

*Radon transformation on reductive
symmetric spaces: support theorems*

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Cover illustration:

The illustration on the cover visualizes a single horosphere in the Lie group $G = SL(2, \mathbf{R})$, considered here as the symmetric space $(G \times G)/\text{diag}(G)$. The group G is depicted as an open solid torus via the diffeomorphism described in [DK00, p. 13 – 16]. The surface inside the torus represents the image of the horosphere

$$\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} : x, y \in \mathbf{R} \right\}$$

under this diffeomorphism.

Radon transformation on reductive symmetric spaces: support theorems

Radontransformatie op reductieve symmetrische ruimten: dragerstellingen

(met een samenvatting in het Nederlands)

Proefschrift

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Voor mijn grootouders

**Al's menschen heerlickheyt, al's menschen pracht en roem,
Is niet als gras en hoy, oft als een veldsche bloem.**

Joost van den Vondel, *De vernieuwde gulden winckel der kunstlievende Nederlanders*, naar de eerste brief van Petrus; Paulus van Ravensteyn voor Dirck Pietersz. Pers, Amsterdam, 1622.

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Introduction

Let G be a connected semisimple Lie group with finite center, let $G = KAN$ be an Iwasawa decomposition of G and let M be the centralizer of A in K . A horosphere in the Riemannian symmetric space $X = G/K$ of non-compact type is a submanifold of the form $gN \cdot K$, with $g \in G$. The set of horospheres is in bijection with the manifold $\mathcal{E} = G/MN$ via the map

$$E : g \cdot MN \mapsto gN \cdot K.$$

The *horospherical transform* on X is the Radon transform \mathcal{R} mapping a compactly supported smooth function ϕ on X to the function

$$\mathcal{R}\phi : g \cdot MN \mapsto \int_N \phi(gn \cdot K)$$

on \mathcal{E} . In [Hel73, Lemma 8.1] S. Helgason proved the following support theorem for this transform.

Let ϕ be a compactly supported smooth function and let V be a closed ball in X . Assume that

$$\mathcal{R}\phi(\xi) = 0 \quad \text{whenever} \quad E(\xi) \cap V = \emptyset.$$

Then

$$\phi(x) = 0 \quad \text{for} \quad x \notin V.$$

Note that this theorem implies that the horospherical transform is injective on the space of compactly supported smooth functions.

In this thesis, Helgason's result is generalized to a support theorem for a class of Radon transforms (including the horospherical transforms) on a reductive symmetric space $X = G/H$ with G a real reductive Lie group of the Harish-Chandra class

and H an essentially connected open subgroup of the fixed-point subgroup G^σ of an involution σ on G .

Let θ be a Cartan involution of G commuting with σ . For each $\sigma \circ \theta$ -stable parabolic subgroup P with Langlands decomposition $M_P A_P N_P$ we consider the Radon transform \mathcal{R}_P mapping a function ϕ on X to the function on the homogeneous space $\mathcal{E}_P = G/(M_P A_P \cap H)N_P$ given by

$$\mathcal{R}_P \phi(g \cdot \xi_P) = \int_{N_P} \phi(gn \cdot H) dn.$$

Here ξ_P denotes the coset $e \cdot (M_P A_P \cap H)N_P$ containing the unit element e . This Radon transform, which is initially defined for compactly supported smooth functions, can be extended to a large class of distributions on X .

If P_0 is a minimal $\sigma \circ \theta$ -stable parabolic subgroup of G contained in P , then A_P is contained in A_{P_0} . The Lie algebra \mathfrak{a}_{P_0} of A_{P_0} is σ -stable and decomposes as the direct sum of the $+1$ and -1 eigenspace for σ . The latter space is denoted by \mathfrak{a}_q . The connected abelian Lie subgroup of G with Lie algebra \mathfrak{a}_q is denoted by A_q .

The maps

$$K \times A_q \rightarrow X; \quad (k, a) \mapsto ka \cdot H$$

and

$$K \times A_q \rightarrow \mathcal{E}_P; \quad (k, a) \mapsto ka \cdot \xi_P$$

are surjective, just as in the Riemannian case. For a subset B of \mathfrak{a}_q , we define

$$X(B) = K \exp(B) \cdot H \quad \text{and} \quad \mathcal{E}_P(B) = K \exp(B) \cdot \xi_P.$$

The support of a transformed function or distribution is non-compact in general. In fact, if the support of a distribution μ is contained in $X(B)$ for some compact convex subset B of \mathfrak{a}_q that is invariant under the action of the normalizer of \mathfrak{a}_q in $K \cap H$, then the support of the Radon transformed distribution $\mathcal{R}_P \mu$ is contained in $\mathcal{E}_P(B + \Gamma_P)$, where Γ_P is the cone in \mathfrak{a}_q spanned by the root vectors corresponding to roots that are positive with respect to P . The support theorem that we prove in this thesis is a partial converse to this statement for distributions μ in a suitable class of distributions, containing the compactly supported ones:

Theorem 5.2.1 *Let B be a convex compact subset of \mathfrak{a}_q that is invariant under the action of the normalizer of \mathfrak{a}_q in $M_P \cap K \cap H$. If*

$$\text{supp}(\mathcal{R}_P \mu) \subseteq \mathcal{E}_P(B + \Gamma_P),$$

then

$$\text{supp}(\mu) \subseteq X(C),$$

where C is the maximal subset of $B + \Gamma_P$ that is invariant under the action of the normalizer of \mathfrak{a}_q in $K \cap H$.

If $K = H$ and P is a minimal parabolic subgroup of G , then C equals B . Our theorem reduces then to the support theorem of Helgason for the horospherical transform on a Riemannian symmetric space of non-compact type. Just as in the Riemannian case, the support theorem implies injectivity of the Radon transform.

After a short introduction to the theory of Radon transformation in Chapter 1 and a short introduction to the theory of reductive Lie groups of the Harish-Chandra class, their parabolic subgroups and reductive symmetric spaces in Chapter 2, we introduce the transforms under consideration and establish some of their properties in Chapter 3. In Chapter 4 we consider the horospherical transform. A horospherical transform is a Radon transform \mathcal{R}_{P_0} as above, with P_0 a minimal $\sigma \circ \theta$ -stable parabolic subgroup. The horospherical transform is related to one component of the so called unnormalized Fourier transform on X corresponding to the parabolic subgroup P_0 . As in the proof for [Hel73, Lemma 8.1], given the support of the horospherical transform of a function, it is possible to derive Paley-Wiener type estimates for this component of the unnormalized Fourier transform. The unnormalized Fourier transform is in turn related to the normalized Fourier transform corresponding to the opposite parabolic subgroup \overline{P}_0 for K -finite functions on X . The Paley-Wiener estimates for the one component of the unnormalized Fourier transform yield Paley-Wiener estimates for one component of the normalized Fourier transform. A first support theorem (Proposition 4.6.2) is then obtained from these estimates by using the Fourier inversion formula of Van den Ban and Schlichtkrull ([BS99, Theorem 4.7]) and then by mimicking the proof for the classical Paley-Wiener theorem for the Euclidean Fourier transform. However, this result gives information only on the intersection of the support of the function with a certain subset of X related to P_0 . Using the equivariance of the horospherical transform, we then obtain the full support theorem (Theorem 4.6.4) for this transform. In Chapter 5 we generalize this (in Theorem 5.2.1) to a support theorem for the Radon transform \mathcal{R}_P on distributions for P an arbitrary $\sigma \circ \theta$ -stable parabolic subgroup.

Chapter 1

Radon transformation and support theorems

This chapter is meant as a short introduction to the theory of Radon transformation. After a few historical comments, we first introduce the concept of a double fibration in Section 1.2. For each double fibration satisfying a few technical conditions we then define two integral transforms, called Radon transforms. This is described in Section 1.3. In Section 1.4 the principal questions that are raised in the theory of Radon transformation are discussed. Finally in Section 1.5 we describe a class of theorems that describe the support of a function in terms of the support of its Radon transform: the support theorems.

An example of a Radon transform is the integral transform obtained by mapping a suitable function ϕ on \mathbf{R}^n to the function $\mathcal{R}_k\phi$ on the manifold of k -dimensional affine subspaces ξ in \mathbf{R}^n , given by

$$\mathcal{R}_k\phi(\xi) = \int_{\xi} \phi(x) dx.$$

Throughout this chapter we use this example as an illustration of the general theory.

1.1 Integral transforms

In 1906 H.B.A. Bockwinkel published an article ([Boc06]) in the proceedings of the Koninklijke Akademie van Wetenschappen (the Royal Netherlands Academy of Sciences) on electromagnetic fields in a crystal, in which he wrote the following passage on the determination of the electromagnetic field in a crystal due to a periodic electromotive force (E.M.F.).

Our method will consist in reducing the question to one of plane waves, by using a formula, proved by Prof. LORENTZ. In this formula a continuous function of the coordinates is represented by an integral over the solid angles of all cones having their vertices in O and filling the whole space. If the $E. M. F.$ is \mathfrak{E}^e then

$$\mathfrak{E}^e = - \int \frac{1}{8\pi^2} \frac{\partial^2 \mathfrak{B}}{\partial n^2} d\omega, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where dn is the element of a line of arbitrary direction within the cone $d\omega$ and \mathfrak{B} a vector given by

$$\mathfrak{B} = \int \mathfrak{C}^e d\sigma, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the integral being taken over the plane, passing through the point considered, perpendicularly to n . Hence, \mathfrak{B} depends on the coordinates, but in such a way as to be constant in every plane perpendicular to n .

More explicitly, Bockwinkel considered the transform mapping a function on \mathbf{R}^3 to a function on the manifold of affine 2-dimensional planes by integrating over these planes as is stated in formula (2). Formula (1), which according to Bockwinkel was proved by H.A. Lorentz, is an inversion formula for this transform.

In 1917 J. Radon considered in his article [Rad17] a generalization of this transform: a suitable function ϕ on \mathbf{R}^n is mapped to the function $\mathcal{R}\phi$ on the manifold of affine hyperplanes ξ in \mathbf{R}^n by defining

$$\mathcal{R}\phi(\xi) = \int_{\xi} \phi(x) dx.$$

In his article Radon derived an inversion formula for this transform, generalizing the formula obtained by Lorentz.

1.2 Double fibrations

In [Hel66] Helgason used the concept of incidence, introduced by Chern in 1942 ([Che42]), to generalize the integral transform considered by Radon and several other transforms to a class of integral transforms named after Radon.

In the theory of Radon transformation as introduced by Helgason one considers a *double fibration of homogeneous spaces*

$$\begin{array}{ccc} & Z = G/(S \cap T) & \\ \swarrow \Pi_X & & \searrow \Pi_E \\ G/S = X & & E = G/T \end{array} \quad (1.2.1)$$

where G is a Lie group, S and T are two closed subgroups of G and Π_X and Π_E are the canonical projections. The map from Z to $X \times E$

$$g \cdot (S \cap T) \mapsto (g \cdot S, g \cdot T)$$

is an embedding. Via this embedding points in Z are identified with points in $X \times E$.

A double fibration defines an incidence relation: $x \in X$ and $\xi \in E$ are said to be *incident* if $(x, \xi) \in Z$. Note that x and ξ are incident if and only if $\xi \in \Pi_E(\Pi_X^{-1}(x))$, or equivalently, $x \in \Pi_X(\Pi_E^{-1}(\xi))$. If the set-valued maps

$$X \ni x \mapsto \Pi_E(\Pi_X^{-1}(x)) \quad \text{and} \quad E \ni \xi \mapsto \Pi_X(\Pi_E^{-1}(\xi))$$

are both injective, then following Helgason, we say that S and T are *transversal*.

Example 1.2.1. Let $n \in \mathbf{N}$. Let $G = \mathbf{O}(n) \ltimes \mathbf{R}^n$ be the semidirect product of the orthogonal group $\mathbf{O}(n)$ and \mathbf{R}^n , i.e. G is the group consisting of pairs $(O, t) \in \mathbf{O}(n) \times \mathbf{R}^n$ equipped with the multiplication given by

$$(O_1, t_1)(O_2, t_2) = (O_1 O_2, O_2^{-1} t_1 + t_2) \quad ((O_{1,2}, t_{1,2}) \in G).$$

Let $k \in \mathbf{Z}$, with $0 < k < n$. We define E_k to be the manifold of k -dimensional affine subspaces of \mathbf{R}^n . The group G has a natural transitive action on \mathbf{R}^n given by

$$(O, t) \cdot x = O(x + t) \quad ((O, t) \in G, x \in \mathbf{R}^n).$$

In fact G is the group of isometries of \mathbf{R}^n . The action of G on \mathbf{R}^n induces in an obvious way a transitive action of G on E_k . We fix an element $\xi_0 \in E_k$ containing

the origin $0 \in \mathbf{R}^n$ and define S to be the stabilizer in G of the origin and T to be the stabilizer in G of ξ_0 . Then

$$S = \mathbf{O}(n) \ltimes \{0\} \quad \text{and} \quad T = (\mathbf{O}(\xi_0) \times \mathbf{O}(\xi_0^\perp)) \ltimes \xi_0,$$

where ξ_0^\perp denotes the orthocomplement of ξ_0 . Note that $\mathbf{O}(\xi_0) \times \mathbf{O}(\xi_0^\perp)$ equals the stabilizer of ξ_0 in $\mathbf{O}(n)$. Now $\mathbf{R}^n \simeq G/S$ and $\mathcal{E}_k \simeq G/T$. Accordingly, the homogeneous space $G/(S \cap T)$ is diffeomorphic to the submanifold of $\mathbf{R}^n \times \mathcal{E}_k$ given by

$$Z = \{(x, \xi) \in \mathbf{R}^n \times \mathcal{E}_k : x \in \xi\}.$$

The double fibration

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ \mathbf{R}^n & & \mathcal{E}_k \end{array} \quad (1.2.2)$$

describes the incidence relations between points in \mathbf{R}^n and k -dimensional affine subspaces in \mathbf{R}^n : $x \in \mathbf{R}^n$ and $\xi \in \mathcal{E}_k$ are incident if and only if $x \in \xi$. Since an affine subspace is determined by the points in it and a point in \mathbf{R}^n is determined by the k -dimensional affine subspaces through it, the subgroups S and T are transversal.

1.3 Radon transforms

Following L. Schwartz, we denote spaces of compactly supported smooth functions and spaces of smooth functions by \mathcal{D} and \mathcal{E} , respectively. Spaces of distributions and spaces of compactly supported distributions we denote by \mathcal{D}' and \mathcal{E}' , respectively.

To be able to define the Radon transforms for a double fibration (1.2.1), we need the following assumptions

- (A) There exist non-zero Radon measures $d_{S \cap T} s$ and $d_{S \cap T} t$ on $S/(S \cap T)$ and $T/(S \cap T)$ invariant under S and T , respectively.
- (B1) For every $\phi \in \mathcal{D}(X)$ the function $T/(S \cap T) \rightarrow \mathbf{C}$ given by

$$t \cdot (S \cap T) \mapsto \phi(t \cdot S)$$

is absolutely integrable with respect to $d_{(S \cap T)} t$.

(B2) For every $\psi \in \mathcal{D}(\mathcal{E})$ the function $S/(S \cap T) \rightarrow \mathbb{C}$ given by

$$s \cdot (S \cap T) \mapsto \phi(s \cdot T)$$

is absolutely integrable with respect to $d_{(S \cap T)s}$.

The *Radon transforms* \mathcal{R} and \mathcal{S} for the double fibration (1.2.1) are defined to be the transforms mapping functions $\phi \in \mathcal{D}(X)$ and $\psi \in \mathcal{D}(\mathcal{E})$ to the functions

$$\mathcal{R}\phi : \mathcal{E} \rightarrow \mathbb{C} \quad \text{and} \quad \mathcal{S}\psi : X \rightarrow \mathbb{C}$$

given by

$$\mathcal{R}\phi(g \cdot T) = \int_{T/(S \cap T)} \phi(gt \cdot S) d_{S \cap T} t$$

and

$$\mathcal{S}\psi(g \cdot S) = \int_{S/(S \cap T)} \psi(gs \cdot T) d_{S \cap T} s,$$

respectively. If the condition

(B) ST is a closed subset of G

is satisfied, then (B1) and (B2) hold and \mathcal{R} and \mathcal{S} are continuous operators from $\mathcal{D}(X)$ to $\mathcal{E}(\mathcal{E})$ and from $\mathcal{D}(\mathcal{E})$ to $\mathcal{E}(X)$, respectively. (See [Hel99, Section II.1].) Note that the Radon transforms are G -equivariant.

If moreover

(C) There exist non-zero G -invariant Radon measures dx and $d\xi$ on X and \mathcal{E} ,

then \mathcal{S} and \mathcal{R} are dual to each other in the sense that for all $\phi \in \mathcal{D}(X)$ and $\psi \in \mathcal{D}(\mathcal{E})$

$$\int_{\mathcal{E}} \mathcal{R}\phi(\xi) \psi(\xi) d\xi = \int_X \phi(x) \mathcal{S}\psi(x) dx, \quad (1.3.1)$$

assuming the measures are suitably normalized. This duality allows to extend the Radon transform to the space of compactly supported distributions: for $\mu \in \mathcal{E}'(X)$ and $\nu \in \mathcal{E}'(\mathcal{E})$, we define $\mathcal{R}\mu \in \mathcal{D}'(\mathcal{E})$ and $\mathcal{S}\nu \in \mathcal{D}'(X)$ to be the distributions

$$\mathcal{R}\mu : \mathcal{D}(\mathcal{E}) \ni \psi \mapsto \mu(\mathcal{S}\psi) \quad \text{and} \quad \mathcal{S}\nu : \mathcal{D}(X) \ni \phi \mapsto \nu(\mathcal{R}\phi). \quad (1.3.2)$$

Example 1.3.1. We continue in the setting of Example 1.2.1.

The homogeneous space $T/(S \cap T)$ is diffeomorphic to ξ_0 . The Lebesgue measure on ξ_0 is T -invariant. Since S is compact, there exists an S -invariant measure on $S/(S \cap T)$. Therefore (A) holds in this case. Furthermore, since S is compact and T is closed in G , the set ST is closed in G and hence (B) holds. The Lebesgue measure on \mathbf{R}^n is G -invariant, just as the measure on \mathcal{E}_k defined by

$$\mathcal{D}(\mathcal{E}_k) \ni f \mapsto \int_{\mathbf{O}(n)} \int_{\xi_0^\perp} f(O(t + \xi_0)) dt dO,$$

where dO is the Haar measure on $\mathbf{O}(n)$ and dt the Lebesgue measure on ξ_0^\perp . We therefore finally conclude that (C) is satisfied in this case. The Radon transforms for the double fibration (1.2.2) are given by

$$\mathcal{R}_k \phi(\xi) = \int_{\xi} \phi(x) d_{\xi} x \quad (\phi \in \mathcal{D}(\mathbf{R}^n), \xi \in \mathcal{E}_k), \quad (1.3.3)$$

where $d_{\xi} x$ is the Lebesgue measure on ξ . The group $\mathbf{O}(n)$ acts on the compact submanifold $\{\xi \in \mathcal{E}_k : x \in \xi\}$ of \mathcal{E}_k in a natural way. Let $d_x \xi$ be the normalized invariant measure on this submanifold. Then

$$\mathcal{S}_k \psi(x) = \int_{\xi \ni x} \psi(\xi) d_x \xi \quad (\psi \in \mathcal{D}(\mathcal{E}_k), x \in \mathbf{R}^n). \quad (1.3.4)$$

1.4 Principal problems

Although many rather strong theorems have been proved for particular examples of Radon transforms, it seems to be impossible to obtain strong results in the general abstract setup described in the previous section. As Helgason mentions in [Hel99, Section 2.2] it may be better to regard this setup as a framework for examples rather than as a set of axioms for a general theory.

When dealing with a certain example one often considers the following problems.

Domain. When considering a double fibration (1.2.1) satisfying assumption (A), the first problem is to find a suitable space of functions or distributions on which the Radon transforms can be defined. If assumption (B) is satisfied then at least the integrals are absolutely convergent for compactly supported continuous functions, but in important examples the same holds for larger spaces of functions.

Range. Once a domain for the Radon transform has been determined, one often tries to determine the image under the transform of the whole domain or of certain subspaces. More generally, one tries to relate function spaces on X and \mathcal{E} to each other by means of the Radon transform. In particular it is usually interesting to find the kernel of the transform.

Inversion formulae. If the transform is injective on the domain or on a certain subspace of the domain, then one of the main problems is to invert the transform.

Invariant differential operators. For the analysis of G -invariant differential operators on the homogeneous space $X = G/S$, it may be useful to consider the question if there exists a map $D \mapsto \widehat{D}$ from the space $\mathbf{D}(X)$ of invariant differential operators on X to the space $\mathbf{D}(\mathcal{E})$ of invariant differential operators on \mathcal{E} with the property that for every function ϕ in the domain of \mathcal{R} and every $D \in \mathcal{D}(X)$

$$\mathcal{R}(D\phi) = \widehat{D}\mathcal{R}\phi.$$

As mentioned before, it is hard to obtain any results that hold in the general setting introduced in Section 1.2 and 1.3, but the formalism can help to prove results for specific Radon transforms. As an example we now derive an inversion formula for the k -plane transform introduced in Example 1.3.1, which is due to Helgason (see [Hel99, Theorem 6.2]).

Example 1.4.1. We continue in the setting of Example 1.2.1 and Example 1.3.1. For the domain of the Radon transform \mathcal{R}_k we choose the space $\mathcal{S}(\mathbf{R}^n)$ of Schwartz functions on \mathbf{R}^n and for the domain of \mathcal{S}_k we choose $\mathcal{E}(\mathcal{E}_k)$. We denote the space of tempered distributions on \mathbf{R}^n by $\mathcal{S}'(\mathbf{R}^n)$.

To be able to formulate the inversion formula, we first need to introduce fractional powers of the laplacian. Let \mathcal{V} be the subspace of $\mathcal{S}'(\mathbf{R}^n)$ consisting of tempered distributions u with the property that the Fourier transform $\mathcal{F}u$ of u is a locally integrable function. Let $\alpha \geq 0$. We define the α -th power $(-\Delta)^\alpha$ of the differential operator $-\Delta$ to be the operator on \mathcal{V} given by

$$(-\Delta)^\alpha u = \mathcal{F}^{-1}(\|\cdot\|^{2\alpha} \mathcal{F}u) \quad (u \in \mathcal{V}).$$

Note that if $\mathcal{F}u$ is a locally integrable function, then $\|\cdot\|^{2\alpha} \mathcal{F}u$ is locally integrable as well and therefore a distribution. In fact it is a tempered distribution and thus the inverse Fourier transform of $\|\cdot\|^{2\alpha} \mathcal{F}u$ is well defined. Note further that if α is a nonnegative integer, then $(-\Delta)^\alpha$ is the ordinary α -th power of $-\Delta$.

Theorem 1.4.2. *Let $\phi \in \mathcal{S}(\mathbf{R}^n)$. Then $S_k(\mathcal{R}_k\phi)$ is a tempered distribution of which the Fourier transform is a locally integrable function. Moreover,*

$$\phi = \frac{\Gamma\left(\frac{n-k}{2}\right)}{(4\pi)^{\frac{k}{2}} \Gamma\left(\frac{n}{2}\right)} (-\Delta)^{\frac{k}{2}} S_k(\mathcal{R}_k\phi). \quad (1.4.1)$$

To prove the theorem we use some elementary distribution theory. For completeness for several well known results, we refer to the recently published book [DK10] by the authors former teachers, J.J. Duistermaat and J.A.C. Kolk.

The plan of the proof is as follows. We analyze the function $S_k(\mathcal{R}_k\phi)$ using the equivariance of the Radon transforms and the explicit formulas (1.3.3) and (1.3.4). We then determine the precise form of the Fourier transform of $S_k(\mathcal{R}_k\phi)$ up to a multiplicative constant and show that up to the prefactor the inversion formula is correct. Finally, we perform some explicit calculations on one specific Schwartz function to determine the constant.

We start with a lemma.

Lemma 1.4.3. *Let $m > -n$ and let $\mu \in \mathcal{D}'(\mathbf{R}^n)$. If μ is $O(n)$ -invariant and homogeneous of degree m , then there exists a constant $c \in \mathbf{C}$, such that*

$$\mu = c \|\cdot\|^m.$$

Proof. Let $p \leq n$ and let $\Phi : \mathbf{R}^n \rightarrow \mathbf{R}^p$ be a smooth submersion. We recall that the pull-back along Φ

$$\mathcal{E}(\mathbf{R}^p) \rightarrow \mathcal{E}(\mathbf{R}^n); \quad \phi \mapsto \Phi^*\phi = \phi \circ \Phi$$

extends to a continuous linear mapping $\mathcal{D}'(\mathbf{R}^p) \rightarrow \mathcal{D}'(\mathbf{R}^n)$. Here smooth functions on \mathbf{R}^n and \mathbf{R}^p are identified with distributions via the Lebesgue measures on \mathbf{R}^n and \mathbf{R}^p , respectively. If Φ is a diffeomorphism, then

$$\Phi^*\mu = (j_{\Phi^{-1}}\mu) \circ (\Phi^{-1})^*,$$

where $j_{\Phi^{-1}}$ is the absolute value of the Jacobian of Φ^{-1} .

Using the method of [DK10, Exercise 13.12] to describe the distributions on $\mathbf{R}^{n+1} \setminus \{0\}$ that are invariant under Lorentz-transformations, one shows that the restriction of μ to $\mathbf{R}^n \setminus \{0\}$ equals the pullback of a distribution ν on $\mathbf{R}_{>0}$ along the norm-function $\|\cdot\|$. Since μ is homogeneous of degree m , the distribution ν has to be homogeneous of the same degree, hence up to a multiplicative constant ν equals

the function $\mathbf{R}_{>0} \ni x \mapsto x^m$. This implies that outside the origin μ is equal to the function $c\|\cdot\|^m$ for some $c \in \mathbf{C}$. Since this function is locally integrable on \mathbf{R}^n , it follows that $\mu = c\|\cdot\|^m + \rho$ for some distribution $\rho \in \mathcal{E}'(\mathbf{R}^n)$ supported in the origin. Since ρ has support contained in $\{0\}$, it follows that ρ is of finite order, say d , and is of the form

$$\rho = \sum_{|\beta| \leq d} c_\beta \partial^\beta \delta.$$

(See e.g. [DK10, Theorem 8.10].) Here δ denotes the Dirac delta distribution supported in the origin and the sum is taken over all multi-indices $\beta = (\beta_1, \dots, \beta_n) \in \mathbf{Z}_{\geq 0}^n$ of length $|\beta| = \sum_{j=1}^n \beta_j$ at most d . Both μ and $c\|\cdot\|^m$ are homogeneous of degree m , hence the same has to hold for ρ . Since $\partial^\beta \delta$ is homogeneous of degree $-n - |\beta|$, it follows that $\rho = 0$. We conclude that there exists a $c \in \mathbf{C}$ such that $\mu = c\|\cdot\|^m$. \square

Proof of Theorem 1.4.2. Let μ be the tempered distribution

$$\mu = \delta \circ \mathcal{S}_k \circ \mathcal{R}_k = \mathcal{S}_k(\mathcal{R}_k \delta),$$

where δ denotes the Dirac delta distribution supported in the origin. Since both \mathcal{S}_k and \mathcal{R}_k are G -equivariant transforms, and δ is $\mathbf{O}(n)$ -invariant, we see that for every $O \in \mathbf{O}(n)$

$$O^* \mu = \delta \circ \mathcal{S}_k \circ \mathcal{R}_k \circ (O^{-1})^* = \delta \circ (O^{-1})^* \circ \mathcal{S}_k \circ \mathcal{R}_k = O^*(\delta) \circ \mathcal{S}_k \circ \mathcal{R}_k = \mu.$$

Therefore μ is $\mathbf{O}(n)$ -invariant. Moreover, if $c > 0$, then

$$c^* \mu(\phi) = c^{-n} \mu((c^{-1})^* \phi) = c^{-n} \int_{\xi \geq 0} \int_{\xi} \phi(c^{-1}x) d_\xi x d_0 \xi = c^{k-n} \mu(\phi)$$

for every $\phi \in \mathcal{D}(\mathbf{R}^n)$, hence μ is homogeneous of degree $k - n$. By Lemma 1.4.3 there exists a constant $c \in \mathbf{C}$ such that

$$\mu = c\|\cdot\|^{k-n}.$$

Note that μ is a tempered distribution.

Now let $\phi \in \mathcal{S}(\mathbf{R}^n)$. For $x \in \mathbf{R}^n$ we define T_x to be the translation $y \mapsto y + x$. Furthermore, we define $\sigma : \mathbf{R}^n \rightarrow \mathbf{R}^n$ to be the reflection $y \mapsto -y$ in the origin. Since both \mathcal{R}_k and \mathcal{S}_k are G -equivariant,

$$\mathcal{S}_k(\mathcal{R}_k \phi)(x) = T_x^*(\mathcal{S}_k(\mathcal{R}_k \phi))(0) = \delta \circ \mathcal{S}_k \circ \mathcal{R}_k(T_x^* \phi) = \mu(T_x^* \phi).$$

Furthermore, since μ is $O(n)$ -invariant, $\sigma^* \mu = \mu$ and it follows that

$$\mathcal{S}_k(\mathcal{R}_k \phi)(x) = \mu(\sigma^*(T_x^* \phi)) = \mu * \phi(x),$$

where $*$ denotes the convolution product. Since μ is a tempered distribution, the convolution $\mu * \phi$ defines a tempered distribution. The Fourier transform of $\mu * \phi$ equals

$$\mathcal{F}(\mu * \phi) = (\mathcal{F}\phi)\mathcal{F}\mu. \quad (1.4.2)$$

A tempered distribution is $O(n)$ -invariant if and only if its Fourier transform is $O(n)$ -invariant and it is homogeneous of degree $k - n$ if and only if its Fourier transform is homogeneous of degree $-k$. (See e.g. [DK10, Exercise 14.30].) Since μ is $O(n)$ -invariant and homogeneous of degree $k - n$, it follows from Lemma 1.4.3 that there exists a constant $c' \in \mathbb{C}$ such that

$$\mathcal{F}\mu = c' \|\cdot\|^{-k}. \quad (1.4.3)$$

Combining (1.4.2) and (1.4.3), we obtain that the Fourier transform of $\mathcal{S}_k(\mathcal{R}_k \phi)$ equals $(\mathcal{F}\phi)\mathcal{F}\mu = c' \|\cdot\|^{-k} \mathcal{F}\phi$, which is a locally integrable function. Furthermore,

$$(-\Delta)^{\frac{k}{2}} \mathcal{S}_k(\mathcal{R}_k \phi) = \mathcal{F}^{-1}(\|\cdot\|^k \mathcal{F}(\mu * \phi)) = c' \mathcal{F}^{-1}(\|\cdot\|^k \|\cdot\|^{-k} \mathcal{F}\phi) = c' \phi.$$

It remains to determine the constant c' . To do this, we calculate both sides of the identity

$$\mu(\phi_0) = c' \mathcal{F}^{-1}(\|\cdot\|^{-k})(\phi_0), \quad (1.4.4)$$

where $\phi_0 : \mathbf{R}^n \rightarrow \mathbf{R}$ is the function given by

$$\phi_0(x) = e^{-\frac{\|x\|^2}{2}} \quad (x \in \mathbf{R}^n).$$

We first compute the left-hand side:

$$\mu(\phi_0) = \mathcal{S}_k(\mathcal{R}_k \phi_0)(0) = \int_{\xi \geq 0} \int_{\xi} \phi_0(x) dx d\xi.$$

Since the integrand is $O(n)$ -invariant, the inner integral is independent of ξ and equal to

$$\int_{\mathbf{R}^k} e^{-\frac{\|x\|^2}{2}} dx = \left(\int_{\mathbf{R}} e^{-\frac{y^2}{2}} dy \right)^k = (2\pi)^{\frac{k}{2}}$$

hence $\mu(\phi_0) = (2\pi)^{\frac{k}{2}}$. The right-hand side of (1.4.4) equals

$$\begin{aligned} c' \int_{\mathbf{R}^n} \|x\|^{-k} \mathcal{F}^{-1} \phi_0(x) dx &= c' (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} \|x\|^{-k} e^{-\frac{\|x\|^2}{2}} dx \\ &= c' \frac{2^{\frac{2-n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_{\mathbf{R}_{>0}} r^{n-k-1} e^{-\frac{r^2}{2}} dr = c' \frac{\Gamma\left(\frac{n-k}{2}\right)}{2^{\frac{k}{2}} \Gamma\left(\frac{n}{2}\right)}. \end{aligned}$$

We conclude that

$$c' = \frac{(4\pi)^{\frac{k}{2}} \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)}.$$

□

An easy computation shows that

$$\Delta \circ \mathcal{S}_{n-1} = \mathcal{S}_{n-1} \circ \square, \quad (1.4.5)$$

where \square is the differential operator on \mathcal{E}_{n-1} given by

$$\square \psi(p\omega + \omega^\perp) = \frac{\partial^2}{\partial p^2} \psi(p\omega + \omega^\perp) \quad (\psi \in \mathcal{E}(\mathcal{E}_{n-1}), p \in \mathbf{R}, \omega \in S^{n-1}).$$

The inversion formula (1) in [Boc06] for the Radon transform \mathcal{R}_2 on \mathbf{R}^3 is obtained from the inversion formula in Theorem 1.4.2 by applying (1.4.5). The difference in the prefactor is due to the fact that Bockwinkel used the (unnormalized) Euclidean measure on the sphere, while we use normalized measures on the submanifolds $\{\xi \in \mathcal{E}_2 : \xi \ni x\}$ of \mathcal{E}_2 for $x \in \mathbf{R}^3$.

1.5 Support theorems

As mentioned in Section 1.4, when considering a specific Radon transform, one of the principal problems is to determine its range and the image of certain subspaces of the domain. Related to these problems are the so called support theorems. Given the support of a function, it is usually not very difficult to give bounds on the support of the Radon transform of that function. A support theorem is the converse to such a result: it is a description of the support of a function in terms of the support of the Radon transformed function.

Example 1.5.1. We continue in the setting of Example 1.2.1. See also Example 1.3.1 and 1.4.1.

Let V be a compact convex subset of \mathbf{R}^n and let $\phi \in \mathcal{S}(\mathbf{R}^n)$. It is clear that if $\text{supp}(\phi) \subseteq V$, then $\mathcal{R}_k \phi(\xi) = 0$ whenever $\xi \in \mathcal{E}_k$ satisfies $\xi \cap V = \emptyset$. The support theorem of Helgason ([Hel65, Theorem 2.1]) is the converse of this statement.

Theorem 1.5.2. *Let $\phi \in \mathcal{S}(\mathbf{R}^n)$ and let V be a compact convex subset of \mathbf{R}^n . Assume that*

$$\mathcal{R}_k \phi(\xi) = 0 \quad \text{whenever} \quad \xi \cap V = \emptyset.$$

Then

$$\text{supp}(\phi) \subseteq V.$$

We sketch the proof given by J.J.O.O. Wiegnerinck in [Wie85]. Although the Radon transform \mathcal{R}_k does not belong to the class of Radon transforms we consider in the later Chapters 3, 4 and 5, there are many similarities between Wiegnerinck's proof and the proof of the support theorem (Theorem 5.2.1) that we give in Chapter 4 and Chapter 5.

Since every affine hyperplane that does not intersect with a given subset V of \mathbf{R}^n is a union of k -dimensional affine subspaces that do not intersect with V either, the following statement holds.

Observation 1.5.3. *Let $\phi \in \mathcal{S}(\mathbf{R}^n)$ and let V be a closed convex subset in \mathbf{R}^n . Assume that*

$$\mathcal{R}_k \phi(\xi) = 0 \quad \text{for every } \xi \in \mathcal{E}_k \text{ with } \xi \cap V = \emptyset.$$

Then

$$\mathcal{R}_{n-1} \phi(\zeta) = 0 \quad \text{for every } \zeta \in \mathcal{E}_{n-1} \text{ with } \zeta \cap V = \emptyset.$$

Because of this observation we only need to consider the hyperplane transform \mathcal{R}_{n-1} . Furthermore, since a convex compact subset V of \mathbf{R}^n equals the intersection of all closed balls containing V , it suffices to only consider the case in which V is a closed ball. Moreover, since \mathcal{R}_{n-1} intertwines the actions of the group of translations on \mathcal{R}^n and \mathcal{E}_{n-1} respectively, without loss of generality we can assume that the ball is centered at the origin. It thus remains to prove the following (seemingly weaker) theorem.

Theorem 1.5.4. *Let $\phi \in \mathcal{S}(\mathbf{R}^n)$ and let $r > 0$. Let B_r be the closed ball in \mathbf{R}^n centered at the origin with radius r . Assume that*

$$\mathcal{R}_{n-1} \phi(\xi) = 0 \quad \text{whenever} \quad \xi \cap B_r = \emptyset.$$

Then

$$\text{supp}(\phi) \subseteq B_r.$$

We denote the Fourier transforms on \mathbf{R}^n and \mathbf{R} by \mathcal{F}_n and \mathcal{F}_1 , respectively. The proof for Theorem 1.5.4 is based on the following observation.

Observation 1.5.5. *Let $\phi \in \mathcal{S}(\mathbf{R}^n)$. Then for every $r \in \mathbf{R}$ and every $\omega \in S^{n-1}$*

$$\begin{aligned} \mathcal{F}_n \phi(r\omega) &= \int_{\mathbf{R}^n} \phi(x) e^{ir\langle \omega, x \rangle} dx = \int_{\mathbf{R}} \int_{\omega^\perp} \phi(p\omega + y) dy e^{-irp} dp \\ &= \mathcal{F}_1 \left(p \mapsto \mathcal{R}_{n-1} \phi(p\omega + \omega^\perp) \right)(r). \end{aligned} \quad (1.5.1)$$

Sketch of the proof for Theorem 1.5.4. We sketch the proof given by Wiegnerinck in [Wie85]. Let $\omega \in S^{n-1}$. Since $\mathcal{R}_{n-1} \phi$ is compactly supported, it follows from (1.5.1) that $\mathbf{C} \ni z \mapsto \mathcal{F}_n \phi(z\omega)$ is holomorphic. Furthermore, since the support of $\mathbf{R} \ni p \mapsto \mathcal{R}_{n-1} \phi(p\omega + \omega^\perp)$ is contained in the interval $[-r, r]$, the directional derivatives of $\mathcal{F}_n \phi$ in the origin satisfy

$$|\partial_\omega^k \mathcal{F}_n \phi(0)| = \left| \int_{\mathbf{R}} \mathcal{R}_{n-1} \phi(p\omega + \omega^\perp) (-ip)^k dp \right| \leq cr^k \quad (k \in \mathbf{Z}_{\geq 0}),$$

where c equals the $L^1(\mathbf{R}^n)$ -norm of ϕ . Using the main lemma in [WK85] one can estimate for any multi-index α the mixed derivative $\partial^\alpha \mathcal{F}_n \phi$ in the origin in terms of the directional derivatives of $\mathcal{F}_n \phi$. We write $\sum_\alpha c_\alpha \zeta^\alpha$ for the Taylor series of $\mathcal{F}_n \phi(\zeta)$ in the origin. From the estimates it follows that there exists a constant $a > 0$ such that for every $\epsilon > 0$ and every multi-index α

$$|c_\alpha| \leq a \frac{(r + \epsilon)^{|\alpha|}}{|\alpha|!}.$$

From these estimates it follows that the Taylor series of $\mathcal{F}_n \phi$ in the origin defines a holomorphic function F . Since for every $\omega \in S^{n-1}$ the restriction of $\mathcal{F}_n \phi$ to the complex line through ω is entire, it follows that $\mathcal{F}_n \phi$ is real analytic on \mathbf{R}^n and equal to F on \mathbf{R}^n . Moreover,

$$|F(\zeta)| \leq ae^{(r+\epsilon) \sum_{j=1}^n |\zeta_j|}.$$

From this estimate, the Fourier inversion formula and the Plancherel-Pólya theorem ([Ron74, Theorem 3.4.2]) it then follows that $\text{supp}(\phi)$ is contained in B_r . \square

Chapter 2

Lie groups and symmetric spaces

This chapter is meant as a short introduction to the Lie theory that we need in later chapters. In Section 2.1 we define the so called reductive Lie groups of the Harish-Chandra class and in Section 2.2 we give a description of their parabolic subgroups. Then in Section 2.3 we give the definition of a reductive symmetric space. A reductive symmetric space is a homogeneous space G/H of a specific kind, where G is a reductive Lie group of the Harish-Chandra class. In Section 2.4, we consider a special class of parabolic subgroups of a reductive Lie group G of the Harish-Chandra class related to a reductive symmetric space G/H . Finally, in Section 2.5, we introduce some notation that we need in later chapters.

The theory discussed in this chapter can be found in [Var77], [Kna02] and [BSD05].

2.1 Reductive Lie groups of the Harish-Chandra class

A Lie group G is said to be a reductive Lie group of the Harish-Chandra class if

- (i) G is a real Lie group and its Lie algebra \mathfrak{g} is reductive.
- (ii) G has finitely many connected components.
- (iii) The image of G under the adjoint representation $\text{Ad} : G \rightarrow \mathbf{GL}(\mathfrak{g}_{\mathbb{C}})$ is contained in the identity component of $\text{Aut}(\mathfrak{g}_{\mathbb{C}})$.
- (iv) The center of the analytic subgroup with Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ is finite.

Note that a connected semisimple Lie group is a reductive Lie group of the Harish-Chandra class if and only if it has finite center.

Let G be a reductive Lie group of the Harish-Chandra class. A Cartan involution on G is an involution θ of G whose set of fixed points $K = G^{\theta}$ is a maximal compact subgroup of G . It is known that all Cartan involutions are conjugate to a given Cartan involution θ via an element of $\text{Ad}(G)$.

If θ is a Cartan involution of G , then we denote the corresponding involution on \mathfrak{g} also by θ . The latter involution is called a Cartan involution of \mathfrak{g} . The Lie algebra of \mathfrak{g} decomposes as a direct sum of vector spaces

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad (2.1.1)$$

where \mathfrak{k} and \mathfrak{p} are the $+1$ and -1 -eigenspace of θ respectively. Note that since $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$, the subspace \mathfrak{p} is a Lie subalgebra of \mathfrak{g} if and only if \mathfrak{p} is abelian. The $+1$ -eigenspace \mathfrak{k} on the other hand is a Lie subalgebra; it is the Lie algebra of K . Note furthermore that K acts on \mathfrak{p} via the adjoint representation and $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$. The decomposition (2.1.1) is called the Cartan decomposition of \mathfrak{g} associated to θ . This decomposition has a global counterpart: the map

$$K \times \mathfrak{p} \rightarrow G; \quad (k, Y) \mapsto k \exp(Y)$$

is a diffeomorphism. The corresponding decomposition of G is called the Cartan decomposition of G associated to θ .

2.2 Parabolic subgroups

Let G be a reductive Lie group of the Harish-Chandra class and let θ be a Cartan involution of G .

We fix a maximal abelian subspace \mathfrak{a} of \mathfrak{p} and write A for $\exp(\mathfrak{a})$. (It can be shown that every other maximal abelian subspace of \mathfrak{p} is conjugate to \mathfrak{a} via K .) The linear operators $\text{ad}(Y) \in \text{End}(\mathfrak{g})$, $Y \in \mathfrak{a}$, are simultaneously diagonalizable. For $\alpha \in \mathfrak{a}^*$ we write

$$\mathfrak{g}_\alpha = \{Y \in \mathfrak{g} : \text{ad}(Z)Y = \alpha(Z)Y \text{ for all } Z \in \mathfrak{a}\}.$$

The subset $\Sigma(\mathfrak{g}, \mathfrak{a})$ of \mathfrak{a}^* , consisting of non-zero elements α with $\mathfrak{g}_\alpha \neq \{0\}$, is a (possibly non-reduced) root system and \mathfrak{g} decomposes as

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}_\alpha.$$

Here \mathfrak{m} is the centralizer of \mathfrak{a} in \mathfrak{k} . Note that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

We choose a set of positive roots $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ in the following manner. Let $Z_0 \in \mathfrak{a}$ be such that $\alpha(Z_0) \neq 0$ for every $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$. We choose $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ to be

$$\Sigma^+(\mathfrak{g}, \mathfrak{a}) = \{\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}) : \alpha(Z_0) > 0\}.$$

We write

$$\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}_\alpha.$$

The Lie algebra \mathfrak{n} is nilpotent. Since $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ for $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$, it follows that

$$\mathfrak{g} = \theta(\mathfrak{n}) \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Let $N = \exp(\mathfrak{n})$ and let $M = \mathcal{Z}_K(\mathfrak{a})$ be the centralizer of \mathfrak{a} in K . Then MAN is a subgroup of G . We define a minimal parabolic subgroup of G to be a subgroup of G that is conjugate to MAN . Furthermore, we define a parabolic subgroup of G to be a subgroup of G that contains a minimal parabolic subgroup.

For a parabolic subgroup P of G , we define L_P to be the subgroup $P \cap \theta(P)$ of P and \mathfrak{l}_P to be its Lie algebra. We further define

$$\mathfrak{a}_P = \mathfrak{p} \cap \mathcal{Z}(\mathfrak{l}_P) \quad \text{and} \quad M_P = \mathcal{Z}_K(\mathfrak{a}_P) \exp(\mathfrak{p} \cap [\mathcal{Z}_{\mathfrak{g}}(\mathfrak{a}_P), \mathcal{Z}_{\mathfrak{g}}(\mathfrak{a}_P)])$$

where $\mathcal{Z}_K(\mathfrak{a}_P)$ denotes the centralizer of \mathfrak{a}_P in K and $\mathcal{Z}_{\mathfrak{g}}(\mathfrak{a}_P)$ denotes the centralizer of \mathfrak{a}_P in \mathfrak{g} . Let $A_P = \exp(\mathfrak{a}_P)$. The multiplication map

$$M_P \times A_P \rightarrow L_P; \quad (m, a) \mapsto ma$$

is a diffeomorphism. We write \mathfrak{m}_P for the Lie algebra of M_P .

We write \mathfrak{n}_P for the nilpotent radical of the Lie algebra of P , i.e. the maximal ideal \mathfrak{n}_P with the property that there exists a $k \in \mathbb{N}$ such that for all $Y_1, \dots, Y_k \in \mathfrak{n}_P$

$$\mathrm{ad}(Y_1) \circ \mathrm{ad}(Y_2) \circ \dots \circ \mathrm{ad}(Y_k) = 0.$$

We define $N_P = \exp(\mathfrak{n}_P)$. Then the multiplication map

$$L_P \times N_P \rightarrow P; \quad (l, n) \mapsto ln$$

is a diffeomorphism. This implies that the map

$$M_P \times A_P \times N_P \rightarrow P; \quad (m, a, n) \mapsto man$$

is a diffeomorphism as well. The corresponding decomposition $P = M_P A_P N_P$ is called the Langlands decomposition of P .

If P_m is a minimal parabolic subgroup, then its Langlands decomposition $P_m = M_{P_m} A_{P_m} N_{P_m}$ has the following properties. The Lie algebra \mathfrak{a}_{P_m} of A_{P_m} is a maximal abelian subspace of \mathfrak{p} and M_{P_m} is the centralizer in K of \mathfrak{a}_{P_m} . Furthermore, there exists a unique choice of a set $\Sigma^+(\mathfrak{g}, \mathfrak{a}_{P_m}; P_m)$ of positive roots for $\Sigma(\mathfrak{g}, \mathfrak{a}_{P_m})$ such that $N_{P_m} = \exp(\mathfrak{n}_{P_m})$, where \mathfrak{n}_{P_m} equals the direct sum of all root spaces \mathfrak{g}_α with $\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a}_{P_m}; P_m)$.

Related to each minimal parabolic subgroup P_m of G there exists a so called Iwasawa decomposition: the map

$$K \times A_{P_m} \times N_{P_m} \rightarrow G; \quad (k, a, n) \mapsto kan$$

is a diffeomorphism. The corresponding decomposition of G is called an Iwasawa decomposition of G . Note that since $KP_m = G$ for every minimal parabolic subgroup P_m , we have $KP = G$ for every parabolic subgroup P of G . It follows from the Iwasawa decomposition that the minimal parabolic subgroups are conjugate to each other via K .

In the next lemma we list some elementary properties of parabolic subgroups and their Langlands decomposition.

Lemma 2.2.1. *Let P be a parabolic subgroup of G .*

(i) *L_P equals the centralizer of \mathfrak{a}_P in G .*

(ii) *$A \subseteq P$ if and only if $\mathfrak{a}_P \subseteq \mathfrak{a}$.*

(iii) Assume $P_m = MAN_{P_m}$ is a minimal parabolic subgroup contained in P .
Then

$$\mathfrak{n}_P = \bigoplus_{\substack{\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a}; P_m) \\ \alpha|_{\mathfrak{a}_P} \neq 0}} \mathfrak{g}_\alpha.$$

(iv) L_P normalizes N_P .

(v) L_P is a reductive Lie group of the Harish-Chandra class.

Proof. For (i), see e.g. [Kna02, Proposition 7.82 and Proposition 7.83].

Assume $A \subseteq P$. Since A is θ -stable, it follows that A is contained in $L_P = P \cap \theta(P)$. By (i) the group A centralizes \mathfrak{a}_P or equivalently \mathfrak{a} centralizes \mathfrak{a}_P . Since \mathfrak{a} is a maximal abelian subspace of \mathfrak{p} and \mathfrak{a}_P is contained in \mathfrak{p} , it follows that $\mathfrak{a}_P \subseteq \mathfrak{a}$. For the converse, assume $\mathfrak{a}_P \subseteq \mathfrak{a}$. Then \mathfrak{a} centralizes \mathfrak{a}_P and is therefore contained in \mathfrak{l}_P by (i). This proves (ii).

For (iii), see e.g. [Kna02, Section VII.7].

Statement (iv) holds for P if and only if it holds for every K -conjugate of P . Without loss of generality we can therefore assume that a minimal parabolic subgroup P_m of the form $P_m = MAN_{P_m}$ is contained in P . Let $\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a}; P_m)$ be a root such that $\alpha|_{\mathfrak{a}_P} \neq 0$. Since L_P centralizes \mathfrak{a}_P , it follows that for every $l \in L_P$, $Y \in \mathfrak{a}$ and $Z \in \mathfrak{g}_\alpha$

$$[Y, \text{Ad}(l)Z] = \text{Ad}(l)[Y, Z] = \alpha(Y)\text{Ad}(l)Z.$$

Therefore L_P normalizes for each $\beta \in \mathfrak{a}_P^* \setminus \{0\}$ the space

$$\bigoplus_{\substack{\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a}; P_m) \\ \alpha|_{\mathfrak{a}_P} = \beta}} \mathfrak{g}_\alpha.$$

This proves in particular that L_P normalizes \mathfrak{n}_P and thus that L_P normalizes N_P .

For (v), see e.g. [Var77, Section II.6.3, Theorem 13]. \square

Lemma 2.2.2. *Let P and Q be two parabolic subgroups of G . Assume that $P \subseteq Q$. Then*

$$L_P \subseteq L_Q, \quad A_Q \subseteq A_P \quad \text{and} \quad N_Q \subseteq N_P.$$

Proof. Since $P \subseteq Q$, we have $\theta(P) \subseteq \theta(Q)$. Hence $L_P \subseteq L_Q$ and $\mathfrak{l}_P \subseteq \mathfrak{l}_Q$. The latter implies that the center $\mathcal{Z}(\mathfrak{l}_Q)$ of \mathfrak{l}_Q centralizes \mathfrak{l}_P . Since $\mathcal{Z}(\mathfrak{l}_Q)$ centralizes \mathfrak{a}_P in particular, it is contained in \mathfrak{l}_P by Lemma 2.2.1 (i). This implies that $\mathcal{Z}(\mathfrak{l}_Q) \subseteq \mathcal{Z}(\mathfrak{l}_P)$ and therefore $\mathfrak{a}_Q \subseteq \mathfrak{a}_P$ and $A_Q \subseteq A_P$. The last claim follows from Lemma 2.2.1(iii). \square

Assume that P and Q are parabolic subgroups of G and $P \subseteq Q$. We write N_P^Q for the intersection $N_P \cap L_Q$. The Lie algebra of N_P^Q is denoted by \mathfrak{n}_P^Q . Note that the multiplication map

$$N_Q \times N_P^Q \rightarrow N_P$$

is a diffeomorphism.

2.3 Reductive Symmetric spaces

A symmetric space is a homogeneous space G/H for a Lie group G , where H is an open subgroup of the fixed point subgroup of some involution σ of G .

Let G be a reductive Lie group of the Harish-Chandra class, let σ be an involution of G and let X be the symmetric space G/H , where H is an open subgroup of G^σ . We denote the corresponding involution on \mathfrak{g} by σ as well.

Lemma 2.3.1. *There exists a Cartan involution θ that commutes with σ , i.e.,*

$$\sigma \circ \theta = \theta \circ \sigma.$$

For a proof, see for example [BSD05, Lemma 3.1, p.19].

The Lie algebra \mathfrak{g} has an eigenspace decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$$

for σ . Here the first component is the $+1$ and the second the -1 eigenspace. Note that \mathfrak{h} is the Lie algebra of H . Since $[\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{h}$, the subspace \mathfrak{q} is a Lie subalgebra of \mathfrak{g} if and only if \mathfrak{q} is abelian. Since σ and θ commute, the Lie algebra \mathfrak{g} decomposes also as

$$\mathfrak{g} = (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{k} \cap \mathfrak{q}) \oplus (\mathfrak{p} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{q}).$$

We fix a maximal abelian subspace \mathfrak{a}_q of $\mathfrak{p} \cap \mathfrak{q}$. The subgroup H is called essentially connected if

$$H = \mathcal{Z}_{K \cap H}(\mathfrak{a}_q) H^0,$$

where H^0 is the identity component of H and $\mathcal{Z}_{K \cap H}(\mathfrak{a}_q)$ the centralizer of \mathfrak{a}_q in $K \cap H$. (See [Ban86, p. 24].) Every Cartan involution commuting with σ is conjugate to θ via an element of $\text{Ad}(H)$ and every maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$ is conjugate to \mathfrak{a}_q via an element of $K \cap H$. This implies that the notion of H being essentially connected is independent of the choice for \mathfrak{a}_q . If H is essentially connected, then $X = G/H$ is called a reductive symmetric space of the Harish-Chandra class, or for short a reductive symmetric space.

Examples of reductive symmetric spaces are spheres, Euclidean spaces, pseudo-Riemannian hyperbolic spaces and the De Sitter and the anti De Sitter space. Also a Lie group G of the Harish-Chandra class may be viewed as a reductive symmetric space. In fact

$$(G \times G)/\text{diag}(G) \rightarrow G; \quad (g_1, g_2) \cdot \text{diag}(G) \mapsto g_1 g_2^{-1}$$

is a diffeomorphism and $\text{diag}(G)$ is the fixed point subgroup of the involution $(g_1, g_2) \mapsto (g_2, g_1)$ of G . Therefore $(G \times G)/\text{diag}(G)$ is a symmetric space. Since $\mathcal{Z}_K(\mathfrak{a})$ meets every connected component of G (see [Kna02, 7.33]), the subgroup $\text{diag}(G)$ of $G \times G$ is essentially connected, hence $(G \times G)/\text{diag}(G)$ is of the Harish-Chandra class. Another important class of symmetric spaces is the class of Riemannian symmetric spaces of non-compact type. These symmetric spaces are obtained by taking G to be a non-compact connected semi-simple Lie group and σ to be equal to a Cartan involution θ of G . The symmetric space is then of the form $X = G/K$.

From now on we will always assume the following.

- (i) G is a reductive Lie group of the Harish-Chandra class.
- (ii) σ is an involution of G and H is an open subgroup of the fixed point subgroup G^σ .
- (iii) θ is a Cartan involution commuting with σ and $K = G^\theta$.
- (iv) H is a essentially connected.
- (v) X is the reductive symmetric space G/H .

2.4 The class of $\sigma \circ \theta$ -stable parabolic subgroups

In this section we describe the $\sigma \circ \theta$ -stable parabolic subgroups, i.e., the parabolic subgroups P with the property that $\sigma \circ \theta(P) = P$.

We fix a maximal abelian subspace \mathfrak{a}_q of $\mathfrak{p} \cap \mathfrak{q}$ and a σ -stable maximal abelian subspace \mathfrak{a} of \mathfrak{p} containing \mathfrak{a}_q . Being σ -stable, \mathfrak{a} decomposes as

$$\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{h}) \oplus \mathfrak{a}_q.$$

We write A and A_q for the connected abelian subgroups of G with Lie algebras \mathfrak{a} and \mathfrak{a}_q , respectively.

We define $\Sigma(\mathfrak{g}, \mathfrak{a}_q)$ to be the set consisting of non-zero elements $\beta \in \mathfrak{a}_q^*$ such that $\beta = \alpha|_{\mathfrak{a}_q}$ for some root $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$. The set $\Sigma(\mathfrak{g}, \mathfrak{a}_q)$ is a root-system in \mathfrak{a}_q^* . For $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}_q)$ we define

$$\mathfrak{g}_\alpha = \{Y \in \mathfrak{g} : \text{ad}(Z)Y = \alpha(Z)Y \text{ for all } Z \in \mathfrak{a}_q\}.$$

Let $\Sigma^+(\mathfrak{g}, \mathfrak{a}_q)$ be a choice of a positive system for $\Sigma(\mathfrak{g}, \mathfrak{a}_q)$. Let

$$\mathfrak{n}_0 = \bigoplus_{\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a}_q)} \mathfrak{g}_\alpha$$

and let $N_0 = \exp(\mathfrak{n}_0)$. Then $\mathcal{Z}_G(\mathfrak{a}_q)$ normalizes N_0 and therefore $P_0 = \mathcal{Z}_G(\mathfrak{a}_q)N_0$ is a subgroup of G . In fact P_0 is a $\sigma \circ \theta$ -stable parabolic subgroup that is minimal in the sense that if Q is a $\sigma \circ \theta$ -stable parabolic subgroup contained in P_0 , then $Q = P_0$. Every other minimal $\sigma \circ \theta$ -stable parabolic subgroup of G is conjugate to P_0 via an element of K .

If P_0 is a minimal $\sigma \circ \theta$ -stable parabolic subgroup, then $\mathfrak{a}_{P_0} \cap \mathfrak{q}$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$ and L_{P_0} equals the centralizer in G of $\mathfrak{a}_{P_0} \cap \mathfrak{q}$. Furthermore, there exists a unique choice of a positive system $\Sigma^+(\mathfrak{g}, \mathfrak{a}_{P_0} \cap \mathfrak{q}; P_0)$ for $\Sigma(\mathfrak{g}, \mathfrak{a}_{P_0} \cap \mathfrak{q})$ such that \mathfrak{n}_{P_0} equals the direct sum of root spaces \mathfrak{g}_α with $\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a}_{P_0} \cap \mathfrak{q}; P_0)$.

Lemma 2.4.1. *Let P be a $\sigma \circ \theta$ -stable parabolic subgroup of G .*

- (i) *L_P equals the centralizer of $\mathfrak{a}_P \cap \mathfrak{q}$ in G .*
- (ii) *$A_q \subseteq P$ if and only if $\mathfrak{a}_P \cap \mathfrak{q} \subseteq \mathfrak{a}_q$.*
- (iii) *Assume P_0 is a minimal $\sigma \circ \theta$ -stable parabolic subgroup contained in P and $A_q \subseteq P_0$. Then*

$$\mathfrak{n}_P = \bigoplus_{\substack{\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a}_q; P_0) \\ \alpha|_{\mathfrak{a}_P \cap \mathfrak{a}_q} \neq 0}} \mathfrak{g}_\alpha.$$

Proof. For (i), see [Ban88, Lemma 2.2].

Assume $A_q \subseteq P$. Since A_q is θ -stable, it is contained in L_P . This implies that the center $\mathcal{Z}(\mathfrak{l}_P)$ of \mathfrak{l}_P is contained in the centralizer $\mathcal{Z}_{\mathfrak{g}}(\mathfrak{a}_q)$ of \mathfrak{a}_q in \mathfrak{g} . Therefore

$$\mathfrak{a}_P \cap \mathfrak{q} = \mathcal{Z}(\mathfrak{l}_P) \cap \mathfrak{p} \cap \mathfrak{q} \subseteq \mathcal{Z}_{\mathfrak{g}}(\mathfrak{a}_q) \cap \mathfrak{p} \cap \mathfrak{q}.$$

Since \mathfrak{a}_q is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$, the latter equals \mathfrak{a}_q .

For the converse, assume that $\mathfrak{a}_P \cap \mathfrak{q} \subseteq \mathfrak{a}_q$. Since \mathfrak{a}_q is abelian, it centralizes $\mathfrak{a}_P \cap \mathfrak{q}$. By (i) the subalgebra \mathfrak{a}_q is contained in \mathfrak{l}_P . Therefore $A_q = \exp(\mathfrak{a}_q)$ is contained in L_P , which in turn is contained in P . This proves (ii).

Since P is $\sigma \circ \theta$ -stable, the Lie algebra \mathfrak{n}_P , being the nilpotent radical of the Lie algebra of P , is $\sigma \circ \theta$ -stable as well. This implies that \mathfrak{g}_{α} is not contained in \mathfrak{n}_P if $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ vanishes on $\mathfrak{a}_P \cap \mathfrak{q}$. The result now follows from Lemma 2.2.1(iii). \square

We write $\mathcal{P}(\mathfrak{a})$ for the collection of parabolic subgroups P of G with $\mathfrak{a}_P = \mathfrak{a}$, and $\mathcal{P}_{\sigma}(\mathfrak{a}_q)$ for the collection of $\sigma \circ \theta$ -stable parabolic subgroups P of G with $\mathfrak{a}_P \cap \mathfrak{a}_q = \mathfrak{a}_q$. Note that $\mathcal{P}(\mathfrak{a})$ and $\mathcal{P}_{\sigma}(\mathfrak{a}_q)$ consist of the minimal parabolic subgroups containing A and the minimal $\sigma \circ \theta$ -stable parabolic subgroups containing A_q , respectively.

Lemma 2.4.2. *Let $P_m \in \mathcal{P}(\mathfrak{a})$ and let P_0 be a minimal $\sigma \circ \theta$ -stable parabolic subgroup containing P_m . Then*

$$\mathfrak{n}_{P_m}^{P_0} = \mathfrak{n}_{P_m} \cap \mathfrak{h} \quad \text{and} \quad N_{P_m}^{P_0} = N_{P_m} \cap H.$$

Proof. We first note that

$$\mathfrak{n}_{P_m} \cap \mathfrak{h} \subseteq \mathfrak{n}_{P_m} \cap \sigma(\mathfrak{n}_{P_m}) = \mathfrak{n}_{P_m} \cap \sigma(\mathfrak{n}_{P_0} \oplus \mathfrak{n}_{P_m}^{P_0}).$$

As $\sigma(\mathfrak{n}_{P_0}) = \theta \mathfrak{n}_{P_0}$ and $\sigma(\mathfrak{n}_{P_m}^{P_0}) \subseteq \sigma(\mathfrak{l}_{P_0}) = \mathfrak{l}_{P_0}$, it follows that

$$\mathfrak{n}_{P_m} \cap \mathfrak{h} \subseteq \mathfrak{n}_{P_m}^{P_0}.$$

For the converse inclusion, it suffices to show that $\mathfrak{n}_{P_m}^{P_0} \subseteq \mathfrak{h}$. For this we note that $\mathfrak{n}_{P_m}^{P_0}$ is a sum of root spaces \mathfrak{g}_{α} with $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$. Assume that $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ is such that \mathfrak{g}_{α} occurs in this sum. We will show that then $\mathfrak{g}_{\alpha} \subseteq \mathfrak{n}_{P_m} \cap \mathfrak{h}$. Since $\mathfrak{n}_{P_m}^{P_0}$ is contained in \mathfrak{l}_{P_0} , it centralizes \mathfrak{a}_q . It follows that the restriction of α to \mathfrak{a}_q vanishes, hence $\sigma^* \alpha = \alpha$ and \mathfrak{g}_{α} is σ -stable. Therefore \mathfrak{g}_{α} decomposes as a direct sum of vector spaces

$$\mathfrak{g}_{\alpha} = \mathfrak{g}_{\alpha}^{+} \oplus \mathfrak{g}_{\alpha}^{-},$$

where \mathfrak{g}_α^\pm is the ± 1 -eigenspace of σ . It suffices to show that $\mathfrak{g}_\alpha^- = \{0\}$. For this, let $Y \in \mathfrak{g}_\alpha^-$. Then $Y - \theta Y$ is contained in $\mathfrak{p} \cap \mathfrak{q}$ and centralizes \mathfrak{a}_q . By maximality of \mathfrak{a}_q it follows that Y belongs to this space. Since the intersection of \mathfrak{a}_q and $\theta \mathfrak{n}_P \oplus \mathfrak{n}_P$ is zero, Y must be zero. This establishes the first identity, and we turn to the second.

Since $\exp : \mathfrak{n}_{P_m} \rightarrow N_{P_m}$ is a diffeomorphism, it follows that $N_{P_m} \cap H \subseteq N_{P_m} \cap G^\sigma = \exp(\mathfrak{n}_{P_m} \cap \mathfrak{h})$. This implies that $N_{P_m} \cap H = \exp(\mathfrak{n}_{P_m} \cap \mathfrak{h})$. The second identity now follows from the first. \square

If P is a $\sigma \circ \theta$ -stable parabolic subgroup containing A_q , then

$$L_P = (M_P \cap K)A_q(L_P \cap H)$$

Furthermore, if $Y, Y_0 \in \mathfrak{a}_q$ and $\exp(Y) \in (M_P \cap K)\exp(Y_0)(L_P \cap H)$, then $Y = \text{Ad}(k)Y_0$ for some k in the normalizer $\mathcal{N}_{M_P \cap K \cap H}(\mathfrak{a}_q)$ of \mathfrak{a}_q in $M_P \cap K \cap H$. This decomposition of L_P is often called the polar decomposition of L_P . Note that in particular G is a $\sigma \circ \theta$ -stable parabolic subgroup and the polar decomposition of G is

$$G = KA_qH.$$

To conclude this section, we describe a generalization of the Iwasawa decomposition. Let P_0 be a minimal $\sigma \circ \theta$ -stable parabolic subgroup. Then the double coset space $P_0 \backslash G/H$ is finite. Furthermore, the sets $P_0 wH$ with $w \in \mathcal{N}_K(\mathfrak{a}_q)$ are open subsets of G . In fact, these are all the open $P \times H$ -orbits in G . The union of these open orbits is dense in G . Finally the map

$$N_{P_0} \times A_q \times (M_{P_0} \cap K) \times_{M_{P_0} \cap K \cap H} H \rightarrow P_0 H$$

is a diffeomorphism.

2.5 Notation

We recall that $\Sigma(\mathfrak{g}, \mathfrak{a}_q)$ is a possibly unreduced root system. We write $\Sigma_\pm^+(\mathfrak{g}, \mathfrak{a}_q; P)$ for the set of positive roots $\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a}_q; P)$ for which the ± 1 -eigenspace of $\sigma \circ \theta$ in the root space \mathfrak{g}_α is non-trivial. We denote by W the Weyl group of the root system $\Sigma(\mathfrak{g}, \mathfrak{a}_q)$. Note that there is a natural isomorphism

$$W \simeq \mathcal{N}_K(\mathfrak{a}_q)/\mathcal{Z}_K(\mathfrak{a}_q).$$

If S is a subgroup of G , then we define W_S to be the subgroup consisting of elements that can be realized as $\text{Ad}(s)|_{\mathfrak{a}_q}$ for $s \in \mathcal{N}_{K \cap S}(\mathfrak{a}_q)$. We write \mathcal{W} for a set

of representatives in $\mathcal{N}_K(\mathfrak{a}_q)$ of $W/W_{K \cap H}$. The Weyl group of $\Sigma(l_P, \mathfrak{a}_q)$ equals $W_{M_P \cap K}$. We write \mathcal{W}_{M_P} for a set of representatives in $\mathcal{N}_{M_P \cap K}(\mathfrak{a}_q)$ for

$$W_{M_P \cap K} / W_{M_P \cap K \cap H}.$$

We write B for an $\text{Ad}(G)$ -invariant, θ -invariant, symmetric, non-degenerate bilinear form on \mathfrak{g} such that

- (i) $-B(\cdot, \theta \cdot)$ is a positive definite inner product on \mathfrak{g} ,
- (ii) \mathfrak{k} and \mathfrak{p} are orthogonal with respect to $-B(\cdot, \theta \cdot)$,
- (iii) the semisimple part of \mathfrak{g} and the center of \mathfrak{g} are orthogonal with respect to $-B(\cdot, \theta \cdot)$
- (iv) the restriction of B to the semisimple part of \mathfrak{g} equals the Killing form.

For a root $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}_q)$, we define $H_\alpha \in \mathfrak{a}_q$ to be the element given by

$$\alpha(Y) = B(H_\alpha, Y) \quad (Y \in \mathfrak{a}_q). \quad (2.5.1)$$

For $g \in G$ and P a parabolic subgroup of G we denote the parabolic subgroup $g^{-1}Pg$ by P^g .

If P is a $\sigma \circ \theta$ -stable parabolic subgroup containing A_q , then by $\mathfrak{a}_q^+(P)$ we denote the set consisting of elements $H \in \mathfrak{a}_q$ such that $\alpha(H) > 0$ for all $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}_q; P)$. Furthermore, for $R \in \mathbf{R}$, we define

$$\mathfrak{a}_q^*(P, R) = \{\lambda \in \mathfrak{a}_{q, \mathbf{C}}^* : \text{Re } \lambda(H_\alpha) < R \text{ for all } \alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a}_q; P)\}.$$

Finally if V is a Fréchet space and (π, V) a continuous representation of G on V , then we denote the space of smooth vectors for π by V^∞ .

Chapter 3

Radon transformation on a reductive symmetric space

In Sections 3.1 – 3.4 we first introduce for each pair of $\sigma \circ \theta$ -stable parabolic subgroups P and Q , with $P \subseteq Q$, a class of closed submanifolds (related to P) of a certain homogeneous space E_Q (related to Q) and describe the Radon transforms that are obtained by integrating over these submanifolds. Then in Section 3.6 some basic estimates are derived. It follows from these estimates that the Radon transforms, which are initially defined on the space of compactly supported smooth functions, extend to larger spaces of functions and distributions defined in Section 3.5. In Section 3.7 we describe certain relations between Radon transforms related to different parabolic subgroups. Finally, in Section 3.9, we derive some properties of the support of the Radon transform of a function or distribution, in terms of the support of that function or distribution. For this we need a few results from convex analysis that we describe in Section 3.8.

3.1 Horospheres

For a $\sigma \circ \theta$ -stable parabolic subgroup P , we define

$$\mathcal{E}_P = G/(L_P \cap H)N_P.$$

Since $(L_P \cap H)N_P$ is closed in P , hence in G , it follows that \mathcal{E}_P is a smooth homogeneous space for G . Note that G is a $\sigma \circ \theta$ -stable parabolic subgroup of G and

$$\mathcal{E}_G = X.$$

The cosets $e \cdot H$ and $e \cdot (L_P \cap H)N_P$ are denoted by x_0 and ξ_P , respectively.

Let P_0 be a minimal $\sigma \circ \theta$ -stable parabolic subgroup of G . A *horosphere* in X is an orbit of a subgroup of G conjugate to N_{P_0} in X of maximal dimension, i.e., a submanifold of X (see Proposition 3.2.2) of the form $g_1 N_{P_0} g_2 \cdot x_0$ with dimension equal to the dimension of N_{P_0} . The set of all horospheres in X is denoted by $\text{Hor}(X)$.

According to [Ros79, Theorem 13] and [Mat79] the group G equals the union of subsets $P_0 k H$, where k runs over a finite subset of K . This implies that $\text{Hor}(X)$ is the union of finitely many G -orbits. The dimension of these orbits need not be constant. It follows from the same theorem in [Ros79] that the set of orbits of maximal dimension is in bijection with \mathcal{W} via the map $w \mapsto G \cdot (N_{P_0} w \cdot x_0) = G \cdot (w^{-1} N_{P_0}^w \cdot x_0)$. Here the superscript w denotes conjugation with w^{-1} .

The stabilizer in G of ξ_{P_0} equals $(L_{P_0} \cap H)N_{P_0}$. (See Proposition A.2.) Therefore the G -orbits in $\text{Hor}(X)$ of maximal dimension, i.e., the sets $G \cdot \xi_{P_0}^w$ for $w \in \mathcal{W}$, are parametrized by the homogeneous spaces $\mathcal{E}_{P_0}^w$.

The double fibration

$$\begin{array}{ccc} & G/(L_{P_0} \cap H) & \\ \swarrow \Pi_X & & \searrow \Pi_{\mathcal{E}_{P_0}} \\ X & & \mathcal{E}_{P_0} \end{array} \quad (3.1.1)$$

describes the incidence relation between points in X and horospheres in X which are parametrized by \mathcal{E}_{P_0} : a point x is contained in a horosphere parametrized by $\xi \in \mathcal{E}_{P_0}$ if and only if

$$x \in \Pi_X(\Pi_{\mathcal{E}_{P_0}}^{-1}(\xi)).$$

If $H = K$, then X is a Riemannian symmetric space. In that case, P_0 is a minimal parabolic subgroup. If $G = KAN$ is an Iwasawa decomposition for G ,

then by a suitable conjugation we may arrange that $P_0 = MAN$, where M is the centralizer of \mathfrak{a} in K . Then (3.1.1) reduces to

$$\begin{array}{ccc} & G/M & \\ \swarrow \Pi_X & & \searrow \Pi_\Xi \\ G/K = X & & \Xi = MN \end{array}$$

The Radon transform corresponding to this double fibration is the horospherical transform described in the introduction.

3.2 Double fibration

In this and the following sections we will assume that P and Q are $\sigma \circ \theta$ -stable parabolic subgroups such that $A \subseteq P \subseteq Q$. Then $A_Q \subseteq A_P \subseteq A$. Since L_P is a closed subgroup of L_Q , whereas $Q \simeq L_Q \times N_Q$, with L_Q normalizing N_Q , it follows that $(L_P \cap H)N_Q$ is a closed subgroup of G . We recall that $\Xi_P = G/(L_P \cap H)N_P$ and $\Xi_Q = G/(L_Q \cap H)N_Q$ and consider the following generalization of (3.1.1).

$$\begin{array}{ccc} & G/(L_P \cap H)N_Q & \\ \swarrow \Pi_{\Xi_Q} & & \searrow \Pi_{\Xi_P} \\ \Xi_Q & & \Xi_P \end{array} \tag{3.2.1}$$

In view of the following proposition, this is a double fibration of the form (1.2.1).

Proposition 3.2.1. *Let P, Q be $\sigma \circ \theta$ -stable parabolic subgroups with $P \subseteq Q$. Then*

$$(L_Q \cap H)N_Q \cap (L_P \cap H)N_P = (L_P \cap H)N_Q.$$

Proof. As $L_P \subseteq L_Q$ and $N_Q \subseteq N_P$, it follows that the set on the right-hand side of the equality is contained in the intersection on the left-hand side. We turn to the converse inclusion. Assume that g belongs to the intersection on the left-hand side. Using that $N_P = N_P^Q N_Q$, that $N_P^Q \subseteq L_Q$ and that $L_P \subseteq L_Q$ we see that $g = ln$ for certain elements $n \in N_Q$ and

$$l \in (L_Q \cap H) \cap (L_P \cap H)N_P^Q.$$

Since P is $\sigma \circ \theta$ -stable, $\sigma(N_P) = \theta N_P$, so that N_P has trivial intersection with H . Therefore, $l \in L_P \cap H$. \square

The double fibration (3.2.1) describes the incidence relations between points in Ξ_Q and subsets of Ξ_Q of the form $gN_P \cdot \xi_Q$ with $g \in G$. Note that for $Q = G$ and $P = P_0$ a minimal $\sigma \circ \theta$ -stable parabolic subgroup, (3.2.1) reduces to (3.1.1).

For $\xi \in \Xi_P$ we define $E_P^Q(\xi)$ to be the subset of Ξ_Q given by

$$E_P^Q(\xi) = \Pi_{\Xi_Q}(\Pi_{\Xi_P}^{-1}(\{\xi\})).$$

Moreover, we agree to write $E_P(\xi)$ for $E_P^G(\xi)$.

Note that for $g \in G$,

$$E_P^Q(g \cdot \xi_P) = g \cdot E_P^Q(\xi_P) = gN_P \cdot \xi_Q = gN_P^Q \cdot \xi_Q.$$

Proposition 3.2.2. *Let $g \in G$. Then*

(i) $E_P^Q(g \cdot \xi_P)$ is a closed submanifold of Ξ_Q .

(ii) The map $n \mapsto gn \cdot \xi_Q$ is a diffeomorphism from N_P^Q onto $E_P^Q(g \cdot \xi_P)$.

Proof. Without loss of generality we may assume that $g = e$. The multiplication map defines a surjective submersion $K \times Q \rightarrow G$. As $K \cap Q = K \cap M_Q$, this submersion induces a diffeomorphism $K \times_{K \cap M_Q} Q \rightarrow G$ which is equivariant for the right action of the closed subgroup $(L_Q \cap H)N_Q$ of Q . Now $Q \simeq L_Q \times N_Q$ and L_Q normalizes N_Q . Therefore, the above diffeomorphism factors through a diffeomorphism $K \times_{K \cap M_Q} L_Q / L_Q \cap H \rightarrow G / (L_Q \cap H)N_Q$. It follows that the map

$$K \times_{(M_Q \cap K)} L_Q / (L_Q \cap H) \rightarrow \Xi_Q; \quad (k, l \cdot (L_Q \cap H)) \mapsto kl \cdot \xi_Q \quad (3.2.2)$$

is a diffeomorphism. Hence, $l \mapsto l \cdot \xi_P$ is a diffeomorphism from $L_Q / (L_Q \cap H)$ onto the closed submanifold $L_Q \cdot \xi_Q$ of Ξ_Q .

Let now P_0 be a minimal $\sigma \circ \theta$ -stable parabolic subgroup contained in P . From [Ban86, Lemma 3.4] applied to L_Q and the minimal $\sigma \circ \theta$ -stable parabolic subgroup $P_0 \cap L_Q$ of L_Q , it follows that the multiplication map

$$N_{P_0}^Q \times_{L_{P_0} \times_{(L_{P_0} \cap H)} (L_Q \cap H)} L_Q \rightarrow L_Q \quad (3.2.3)$$

is a diffeomorphism onto the open subset $(P_0 \cap L_Q)(L_Q \cap H)$ of L_Q . Therefore

$$N_{P_0}^Q \times_{L_{P_0} / (L_{P_0} \cap H)} L_Q \rightarrow L_Q \cdot \xi_Q$$

is a diffeomorphism onto the open subset $(P_0 \cap L_Q) \cdot \xi_Q$ of the submanifold $L_Q \cdot \xi_Q$ of Ξ_Q . As the multiplication map

$$N_{P_0}^P \times N_P^Q \rightarrow N_{P_0}^Q$$

is a diffeomorphism, the map $N_P^Q \ni n \mapsto n \cdot \xi_Q$ is a diffeomorphism onto $N_P^Q \cdot \xi_Q$ and this set is a submanifold of $L_Q \cdot \xi_Q$ and therefore a submanifold of Ξ_Q . Furthermore, it follows that $N_P^Q \cdot \xi_Q$ is closed in $(P_0 \cap L_Q) \cdot \xi_Q$ equipped with the subspace topology. It remains to be shown that $N_P^Q \cdot \xi_Q$ is closed in Ξ_Q .

Let $(n_j)_{j \in \mathbb{N}}$, $(a_j)_{j \in \mathbb{N}}$ and $(m_j)_{j \in \mathbb{N}}$ be sequences in $N_{P_0}^Q$, A_q and M_{P_0} , respectively, such that $(n_j a_j m_j \cdot \xi_Q)_{j \in \mathbb{N}}$ converges to a point in the boundary of $(P_0 \cap L_Q) \cdot \xi_Q$. Then, in view of [Ban86, Lemma 3.4], the set $\{a_j : j \in \mathbb{N}\}$ is not relatively compact in A_q . It follows that if $(n_j \cdot \xi_Q)_{j \in \mathbb{N}}$ is a sequence in $N_P^Q \cdot \xi_Q$ converging to ξ in Ξ_Q , then ξ cannot be an element of the boundary of $(P_0 \cap L_Q) \cdot \xi_Q$, hence $\xi \in (P_0 \cap L_Q) \cdot \xi_Q$. Since $N_P^Q \cdot \xi_Q$ is closed in $(P_0 \cap L_Q) \cdot \xi_Q$, we conclude that $\xi \in N_P^Q \cdot \xi_Q$. This proves that $N_P^Q \cdot \xi_Q$ is closed in Ξ_Q . \square

Corollary 3.2.3. *The set $(L_Q \cap H)N_Q(L_P \cap H)N_P$ equals $(L_Q \cap H)N_P$. The latter is a closed submanifold of G .*

Proof. Since $L_P \subseteq L_Q$ and $N_Q \subseteq N_P$,

$$\begin{aligned} (L_Q \cap H)N_Q(L_P \cap H)N_P &= (L_Q \cap H)N_Q N_P(L_P \cap H) = \\ (L_Q \cap H)N_P(L_P \cap H) &= (L_Q \cap H)(L_P \cap H)N_P = (L_Q \cap H)N_P. \end{aligned}$$

The set $N_P(L_Q \cap H)$ equals the pre-image of $E_P^Q(\xi_P)$ under the projection $G \rightarrow \Xi_Q$. Since $E_P^Q(\xi_P)$ is a closed submanifold according to Proposition 3.2.2 and G is a fiber bundle over Ξ_Q , it follows that $N_P(L_Q \cap H)$ is a closed submanifold of G . The same holds for its image under the map $g \mapsto g^{-1}$, which is $(L_Q \cap H)N_P$. \square

As a consequence of Corollary 3.2.3, the double fibration (3.2.1) satisfies condition (B) of Section 1.3.

According to Corollary A.4 of the appendix, the groups $(L_P \cap H)N_P$ and $(L_Q \cap H)N_Q$ are transversal. We recall from Section 1 that this means in particular that the map

$$\xi \mapsto E_P^Q(\xi)$$

is injective.

3.3 Invariant measures

Properties (A) and (C) of Section 1.3 are satisfied for the double fibration 3.2.1 if certain invariant measures exist. In this section we will prove the existence of such measures.

Lemma 3.3.1. *Let \mathfrak{r} be a reductive Lie algebra with Cartan involution θ . Every θ -stable Lie subalgebra of \mathfrak{r} is reductive.*

Proof. Let \mathfrak{s} be a θ -stable Lie subalgebra of \mathfrak{r} . The restriction of the positive definite inner product $\langle \cdot, \cdot \rangle = -B(\cdot, \theta \cdot)$ to $\mathfrak{s} \times \mathfrak{s}$ is a positive definite inner product on \mathfrak{s} . Assume that \mathfrak{i} is an ideal of \mathfrak{s} , then $\mathfrak{s} = \mathfrak{i} \oplus \mathfrak{i}^\perp$ as vector spaces. Here the orthocomplement is taken with respect to $\langle \cdot, \cdot \rangle$. Let $X \in \mathfrak{i}^\perp$ and $Y \in \mathfrak{s}$. Then for every $Z \in \mathfrak{i}$ we have

$$\begin{aligned} \langle [X, Y], Z \rangle &= -B([X, Y], \theta Z) = -B(X, [Y, \theta Z]) \\ &= -B(X, \theta[\theta Y, Z]) = \langle X, [\theta Y, Z] \rangle = 0. \end{aligned}$$

For the last equality it was used that $[\theta Y, Z] \in \mathfrak{i}$. This shows that $[X, Y] \in \mathfrak{i}^\perp$, and we conclude that \mathfrak{i}^\perp is an ideal of \mathfrak{s} complementary to \mathfrak{i} . It follows that \mathfrak{s} is reductive. □

We retain the notation of the previous sections. In particular, P and Q are $\sigma \circ \theta$ -stable parabolic subgroups containing A and such that $P \subset Q$.

Proposition 3.3.2. *The group $(L_P \cap H)N_Q$ is unimodular.*

Proof. Let Δ be the modular function of $(L_P \cap H)N_Q$. Note that

$$(L_P \cap H)N_Q = (L_P \cap H \cap K)(L_P \cap H)^0 N_Q,$$

where $(L_P \cap H)^0$ denotes the identity component of $(L_P \cap H)$. Since $(L_P \cap H \cap K)$ is compact, $\Delta(kg) = \Delta(g)$ for all $k \in L_P \cap H \cap K$ and $g \in (L_P \cap H)^0 N_Q$. If $n \in N_Q$, then $\text{ad}(\log(n))$ is upper triangular with respect to the usual basis of $(\mathfrak{l}_P \cap \mathfrak{h}) \oplus \mathfrak{n}_Q$, hence

$$\Delta(n) = |\det(\text{Ad}(n)|_{(\mathfrak{l}_P \cap \mathfrak{h}) \oplus \mathfrak{n}_Q})| = \exp\left(\text{tr}(\text{ad}(\log(n))|_{(\mathfrak{l}_P \cap \mathfrak{h}) \oplus \mathfrak{n}_Q})\right) = 1.$$

The group $(L_P \cap H)^0$ normalizes both $\mathfrak{l}_P \cap \mathfrak{h}$ and \mathfrak{n}_Q . Let $l \in (L_P \cap H)^0$, then it follows that

$$\Delta(l) = |\det(\text{Ad}(l)|_{\mathfrak{l}_P \cap \mathfrak{h}})| |\det(\text{Ad}(l)|_{\mathfrak{n}_Q})|.$$

Since $(L_P \cap H)^0$ is reductive, the modular function of this group is trivial, hence the first factor in the product is equal to 1. Similarly, the modular function Δ_Q of Q evaluated in l equals

$$\Delta_Q(l) = |\det(\text{Ad}(l)|_{\mathfrak{l}_Q})| |\det(\text{Ad}(l)|_{\mathfrak{n}_Q})|.$$

Again the first factor in the product equals 1, this time by reductivity of L_Q^0 . We conclude that

$$\Delta(l) = \Delta_Q(l).$$

If $m \in M_Q$, $a \in A_Q$ and $n \in N_Q$, then it is known that $\Delta_Q(man) = a^{2\rho_Q}$, where ρ_Q is the element of \mathfrak{a}_Q^* given by

$$\rho_Q(Y) = \frac{1}{2} \text{tr}(\text{ad}(Y)|_{\mathfrak{n}_Q}), \quad (Y \in \mathfrak{a}_Q). \quad (3.3.1)$$

Let $m \in M_Q$ and $a \in A_Q$ be such that $l = ma$. Since both M_Q and A_Q are σ -stable, it follows that $a \in A_Q \cap H$. Now Q is $\sigma \circ \theta$ -stable, hence $\sigma \circ \theta(\rho_Q) = \rho_Q$ and we see that $\rho_Q = 0$ on $\mathfrak{a}_Q \cap \mathfrak{h}$, so that $a^{2\rho_Q} = 1$. Therefore, $\Delta(l) = \Delta_Q(l) = 1$ and we conclude that $(L_P \cap H)N_Q$ is unimodular. \square

Corollary 3.3.3.

- (i) *There exists a non-zero G -invariant Radon measure on $G/(L_P \cap H)N_Q$. In particular there exists a non-zero G -invariant Radon measure on Ξ_P .*
- (ii) *There exists a non-zero $(L_Q \cap H)N_Q$ -invariant Radon measure on*

$$(L_Q \cap H)N_Q / (L_P \cap H)N_Q.$$

- (iii) *There exists a non-zero $(L_P \cap H)N_P$ -invariant Radon measure on*

$$(L_P \cap H)N_P / (L_P \cap H)N_Q.$$

Proof. By Proposition 3.3.2, also applied with $P = Q$, all occurring groups are unimodular. \square

As a consequence of Corollary 3.3.3, the double fibration (3.2.1) satisfies properties (A) and (C) of Section 1.3.

The groups $(L_P \cap H)N_Q$, N_P^Q and $(L_P \cap H)N_P$ are unimodular (see Proposition 3.3.2) and the multiplication map

$$(L_P \cap H)N_Q \times N_P^Q \rightarrow (L_P \cap H)N_P$$

is a diffeomorphism. Therefore,

$$\begin{aligned} \int_{(L_P \cap H)N_P / (L_P \cap H)N_Q} \psi(ln \cdot (L_P \cap H)N_Q) d_{(L_P \cap H)N_Q}(ln) = \\ \int_{N_P^Q} \psi(n \cdot (L_P \cap H)N_Q) dn \end{aligned}$$

for every $\psi \in L^1((L_P \cap H)N_P / (L_P \cap H)N_Q)$ if the measures are suitably normalized. Similarly,

$$\begin{aligned} \int_{(L_Q \cap H)N_Q / (L_P \cap H)N_Q} \phi(ln \cdot (L_P \cap H)N_Q) d_{(L_P \cap H)N_Q}(ln) = \\ \int_{(L_Q \cap H) / (L_P \cap H)} \phi(l \cdot (L_P \cap H)N_Q) dl \end{aligned}$$

for every $\phi \in L^1((L_Q \cap H)N_Q / (L_P \cap H)N_Q)$ if the measures are suitably normalized.

If N is a simply connected subgroup of G with nilpotent Lie algebra \mathfrak{n} , then the Haar measure on N is related to the Lebesgue measure on \mathfrak{n} by

$$\int_N \phi(n) dn = c \int_{\mathfrak{n}} \phi(\exp(Y)) dY \quad (\phi \in L^1(N))$$

for some constant $c > 0$. Here dY denotes the unit Lebesgue measure of \mathfrak{n} relative to inner product $\langle \cdot, \cdot \rangle$. We will choose the normalization of measures on the groups A_q and N_P^Q always such that $c = 1$. If P , Q and S are parabolic subgroups with $P \subseteq Q \subseteq S$, then because of this choice for the normalization of the measures

$$\int_{N_P^S} \phi(n) dn = \int_{N_P^Q} \int_{N_Q^S} \phi(nn') dn' dn \quad (\phi \in L^1(N_P^S)). \quad (3.3.2)$$

Furthermore, we normalize the Haar measure on any compact group such that it is a probability measure.

3.4 Radon transforms

The Radon transforms \mathcal{R}_P and \mathcal{S}_P for the double fibration (3.2.1) are given by

$$\mathcal{R}_P^Q \phi(g \cdot \xi_P) = \int_{N_P^Q} \phi(gn \cdot \xi_Q) dn \quad (\phi \in \mathcal{D}(\Xi_Q), g \in G) \quad (3.4.1)$$

$$\mathcal{S}_P^Q \psi(g \cdot \xi_Q) = \int_{(L_Q \cap H)/(L_P \cap H)} \psi(gh \cdot \xi_P) d_{L_P \cap H} h \quad (\psi \in \mathcal{D}(\Xi_P), g \in G). \quad (3.4.2)$$

We write \mathcal{R}_P for \mathcal{R}_P^G and \mathcal{S}_P for \mathcal{S}_P^G . These Radon transforms are given by

$$\begin{aligned} \mathcal{R}_P \phi(g \cdot \xi_P) &= \int_{N_P} \phi(gn \cdot x_0) dn \quad (\phi \in \mathcal{D}(X), g \in G) \\ \mathcal{S}_P \psi(g \cdot x_0) &= \int_{H/(L_P \cap H)} \psi(gh \cdot \xi_P) d_{L_P \cap H} h \quad (\psi \in \mathcal{D}(\Xi_P), g \in G). \end{aligned}$$

If $P = P_0$ is a minimal $\sigma \circ \theta$ -stable parabolic subgroup, then \mathcal{R}_{P_0} is called the *horospherical transform associated to P_0*

Remark 3.4.1. In [Krö09, Section 2] it is claimed that the set $\text{Hor}(X)$ of horospheres in X can be given the structure of a connected real analytic manifold. However, the real analytic atlas for $\text{Hor}(X)$ given there is not an atlas in the proper sense. In fact, each of the finitely many G -orbits in $\text{Hor}(X)$ serves as the domain for a chart, but it is easily seen that not all of these orbits need to have maximal dimension. In Section 5.4 of Appendix B we discuss an example of this phenomena.

In [Krö09, Remark 3.3] it is furthermore claimed that the horospherical transforms \mathcal{R}_{P^w} with $w \in \mathcal{W}$ induce a transform from the space of real analytic vectors for the left regular representation of G on $L^1(X)$ to the space of real analytic functions on $\text{Hor}(X)$. Even if $\text{Hor}(X)$ could be equipped with a canonical structure of a real analytic manifold, then it is not clear to us why this should be true.

3.5 Spaces of functions and distributions

From (3.2.2) and the fact that

$$L_P/(L_P \cap H) \simeq M_P/(M_P \cap H) \times A_P/(A_P \cap H),$$

it follows that there is a unique map $\mathfrak{A}_P : \mathfrak{E}_P \rightarrow \mathfrak{a}_P \cap \mathfrak{a}_q$ such that

$$\mathfrak{A}_P(kman \cdot \xi_P) = \pi_q(\log a) \quad (k \in K, m \in M_P, a \in A_P, n \in N_P),$$

where π_q denotes the orthogonal projection $\mathfrak{g} \rightarrow \mathfrak{q}$. Note that the map \mathfrak{A}_P is real analytic.

We define the function $J_P : \mathfrak{E}_P \rightarrow \mathbf{R}$ by

$$J_P = e^{-2\rho_P \circ \mathfrak{A}_P}$$

where ρ_P is defined as in (3.3.1) with P in place of Q . Note that J_G equals the constant function $\mathbf{1}$ on X .

Let

$$L^1(\mathfrak{E}_P, J_P) = \{\phi : \mathfrak{E}_P \rightarrow \mathbf{C} : \phi J_P \in L^1(\mathfrak{E}_P)\}.$$

Endowed with the norm

$$\phi \mapsto \int_{\mathfrak{E}_P} |\phi(\xi)| J_P(\xi) d\xi, \quad (3.5.1)$$

$L^1(\mathfrak{E}_P, J_P)$ is a Banach space.

Lemma 3.5.1. *For every compact subset C of G , there exists a constant $c > 0$ such that for every $g \in C$ and $\xi \in \mathfrak{E}_P$*

$$c^{-1} J_P(g \cdot \xi) \leq J_P(\xi) \leq c J_P(g \cdot \xi). \quad (3.5.2)$$

Proof. Let $P_m \in \mathcal{P}(\mathfrak{a})$ be a minimal parabolic subgroup contained in P . Let $k \in K$, $a \in A$ and $n \in N_{P_m}$. Note that there exist unique $a_{M_P} \in A \cap M_P$, $a_{A_P} \in A_P$ such that $a = a_{M_P} a_{A_P}$. Furthermore, there exist unique $n_P \in N_P$ and $n^P \in N_{P_m}^P = N_{P_m} \cap M_P$ such that $n = n^P n_P$. Hence

$$\begin{aligned} \mathfrak{A}_P(kan \cdot \xi_P) &= \mathfrak{A}_P(k a_{M_P} a_{A_P} n^P n_P \cdot \xi_P) \\ &= \mathfrak{A}_P(k(a_{M_P} n^P) a_{A_P} n_P \cdot \xi_P) = \pi_q(a_{A_P}). \end{aligned}$$

We conclude that

$$\mathfrak{A}_P(g \cdot \xi_P) = \pi_{\mathfrak{a}_P \cap \mathfrak{q}} \circ \mathfrak{A}_{KAN_{P_m}}(g),$$

where $\pi_{\mathfrak{a}_P \cap \mathfrak{q}}$ is the orthogonal projection (with respect to $-B(\cdot, \theta \cdot)$) onto $\mathfrak{a}_P \cap \mathfrak{q}$ and $\mathfrak{A}_{KAN_{P_m}}$ is the function $G \rightarrow \mathfrak{a}$ given by

$$g \in K \exp(\mathfrak{A}_{KAN_{P_m}}(g)) N_{P_m} \quad (g \in G). \quad (3.5.3)$$

Let $k_1, k'_1, k_2 \in K$, $a_1, a_2 \in A$ and $n_2 \in N_{P_m}$. Then, by [Kos73, Theorem 4.1],

$$\begin{aligned} \mathfrak{A}_{KAN_{P_m}}(k_1 a_1 k'_1 k_2 a_2 n_2) &\in \mathfrak{A}_{KAN_{P_m}}(a_1 K) + \log a_2 \\ &= \text{ch}(W(\mathfrak{a}) \cdot \log a_1) + \log a_2, \end{aligned}$$

where $W(\mathfrak{a})$ denotes the Weyl-group $\mathcal{N}_K(\mathfrak{a})/\mathcal{Z}_K(\mathfrak{a})$. Therefore, if $g_1, g_2 \in G$,

$$\mathfrak{A}_{KAN_{P_m}}(g_1 g_2) = \text{ch}(W(\mathfrak{a}) \cdot \mathfrak{A}_{KAK}(g_1)) + \mathfrak{A}_{KAN_{P_m}}(g_2).$$

Here \mathfrak{A}_{KAK} is the map from G to the closed positive Weyl chamber $\overline{\mathfrak{a}^+}(P_m)$ given by

$$g \in K \exp(\mathfrak{A}_{KAK}(g))K \quad (g \in G).$$

For $g \in G$ and $\xi \in \mathcal{E}_P$ we now find

$$\mathfrak{A}_P(g \cdot \xi) = Y(g) + \mathfrak{A}_P(\xi),$$

with $Y(g) \in \pi_{\mathfrak{a}_P \cap \mathfrak{a}_q}(\text{ch}(W(\mathfrak{a}) \cdot \mathfrak{A}_{KAK}(g)))$. The estimate (3.5.2) now follows from the observation that $Y(g)$ is a locally bounded function of g . \square

It follows from Lemma 3.5.1 that the space $L^1(\mathcal{E}_P, J_P)$ is invariant under left translation by elements of g . Accordingly, we define the representation π of G in this space by

$$[\pi(g)\phi](\xi) := \phi(g^{-1} \cdot \xi), \quad (\pi \in L^1(\mathcal{E}_P, J_P), g \in G). \quad (3.5.4)$$

Proposition 3.5.2. *The representation π is a continuous Banach-representation.*

Proof. Put $V := L^1(\mathcal{E}_P, J_P)$, and write $\|\cdot\|_V$ for the norm given in (3.5.1). For a compact subset $C \subseteq G$, let $c > 0$ be the constant of Lemma 3.5.1. Then for all $\phi \in V$ and $g \in C$ we have

$$\|\pi(g)\phi\|_V = \int_{\mathcal{E}_P} |\phi(g^{-1} \cdot \xi) J_P(\xi)| d\xi = \int_{\mathcal{E}_P} |\phi(\xi) J(g \cdot \xi)| d\xi \leq c \|\phi\|_V.$$

This shows that each map $\pi(g) : V \rightarrow V$ is bounded, and that the family $\{\pi(g)|g \in C\}$ is equicontinuous.

We will now show that $\lim_{g \rightarrow e} \pi(g)\phi = \phi$ for each $\phi \in V$. By the above mentioned equicontinuity, it suffices to do this for a dense subspace of V . We take the dense subspace $V_0 := C_c(\mathcal{E}_P)$. Then for each $\phi \in V_0$ we have by the principle

of uniform continuity that $\pi(g)\phi \rightarrow \phi$ uniformly and with supports in a compact set as $g \rightarrow e$. This in turn implies that $\pi(g)\phi \rightarrow \phi$ in V .

It now follows by application of the principle of uniform boundedness that π is a continuous representation. By what we proved above, we can also give the following direct argument.

Let $g_0 \in G$ and $\phi_0 \in V$. We will prove the continuity of the map $\pi : G \times V \rightarrow V$ at (g_0, ϕ_0) . Let $\epsilon > 0$. Then there exists a compact neighborhood C of e in G such that $\|\pi(g)\phi_0 - \phi_0\| < \epsilon/2$ for $g \in C$. Let $M > 0$ be such that $\|\pi(g)\| \leq M$ for all $g \in Cg_0$ and let $\delta < \epsilon/2(M + 1)$. Then for all $g \in Cg_0$ and all $\phi \in V$ with $\|\phi - \phi_0\| < \delta$ we have:

$$\begin{aligned} \|\pi(g)\phi - \pi(g_0)\phi_0\| &\leq \|\pi(g)(\phi - \phi_0)\| + \|\pi(g)\phi_0 - \phi_0\| + \|\phi_0 - \phi\| \\ &< M\delta + \epsilon/2 + \delta < \epsilon. \end{aligned}$$

□

Lemma 3.5.3. *Let P be a $\sigma \circ \theta$ -stable parabolic subgroup containing A . Then with suitably normalized measures, we have*

$$\int_{\Xi_P} \phi(\xi) J_P(\xi) d\xi = \int_K \int_{L_P/(L_P \cap H)} \phi(kl \cdot \xi_P) d_{L_P \cap H} l dk$$

for every $\phi \in L^1(\Xi_P, J_P)$.

Proof. We leave it to the reader to check that all measures that appear in this proof may be normalized such that the equalities hold.

If $\chi \in L^1(G)$, then

$$\begin{aligned} \int_G \chi(g) dg &= \int_K \int_P \chi(kp) \Delta_P(p) dp dk \\ &= \int_K \int_{L_P/(L_P \cap H)} \int_{(L_P \cap H)N_P} \chi(kls) \Delta_P(ls) ds d_{(L_P \cap H)} l dk. \end{aligned}$$

Here dp denotes left invariant measure, and Δ_P the modular function of P . If $m \in M_P \cap K$, $a \in A_q$, $h \in L_P \cap H$ and $n \in N_P$, then $\Delta_P(mahn) = a^{2\rho_P}$. Therefore Δ_P is right $(L_P \cap H)N_P$ -invariant. Hence

$$\int_K \int_{L_P/(L_P \cap H)} \eta(kl \cdot \xi_P) \Delta_P(l) d_{(L_P \cap H)} l dk \quad (\eta \in L^1(\Xi_P))$$

defines a G -invariant Radon measure on \mathcal{E}_P . Note that G -invariant Radon measures on \mathcal{E}_P are unique up to multiplication by a constant. Therefore, under the assumption that the measure are suitably normalized,

$$\int_{\mathcal{E}_P} \phi(\xi) J_P(\xi) d\xi = \int_K \int_{L_P/(L_P \cap H)} \Delta_P(l) \phi(kl \cdot \xi_P) J_P(kl \cdot \xi_P) d_{L_P \cap H} l dk.$$

for every $\phi \in L^1(\mathcal{E}_P, J_P)$.

The pull-back of J_P under the map

$$K \times L_P/(L_P \cap H) \rightarrow \mathcal{E}_P; \quad (k, l \cdot (L_P \cap H)) \mapsto kl \cdot \xi_P$$

equals

$$K \times L_P/(L_P \cap H) \rightarrow \mathbf{R}; \quad (k, l \cdot (L_P \cap H)) \mapsto \frac{1}{\Delta_P(l)}.$$

Hence,

$$\int_{\mathcal{E}_P} \phi(\xi) J_P(\xi) d\xi = \int_K \int_{L_P/(L_P \cap H)} \phi(kl \cdot \xi_P) d_{L_P \cap H} l dk.$$

□

We define $\mathcal{E}^1(\mathcal{E}_P, J_P)$ to be the subspace of $\mathcal{E}(\mathcal{E}_P)$ consisting of functions that represent a smooth vector for the representation π of G on $L^1(\mathcal{E}_P, J_P)$ defined in (3.5.4). We endow this space with the Fréchet topology induced by the natural bijection

$$\mathcal{E}^1(\mathcal{E}_P, J_P) \rightarrow L^1(\mathcal{E}_P, J_P)^\infty.$$

The space $\mathcal{E}^1(X, J_G)$ is denoted by $\mathcal{E}^1(X)$. (Recall that $J_G = \mathbf{1}_X$.)

Proposition 3.5.4. *If $\phi \in \mathcal{E}^1(\mathcal{E}_P, J_P)$, then ϕ vanishes at infinity.*

Proof. Let Δ_K be the diagonal in $K \times K$ and let $S = ((K \times K)/\Delta_K) \times (L_P/(L_P \cap H))$. The map

$$\Phi : S \rightarrow \mathcal{E}_P; \quad ((k_1, k_2) \cdot \Delta_K, l \cdot (L_P \cap H)) \mapsto k_1 k_2^{-1} l \cdot \xi_P$$

is a surjective, smooth submersion. Since we take the measure of K to be normalized, pullback under Φ defines an isometric embedding

$$\Phi^* : L^1(\mathcal{E}_P, J_P) \rightarrow L^1(S).$$

by Lemma 3.5.3.

Let $n \in \mathbf{N}$ and let $v \in \mathcal{U}(\mathfrak{g})$ be of degree smaller than or equal to n . If $k \in K$, then $\text{Ad}(k)v$ can be written as a finite sum

$$\text{Ad}(k)v = \sum_j c_j(k)u_j,$$

where the c_j are continuous functions from K to \mathbf{C} and the u_j form a basis for the subspace of $\mathcal{U}(\mathfrak{g})$ consisting of the elements of order at most n . Since K is compact, the functions c_j are bounded. Therefore, if $\phi \in \mathcal{E}^1(\mathcal{E}_P, J_P)$, $u, v_1 \in \mathcal{U}(\mathfrak{k})$ and $v_2 \in \mathcal{U}(\mathfrak{l}_P)$, then the $L^1(S)$ -norm of $(u \otimes v_1 \otimes v_2)\Phi^*\phi$ can be estimated by a constant times

$$\sum_j \int_{\mathcal{E}_P} |uu_j\phi(\xi)|J_P(\xi) d\xi.$$

This proves that pullback under Φ maps $\mathcal{E}^1(\mathcal{E}_P, J_P)$ to the space $\mathcal{E}^1(S)$ of smooth representatives for elements in $L^1(S)^\infty$. Note that S is a symmetric space of the Harish-Chandra class. According to [KS07, Theorem 3.1] every function $\psi \in \mathcal{E}^1(S)$ vanishes at infinity. Since pullback under Φ maps $\mathcal{E}^1(\mathcal{E}_P, J_P)$ to $\mathcal{E}^1(S)$ and Φ is a continuous surjection, it follows that every function $\phi \in \mathcal{E}^1(\mathcal{E}_P, J_P)$ vanishes at infinity. \square

Let $L^\infty(\mathcal{E}_P, J_P)$ be the space of equivalence classes (modulo differences on sets of measure 0) of measurable functions ϕ such that ϕ/J_P is essentially bounded, i.e.,

$$L^\infty(\mathcal{E}_P, J_P) = \left\{ \phi : \frac{\phi}{J_P} \in L^\infty(\mathcal{E}_P) \right\}.$$

We endow this space with the norm

$$\phi \mapsto \left\| \frac{\phi}{J_P} \right\|_{L^\infty(\mathcal{E}_P)}.$$

Lemma 3.5.5. *The pairing*

$$(\psi, \phi) \mapsto \int_{\mathcal{E}_P} \psi \phi d\xi \tag{3.5.5}$$

induces an isometric isomorphism $L^\infty(\mathcal{E}_P, J_P) \rightarrow L^1(\mathcal{E}_P, J_P)'$.

Proof. Since $(\mathcal{E}_P, d\xi)$ is a σ -finite measure space, the pairing (3.5.5) induces an isometric isomorphism from $L^\infty(\mathcal{E}_P)$ onto $L^1(\mathcal{E}_P)'$. (See for example [Fri82, Theorem 4.14.6].) Now use that $\phi \mapsto \phi J_P$ defines an isometric isomorphism from $L^1(\mathcal{E}_P, J_P)$ onto $L^1(\mathcal{E}_P)$, whereas $\psi \mapsto \psi/J_P$ induces an isometric isomorphism from $L^\infty(\mathcal{E}_P, J_P)$ onto $L^\infty(\mathcal{E}_P)$. \square

Let $C_b(\mathcal{E}_P, J_P)$ be the space of continuous functions ϕ such that ϕ/J_P is bounded and let $\mathcal{E}_b(\mathcal{E}_P, J_P)$ be the space of functions $\phi \in \mathcal{E}(\mathcal{E}_P)$ such that

$$u\phi \in C_b(X_P, J_P) \quad (u \in \mathcal{U}(\mathfrak{g})).$$

The space $\mathcal{E}_b(\mathcal{E}_P, J_P)$ is a Fréchet space with the topology induced from the obvious set of seminorms. Note that $\mathcal{E}_b(X, J_G) = \mathcal{E}_b(X)$.

Although the left-regular representation of G on $C_b(\mathcal{E}_P, J_P)$ is not continuous unless \mathcal{E}_P is compact (which is a rather uninteresting case), we do have the following result.

Proposition 3.5.6. *The left regular representation of G on the space $\mathcal{E}_b(\mathcal{E}_P, J_P)$ is a smooth Fréchet representation.*

Proof. We denote the left regular representation of G on $\mathcal{E}_b(\mathcal{E}_P, J_P)$ by π . Let $u \in \mathcal{U}(\mathfrak{g})$ be of order n . Let $\{u_k : 1 \leq k \leq m\}$ be a basis for the subspace of $\mathcal{U}(\mathfrak{g})$ consisting of elements of order at most n . Then there exist continuous functions $c_k : G \rightarrow \mathbb{C}$ such that $\text{Ad}(g)u = \sum_{k=1}^m c_k(g)u_k$. Let C be a compact subset of G and $\phi \in \mathcal{E}_b(\mathcal{E}_P, J_P)$. By Lemma 3.5.1 there exists a constant $c > 0$ such that for every $g \in C$

$$\sup_{\xi \in \mathcal{E}_P} \frac{|u(\pi(g)\phi)(\xi)|}{J_P(\xi)} \leq \sup_{\xi \in \mathcal{E}_P} c \sum_{k=1}^m \frac{|u_k\phi(g \cdot \xi)|}{J_P(g \cdot \xi)}$$

This implies in particular that $\pi(g)$ is continuous for every $g \in G$ and every seminorm of $\pi(g)\phi$ is locally uniformly bounded in g . Furthermore, if $g = \exp(Y)$ for some element $Y \in \mathfrak{g}$, then again by Lemma 3.5.1 there exists a constant c such that

$$\begin{aligned} \sup_{\mathcal{E}_P} \frac{|\pi(g)\phi - \phi|}{J_P} &\leq \sup_{\mathcal{E}_P} \frac{1}{J_P} \int_0^1 \left| \frac{d}{dt} \pi(\exp(tY))\phi \right| dt \\ &\leq \int_0^1 \sup_{\mathcal{E}_P} \frac{|\pi(\exp(tY))(Y\phi)|}{J_P} dt \leq c \sup_{\mathcal{E}_P} \frac{|Y\phi|}{J_P}. \end{aligned}$$

Therefore $\lim_{g \rightarrow e} \pi(g)\phi = \phi$. We conclude that the assumptions in [War72, Proposition 4.1.1.1] are satisfied and hence that π is a continuous representation.

In order to prove that the representation is smooth, it suffices to show that for every $\phi \in \mathcal{E}_b(\Xi_P, J_P)$ and $Y \in \mathfrak{g}$ we have

$$\lim_{t \rightarrow 0} \sup_{\Xi_P} \frac{1}{J_P} \left| \frac{\pi(\exp(tY))\phi - \phi}{t} - Y\phi \right| = 0. \quad (3.5.6)$$

Since

$$\int_{s=0}^t (t-s) \frac{d^2}{ds^2} \pi(\exp(sY))\phi \, ds = \pi(\exp(tY))\phi - \phi - tY\phi,$$

it follows that

$$\begin{aligned} & \sup_{\Xi_P} \frac{1}{J_P} \left| \frac{\pi(\exp(tY))\phi - \phi}{t} - Y\phi \right| \\ & \leq \sup_{\Xi_P} \frac{1}{J_P} \int_{s=0}^t \frac{t-s}{t} \left| \frac{d^2}{ds^2} \pi(\exp(sY))\phi \right| ds \\ & \leq \int_{s=0}^t \frac{t-s}{t} \sup_{\Xi_P} \frac{\left| \frac{d^2}{ds^2} \pi(\exp(sY))\phi \right|}{J_P} ds. \end{aligned}$$

By Lemma 3.5.1 there exists a constant c such that the latter is smaller than or equal to

$$c \sup_{\Xi_P} \frac{|Y^2\phi|}{J_P} \int_{s=0}^t \frac{t-s}{t} ds = c \sup_{\Xi_P} \frac{|Y^2\phi|}{J_P} \frac{t}{2}.$$

This implies (3.5.6). □

3.6 Extensions of the Radon transforms

Recall that for a parabolic subgroup P of G and an element $g \in G$ the conjugate parabolic subgroup $g^{-1}Pg$ is denoted by P^g . Furthermore, recall that \mathcal{W}_{M_P} denotes a set of representatives for the quotient of the Weyl groups $W_{M_P \cap K}$ and $W_{M_P \cap K \cap H}$. (See Section 2.2.)

Lemma 3.6.1. *Let $P_0 \in \mathcal{P}_\sigma(\mathfrak{a}_q)$ and let P be a parabolic subgroup containing P_0 . Then with a suitable normalization of the measure $d\xi$ on Ξ_P ,*

$$\int_{\Xi_P} \phi(\xi) J_P(\xi) \, d\xi = \sum_{w \in \mathcal{W}_{M_P}} \int_K \int_{A_q} \int_{N_{P_0^w}^P} \phi(kan \cdot \xi_P) \, dn \, da \, dk$$

for all $\phi \in L^1(\Xi_P, J_P)$.

Proof. From [Óla87, Theorem 1.2] applied to $L_P/(L_P \cap H)$ and the minimal $\sigma \circ \theta$ -stable parabolic subgroup

$$P_0 \cap L_P = M_{P_0} A_{P_0} N_{P_0}^P$$

of L_P , it follows that

$$\begin{aligned} & \int_{L_P/(L_P \cap H)} \phi(kl \cdot \xi_P) d_{L_P \cap H} l \\ &= c \sum_{w \in \mathcal{W}_{M_P}} \int_{M_{P_0} \cap K} \int_{A_q} \int_{N_{P_0}^P w} \phi(mwan \cdot \xi_P) dn da dm, \end{aligned}$$

for some constant $c > 0$. Using Lemma 3.5.3 and the invariance of the measure on K , we obtain

$$\begin{aligned} & \int_{\mathcal{E}_P} \phi(\xi) J_P(\xi) d\xi \\ &= c \sum_{w \in \mathcal{W}_{M_P}} \int_K \int_{M_{P_0} \cap K} \int_{A_q} \int_{N_{P_0}^P w} \phi(kmwan \cdot \xi_P) dn da dm dk \\ &= c \sum_{w \in \mathcal{W}_{M_P}} \int_K \int_{A_q} \int_{N_{P_0}^P w} \phi(kan \cdot \xi_P) dn da dk. \end{aligned}$$

The measure $d\xi$ can be normalized such that $c = 1$. □

From now on we assume that the measure on \mathcal{E}_P is normalized such that the identity in Lemma 3.6.1 holds.

As before, let P and Q be $\sigma \circ \theta$ -stable parabolic subgroups of G with $A \subseteq P \subseteq Q$.

Lemma 3.6.2. *If $\phi \in L^1(\mathcal{E}_Q, J_Q)$, then*

$$\int_{\mathcal{E}_P} \left(\int_{N_P^Q} |\phi(gn \cdot \xi_Q)| dn \right) J_P(g \cdot \xi_P) d_{(L_P \cap H)N_P} g \leq \int_{\mathcal{E}_Q} |\phi(\xi)| J_Q(\xi) d\xi.$$

Proof. Since $A_q \subseteq L_P \subseteq L_Q$, we can choose the sets of representatives \mathcal{W}_{M_P} and \mathcal{W}_{M_Q} such that $\mathcal{W}_{M_P} \subseteq \mathcal{W}_{M_Q}$. Let $P_0 \in \mathcal{P}_\sigma(\mathfrak{a}_q)$ be a minimal $\sigma \circ \theta$ -stable

parabolic subgroup with $P_0 \subseteq P$. By Lemma 3.6.1,

$$\begin{aligned} \int_{\mathcal{E}_P} \left(\int_{N_P^Q} |\phi(gn \cdot \xi_Q)| dn \right) J_P(g \cdot \xi_P) d_{(L_P \cap H)N_P} g = \\ \sum_{w \in \mathcal{W}_{M_P}} \int_K \int_{A_q} \int_{N_{P_0^w}^P} \int_{N_P^Q} |\phi(kann_P \cdot \xi_Q)| dn_P dn da dk = \\ \sum_{w \in \mathcal{W}_{M_P}} \int_K \int_{A_q} \int_{N_{P_0^w}^Q} |\phi(kan \cdot \xi_Q)| dn da dk. \end{aligned}$$

Here we used (3.3.2). Now we use that $\mathcal{W}_{M_P} \subseteq \mathcal{W}_{M_Q}$ and apply Lemma 3.6.1 once more, to obtain

$$\begin{aligned} \sum_{w \in \mathcal{W}_{M_P}} \int_K \int_{A_q} \int_{N_{P_0^w}^Q} |\phi(kan \cdot \xi_Q)| dn da dk \leq \\ \sum_{w \in \mathcal{W}_{M_Q}} \int_K \int_{A_q} \int_{N_{P_0^w}^Q} |\phi(kan \cdot \xi_Q)| dn da dk = \int_{\mathcal{E}_Q} |\phi(\xi)| J_Q(\xi) d\xi. \end{aligned}$$

□

Recall the definition of the map $\mathcal{R}_P^Q : \mathcal{D}(\mathcal{E}_Q) \rightarrow \mathcal{E}(\mathcal{E}_P)$ from (3.4.1).

Proposition 3.6.3. *The transform \mathcal{R}_P^Q defines a continuous map from $\mathcal{D}(\mathcal{E}_Q)$ to $\mathcal{E}^1(\mathcal{E}_P, J_P)$.*

Proof. By Lemma 3.6.2, the Radon transform \mathcal{R}_P^Q defines a continuous map from $\mathcal{D}(\mathcal{E}_Q)$ to $L^1(\mathcal{E}_P, J_P)$. Due to continuity and equivariance, it is a continuous transform between the spaces of smooth vectors for the left-regular representation of G , i.e., it is a continuous map from $\mathcal{D}(\mathcal{E}_Q)$ to $L^1(\mathcal{E}_P, J_P)^\infty$. The proposition now follows from the fact that \mathcal{R}_P^Q maps elements in $\mathcal{D}(X)$ to smooth functions. □

The image of the injection

$$\mathcal{D}(\mathcal{E}_Q) \hookrightarrow L^1(\mathcal{E}_Q, J_Q)$$

is dense. Hence, by Lemma 3.6.2 and Proposition 3.6.3 there exists a unique continuous transform

$$\mathcal{T}_P^Q : L^1(\mathcal{E}_Q, J_Q) \rightarrow L^1(\mathcal{E}_P, J_P)$$

such that

$$\begin{array}{ccc} \mathcal{D}(\mathcal{E}_Q) & \xrightarrow{\mathcal{R}_P^Q} & \mathcal{E}^1(\mathcal{E}_P, J_P) \\ \downarrow & & \downarrow \\ L^1(\mathcal{E}_Q, J_Q) & \xrightarrow{\mathcal{T}_P^Q} & L^1(\mathcal{E}_P, J_P) \end{array}$$

is a commuting diagram. Note that \mathcal{T}_P^Q is equivariant and if $\phi \in L^1(\mathcal{E}_Q, J_Q)$, then

$$\mathcal{T}_P^Q \phi(g \cdot \xi_P) = \int_{N_P^Q} \phi(gn \cdot \xi_Q) dn$$

for almost every $g \cdot \xi_P \in \mathcal{E}_P$. Since \mathcal{T}_P^Q is an equivariant and continuous transform, it maps $L^1(\mathcal{E}_Q, J_Q)^\infty$ continuously to $L^1(\mathcal{E}_P, J_P)^\infty$.

Proposition 3.6.4. *The map $\mathcal{R}_P^Q : \mathcal{D}(\mathcal{E}_Q) \rightarrow \mathcal{E}^1(\mathcal{E}_P)$ has a unique continuous linear extension to a map*

$$\mathcal{R}_P^Q : \mathcal{E}^1(\mathcal{E}_Q, J_Q) \rightarrow \mathcal{E}^1(\mathcal{E}_P, J_P)$$

Furthermore, if $\phi \in \mathcal{E}^1(\mathcal{E}_Q, J_Q)$, then for every $g \in G$

$$\mathcal{R}_P^Q \phi(g \cdot \xi_P) = \int_{N_P^Q} \phi(gn \cdot \xi_Q) dn \quad (3.6.1)$$

with absolutely convergent integral.

Proof. There exists a unique transform \mathcal{R}_P^Q such that the diagram

$$\begin{array}{ccc} L^1(\mathcal{E}_Q, J_Q)^\infty & \xrightarrow{\mathcal{T}_P^Q} & L^1(\mathcal{E}_P, J_P)^\infty \\ \downarrow \simeq & & \downarrow \simeq \\ \mathcal{E}^1(\mathcal{E}_Q, J_Q) & \xrightarrow{\mathcal{R}_P^Q} & \mathcal{E}^1(\mathcal{E}_P, J_P) \end{array}$$

is a commuting diagram. This transform clearly is continuous and extends the transform $\mathcal{R}_P^Q : \mathcal{D}(\mathcal{E}_Q) \rightarrow \mathcal{E}^1(\mathcal{E}_P, J_P)$. It remains to be shown that the integrals in (3.6.1) are absolutely convergent and that \mathcal{R}_P^Q is given by these integrals.

Let $\chi \in \mathcal{D}(G)$ and $\varphi \in \mathcal{E}^1(\mathcal{E}_Q, J_Q)$. Since $\mathcal{T}_P^Q \varphi \in L^1(\mathcal{E}_P, J_P)^\infty$ and since J_P is smooth and non-vanishing, the integral

$$\psi(g \cdot \xi_P) := \int_G \chi(\gamma) \mathcal{T}_P^Q \varphi(\gamma^{-1} g \cdot \xi_P) d\gamma = \int_G \chi(g\gamma) \mathcal{T}_P^Q \varphi(\gamma^{-1} \cdot \xi_P) d\gamma$$

is absolutely convergent for every $g \in G$. Furthermore, by Fubini,

$$\psi(g \cdot \xi_P) = \int_{N_P^Q} (\chi * \varphi)(gn \cdot \xi_Q) dn,$$

where $\chi * \varphi$ denotes the convolution of φ with χ , i.e.,

$$\chi * \varphi(g \cdot \xi_Q) = \int_G \chi(g\gamma) \varphi(\gamma^{-1} \cdot \xi_Q) d\gamma.$$

By application of Lebesgue's dominated convergence theorem one sees that ψ is a smooth function on \mathcal{E}_P . The function $\chi * \varphi$ is an element of $\mathcal{E}^1(\mathcal{E}_Q, J_Q)$, hence ψ is a smooth representative for the smooth vector $\mathcal{T}_P^Q(\chi * \varphi) \in L^1(\mathcal{E}_P, J_P)^\infty$. This implies that $\psi = \mathcal{R}_P^Q(\chi * \varphi)$. We conclude that the second statement in the proposition holds for functions $\phi = \chi * \varphi$ with $\chi \in \mathcal{D}(G)$ and $\varphi \in \mathcal{E}^1(\mathcal{E}_P, J_P)$.

By Proposition 3.5.2 the left regular representation of G on $L^1(\mathcal{E}_Q, J_Q)$ is a Banach representation. By [DM78, Théorème 3.3] the space of smooth vectors for this representation equals the Gårding subspace. This implies that every element of $\mathcal{E}^1(\mathcal{E}_Q, J_Q)$ can be written as a finite sum of convolutions $\chi * \phi$, with $\chi \in \mathcal{D}(G)$ and $\phi \in \mathcal{E}^1(\mathcal{E}_Q, J_Q)$. This proves the proposition. \square

Lemma 3.6.5. *There exists a normalization of the invariant measure on $(L_Q \cap H)/(L_P \cap H)$ such that for every $\phi \in L^1(\mathcal{E}_Q, J_Q)$ and $\psi \in C_b(\mathcal{E}_P, J_P)$ the integral*

$$\int_{\mathcal{E}_Q} \int_{(L_Q \cap H)/(L_P \cap H)} \psi(gl \cdot \xi_P) \phi(g \cdot \xi_Q) d_{(L_P \cap H)} l d_{(L_Q \cap H)N_Q} g$$

is absolutely convergent and equal to

$$\int_{\mathcal{E}_P} \int_{N_P^Q} \psi(g \cdot \xi_P) \phi(gn \cdot \xi_Q) dn d_{(L_P \cap H)N_P} g.$$

Proof. The distributions on $G/(L_P \cap H)N_Q$, for $\theta \in \mathcal{D}(G/(L_P \cap H)N_Q)$ given by

$$\int_{\mathcal{E}_Q} \int_{(L_Q \cap H)/(L_P \cap H)} \theta(gl \cdot (L_P \cap H)N_Q) d_{(L_P \cap H)} l d_{(L_Q \cap H)N_Q} g$$

and

$$\int_{\mathcal{E}_P} \int_{N_P^Q} \theta(gn \cdot (L_P \cap H)N_Q) dn d_{(L_P \cap H)N_P} g,$$

respectively, both define a G -invariant Radon measure on $G/(L_P \cap H)N_Q$. These measures can therefore only differ by a multiplicative positive constant (see for example [Kna02, Theorem 8.36]), and for a suitable normalization for the invariant measure on $(L_Q \cap H)/(L_P \cap H)$ they are equal. Since $\psi \in C_b(\mathcal{E}_P, J_P)$ the function $|\frac{\psi}{J_P}|$ is bounded on \mathcal{E}_P . Hence, by Lemma 3.6.2

$$\begin{aligned} \int_{\mathcal{E}_P} \int_{N_P^Q} \psi(g \cdot \xi_P) \phi(gn \cdot \xi_Q) dn d_{(L_P \cap H)N_P} g = \\ \int_{\mathcal{E}_P} \int_{N_P^Q} \frac{\psi(g \cdot \xi_P)}{J_P(g \cdot \xi_P)} \phi(gn \cdot \xi_Q) J_P(g \cdot \xi_P) dn d_{(L_P \cap H)N_P} g \end{aligned}$$

is absolutely convergent. By the first part of the proof combined with Fubini's Theorem,

$$\int_{\mathcal{E}_Q} \int_{(L_Q \cap H)/(L_P \cap H)} \psi(gl \cdot \xi_P) \phi(g \cdot \xi_Q) d_{(L_P \cap H)} l d_{(L_Q \cap H)N_Q} g$$

is absolutely convergent as well and we see that the claimed equality holds. \square

From now on we assume that the $L_Q \cap H$ -invariant measure on the homogeneous space $(L_Q \cap H)/(L_P \cap H)$ is normalized such that the equality in Lemma 3.6.5 holds.

Proposition 3.6.6. *If $\psi \in \mathcal{E}_b(\mathcal{E}_P, J_P)$, then for every $g \in G$ the integral*

$$\mathcal{S}_P^Q \psi(g \cdot \xi_Q) := \int_{(L_Q \cap H)/(L_P \cap H)} \psi(gl \cdot \xi_P) d_{(L_P \cap H)} l \quad (3.6.2)$$

is absolutely convergent and the associated function $\mathcal{S}_P^Q \psi : \mathcal{E}_Q \rightarrow \mathbb{C}$ belongs to the space $\mathcal{E}_b(\mathcal{E}_Q, J_Q)$. Furthermore, the transform

$$\mathcal{S}_P^Q : \mathcal{E}_b(\mathcal{E}_P, J_P) \rightarrow \mathcal{E}_b(\mathcal{E}_Q, J_Q) \quad (3.6.3)$$

thus obtained, is continuous.

Proof. Let $\eta \in \mathcal{E}_b(\mathcal{E}_P, J_P)$ and $\chi \in \mathcal{D}(G)$. It follows from Lemma 3.6.5 that the integral (3.6.2) is absolutely convergent for almost every $g \in G$ and the associated almost everywhere defined function $g \mapsto \mathcal{S}_P^Q \eta(g \cdot \xi_Q)$ on G is locally integrable. Therefore, for every $g \in G$, the integral

$$\int_G \chi(g\gamma) \int_{(L_Q \cap H)/(L_P \cap H)} \eta(\gamma^{-1}l \cdot \xi_P) d_{(L_P \cap H)} l d\gamma$$

is absolutely convergent. Furthermore, the integral depends smoothly on g and by Fubini's Theorem it is equal to $\mathcal{S}_P^Q(\chi * \eta)(l \cdot \xi_Q)$. Here $\chi * \eta$ denotes the convolution product between χ and η , i.e., $\chi * \eta$ is the function on \mathcal{E}_P given by

$$\chi * \eta(\xi) = \int_G \chi(g) \eta(g^{-1} \cdot \xi) dg \quad (\xi \in \mathcal{E}_P).$$

This proves that for every $\chi \in \mathcal{D}(G)$ and $\eta \in \mathcal{E}_b(\mathcal{E}_P, J_P)$ the function $\mathcal{S}_P^Q(\chi * \eta)$ is defined by absolutely convergent integrals (3.6.2) and is smooth.

Let $\psi \in \mathcal{E}_b(\mathcal{E}_P, J_P)$. By [DM78, Théorème 3.3] the space of smooth vectors for the left regular representation of G on $C_b(\mathcal{E}_P, J_P)$ equals the space of Gårding vectors. Therefore ψ equals a finite sum of convolutions $\chi * \eta$, with $\chi \in \mathcal{D}(G)$ and $\eta \in \mathcal{E}_b(\mathcal{E}_P, J_P)$. We conclude from the above argument that $\mathcal{S}_P^Q \psi$ is a smooth function on \mathcal{E}_Q defined by absolutely convergent integrals (3.6.2).

From Lemma 3.6.5 and 3.6.2 it follows that for every $\phi \in L^1(\mathcal{E}_Q, J_Q)$

$$\int_{\mathcal{E}_Q} |\mathcal{S}_P^Q \psi(\xi) \phi(\xi)| d\xi \leq \sup_{\mathcal{E}_P} \frac{|\psi|}{J_P} \int_{\mathcal{E}_Q} |\phi(\xi)| J_Q(\xi) d\xi.$$

Therefore \mathcal{S}_P^Q defines a continuous map from $\mathcal{E}_b(\mathcal{E}_P, J_P)$ to the dual space of $L^1(\mathcal{E}_Q, J_Q)$. By Lemma 3.5.5, this dual space equals $L^\infty(\mathcal{E}_Q, J_Q)$. The space $C_b(\mathcal{E}_Q, J_Q)$ naturally embeds onto a closed subspace of $L^\infty(\mathcal{E}_Q, J_Q)$. As \mathcal{S}_P^Q maps $\mathcal{E}_b(\mathcal{E}_P, J_P)$ into $C_b(\mathcal{E}_Q, J_Q)$ and is continuous as a map from that space into $L^\infty(\mathcal{E}_Q, J_Q)$, it follows that \mathcal{S}_P^Q is continuous as a map $\mathcal{E}_b(\mathcal{E}_P, J_P) \rightarrow C_b(\mathcal{E}_Q, J_Q)$. Since \mathcal{S}_P^Q is equivariant and the left regular representation of G on $\mathcal{E}_b(\mathcal{E}_P, J_P)$ is smooth, it follows that \mathcal{S}_P^Q is a continuous map $\mathcal{E}_b(\mathcal{E}_P, J_P) \rightarrow \mathcal{E}_b(\mathcal{E}_Q, J_Q)$. \square

Note that the transform (3.6.3) is an extension of the earlier defined transform (3.4.2) for compactly supported smooth functions. Thus, the notation is unambiguous.

Lemma 3.6.5 has the following corollary.

Corollary 3.6.7. *If $\phi \in \mathcal{E}^1(\Xi_Q, J_Q)$ and $\psi \in \mathcal{E}_b(\Xi_P, J_P)$, then*

$$\int_{\Xi_P} \mathcal{R}_P^Q \phi(\xi) \psi(\xi) d\xi = \int_{\Xi_Q} \phi(\zeta) S_P^Q \psi(\zeta) d\zeta.$$

Remark 3.6.8. Let $S_P^{Q^t}$ be the adjoint transform of S_P^Q . By Corollary 3.6.7, the following diagram commutes.

$$\begin{array}{ccc} \mathcal{E}^1(\Xi_Q, J_Q) & \xrightarrow{\mathcal{R}_P^Q} & \mathcal{E}^1(\Xi_P, J_P) \\ \downarrow & & \downarrow \\ \mathcal{E}_b'(\Xi_Q, J_Q) & \xrightarrow{S_P^{Q^t}} & \mathcal{E}_b'(\Xi_P, J_P) \end{array}$$

This allows to extend the definition (1.3.2) of the Radon transform $\mathcal{R}_P^Q \mu$ of a compactly supported distribution $\mu \in \mathcal{E}'(\Xi_Q)$ to the Radon transform $\mathcal{R}_P^Q \mu$ of a distribution $\mu \in \mathcal{E}_b'(\Xi_Q, J_Q)$. Accordingly, from now on we will write \mathcal{R}_P^Q for $S_P^{Q^t}$. If $\mu \in \mathcal{E}_b'(\Xi_Q, J_Q)$, then $\mathcal{R}_P^Q \mu$ is the distribution in $\mathcal{E}_b'(\Xi_P, J_P)$ given by

$$\mathcal{R}_P^Q \mu(\psi) = \mu(S_P^Q \psi) \quad (\psi \in \mathcal{E}_b(\Xi_P, J_P)).$$

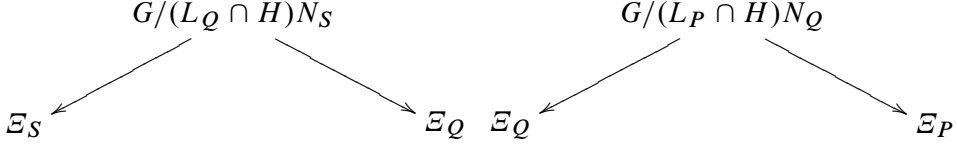
3.7 Relations between Radon transforms

Let P , Q and S be three $\sigma \circ \theta$ -stable parabolic subgroups such that $P \subseteq Q \subseteq S$.

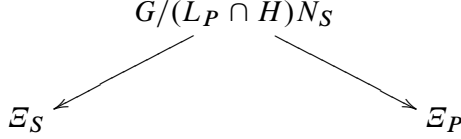
We now consider the following diagram.

$$\begin{array}{ccccc} & & G/(L_P \cap H)N_S & & \\ & \swarrow & & \searrow & \\ G/(L_Q \cap H)N_S & & \circlearrowleft & & G/(L_P \cap H)N_Q \\ \swarrow & & & & \searrow \\ \Xi_S & & \Xi_Q & & \Xi_P \end{array}$$

Here every map is a canonical projection. This diagram describes four double fibrations of the type considered in Section 1.3. Only three of these are relevant for our purposes:



and



The Radon transforms for these double fibrations are related to each other in the following way.

Proposition 3.7.1.

(i) The Radon transforms of functions \mathcal{R}_P^Q and \mathcal{R}_Q^S compose to a Radon transform of functions as follows

$$\mathcal{R}_P^Q \circ \mathcal{R}_Q^S = \mathcal{R}_P^S : \mathcal{C}^1(\mathcal{E}_S, J_S) \rightarrow \mathcal{C}^1(\mathcal{E}_P, J_P).$$

(ii) The dual Radon transforms of functions \mathcal{S}_P^Q and \mathcal{S}_Q^S compose to a Radon transform of functions as follows

$$\mathcal{S}_Q^S \circ \mathcal{S}_P^Q = \mathcal{S}_P^S : \mathcal{C}_b(\mathcal{E}_P, J_P) \rightarrow \mathcal{C}_b(\mathcal{E}_S, J_S).$$

(iii) The Radon transforms of distributions \mathcal{R}_P^Q and \mathcal{R}_Q^S compose to a Radon transform of distributions as follows

$$\mathcal{R}_P^Q \circ \mathcal{R}_Q^S = \mathcal{R}_P^S : \mathcal{C}'_b(\mathcal{E}_S, J_S) \rightarrow \mathcal{C}'_b(\mathcal{E}_P, J_P).$$

Proof.

(i): The multiplication map

$$N_P^Q \times N_Q^S \rightarrow N_P^S$$

is a diffeomorphism with Jacobian equal to the constant function 1. Therefore the identity follows from the definitions and by application of Fubini's Theorem.

(ii): The continuous linear functional on $\mathcal{D}((L_S \cap H)/(L_P \cap H))$ mapping a function ψ to

$$\int_{(L_S \cap H)/(L_Q \cap H)} \int_{(L_Q \cap H)/(L_P \cap H)} \psi(l_S l_Q \cdot (L_P \cap H)) d_{(L_P \cap H)} l_Q d_{(L_Q \cap H)} l_S$$

defines a $L_S \cap H$ invariant measure on $(L_S \cap H)/(L_P \cap H)$. If the measures are normalized such that the equality in Lemma 3.6.5 holds, as we assumed, then this measure and the invariant measure on $(L_S \cap H)/(L_P \cap H)$ are equal. This proves the claim.

(iii): This is a direct corollary of (ii). \square

3.8 Some convex geometry

In this section we prove some results in convex geometry that are needed in the next section.

Let V be a finite dimensional vector space. If B is a subset of V , we denote the convex hull of B by $\text{ch}(B)$, i.e., $\text{ch}(B)$ is the smallest convex set containing B . We call a subset of V a cone if it is closed under the action of the multiplicative group $\mathbf{R}_{>0}$.

Let S be a subset of V . The function

$$H_S : V^* \rightarrow \mathbf{R} \cup \{\pm\infty\}; \quad \lambda \mapsto \sup_{x \in S} \lambda(x)$$

is called the support function of S . Note that the image of H_S contains $-\infty$ if and only if $S = \emptyset$. We define

$$\mathcal{C}_S = \{\lambda \in V^* : H_S(\lambda) < \infty\}. \quad (3.8.1)$$

Note that \mathcal{C}_S is a convex cone.

It is a well known result from convex geometry that if S is a subset of V , then

$$x \in \overline{\text{ch}(S)} \quad \text{if and only if} \quad \lambda(x) \leq H_S(\lambda) \text{ for all } \lambda \in \mathcal{C}_S.$$

Lemma 3.8.1. *Let B be a non-empty subset of V and let Γ be a cone in V containing 0. Then the following statements hold.*

$$(i) \quad H_\Gamma - H_{-B} \leq H_{B+\Gamma} \leq H_\Gamma + H_B.$$

$$(ii) \quad \mathcal{C}_{B+\Gamma} = \mathcal{C}_\Gamma \cap \mathcal{C}_B = \{\lambda \in \mathcal{C}_B : \lambda(x) \leq 0 \text{ for every } x \in \Gamma\}.$$

(iii) The functions $H_{B+\Gamma}$ and H_B have equal restriction to $\mathcal{C}_{B+\Gamma}$. In particular,

$$H_\Gamma|_{\mathcal{C}_\Gamma} = 0.$$

Proof.

(i): Let $\lambda \in V^*$. Then $-H_{-B}(\lambda) = \inf_B \lambda$. Moreover, for every $x \in B$ and $y \in \Gamma$

$$\lambda(y) + \inf_B \lambda \leq \lambda(x + y) \leq \lambda(y) + \sup_B \lambda$$

because $B \neq \emptyset$. The required estimate at the point λ now follows by taking suprema over $x \in B$ and $y \in \Gamma$.

(ii): By (i) we have $H_{B+\Gamma} \leq H_\Gamma + H_B$, hence

$$\mathcal{C}_B \cap \mathcal{C}_\Gamma \subseteq \mathcal{C}_{B+\Gamma}.$$

To prove the converse inclusion, let $\lambda \in \mathcal{C}_{B+\Gamma}$. From $B \subseteq B + \Gamma$ we see that $H_B(\lambda) \leq H_{B+\Gamma}(\lambda)$, hence $\lambda \in \mathcal{C}_B$. If $\lambda \notin \mathcal{C}_\Gamma$, then there exists an $x \in \Gamma$ such that $\lambda(x) > 0$, hence, because Γ is a cone,

$$H_{B+\Gamma}(\lambda) \geq \sup_{b \in B, r \in \mathbf{R}_{>0}} \lambda(b + rx) = \infty.$$

This contradicts the assumption $\lambda \in \mathcal{C}_{B+\Gamma}$, and we see that $\lambda \in \mathcal{C}_\Gamma$. We have now established the first equality of (ii).

Let $\lambda \in V^*$. If there exists an element $x \in \Gamma$ such that $\lambda(x) > 0$, then, because Γ is a cone,

$$H_\Gamma(\lambda) \geq \sup_{r \in \mathbf{R}_{>0}} \lambda(rx) = \infty.$$

On the other hand, if $\lambda|_\Gamma \leq 0$, then clearly $H_\Gamma(\lambda) = 0$. We thus see that

$$\mathcal{C}_\Gamma = \{\lambda \in V^* : \lambda(x) \leq 0 \text{ for every } x \in \Gamma\}.$$

(iii): Let $\lambda \in \mathcal{C}_{B+\Gamma}$. Then by (ii) we have $\lambda(x) \leq 0$ for every $x \in \Gamma$. Since Γ is a cone, it follows that $H_\Gamma|_{\mathcal{C}_{B+\Gamma}} = 0$. Using subsequently (i) and the fact that $B \subseteq B + \Gamma$, we find

$$H_{B+\Gamma}(\lambda) \leq (H_B + H_\Gamma)(\lambda) = H_B(\lambda) \leq H_{B+\Gamma}(\lambda).$$

This establishes the equality of the restrictions. The final assertion now follows by taking $B = \{0\}$. \square

Lemma 3.8.2. *Let \mathcal{C} be a collection of cones Γ in V containing 0 and let S be a closed convex subset of V .*

$$\text{If } \mathcal{C}_S \subseteq \bigcup_{\Gamma \in \mathcal{C}} \mathcal{C}_\Gamma, \text{ then } S = \bigcap_{\Gamma \in \mathcal{C}} (S + \Gamma).$$

Proof. Assume that the hypothesis is fulfilled. If $\Gamma \in \mathcal{C}$ then $0 \in \Gamma$ hence $S \subseteq S + \Gamma$. It follows that $S \subseteq \bigcap_{\Gamma \in \mathcal{C}} (S + \Gamma)$.

Conversely, suppose $x \in \bigcap_{\Gamma \in \mathcal{C}} (S + \Gamma)$. Let $\lambda \in \mathcal{C}_S$. By assumption there exists a $\Gamma \in \mathcal{C}$ such that $\lambda \in \mathcal{C}_\Gamma$. According to Lemma 3.8.1 (ii), $\lambda \in \mathcal{C}_{S+\Gamma}$. Since $x \in S + \Gamma$,

$$\lambda(x) \leq H_{S+\Gamma}(\lambda).$$

By Lemma 3.8.1 (iii), $H_{S+\Gamma}(\lambda) = H_S(\lambda)$, hence $\lambda(x) \leq H_S(\lambda)$. As S is closed and convex, this implies that $x \in S$. \square

3.9 Support of a transformed function or distribution

Let P and Q be $\sigma \circ \theta$ -stable parabolic subgroups of G with $A \subseteq P \subseteq Q$.

The support of the transform $\mathcal{R}_P^Q \phi$ of a function $\phi \in \mathcal{D}(\mathcal{E}_Q)$ need not be compact in general. In Section 5.4 of Appendix B we give an example of this phenomena. The aim of the present section is to give a description of the support of a transformed function or distribution in terms of the support of that function or distribution.

For a subset T of $\Sigma(\mathfrak{g}, \mathfrak{a}_q)$, we define

$$\Gamma(T) = \sum_{\alpha \in T} \mathbf{R}_{\geq 0} H_\alpha.$$

Here H_α is given by (2.5.1). We define the cone

$$\Gamma_P = \Gamma(\Sigma^+(\mathfrak{g}, \mathfrak{a}_q; P)).$$

Since $G = KL_P N_P$ and $L_P = (L_P \cap K)A_q(L_P \cap H)$, the map

$$K \times A_q \rightarrow \mathcal{E}_P; \quad (k, a) \mapsto ka \cdot \xi_P$$

is surjective. In case $P = G$, this statement is equivalent to $X = KA_q \cdot x_0$.

If B is a subset of \mathfrak{a}_q , then we define

$$X(B) = K \exp(B) \cdot x_0 \quad \text{and} \quad \Xi_P(B) = K \exp(B) \cdot \xi_P.$$

Note that

$$X(B) = \Xi_G(B).$$

We recall from Section 2.2 that $\mathcal{P}_\sigma(\mathfrak{a}_q)$ denotes the collection of minimal $\sigma \circ \theta$ -stable parabolic subgroups containing A .

Lemma 3.9.1. *Let \mathcal{C} be the collection of $P_0 \in \mathcal{P}_\sigma(\mathfrak{a}_q)$ with $P_0 \subseteq P$ and let B be a closed and convex subset of \mathfrak{a}_q . Then*

$$B + \Gamma_P = \bigcap_{P_0 \in \mathcal{C}} (B + \Gamma_{P_0}).$$

Proof. There is a one to one correspondence between the minimal $\sigma \circ \theta$ -stable parabolic subgroups of G contained in P and the minimal $\sigma \circ \theta$ -stable parabolic subgroups of L_P : if P_0 is a minimal $\sigma \circ \theta$ -stable parabolic subgroup of G contained in P , then $R = L_P \cap P_0$ is a minimal $\sigma \circ \theta$ -stable parabolic subgroup of L_P and, vice versa, if R is a minimal $\sigma \circ \theta$ -stable parabolic subgroup of L_P , then $P_0 = \mathcal{Z}_G(\mathfrak{a}_{P_0})N_R N_P$ is a minimal $\sigma \circ \theta$ -stable parabolic subgroup of G contained in P . Furthermore, let \mathcal{C} be as above, and let \mathcal{C}' be the set of $\sigma \circ \theta$ -stable parabolic subgroups R of L_P containing A . Then the map $P_0 \mapsto P_0 \cap L_P$ is a bijection from \mathcal{C} onto \mathcal{C}' . If $P_0 \in \mathcal{C}$ and $R = L_P \cap P_0$ is the corresponding parabolic subgroup of L_P , then

$$\Gamma_{P_0} = \Gamma_P + \Gamma_{P;R},$$

where

$$\Gamma_{P;R} = \Gamma(\Sigma^+(\mathfrak{l}_P, \mathfrak{a}_q; R)).$$

The cone $\mathcal{C}_{\Gamma_{P;R}}$ equals the closure of the dual Weyl chamber corresponding to θR , the opposite of R . Hence,

$$\bigcup_{R \in \mathcal{C}'} \mathcal{C}_{\Gamma_{P;R}} = \mathfrak{a}_q^*.$$

Application of Lemma 3.8.2 with $S = B + \Gamma_P$ now yields

$$\bigcap_{P_0 \in \mathcal{C}} (B + \Gamma_{P_0}) = \bigcap_{R \in \mathcal{C}'} (B + \Gamma_P + \Gamma_{P;R}) = B + \Gamma_P.$$

□

We recall that if KAN is an Iwasawa decomposition for G , then the map $\mathfrak{A}_{KAN} : G \rightarrow \mathfrak{a}$ is given by (3.5.3). We further recall that $W_{M_P \cap K \cap H}$ denotes the subgroup of W consisting of all elements that can be realized in $M_P \cap K \cap H$.

Lemma 3.9.2. *Let B be a $W_{M_P \cap K \cap H}$ -invariant convex subset of \mathfrak{a}_q and $g \in G$. Then the following two assertions are equivalent.*

$$(i) \quad g \cdot \xi_P \in \Xi_P(B + \Gamma_P)$$

$$(ii) \quad \text{For all } P_0 \in \mathcal{P}_\sigma(\mathfrak{a}_q) \text{ and } P_m \in \mathcal{P}(\mathfrak{a}) \text{ with } P_m \subseteq P_0 \subseteq P,$$

$$\pi_q \circ \mathfrak{A}_{KAN_{P_m}}(g(L_P \cap H)) \subseteq B + \Gamma_{P_0}.$$

Proof. Let $g \in G$ and write $g = k \exp Y h n$, where $k \in K$, $Y \in \mathfrak{a}_q$, $h \in L_P \cap H$ and $n \in N_P$. Let \mathcal{C} be defined as in the previous lemma. Let $P_0 \in \mathcal{C}$ and let P_m be a minimal parabolic subgroup of G with $A \subseteq P_m \subseteq P_0$. We now observe that by Lemma 2.4.2

$$\pi_q \circ \mathfrak{A}_{KAN_{P_m}}(g(L_P \cap H)) = \pi_q \circ \mathfrak{A}_{(M_P \cap K)AN_{P_m}^P}(\exp Y \cdot (L_P \cap H)), \quad (3.9.1)$$

where $\mathfrak{A}_{(M_P \cap K)AN_{P_m}^P} : L_P \rightarrow \mathfrak{a}$ is defined as in (3.5.3) for the Iwasawa decomposition

$$L_P = (K \cap M_P)AN_{P_m}^P. \quad (3.9.2)$$

Applying the convexity theorem of Van den Ban ([Ban86, Theorem 1.1]) to L_P , $L_P \cap H$ and the Iwasawa decomposition (3.9.2), we obtain from (3.9.1) that

$$\pi_q \circ \mathfrak{A}_{KAN_{P_m}}(g(L_P \cap H)) = \text{ch}(W_{M_P \cap K \cap H} \cdot Y) + \Gamma(\Sigma_-^+(\mathfrak{l}_P, \mathfrak{a}_q; P_0 \cap L_P)). \quad (3.9.3)$$

Note that

$$W_{M_P \cap K \cap H} \cdot \Gamma_P = \Gamma_P.$$

Now assume that (i) holds, then $Y \in B + \Gamma_P$. The latter set is $W_{M_P \cap K \cap H}$ -invariant, so if P_0, P_m are as in (ii), then it follows from (3.9.3) that

$$\pi_q \circ \mathfrak{A}_{KAN_{P_m}}(g(L_P \cap H)) \subseteq B + \Gamma_P + \Gamma(\Sigma_-^+(\mathfrak{l}_P, \mathfrak{a}_q; P_0 \cap L_P)) \subseteq B + \Gamma_{P_0},$$

and (ii) follows.

Conversely, assume that (ii) holds. Then it follows from (3.9.3) that $Y \in B + \Gamma_{P_0}$ for all $P_0 \in \mathcal{C}$. In view of Lemma 3.9.1, this implies that $Y \in B + \Gamma_P$, so that (i) follows. \square

We recall that the map E_P^Q from \mathcal{E}_P to the power set of \mathcal{E}_Q is given by

$$E_P^Q(g \cdot \xi_P) = gN_P^Q \cdot \xi_Q \quad (\xi \in \mathcal{E}_P).$$

Proposition 3.9.3. *Let B be a $W_{M_Q \cap K \cap H}$ -invariant convex subset of \mathfrak{a}_q . Then the following statements hold.*

(i) *If $g \in G$ and $g \cdot \xi_Q \in \mathcal{E}_Q(B + \Gamma_Q)$, then $g \cdot \xi_P \in \mathcal{E}_P(B + \Gamma_P)$.*

(ii) *$\{\xi \in \mathcal{E}_P : E_P^Q(\xi) \cap \mathcal{E}_Q(B) \neq \emptyset\} \subseteq \mathcal{E}_P(B + \Gamma_P)$.*

Proof. (i): Let $g \cdot \xi_Q \in \mathcal{E}_Q(B + \Gamma_Q)$. Then assertion (ii) of Lemma 3.9.2 holds with Q in place of P . Since $P \subseteq Q$, every minimal $\sigma \circ \theta$ -stable parabolic subgroup P_0 that is contained in P is also contained in Q so that assertion (ii) of Lemma 3.9.2 also holds with P . By the mentioned lemma it then follows that $g \cdot \xi_P \in \mathcal{E}_P(B + \Gamma_P)$.

(ii): Assume $E_P^Q(g \cdot \xi_P) \cap \mathcal{E}_Q(B) \neq \emptyset$. There exists an $n \in N_P$ such that $gn \cdot \xi_Q \in \mathcal{E}_Q(B)$. Therefore there exist $k \in K$ and $n \in N_P$ such that

$$kgn \in \exp(B)(L_Q \cap H).$$

Let $P_0 \in \mathcal{P}_\sigma(\mathfrak{a}_q)$ and $P_m \in \mathcal{P}(\mathfrak{a})$ be such that $P_m \subseteq P_0 \subseteq P$. Since $L_P \subseteq L_Q$,

$$\begin{aligned} \pi_q \circ \mathfrak{A}_{KAN_{P_m}}(g(L_P \cap H)) &= \pi_q \circ \mathfrak{A}_{KAN_{P_m}}(kgn(L_P \cap H)) \\ &\subseteq \pi_q \circ \mathfrak{A}_{KAN_{P_m}}(\exp(B)(L_Q \cap H)). \end{aligned}$$

The last of these sets is contained in

$$\pi_q \circ \mathfrak{A}_{(K \cap L_Q)AN_{P_m}^Q}(\exp(B)(L_Q \cap H)),$$

which is equal to $B + \Gamma(\Sigma_\pm^+(\mathfrak{l}_Q, \mathfrak{a}_q; P_0 \cap L_Q))$ by the convexity theorem [Ban86, Theorem 1.1] of Van den Ban. As the latter is contained in $B + \Gamma_{P_0}$, the statement now follows by application of Lemma 3.9.2. \square

The first part of Proposition 3.9.3 has the following corollary.

Corollary 3.9.4. *Assume that B is a compact $W_{M_Q \cap K \cap H}$ -invariant convex subset of \mathfrak{a}_q and that $\mu \in \mathcal{E}_b^l(\mathcal{E}_Q, J_Q)$. Then*

$$\text{supp}(\mu) \subseteq \mathcal{E}_Q(B + \Gamma_Q) \quad \text{implies} \quad \text{supp}(\mathcal{R}_P^Q \mu) \subseteq \mathcal{E}_P(B + \Gamma_P).$$

In particular, if $B \subseteq \mathfrak{a}_q$ is compact, convex and $W_{K \cap H}$ -invariant and $\mu \in \mathcal{E}_b^l(X)$, then

$$\text{supp}(\mu) \subseteq X(B) \quad \text{implies} \quad \text{supp}(\mathcal{R}_P \mu) \subseteq \mathcal{E}_P(B + \Gamma_P).$$

Proof. Let $\phi \in \mathcal{E}^1(\mathcal{E}_Q, J_Q)$ and assume that $\text{supp}(\phi) \subseteq \mathcal{E}_Q(B + \Gamma_Q)$. Let $g \in G$ and assume that $\mathcal{R}_P^Q \phi$ is non-zero at $g \cdot \xi_P$. As

$$\mathcal{R}_P^Q \phi(g \cdot \xi_P) = \int_{N_P^Q} \phi(gn \cdot \xi_Q) dn,$$

there exists an $n \in N_P^Q$ such that $\phi(gn \cdot \xi_Q) \neq 0$. By assumption $gn \cdot \xi_Q \in \mathcal{E}_Q(B + \Gamma_Q)$, hence by part (i) of Proposition 3.9.3, the element $g \cdot \xi_P = gn \cdot \xi_P$ is contained in $\mathcal{E}_P(B + \Gamma_P)$. Since B is compact, the set $\mathcal{E}_P(B + \Gamma_P)$ is closed. Therefore, the support of $\mathcal{R}_P^Q \phi$ is contained in $\mathcal{E}_P(B + \Gamma_P)$. We conclude that the following implication holds for every $\phi \in \mathcal{E}^1(\mathcal{E}_Q, J_Q)$,

$$\text{supp}(\phi) \subseteq \mathcal{E}_Q(B + \Gamma_Q) \implies \text{supp}(\mathcal{R}_P^Q \phi) \subseteq \mathcal{E}_P(B + \Gamma_P). \quad (3.9.4)$$

Let now $\psi \in \mathcal{E}_b(\mathcal{E}_P, J_P)$ and assume that $\text{supp}(\psi) \cap \mathcal{E}_P(B + \Gamma_P) = \emptyset$. Then by (3.9.4) and Corollary 3.6.7

$$\int_{\mathcal{E}_Q} \mathcal{S}_P^Q \psi(\xi) \phi(\xi) d\xi = \int_{\mathcal{E}_P} \psi(\zeta) \mathcal{R}_P^Q \phi(\zeta) d\zeta = 0$$

for all $\phi \in \mathcal{E}^1(\mathcal{E}_Q, J_Q)$ with $\text{supp}(\phi) \subseteq \mathcal{E}_Q(B + \Gamma_Q)$. Therefore, $\text{supp}(\mathcal{S}_P^Q \psi) \cap \mathcal{E}_Q(B + \Gamma_Q) = \emptyset$. We conclude that for all $\psi \in \mathcal{E}_b(\mathcal{E}_P, J_P)$ the following implication is valid:

$$\text{supp}(\psi) \cap \mathcal{E}_P(B + \Gamma_P) = \emptyset \implies \text{supp}(\mathcal{S}_P^Q \psi) \cap \mathcal{E}_Q(B + \Gamma_Q) = \emptyset.$$

Finally, let $\mu \in \mathcal{E}'_b(\mathcal{E}_Q, J_Q)$ and assume that $\text{supp}(\mu) \subseteq \mathcal{E}_Q(B + \Gamma_Q)$. If $\psi \in \mathcal{E}_b(\mathcal{E}_P, J_P)$ is supported in the complement of $\mathcal{E}_P(B + \Gamma_P)$, then $\text{supp}(\mathcal{S}_P^Q \psi)$ is disjoint from $\mathcal{E}_Q(B + \Gamma_Q)$, hence

$$\mathcal{R}_P^Q \mu(\psi) = \mu(\mathcal{S}_P^Q \psi) = 0.$$

Therefore $\text{supp}(\mu)$ is contained in $\mathcal{E}_P(B + \Gamma_P)$. This proves the first statement.

The second statement is obtained from the first by taking Q equal to G . \square

The following proposition gives a characterization of the support of the horospherical transform of a non-negative compactly supported smooth function on X .

Proposition 3.9.5. *Let $P_0 \in \mathcal{P}_\sigma(\mathfrak{a}_q)$ and let $B \subseteq \mathfrak{a}_q$. Then*

$$\{\xi \in \mathcal{E}_{P_0} : E_{P_0}(\xi) \cap X(B) \neq \emptyset\} = \bigcup_{b \in B} \mathcal{E}_{P_0}(\text{ch}(W_{K \cap H} \cdot b) + \Gamma(\Sigma_-^+(\mathfrak{g}, \mathfrak{a}_q; P_0))).$$

If $\phi \in \mathcal{D}(X)$ is non-negative and satisfies $\text{supp}(\phi) = X(B)$, then

$$\text{supp}(\mathcal{R}_{P_0}\phi) = \{\xi \in \mathcal{E}_{P_0} : E_{P_0}(\xi) \cap X(B) \neq \emptyset\}. \quad (3.9.5)$$

We first prove the following lemma.

Lemma 3.9.6. *Let $P_0 \in \mathcal{P}_\sigma(\mathfrak{a}_q)$ and let $a, b \in A_q$. Then*

$$KaN_{P_0} \cap KbH \neq \emptyset \iff \log a \in \text{ch}(W_{K \cap H} \log b) + \Gamma(\Sigma_-^+(\mathfrak{g}, \mathfrak{a}_q; P_0)). \quad (3.9.6)$$

Proof. Let $P_m \in \mathcal{P}(\mathfrak{a})$ be a minimal parabolic subgroup contained in P_0 . Then, by Lemma 2.4.2, N_{P_m} decomposes as $N_{P_m} = N_{P_0}N_H$, where $N_H = N_{P_m} \cap H$. It follows that the first assertion of (3.9.6) is equivalent to

$$KaN_{P_m} \cap KbH \neq \emptyset.$$

As $A \cap H$ normalizes N_{P_m} , the latter assertion is equivalent to

$$Ka(A \cap H)N_{P_m} \cap KbH \neq \emptyset. \quad (3.9.7)$$

Finally, the assertion (3.9.7) is equivalent to the assertion on the right-hand side of (3.9.6) by van den Ban's convexity theorem [Ban86, Theorem 1.1]. \square

Proof for Proposition 3.9.5. If $P_m \in \mathcal{P}(\mathfrak{a})$ is a minimal parabolic subgroup contained in P_0 , then

$$L_{P_0} = (M_{P_0} \cap K)AN_{P_m}^{P_0}$$

is an Iwasawa decomposition for L_{P_0} . By Lemma 2.4.2 the subgroup $N_{P_m}^{P_0}$ is contained in H , hence the map $(k, a) \mapsto ka \cdot \xi_{P_0}$ induces a diffeomorphism

$$K/K \cap M_{P_0} \times A_q \rightarrow \mathcal{E}_{P_0}. \quad (3.9.8)$$

Let $\xi \in \mathcal{E}_{P_0}$. Then we may write $\xi = ka \cdot \xi_{P_0}$ with $k \in K$ and $a \in A_q$. Then $E_{P_0}(\xi) \cap X(B) \neq \emptyset$ is equivalent to the existence of an element $Y \in B$ such that

$KaN_{P_0} \cap K \exp(Y)H \neq \emptyset$. By Lemma 3.9.6, the latter assertion is equivalent to the existence of an element $Y \in B$ such that

$$\log a \in \text{ch}(W_{K \cap H} Y) + \Gamma(\Sigma_-^+(\mathfrak{g}, \mathfrak{a}_q; P_0)).$$

As (3.9.8) is a diffeomorphism, this assertion is in turn equivalent to the existence of a $Y \in B$ such that

$$ka \cdot \xi_{P_0} \in \Xi_{P_0} \left(\text{ch}(W_{K \cap H} Y) + \Gamma(\Sigma_-^+(\mathfrak{g}, \mathfrak{a}_q; P_0)) \right).$$

This completes the proof of the first statement of Proposition 3.9.5.

We now turn to the proof of the second assertion. Assume that $\phi \in \mathcal{D}(X)$ is as stated. Let $g \in G$ and assume that

$$E_{P_0}(g \cdot \xi_{P_0}) \cap X(B) \neq \emptyset.$$

Then there is an $n \in N_{P_0}$ such that $gn \cdot x_0 \in X(B)$.

As the map $p : G \rightarrow X$, $g \mapsto g \cdot x_0$ defines a fiber bundle, $\text{supp}(p^* \phi) = p^{-1}(\text{supp } \phi)$, hence $gn \in \text{supp}(p^* \phi)$. Since $p^* \phi \geq 0$ it now follows that there exists a sequence $(g_j)_{j \in \mathbb{N}}$ in G such that $\phi(g_j \cdot x_0) > 0$ and $g_j \rightarrow gn$ if $j \rightarrow \infty$. Note that

$$g_j \cdot \xi_{P_0} \rightarrow gn \cdot \xi_{P_0} = g \cdot \xi_{P_0}$$

for $j \rightarrow \infty$. Since ϕ is continuous, for every $j \in \mathbb{N}$ there exists an open neighborhood U_j of g_j in G such that $\phi(g \cdot x_0) > 0$ if $g \in U_j$. This implies that $\mathcal{R}_{P_0} \phi(g_j \cdot \xi_{P_0}) > 0$ and thus we conclude that $g \cdot \xi_{P_0} \in \text{supp}(\mathcal{R}_{P_0} \phi)$. This implies that the set on the left-hand side of (3.9.5) is contained in the set on the right-hand side. We will finish the proof by establishing the converse inclusion.

Let $g \in G$ and assume

$$g \cdot \xi_{P_0} \in \text{supp}(\mathcal{R}_{P_0} \phi).$$

Since $g \mapsto g \cdot \xi_{P_0}$ defines a fiber bundle $G \rightarrow \Xi_{P_0}$ we may apply a similar argument as above to see that there exists a sequence $(g_j)_{j \in \mathbb{N}}$ in G , converging to g such that for every $j \in \mathbb{N}$

$$0 < \mathcal{R}_{P_0} \phi(g_j \cdot \xi_{P_0}) = \int_{N_{P_0}} \phi(g_j n \cdot \xi_{P_0}) dn.$$

In particular, there exists a sequence $(n_j)_{j \in \mathbb{N}}$ in N_{P_0} such that

$$\phi(g_j n_j \cdot x_0) > 0.$$

Note that $(g_j n_j \cdot x_0)_{j \in \mathbb{N}}$ is a sequence in the support of ϕ , which equals $X(B)$. As this support is compact, there exists a convergent subsequence. Without loss of generality we may therefore assume that $g_j n_j \cdot x_0$ converges to a point $g_0 \cdot x_0 \in X(B)$ if $j \rightarrow \infty$. Now

$$\lim_{j \rightarrow \infty} n_j \cdot x_0 = \lim_{j \rightarrow \infty} g_j^{-1} g_j n_j \cdot x_0 = g^{-1} g_0 \cdot x_0.$$

Using a local section of the bundle $p : G \rightarrow X$ around the point $g^{-1} g_0 \cdot x_0$, we find that there exist $h_j \in H$ such that

$$\lim_{j \rightarrow \infty} n_j h_j = g^{-1} g_0.$$

According to [Ban86, Lemma 3.4], if a sequence $(m_j a_j n_j h_j)_{j \in \mathbb{N}}$ in G , with $m_j \in M_{P_0}$, $a_j \in A_q$, $n_j \in N_{P_0}$ and $h_j \in H$, converges to a point in the boundary of the open subset $P_0 H$ of G , then $\{a_j : j \in \mathbb{N}\}$ is not relatively compact in A_q . Furthermore, by the same lemma

$$N_{P_0} \times L_{P_0} \times_{L_{P_0} \cap H} H \rightarrow P_0 H; \quad (n, l, h) \mapsto n l h$$

is a diffeomorphism. Applying these results to the sequence $(n_j h_j)$, we see that it converges to a point in $N_{P_0} L_{P_0} H \cap \overline{N_{P_0} H} = N_{P_0} H$. We now conclude that there exist $n \in N_{P_0}$ and $h \in H$ such that $n_j \rightarrow n$ and $h_j \rightarrow h$, for $j \rightarrow \infty$. Since $(g_j n_j \cdot x_0)$ is a sequence in the compact set $X(B)$ it follows that the limit $g n \cdot x_0$ is contained in $X(B)$ as well. On the other hand,

$$g n \cdot x_0 \in g N_{P_0} \cdot x_0 = E_{P_0}(g \cdot \xi_{P_0}),$$

and we see that the intersection of $E_{P_0}(g \cdot \xi_{P_0})$ and $X(B)$ is non-empty. \square

Remark 3.9.7. Note that if $\Sigma_-^+(\mathfrak{g}, \mathfrak{a}_q; P_0) = \emptyset$, the Radon transform $\mathcal{R}_{P_0} \phi$ of a compactly supported smooth function $\phi \in \mathcal{D}(X)$ is again a compactly supported smooth function. This holds in particular if X is a Riemannian symmetric space.

Chapter 4

Support theorem for the horospherical transform

The aim of this chapter is to prove a support theorem for the horospherical transform for functions.

In Section 4.1 we derive Paley-Wiener type estimates for the Fourier transform on a Euclidean space for Schwartz functions with a certain type of support. The horospherical transform is related to the so-called unnormalized Fourier transform on X . Given the support of the horospherical transform of a function, the theory from Section 4.1 yields a Paley-Wiener estimate for one component of the unnormalized Fourier transform. This is described in Section 4.2. In Section 4.3 Paley-Wiener estimates for one component of the normalized τ -spherical Fourier transform are deduced from the estimates for the unnormalized Fourier transform. Then in Section 4.4 we introduce some subspaces of $\mathcal{E}^1(X)$ that will be used in the last two sections. For the normalized τ -spherical Fourier transform there exists an inversion formula due to Van den Ban and Schlichtkrull, that we describe in Section 4.5. Finally, in Section 4.6, we use the inversion formula and the Paley-Wiener estimates to obtain a support theorem for the horospherical transform.

Throughout this chapter, P_0 denotes a *minimal* $\sigma \circ \theta$ -stable parabolic subgroup containing A .

4.1 The Euclidean Fourier transform and Paley-Wiener estimates

Let V be a finite dimensional real vector space equipped with a positive definite inner product.

Let $u \in \mathcal{E}(V)$. For each functional $\nu \in V^*$ we define $u_\nu := e^{-\nu}u$. Moreover, we define the set

$$\mathcal{C}(u) := \{\nu \in V^* : u_\nu \in \mathcal{S}(V)\}.$$

This set is a convex cone in V^* . (See [Hör03, Section 7.4].) The Fourier transform of u is defined to be the function $\mathcal{F}u$ on $\mathcal{C}(u) + iV^*$ given by

$$\mathcal{F}u(\lambda) = \int_V e^{-\lambda(x)} u(x) dx = \int_V e^{-i\zeta(x)} u_\nu(x) dx \quad (\lambda = \nu + i\zeta \in \mathcal{C}(u) + iV^*).$$

Let $P(V^*)$ denote the ring of polynomial functions $V^* \rightarrow \mathbf{C}$. If $p \in P(V^*)$, we use the notation $p(\partial)$ for the linear partial differential operator with constant coefficients on V determined by

$$p(\partial)e^\nu = p(\nu)e^\nu \quad (\nu \in V^*).$$

In a similar fashion, we associate to each $p \in P(V)$ a differential operator $p(\partial)$ on V^* .

Since for every homogeneous polynomial $p_h \in P(V)$ of degree 1

$$(p_h(\partial)u)_\nu = p_h(\partial)u_\nu + p_h(\nu)u_\nu$$

is a Schwartz function if u_ν is Schwartz, we see that

$$\mathcal{C}(u) \subseteq \mathcal{C}(p(\partial)u)$$

for every polynomial function $p : V^* \rightarrow \mathbf{C}$.

The function $\zeta \mapsto \mathcal{F}u(\nu + i\zeta)$ is a Schwartz function on V^* for each $\nu \in \mathcal{C}(u)$. Furthermore, $\mathcal{F}u$ is holomorphic on the interior of $\mathcal{C}(u) + iV^*$ (see [Hör03, Theorem 7.4.2]) and there

$$p(\partial)\mathcal{F}u = \mathcal{F}(x \mapsto p(-x)u(x)), \quad \text{and} \quad \mathcal{F}(q(\partial)u) = q\mathcal{F}(u),$$

for all $p \in P(V)$ and $q \in P(V^*)$.

Lemma 4.1.1. *Let S be a closed convex subset of V and let the cone $-\mathcal{C}_S \subseteq V^*$ be defined as in (3.8.1). Let $u \in \mathcal{S}(V)$. Then*

$$\text{supp}(u) \subseteq S \implies -\mathcal{C}_S \subseteq \mathcal{C}(u).$$

Proof. Let $v \in -\mathcal{C}_S$. Then $(p(\partial)e^{-v})\mathbf{1}_S$ is bounded for every polynomial $p \in P(V^*)$. By application of the Leibniz rule we now see that $p(\partial)u_v = \mathcal{O}(1 + \|x\|)^{-N}$ for all $p \in P(V^*)$ and $N \in \mathbb{N}$. Hence u_v is Schwartz. \square

Proposition 4.1.2 (Paley-Wiener estimate). *Let S be a closed, convex subset of V . If $u \in \mathcal{S}(V)$ with $\text{supp}(u) \subseteq S$, then for every $N \in \mathbb{N}$ and $\lambda \in -\mathcal{C}_S + iV^*$*

$$|\mathcal{F}u(\lambda)| \leq 2^N \|(1 + \Delta)^N u\|_{L^1} (1 + \|\lambda\|)^{-N} e^{H_S(-\text{Re}(\lambda))}.$$

Proof. If $v \in \mathcal{S}(V)$ satisfies $\text{supp}(v) \subseteq S$ and $\lambda \in -\mathcal{C}_S + iV^*$, then

$$|\mathcal{F}v(\lambda)| \leq \int_V |e^{-\lambda(x)}| |v(x)| dx \leq \|v\|_{L^1} e^{H_S(-\text{Re}(\lambda))}$$

Let $N \in \mathbb{N}$. Then

$$(1 + \|\lambda\|)^N |\mathcal{F}u(\lambda)| \leq 2^N (1 + \|\lambda\|^2)^N |\mathcal{F}u(\lambda)| = 2^N |\mathcal{F}((1 + \Delta)^N u)(\lambda)|,$$

hence, by taking $v = (1 + \Delta)^N u$, we obtain

$$(1 + \|\lambda\|)^N |\mathcal{F}u(\lambda)| \leq 2^N \|(1 + \Delta)^N u\|_{L^1} e^{H_S(-\text{Re}(\lambda))}.$$

\square

Lemma 4.1.3. *Let B be a compact subset of V and Γ a cone in V containing 0. For every $v_0 \in -\mathcal{C}_{B+\Gamma}$ and for every $\lambda \in v_0 - \mathcal{C}_{B+\Gamma}$*

$$H_B(-\lambda) - H_{B+\Gamma}(-v_0) \leq H_{B+\Gamma}(-\lambda + v_0) \leq H_B(-\lambda) + H_B(v_0).$$

Proof. Let $v_0 \in -\mathcal{C}_{B+\Gamma}$ and suppose $\lambda \in v_0 - \mathcal{C}_{B+\Gamma}$. Then

$$-\lambda \in -v_0 + \mathcal{C}_{B+\Gamma} \subseteq \mathcal{C}_{B+\Gamma}.$$

By the triangle inequality for suprema

$$H_{B+\Gamma}(-\lambda) \leq H_{B+\Gamma}(-\lambda + v_0) + H_{B+\Gamma}(-v_0),$$

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hence

$$H_{B+\Gamma}(-\lambda) - H_{B+\Gamma}(-v_0) \leq H_{B+\Gamma}(-\lambda + v_0).$$

Note that $H_{B+\Gamma}(-v_0)$ is finite because $-v_0 \in \mathcal{C}_{B+\Gamma}$ and $H_{B+\Gamma}(-\lambda) = H_B(-\lambda)$ by Lemma 3.8.1 (iii).

By the same lemma $H_{B+\Gamma}(-\lambda + v_0) = H_B(-\lambda + v_0)$. Using the triangle inequality for suprema, we find

$$H_B(-\lambda + v_0) \leq H_B(-\lambda) + H_B(v_0).$$

Note that $H_B(v_0)$ is finite because B is compact. □

Lemma 4.1.4. *For all $\lambda, v_0 \in V^*$ we have*

$$\frac{1 + \|\lambda\|}{1 + \|v_0\|} \leq (1 + \|\lambda + v_0\|) \leq (1 + \|\lambda\|)(1 + \|v_0\|).$$

Proof. The estimate on the right is a straightforward consequence of the triangle inequality. It follows that

$$(1 + \|\lambda\|) = (1 + \|\lambda + v_0 + (-v_0)\|) \leq (1 + \|\lambda + v_0\|)(1 + \|v_0\|).$$

This implies the required estimate on the left. □

Proposition 4.1.5 (Paley-Wiener estimate). *Let B be a compact subset of V and let $\Gamma \subseteq V$ be a closed cone. Let $v_0 \in -\mathcal{C}_\Gamma$. Then for every $N \in \mathbf{N}$ there exists a constant $C_{v_0, N} > 0$ with the following property.*

If u is a smooth function on V such that $u_{v_0} \in \mathcal{S}(V)$ and $\text{supp}(u) \subseteq B + \Gamma$, then for every $\lambda \in v_0 - \mathcal{C}_\Gamma + iV^$,*

$$|\mathcal{F}u(\lambda)| \leq C_{v_0, N} \|(1 + \Delta)^N u_{v_0}\|_{L^1} (1 + \|\lambda\|)^{-N} e^{H_B(-\text{Re}(\lambda))}.$$

Proof. Let $N \in \mathbf{N}$. Since $u_{v_0} \in \mathcal{S}(V)$ and $\text{supp}(u_{v_0}) = \text{supp}(u) \subseteq B + \Gamma$, it follows by application of Proposition 4.1.2 that

$$\begin{aligned} |\mathcal{F}u(\lambda)| &= |\mathcal{F}u_{v_0}(\lambda - v_0)| \\ &\leq 2^N \|(1 + \Delta)^N u_{v_0}\|_{L^1} (1 + \|\lambda - v_0\|)^{-N} e^{H_{B+\Gamma}(-\text{Re}(\lambda - v_0))} \end{aligned} \tag{4.1.1}$$

for all $\lambda \in v_0 - \mathcal{C}_{B+\Gamma} + iV^*$. Since $-\text{Re}(\lambda - v_0) \in \mathcal{C}_{B+\Gamma}$, it follows by application of Lemma 4.1.3 that

$$H_{B+\Gamma}(-\text{Re}(\lambda - v_0)) \leq H_B(-\text{Re}(\lambda)) + H_B(v_0). \tag{4.1.2}$$

Finally, by application of Lemma 4.1.4 we see that

$$(1 + \|\lambda - \nu_0\|)^{-N} \leq (1 + \|\lambda\|)^{-N} (1 + \|\nu_0\|)^N. \quad (4.1.3)$$

Substituting the estimates (4.1.2) and (4.1.3) in (4.1.1), we obtain the required estimate with $C_{\nu_0, N} = 2^N (1 + \|\nu_0\|)^N e^{H_B(\nu_0)}$. \square

Remark 4.1.6. Proposition 4.1.5 is part of the following Paley-Wiener Theorem that we state here for the sake of completeness.

Let B be a compact subset of V and let $\Gamma \subseteq V$ be a closed cone. Assume that $\nu_0 \in -C_\Gamma$ and that v is a function from $\nu_0 - C_\Gamma + iV^$ to \mathbb{C} . Then the following two statements are equivalent.*

- (I) *v equals the restriction to $\nu_0 - C_\Gamma + iV^*$ of the Fourier transform $\mathcal{F}u$ of a function $u \in \mathcal{E}(V)$ such that $u_{\nu_0} \in \mathcal{S}(V)$ and $\text{supp}(u) \subseteq B + \Gamma$*
- (II) *The function v is continuous and its restriction to $\nu_0 + iV^*$ is Schwartz. For every $\nu \in -C_\Gamma$ and $\lambda \in \nu_0 - C_\Gamma + iV^*$ the function*

$$z \mapsto v(z\nu + \lambda)$$

is holomorphic on $\{z \in \mathbb{C} : \text{Re } z > 0\}$ and for every $N \in \mathbb{N}$ there exists a positive constant C_N such that for all $\lambda \in \nu_0 - C_\Gamma + iV^$*

$$|v(\lambda)| \leq C_N (1 + \|\lambda\|)^{-N} e^{H_B(-\text{Re}(\lambda))}.$$

For every N there exists a constant $C_{\nu_0, N}$, depending on ν_0 and N only, such that if (I) holds, then (II) holds with C_N smaller than or equal to $C_{\nu_0, N} \|(1 + \Delta)^N u_{\nu_0}\|_{L^1}$.

The proof for the case $\nu_0 = 0$ is similar to the usual proof for the Paley-Wiener theorem for $\mathcal{D}(V)$. See for example [Rud73, Theorem 7.22]. For $\nu_0 \neq 0$ the theorem then follows by application of Lemmas 4.1.3 and 4.1.4.

4.2 The unnormalized Fourier transform

We start with recalling several definitions and results from [Ban88], and [BS97b].

Let $(\zeta, \mathcal{H}_\zeta)$ be a unitary representation of M_{P_0} in a finite dimensional Hilbert space \mathcal{H}_ζ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. The space $\mathcal{E}(P_0 : \zeta : \lambda)$ of smooth vectors for the induced

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representation $\text{Ind}_{P_0}^G(\zeta \otimes e^\lambda \otimes 1)$ of G (by left-induction) from the representation $\zeta \otimes e^\lambda \otimes 1$ of P_0 consists of the smooth functions $f : G \rightarrow \mathcal{H}_\zeta$ satisfying

$$f(mang) = a^{\lambda + \rho_{P_0}} \zeta(m) f(g) \quad (m \in M_{P_0}, a \in A_{P_0}, n \in N_{P_0}, g \in G). \quad (4.2.1)$$

Here ρ_{P_0} is defined as in (3.3.1) with P_0 in place of Q .

We define $V(\zeta)$ to be the formal direct sum of Hilbert spaces

$$V(\zeta) = \bigoplus_{w \in \mathcal{W}} V(\zeta, w), \quad V(\zeta, w) = \mathcal{H}_\zeta^{w(H \cap M_{P_0})w^{-1}},$$

where $\mathcal{H}_\zeta^{w(H \cap M_{P_0})w^{-1}}$ is the subspace of $w(H \cap M_{P_0})w^{-1}$ -fixed vectors in \mathcal{H}_ζ .

Let $(\widehat{M}_{P_0})_H$ be the set of equivalence classes of finite dimensional unitary representations $(\zeta, \mathcal{H}_\zeta)$ of M_{P_0} such that $V(\zeta) \neq \{0\}$. The principal series of representations for X is the series of representations $\text{Ind}_{P_0}^G(\zeta \otimes e^\lambda \otimes 1)$ with $\lambda \in \mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*$ and $(\zeta, \mathcal{H}_\zeta) \in (\widehat{M}_{P_0})_H$.

Let $\text{Ind}_{P_0}^G(\zeta \otimes e^\lambda \otimes 1)$ be a principal series representation. The space of generalized functions $G \rightarrow \mathcal{H}_\zeta$ satisfying (4.2.1) is denoted by $C^{-\infty}(P_0 : \zeta : \lambda)$. Following [Ban88, Section 5] we define the map $j(P_0 : \zeta : \lambda)$, from $V(\zeta)$ to the space $C^{-\infty}(P_0 : \zeta : \lambda)^H$ of H -fixed elements in $C^{-\infty}(P_0 : \zeta : \lambda)$ as follows. The sets $P_0 w H$, for $w \in \mathcal{W}$ are disjoint and open in G and their union

$$\Omega(P_0) = \bigcup_{w \in \mathcal{W}} P_0 w H$$

is dense in G . For $\lambda \in \mathfrak{a}_{\mathbb{Q}}^*(P_0, 0) - \rho_{P_0}$ the function is given by

$$j(P_0 : \zeta : \lambda)(\eta)(x) := \begin{cases} a^{\lambda + \rho_{P_0}} \zeta(m) \eta_w & \text{for } x = manwh \in \Omega(P_0) \text{ with} \\ & m \in M_{P_0}, a \in A_{P_0}, n \in N_{P_0}, \\ & w \in \mathcal{W} \text{ and } h \in H \\ 0 & \text{for } x \notin \Omega(P_0). \end{cases} \quad (4.2.2)$$

It is known that for $\lambda \in \mathfrak{a}_{\mathbb{Q}}^*(P_0, 0) - \rho_{P_0}$ the function $j(P_0 : \zeta : \lambda)(\eta)$ thus defined is continuous, see [Ban88, Proposition 5.6]. For the remaining $\lambda \in \mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*$ the map is defined by meromorphic continuation. For generic $\lambda \in \mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*$, the map $j(P_0 : \zeta : \lambda)$ is known to be a bijection $V(\zeta) \rightarrow C^{-\infty}(P_0 : \zeta : \lambda)^H$. See [Ban88, Theorem 5.10].

Lemma 4.2.1. *Let B be a compact subset of \mathfrak{a}_q and let $\psi \in \mathcal{E}^1(\mathcal{E}_{P_0}, J_{P_0})$ be such that*

$$\text{supp}(\psi) \subseteq \mathcal{E}_{P_0}(B + \Gamma_{P_0}).$$

For $k \in K$, let ψ_k be the function on \mathfrak{a}_q given by

$$\psi_k(Y) = \psi(k \exp(Y) \cdot \xi_{P_0}) \quad (Y \in \mathfrak{a}_q).$$

If $v \in \mathfrak{a}_q^(P_0, 0)$ then $e^v \psi_k$ is a Schwartz function for every $k \in K$. The map*

$$K \rightarrow \mathcal{S}(\mathfrak{a}_q); \quad k \mapsto e^v \psi_k$$

thus defined is continuous.

Proof. Let ψ be fixed as above. According to Proposition 3.5.4, the function $u\psi$ vanishes at infinity for every $u \in \mathcal{U}(\mathfrak{g})$. In particular, it follows that each of the functions $u\psi$ is bounded and uniformly continuous on \mathcal{E}_{P_0} .

For every $u \in \mathcal{U}(\mathfrak{g})$ there exists a finite set $F_u \subseteq \mathcal{U}(\mathfrak{g})$, consisting of linearly independent elements, such that $\text{Ad}(k)u \in \text{span}(F_u)$, for all $k \in K$. Write $\text{Ad}(k)u = \sum_{v \in F_u} c_{u,v}(k)v$, then the $c_{u,v}$ are continuous, hence bounded functions on K . It follows that there exists a constant $C_u > 0$ such that

$$\begin{aligned} \sup_{k \in K} \sup_{\mathfrak{a}_q} |u \psi_k| &= \sup_{k \in K} \sup_{Y \in \mathfrak{a}_q} |(\text{Ad}(k)u) \psi(k \exp(Y) \cdot \xi_{P_0})| \\ &= \sum_{v \in F_u} \sup_{k \in K} \sup_{Y \in \mathfrak{a}_q} |c_{u,v}(k)| |v \psi(k \exp(Y) \cdot \xi_{P_0})| \\ &< C_u. \end{aligned}$$

Since

$$K/(M_{P_0} \cap K \cap H) \times \mathfrak{a}_q \rightarrow \mathcal{E}_{P_0}; \quad (k \cdot (M_{P_0} \cap K \cap H), Y) \mapsto k \exp(Y) \cdot \xi_{P_0}$$

is a diffeomorphism and $\text{supp}(\psi)$ is contained in $\mathcal{E}(B + \Gamma_{P_0})$, the support of ψ_k is contained in $B + \Gamma_{P_0}$. Let $v \in \mathfrak{a}_q(P_0, 0)$; then $v < 0$ on $\Gamma_{P_0} \setminus \{0\}$. Let p be a polynomial function on \mathfrak{a}_q and let $u \in \mathbf{S}(\mathfrak{a}_q)$. Then by the Leibniz rule there exist finitely many elements $u_j \in \mathbf{S}(\mathfrak{a}_q)$ (independent of k) such that

$$\sup_{\mathfrak{a}_q} |p u(e^v \psi_k)| \leq \sum_j \sup_{\mathfrak{a}_q} |p e^v u_j \psi_k| \leq \sup_{B + \Gamma_{P_0}} |p e^v| \sum_j C_{u_j} < \infty.$$

This proves that the functions $e^v \psi_k$ are Schwartz functions on \mathfrak{a}_q .

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If $k, k' \in K$, then

$$\sup_{\mathfrak{a}_q} |p u(\psi_k - \psi_{k'})| \leq \left(\sup_{B + \Gamma_{P_0}} |p e^v| \right) \left(\sum_j \sup_{\mathfrak{a}_q} |u_j \psi_k - u_j \psi_{k'}| \right).$$

Now

$$\sup_{\mathfrak{a}_q} |u_j \psi_k - u_j \psi_{k'}| \leq \sum_{v \in F_{u_j}} |c_{u_j, v}(k)| \sup_{\Xi_{P_0}} |l_k^*(v\psi) - l_{k'}^*(v\psi)|. \quad (4.2.3)$$

Here l_g ($g \in G$) denotes the left translation $\Xi_{P_0} \rightarrow \Xi_{P_0}$ given by $\xi \mapsto g \cdot \xi$.

The second statement of the proposition now follows since the right-hand side of (4.2.3) converges to 0 if $k \rightarrow k'$ by the uniform continuity of the functions $v\psi$ and the boundedness of the functions $c_{u_j, v} : K \rightarrow \mathbb{C}$. \square

Let \mathcal{F}_{A_q} be the (Euclidean) Fourier transform on A_q , i.e., the transform mapping a function $\psi \in L^1(A_q)$ to the function on $i\mathfrak{a}_q^*$ given by

$$\mathcal{F}_{A_q} \psi(\lambda) = \int_{A_q} \psi(a) a^{-\lambda} da \quad (\lambda \in i\mathfrak{a}_q^*).$$

Note that \mathcal{F}_{A_q} is related to the Fourier transform $\mathcal{F}_{\mathfrak{a}_q}$ on \mathfrak{a}_q by

$$\mathcal{F}_{A_q} \psi = \mathcal{F}_{\mathfrak{a}_q}(\psi \circ \exp).$$

Proposition 4.2.2. *Let B be a compact subset of \mathfrak{a}_q , let $\eta \in V(\zeta, e)$ and let $\lambda \in \mathfrak{a}_q^*(\overline{P_0}, 0) + \rho_{P_0}$. If $\phi \in \mathcal{E}_{P_0}^1(X)$ satisfies*

$$\text{supp}(\mathcal{R}_{P_0} \phi) \subseteq \Xi_{P_0}(B + \Gamma_{P_0}),$$

then for every $g \in G$ the integral

$$\int_X \phi(x) j(P_0 : \zeta : -\lambda)(\eta)(g \cdot x) dx \quad (4.2.4)$$

is absolutely convergent and equals

$$\int_{M_{P_0} \cap K} \mathcal{F}_{A_q} \left(a \mapsto a^{\rho_{P_0}} \mathcal{R}_{P_0} \phi(g^{-1} m a \cdot \xi_{P_0}) \right) (\lambda) \zeta(m) \eta dm. \quad (4.2.5)$$

Proof. Let ϕ satisfy the above hypotheses. For λ as stated, the function $j(P_0 : \zeta : -\lambda)(\eta)$ is continuous. We will first prove the assertions under the assumption that $g = e$.

By Proposition 3.6.4 and Proposition 3.5.4, the function $\mathcal{R}_{P_0}\phi$ is an element of $\mathcal{E}_b(\mathcal{E}_{P_0})$. In view of the condition on the support of this Radon transform it now follows by application of Lemma 4.2.1 that

$$\mathfrak{a}_q \ni Y \mapsto e^{-\lambda(Y) + \rho_{P_0}(Y)} \mathcal{R}_{P_0}\phi(k \exp(Y) \cdot \xi_{P_0}) \quad (4.2.6)$$

is a continuous family (with family parameter $k \in K$) of functions in the Schwartz space $\mathcal{S}(\mathfrak{a}_q)$. Therefore, the integral

$$\int_{A_q} a^{-\lambda + \rho_{P_0}} \mathcal{R}_{P_0}\phi(k^{-1}ma \cdot \xi_{P_0}) da$$

is absolutely convergent and depends continuously on $m \in M_{P_0} \cap K$. Since $M_{P_0} \cap K$ is compact, the integral

$$\int_{M_{P_0} \cap K} \int_{A_q} a^{-\lambda + \rho_{P_0}} \mathcal{R}_{P_0}\phi(k^{-1}ma \cdot \xi_{P_0}) \zeta(m) \eta da dm \quad (4.2.7)$$

is absolutely convergent as well. Clearly, this integral equals (4.2.5), with $g = k$.

We will proceed to show that the integral also equals (4.2.4). Indeed, substituting the definition of the Radon transform in (4.2.7), we see that (4.2.5) equals the absolutely convergent integral

$$\begin{aligned} & \int_{M_{P_0} \cap K} \int_{A_q} \int_{N_{P_0}} a^{-\lambda + \rho_{P_0}} \phi(k^{-1}man \cdot \xi_{P_0}) \zeta(m) \eta dn da dm \\ &= \int_{M_{P_0} \cap K} \int_{A_q} \int_{N_{P_0}} \phi(k^{-1}man \cdot \xi_{P_0}) j(P_0 : \zeta : -\lambda)(\eta)(man \cdot x_0) dn da dm. \end{aligned} \quad (4.2.8)$$

For the last equality we have used that $-\lambda \in \mathfrak{a}_q^*(P_0, 0) - \rho_{P_0}$, so that $j(P_0 : \zeta : -\lambda)(\eta)$ is the continuous function given by (4.2.2). As $\eta \in V(\zeta, e)$, this continuous function is supported by the closure of the set $P_0 H$. As before, we denote $w^{-1}P_0 w$ by P_0^w . We now recall [Óla87, Theorem 1.2] According to this result,

$$\int_{P_0 \cdot x_0} \psi(x) dx = \int_{M_{P_0} \cap K} \int_{A_q} \int_{N_{P_0}} \psi(man \cdot x_0) dn da dm, \quad (4.2.9)$$

for every $\psi \in C_c(X)$. As the multiplication map

$$(M_{P_0} \cap K)/(M_{P_0} \cap K \cap H) \times A_q \times N_{P_0} \rightarrow P_0 \cdot x_0$$

is a diffeomorphism onto the open subset $P_0 \cdot x_0$ of X , it follows that (4.2.9) is valid for any measurable function $\psi : X \rightarrow \mathbb{C}$, provided the integral on either one of the two sides of the equation is absolutely convergent; and in that case the integral is absolutely convergent as well. Applying this result to (4.2.8) we find that (4.2.5) equals the integral

$$\int_{P_0 \cdot x_0} \phi(x) j(P_0 : \zeta : -\lambda)(\eta)(x) dx.$$

As $j(P_0 : \zeta : -\lambda)(\eta)(x)$ is supported by $P_0 \cdot x_0$, the latter integral equals (4.2.4). This completes the proof for $g = e$.

Next assume that $g = k \in K$. Then $L_k \phi : x \mapsto \phi(k^{-1} \cdot x)$ satisfies the same conditions as ϕ . Hence (4.2.5) is absolute convergent and equals the absolute convergent integral

$$\int_X \phi(k^{-1} \cdot x) j(P_0 : \zeta : -\lambda)(\eta)(x) dx = \int_X \phi(x) j(P_0 : \zeta : -\lambda)(\eta)(k \cdot x) dx;$$

for the last equality we have used invariance of the measure dx . This establishes the result for $g \in K$. We now assume that g is an arbitrary element of G . Write $g = m_g a_g n_g k_g$, with $m_g \in M_{P_0}$, $a_g \in A_{P_0}$, $n_g \in N_{P_0}$ and $k_g \in K$. Then by the transformation properties of j , the integral (4.2.4) equals

$$a_g^{-\lambda + \rho_{P_0}} \zeta(m_g) \int_X \phi(x) j(P_0 : \zeta : -\lambda)(\eta)(k_g \cdot x) dx.$$

By what we proved above, this expression equals

$$\begin{aligned} & a_g^{-\lambda + \rho_{P_0}} \zeta(m_g) \int_{M_{P_0} \cap K} \mathcal{F}_{A_q} \left(a \mapsto a^{\rho_{P_0}} \mathcal{R}_{P_0} \phi(k_g^{-1} m a \cdot \xi_{P_0}) \right) (\lambda) \zeta(m) \eta dm. \\ &= \int_{M_{P_0} \cap K} \mathcal{F}_{A_q} \left(a \mapsto a^{\rho_{P_0}} \mathcal{R}_{P_0} \phi(k_g^{-1} m a_g^{-1} a \cdot \xi_{P_0}) \right) (\lambda) \zeta(m_g m) \eta dm \\ &= \int_{M_{P_0} \cap K} \mathcal{F}_{A_q} \left(a \mapsto a^{\rho_{P_0}} \mathcal{R}_{P_0} \phi(k_g^{-1} n_g^{-1} a_g^{-1} m_g^{-1} m a \cdot \xi_{P_0}) \right) (\lambda) \zeta(m) \eta dm. \end{aligned}$$

For the last equality we have used that the measures dm and da are left-invariant, that A_{P_0} and M_{P_0} commute, and that N_{P_0} is normal in P_0 and stabilizes ξ_{P_0} . We finally observe that the last obtained integral equals (4.2.5). \square

Following [BS97b], the unnormalized Fourier transform $\mathcal{F}_{P_0}^{\text{un}}\phi(\zeta : \lambda)$ of a function $\phi \in \mathcal{D}(X)$ is defined to be the element of $\text{Hom}(V(\zeta), C^{-\infty}(P_0 : \zeta : \lambda))$ given by

$$\mathcal{F}_{P_0}^{\text{un}}\phi(\zeta : \lambda)\eta : g \mapsto \int_X \phi(x)j(P_0 : \zeta : -\lambda)(\eta)(g \cdot x) dx, \quad (4.2.10)$$

for $\eta \in V(\zeta)$. This Fourier transform depends meromorphically on $\lambda \in \mathfrak{a}_{\mathfrak{q}}$. For $\lambda \in \mathfrak{a}_{\mathfrak{q}}^*(\overline{P}_0, 0) + \rho_{P_0}$, the dependence is holomorphic, and the integral in (4.2.10) is absolutely convergent.

If $\phi \in \mathcal{E}_{P_0}^1(X)$, satisfies

$$\text{supp}(\mathcal{R}_{P_0}\phi) \subseteq \Xi_{P_0}(B + \Gamma_{P_0})$$

for some compact subset B of $\mathfrak{a}_{\mathfrak{q}}$, then we define (the first component of) the unnormalized Fourier transform $\mathcal{F}_{P_0,e}^{\text{un}}\phi(\zeta : \lambda)$, for $\lambda \in \mathfrak{a}_{\mathfrak{q}}^*(\overline{P}_0, 0) + \rho_{P_0}$, to be the homomorphism $V(\zeta, e) \rightarrow \mathcal{E}(P_0 : \zeta : \lambda)$ given by the absolutely convergent integral (4.2.10), for $\eta \in V(\zeta, e)$.

We note that for $\phi \in \mathcal{D}(X)$ the Radon transform $\mathcal{R}_{P_0}\phi$ has support in a set of the mentioned form, by Corollary 3.9.4. In that case,

$$\mathcal{F}_{P_0,e}^{\text{un}}\phi(\zeta : \lambda) = \mathcal{F}_{P_0}^{\text{un}}\phi(\zeta : \lambda)|_{V(\zeta,e)},$$

for all $\lambda \in \mathfrak{a}_{\mathfrak{q}}^*(\overline{P}_0, 0) + \rho_{P_0}$.

Proposition 4.2.3. *Let B be a compact subset of $\mathfrak{a}_{\mathfrak{q}}$ and let Γ be a cone in $\mathfrak{a}_{\mathfrak{q},\mathbb{C}}^*$ generated by a compact subset of $\mathfrak{a}_{\mathfrak{q}}^*(\overline{P}_0, 0)$. For sufficiently large $R > 0$ there exist for every $N \in \mathbb{N}$ a constant $C_N > 0$, a finite set $F_N \subset \mathcal{U}(\mathfrak{g})$, such that the following holds. For all $\phi \in \mathcal{E}^1(X)$ satisfying*

$$\text{supp}(\mathcal{R}_{P_0}\phi) \subseteq \Xi_{P_0}(B + \Gamma_{P_0}),$$

for all $k \in K$, $\eta \in V(\zeta, e)$ and all $\lambda \in \Gamma$ with $\|\lambda\| > R$,

$$\|\mathcal{F}_{P_0,e}^{\text{un}}\phi(\zeta : \lambda)\eta(k)\| \leq C_N \sum_{u \in F_N} \|u\phi\|_{L^1(X)} (1 + \|\lambda\|)^{-N} e^{H_B(-\text{Re}(\lambda))} \|\eta\|.$$

Let $\eta \in V(\zeta, e)$. The function

$$K \times (\mathfrak{a}_{\mathfrak{q}}^*(\overline{P}_0, 0) + \rho_{P_0}) \ni (k, \lambda) \mapsto \mathcal{F}_{P_0,e}^{\text{un}}\phi(\zeta : \lambda)\eta(k) \quad (4.2.11)$$

is smooth, and in addition holomorphic in the second variable.

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Proof. Let $\nu_0 \in \mathfrak{a}_q^*(\overline{P}_0, 0) + \rho_{P_0}$. Then $-\nu_0 + \rho_{P_0} \in \mathfrak{a}_q^*(P_0, 0)$, so that by Lemma 4.2.1 the function

$$\mathfrak{a}_q \ni Y \mapsto e^{-\nu_0(Y)} e^{\rho_{P_0}(Y)} \mathcal{R}_{P_0} \phi(\exp(Y) \cdot \xi_{P_0})$$

belongs to $\mathcal{S}(\mathfrak{a}_q)$ and is supported in $B + \Gamma_{P_0}$. We now apply Proposition 4.1.5 with Γ_{P_0} in place of Γ , so that $-C_\Gamma = \mathfrak{a}_q^*(\overline{P}_0, 0)$, and with $u = e^{\rho_{P_0}} \mathcal{R}_{P_0} \phi(\exp(\cdot) \cdot \xi_{P_0})$. This gives the existence of a constant $C_{\nu_0, N} > 0$, for each $N \in \mathbb{N}$, such that for all $\lambda \in \nu_0 + \mathfrak{a}_q^*(\overline{P}_0, 0)$

$$\left| \mathcal{F}_{A_q}(a \mapsto a^{\rho_{P_0}} \mathcal{R}_{P_0} \phi(a \cdot \xi_{P_0}))(\lambda) \right| \leq C_N(\phi) (1 + \|\lambda\|)^{-N} e^{H_B(-\operatorname{Re}(\lambda))}.$$

Here

$$C_N(\phi) = C_{\nu_0, N} \|(1 + \Delta_{A_q})^N (a \mapsto a^{\rho_{P_0} - \nu_0} \mathcal{R}_{P_0} \phi(a \cdot \xi_{P_0}))\|_{L^1(A_q)}.$$

We note that the constant $C_{\nu_0, N}$ is independent of the function ϕ .

For each $k \in K$, the function $L_k \phi : x \mapsto \phi(k^{-1}x)$ satisfies the same hypothesis as ϕ , so that the above estimates apply. In view of the definition of the unnormalized Fourier transform, (4.2.10), and Proposition 4.2.2 we now obtain the estimate

$$\begin{aligned} & \|\mathcal{F}_{P_0, e}^{\text{un}} \phi(\zeta : \lambda) \eta(k)\| \\ &= \left\| \int_{M_{P_0} \cap K} \mathcal{F}_{A_q}(a \mapsto a^{\rho_{P_0}} \mathcal{R}_{P_0} \phi(k^{-1}ma \cdot \xi_{P_0}))(\lambda) \zeta(m) \eta \, dm \right\| \\ &\leq \int_{M_{P_0} \cap K} \left| \mathcal{F}_{A_q}(a \mapsto a^{\rho_{P_0}} \mathcal{R}_{P_0} \phi(k^{-1}ma \cdot \xi_{P_0}))(\lambda) \right| \|\zeta(m) \eta\| \, dm \\ &\leq \widetilde{C}_{N, k}(\phi) (1 + \|\lambda\|)^{-N} e^{H_B(-\operatorname{Re}(\lambda))} \|\eta\| \end{aligned}$$

for all $k \in K$ and $\lambda \in \nu_0 + \mathfrak{a}_q^*(\overline{P}_0, 0)$. For the last inequality we used that ζ is unitary and we wrote

$$\widetilde{C}_{N, k}(\phi) := \int_{M_{P_0} \cap K} C_N(L_{m^{-1}k} \phi) \, dm.$$

Using Leibniz' rule, the fact that $M_{P_0} \cap K$ centralizes A_q , and the fact that the function $a^{\rho_{P_0} - \nu_0}$ is bounded on $\exp(B + \Gamma_{P_0})$, we infer that there exists a finite

subset $F'_N \subseteq S_{2N}(\mathfrak{a}_q)$, such that

$$\begin{aligned} \widetilde{C}_{N,k}(\phi) &\leq \widetilde{C}_{v_0,N} \sum_{u \in F'_N} \int_{M_{P_0} \cap K} \|u(a \mapsto \mathcal{R}_{P_0}(L_k \phi)(ma \cdot \xi_{P_0}))\|_{L^1(A_q)} dm \\ &\leq \widetilde{C}_{v_0,N} \sum_{u \in F'_N} \int_{M_{P_0} \cap K} \int_{A_q} \int_{N_{P_0}} |u(L_k \phi)(man \cdot x_0)| dn da dm. \end{aligned}$$

We now note that $u(L_k \phi) = L_k([\text{Ad}(k)^{-1}u]\phi)$ and that $\text{Ad}(k)^{-1}u$ is expressible in terms of a basis of $\mathcal{U}_{2N}(\mathfrak{g})$, with coefficients that are continuous, hence bounded, functions of $k \in K$. Combining this observation with (4.2.9) we see that there exists a finite subset $F_N \subseteq \mathcal{U}(\mathfrak{g})$ such that

$$\widetilde{C}_{N,k}(\phi) \leq \widetilde{C}_{v_0,N} \sum_{u \in F_N} \|u\phi\|_{L^1(X)}.$$

In view of the previous estimates, we now conclude that for all $\lambda \in \nu_0 + \mathfrak{a}_q^*(\overline{P}_0, 0)$

$$\|\mathcal{F}_{P_0,e}^{\text{un}}\phi(\zeta : \lambda)\eta(k)\| \leq \widetilde{C}_{v_0,N} \sum_{u \in F_N} \|u\phi\|_{L^1(X)} (1 + \|\lambda\|)^{-N} e^{H_B(-\text{Re}(\lambda))}.$$

Since $\Gamma \subset \mathfrak{a}_q^*(\overline{P}_0, 0)$ is a cone in $\mathfrak{a}_{q,\mathbb{C}}^*$ generated by a compact subset, we have, for sufficiently large $R > 0$, that

$$\{\lambda \in \Gamma : \|\lambda\| > R\} \subseteq \nu_0 + \mathfrak{a}_q^*(\overline{P}_0, 0).$$

The first statement now follows.

We address the second statement. Let U be an open subset of $\mathfrak{a}_{q,\mathbb{C}}^*$ with compact closure in $\mathfrak{a}_q^*(\overline{P}_0, 0) + \rho_{P_0}$. Then it suffices to prove the smoothness and holomorphy of (4.2.11) on $K \times U$. Fix $\lambda_0 \in \mathfrak{a}_q^*(\overline{P}_0, 0) + \rho_{P_0}$ such that $U \subseteq \mathfrak{a}_q^*(\overline{P}_0, 0) + \lambda_0$.

It follows from (4.2.10) and Proposition 4.2.2 that

$$\begin{aligned} \mathcal{F}_{P_0,e}^{\text{un}}\phi(\zeta : \lambda)\eta(k) & \\ = \int_{M_{P_0} \cap K} \mathcal{F}_{A_q}(a \mapsto a^{\rho_{P_0}} \mathcal{R}_{P_0}\phi(k^{-1}ma \cdot \xi_0))(\lambda)\zeta(m)\eta dm, & \end{aligned} \quad (4.2.12)$$

for all $\lambda \in \rho_{P_0} + \mathfrak{a}_q^*(\overline{P}_0, 0)$.

Put $S = B + \Gamma_{P_0}$. Then $\mathfrak{a}_q^*(\overline{P}_0, 0) \subseteq -\mathcal{C}_S$. We note that $\mathcal{S}_S(A_q) := \{\psi \in \mathcal{S}(A_q) : \text{supp}(\psi) \subseteq S\}$ is a closed subspace of $\mathcal{S}(A_q)$. As in the proof of

Proposition 4.2.2, it follows that (4.2.6) is a continuous family in $\mathcal{S}_S(A_q)$, with family parameter $k \in K$. As this also applies to $u\phi$, for every $u \in \mathcal{U}(\mathfrak{k})$, it actually follows that (4.2.6) is a smooth family in $\mathcal{S}_S(A_q)$. We now note that the Euclidean Fourier transform defines a continuous linear map

$$\mathcal{F}_{A_q} : \mathcal{S}_S(A_q) \rightarrow \mathcal{O}(-\mathcal{C}_S) = \mathcal{O}(\mathfrak{a}_q^*(\overline{P}_0, 0)).$$

Here $\mathcal{O}(-\mathcal{C}_S)$ denotes the space of holomorphic functions on $-\mathcal{C}_S$, equipped with the usual Fréchet topology. Combining this with the above assertion about smooth families we infer that

$$k \mapsto \mathcal{F}_{A_q}(a \mapsto a^{\rho_{P_0} - \lambda_0} \mathcal{R}_{P_0} \phi(ka \cdot \xi_0))$$

is a smooth function on K with values in the Fréchet space $\mathcal{O}(\mathfrak{a}_q^*(\overline{P}_0, 0))$. This in turn implies that

$$\begin{aligned} (k, \lambda) &\mapsto \mathcal{F}_{A_q}(a \mapsto a^{\rho_{P_0} - \lambda_0} \mathcal{R}_{P_0} \phi(ka \cdot \xi_{P_0}))(\lambda) \\ &= \mathcal{F}_{A_q}(a \mapsto a^{\rho_{P_0}} \mathcal{R}_{P_0} \phi(ka \cdot \xi_{P_0}))(\lambda + \lambda_0) \end{aligned}$$

is smooth on $K \times \mathfrak{a}_q^*(\overline{P}_0, 0)$ and in addition holomorphic in the second variable. As $U \subseteq \lambda_0 + \mathfrak{a}_q^*(\overline{P}_0, 0)$, this in turn implies that (4.2.12) is smooth in $(k, \lambda) \in K \times U$ and in addition holomorphic in $\lambda \in U$. \square

4.3 The τ -spherical Fourier transform $\mathcal{F}_{\overline{P}_0, \tau}$

Let $\vartheta \subset \widehat{K}$ be a finite set of K -types. If (π, V) is a G or K representation, then we write V_ϑ for the space of K -finite vectors with isotypes contained in ϑ .

Let $V_\tau = C(K)_\vartheta$, where the set ϑ of isotypes is taken with respect to the left-regular representation of K on $C(K)$, and let $\tau = \tau_\vartheta$ be the representation of K on V_τ obtained from the right-action. We equip V_τ with the inner product induced from $L^2(K)$. With respect to this inner product, τ is unitary.

As before, let $P_0 \in \mathcal{P}_\sigma(\mathfrak{a}_q)$. In this section we will consider the τ -spherical Fourier transform $\mathcal{F}_{\overline{P}_0, \tau}$ as defined in Section 6 of [BS97b]. Before we can write down the definition of this Fourier transform, we need to introduce some notation.

We denote the restriction of τ to $M_{P_0} \cap K$ by $\tau_{M_{P_0}}$. Let ${}^\circ\mathcal{C}(\tau)$ be the formal direct sum of Hilbert spaces

$${}^\circ\mathcal{C}(\tau) = \bigoplus_{w \in \mathcal{W}} {}^\circ\mathcal{C}(\tau)_w.$$

Here

$${}^0\mathcal{C}(\tau)_w = C^\infty(M_{P_0}/w(M_{P_0} \cap H)w^{-1} : \tau_{M_{P_0}})$$

is the finite dimensional Hilbert space of $\tau_{M_{P_0}}$ -spherical functions on the symmetric space $M_{P_0}/w(M_{P_0} \cap H)w^{-1}$, i.e., the Hilbert space of smooth functions

$$f : M_{P_0}/w(M_{P_0} \cap H)w^{-1} \rightarrow V_\tau$$

satisfying

$$f(k \cdot x) = \tau(k)f(x) \quad (k \in M_{P_0} \cap K, x \in M_{P_0}/w(M_{P_0} \cap H)w^{-1}).$$

The inner product on ${}^0\mathcal{C}(\tau)_w$ is induced from the inner product on the space of square integrable functions.

Assume $\zeta \in (\widehat{M_{P_0}})_H$. The space of smooth functions $f : K \rightarrow \mathcal{H}_\zeta \otimes V_\tau$ satisfying

$$f(mk_0k) = (\zeta(m) \otimes \tau(k)^{-1})f(k_0) \quad \text{for } k, k_0 \in K, m \in M_{P_0} \cap K$$

is denoted by $\mathcal{E}(K : \zeta : \tau)$. Note that evaluation at the identity element induces a linear isomorphism

$$f \mapsto f(e), \quad \mathcal{E}(K : \zeta : \tau) \xrightarrow{\cong} (\mathcal{H}_\zeta \otimes V_\tau)^{M_{P_0} \cap K}.$$

Let $\overline{V(\zeta)}$ be the conjugate vector space of $V(\zeta)$. Following [BS97b, p. 528], we define a linear map

$$\mathcal{E}(K : \zeta : \tau) \otimes \overline{V(\zeta)} \rightarrow {}^0\mathcal{C}(\tau); \quad T \mapsto \psi_T$$

by

$$(\psi_{f \otimes \eta})_w(m) = \langle f(e) | \zeta(m) \eta_w \rangle_{\mathcal{H}_\zeta} \quad (m \in M_{P_0}/w(M_{P_0} \cap H)w^{-1}),$$

for $f \in \mathcal{E}(K : \zeta : \tau)$ and $\eta \in \overline{V(\zeta)}$. Here $\langle \cdot | \cdot \rangle_{\mathcal{H}_\zeta}$ denotes the inner product on \mathcal{H}_ζ . This inner product is taken to be anti-linear in the second variable. Let $\mathcal{D}(X : \tau)$ be the space of compactly supported smooth functions $f : X \rightarrow V_\tau$ satisfying $f(k \cdot x) = \tau(k)f(x)$. We define $\varsigma : \mathcal{D}(X)_\vartheta \rightarrow \mathcal{D}(X : \tau)$ by

$$\varsigma(\phi)(x)(k) = \phi(kx).$$

This map is a bijection. (See [BS97b, Lemma 5].)

Restriction to K induces a bijection from $\mathcal{E}(P_0 : \zeta : \lambda)_\vartheta$ onto $\mathcal{E}(K : \zeta : \tau)$. Using this linear isomorphism and the linear isomorphism $\overline{V(\zeta)} \rightarrow V(\zeta)^*$ (defined via the Hermitian inner product on $V(\zeta)$) we may view $\mathcal{F}_{P_0}^{\text{un}}(\phi)(\zeta : \lambda)$ as an element of $\mathcal{E}(K : \zeta : \tau) \otimes \overline{V(\zeta)}$, for all $\lambda \in \mathfrak{a}_{\mathfrak{q}}^*(\overline{P}_0, 0) + \rho_{P_0}$. It thus makes sense to consider inner products between $\mathcal{F}_{P_0}^{\text{un}}\phi(\zeta : \lambda)$ and elements in $\mathcal{E}(K : \zeta : \tau) \otimes \overline{V(\zeta)}$.

The τ -spherical Fourier Transform $\mathcal{F}_{\overline{P}_0, \tau}$ is the linear transform from the space $\mathcal{D}(X : \tau)$ to the space of meromorphic ${}^0\mathcal{C}(\tau)$ -valued functions on $\mathfrak{a}_{\mathfrak{q}, \mathbb{C}}^*$, defined in [BS97b, (59)]. For our purposes, it is sufficient to use the following characterization in terms of the unnormalized Fourier transform discussed in the previous section. Let $(\widehat{M}_{P_0})_H(\tau)$ denote the finite collection of representations $\zeta \in (\widehat{M}_{P_0})_H$ such that $\zeta|_{M_{P_0} \cap K}$ and $\tau|_{M_{P_0} \cap K}$ have a $(M_{P_0} \cap K)$ -type in common. Then the spherical Fourier transform is completely determined by the requirement that

$$\langle \mathcal{F}_{\overline{P}_0, \tau} \varsigma(\phi)(\lambda) | \psi_{f \otimes \eta} \rangle = \langle \mathcal{F}_{P_0}^{\text{un}} \phi(\zeta : \lambda) | (A(\overline{P}_0 : P_0 : \zeta : \bar{\lambda})^{-1} f) \otimes \eta \rangle \quad (4.3.1)$$

for $\phi \in \mathcal{D}(X)_\vartheta$, $\zeta \in (\widehat{M}_{P_0})_H(\tau)$, $f \in \mathcal{E}(K : \zeta : \tau)$, $\eta \in \overline{V(\zeta)}$ and generic $\lambda \in \mathfrak{a}_{\mathfrak{q}, \mathbb{C}}^*$. Here

$$A(\overline{P}_0 : P_0; \zeta : \lambda) : \mathcal{E}(P_0 : \zeta : \lambda) \rightarrow \mathcal{E}(\overline{P}_0 : \zeta : \lambda)$$

is the so-called standard intertwining operator. It is initially defined for elements λ of $\mathfrak{a}_{\mathfrak{q}}^*(\overline{P}_0, -R)$ with $R > 0$ sufficiently large by the absolutely convergent integral

$$A(\overline{P}_0 : P_0; \zeta : \lambda) \phi(g) = \int_{N_{P_0}} \phi(ng) \, dn, \quad (\phi \in \mathcal{E}(P_0 : \zeta : \lambda), \, g \in G).$$

For the remaining $\lambda \in \mathfrak{a}_{\mathfrak{q}, \mathbb{C}}^*$ it is defined by meromorphic continuation. In (4.3.1) the topological linear isomorphisms

$$\mathcal{E}(Q : \zeta : \lambda)_\vartheta \rightarrow \mathcal{E}(K : \zeta : \tau),$$

given by $f \mapsto \varsigma(f|_K)$, were used for $Q = P_0$ and $Q = \overline{P}_0$ to view $A(\overline{P}_0 : P_0; \zeta : \lambda)$ as an endomorphism of the space $\mathcal{E}(K : \zeta : \tau)$.

Remark 4.3.1. To see that the definition for $\mathcal{F}_{\overline{P}_0, \tau}$ is in fact equivalent to the defining identity [BS97b, (59)], use subsequently loc. cit. (59), (50) with P and P' replaced by \overline{P}_0 and P_0 respectively, (47) and the identity similar to the one in Proposition 3 for the unnormalized Fourier transforms. The last mentioned identity is obtained from the proof of Proposition 3 by using (30) instead of (53).

If $\phi \in \mathcal{E}^1(X)_\vartheta$ satisfies

$$\text{supp}(\mathcal{R}_{P_0}\phi) \subseteq \Xi_{P_0}(B + \Gamma_{P_0}),$$

then by definition the unnormalized Fourier transform $\mathcal{F}_{P_0, e}^{\text{un}}\phi(\zeta : \lambda)$ is an element of $\text{Hom}(V(\zeta, e), \mathcal{E}(P_0 : \zeta : \lambda))$ for $\lambda \in \mathfrak{a}_q^*(\overline{P}_0, 0) + \rho_{P_0}$. In accordance with (4.3.1) we now define the (first component of the) spherical Fourier transform $\mathcal{F}_{\overline{P}_0, \tau, e}\mathcal{S}(\phi)(\lambda)$ of such a function ϕ to be the meromorphic ${}^0\mathcal{C}(\tau)_e$ -valued function on $\mathfrak{a}_q^*(\overline{P}_0, 0) + \rho_{P_0}$ which is given by

$$\langle \mathcal{F}_{\overline{P}_0, \tau, e}\mathcal{S}(\phi)(\lambda) | \psi f \otimes \eta \rangle = \langle \mathcal{F}_{P_0, e}^{\text{un}}\phi(\zeta : \lambda) | (A(\overline{P}_0 : P_0 : \zeta : \bar{\lambda})^{-1} f) \otimes \eta \rangle \quad (4.3.2)$$

for $\zeta \in (\widehat{M}_{P_0})_H(\tau)$, $f \in \mathcal{E}(K : \zeta : \tau)$, $\eta \in \overline{V(\zeta, e)}$ and generic $\lambda \in \mathfrak{a}_q^*(\overline{P}_0, 0) + \rho_{P_0}$.

Note that this definition is compatible with (4.3.1). If $\phi \in \mathcal{D}(X)_\vartheta$, then

$$\mathcal{F}_{\overline{P}_0, \tau, e}\mathcal{S}(\phi)(\lambda) = \text{pr}_e \mathcal{F}_{\overline{P}_0, \tau}\mathcal{S}(\phi)(\lambda),$$

for generic $\lambda \in \mathfrak{a}_q^*(\overline{P}_0, 0) + \rho_{P_0}$. Here pr_e denotes the projection ${}^0\mathcal{C}(\tau) \rightarrow {}^0\mathcal{C}(\tau)_e$.

Let $A(\overline{P}_0 : P_0 : \zeta : \lambda)_\tau$ denote the standard intertwining operator, viewed as an endomorphism of $\mathcal{E}(K : \zeta : \tau)$. Then it follows from [Ban92, Lemma 16.6] that the $\text{End}(\mathcal{E}(K : \zeta : \tau))$ -valued meromorphic function $\lambda \mapsto A(\overline{P}_0 : P_0 : \zeta : \bar{\lambda})_\tau^{-1}$ is of $\Sigma(\mathfrak{g}, \mathfrak{a}_q)$ -polynomial growth on $\mathfrak{a}_q^*(\overline{P}_0, 0)$. This means that there exists a polynomial function $\pi : \mathfrak{a}_{q, \mathbf{C}}^* \rightarrow \mathbf{C}$, which is a product of functions of the form $\lambda \mapsto \langle \lambda, \alpha \rangle - c$ with $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}_q)$ and $c \in \mathbf{R}$, such that

$$\lambda \mapsto \pi(\lambda) A(\overline{P}_0 : P_0 : \zeta : \bar{\lambda})_\tau^{-1}$$

is a holomorphic $\text{End}(\mathcal{E}(K : \zeta : \tau))$ -valued function on $\mathfrak{a}_q^*(\overline{P}_0, 0)$ and satisfies a polynomial estimate of the form

$$\|\pi(\lambda) A(\overline{P}_0 : P_0 : \zeta : \bar{\lambda})_\tau^{-1}\| \leq C(1 + \|\lambda\|)^N \quad (\lambda \in \mathfrak{a}_q^*(\overline{P}_0 : 0)),$$

with $C > 0$ and $N \in \mathbf{N}$. Here we note that the space $\text{End}(\mathcal{E}(K : \zeta : \tau))$ is finite dimensional.

The following proposition is now a direct corollary of Proposition 4.2.3 and (4.3.2).

Proposition 4.3.2. *Let B be a compact subset of \mathfrak{a}_q . If $\phi \in \mathcal{E}^1(X)_\theta$ satisfies*

$$\text{supp}(\mathcal{R}_{P_0}\phi) \subseteq \Xi_{P_0}(B + \Gamma_{P_0}). \quad (4.3.3)$$

Then the map

$$\mathfrak{a}_q^*(\overline{P}_0, 0) + \rho_{P_0} \rightarrow {}^0\mathcal{C}(\tau)_e; \quad \lambda \mapsto \pi(\lambda)\mathcal{F}_{\overline{P}_0, \tau, e}\mathcal{S}(\phi)(\lambda)$$

is holomorphic.

Furthermore, let Γ be any cone in $\mathfrak{a}_{q, \mathbb{C}}^$ generated by a compact subset of $\mathfrak{a}_q^*(\overline{P}_0, 0)$. Then there exists a constant $R > 0$ and for every $N \in \mathbb{N}$ a constant $C_N > 0$ and a finite subset $F_N \subseteq \mathcal{U}(\mathfrak{g})$ such that for all $\phi \in \mathcal{E}^1(X)_\theta$ satisfying (4.3.3) the estimate*

$$\|\pi(\lambda)\mathcal{F}_{\overline{P}_0, \tau, e}\mathcal{S}(\phi)(\lambda)\| \leq C_N \sum_{u \in F_N} \|u\phi\|_{L^1(X)} (1 + \|\lambda\|)^{-N} e^{H_B(-\text{Re}(\lambda))}$$

is valid for all $\lambda \in \Gamma$ with $\|\lambda\| > R$.

4.4 Function Spaces

As before, we assume that P_0 is a minimal $\sigma \circ \theta$ -stable parabolic subgroup containing A . Let $\Gamma_{P_0}^{W_{K \cap H}}$ be the maximal $W_{K \cap H}$ -invariant subcone of Γ_{P_0} , i.e.,

$$\Gamma_{P_0}^{W_{K \cap H}} = \bigcap_{w \in W_{K \cap H}} \Gamma_{P_0}^w.$$

For a subset S of \mathfrak{a}_q , we define

$$\mathcal{E}^1(X; S) = \{\phi \in \mathcal{E}^1(X) : \text{supp}(\phi) \subseteq X(S)\}.$$

We further define $\mathcal{E}_{P_0}^1(X)$ to be the subspace of $\mathcal{E}^1(X)$ given by

$$\mathcal{E}_{P_0}^1(X) = \mathcal{S}(X) + \bigcup_{\substack{B \subset \mathfrak{a}_q \\ B \text{ compact}}} \mathcal{E}^1(X; B + \Gamma_{P_0}^{W_{K \cap H}}).$$

Here $\mathcal{S}(X)$ denotes the space of rapidly decreasing functions on X , which is the intersection of the Harish-Chandra L^p -Schwartz spaces $\mathcal{C}^p(X)$ for $p > 0$. (See [Ban92, Chapter 17].)

Remark 4.4.1. If X is a Riemannian symmetric space (hence $H = K$) or X is a Lie group (i.e., $G = G_0 \times G_0$ for some reductive Lie group G_0 of the Harish-Chandra class and $H = \text{diag}(G_0)$), then $W_{K \cap H}$ equals the full Weyl group W . In these cases the cone $\Gamma_{P_0}^{W_{K \cap H}}$ is the trivial cone $\{0\}$ so that $\mathcal{E}_{P_0}^1(X) = \mathcal{S}(X)$ is independent of P_0 .

We say that a cone Γ in a finite dimensional real vector space V is finitely generated if there exists a finite set $\{\omega_k \in V : 1 \leq k \leq n\}$ such that

$$\Gamma = \sum_{k=1}^n \mathbf{R}_{\geq 0} \omega_k. \quad (4.4.1)$$

A cone is said to be polyhedral if it equals the intersection of finitely many closed halfspaces. According to [Roc70, Theorem 19.1] every finitely generated cone is polyhedral and, vice versa, every polyhedral cone is finitely generated. If Γ is a finitely generated cone (4.4.1), then \mathcal{C}_Γ equals the polyhedral cone

$$\mathcal{C}_\Gamma = \{\lambda \in V^* : \lambda(\omega_k) \leq 0 \text{ for all } 1 \leq k \leq n\}.$$

Therefore \mathcal{C}_Γ is finitely generated as well.

Note that every finitely generated cone is closed and convex.

Lemma 4.4.2. *Let V be a finite dimensional real vector space and let $n \in \mathbf{N}$. For $k \in \mathbf{N}$ with $k \leq n$, let Γ_k be a finitely generated cone in V and let B_k be a compact subset of V . Then there exists a compact subset B of V such that*

$$\bigcap_{k=1}^n (B_k + \Gamma_k) \subseteq B + \bigcap_{k=1}^n \Gamma_k.$$

Proof. Fix a positive definite inner product on V . This inner product induces a dual inner product on V^* in the usual manner. We write $\Gamma_0 = \bigcap_{k=1}^n \Gamma_k$.

Since the cones Γ_k are closed and convex,

$$\Gamma_0 = \{x \in V : \lambda(x) \leq 0 \text{ for all } \lambda \in \mathcal{C}_{\Gamma_1} + \cdots + \mathcal{C}_{\Gamma_n}\}.$$

Hence, \mathcal{C}_{Γ_0} equals the closure of $\sum_{k=1}^n \mathcal{C}_{\Gamma_k}$. As the cones \mathcal{C}_{Γ_k} are finitely generated, so is their sum. In particular the sum is closed and we conclude that

$$\mathcal{C}_{\Gamma_0} = \sum_{k=1}^n \mathcal{C}_{\Gamma_k}.$$

4. Support theorem for the horospherical transform

We define the continuous functions

$$s : \mathcal{C}_{\Gamma_1} \times \cdots \times \mathcal{C}_{\Gamma_n} \rightarrow \mathcal{C}_{\Gamma_0}; \quad (\lambda_k)_{k=1}^n \mapsto \sum_{k=1}^n \lambda_k$$

and

$$\nu : (V^*)^n \rightarrow \mathbf{R}; \quad (\lambda_k)_{k=1}^n \mapsto \sum_{k=1}^n \|\lambda_k\|.$$

Note that s is a surjection. Since ν is proper and non-negative, ν has a minimum on $s^{-1}(\lambda)$. We define the function

$$H_0 : \mathcal{C}_{\Gamma_0} \rightarrow \mathbf{R}; \quad \lambda \mapsto \min_{s^{-1}(\{\lambda\})} \nu.$$

If $\kappa, \lambda \in \mathcal{C}_{\Gamma_0}$, then $s^{-1}(\kappa) + s^{-1}(\lambda) \subseteq s^{-1}(\kappa + \lambda)$. Since ν is subadditive, we deduce that

$$H_0(\kappa + \lambda) \leq \min_{s^{-1}(\kappa) + s^{-1}(\lambda)} \nu \leq \min_{s^{-1}(\kappa)} \nu + \min_{s^{-1}(\lambda)} \nu = H_0(\kappa) + H_0(\lambda).$$

Hence H_0 is subadditive. Furthermore, if $r > 0$ and $\lambda \in \mathcal{C}_{\Gamma_0}$, then $s^{-1}(r\lambda) = rs^{-1}(\lambda)$ and thus it follows that H_0 is positively homogeneous of degree 1. This implies in particular that H_0 is a convex function on \mathcal{C}_{Γ_0} .

Since \mathcal{C}_{Γ_0} is finitely generated, the intersection of \mathcal{C}_{Γ_0} with any finitely generated cone is again finitely generated. If we fix an orthonormal basis for V , then this is in particular the case for the intersection of \mathcal{C}_{Γ_0} with any orthant (hyperoctant). Such an intersection is a proper cone in the sense that if λ is a non-zero element of the cone, then $-\lambda$ is not. We can thus conclude that there exists a finite collection $\{\mathcal{C}_j : 1 \leq j \leq m\}$ of proper finitely generated cones \mathcal{C}_j such that

$$\mathcal{C}_{\Gamma_0} = \bigcup_{j=1}^m \mathcal{C}_j.$$

For $1 \leq j \leq m$, let $\{\omega_k^j \subseteq \mathcal{C}_j : 1 \leq k \leq n_j\}$ be a finite set such that

$$\mathcal{C}_j = \sum_{k=1}^{n_j} \mathbf{R}_{\geq 0} \omega_k^j.$$

Since \mathcal{C}_j is a proper closed cone, $\mathcal{C}_j \setminus \{0\}$ is contained in an open halfspace. This implies that

$$s_j : \prod_{k=1}^{n_j} \mathbf{R}_{\geq 0} \omega_k^j \rightarrow \mathcal{C}_j; \quad (r_k \omega_k^j)_{k=1}^{n_j} \mapsto \sum_{k=1}^{n_j} r_k \omega_k^j$$

is a proper map. Therefore, there exist $r_k > 0$, for $1 \leq k \leq n_j$, such that the intersection of the unit sphere with \mathcal{C}_j is contained in

$$\text{ch} \left(\bigcup_{k=1}^{n_j} [0, r_k] \omega_k^j \right).$$

Since H_0 is convex, the supremum of H_0 over the intersection of the unit sphere with \mathcal{C}_j is by [Roc70, Theorem 32.2] smaller than or equal to the supremum of H_0 over the sets $[0, r_k] \omega_k^j$. The latter is finite because H_0 is homogeneous and $H_0(\omega_k^j)$ is finite for every $1 \leq k \leq n_j$. Therefore, there exists an $R_j > 0$ such that $H_0(\lambda) \leq R_j \|\lambda\|$ for every $\lambda \in \mathcal{C}_j$. Let R be the maximum of the R_j . Then

$$H_0(\lambda) \leq R \|\lambda\| \quad (\lambda \in \mathcal{C}_{\Gamma_0}).$$

Let now $x \in \bigcap_k (B_k + \Gamma_k)$. We will use that by compactness of each set B_k , for $1 \leq k \leq n$, we have $\mathcal{C}_{\Gamma_k} = \mathcal{C}_{B_k + \Gamma_k}$ and $H_{B_k + \Gamma_k} = H_{B_k}$ on the latter set (see Lemma 3.8.1 (iii)). Let $\lambda \in \mathcal{C}_{\Gamma_0}$; then we may write $\lambda = \sum_{k=1}^n \lambda_k$ with $\lambda_k \in \mathcal{C}_{\Gamma_k}$, and we see that

$$\lambda(x) = \sum_{k=1}^n \lambda_k(x) \leq \sum_{k=1}^n H_{B_k}(\lambda_k).$$

Again, by compactness of the sets B_k , there exists an $r > 0$ such that $H_{B_k} \leq r \|\cdot\|$ and we finally see that

$$\lambda(x) \leq r \sum_{k=1}^n \|\lambda_k\|.$$

This inequality holds for every n -tuple $(\lambda_k)_{k=1}^n \in \mathcal{C}_{\Gamma_1} \times \cdots \times \mathcal{C}_{\Gamma_n}$ such that $s((\lambda_k)_{k=1}^n)$ is equal to λ . Therefore,

$$\lambda(x) \leq r H_0(X) \leq r R \|\lambda\|.$$

Let $B(0, rR)$ is the closed ball centered at the origin with radius rR . From Lemma 3.8.1 it follows that $\mathcal{C}_{B(0, rR) + \Gamma_0} = \mathcal{C}_{\Gamma_0}$ and the restriction of $H_{B(0, rR) + \Gamma_0}$ to \mathcal{C}_{Γ_0} equals $H_{B(0, rR)}$. The latter support function is given by

$$H_{B(0, rR)}(\lambda) = rR \|\lambda\| \quad (\lambda \in V^*).$$

4. Support theorem for the horospherical transform

We can thus conclude that $x \in B(0, rR) + \Gamma_0$. This establishes the desired inclusion with $B = B(0, rR)$. \square

Remark 4.4.3. The lemma does not hold true if “finitely generated” is replaced by “closed and convex”. Bart van den Dries showed us the following counterexample. Let

$$\begin{aligned}\Gamma_1 &= \{(x, 0, z) \in \mathbf{R}^3 : 0 \leq x \leq z\}, \\ \Gamma_2 &= \{(x, y, z) \in \mathbf{R}^3 : 0 \leq x \leq z, \frac{x^2}{z} \leq y \leq x\}.\end{aligned}$$

If $B = \{(0, y, 0) : -\frac{1}{2} \leq y \leq \frac{1}{2}\}$, then for every $t > 1$,

$$(t, \frac{1}{2}, t^2) = \begin{cases} (t, 0, t^2) + (0, \frac{1}{2}, 0) \\ (t, 1, t^2) + (0, -\frac{1}{2}, 0) \end{cases}$$

is contained in the intersection of $B + \Gamma_1$ and $B + \Gamma_2$, but there exists no compact subset B' of \mathbf{R}^3 such that

$$\{(t, \frac{1}{2}, t^2) : t > 1\} \subseteq B' + \{(0, 0, z) : z \geq 0\} = B' + (\Gamma_1 \cap \Gamma_2).$$

When going through the proof for Lemma 4.4.2 in this particular case, the first serious obstruction encountered is that the sum of \mathcal{C}_{Γ_1} and \mathcal{C}_{Γ_2} is not closed and therefore

$$\mathcal{C}_{\Gamma_1 \cap \Gamma_2} \neq \mathcal{C}_{\Gamma_1} + \mathcal{C}_{\Gamma_2}.$$

Proposition 4.4.4. $\mathcal{E}_{P_0}^1(X)$ is a G -invariant subspace of $\mathcal{E}^1(X)$.

Proof. Let $B \subseteq \mathfrak{a}_q$ be a compact subset. As $\mathcal{S}(X)$ and $\mathcal{E}^1(X)$ are G -invariant, it suffices to show that for every $g \in G$ there exists a compact subset $B'' \subseteq \mathfrak{a}_q$ such that

$$g \cdot X(B + \Gamma_{P_0}^{W_{K \cap H}}) \subseteq X(B'' + \Gamma_{P_0}^{W_{K \cap H}}). \quad (4.4.2)$$

As $Kg \cdot X(B + \Gamma_{P_0}^{W_{K \cap H}})$ is a K -invariant subset of X , there exists a unique $W_{K \cap H}$ -invariant compact subset $C \subseteq \mathfrak{a}_q$ such that

$$Kg \cdot X(B + \Gamma_{P_0}^{W_{K \cap H}}) = X(C).$$

We will finish the proof by showing that C is contained in a set of the form $B'' + \Gamma_{P_0}^{W_{K \cap H}}$.

Let $a \in A$ be such that $g \in KaK$. Furthermore, let $P_m \in \mathcal{P}(\mathfrak{a})$ be a minimal parabolic subgroup contained in P_0 and let $\mathfrak{A}_{KAN_{P_m}} : G \rightarrow \mathfrak{a}$ be defined as in (3.5.3).

Then by the convexity theorem ([Ban86, Theorem 1.1]) of Van den Ban,

$$\text{ch}(C) + \Gamma(\Sigma_-^+(\mathfrak{g}, \mathfrak{a}_q; P_0)) = \pi_q \circ \mathfrak{A}_{KAN_{P_m}}(aK \exp(B + \Gamma_{P_0}^{W_{K \cap H}})H).$$

The set on the right-hand side equals

$$\pi_q(\mathfrak{A}_{KAN_{P_m}}(aK)) + \pi_q(\mathfrak{A}_{KAN_{P_m}}(\exp(B + \Gamma_{P_0}^{W_{K \cap H}})H)).$$

If we apply the convexity theorem of Kostant ([Kos73, Theorem 4.1]) to the first and the convexity theorem of Van den Ban ([Ban86, Theorem 1.1]) to the second term, we obtain

$$\begin{aligned} & \text{ch}(C) + \Gamma(\Sigma_-^+(\mathfrak{g}, \mathfrak{a}_q; P_0)) \\ &= \pi_q(W(\mathfrak{a}) \cdot \log a) + \text{ch}(W_{K \cap H} \cdot B) + \Gamma_{P_0}^{W_{K \cap H}} + \Gamma(\Sigma_-^+(\mathfrak{g}, \mathfrak{a}_q; P_0)). \end{aligned} \tag{4.4.3}$$

Here $W(\mathfrak{a})$ denotes the Weyl-group $\mathcal{N}_K(\mathfrak{a})/\mathcal{Z}_K(\mathfrak{a})$. Put

$$B' = \pi_q(\text{ch}(W(\mathfrak{a}) \cdot \log a)) + \text{ch}(W_{K \cap H} \cdot B)$$

and note that each cone on the right-hand side of (4.4.3) is contained in Γ_{P_0} . Hence, it follows from (4.4.3) that

$$C \subseteq B' + \Gamma_{P_0}.$$

By $W_{K \cap H}$ -invariance of C this implies that

$$C \subseteq \bigcap_{w \in W_{K \cap H}} w \cdot (B' + \Gamma_{P_0}) = \bigcap_{w \in W_{K \cap H}} (B' + \Gamma_{P_0}^w).$$

The cones $\Gamma_{P_0}^w$ are finitely generated and B' is compact, so that we may complete the proof by application of Lemma 4.4.2. \square

Remark 4.4.5. By inspection of the above proof, one readily sees that for every compact subset $G_0 \subseteq G$ there exists a compact $B'' \subseteq \mathfrak{a}_q$ such that (4.4.2) is valid for all $g \in G_0$.

4.5 Inversion formula

We continue in the setting of Section 4.3. Let X_+ be the union of disjoint open subsets of X

$$X_+ = \bigcup_{w \in \mathcal{W}} X(\mathfrak{a}_q^+(P_0^w)).$$

Let

$$E_+(\overline{P}_0 : \lambda : \cdot) : X_+ \rightarrow \text{Hom}({}^0\mathcal{C}(\tau), V_\tau)$$

be defined by

$$E_+(\overline{P}_0 : \lambda : kaw \cdot x_0) : \psi \mapsto \tau(k)\Phi_{\overline{P}_0, w}(\lambda : a)\psi_w(e)$$

where $\Phi_{\overline{P}_0, w}(\lambda, \cdot)$ is the $\text{End}(V_\tau^{K \cap M_{P_0} \cap wHw^{-1}})$ -valued function on $A_q^+(\overline{P}_0)$ defined in [BS97a, Section 10]. Let $\nu \in \mathfrak{a}_q^*(\overline{P}_0, 0)$, with $\|\nu\| = 1$. By [BS99, Theorem 4.7],

$$\phi(x) = |W| \int_{t\nu + i\mathfrak{a}_q^*} E_+(\overline{P}_0 : \lambda : x) \mathcal{F}_{\overline{P}_0, \tau} \zeta(\phi)(\lambda) d\lambda$$

if $\phi \in \mathcal{S}(X)_\vartheta$, $x \in X_+$ and if $t > 0$ is sufficiently large. This result can be partially extended to the K -finite functions in the larger space $\mathcal{E}_{P_0}^1(X)$.

Proposition 4.5.1 (Inversion Formula). *If $\phi \in \mathcal{E}_{P_0}^1(X)_\vartheta$ and $\nu \in \mathfrak{a}_q^*(\overline{P}_0, 0)$, then for $k \in K$, $a \in A_q^+(\overline{P}_0)$ and sufficiently large $t > 0$*

$$\phi(ka \cdot x_0) = |W| \int_{t\nu + i\mathfrak{a}_q^*} \tau(k)\Phi_{\overline{P}_0, e}(\lambda : a) \left(\mathcal{F}_{\overline{P}_0, \tau, e} \zeta(\phi)(\lambda)(e) \right) d\lambda.$$

Proof. It suffices to prove the proposition for $\phi \in \mathcal{E}^1(X; B + \Gamma_{P_0}^{W_{K \cap H}})_\vartheta$, where B is a compact subset of \mathfrak{a}_q . As $\mathcal{E}^1(X; B + \Gamma_{P_0}^{W_{K \cap H}})_\vartheta \cap \mathcal{D}(X)_\vartheta$ is dense in $\mathcal{E}^1(X; B + \Gamma_{P_0}^{W_{K \cap H}})_\vartheta$, there exists a sequence $(\phi_j)_{j \in \mathbb{N}}$ in $\mathcal{E}^1(X; B + \Gamma_{P_0}^{W_{K \cap H}})_\vartheta \cap \mathcal{D}(X)_\vartheta$ converging to ϕ in $\mathcal{E}^1(X; B + \Gamma_{P_0}^{W_{K \cap H}})_\vartheta$. According to [BS97a, Theorem 9.1], the function

$$i\mathfrak{a}_q^* \ni \lambda \mapsto \Phi_{\overline{P}_0, e}(t\nu + \lambda : a)$$

is bounded if $t > 0$ is sufficiently large. The Paley-Wiener estimate in Proposition 4.3.2 therefore implies that, for $t > 0$ sufficiently large,

$$\lim_{j \rightarrow \infty} \int_{t\nu + i\mathfrak{a}_q^*} \left| \tau(k)\Phi_{\overline{P}_0, e}(\lambda : a) \left(\mathcal{F}_{\overline{P}_0, \tau, e} \zeta(\phi - \phi_j)(\lambda)(e) \right) \right| d\lambda = 0.$$

□

4.6 A support theorem for the horospherical transform for functions

Lemma 4.6.1. *Let B be a compact subset of \mathfrak{a}_q . Then the set*

$$\{Y \in \mathfrak{a}_q : \nu(Y) + H_B(-\nu) \geq 0 \text{ for all } \nu \in \mathfrak{a}_q^*(\overline{P}_0, 0) \cap \mathfrak{a}_q^*\} = \text{ch}(B) + \Gamma_{P_0}.$$

Proof. The set $\text{ch}(B) + \Gamma_{P_0}$ is convex and closed. Therefore it equals

$$\{Y \in \mathfrak{a}_q : \lambda(Y) \leq H_{\text{ch}(B) + \Gamma_{P_0}}(\lambda) \text{ for all } \lambda \in \mathcal{C}_{\text{ch}(B) + \Gamma_{P_0}}\}.$$

By Lemma 3.8.1 (ii) the cone $\mathcal{C}_{\text{ch}(B) + \Gamma_{P_0}}$ equals $\mathcal{C}_{\Gamma_{P_0}}$, which in turn equals the closure of $-\mathfrak{a}_q^*(\overline{P}_0, 0) \cap \mathfrak{a}_q^*$. Furthermore, by Lemma 3.8.1 (iii) the support function $H_{\text{ch}(B) + \Gamma_{P_0}}$ and $H_{\text{ch}(B)} = H_B$ coincide on that set. We thus obtain that $\text{ch}(B) + \Gamma_{P_0}$ equals

$$\{Y \in \mathfrak{a}_q : \lambda(Y) \leq H_B(\lambda) \text{ for all } \lambda \text{ in the closure of } -\mathfrak{a}_q^*(\overline{P}_0, 0) \cap \mathfrak{a}_q^*\}.$$

Since B is compact, the function H_B is continuous, hence the last equals

$$\{Y \in \mathfrak{a}_q : -\lambda(Y) + H_B(\lambda) \geq 0 \text{ for all } \lambda \in -\mathfrak{a}_q^*(\overline{P}_0, 0) \cap \mathfrak{a}_q^*\}.$$

This proves the lemma. □

We can now prove a preliminary support theorem for the Radon transform \mathcal{R}_{P_0} associated with the minimal $\sigma \circ \theta$ -stable parabolic subgroup $P_0 \in \mathcal{P}_\sigma(\mathfrak{a}_q)$. The proof is based on a Paley-Wiener type shift argument. At the end of this section we will sharpen the preliminary result by invoking the equivariance of the Radon transform.

Proposition 4.6.2. *Let B be a convex compact subset of \mathfrak{a}_q and let $\phi \in \mathcal{E}_{P_0}^1(X)$. If*

$$\text{supp}(\mathcal{R}_{P_0}\phi) \subseteq \Xi_{P_0}(B + \Gamma_{P_0}),$$

then

$$\text{supp}(\phi) \cap A_q^+(\overline{P}_0) \cdot x_0 \subseteq \left(\exp(B + \Gamma_{P_0}) \cap A_q^+(\overline{P}_0) \right) \cdot x_0.$$

Proof. Because of equivariance and continuity of the Radon transform it suffices to prove the claim for K -finite functions ϕ . We will therefore assume that ϕ is K -finite with isotypes contained in a finite subset ϑ of \widehat{K} .

4. Support theorem for the horospherical transform

Assume that $a \in A_q^+(\overline{P}_0)$, but $\log a \notin B + \Gamma_{P_0}$. By Lemma 4.6.1 there exists a $\nu \in \mathfrak{a}_q^*(\overline{P}_0, 0) \cap \mathfrak{a}_q^*$ such that

$$\nu(\log a) + H_{\log(B)}(-\nu) < 0.$$

According to [BS97a, Theorem 9.1] there exists a constant $c > 0$ such that

$$\|\Phi_{\overline{P}_0, e}(\lambda : a)\| < ca^{t\nu}$$

for all sufficiently large $t > 0$ and $\lambda \in t\nu + i\mathfrak{a}_q^*$. The Paley-Wiener estimate for $\mathcal{F}_{\overline{P}_0, \tau}\phi$ (Proposition 4.3.2) and the inversion formula (Proposition 4.5.1) imply that for every integer N there exists a constant $C_N > 0$ such that for sufficiently large $t > 0$

$$|\phi(a \cdot x_0)| \leq C_N e^{t(\nu(\log a) + H_B(-\nu))} \int_{t\nu + i\mathfrak{a}_q^*} (1 + \|\lambda\|)^{-N} d\lambda.$$

Let $N \geq \dim \mathfrak{a}_q + 1$; then by taking the limit for $t \rightarrow \infty$ we find

$$\phi(a \cdot x_0) = 0.$$

□

Lemma 4.6.3. *Let $S \subseteq \mathfrak{a}_q$, let $a \in A$ and let $g \in KaK$. Then*

$$g \cdot \mathcal{E}_{P_0}(S) \subseteq \mathcal{E}_{P_0}(\pi_q \operatorname{ch}(W(\mathfrak{a}) \cdot \log a) + S).$$

Proof. Let $P_m \in \mathcal{P}(\mathfrak{a})$ be a minimal parabolic subgroup contained in P_0 . By Kostant's convexity theorem ([Kos73, Theorem 4.1]),

$$gK \subseteq K \exp(\operatorname{ch}(W(\mathfrak{a}) \cdot \log a)) N_{P_m}.$$

Using that $N_{P_m} = N_{P_m}^{P_0} N_{P_0}$ and that $N_{P_m}^{P_0} \subseteq M_{P_0} \cap H$ (see Lemma 2.4.2) we now find that

$$\begin{aligned} g \cdot \mathcal{E}_{P_0}(S) &= gK \exp(S) \cdot \xi_{P_0} \subseteq K \exp(\operatorname{ch}(W(\mathfrak{a}) \cdot \log a)) N_{P_m} \exp(S) \cdot \xi_{P_0} \\ &= K \exp(\operatorname{ch}(W(\mathfrak{a}) \cdot \log a) + S) \cdot \xi_{P_0} \\ &= \mathcal{E}_{P_0}(\pi_q \operatorname{ch}(W(\mathfrak{a}) \cdot \log a) + S). \end{aligned}$$

□

We can now sharpen Proposition 4.6.2 by using the equivariance of the Radon transform.

Theorem 4.6.4 (Support theorem for the horospherical transform). *Let B be a convex compact subset of \mathfrak{a}_q and let $\phi \in \mathcal{E}_{P_0}^1(X)$. If*

$$\text{supp}(\mathcal{R}_{P_0}\phi) \subseteq \mathcal{E}_{P_0}(B + \Gamma_{P_0}),$$

then

$$\text{supp}(\phi) \subseteq X \left(\bigcap_{w \in W_{K \cap H}} w \cdot (B + \Gamma_{P_0}) \right). \quad (4.6.1)$$

Proof. Assume that ϕ satisfies the hypothesis. We will first show that

$$\text{supp}(\phi) \cap A_q \cdot x_0 \subseteq \exp(B + \Gamma_{P_0}) \cdot x_0. \quad (4.6.2)$$

Let $Y_0 \in \mathfrak{a}_q$ be such that $\exp Y_0 \in \text{supp}(\phi)$. Then there exists a $Y \in \mathfrak{a}_q^+(\overline{P_0})$ such that $Y_0 + Y \in \mathfrak{a}_q^+(\overline{P_0})$. By equivariance of \mathcal{R}_{P_0} and by application of Lemma 4.6.3 we find that

$$\begin{aligned} \text{supp}(\mathcal{R}_{P_0}(l_{\exp(-Y)}^*\phi)) &= \exp(Y) \cdot \text{supp}(\mathcal{R}_{P_0}\phi) \subseteq \exp(Y) \cdot \mathcal{E}_{P_0}(B + \Gamma_{P_0}) \\ &\subseteq \mathcal{E}_{P_0}(\pi_q \text{ch}(W(\mathfrak{a}) \cdot Y) + B + \Gamma_{P_0}). \end{aligned}$$

From Propositions 4.4.4 and 4.6.2 it now follows that

$$\begin{aligned} \exp(Y_0 + Y) \cdot x_0 &\in \text{supp}(l_{\exp(-Y)}^*\phi) \cap A_q^+(\overline{P_0}) \cdot x_0 \\ &\subseteq \exp(\pi_q \text{ch}(W(\mathfrak{a}) \cdot Y) + B + \Gamma_{P_0}) \cdot x_0. \end{aligned}$$

We conclude that the set $\pi_q \text{ch}(W(\mathfrak{a}) \cdot Y) + B + \Gamma_{P_0}$ contains $Y_0 + Y$. On the other hand, since $Y \in \mathfrak{a}_q^+(\overline{P_0})$,

$$\pi_q \text{ch}(W(\mathfrak{a}) \cdot Y) \subseteq Y + \Gamma_{P_0}.$$

We thus see that $Y_0 + Y \in Y + B + \Gamma_{P_0}$, so that $Y_0 \in B + \Gamma_{P_0}$. We have proved (4.6.2).

If $k \in K$, then by equivariance of the Radon transform, the function $l_k^*\phi$ satisfies the same hypotheses as ϕ , so that (4.6.2) is valid with $l_k^*\phi$ in place of ϕ . This implies that

$$\begin{aligned} \text{supp}(\phi) \cap A_q \cdot x_0 &\subseteq \bigcap_{w \in \mathcal{N}_{K \cap H}(\mathfrak{a}_q)} w \cdot \exp(B + \Gamma_{P_0}) \cdot x_0 \\ &= \exp \left(\bigcap_{w \in W_{K \cap H}} w \cdot (B + \Gamma_{P_0}) \right) \cdot x_0. \end{aligned}$$

4. Support theorem for the horospherical transform

Invoking the K -equivariance of the Radon transform once more in a similar way, we conclude that (4.6.1) holds. \square

Remark 4.6.5. Let B be a $W_{K \cap H}$ -invariant closed convex subset of \mathfrak{a}_q . If

$$\mathcal{C}_{\Gamma_{P_0}^{W_{K \cap H}}} = \bigcup_{w \in W_{K \cap H}} \mathcal{C}_{\Gamma_{P_0}^w} \quad (4.6.3)$$

then

$$\bigcap_{w \in W_{K \cap H}} B + \Gamma_{P_0}^w = B + \Gamma_{P_0}^{W_{K \cap H}} \quad (4.6.4)$$

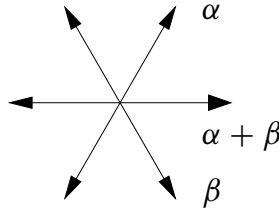
by Lemma 3.8.2. If $W = W_{K \cap H}$, then (4.6.3) holds. (See Lemma 3.9.1.) In general there exists a compact subset B' of \mathfrak{a}_q such that

$$B + \Gamma_{P_0}^{W_{K \cap H}} \subseteq \bigcap_{w \in W_{K \cap H}} B + \Gamma_{P_0}^w \subseteq B' + \Gamma_{P_0}^{W_{K \cap H}}$$

(see Lemma 4.4.2), but if (4.6.3) does not hold, then (4.6.4) is not necessarily true. The following is a counterexample.

Let G be the universal covering group of $SL(3, \mathbf{R})$. Let θ be a Cartan involution for G and $G = KAN$ an Iwasawa decomposition such that $G^\theta = K$. The root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ is of type A_2 . Let $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ be a system of positive roots and let α and β be the simple roots in that system. Then

$$\Sigma^+(\mathfrak{g}, \mathfrak{a}) = \{\alpha, \beta, \alpha + \beta\}.$$



Let $\epsilon : \Sigma(\mathfrak{g}, \mathfrak{a}) \rightarrow \{\pm 1\}$ be given by

$$\epsilon(\pm\alpha) = \epsilon(\pm\beta) = -1 \quad \text{and} \quad \epsilon(\pm(\alpha + \beta)) = 1.$$

Let $\theta_\epsilon : \mathfrak{g} \rightarrow \mathfrak{g}$ be the Lie algebra involution given by

$$\theta_\epsilon(Y) = \begin{cases} -Y & (Y \in \mathfrak{a}) \\ \epsilon(\gamma)\theta(Y) & (Y \in \mathfrak{g}_\gamma, \gamma \in \Sigma(\mathfrak{g}, \mathfrak{a})). \end{cases}$$

Since G is simply connected, θ_ϵ lifts to a Lie group involution of G , which we also denote by θ_ϵ . Let $K_\epsilon = G^{\theta_\epsilon}$ and let $X = G/K_\epsilon$. We claim that (4.6.4) does not hold for every compact convex Weyl-group invariant subset B of \mathfrak{a}_q in this case.

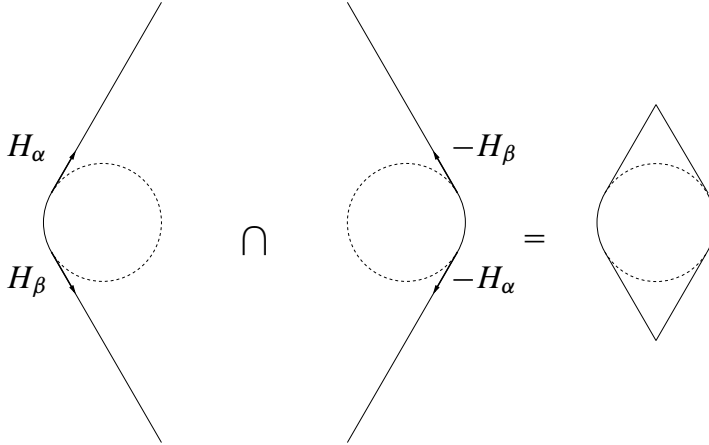
The group $W_{K \cap K_\epsilon}$ equals the Weyl group for the root system $\Sigma_+(\mathfrak{g}^{\theta \circ \theta_\epsilon}, \mathfrak{a}) = \{\pm(\alpha + \beta)\}$. The reflection s in $\alpha + \beta$ maps α to $-\beta$ and β to $-\alpha$. Let P be the minimal parabolic subgroup of G such that $A_P = A$ and $\Sigma^+(\mathfrak{g}, \mathfrak{a}; P) = \{\alpha, \beta, \alpha + \beta\}$. Then P is $\theta_\epsilon \circ \theta$ -stable and $P^s = \overline{P}$. Therefore,

$$\Gamma_P^{W_{K \cap K_\epsilon}} = \Gamma_P \cap \Gamma_{\overline{P}} = \{0\}.$$

Let B be the closed ball in \mathfrak{a}_q with radius r , centered at the origin. The angle between the root vectors H_α and $H_{\alpha+\beta}$ equals the angle between H_β and $H_{\alpha+\beta}$; both are equal to $\frac{\pi}{3}$. Let v be a vector perpendicular to $H_{\alpha+\beta}$ and with length r , then a straightforward calculation shows that

$$(B + \Gamma_P) \cap (B + \Gamma_{\overline{P}}) = \text{ch}(B \cup \{\pm 2v\}).$$

In pictures:



Chapter 5

Support theorems

The support theorem (Theorem 4.6.4) for the horospherical transform for functions can be generalized to a support theorem for the Radon transform \mathcal{R}_P corresponding to a (possibly non-minimal) $\sigma \circ \theta$ -stable parabolic subgroup P for distributions in a suitable subspace of the distribution space $\mathcal{E}'_b(X)$. In Section 5.1 we describe the spaces of distributions needed to formulate the support theorem in Section 5.2. The support theorem implies injectivity of the Radon transform on these spaces of distributions. In Section 5.3 we discuss some implications of this for generalizing the support theorem to even larger spaces of distributions. In Section 5.4 some final remarks are made.

Throughout this chapter P is assumed to be a $\sigma \circ \theta$ -stable parabolic subgroup of G containing A .

5.1 Spaces of distributions

We define the convolution product $\theta * \phi$ of $\theta \in \mathcal{D}(G)$ and $\phi \in \mathcal{E}_b(X)$ to be the function on X given by

$$\theta * \phi(x) = \int_G \theta(g) \phi(g^{-1} \cdot x) dg \quad (x \in X).$$

Since the left-regular representation of G on the Fréchet space $\mathcal{E}_b(X)$ is continuous, it follows from standard representation theory that convolution with a compactly supported smooth function θ on G defines a continuous operator from $\mathcal{E}_b(X)$ to itself.

For $\theta \in \mathcal{D}(G)$ we define $\check{\theta}$ to be the compactly supported smooth function given by

$$\check{\theta}(g) = \theta(g^{-1}) \quad (g \in G).$$

Since

$$\langle \theta * \chi, \phi \rangle = \langle \chi, \check{\theta} * \phi \rangle \quad (\theta \in \mathcal{D}(G), \chi \in \mathcal{D}(X), \phi \in \mathcal{E}_b(X)),$$

it makes sense to define the convolution product $\theta * \mu$ of a function $\theta \in \mathcal{D}(G)$ and a distribution $\mu \in \mathcal{E}'_b(X)$ by

$$\theta * \mu(\phi) = \mu(\check{\theta} * \phi) \quad (\phi \in \mathcal{E}_b(X)).$$

Convolution with a compactly supported smooth function θ on G defines a continuous map from the distribution space $\mathcal{E}'_b(X)$ to itself. Note that the distribution $\theta * \mu$, with $\theta \in \mathcal{D}(G)$ and $\mu \in \mathcal{E}'_b(X)$, defines a smooth function.

For $\psi \in \mathcal{E}_b(\mathcal{E}_P, J_P)$ and $\theta \in \mathcal{D}(G)$ we furthermore define the convolution product $\theta * \psi$ to be the function on \mathcal{E}_P given by

$$\theta * \psi(\xi) = \int_G \theta(g) \psi(g^{-1} \cdot \xi) dg.$$

Since the left regular representation of G on the Fréchet space $\mathcal{E}_b(\mathcal{E}_P, J_P)$ is continuous, it follows again from standard representation theory that convolution by a compactly supported smooth function on G defines a continuous map from the space $\mathcal{E}_b(\mathcal{E}_P, J_P)$ to itself.

Lemma 5.1.1. *Let $\theta \in \mathcal{D}(G)$, $\mu \in \mathcal{E}'_b(X)$ and $\psi \in \mathcal{E}_b(\mathcal{E}_P, J_P)$, then*

$$\mathcal{R}_P(\theta * \mu)(\psi) = \mathcal{R}_P \mu(\check{\theta} * \psi).$$

Proof. We denote the left regular representation on $\mathcal{E}_b(X)$ and $\mathcal{E}_b(\Xi_P, J_P)$ both by L . Using equivariance and continuity of \mathcal{S}_P , we obtain

$$\begin{aligned} \mathcal{R}_P(\theta * \mu)(\psi) &= (\theta * \mu)(\mathcal{S}_P \psi) = \mu(\check{\theta} * (\mathcal{S}_P \psi)) = \int_G \theta(g^{-1}) \mu(L_g \mathcal{S}_P \psi) dg \\ &= \int_G \theta(g^{-1}) \mathcal{R}_P \mu(L_g \psi) dg = \mathcal{R}_P \mu(\check{\theta} * \psi). \end{aligned}$$

□

Let $\Gamma_P^{W_{K \cap H}}$ be the maximal $W_{K \cap H}$ -invariant subcone of Γ_P and let $\mathcal{E}_P^1(X)$ be the subspace of $\mathcal{E}^1(X)$ given by

$$\mathcal{E}_P^1(X) = \mathcal{S}(X) + \bigcup_{\substack{B \subset \mathfrak{a}_q \\ B \text{ compact}}} \mathcal{E}^1(X; B + \Gamma_P^{W_{K \cap H}}).$$

For $P \in \mathcal{P}_\sigma(\mathfrak{a}_q)$ these definitions agree with the definitions given in the beginning of Section 4.4.

Proposition 5.1.2. *Let \mathcal{C} denote the (finite) collection of $P_0 \in \mathcal{P}_\sigma(\mathfrak{a}_q)$ contained in P . Then the space $\mathcal{E}_P^1(X)$ equals the intersection*

$$\mathcal{E}_P^1(X) = \bigcap_{P_0 \in \mathcal{C}} \mathcal{E}_{P_0}^1(X).$$

In particular, $\mathcal{E}_P^1(X)$ is invariant under the left action by G .

Proof. If $P_0 \in \mathcal{C}$ then $\Gamma_P^{W_{K \cap H}} \subseteq \Gamma_{P_0}^{W_{K \cap H}}$, hence $\mathcal{E}_P^1(X) \subseteq \mathcal{E}_{P_0}^1(X)$. It follows that $\mathcal{E}_P^1(X)$ is contained in the given intersection.

For the remaining inclusion, assume that

$$\phi \in \bigcup_{\substack{B \subset \mathfrak{a}_q \\ B \text{ compact}}} \mathcal{E}^1(X; B + \Gamma_{P_0}^{W_{K \cap H}})$$

for each $P_0 \in \mathcal{C}$. Then for every such P_0 there exists a compact subset B_{P_0} of \mathfrak{a}_q such that

$$\text{supp}(\phi) \subseteq X(B_{P_0} + \Gamma_{P_0}^{W_{K \cap H}}).$$

Let B be a $W_{K \cap H}$ -invariant compact subset of \mathfrak{a}_q containing the (finite) union of the sets B_{P_0} . Then

$$\text{supp}(\phi) \subseteq X \left(\bigcap_{P_0 \subseteq P} (B + \Gamma_{P_0}^{W_{K \cap H}}) \right) \subseteq X \left(\bigcap_{P_0 \subseteq P} \bigcap_{w \in W_{K \cap H}} (B + \Gamma_{P_0^w}) \right).$$

5. Support theorems

In view of Lemma 3.9.1 applied to P^w and \mathcal{C}^w it follows that

$$\text{supp}(\phi) \subseteq X \left(\bigcap_{w \in W_{K \cap H}} (B + \Gamma_{P^w}) \right).$$

According to Lemma 4.4.2 there exists a compact subset B' of $\mathfrak{a}_{\mathfrak{q}}$ such that the support of ϕ is contained in $X(B' + \Gamma_P^{W_{K \cap H}})$ and thus we conclude that $\phi \in \mathcal{E}_P^1(X)$.

The last assertion follows from the fact that each of the spaces $\mathcal{E}_{P_0}^1(X)$ is G -invariant by Proposition 4.4.4. \square

We define

$$\mathcal{V}_P(X) = \{\mu \in \mathcal{E}'_b(X) : \theta * \mu \in \mathcal{E}_P^1(X) \text{ for every } \theta \in \mathcal{D}(G)\}.$$

Proposition 5.1.3. *The space $\mathcal{V}_P(X)$ is a G -invariant subspace of $\mathcal{E}'_b(X)$. Furthermore, let \mathcal{C} be as in Proposition 5.1.2. Then*

$$\mathcal{V}_P(X) = \bigcap_{P_0 \in \mathcal{C}} \mathcal{V}_{P_0}(X).$$

Proof. Let $\mu \in \mathcal{V}_P(X)$ and let $g_0 \in G$. We will prove that $\theta * (l_{g_0}^* \mu) \in \mathcal{E}_P^1(X)$ for every $\theta \in \mathcal{D}(G)$. To do so, let $\theta \in \mathcal{D}(G)$. If $\phi \in \mathcal{E}_b(X)$, then by unimodularity of G

$$\begin{aligned} l_{g_0}^*(\check{\theta} * \phi) &= \int_G \theta(g^{-1}) l_{g^{-1}g_0}^* \phi \, dg \\ &= \int_G \theta(g_0^{-1}g^{-1}g_0) l_{g_0^{-1}g^{-1}}^* \phi \, dg = \check{\theta}^{g_0} * (l_{g_0}^* \phi), \end{aligned}$$

where θ^{g_0} is the compactly supported smooth function on G given by $\theta^{g_0}(g) = \theta(g_0^{-1}gg_0)$. Hence for every $\phi \in \mathcal{E}_b(X)$

$$\begin{aligned} (\theta * (l_{g_0}^* \mu))(\phi) &= \mu(l_{g_0}^*(\check{\theta} * \phi)) = \mu(\check{\theta}^{g_0} * (l_{g_0}^* \phi)) \\ &= (\theta^{g_0} * \mu)(l_{g_0}^* \phi) = l_{g_0}^*(\theta^{g_0} * \mu)(\phi). \end{aligned}$$

Since $\theta^{g_0} \in \mathcal{D}(G)$ and $\mu \in \mathcal{V}_P(X)$, we have $\theta^{g_0} * \mu \in \mathcal{E}_P^1(X)$. The latter space is G -invariant by Proposition 5.1.2. This proves the first statement of the proposition.

The second statement is a direct corollary of Proposition 5.1.2. \square

We finally define

$$\mathcal{V}(X) = \{\mu \in \mathcal{E}'_b(X) : \theta * \mu \in \mathcal{S}(X) \text{ for every } \theta \in \mathcal{D}(G)\}.$$

Let \mathcal{W}_P be a set of representatives in K for the double cosets in the double quotient $W_{M_P \cap K} \backslash W / W_{K \cap H}$.

Proposition 5.1.4. *The space $\mathcal{V}(X)$ equals the intersection*

$$\mathcal{V}(X) = \bigcap_{w \in \mathcal{W}_P} \mathcal{V}_{P^w}(X). \quad (5.1.1)$$

In particular, $\mathcal{V}(X)$ is a G -invariant subspace of $\mathcal{E}'_b(X)$.

Proof. It is clear that $\mathcal{V}(X)$ is contained in each of the spaces $\mathcal{V}_{P^w}(X)$. It remains to prove that intersection on the right-hand side of (5.1.1) is contained in the left-hand side. To do this it suffices to show that if B_w is a compact subset of \mathfrak{a}_q for $w \in \mathcal{W}_P$, then the intersection

$$\bigcap_{w \in \mathcal{W}_P} (B_w + \Gamma_{P^w}^{W_{K \cap H}}) \quad (5.1.2)$$

is compact.

Using that $M_P \cap K$ normalizes, we obtain

$$\bigcap_{w \in \mathcal{W}_P} \Gamma_{P^w}^{W_{K \cap H}} = \bigcap_{w'' \in W_{M_P \cap K}} \bigcap_{w \in \mathcal{W}_P} \bigcap_{w' \in W_{K \cap H}} \Gamma_{P^{w''ww'}} = \bigcap_{w \in W} \Gamma_{P^w} = \{0\}.$$

By Lemma 4.4.2, the intersection (5.1.2) is compact. This proves the first statement

The second statement is now a direct corollary of Proposition 5.1.3. \square

Remark 5.1.5. Note that the spaces $\mathcal{E}'(X)$ and $\mathcal{S}(X)$ are contained in both $\mathcal{V}_P(X)$ and $\mathcal{V}(X)$. Furthermore, the spaces $\mathcal{V}_P(X)$ and $\mathcal{V}(X)$ contain all integrable functions ϕ on X that are of rapid decay, i.e., the functions ϕ with the property that if C is a compact subset of G , then for every $n \in \mathbf{N}$

$$\sup_{x \in X} \|x\|^n \int_C |l_g^* \phi(x)| dg < \infty.$$

Here $\|\cdot\| : X \rightarrow \mathbf{R}$ denotes the function given by

$$\|ka \cdot x_0\| = e^{\|\log a\|} \quad (k \in K, a \in A_q).$$

The subspace of $L^1(X)$ consisting of the functions with support contained in $X(B + \Gamma_P^{W_{K \cap H}})$ for some compact subset B of \mathfrak{a}_q , is a subspace of $\mathcal{V}_P(X)$ as well.

5.2 Support theorems

Theorem 5.2.1 (Support Theorem). *Let B be a $W_{M_P \cap K \cap H}$ -invariant convex compact subset of \mathfrak{a}_q and let $\mu \in \mathcal{V}_P(X)$. If*

$$\text{supp}(\mathcal{R}_P \mu) \subseteq \Xi_P(B + \Gamma_P),$$

then

$$\text{supp}(\mu) \subseteq X \left(\bigcap_{w \in W_{K \cap H}} w \cdot (B + \Gamma_P) \right).$$

Remark 5.2.2. Note that if $P = P_0$ is a minimal $\sigma \circ \theta$ -stable parabolic subgroup, then any subset B of \mathfrak{a}_q is $W_{M_{P_0} \cap K \cap H}$ -invariant since M_{P_0} centralizes \mathfrak{a}_q .

If $P = G$, then $\mathcal{R}_P = \mathcal{R}_G$ equals the identity operator $\mathcal{V}_G(X) \rightarrow \mathcal{V}_G(X)$. In this case the support theorem reduces to the following tautology. Let B be a $W_{K \cap H}$ -invariant convex compact subset of \mathfrak{a}_q and let $\mu \in \mathcal{V}_P(X)$. Then

$$\text{supp}(\mu) \subseteq X(B) \implies \text{supp}(\mu) \subseteq X(B).$$

Proof for Theorem 5.2.1. First assume that $P = P_0$ is a minimal $\sigma \circ \theta$ -stable parabolic subgroup. Let B_U be a closed ball in \mathfrak{a} centered at 0 and let U be the subset $K \exp(B_U)K$ of G . Note that U is symmetric in the sense that $U^{-1} = U$. Let $\theta \in \mathcal{D}(G)$ and assume that $\text{supp}(\theta) \subseteq U$. If $\psi \in \mathcal{D}(\Xi_{P_0})$ satisfies

$$\text{supp}(\psi) \cap U \cdot \Xi_{P_0}(B + \Gamma_{P_0}) = \emptyset,$$

then the support of $\check{\theta} * \psi$ does not intersect $\Xi_{P_0}(B + \Gamma_{P_0})$ and thus we find

$$\mathcal{R}_{P_0}(\theta * \mu)(\psi) = \mathcal{R}_{P_0} \mu(\check{\theta} * \psi) = 0.$$

As this holds for all ψ as above,

$$\text{supp}(\mathcal{R}_{P_0}(\check{\theta} * \mu)) \subseteq U \cdot \Xi_{P_0}(B + \Gamma_{P_0}).$$

(Here we used that U is compact, so that the set on the right-hand side is closed.) Let $P_m \in \mathcal{P}(\mathfrak{a})$ be a minimal parabolic subgroup contained in P_0 and let $\mathfrak{A}_{KAN_{P_m}}$ be the map $G \rightarrow \mathfrak{a}$ as defined in (3.5.3). By Kostant's convexity theorem ([Kos73, Theorem 4.1]),

$$\mathfrak{A}_{KAN_{P_m}}(\exp(B_U)K) = B_U.$$

Using that N_{P_m} is contained in $(L_{P_0} \cap H)N_{P_0}$ (see Lemma 2.4.2), we now find that

$$\begin{aligned} U \cdot \Xi_{P_0}(B + \Gamma_{P_0}) &= K \exp(B_U) K \exp(B + \Gamma_{P_0}) \cdot \xi_{P_0} \\ &= K \exp(B_U + B + \Gamma_{P_0}) \cdot \xi_{P_0} \\ &= \Xi_{P_0}((B_U \cap \mathfrak{a}_q) + B + \Gamma_{P_0}). \end{aligned}$$

Note that $(B_U \cap \mathfrak{a}_q) + B$ is a compact convex subset of \mathfrak{a}_q , hence Theorem 4.6.4 can be applied and thus we conclude that

$$\text{supp}(\theta * \mu) \subseteq X \left(\bigcap_{w \in W_{K \cap H}} w \cdot ((B_U \cap \mathfrak{a}_q) + B + \Gamma_{P_0^w}) \right).$$

For each $j \in \mathbb{N}$, let B_j be the ball of radius $1/j$ and center 0 in \mathfrak{a} and let $U_j = K \exp(B_j) K$. Let $(\theta_j \in \mathcal{D}(G))_{j \in \mathbb{N}}$ be a sequence such that

$$\text{supp}(\theta_j) \subseteq U_j$$

and $\theta_j \rightarrow \delta$ in $\mathcal{E}'(G)$ (with respect to the weak topology) for $j \rightarrow \infty$. Since convolution is sequentially continuous with respect to each variable separately, the sequence $\theta_j * \mu$ converges to μ in $\mathcal{D}'(X)$ (with respect to the weak topology) for $j \rightarrow \infty$. Therefore

$$\text{supp}(\mu) \subseteq X \left(\bigcap_{w \in W_{K \cap H}} w \cdot ((B_{U_j} \cap \mathfrak{a}_q) + B + \Gamma_{P_0}) \right)$$

for every $j \in \mathbb{N}$, and we conclude that

$$\text{supp}(\mu) \subseteq X \left(\bigcap_{w \in W_{K \cap H}} w \cdot (B + \Gamma_{P_0}) \right). \quad (5.2.1)$$

This proves the theorem for minimal $\sigma \circ \theta$ -stable parabolic subgroups $P = P_0$.

We now assume P to be an arbitrary $\sigma \circ \theta$ -stable parabolic subgroup. Let $\mu \in \mathcal{V}_P(X)$. Assume the support of $\mathcal{R}_P \mu$ is contained in $\Xi_P(B + \Gamma_P)$. Let \mathcal{C} be the set of minimal $\sigma \circ \theta$ -stable parabolic subgroups $P_0 \in \mathcal{P}_\sigma(\mathfrak{a}_q)$ contained in P . Let $P_0 \in \mathcal{C}$. Proposition 5.1.3 implies that $\mu \in \mathcal{V}_{P_0}(X)$ and from Corollary 3.9.4 and Proposition 3.7.1 it follows that the support of $\mathcal{R}_{P_0} \mu$ is contained in $\Xi_{P_0}(B + \Gamma_{P_0})$. The previous result for minimal $\sigma \circ \theta$ -stable parabolic subgroups implies that (5.2.1) holds. Since this is true for each $P_0 \in \mathcal{C}$, it follows that

$$\text{supp}(\mu) \subseteq X \left(\bigcap_{P_0 \in \mathcal{C}} \bigcap_{w \in W_{K \cap H}} w \cdot (B + \Gamma_{P_0}) \right)$$

The theorem now follows by application of Lemma 3.9.1. □

Remark 5.2.3. With essentially the same proof, it is seen that the support theorem can be generalized to distributions $\mu \in \mathcal{E}'_b(X)$ for which there exist a sequence $(\theta_j \in \mathcal{D}(G))_{j \in \mathbb{N}}$ such that

- (i) $\text{supp}(\theta_j)$ is contained in B_j
- (ii) $\theta_j \rightarrow \delta$ in $\mathcal{E}'(G)$ for $j \rightarrow \infty$ (with respect to the weak topology).
- (iii) $\theta_j * \mu \in \mathcal{E}^1_P(X)$.

It is not clear to us whether the subset of these distributions forms a subspace of $\mathcal{E}'_b(X)$, nor are we certain that the set of these distributions is actually strictly larger than $\mathcal{V}_P(X)$.

Corollary 5.2.4. *Let $\mu \in \mathcal{V}(X)$, let B be a W -invariant convex compact subset of \mathfrak{a}_q and let $g \in G$. Then the following statements are equivalent.*

- (i) $\text{supp}(\mathcal{R}_{P^w} \mu) \subseteq g \cdot \mathcal{E}_{P^w}(B + \Gamma_{P^w})$ for every $w \in \mathcal{W}_P$.
- (ii) $\text{supp}(\mu) \subseteq g \cdot X(B)$.

Proof.

(i) \Rightarrow (ii): If $\text{supp}(\mathcal{R}_{P^w} \mu)$ is contained in $g \cdot \mathcal{E}_{P^w}(B + \Gamma_{P^w})$, then $\text{supp}(\mathcal{R}_{P^w}(l_g^* \mu))$ is contained in $\mathcal{E}_{P^w}(B + \Gamma_{P^w})$, hence

$$\text{supp}(l_g^* \mu) \subseteq X \left(\bigcap_{w' \in W_{K \cap H}} (B + \Gamma_{P^{ww'}}) \right)$$

by Theorem 5.2.1. Since this holds for all $w \in \mathcal{W}_P$, it follows that $\text{supp}(l_g^* \mu)$ is contained in

$$X \left(\bigcap_{w \in \mathcal{W}_P} \bigcap_{w' \in W_{K \cap H}} (B + \Gamma_{P^{ww'}}) \right).$$

Since P is stable under $W_{M_P \cap K}$, it follows that the last equals

$$X \left(\bigcap_{w \in W} (B + \Gamma_{P^w}) \right).$$

According to Proposition 3.8.2 the latter set equals $X(B)$. We thus obtain

$$\text{supp}(\mu) \subseteq g \cdot X(B).$$

(ii) \Rightarrow (i): This is a consequence of Corollary 3.9.4. □

If X is a Riemannian symmetric space (hence $H = K$) and $P = P_0$ is a minimal parabolic subgroup, Theorem 5.2.1 reduces to the support theorem [Hel73, Lemma 8.1] of Helgason for the horospherical transform on X . (See also Theorem 1.1, Corollary 1.2 and the subsequent Remark in chapter IV of [Hel94].) The support theorem can in this case be described in a purely geometrical setting as follows.

Suppose X is a Riemannian symmetric space. A horosphere in X is a closed submanifold of X by Proposition 3.2.2. Therefore the Riemannian structure on X induces a Riemannian structure and thus a measure on every horosphere. Let \mathcal{R} be the transform mapping a function $\phi \in \mathcal{S}(X)$ to the function on the set $\text{Hor}(X)$ of horospheres in X

$$\mathcal{R}\phi : \xi \mapsto \int_{x \in \xi} \phi(x) dx.$$

In this case $\text{Hor}(X)$ is in bijection with \mathcal{E}_{P_0} where P_0 is a minimal parabolic subgroup of the identity component G of the isometry group of X . In this way $\text{Hor}(X)$ can be given the structure of a G -manifold. Let $x \in X$. Using the Iwasawa decomposition for G , it is easily seen that the stabilizer G_x of x in G (i.e., the isotropy group of G at x) acts transitively on the set of horospheres containing x . Therefore this set carries a unique normalized G_x -invariant measure $d\xi$. The dual transform of \mathcal{R} is the transform \mathcal{S} mapping a function $\psi \in \mathcal{C}(X)$ to

$$\mathcal{S}\psi : x \mapsto \int_{\xi \ni x} \psi(\xi) d\xi.$$

The Radon transform \mathcal{R} is defined on $\mathcal{V}(X)$ to be the transpose of \mathcal{S} . Let $d(\cdot, \cdot)$ be the distance-function on X . For $x \in X$ and $R \geq 0$ we define $\beta_R(x) = \{\xi \in \text{Hor}(X) : d(\xi, x) \leq R\}$ and $B_R(x) = \{x' \in X : d(x', x) \leq R\}$.

Corollary 5.2.5 (Riemannian case; [Hel94, Ch. IV, Corollary 1.2]). *Let $\mu \in \mathcal{V}(X)$, $x \in X$ and $R \geq 0$. If*

$$\text{supp}(\mathcal{R}\mu) \subseteq \beta_R(x),$$

then

$$\text{supp}(\mu) \subseteq B_R(x).$$

Proof. Every closed ball in $X = G/K$ is of the form $g \cdot X(B)$, where g is an element of G and B is a closed ball in \mathfrak{a} . The statement is therefore a direct corollary of Proposition 3.9.5 and Corollary 5.2.4. \square

5.3 Injectivity

Theorem 5.2.1 has the following corollary.

Theorem 5.3.1. *The Radon transform*

$$\mathcal{R}_P : \mathcal{V}_P(X) \rightarrow \mathcal{E}'_b(\Xi_P, J_P)$$

is injective.

Remark 5.3.2. In [Krö09, Theorem 5.5] it is claimed that the horospherical transform is injective on a certain subspace of $\mathcal{S}(X)$. The proof for this theorem relies in an essential way on the assumption that $\text{Hor}(X)$ admits the structure of an analytic manifold and the horospherical transforms \mathcal{R}_{P^w} for $w \in \mathcal{W}$ together induce a transform \mathcal{R} on $\text{Hor}(X)$ with the property that the transform $\mathcal{R}\phi$ of a real analytic vector ϕ for the left-regular representation of G on $L^1(X)$ is a real analytic function on $\text{Hor}(X)$. As stated in Remark 3.4.1 we believe that there are some problems with this kind of reasoning.

A natural question is whether Theorem 5.2.1 can be generalized to a support theorem for a larger subspace of $\mathcal{D}'(X)$. If so, the Radon transform \mathcal{R}_P would be injective on that larger subspace as well. We will now show that the support theorem does not hold in general on the Harish-Chandra Schwartz spaces $\mathcal{C}^p(X)$ for $0 < p \leq 1$.

We will use the notations introduced in Chapter 4. Let $P_0 \in \mathcal{P}_\sigma(\mathfrak{a}_q)$ and let $0 < p < 1$.

Lemma 5.3.3. *Let $\zeta \in (\widehat{M}_{P_0})_H$ and let $1 < c < \frac{2}{p} - 1$ and $\eta \in V(\zeta)$. If $\phi \in \mathcal{C}^p(X)$ then for every $g \in G$ and $\lambda \in c\rho_{P_0} + i\mathfrak{a}_q^*$ the integral*

$$\int_X \phi(x) j(P_0 : \zeta : -\lambda)(\eta)(g \cdot x) dx$$

is absolutely convergent. If $\eta \in V(\zeta, e)$, then the integral is equal to

$$\int_{M_{P_0} \cap K} \mathcal{F}_{A_q} \left(a \mapsto a^{\rho_{P_0}} \mathcal{R}_{P_0} \phi(g^{-1}ma \cdot \xi_{P_0}) \right) (\lambda) \zeta(m) \eta dm. \quad (5.3.1)$$

Proof. Since $\mathcal{C}^p(X)$ is G -invariant, it suffices to prove the claim for $g = e$.

Let λ be as in the lemma. Let $P_m \in \mathcal{P}(\mathfrak{a})$ be contained in P_0 . If

$$kah = ma'nwh'$$

where $k \in K$, $a, a' \in A_q$, $h, h' \in H$, $m \in M_{P_0} \cap K$, $w \in W$ and $n' \in N_{P_0}$, then

$$w \cdot \log(a') = \pi_q \circ \mathfrak{A}_{KAN_{P_0^w}}(ahh'^{-1}).$$

The last is contained in $\text{ch}(W_{K \cap H} \cdot \log(a)) + \Gamma_{P_0^w}$ by the convexity theorem ([Ban86, Theorem 1.1]) of Van den Ban. Hence

$$\log(a') \in \text{ch}(W \cdot \log(a)) + \Gamma_{P_0}$$

and therefore

$$|(a')^{-\lambda + \rho_{P_0}}| \leq \max_{w \in W} (waw^{-1})^{(1-c)\rho_{P_0}}.$$

We define the function

$$J : A_q \rightarrow \mathbf{R}_{\geq 0}; \quad a \mapsto \prod_{\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a}_q; P_0)} |a^\alpha - a^{-\alpha}|^{m_\alpha^+} (a^\alpha + a^{-\alpha})^{m_\alpha^-},$$

where m_α^\pm is the dimension of the ± 1 -eigenspace for $\sigma \circ \theta$ in \mathfrak{g}_α . Since $\lambda \in \mathfrak{a}_q^*(\overline{P_0}, 0) + \rho_{P_0}$, it follows by [Ban88, Proposition 5.6] that $j(P_0 : \zeta : \lambda)$ is continuous. In view of [Sch84, p. 149], there exists a normalization of the measure on X such that

$$\begin{aligned} & \int_X \|\phi(x) j(P_0 : \zeta : -\lambda)(\eta)(x)\| dx \\ &= \int_K \int_{A_q} \|\phi((ka \cdot x_0)) j(P_0 : \zeta : -\lambda)(\eta)((ka \cdot x_0))\| J(a) da dk \\ &\leq \int_K \int_{A_q} |\phi(ka \cdot x_0)| J(a) \max_{w \in W} (waw^{-1})^{(1-c)\rho_{P_0}} da dk. \end{aligned}$$

Following Harish-Chandra, we use the notation \mathcal{E} for the elementary spherical function on G with spectral parameter 0, and we put

$$\Theta : X \rightarrow \mathbf{R}_{>0}; \quad x \mapsto \sqrt{\mathcal{E}(x\sigma(x)^{-1})}.$$

By [Ban92, Theorem 17.1] there exists a constant $C > 0$ such that

$$|\phi| \leq C \Theta^{\frac{2}{p}}.$$

Furthermore, by [Ban92, Corollary 17.6] it follows that for sufficiently small $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that

$$J(a) \leq C_\epsilon \Theta(ka \cdot x_0)^{-2-\epsilon}.$$

We infer that there exists a constant $\widetilde{C}_\epsilon > 0$ such that

$$\begin{aligned}
 & \int_X \|\phi(x)j(P_0 : \zeta : -\lambda)(\eta)(x)\| dx \\
 & \leq \widetilde{C}_\epsilon \int_K \int_{A_q} \Theta^{\frac{2}{p}-2-\epsilon}(ka \cdot x_0) \max_{w \in W_{K \cap H}} (waw^{-1})^{-(1-c)\rho_{P_0}} da dk \\
 & \leq \widetilde{C}_\epsilon |W| \int_{A_q^+(P_0)} \Theta^{\frac{2}{p}-2-\epsilon}(a \cdot x_0) a^{-(1-c)\rho_{P_0}} da.
 \end{aligned}$$

By [Ban92, Corollary 17.6], for every $\delta > 0$ there exists a constant $c_\delta > 0$ such that

$$\Theta(a \cdot x_0) \leq c_\delta a^{(\delta-1)\rho_{P_0}} \quad (a \in A_q^+(P_0)),$$

hence for $\epsilon < \frac{2}{p} - 1 - c$ the last integral is convergent.

The claimed equality follows from equation (4.2.9). \square

Let ϑ be a finite subset of \widehat{K} and put $\tau = \tau_\vartheta$. For $x \in X$ and $\lambda \in \mathfrak{a}_{q,\mathbf{C}}^*$, let $E_{P_0}(\cdot : \lambda : x)$ denote the (unnormalized) τ -spherical Eisenstein integral defined in [BS97b, Section 2], i.e., the element of $\text{Hom}({}^\circ\mathcal{C}(\tau), V_\tau)$ given by

$$E_{P_0}(\psi_{f \otimes \eta} : \lambda : x)(k) = \int_K \langle f(l)(k), j(P_0 : \zeta : \bar{\lambda})(\eta)(l \cdot x) \rangle dl$$

for $\zeta \in (\widehat{M}_{P_0})_H$, $f \in C(K : \zeta : \tau)$, $\eta \in \overline{V(\zeta)}$ and $x \in X$.

Using the K -invariance of the measure on X , we obtain the following immediate corollary of Lemma 5.3.3.

Corollary 5.3.4. *Let $\zeta \in (\widehat{M}_{P_0})_H$, let $\eta \in \overline{V(\zeta)}$ and let $f \in C(K : \zeta : \tau)$. Furthermore, let $1 < c < \frac{2}{p} - 1$. Then the τ -spherical Eisenstein integral $E_{P_0}(\psi_{f \otimes \eta} : \cdot : x)$ is regular on $c\rho_{P_0} + i\mathfrak{a}_q^*$. Moreover, for every $\phi \in \mathcal{C}^p(X)_\vartheta$ the integral*

$$\int_X \int_K \varsigma(\phi)(x)(k) \overline{E_{P_0}(\psi_{f \otimes \eta} : -\bar{\lambda} : x)(k)} dk dx$$

is absolutely convergent for every $x \in X$ and equals

$$\int_X \phi(x) \langle j(P_0 : \zeta : -\lambda)(\eta)(x), f \rangle dx.$$

For $x \in X$ and $\lambda \in \mathfrak{a}_{\mathfrak{q}, \mathbb{C}}$, let $E_{\overline{P}_0}^\circ(\cdot : -\bar{\lambda} : x)$ be the normalized τ -spherical Eisenstein integral for the minimal $\sigma \circ \theta$ -stable parabolic subgroup \overline{P}_0 defined in [BS97b, Section 5], i.e., the element of $\text{Hom}({}^\circ\mathcal{C}(\tau), V_\tau)$ given by

$$E_{\overline{P}_0}^\circ(\psi_{f \otimes \eta} : \lambda : x) = E_{P_0}(\psi_{A(\overline{P}_0 : P_0 : \zeta : -\lambda)^{-1} f \otimes \eta} : \lambda : x). \quad (5.3.2)$$

For $r \in \mathbf{R}$ we define $C_r(X, \tau)$ to be the space of continuous functions $f : X \rightarrow V_\tau$ satisfying the identity

$$f(k \cdot x) = \tau(k)f(x) \quad (k \in K)$$

and the estimate

$$\sup_{k \in K, a \in A_{\mathfrak{q}}} e^{-r|\log a|} |f(ka \cdot x_0)| < \infty.$$

Let $R \in \mathbf{R}$ be such that $\rho_{P_0} \in \mathfrak{a}_{\mathfrak{q}}^*(\overline{P}_0, R)$ and let ω be a connected and bounded open subset of $\mathfrak{a}_{\mathfrak{q}}^*(\overline{P}_0, R)$ containing both 0 and ρ_{P_0} . Let $E_{P_0}^\circ(\lambda : x)^*$ be the dual of $E_{P_0}^\circ(\cdot : -\bar{\lambda} : x)$. Then, according to [BS99, Lemma 12.2] there exists an $r \in \mathbf{R}$ such that the ${}^\circ\mathcal{C}(\tau)$ -valued integral

$$\int_X E^\circ(-\bar{\lambda} : x)^* f(x) dx \quad (5.3.3)$$

is absolutely convergent for every $f \in C_r(X : \tau)$ and generic $\lambda \in \omega + i\mathfrak{a}_{\mathfrak{q}}^*$, and (5.3.3) depends meromorphically on λ in that region. Following [BS99, Section 12] we define the (normalized) τ -spherical Fourier transform $\mathcal{F}_{\overline{P}_0, \tau} f(\lambda)$ of a function $f \in C_r(X : \tau)$ for generic $\lambda \in \omega + i\mathfrak{a}_{\mathfrak{q}}^*$ given by (5.3.3). This definition coincides with the definition for compactly supported smooth functions ϕ given in Section 4.3.

If $0 < p < 1$ is sufficiently small, then ς maps $\mathcal{C}^p(X)_\theta$ into $C_r(X : \tau)$. Fix such a p .

Proposition 5.3.5. *Let $\phi \in \mathcal{C}^p(X)_\theta$. Then*

$$\text{pr}_e \mathcal{F}_{\overline{P}_0, \tau} \varsigma(\phi)|_{i\mathfrak{a}_{\mathfrak{q}}^*} = 0 \quad \text{if and only if} \quad \mathcal{R}_{P_0} \phi = 0.$$

Proof. Fix a $1 < c < \frac{2}{p} - 1$ such that $c\rho_{P_0} + i\mathfrak{a}_{\mathfrak{q}}^* \subseteq \omega + i\mathfrak{a}_{\mathfrak{q}}^*$ and

$$\{A(\overline{P}_0 : P_0 : \zeta : \bar{\lambda})^{-1} : \lambda \in c\rho_{P_0} + i\mathfrak{a}_{\mathfrak{q}}^*\}$$

is a regular family of bijective operators. As the τ -spherical Fourier transform $\mathcal{F}_{\overline{P}_0, \tau} \phi(\lambda)$ of ϕ depends meromorphically on λ , it follows that the restriction to $i\mathfrak{a}_q^*$ of the first component of $\mathcal{F}_{\overline{P}_0, \tau} \phi$ vanishes if and only if it vanishes on $c\rho_{P_0} + i\mathfrak{a}_q^*$. From (5.3.2) and Corollary 5.3.4 it follows that the latter is the case if and only if (5.3.1) vanishes for every $g \in G$, $\lambda \in \mathfrak{a}_{q, \mathbb{C}}^*$, $\zeta \in (\widehat{M}_{P_0})_H$ and $\eta \in V(\zeta, e)$. We will now show that the last is equivalent to $\mathcal{R}_{P_0} \phi = 0$. For this we observe that (5.3.1) is basically equal to the Fourier transform on the compact homogeneous space $M_{P_0}/(M_{P_0} \cap H)$ applied to the Euclidean Fourier transform on A_q of the horospherical transform of ϕ .

For fixed $\lambda \in c\rho_{P_0} + i\mathfrak{a}_q^*$, the function $(M_{P_0} \cap K)/(M_{P_0} \cap K \cap H) \rightarrow \mathbb{C}$;

$$m \cdot (M_{P_0} \cap K \cap H) \mapsto \mathcal{F}_{A_q} \left(A_q \ni a \mapsto a^{\rho_{P_0}} \mathcal{R}_{P_0} \phi(ma \cdot \xi_{P_0}) \right) (\lambda)$$

is continuous and hence square integrable. Let $P_m \in \mathcal{P}(\mathfrak{a})$ be a minimal parabolic subgroup contained in P_m . The Iwasawa decomposition of M_{P_0} equals

$$M_{P_0} = (M_{P_0} \cap K)(M_{P_0} \cap A)(N_{P_m} \cap H)$$

and $(M_{P_0} \cap A)$ is contained in $A \cap H$, the pull-back along the equivariant diffeomorphism

$$\begin{aligned} (M_{P_0} \cap K)/(M_{P_0} \cap K \cap H) &\rightarrow M_{P_0}/(M_{P_0} \cap H); \\ m \cdot (M_{P_0} \cap K \cap H) &\mapsto m \cdot (M_{P_0} \cap H) \end{aligned}$$

defines an isometric isomorphism (if the measures are suitably normalized) between the Hilbert spaces $L^2(M_{P_0}/(M_{P_0} \cap H))$ and $L^2((M_{P_0} \cap K)/(M_{P_0} \cap K \cap H))$. According to the Plancherel theorem for compact homogeneous spaces, the last vector space decomposes as a direct sum of finite dimensional irreducible $M_{P_0} \cap K$ -subspaces and hence the same holds for the first. Since $\mathfrak{m}_{P_0} \cap \mathfrak{p}$ is contained in \mathfrak{h} , the elements $\exp(Y)$, with $Y \in \mathfrak{m}_{P_0} \cap \mathfrak{p}$, act trivially on each of these subspaces, which are therefore in fact irreducible M_{P_0} -subspaces. The corresponding Fourier transform on $L^2(M_{P_0}/(M_{P_0} \cap H))$ is injective as the one on $(M_{P_0} \cap K)/(M_{P_0} \cap K \cap H)$ is injective. Furthermore, the Euclidean Fourier transform \mathcal{F}_{A_q} is injective on $c\rho_{P_0} + i\mathfrak{a}_q^*$. It now follows that $\mathcal{R}_{P_0} \phi = 0$ if and only if the restriction of $\text{pr}_e \mathcal{F}_{\overline{P}_0, \tau} \phi$ to $i\mathfrak{a}_q^*$ vanishes. \square

By the Plancherel decomposition ([Del98, Théorème 3] and [BS05, Theorem 23.1]) the kernel of $\mathcal{F}_{\overline{P}_0, \tau}$ is non-trivial if and only if there are discrete series or

intermediate series of representations present. If this kernel has a non-trivial intersection with $\mathcal{C}^p(X)_\vartheta$, then the kernel of \mathcal{R}_{P_0} in $\mathcal{C}^p(X)$ is non-trivial and the support theorem cannot hold on this space. The estimates in the proof of [FJ80, Theorem 4.8] together with [Ban87, Theorem 7.3] show that this is in particular the case if the rank condition

$$\text{rank}(G/H) = \text{rank}(K/(K \cap H)) \quad (5.3.4)$$

is satisfied, because in that case there exists a finite subset ϑ of \widehat{K} such that the subspace of $\mathcal{C}^p(X)_\vartheta$ corresponding to the discrete series of representations is non-trivial. We thus obtain the following proposition.

Proposition 5.3.6. *Let $0 < p < 1$. Assume that (5.3.4) holds. Then the horospherical transform \mathcal{R}_{P_0} is not injective on $\mathcal{C}^p(X)$ and the support theorem for the horospherical transform for functions (Theorem 4.6.4) is not valid with $\mathcal{E}_{P_0}^1(X)$ replaced by $\mathcal{C}^p(X)$.*

A similar result, but with a different proof, can be found in [Krö09, Theorems 4.1 and 4.2].

Finally, we note that Theorem 5.3.1 and Proposition 5.3.5 imply that $\text{pr}_e \mathcal{F}_{\overline{P_0}, \tau}$ is injective on $\mathcal{S}(X)_\vartheta$. However, the following stronger result already follows from the inversion formula (Theorem 4.5.1), together with (4.3.2) and the equivariance of $\mathcal{F}_{P_0, e}^{\text{un}}$.

Theorem 5.3.7. *Assume that P_0 is a minimal $\sigma \circ \theta$ -stable parabolic subgroup, ϑ is a finite subset of K -types and $\tau = \tau_\vartheta$. Then the map*

$$\mathcal{E}_{P_0}^1(X)_\vartheta \ni \phi \mapsto \mathcal{F}_{\overline{P_0}, \tau, e} \phi$$

is injective.

This theorem might be a consequence of the symmetry relations between the components of the Fourier transform. (See [BS97c, Section 16].) It is however not clear to us how to prove this result using only those symmetries.

5.4 Further remarks

Theorem 5.2.1 partially generalizes [Qui93, Theorem 4.1] and [GQ94, Theorem 4.1] to a larger class of distributions and to non-Riemannian reductive symmetric

spaces. However, given the support of the Radon transform of a compactly supported distribution, the micro-local techniques used by Gonzales and Quinto allow to give more precise statements about the support of that distribution. It is essential for their arguments that $\mathcal{S}_P \circ \mathcal{R}_P$ is defined on the space of functions or distributions under consideration. In the setting of a non-Riemannian symmetric space, $\mathcal{S}_P \circ \mathcal{R}_P$ is in general not defined on the space of compactly supported smooth functions. Therefore, the methods described in those articles cannot straightforwardly be applied to non-Riemannian symmetric spaces.

Appendix A: Transversality

In this appendix we show that the subgroups $(L_Q \cap H)N_Q$ and $(L_P \cap H)N_P$ of G that play an important role in the double fibration (3.2.1) are transversal.

Throughout this appendix we assume that P and Q are $\sigma \circ \theta$ -stable parabolic subgroups such that $A \subseteq P \subseteq Q$. Then $\mathfrak{a}_Q \subseteq \mathfrak{a}_P \subseteq \mathfrak{a}$.

Lemma A.1.

$$\mathcal{N}_{K \cap H}(\mathfrak{l}_P \cap \mathfrak{q}) = \mathcal{N}_{K \cap H}(\mathfrak{l}_P) = \mathcal{N}_{K \cap H}(\mathfrak{a}_P) = \mathcal{N}_{K \cap H}(\mathfrak{a}_P \cap \mathfrak{a}_q)$$

Proof. We will first show that

$$\mathcal{Z}_{\mathfrak{g}}(\mathfrak{l}_P \cap \mathfrak{q}) \cap \mathfrak{p} \cap \mathfrak{q} = \mathfrak{a}_P \cap \mathfrak{a}_q. \quad (\text{A.1})$$

Clearly the set on the right-hand side is contained in the set on the left-hand side. Conversely, as $\mathfrak{l}_P \cap \mathfrak{q}$ contains \mathfrak{a}_q , it follows that

$$\mathcal{Z}_{\mathfrak{g}}(\mathfrak{l}_P \cap \mathfrak{q}) \cap \mathfrak{p} \cap \mathfrak{q} \subseteq \mathcal{Z}_{\mathfrak{g}}(\mathfrak{a}_q) \cap \mathfrak{p} \cap \mathfrak{q} = \mathfrak{a}_q.$$

Here we have used that \mathfrak{a}_q is maximal abelian in $\mathfrak{p} \cap \mathfrak{q}$. Let $Y \in \mathfrak{a}_q$ and assume that Y centralizes $\mathfrak{l}_P \cap \mathfrak{q}$. Since $\mathfrak{a}_q \subset \mathfrak{l}_P$ it follows that Y normalizes \mathfrak{l}_P , hence also the B -orthocomplement of $\mathfrak{l}_P \cap \mathfrak{q}$ in \mathfrak{l}_P , which is $\mathfrak{l}_P \cap \mathfrak{h}$. On the other hand, $[\mathfrak{a}_q, \mathfrak{l}_P \cap \mathfrak{h}] \subseteq [\mathfrak{q}, \mathfrak{h}] \subseteq \mathfrak{q}$, and we see that \mathfrak{a}_q centralizes $\mathfrak{l}_P \cap \mathfrak{h}$. It follows that Y centralizes \mathfrak{l}_P , hence belongs to $\mathfrak{a}_P \cap \mathfrak{a}_q$. This establishes (A.1)

For the proof of the actual lemma, assume that $k \in K \cap H$. Then k normalizes both \mathfrak{p} and \mathfrak{q} . Now assume that k normalizes $\mathfrak{l}_P \cap \mathfrak{q}$. Then k also normalizes $\mathcal{Z}_{\mathfrak{g}}(\mathfrak{l} \cap \mathfrak{q})$, hence also $\mathcal{Z}_{\mathfrak{g}}(\mathfrak{l} \cap \mathfrak{q}) \cap \mathfrak{p} \cap \mathfrak{q} = \mathfrak{a}_P \cap \mathfrak{a}_q$. If k normalizes $\mathfrak{a}_P \cap \mathfrak{a}_q$, then it also normalizes the centralizer of this space in \mathfrak{g} , which is \mathfrak{l}_P . If k normalizes \mathfrak{l}_P , then it also normalizes the center of \mathfrak{l}_P , hence also the \mathfrak{p} -part of the center of \mathfrak{l}_P , which is \mathfrak{a}_P . If k normalizes \mathfrak{a}_P then also $\mathfrak{a}_P \cap \mathfrak{q} = \mathfrak{a}_P \cap \mathfrak{a}_q$. Finally, if k normalizes \mathfrak{l}_P then also $\mathfrak{l}_P \cap \mathfrak{q}$. The lemma follows. \square

Proposition A.2. *The stabilizer in G of $N_P^Q \cdot \xi_Q$ equals $(L_P \cap H)N_P$.*

Proof. It is clear that $(L_P \cap H)N_P$ stabilizes $N_P^Q \cdot \xi_Q$, hence it remains to prove that the stabilizer is contained in $(L_P \cap H)N_P$.

Assume that

$$gN_P^Q \cdot \xi_Q = N_P^Q \cdot \xi_Q.$$

Then

$$gN_P(L_Q \cap H) = N_P(L_Q \cap H),$$

hence

$$gN_P H = N_P H.$$

This implies that $g = nh$, with $n \in N_P$ and $h \in H$ satisfying

$$hN_P H = N_P H.$$

We will finish the proof by showing that $h \in L_P \cap H$. We first note that $N_P H$ is submanifold of G containing e (see Corollary 3.2.3) and that

$$hN_P h^{-1} \subseteq N_P H.$$

Differentiating at e and using that $\text{Ad}(h)$ is a linear isomorphism mapping \mathfrak{h} onto itself, we see that

$$\text{Ad}(h)(\mathfrak{n}_P \oplus \mathfrak{h}) = \mathfrak{n}_P \oplus \mathfrak{h}.$$

Note that \mathfrak{g} decomposes as $\mathfrak{g} = (\mathfrak{h} \oplus \mathfrak{n}_P) \oplus (\mathfrak{l}_P \cap \mathfrak{q})$. In fact, $\mathfrak{l}_P \cap \mathfrak{q}$ is the orthocomplement of $\mathfrak{h} \oplus \mathfrak{n}_P$ with respect to the non-degenerate bilinear form B . Therefore h normalizes $\mathfrak{l}_P \cap \mathfrak{q}$.

Write $h = k \exp Y$ with $k \in K$ and $Y \in \mathfrak{p}$. As H is θ -stable, it follows that $k \in K \cap H$ and $Y \in \mathfrak{p} \cap \mathfrak{h}$. Moreover, since $\mathfrak{l}_P \cap \mathfrak{q}$ is θ -stable, so is the normalizer of this space in G and it follows that both k and Y normalize $\mathfrak{l}_P \cap \mathfrak{q}$. We will finish the proof by showing that both $\exp Y$ and k belong to $L_P \cap H$.

We may write $Y = Y_0 + (U + \sigma(U))$, with $Y_0 \in \mathfrak{h} \cap \mathfrak{l}_P$ and $U \in \mathfrak{n}_P$. Since Y_0 normalizes $\mathfrak{l}_P \cap \mathfrak{q}$, we see that $U + \sigma(U)$ normalizes $\mathfrak{l}_P \cap \mathfrak{q}$. Let $Z \in \mathfrak{l}_P \cap \mathfrak{q}$. Then $[U, Z] \in \mathfrak{n}_P$ and $[\sigma(U), Z] \in \bar{\mathfrak{n}}_P$, so $[U + \sigma(U), Z] \in \mathfrak{l}_P \cap (\mathfrak{n}_P \oplus \bar{\mathfrak{n}}_P) = \{0\}$. Thus, we see that $U + \sigma(U)$ centralizes $\mathfrak{l}_P \cap \mathfrak{q}$. In particular, $U + \sigma(U)$ centralizes \mathfrak{a}_P , which in turn implies that $U = 0$. We now see that $Y \in \mathfrak{l}_P \cap \mathfrak{h} \cap \mathfrak{p}$.

In particular, it follows that $\exp Y \in L_P \cap H$ so that it remains to prove the same statement about k .

Since both h and $\exp Y$ stabilize $N_P \cdot x_0$ it follows $k = h(\exp Y)^{-1}$ stabilizes $N_P \cdot x_0$ as well. We thus obtain

$$N_P H = k N_P H = k N_P k^{-1} H = k N_P H k^{-1}.$$

Hence, the closed submanifold $N_P H$ of G is stable under conjugation by k . It follows that its tangent space $\mathfrak{n}_P \oplus \mathfrak{h}$ at e is stable under $\text{Ad}(k)$ and therefore that the B -orthocomplement of this tangent space, $\mathfrak{l}_P \cap \mathfrak{q}$, is stable under $\text{Ad}(k)$ as well.

As $k \in \mathcal{N}_{K \cap H}(\mathfrak{l}_P \cap \mathfrak{q})$, it follows from Lemma A.1 that $k \in \mathcal{N}_{K \cap H}(\mathfrak{a}_P)$. We will show that, in fact, $k \in \mathcal{Z}_{K \cap H}(\mathfrak{a}_P)$. Aiming at a contradiction, assume this not to be the case. The group $k P k^{-1}$ is a $\sigma \circ \theta$ -stable parabolic subgroup of G with split component \mathfrak{a}_P . By [Kna02, Proposition 7.86] there exists an $\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a}; P)$ such that $-\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a}; k P k^{-1})$. Fix $Y_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$. Then $\theta Y_\alpha \in \mathfrak{g}_{-\alpha} \subset \text{Ad}(k) \mathfrak{n}_P$, hence $[Y_\alpha, \theta Y_\alpha] \subset \mathfrak{n}_P \oplus \mathfrak{h}$. This implies that the orthogonal projection of $[Y_\alpha, \theta Y_\alpha]$ onto $\mathfrak{a}_\mathfrak{q}$ is zero. On the other hand, the commutator $[Y_\alpha, \theta Y_\alpha]$ is an element of \mathfrak{a} and therefore for every $X \in \mathfrak{a}$

$$B([Y_\alpha, \theta Y_\alpha], X) = B(\theta Y_\alpha, [X, Y_\alpha]) = -\|Y_\alpha\|^2 \alpha(X) = -\|Y_\alpha\|^2 B(H_\alpha, X).$$

Here H_α is the element of $\mathfrak{a}_\mathfrak{q}$ given by (2.5.1). This implies

$$[Y_\alpha, \theta Y_\alpha] = -\|Y_\alpha\|^2 H_\alpha.$$

The orthogonal projection of H_α on $\mathfrak{a}_\mathfrak{q}$ is non-zero, which gives a contradiction. We conclude that $k \in \mathcal{Z}_{K \cap H}(\mathfrak{a}_P) = L_P \cap K \cap H$. \square

Proposition A.3. *The stabilizer of $(L_Q \cap H) \cdot \xi_P$ equals $(L_Q \cap H) N_Q$.*

Proof. Since $N_Q \subseteq N_P$ it is clear that $(L_Q \cap H) N_Q$ normalizes $(L_Q \cap H) \cdot \xi_P$. It remains to prove that the stabilizer is contained in $(L_Q \cap H) N_Q$.

Let $g \in G$ and suppose

$$g(L_Q \cap H) \cdot \xi_P = (L_Q \cap H) \cdot \xi_P.$$

Since $L_P \subseteq L_Q$, this implies

$$g(L_Q \cap H) N_P = (L_Q \cap H) N_P.$$

Hence there exist $l \in L_Q \cap H$ and $n \in N_P$ such that $g = ln$ and

$$n(L_Q \cap H) N_P = (L_Q \cap H) N_P.$$

We will finish the proof by showing that $n \in N_Q$. By Corollary 3.2.3, the set $(L_Q \cap H)N_P$ is a submanifold of G . Using that $n(L_Q \cap H)n^{-1}$ is contained in $(L_Q \cap H)N_P$, we find by differentiating at e

$$\text{Ad}(n)((l_Q \cap \mathfrak{h}) \oplus \mathfrak{n}_P) = (l_Q \cap \mathfrak{h}) \oplus \mathfrak{n}_P.$$

Here we used that $\text{Ad}(n)$ is a linear isomorphism of \mathfrak{g} mapping \mathfrak{n}_P onto itself. The orthocomplement of $(l_Q \cap \mathfrak{h}) \oplus \mathfrak{n}_P$ with respect to B equals $(l_P \cap \mathfrak{q}) \oplus \mathfrak{n}_Q$. Therefore n normalizes $(l_P \cap \mathfrak{q}) \oplus \mathfrak{n}_Q$. In particular

$$\text{Ad}(n)(\mathfrak{a}_P \cap \mathfrak{a}_Q) \subseteq (l_P \cap \mathfrak{q}) \oplus \mathfrak{n}_Q.$$

Since $n \in N_P$, we also have

$$\text{Ad}(n)(\mathfrak{a}_P \cap \mathfrak{a}_Q) \subseteq (\mathfrak{a}_P \cap \mathfrak{q}) \oplus \mathfrak{n}_P,$$

hence

$$\text{Ad}(n)(\mathfrak{a}_P \cap \mathfrak{a}_Q) \subseteq ((l_P \cap \mathfrak{q}) \oplus \mathfrak{n}_Q) \cap ((\mathfrak{a}_P \cap \mathfrak{q}) \oplus \mathfrak{n}_P) = (\mathfrak{a}_P \cap \mathfrak{a}_Q) \oplus \mathfrak{n}_Q. \quad (\text{A.2})$$

Fix a minimal $\sigma \circ \theta$ -stable parabolic subgroup $P_0 \in \mathcal{P}_\sigma(\mathfrak{a}_Q)$ contained in P . Let S be the collection of simple roots for the positive system $\Sigma^+ := \Sigma^+(\mathfrak{g}, \mathfrak{a}_Q; P_0)$. Let S_0 denote the set of roots $\alpha \in S$ which vanish on $\mathfrak{a}_Q \cap \mathfrak{q}$ and put $S_1 = S \setminus S_0$. Let S_{-1} be a finite subset of $\mathfrak{a}_Q^* \setminus S$ such that $S_{-1} \cup S$ is a basis for \mathfrak{a}_Q^* . Equip this basis with a total ordering $<$ such that $S_{-1} < S_0 < S_1$, and equip \mathfrak{a}_Q^* with the associated lexicographic ordering, also denoted $<$. Since Q is a $\sigma \circ \theta$ -stable parabolic subgroup containing P_0 , a root $\alpha \in \Sigma^+$ vanishes on $\mathfrak{a}_Q \cap \mathfrak{a}_Q$ if and only if it is a sum of simple roots from S_0 .

Thus if Σ_0^+ is the set of roots in Σ^+ vanishing on $\mathfrak{a}_Q \cap \mathfrak{a}_Q$ and Σ_1^+ its complement, then

$$\Sigma_0^+ < \Sigma_1^+.$$

Let

$$\log(n) = \sum_{\alpha \in \Sigma^+} Y_\alpha,$$

where $Y_\alpha \in \mathfrak{g}_\alpha$. Then $Y_\alpha \in \mathfrak{n}_P$ for all α . Indeed, if $\alpha|_{\mathfrak{a}_P \cap \mathfrak{a}_Q} = 0$, then $Y_\alpha = 0$. Let α_0 be the smallest root in Σ^+ such that $Y_{\alpha_0} \neq 0$. Then for every $Y \in \mathfrak{a}_P \cap \mathfrak{q}$

$$Y - \text{Ad}(n)Y = \sum_{\alpha \in \Sigma^+} \alpha(Y)Y_\alpha - \sum_{k=2}^{\infty} \frac{\text{ad}(\log n)^k Y}{k!}.$$

The sum on the right-hand side decomposes as a sum of terms $Z_\beta(Y) \in \mathfrak{g}_\beta$ with $\beta \in \Sigma^+$, $\beta \geq \alpha_0$. If Y is a P -regular element, then the lowest order part of the sum equals $Z_{\alpha_0}(Y) = \alpha_0(Y)Y_{\alpha_0}$ and is different from zero. From (A.2) it now follows that $Y_{\alpha_0} \in \mathfrak{n}_Q$, hence α_0 does not vanish on $\mathfrak{a}_Q \cap \mathfrak{a}_q$, so $\alpha_0 \in \Sigma_1^+$. This implies that any root $\alpha \in \Sigma^+$ with $\alpha \geq \alpha_0$ belongs to Σ_1^+ . Thus, if $Y_\alpha \neq 0$, then $\alpha \in \Sigma_1^+$ and hence $Y_\alpha \in \mathfrak{n}_Q$. We conclude that $n \in N_Q$. \square

Corollary A.4. *$(L_Q \cap H)N_Q$ and $(L_P \cap H)N_P$ are transversal.*

Appendix B: Hyperbolic Space

We use the general notation of Chapter 3.

Let X be the hyperboloid of one sheet $\{x \in \mathbf{R}^3 : x_1^2 + x_2^2 - x_3^2 = 1\}$ with the pseudo-Riemannian structure induced from the Minkowski-metric

$$\langle \cdot, \cdot \rangle : (x, y) \mapsto x_1 y_1 + x_2 y_2 - x_3 y_3$$

on \mathbf{R}^3 . The proper orthochronous Lorentz group $G = SO(2, 1)^0$ acts isometrically and transitively on X . Let $x_0 = (1, 0, 0) \in \mathbf{R}^3$. Then the stabilizer of x_0 equals

$$H = \begin{pmatrix} 1 & 0 \\ 0 & SO(1, 1)^0 \end{pmatrix}.$$

Therefore X equals the pseudo-Riemannian semisimple symmetric space G/H . Let

$$\begin{aligned} K &= \left\{ k_\varphi = \begin{pmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} : \varphi \in \mathbf{R} \right\}, \\ A &= \left\{ a_t = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} : t \in \mathbf{R} \right\} \\ N &= \left\{ n_v = \begin{pmatrix} 1 - \frac{v^2}{2} & -v & \frac{v^2}{2} \\ v & 1 & -v \\ -\frac{v^2}{2} & -v & 1 + \frac{v^2}{2} \end{pmatrix} : v \in \mathbf{R} \right\}. \end{aligned} \tag{B.1}$$

Because $\dim A = 1$, every parabolic subgroup P of G equals either G or a K -conjugate of the minimal parabolic subgroup AN . Note that the centralizer of \mathfrak{a} in K is trivial. Further note that the intersection of K and H is trivial.

B.1 The structure of $\text{Hor}(X)$

Let $P = AN$ and let $Q = k_{\frac{\pi}{2}}^{-1} N k_{\frac{\pi}{2}}$. Then $Q = HN_Q$, where $N_Q = k_{\frac{\pi}{2}}^{-1} N k_{\frac{\pi}{2}}$. (Note that $H = k_{\frac{\pi}{2}}^{-1} A k_{\frac{\pi}{2}}$, hence $K \times H \times N_Q \rightarrow G$ is an Iwasawa decomposition for G .) Let η_0 be the element $(1, 0, 1)$ in the forward light cone

$$\mathcal{C}^+ = \{\eta \in \mathbf{R}^3 : \langle \eta, \eta \rangle = 0, \eta_3 > 0\}.$$

There are four G -orbits in the set $\text{Hor}(X)$ of horospheres in X :

(i) $\text{Hor}_P(X)$ consisting of the G -translates of

$$N \cdot x_0 = \{(1 - \frac{v^2}{2}, v, -\frac{v^2}{2}) : v \in \mathbf{R}\} = \{x \in X : \langle \eta_0, x \rangle = 1\}.$$

(ii) $\text{Hor}_Q(X)$ consisting of the G -translates of the line

$$N k_{\frac{\pi}{2}} \cdot x_0 = \{(v, -1, v) : v \in \mathbf{R}\},$$

i.e. the G -translates of the connected component of $\{x \in X : \langle \eta_0, x \rangle = 0\}$ containing the point $(0, -1, 0)$.

(iii) $\text{Hor}_{\overline{P}}(X)$ consisting of the G -translates of

$$\begin{aligned} N k_{\pi} \cdot x_0 &= k_{\pi} \overline{N} \cdot x_0 = \{(-1 + \frac{v^2}{2}, -v, \frac{v^2}{2}) : v \in \mathbf{R}\} \\ &= \{x \in X : \langle \eta_0, x \rangle = -1\}. \end{aligned}$$

(iv) $\text{Hor}_{\overline{Q}}(X)$ consisting of the G -translates of the line

$$N k_{\frac{3\pi}{2}} \cdot x_0 = k_{\pi} \overline{N} k_{\frac{\pi}{2}} \cdot x_0 = \{(-v, 1, -v) : v \in \mathbf{R}\},$$

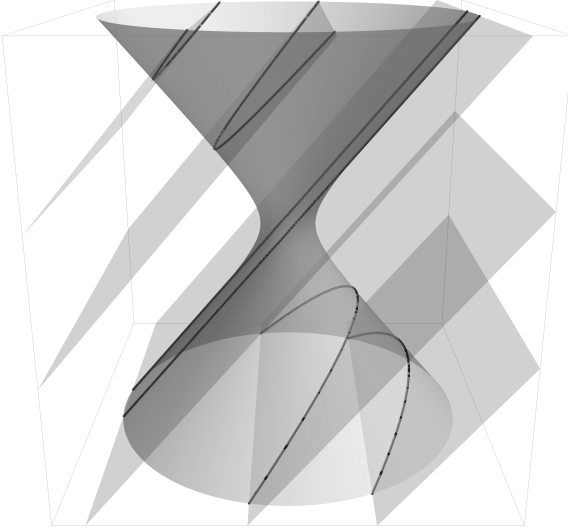
i.e. the G -translates of the connected component of $\{x \in X : \langle \eta_0, x \rangle = 0\}$ containing the point $(0, 1, 0)$.

$\text{Hor}_P(X)$ and $\text{Hor}_{\overline{P}}(X)$ are in bijection with $\mathcal{E}_P = G/N$ and $\mathcal{E}_{\overline{P}} = G/\overline{N}$ respectively via the maps

$$E_P : gN \cdot x_0 \mapsto g \cdot \xi_P \quad \text{and} \quad E_{\overline{P}} : g k_{\pi} \overline{N} \cdot x_0 \mapsto g k_{\pi} \cdot \xi_{\overline{P}}.$$

Similarly, $\text{Hor}_Q(X)$ and $\text{Hor}_{\overline{Q}}(X)$ are in bijection with G/HN_Q and $G/H\overline{N}_Q$ respectively, via the maps

$$E_Q : g N k_{\frac{\pi}{2}} \cdot x_0 \mapsto g k_{\frac{\pi}{2}} \cdot N_Q H \quad \text{and} \quad E_{\overline{Q}} : g N k_{\frac{3\pi}{2}} \cdot x_0 \mapsto g k_{\frac{3\pi}{2}} \cdot \overline{N}_Q H.$$



Hyperboloid of one sheet with 6 horospheres: two of Hor_P , two of $\text{Hor}_{\bar{P}}$, one of Hor_Q and one of $\text{Hor}_{\bar{Q}}$.

The diagonal action of $\mathbf{R}_{>0}$ on the real analytic manifold $\mathcal{C}^+ \times \mathbf{R}$ is free and proper. Therefore

$$\mathbf{R}_{>0} \backslash (\mathcal{C}^+ \times \mathbf{R})$$

has the structure of a real analytic manifold. We denote the cosets $\mathbf{R}_{>0} \cdot (\eta, p)$ by $[\eta, p]$.

Let $\mathcal{P}(X)$ be the set of intersections of X with planes of the form

$$\{x \in \mathbf{R}^3 : \langle \eta, x \rangle = p\}$$

where $\eta \in \mathcal{C}^+$ and $p \in \mathbf{R}$. The map Ψ from $\mathbf{R}_{>0} \backslash (\mathcal{C}^+ \times \mathbf{R})$ to $\mathcal{P}(X)$, given by

$$\Psi([\eta, p]) = \{x \in X : \langle \eta, x \rangle = p\}$$

is a bijection. Via this bijection $\mathcal{P}(X)$ inherits the structure of a real analytic manifold from $\mathbf{R}_{>0} \backslash (\mathcal{C}^+ \times \mathbf{R})$. The G -action on \mathcal{C}^+ induces an action on $\mathbf{R}_{>0} \backslash (\mathcal{C}^+ \times \mathbf{R})$ that is given by

$$g \cdot [\eta, p] = [g \cdot \eta, p].$$

Note that Ψ is G -equivariant with respect to this action and the natural action of G on $\mathcal{P}(X)$ and hence the last is continuous.

There are three G -orbits in $\mathcal{P}(X)$:

$$(i) \mathcal{P}_+(X) = \Psi\left(\mathbf{R}_{>0} \backslash (\mathcal{C}^+ \times \mathbf{R}_{>0})\right) = \text{Hor}_P(X),$$

$$(ii) \mathcal{P}_0(X) = \Psi\left(\mathbf{R}_{>0} \setminus (\mathcal{C}^+ \times \{0\})\right),$$

$$(iii) \mathcal{P}_-(X) = \Psi\left(\mathbf{R}_{>0} \setminus (\mathcal{C}^+ \times \mathbf{R}_{<0})\right) = \text{Hor}_{\overline{P}}(X).$$

Each element of (ii) equals the union of two horospheres: two straight lines, one from each of the two families of generating lines of X . Unlike $\mathcal{P}(X)$, there does not seem to be a natural way to provide $\text{Hor}(X)$ with the structure of a manifold since any reasonable topology on $\text{Hor}(X)$ is non-Hausdorff.

B.2 Incidence between subsets of X and horospheres

Let $t_0 \in \mathbf{R}$. Then $X(\{a_{t_0}\}) = K \cdot (\cosh t_0, 0, \sinh t_0)$. From

$$a_t n_v \cdot x_0 = \left(\cosh t - e^t \frac{v^2}{2}, v, \sinh t - e^t \frac{v^2}{2} \right) \quad (t \in \mathbf{R}, v \in \mathbf{R}),$$

and an elementary computation, we obtain the following result.

Proposition B.1. *Let $t_0 \in \mathbf{R}$. Then*

$$K a_t N \cdot x_0 \cap X(\{a_{t_0}\}) \neq \emptyset \quad \text{if and only if} \quad t \geq t_0.$$

This proposition has the following corollary.

Corollary B.2. *Let $t_0 \in \mathbf{R}$ and let $\phi \in \mathcal{E}^1(X)$. If*

$$\text{supp}(\phi) \subseteq X(\{a_t : t \in t_0 + \mathbf{R}_{\geq 0}\}),$$

then

$$\text{supp}(\mathcal{R}_P \phi) \subseteq \Xi_P(\{a_t : t \in t_0 + \mathbf{R}_{\geq 0}\}).$$

Moreover, if ϕ is non-negative and $\phi(k a_{t_0} \cdot x_0) > 0$ for every $k \in K$, then

$$\mathcal{R}_P \phi(k a_t \cdot \xi_P) > 0 \quad (k \in K, t > t_0).$$

Proof. Let $t < t_0$. By Proposition B.1 the horospheres parametrized by $k a_t \cdot \xi_P$, with $k \in K$, do not intersect with $X(\{a_t : t \in t_0 + \mathbf{R}_{\geq 0}\})$. Therefore these horospheres do not intersect with the support of ϕ , hence $\mathcal{R}_P \phi(k a_t \cdot \xi_P) = 0$ for every $k \in K$. This proves the first statement.

For the second statement, assume that ϕ is non-negative and $\phi(ka_{t_0} \cdot x_0) > 0$ for every $k \in K$. Let $t > t_0$. By Proposition B.1 there exist $n \in N$ such that $\phi(ka_t n \cdot x_0) > 0$. Therefore

$$\mathcal{R}_P \phi(ka_t \cdot \xi_P) = \int_N \phi(ka_t n \cdot x_0) dn > 0.$$

□

Note that the above corollary implies that the horospherical transform of a compactly support smooth function need not compactly supported.

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Index of notation

$*$	92	E_P	30	n_P	18
A	17	E_P^Q	30	N_P^Q	20
\mathfrak{a}	17	\mathcal{F}	62	n_P^Q	20
$A(\overline{P}_0 : P_0; \zeta : \lambda)$..	76	\mathcal{F}_{A_q}	68	$\Omega(P_0)$	66
\mathfrak{A}_{KAK}	37	$\mathcal{F}_{\overline{P}_0, \tau}$	76	\mathfrak{p}	16
\mathfrak{A}_{KAN}	36	$\mathcal{F}_{\overline{P}_0, \tau, e}$	77	$\mathcal{P}(\mathfrak{a})$	23
A_P	17	$\mathcal{F}_{P_0}^{\text{un}}$	71	$\mathcal{P}_\sigma(\mathfrak{a}_q)$	23
\mathfrak{A}_P	36	$\mathcal{F}_{P_0, e}^{\text{un}}$	71	P^g	25
\mathfrak{a}_P	17	$\Gamma(T)$	53	$\Phi_{\overline{P}_0, w}(\lambda, \cdot)$	84
A_q	22	Γ_P	53	\mathfrak{q}	20
\mathfrak{a}_q	22	$\Gamma_{P_0}^{W_{K \cap H}}$	78	\mathcal{R}	5
$\mathfrak{a}_q^*(P, R)$	25	H	20	ρ_P	33
$\mathfrak{a}_q^+(P)$	25	\mathfrak{h}	20	\mathcal{R}_P	35
B	25	H_α	53	\mathcal{R}_P^Q	35, 45
$\mathcal{C}(u)$	62	H_S	51	\mathcal{S}	5
$C^{-\infty}(P_0 : \zeta : \lambda)$..	66	$j(P_0 : \zeta : \lambda)$	66	$\mathcal{S}(X)$	78
${}^\circ\mathcal{C}(\tau)$	74	J_P	36	σ	20
${}^\circ\mathcal{C}(\tau)_w$	75	K	16	$\Sigma(\mathfrak{g}, \mathfrak{a})$	17
\mathcal{C}_S	51	\mathfrak{k}	16	$\Sigma(\mathfrak{g}, \mathfrak{a}_q)$	22
$C_b(\mathcal{E}_P, J_P)$	41	$L^1(\mathcal{E}_P, J_P)$	36	$\Sigma^+(\mathfrak{g}, \mathfrak{a})$	17
ch	51	$L^\infty(\mathcal{E}_P, J_P)$	40	$\Sigma^+(\mathfrak{g}, \mathfrak{a}_q)$	22
$\mathcal{D}, \mathcal{D}'$	4	L_P	17	$\Sigma_\pm^+(\mathfrak{g}, \mathfrak{a}_q; P)$	24
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$\mathcal{E}^1(\mathcal{E}_P, J_P)$	39	\mathfrak{m}_P	17	$V(\zeta, w)$	66
$\mathcal{E}_P^1(X)$	93	$(\widehat{M}_{P_0})_H$	66	V^∞	25
$\mathcal{E}_{P_0}^1(X)$	78	N_P	18	$\mathcal{V}_P(X)$	94
$\mathcal{E}_b(X)$	41			V_τ	74
$\mathcal{E}_b(\mathcal{E}_P, J_P)$	41				

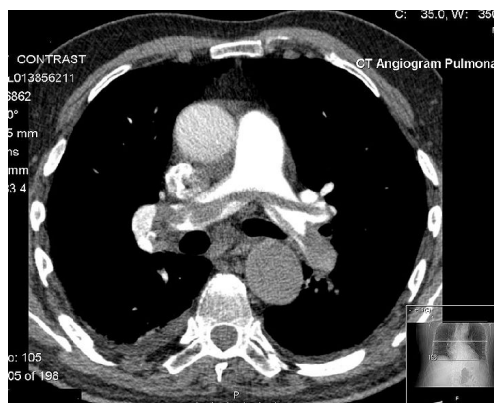
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Samenvatting voor niet-wiskundigen

In dit proefschrift wordt een zogenaamde dragerstelling voor een bepaalde klasse van Radontransformaties geformuleerd en voor deze stelling wordt een wiskundig bewijs gegeven. Het voert wat te ver om in deze samenvatting het bewijs te beschrijven, maar ik zal een poging doen om duidelijk te maken wat de stelling zegt.



Figuur 1: CT-scan van een lichaam

Allereerst zal ik uitleggen wat een Radontransformatie is. De lezer heeft ongetwijfeld wel eens gehoord van een computertomogram of CT-scan. Om een dergelijke scan te maken, worden een Röntgenbron en een Röntgendetector aan weerszijden van een object geplaatst. Vanaf de bron wordt een smalle bundel Röntgenstraling uitgezonden die door het object heengaat en daar gedeeltelijk geabsorbeerd wordt. De detector meet vervolgens de intensiteit van het restant van de straling.

Hiermee wordt bepaald hoeveel straling er geabsorbeerd is door het materiaal in het object op de lijn van de bron naar de detector. Door de bron en de detector te verplaatsen kan dit gedaan worden voor iedere lijn door het object. Op deze manier wordt aan iedere lijn een meetwaarde toegekend. De vraag is nu hoe uit deze data een weergave als in Figuur 1 kan worden verkregen.



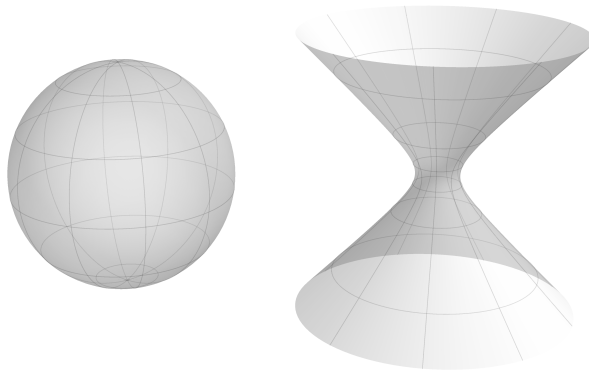
Figuur 2: Johann Radon

Ieder soort materie heeft een eigen Röntgendichtheid. Naarmate de Röntgendichtheid van materie groter is, absorbeert deze meer Röntgenstraling. In een voorstelling als Figuur 1 is de Röntgendichtheid weergegeven: een lage dichtheid correspondeert met zwart of donker grijs, een hoge dichtheid met licht grijs of wit. Wiskundig gezien wordt het getal, dat aan een lijn wordt toegekend door de absorptie te meten, verkregen door de Röntgendichtheid te integreren over die lijn; het is het totaal van de Röntgendichtheid over de lijn. De Röntgendichtheidsfunctie is de functie die aan ieder punt in de ruimte de bijbehorende dichtheid toekent. Deze functie wordt omgevormd tot een functie op de verzameling van lijnen door aan iedere lijn de integraal van de Röntgendichtheidsfunctie over die lijn toe te kennen. Deze omvorming is het schoolvoorbeeld van een Radontransformatie. De

data, die een CT-scanner levert, is in feite de Radontransformatie van de dichtheidsfunctie. Om een weergave als in Figuur 1 uit de meetdata te verkrijgen, moet de vraag worden beantwoord hoe een functie uit zijn Radongetransformeerde kan worden verkregen. Deze vraag werd al beantwoord ver voordat de eerste CT-scanner gemaakt werd, namelijk in 1917 door de Oostenrijkse wiskundige Johann Radon.

Een probleem dat minder relevant is voor de toepassing in de theorie van computertomografie, maar op zichzelf interessant is en toepassingen heeft in de wiskunde, is het volgende. Stel dat B een bol in de ruimte is en stel dat de Radongetransformeerde van een functie gelijk is aan 0 op alle lijnen die niet door de bol B gaan. De vraag is nu of uit deze informatie conclusies getrokken kunnen worden over het gebied waar de oorspronkelijke functie gelijk aan 0 is. Specifieker: kan geconcludeerd worden dat de oorspronkelijke functie buiten de bol gelijk is aan 0?

In de situatie die optreedt bij computertomografie, zoals hierboven beschreven, is het antwoord op deze vraag triviaal: als er straling uitgezonden wordt over een lijn die door materie gaat met een Röntgendichtheid die niet gelijk is aan 0, dan zal er straling geabsorbeerd worden. Als op een lijn geen absorptie wordt gemeten, dan kan derhalve op die lijn geen materie aanwezig zijn met een Röntgendichtheid ongelijk aan 0 en is de dichtheidsfunctie daar dan dus gelijk aan 0. De vraag is in deze situatie zo eenvoudig te beantwoorden omdat de functies onder beschouwing dichtheidsfuncties zijn en daarom nergens negatieve waarden aannemen. Als een functie ook negatieve waarden kan aannemen, dan wordt het een stuk moeilijker om de vraag te beantwoorden. De integraal van een dergelijke functie is de som van de bijdragen van de positieve waarden en de (negatieve) bijdragen van de negatieve waarden. Het totaal kan gelijk zijn aan 0 zonder dat de afzonderlijke delen gelijk zijn aan 0. Toch is het ook in dit geval zo dat de functie gelijk is aan 0 buiten de bol als zijn Radongetransformeerde gelijk is aan 0 op alle lijnen die de bol niet snijden; deze stelling werd bewezen door Sigurdur Helgason in 1965. Het gebied waar een functie ongelijk is aan 0 wordt de drager van die functie genoemd. De stelling doet een uitspraak over de drager van de functie in termen van de drager van de Radongetransformeerde functie en wordt daarom een dragerstelling genoemd.



Figuur 3: Links een bol, rechts een hyperboloïde

In dit proefschrift wordt een dragerstelling bewezen voor een andere Radon-transformatie dan degene die hierboven beschreven is. In de eerste plaats moet de “gewone” driedimensionale ruimte die wij om ons heen zien vervangen worden door een zogenaamde symmetrische ruimte. Zoals de naam al doet vermoeden is een symmetrische ruimte een object met veel symmetrieën. Voorbeelden van sym-

metrische ruimten zijn boloppervlakken en hyperboloïden.

Op een symmetrische ruimte kan meetkunde worden bedreven. In iedere symmetrische ruimte bestaat een bijzondere collectie van meetkundige objecten genaamd horosferen. In veel gevallen zijn deze horosferen interessante objecten. In plaats van lijnen in de “normale” 3-dimensionale ruimte, worden in dit proefschrift horosferen in een symmetrische ruimte beschouwd. De Radontransformatie neemt een functie, die aan ieder punt in de symmetrische ruimte een getal toekent, en vormt die om in de functie op de verzameling van horosferen, die aan iedere horosfeer de waarde van de integraal van de functie over die horosfeer toekent. Deze Radontransformatie wordt ook wel de horosferische transformatie genoemd.

Er is een belangrijke collectie van symmetrische ruimten waarop op een natuurlijke manier over afstanden gesproken kan worden. Dit zijn de zogenaamde Riemannse symmetrische ruimten. Niet alle symmetrische ruimten hebben echter een dergelijk natuurlijk afstandsbe­grip. In 1973 bewees Sigurdur Helgason een dragerstelling voor de horosferische transformatie op een Riemannse symmetrische ruimte. De stelling luidt als volgt. Stel B is een bol in een Riemannse symmetrische ruimte. Als de horosferische transformatie van een functie gelijk is aan 0 op iedere horosfeer die B niet raakt, dan is de oorspronkelijke functie gelijk aan 0 buiten de bol B . De dragerstelling, die in dit proefschrift wordt bewezen, is een generalisatie van deze stelling voor horosferische transformaties op symmetrische ruimten die niet noodzakelijk Riemanns zijn.

Dankwoord

De totstandkoming van dit proefschrift is niet alleen het werk van de auteur; integendeel. Velen hebben bewust of onbewust een bijdrage geleverd. Een aantal van hen wil ik op deze plaats in het bijzonder bedanken.

Allereerst en voornamelijk gaat mijn dank uit naar mijn promotor Erik van den Ban. De vele discussies met jou, Erik, je scherpe blik en het vele commentaar dat je geleverd hebt, hebben mij zeer geholpen. Het was voor mij een eer jou als leermeester te mogen hebben.

Ook mijn vroegere leermeester Joop Kolk ben ik dank verschuldigd. Dankzij de hulp van Joop en Erik heb ik deze promotieplaats kunnen krijgen. Joop, de vele gesprekken die wij gehad hebben kon ik altijd zeer waarderen en ik denk er met genoeg aan terug.

Toen Joop en Erik een subsidie van NWO toegewezen kregen voor het project dat geresulteerd heeft in dit proefschrift, bleek dat dit via NWO zou worden bekostigd door het Van Beuningen-Peterich Fonds, waarvoor ik bij deze mijn dank uitspreek. De afname van investeringen van de overheid in de wetenschap gedurende de afgelopen jaren, maakt dat promotieplaatsen die gefinancierd worden door particuliere instellingen zeer waardevol zijn. Ik hoop dat dit initiatief door velen gevolgd zal worden.

Het werken aan het Mathematisch Instituut in Utrecht is altijd plezierig geweest, mede dankzij mijn collega-AiO's. Hiervoor wil ik hen bedanken. Een aantal AiO's wil ik bij name noemen.

In mijn ochtendritueel kwam gewoonlijk, na het halen van koffie, een bezoek aan de kamer van Bart voor. Onder het genot van een koekje uit de trommel hebben wij het wel en wee van het AiO-schap en vele, vele andere zaken besproken. Ik heb nooit het gevoel gehad een nuttige bijdrage te kunnen leveren aan de wiskundige problemen waar Bart mee worstelt, maar hij heeft mij wel vaak kunnen helpen. De lezer kan een voorbeeld hiervan vinden in Remark 4.4.3. Bart, ik ben blij dat je hebt toegestemd om op te treden als mijn paranimf.

Het is niet altijd makkelijk om met anderen over je onderzoek te spreken, zelfs niet met wiskundigen. Nadat Vincent promoveerde in december 2009 en elders ging werken, merkte ik pas hoe fijn het is vragen te kunnen stellen aan iemand die dezelfde taal spreekt.

When I started as a PhD-student, I shared a room on the sixth floor of the Wiskunde Gebouw with Steven, Liesbeth, Bas and Taoufik. After some time we had to leave this office and I was moved to a different one at the fifth floor. This room I shared with Albert-Jan and Alexander. After Alexander had his defense and started working in Scotland, it was decided that we had to leave and Albert-Jan and I were moved to a room on the fourth floor, next to the one of Vincent and Charlene. A few months later the fourth floor had to be abandoned by the Mathematical institute. Hence it was decided that we had to leave our office and Albert-Jan and I were moved to the seventh floor. I would like to thank all my previous roommates for the great times I had with them in between moving.

Albert-Jan is met bijna 4 jaar het langst mijn kamergenoot geweest. Hij heeft mij in die tijd kennis laten maken met zijn favoriete Chinese restaurant en van hem heb ik geleerd dat de Bruhatdecompositie van $GL(n, \mathbf{R})$ in sommige kringen de LU -decompositie wordt genoemd (hoewel strikt gezien de laatste natuurlijk alleen gedefinieerd is op de open Bruhatcel en dus zeker niet bestaat voor alle inverteerbare matrices). AJ, onze werktijden hadden een niet al te grote overlap, maar het was altijd gezellig als je er was. Hiervoor mijn dank.

Ten slotte wil ik mijn vrienden en mijn familie bedanken voor hun steun de afgelopen jaren. Met name wil ik Nico noemen. Hoewel ik mij moeilijk kan voorstellen dat de werken van Aldegonde, Van den Vondel en Hooft mij ooit zullen gaan vervelen, was het mij toch altijd weer een groot genoegen om mijn boeken op hun plaats in mijn boekenkasten te laten staan en een avond bij Nico en Annet door te brengen. Het verheugt mij dan ook zeer dat Nico heeft toegezegd mij als paranimf bij te staan. Nico, mijn hartelijke dank daarvoor.

Niet alleen de afgelopen jaren, maar zo lang ik mij kan herinneren hebben mijn ouders mij gesteund en gemotiveerd. De wiskunde in het voorgaande gedeelte mag jullie niet veel zeggen, maar aan jullie heb ik het in hoge mate te danken dat ik dit proefschrift heb kunnen schrijven.

Curriculum Vitae

Job Kuit werd op 13 januari 1983 geboren te Ede. Zijn Atheneumdiploma behaalde hij aan het Ichthus college in Veenendaal in 2001. Daarna studeerde hij zowel wiskunde als theoretische natuurkunde aan de universiteit van Utrecht. In 2007 studeerde hij in beide disciplines *cum laude* af. In zijn afstudeerwerk, dat geschreven werd onder het toezien van Joop Kolk, werd de basis gelegd voor zijn latere promotieonderzoek.

Omdat het Job in de laatste jaren van zijn studie zeer duidelijk was geworden dat zijn interesse niet lag in de fysica, maar des te meer in de wiskunde, begon hij in augustus 2001 zijn promotieonderzoek aan het Mathematisch Instituut van Universiteit Utrecht in de theorie van Radontransformatie op symmetrische ruimten. Dit onderzoek werd uitgevoerd onder begeleiding van Erik van den Ban en heeft geresulteerd in dit proefschrift.

Gedurende de maanden augustus en september van 2011 zal Job deelnemen aan de workshop *Analysis on Lie groups* in het Max Planck Institute in Bonn, waarna hij in oktober 2011 zal gaan werken als postdoc aan de universiteit van Kopenhagen.

