

# Dynamics driven by the Trace Anomaly in FLRW Universes

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By means of a semiclassical analysis we show that the trace anomaly does not affect the cosmological constant. We calculate the evolution of the Hubble parameter in quasi de Sitter spacetime, where the Hubble parameter varies slowly in time, and in FLRW spacetimes. We show dynamically that a Universe consisting of matter with a constant equation of state, a cosmological constant and the quantum trace anomaly evolves either to the classical de Sitter attractor or to a quantum trace anomaly driven one. There is no dynamical effect that influences the effective value of the cosmological constant.

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## I. INTRODUCTION

This paper aims at summarising the main results of [1], as presented at the “Invisible Universe Conference” in Paris (2009). We show that the cosmological constant problem is not solved by the trace anomaly.

In semiclassical gravity, one treats all matter fields as quantum fields living on a classical curved background. The fields, i.e.: their classical (background) expectation values and their fluctuations, cause the background to curve according to Einstein’s field equations. The question how these fluctuations affect the background geometry, or in particular the cosmological constant, is a line of research known as quantum backreaction. We need to combine classical general relativity with knowledge of quantum field theory in curved spacetimes [2].

We supplement the classical Einstein-Hilbert action with the trace anomaly or conformal anomaly which quantum field theories are known to exhibit [2, 3, 4, 5, 6, 7, 8, 9]. Suppose a classical action is invariant under conformal transformations of the metric, then the resulting stress-energy tensor is traceless:

$$T^\mu{}_\mu = 0. \quad (1)$$

As an explicit example, consider a conformally coupled scalar field. In quantum field theory the stress-tensor is promoted to an operator. A careful renormalisation procedure renders its expectation value  $\langle \hat{T}^\mu{}_\nu \rangle$  finite. However, inevitably, the renormalisation procedure results in general in a non-vanishing trace of the renormalised stress-energy tensor:

$$\langle \hat{T}^\mu{}_\mu \rangle \neq 0. \quad (2)$$

Classical conformal invariance cannot be preserved at the quantum level. In short, this is the trace anomaly. An important question that immediately arises is how this non-zero trace affects the evolution of the Universe through the Einstein field equations. In particular, one could wonder whether it influences the effective value of the cosmological constant.

It has been argued (see [10] and references therein) that the trace anomaly could potentially provide us with a dynamical explanation of the cosmological constant problem. In brief, the line of reasoning is as follows. Firstly, a new conformal degree of freedom is introduced as  $g_{\mu\nu}(x) = \exp[2\sigma(x)]\bar{g}_{\mu\nu}(x)$ . Furthermore, the trace anomaly stems from a non-local effective action that generates the conformal anomaly by variation with respect to the metric [11]. The authors of [10] then argue that the new conformal field should dynamically screen the cosmological constant, thus solving the cosmological constant problem.

The proposal advocated in [10] is very appealing. Before studying the effect of this new conformal degree of freedom, we feel that firstly a proper complete analysis of the dynamics resulting from the effective action of the trace anomaly should be performed. This is what we pursue in this contribution.

We argue in favour of a semiclassical approach to examining the connection between the cosmological constant and the trace anomaly: we take expectation values of inhomogeneous quantum fluctuations with respect to a certain state

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to study its effect on the background spacetime. Phase transitions aside, quantum fluctuations affect the background *homogeneously*. Moreover, in accordance with the Cosmological Principle, inhomogeneous fluctuations of the metric tensor and in particular of the conformal part of the metric tensor are observed to be small at the largest scales, comparable to the Hubble radius (and expected to be small also in the early Universe). We do not consider a new, conformal degree of freedom. Hence, we do certainly not exclude any possible effect this (inhomogeneous) conformal degree of freedom might have on the cosmological constant. However, it is plausible that in order to address the link between the cosmological constant and the trace anomaly, a semiclassical analysis suffices.

A final note is in order regarding an application of the conformal anomaly: trace anomaly induced inflation. In the absence of a cosmological constant, the trace anomaly could provide us with an effective cosmological constant [12, 13, 14, 15, 16, 17, 18]. If one includes a cosmological constant, the theory of anomaly induced inflation is plagued by instabilities, which we will also come to address. Improving on e.g. [14, 19], we incorporate matter with a constant equation of state in the Einstein field equations.

## II. THE DYNAMICS DRIVEN BY THE TRACE ANOMALY

The trace anomaly or the conformal anomaly in four dimensions is in general curved spacetimes given by [2, 3, 10]:

$$T_Q \equiv \langle \hat{T}^\mu{}_\mu \rangle = bF + b' \left( E - \frac{2}{3} \square R \right) + b'' \square R, \quad (3)$$

where:

$$E \equiv {}^*R_{\mu\nu\kappa\lambda} {}^*R^{\mu\nu\kappa\lambda} = R_{\mu\nu\kappa\lambda} R^{\mu\nu\kappa\lambda} - 4R_{\mu\nu} R^{\mu\nu} + R^2 \quad (4a)$$

$$F \equiv C_{\mu\nu\kappa\lambda} C^{\mu\nu\kappa\lambda} = R_{\mu\nu\kappa\lambda} R^{\mu\nu\kappa\lambda} - 2R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2, \quad (4b)$$

where as usual  $R_{\mu\nu\kappa\lambda}$  is the Riemann curvature tensor,  ${}^*R_{\mu\nu\kappa\lambda} = \varepsilon_{\mu\nu\alpha\beta} R^{\alpha\beta}{}_{\kappa\lambda} / 2$  its dual,  $C_{\mu\nu\kappa\lambda}$  the Weyl tensor and  $R_{\mu\nu}$  and  $R$  the Ricci tensor and scalar, respectively. The Gauss-Bonnet invariant is denoted by  $E$ . The general expression for the trace anomaly can also contain additional contributions if the massless conformal field is coupled to other long range gauge fields (see e.g. [2]). Finally, the parameters  $b$ ,  $b'$  and  $b''$  appearing in (3), dimensionless quantities multiplied by  $\hbar$ , are determined by the (matter) degrees of freedom in a theory as follows:

$$b = \frac{1}{120(4\pi)^2} (N_S + 6N_F + 12N_V) \quad (5a)$$

$$b' = -\frac{1}{360(4\pi)^2} \left( N_S + \frac{11}{2} N_F + 62N_V \right), \quad (5b)$$

where  $N_S$ ,  $N_F$  and  $N_V$  denote the number of fields of spin 0, 1/2 and 1 respectively ( $\hbar = 1$ ). It turns out that the coefficient  $b''$  is regularisation dependent and is therefore not considered to be part of the true conformal anomaly. We take this into account by allowing  $b''$  to vary.

Let us clearly state the assumptions we use to solve for the dynamics that is driven by the conformal anomaly. Firstly, we specialise to flat Friedmann-Lemaître-Robertson-Walker or FLRW spacetimes in which the metric is given by  $g_{\alpha\beta} = \text{diag}(-1, a^2(t), a^2(t), a^2(t))$  where  $a(t)$  is the scale factor of the Universe in cosmic time  $t$ . As usual,  $H = \dot{a}/a$  is the Hubble parameter. We consider a Universe in the presence of a) a non-zero cosmological constant, b) the trace anomaly as a contribution to the quantum stress-energy tensor and c) classical matter with constant equation of state  $\rho_M = \omega p_M$ , where  $\omega > -1$ . Thirdly, we use covariant stress-energy conservation and assume a perfect fluid form for the quantum density  $\rho_Q$  and quantum pressure  $p_Q$  contributing to the trace anomaly:

$$T^\mu{}_{\nu,Q} = (-\rho_Q, p_Q, p_Q, p_Q). \quad (6)$$

The relevant differential equation governing the dynamics driven by the trace anomaly, derived from the Einstein field equations is [1]:

$$9(1 + \omega)H^2(t) + 6\dot{H}(t) - 3(1 + \omega)\Lambda = -8\pi G [T_Q + (1 - 3\omega)\rho_Q]. \quad (7)$$

Here, the trace anomaly in FLRW spacetimes reads:

$$T_Q = 4b' \left\{ \ddot{H} + 7\dot{H}H + 4\dot{H}^2 + 18\dot{H}H^2 + 6H^4 \right\} - 6b'' \left\{ \ddot{H} + 7\dot{H}H + 4\dot{H}^2 + 12\dot{H}H^2 \right\}. \quad (8)$$

The quantum density contributing to (7) is given by (also see [13, 20] for details):

$$\rho_Q = 2b' \left[ -2\ddot{H}H + \dot{H}^2 - 6\dot{H}H^2 - 3H^4 \right] + 3b'' \left[ 2\ddot{H}H - \dot{H}^2 + 6\dot{H}H^2 \right]. \quad (9)$$

To capture the leading order dynamics we work in quasi de Sitter spacetime where  $H(t)$  depends only mildly on time:

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \text{constant} \ll 1, \quad (10)$$

i.e.: we assume that  $\epsilon$  is both small and time independent. This truncates the trace anomaly (8) up to terms linear in  $\dot{H}$ :

$$T_Q = 24b' \left\{ 3\dot{H}H^2 + H^4 \right\} - 72b'' \dot{H}H^2. \quad (11)$$

The expression for the quantum density (9) reduces to:

$$\rho_Q = -6b' \left[ 2\dot{H}H^2 + H^4 \right] + 18b'' \dot{H}H^2. \quad (12)$$

We will examine both the exact forms (8) and (9) in general FLRW spacetimes and their truncated versions (11) and (12) in quasi de Sitter spacetime. Recall that we should allow the  $b''$  parameter to vary. Note that when  $b'' = 2b'/3$  the truncated versions are exact.

Truncating the expression for the trace anomaly is motivated by the following realisation. Generally, higher derivative contributions in an equation of motion have the tendency to destabilise a system unless the initial conditions are highly fine-tuned. Formally, this is known as the theorem of Ostrogradsky and its relevance to Cosmology is outlined, for example, in [21].

In the literature (see e.g. [17]), one only has considered a Universe with radiation, a cosmological term and the trace anomaly (radiation does not contribute classically to the trace of the Einstein field equations). We incorporate matter with constant equation of state parameter  $\omega$ .

Independently on whether one truncates the expressions for the anomalous trace or quantum density, one can straightforwardly solve for the asymptotes of (7):

$$\left( H_0^{C,A} \right)^2 = \frac{-1 \pm \sqrt{1 + 64\pi G b' \Lambda / 3}}{32\pi G b'}. \quad (13)$$

Here,  $H_0^C$  turns out to be the classical de Sitter attractor, whereas  $H_0^A$  is a new, quantum anomaly driven attractor. Note from equation (5b) that  $b' < 0$ . We can write the above expression in a somewhat more convenient form by defining the dimensionless parameter  $\lambda$ :

$$\lambda = \frac{G\Lambda}{3}, \quad (14)$$

that sets the scale for the cosmological constant  $\Lambda$ . We expand (13) as  $\lambda \ll 1$ , finding<sup>1</sup>:

$$H_0^C = \sqrt{\frac{\Lambda}{3}} [1 - 8\pi b' \lambda] \quad (15a)$$

$$H_0^A = \sqrt{\frac{-1}{16\pi G b'} - \frac{\Lambda}{3}}. \quad (15b)$$

In the absence of a cosmological constant, the trace anomaly can thus provide us with an inflationary scenario which has already been appreciated by [12, 13, 16]. Finally, note these asymptotes are independent of  $b''$ .

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<sup>1</sup> Note the nomenclature in the literature is somewhat misleading. Rather than calling (15a) the classical de Sitter attractor, it would be more natural to denote it with the quantum corrected classical attractor. Hence, the quantum attractor (15b) should preferably be denoted by anomaly driven attractor or Planck scale attractor. We will nevertheless adopt the nomenclature existing in the literature.

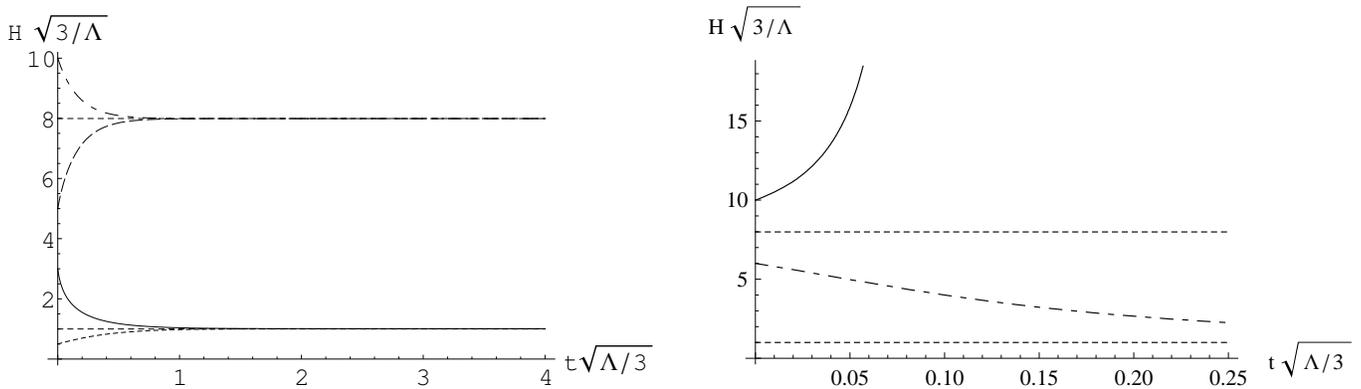


Figure 1: Left plot: dynamics of the Hubble parameter in quasi de Sitter spacetime with a non-zero cosmological constant, the trace anomaly and matter  $\omega = 0$ . Depending on the initial conditions,  $H(t)$  evolves to either the classical de Sitter or the quantum anomaly driven attractor. We have used  $\lambda = 1/50$ ,  $b' = -0.015$  (Standard Model value) and  $b'' = 0$ . Right plot: instability of the quantum anomaly driven attractor in quasi de Sitter spacetimes. We have used  $\lambda = 1/50$ ,  $\omega = 1/3$ ,  $b'' = 7b'/6 < 5b'/6$ .

### III. THE TRACE ANOMALY IN QUASI DE SITTER SPACETIME

In this section we work in quasi de Sitter spacetime, where we treat  $\epsilon$  as a small and time independent constant which allows us to neglect higher order derivative contributions, motivated by the theorem of Ostrogradsky [21].

We have to distinguish two cases separately. In the spirit of [10], the numerical value of the parameter  $b''$  occurring in the trace anomaly is not fixed and we therefore allow it to take different values. First, we consider an unrestricted value of  $b''$ , but where  $b'' \neq 2b'/3$ , and secondly we set  $b'' = 2b'/3$ .

#### A. Case I: unrestricted value of $b''$

We thus insert the truncated expression for the trace anomaly (11) and the quantum density (12) into the Einstein field equation (7). This differential equation can be solved exactly:

$$t - t' = \frac{1}{-48\pi G b'(1 + \omega) \{(H_0^A)^2 - (H_0^C)^2\}} \left[ -\frac{1 + \frac{1}{2}\alpha(H_0^A)^2}{H_0^A} \left\{ \log \left( \frac{H(t) + H_0^A}{H(t) - H_0^A} \right) - \log \left( \frac{H(t') + H_0^A}{H(t') - H_0^A} \right) \right\} \right. \\ \left. + \frac{1 + \frac{1}{2}\alpha(H_0^C)^2}{H_0^C} \left\{ \log \left( \frac{H(t) + H_0^C}{H(t) - H_0^C} \right) - \log \left( \frac{H(t') + H_0^C}{H(t') - H_0^C} \right) \right\} \right]. \quad (16)$$

In the left plot of figure 1, we numerically calculate the dynamics of the Hubble parameter for various initial conditions. The two asymptotes divide this graph into three distinct regions that are not connected for finite time evolution. The region bounded by the two asymptotes contains initial conditions for  $H(t)$  such that  $H(t)$  grows for late times towards  $H_0^A$  and initial conditions such that  $H(t)$  asymptotes to the de Sitter attractor  $H_0^C$ , separated by a branching point [1]:

$$H_{\text{BP}} = \frac{1}{\sqrt{8\pi G \{9(1 + \omega)b'' - 2(5 + 3\omega)b'\}}}. \quad (17)$$

Using the analytical solution (16), we can study the stability of the late time attractors. The quantum anomaly driven attractor is stable, whenever the following inequality is satisfied:

$$b'' > \frac{2}{9} b' \frac{4 + 3\omega + 16\pi\lambda b'(5 + 3\omega)}{(1 + \omega)(1 + 16\pi\lambda b')}. \quad (18)$$

In the right plot of figure 1, we numerically calculate the evolution of the Hubble parameter in a radiation dominated Universe when this inequality is not satisfied. We used  $b'' = 7b'/6 < 5b'/6$ . For initial conditions above  $H_0^A$ , the Hubble parameter increases to even higher energies, whereas for initial conditions below the quantum attractor, the Hubble parameter evolves towards the classical attractor. The low energy limit reveals less surprising behaviour: the classical de Sitter attractor is always stable.

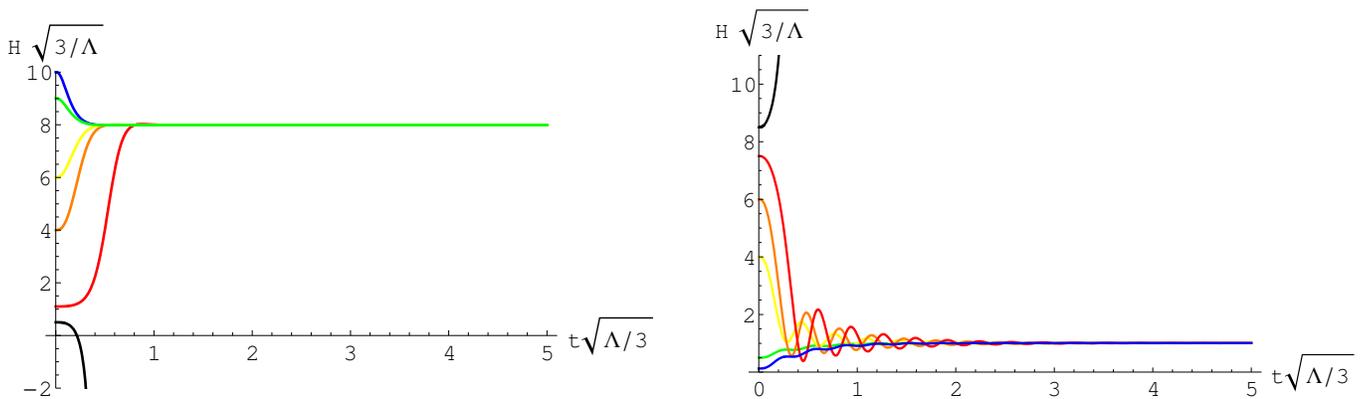


Figure 2: Left plot: dynamics of the Hubble parameter taking the full trace anomaly into account. We took  $b'' = 0$  such that  $b'' - 2b'/3 > 0$  yielding an unstable classical attractor. We have used  $\omega = 0$ ,  $\lambda = 1/50$ ,  $b' = -0.015$  (Standard Model value). Right plot: dynamics of the Hubble parameter taking the full trace anomaly into account. We took  $b'' = b'$  such that  $b'' - 2b'/3 < 0$  yielding a stable classical attractor (the other parameters have not changed). Clearly, the classical attractor is under-damped, resulting in various oscillations around  $H_0^C$ .

### B. Case II: $b'' = 2b'/3$

As indicated earlier, we must consider the case when  $b'' = 2b'/3$  separately because in this particular case the total coefficient in front of the  $\square R$  contribution to the trace anomaly vanishes. All higher derivative contributions precisely cancel and also the  $\dot{H}^2$  contribution happens to cancel, such that we find ourselves immediately situated in quasi de Sitter spacetime. Albeit a simple case, we do take the full trace anomaly into account.

The analytic solution obtained in (16) still applies and moreover, it becomes exact. Most important, the features of the left plot of figure 1, e.g.: two stable attractors and the occurrence of a branching point, do not change.

## IV. THE TRACE ANOMALY IN FLRW SPACETIMES

We turn our attention to solving the full trace equation (7), where we truncate the expression neither for the anomalous trace (8) nor for the quantum density (9). Before turning to numerical methods, let us summarise what we can learn from analytically studying the stability behaviour of the late time attractors. Let us consider small perturbations  $\delta H(t)$  around the two asymptotes and insert:

$$H(t) = H_0^{C,A} + \delta H(t), \quad (19)$$

in equation (7), where  $H_0^{C,A}$  can either denote the classical or the quantum attractor. If we make the ansatz  $\delta H(t) = c \exp[\xi t]$ , we can solve the linearised characteristic equation for the eigenvalues  $\xi$ . Clearly,  $\text{Re}(\xi) < 0$  ensures the stability of an attractor. This yields:

$$\text{If } b'' - 2b'/3 > 0, \quad \text{then } \begin{cases} \text{Classical attractor unstable} \\ \text{Quantum attractor stable} \end{cases} \quad (20a)$$

$$\text{If } b'' - 2b'/3 < 0, \quad \text{then } \begin{cases} \text{Classical attractor stable} \\ \text{Quantum attractor unstable} \end{cases} \quad (20b)$$

Surprisingly, the stability analysis does not depend on the equation of state  $\omega$ . This calculation thus proves the statements about stability made in e.g. [14] using the Routh-Hurwitz method. Our proof is more general because we include a constant, but otherwise arbitrary, equation of state parameter  $\omega > -1$ . Moreover, while the Routh-Hurwitz method can only guarantee stability of a solution (when certain determinants are all strictly positive), it does not tell anything about instability [14]. Furthermore appreciate that the singular point in this analysis,  $b'' - 2b'/3 = 0$ , immediately directs us to the quasi de Sitter spacetime analysis performed in the previous section, where all higher derivative contributions precisely cancel rendering both attractors stable.

Let us compare the two plots in figure 2. In the left plot, we used  $b'' = 0$  such that  $b'' - 2b'/3 > 0$ , yielding an unstable classical attractor. However, if  $H(0) \leq H_0^C$  the quantum anomaly driven asymptote is not an attractor and the Hubble parameter runs away to negative infinity. In the right plot, we set  $b'' = b'$  such that  $b'' - 2b'/3 < 0$

which gives us a stable classical attractor. Likewise, for initial conditions  $H(0) \geq H_0^A$  the de Sitter solution is not an attractor and the Hubble parameter rapidly blows up to positive infinity.

The right plot reveals another interesting phenomenon. In this case, the classical attractor is under-damped, resulting in decaying oscillations around this attractor. Clearly, oscillatory behaviour occurs whenever the eigenvalues  $\xi$  develop an imaginary contribution. We can derive that whenever the classical de Sitter attractor is stable oscillations occur. Oscillatory behaviour around the quantum attractor occurs only when:

$$b'' < -\frac{2}{9}b' \left( \frac{1 + 8\pi\lambda b'}{1 + 8\pi\lambda} \right). \quad (21)$$

## V. CONCLUSION

We have studied the dynamics of the Hubble parameter both in quasi de Sitter and in FLRW spacetimes including matter, a cosmological term and the trace anomaly. We have seen that there is no dynamical effect that influences the effective value of the cosmological constant, i.e.: the classical de Sitter attractor. Based on our semiclassical analysis we thus conclude that the trace anomaly does *not* solve the cosmological constant problem.

Ostrogradsky's theorem merits another remark. Clearly, including the higher derivative contributions in FLRW spacetimes modifies the dynamics of the Hubble parameter significantly: attractors, that were stable in the absence of higher derivatives, under certain conditions destabilise. We do not know which of the two approaches is correct. Discarding these higher derivatives and studying the trace anomaly in quasi de Sitter spacetime would seem plausible.

Finally, one could wonder whether the quantum anomaly driven attractor is physical. The quantum attractor is of the order of the Planck mass  $M_{\text{pl}}$ , so only when the matter in the early Universe is sufficiently dense,  $H \simeq \mathcal{O}(M_{\text{pl}})$ . We then expect to evolve towards the quantum attractor. However, at these early times we also expect perturbative general relativity to break down. Hence, this attractor may be seriously affected by quantum fluctuations or it might even not be there.

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