathfrak{g} \rightarrow TM$  (the *anchor* of  $\mathfrak{g}$ ) satisfying  $[X, fY] = f[X, Y] + \rho(X)(f) \cdot Y$  for all  $X, Y \in \Gamma(\mathfrak{g})$  and  $f \in C^\infty(M)$ . Important examples are tangent bundles, Lie algebras, foliations, and algebroids associated to Poisson manifolds. It is easy to see (and has already been remarked in many other places [9], [3], [6], etc. etc.) that many of the basic constructions involving vector fields have a straightforward  $\mathfrak{g}$ -version (just replace  $\mathcal{X}(M)$  by  $\Gamma(\mathfrak{g})$ ). Let us briefly point out some of them.

- (a) *Cohomology:* the Lie-type formula (4) for the classical De Rham differential makes sense for  $X \in \Gamma \mathfrak{g}$  and defines a differential  $d$  on the space  $C^*(\mathfrak{g}) = \Gamma \Lambda^* \mathfrak{g}^*$ , hence a cohomology theory  $H^*(\mathfrak{g})$ . Particular cases are De Rham cohomology, Lie algebra cohomology, foliated cohomology, and Poisson cohomology.
- (b) *Connections and Chern characters:* According to the general philosophy,  $\mathfrak{g}$ -connections on a vector bundle  $E$  over  $M$  are linear maps  $\Gamma(\mathfrak{g}) \times \Gamma E \rightarrow \Gamma E$  satisfying the usual identities. Using their curvatures, one obtains  $\mathfrak{g}$ -Chern classes  $Ch^{\mathfrak{g}}(E) \in H^*(\mathfrak{g})$  independent of the connection.
- (c) *Representations:* Motivated by the case of Lie algebras, and also by the relation to groupoids (see e.g. [3]), vector bundles  $E$  over  $M$  together with a flat  $\mathfrak{g}$ -connection are called representations of  $\mathfrak{g}$ . This time  $\nabla$  should be viewed as an (infinitesimal) action of  $\mathfrak{g}$  on  $E$ .
- (d) *Flat characteristic classes:* The explicit approach to flat characteristic classes (as e.g. in [1], or as in the previous section) has an obvious  $\mathfrak{g}$ -version. Hence, if  $E$  is a representation

of  $\mathfrak{g}$ , then  $Ch^{\mathfrak{g}}(E) = 0$ , and one obtains the secondary characteristic classes  $u_{2p-1}(E) \in H^{2p-1}(\mathfrak{g})$ . Maybe less obvious is the fact that one can also extend the Chern-Weil type approach, at the level of frame bundles (as e.g. in [8]). This has been explained in [3], and has certain advantages (e.g. for proving “Morita invariance” of the  $u_{2p-1}(E)$ ’s and for relating them to differentiable cohomology).

- (e) *Up to homotopy:* All the constructions and results of the previous sections carry over to algebroids without any problem. As above, a *representation up to homotopy* of  $\mathfrak{g}$  is a supercomplex (10) of vector bundles over  $M$ , together with a flat  $\mathfrak{g}$ -connection up to homotopy.
- (f) *The adjoint representation:* The main reason for working “up to homotopy” is that the adjoint representation of  $\mathfrak{g}$  only makes sense as a representation up to homotopy [5]. Roughly speaking, it is the formal difference  $\mathfrak{g} - TM$ . The precise definition is:

$$\text{Ad}(\mathfrak{g}) : \quad \mathfrak{g} \begin{array}{c} \xrightarrow{0} \\ \xrightarrow{\rho} \end{array} TM \quad , \quad (17)$$

with the flat  $\mathfrak{g}$ -connection up to homotopy  $\nabla^{ad}$  given by  $\nabla_X^{ad}(Y) = [X, Y]$ ,  $\nabla_X^{ad}(V) = [\rho(X), Y]$  (and the homotopies  $H(f, X)(Y) = 0$ ,  $H(f, X)(V) = V(f)X$ ), for all  $X, Y \in \Gamma\mathfrak{g}$ ,  $V \in \mathcal{X}(M)$ .

Let us denote by  $u_{2p-1}^{\mathfrak{g}}$  the characteristic classes  $u_{2p-1}(\text{Ad}(\mathfrak{g}))$  of the adjoint representation. The most useful description from a computational (but not conceptual) point of view is given by (vi) of Theorem 2 (more precisely, its  $\mathfrak{g}$ -version).

**1 Definition** We call *basic  $\mathfrak{g}$ -connection* any  $\mathfrak{g}$ -connection on  $\text{Ad}(\mathfrak{g})$  which is equivalent to the adjoint connection  $\nabla^{ad}$ .

It is not difficult to see that any such connection is also basic in sense of [6] (and the two notions are equivalent at least in the regular case). Hence we have the following possible description of the  $u_{2p-1}^{\mathfrak{g}}$ ’s, which shows the compatibility with Fernandes’ intrinsic characteristic classes [6, 7]:

$$u_{2p-1}^{\mathfrak{g}} = \begin{cases} 0 & \text{if } p = \text{even} \\ \frac{1}{2}(-1)^{\frac{p+1}{2}} c_{S_p}(\nabla_{\text{bas}}, \nabla_{\text{m}}) & \text{if } p = \text{odd} \end{cases} ,$$

where  $\nabla_{\text{bas}}$  is any basic  $\mathfrak{g}$ -connection, and  $\nabla_{\text{m}}$  is any metric connection on  $\mathfrak{g} \oplus TM$ . Hence the conclusion of our discussion is the following (which can also be taken as definition of the characteristic classes of [6, 7]).

**Theorem 3** *If  $E$  is a representation up to homotopy then  $Ch^{\mathfrak{g}}(E) = 0$ , and the secondary characteristic classes  $u_{2p-1}(E) \in H^{2p-1}(\mathfrak{g})$  of representations [4] can be extended to such representations up to homotopy. When applied to the adjoint representation  $\text{Ad}(\mathfrak{g})$ , the resulting classes  $u_{2p-1}^{\mathfrak{g}}$  are (up to a constant) the intrinsic characteristic classes of  $\mathfrak{g}$  [6].*

**More on basic connections:** Let us try to shed some light on the notion of basic  $\mathfrak{g}$ -connection. In our context these are the linear connections which are equivalent to the adjoint connection, while in [6] they appear as a natural extension of Bott’s basic connections

for foliations. Although not flat in general, they are always flat up to homotopy. The existence of such connections is ensured by Lemma 3 and it was also proven in [6]. There is however a very simple and explicit way to produce them out of ordinary connections on the vector bundle  $\mathfrak{g}$ .

**Proposition 2** *Let  $\nabla$  be a connection on the vector bundle  $\mathfrak{g}$ . Then the formulas*

$$\begin{aligned}\check{\nabla}_X^0(Y) &= [X, Y] + \nabla_{\rho(Y)}(X) \\ \check{\nabla}_X^1(V) &= [\rho(X), V] + \rho(\nabla_V(X))\end{aligned}$$

$(X, Y \in \Gamma\mathfrak{g}, V \in \Gamma TM)$  define a basic  $\mathfrak{g}$ -connection  $\check{\nabla} = (\check{\nabla}^0, \check{\nabla}^1)$ .

*Proof:* We have  $\check{\nabla} = \nabla^{ad} + [\theta, \partial]$ , where  $\theta$  is the (non-linear)  $\text{End}(\text{Ad}(\mathfrak{g}))$ -valued form on  $\mathfrak{g}$  given by  $\theta(X)(V) = \nabla_V(X)$ ,  $\theta(X)(Y) = 0$ .  $\square$

Depending on the special properties of  $\mathfrak{g}$ , there are various other useful basic connections. This happens for instance when  $\mathfrak{g}$  is regular, i.e. when the rank of the anchor  $\rho$  is constant. Let us argue that, in this case, the adjoint representation is (up to homotopy) the formal difference  $K - \nu$ , where  $K$  is the kernel of  $\rho$ , and  $\nu$  is the normal bundle  $TM/\mathcal{F}$  of the foliation  $\mathcal{F} = \rho(\mathfrak{g})$ . This time, Bott's formulas [2] trully make sense on  $\nu$  and  $K$ , making them into honest representations of  $\mathfrak{g}$ :

$$\nabla_X(\bar{Y}) = \overline{[X, Y]}, \quad \forall X \in \Gamma\mathfrak{g}, \bar{Y} \in \Gamma\nu \quad (18)$$

$$\nabla_X(Y) = [X, Y], \quad \forall X \in \Gamma\mathfrak{g}, Y \in \Gamma K \quad (19)$$

Now, choosing splittings  $\alpha : \mathcal{F} \rightarrow \mathfrak{g}$  for  $\rho$ , and  $\beta : TM \rightarrow \mathcal{F}$  for the inclusion, we have induced decompositions

$$\mathfrak{g} \cong K \oplus \mathcal{F}, \quad TM \cong \nu \oplus \mathcal{F}.$$

As mentioned above, the formal difference  $K - \nu$  (view it as a graded complex with  $K$  in even degree,  $\nu$  in odd degree, and zero differential) is a representation of  $\mathfrak{g}$ . On the other hand, any  $\mathcal{F}$ -connection  $\nabla$  on  $\mathcal{F}$  defines a  $\mathfrak{g}$ -connection on the super-complex

$$D(\mathcal{F}) : \mathcal{F} \begin{array}{c} \xrightarrow{0} \\ \xrightarrow{id} \end{array} \mathcal{F}$$

(and its homotopy class does not depend on  $\nabla$ ). Hence one has an induced  $\mathfrak{g}$ -connection  $\nabla^{\alpha, \beta}$  on  $\text{Ad}(\mathfrak{g})$ , so that  $(\text{Ad}(\mathfrak{g}), \nabla^{\alpha, \beta})$  is isomorphic to  $(K - \nu) \oplus D(\mathcal{F})$ . Explicitly,

$$\begin{aligned}\nabla_X^{\alpha, \beta}(Y) &= [X, Y - \alpha\rho(Y)] + \alpha\nabla_{\rho(Y)}(\rho X) \\ \nabla_X^{\alpha, \beta}(V) &= [\rho(X), V] - \beta[\rho(X), V] + \nabla_{\rho(X)}(\beta(V))\end{aligned}$$

for all  $X, Y \in \Gamma\mathfrak{g}, V \in \mathcal{X}(M)$ . Note that the second part of the following proposition can also be derived from (iv) of Proposition 1.

**Proposition 3** *Assume that  $\mathfrak{g}$  is regular. For any  $\mathcal{F}$ -connection  $\nabla$  on  $\mathcal{F}$ , and any splittings  $\alpha, \beta$  as above,  $\nabla^{\alpha, \beta}$  is a basic  $\mathfrak{g}$ -connection. In particular*

$$u_{2p-1}^{\mathfrak{g}} = u_{2p-1}(K) - u_{2p-1}(\nu),$$

where  $K$  and  $\nu$  are the representations of  $\mathfrak{g}$  defined by Bott's formulas (18), (19).

*Proof:* We have  $\nabla^{\alpha,\beta} = \nabla^{\text{ad}} + [\theta, \partial]$ , where  $\theta$  is the  $\text{End}(\text{Ad}(\mathfrak{g}))$ -valued non-linear form which is given by

$$\theta(X)(V) = \alpha[\rho(X), \beta(V)] - \alpha\beta[\rho(X), V] - [X, \alpha\beta(V)] + \alpha\nabla_{\rho(X)}\beta(V)$$

for  $V \in \Gamma(TM)$ , while  $\theta(X) = 0$  on  $\mathfrak{g}$  (we leave to the reader to check that the previous formula is  $C^\infty(M)$ -linear on  $V$ ).  $\square$ .

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