

Path integral for inflationary perturbationsTomislav Prokopec¹ and Gerasimos Rigopoulos²¹*Institute for Theoretical Physics and Spinoza Institute, Utrecht University, Leuvenlaan 4, 3584 CE Utrecht, The Netherlands*²*Helsinki Institute of Physics, P.O. Box 64, FIN-00014, University of Helsinki, Finland*

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The quantum theory of cosmological perturbations in single-field inflation is formulated in terms of a path integral. Starting from a canonical formulation, we show how the free propagators can be obtained from the well-known gauge-invariant quadratic action for scalar and tensor perturbations, and determine the interactions to arbitrary order. This approach does not require the explicit solution of the energy and momentum constraints, a novel feature which simplifies the determination of the interaction vertices. The constraints and the necessary imposition of gauge conditions is reflected in the appearance of various commuting and anticommuting auxiliary fields in the action. These auxiliary fields are not propagating physical degrees of freedom but need to be included in internal lines and loops in a diagrammatic expansion. To illustrate the formalism we discuss the tree-level three-point and four-point functions of the inflaton perturbations, reproducing the results already obtained by the methods used in the current literature. Loop calculations are left for future work.

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I. INTRODUCTION

Perhaps the most remarkable aspect of inflation [1] is its ability to imprint fluctuations on cosmic scales through a confluence of quantum mechanics and general relativity. This connection, first realized more than 25 years ago [2–7],¹ has given inflationary theory the impetus which positioned it as the leading paradigm for approaching the physics of the early universe. Since then, the primordial fluctuations have been measured in the cosmic microwave background [8] with ever increasing accuracy and resolution [9,10] and will be scrutinized even further in the near future [11]. It is not surprising then that over these past decades a lot of effort has been devoted to fleshing out the predictions inflation makes for these fluctuations in a variety of theoretical settings. Since these fluctuations are initially small, of order 10^{-5} at $z \approx 1090$, the linearized theory of perturbations has been developed to a significant degree and has been used, rather successfully, to compare theory to observation. Over the past few years, efforts have intensified to explore inflationary perturbations beyond linear order, mostly in the context of the related non-Gaussianity which has become a significant subfield of cosmological research [12]. The amount of work that has been done on the subject is by now rather voluminous with many authors examining various aspects. A definitive calculation was performed by Maldacena [13] showing that single-field inflation leads to small primordial non-Gaussianities (see also [14]) with more complicated single-field and multifield models providing more possibilities for larger non-Gaussianity; see, e.g., [15–31]. On the observational side, a major effort is under way to

develop observational measures of non-Gaussianity [10,32–41].

The foray into nonlinear corrections and the associated non-Gaussianity has been motivated by a number of reasons. If one neglects the running of the spectral indices, at linear order inflationary theories predict four numbers: the amplitude and spectral index of scalar and tensor perturbations, and a variety of different models, can coincide on these predictions. However, non-Gaussianity can be rather discriminatory for different models due to its much richer, and more complicated, structure. As an example, the detection of a significant three-point function would immediately rule out single-field inflation, as well as some simple multifield generalizations, may favor alternative models, or provide evidence for the processes involved in heating up the universe after inflation. Entwined with considerations of testing inflationary theory against observations and non-Gaussianity are considerations of theoretical understanding: calculating and controlling higher order quantum loop corrections to inflationary predictions, back-reaction issues [42], and various divergences which appear at higher orders of perturbation theory—see, for example, [43–56].

So far, all the work on inflationary nonlinear corrections has been done using the operator language in the interaction picture. However, in many branches of modern physics it is the path-integral formulation for quantum mechanical systems which has proved quite useful. For example, it has been of paramount importance in understanding gauge theories and their experimental consequences for particle physics, the theoretical description of condensed matter systems, and is the preferred language in which modern theories of fundamental physics are formulated and quantized. In this spirit, and hoping to throw more light on the understanding of inflation, here we

¹The first rigorous and quantitatively accurate treatments of inflationary perturbations were [2,7].

develop a path-integral formulation for inflationary perturbations to arbitrary order in interaction terms. An early path-integral formulation of linear perturbation theory can be found in [57]. A novel feature of our approach is that there is no need to explicitly solve the energy and momentum constraints to a particular perturbative order as is usually done when working in a particular gauge. This allows us to obtain the interaction terms, and hence the vertices, to arbitrary order in a closed form.

We start from the more fundamental canonical path integral and show how to obtain the configuration phase space path integral, making in the process a connection with the well-known gauge-invariant linear perturbation theory. With a single inflaton there is only one scalar—expressed in terms of the Sasaki-Mukhanov variable—and two tensor degrees of freedom which propagate, as expected, while the vectors completely drop out at quadratic order. Moreover, various (real and commuting) auxiliary fields which do not appear in the in-state, the external lines in the “in-in diagrammatic” expansion, must be included in computations of N -point functions with $N \geq 4$ or calculations involving loops. These fields arise because the constraints are not explicitly solved. Furthermore, anti-commuting ghosts arise from the path-integral measure which must also be taken into account at a certain loop order. Here we focus on standard potential-energy-dominated single-field inflation, but the generalization to more fields and more complex theories is in principle straightforward as long as a canonical formulation is available.

The paper is organized as follows: In Sec. II we derive the action for perturbations in a form that includes interactions to all orders in a closed expression, discuss the role of constraints and gauge conditions, and formulate the transition amplitude between two quantum states separated by (background) time t in terms of a path integral. In the process the action is written in a form that makes contact with known results from gauge-invariant linear perturbation theory [58] and simplifies the computation of the propagators. In Sec. III we discuss the “in-in” generating functional which provides the appropriate diagrammatic rules for the computation of the N -point functions relevant to cosmology. In Sec. IV we apply the above formalism to give expressions for the tree-level three-point and four-point functions in terms of propagators, reproducing the results already obtained using an operator formalism and a different methodology. We close in Sec. V with a discussion of our results.

II. QUANTIZATION OF INFLATIONARY PERTURBATIONS

We will consider a single scalar field χ minimally coupled to Einstein gravity. The action in the canonical form [59] is ($\kappa^2 = 16\pi G_N = 1$)

$$S = \int d^4x \{ p^{ij} \partial_t g_{ij} + p_\chi \partial_t \chi - N \mathcal{H} - N_i \mathcal{H}^i \}, \quad (1)$$

with the Hamiltonian \mathcal{H} and momentum density \mathcal{H}^i

$$\begin{aligned} \mathcal{H} = & \frac{1}{\sqrt{g}} \left(p^{ij} g_{ik} g_{jl} p^{kl} - \frac{1}{2} p^2 \right) - \sqrt{g} R + \frac{1}{2} \frac{p_\chi^2}{\sqrt{g}} \\ & + \frac{1}{2} \sqrt{g} g^{ij} \nabla_i \chi \nabla_j \chi + \sqrt{g} V(\chi), \end{aligned} \quad (2)$$

$$\mathcal{H}^i = p_\chi \partial^i \chi - 2 \nabla_j p^{ij}, \quad (3)$$

where p^{ij} and p_χ are the canonical momenta of the spatial metric g_{ij} and scalar field χ . They are tensor densities of weight one, i.e. p^{ij}/\sqrt{g} and p_χ/\sqrt{g} are true tensors. In particular, the covariant derivative in (3) is to be understood as $\nabla_j p^{ij} = \partial_j p^{ij} + \Gamma_{jl}^i p^{jl}$, where Γ_{jl}^i is the Levi-Civita connection. Finally, N and N^i are the lapse and shift functions, related to the temporal components of the metric tensor as $g_{0i} = g_{ij} N^j$ and $g_{00} = -N^2 + N^i g_{ij} N^j$, R denotes the spatial Ricci scalar, $g = \det[g_{ij}]$ and $p \equiv g_{ij} p^{ij}$. We will be interested in quantizing perturbations around a classical inflationary background, so let us set

$$p^{ij} = \frac{\mathcal{P}(t)}{6a(t)} (\delta^{ij} + \pi^{ij}(t, \mathbf{x})), \quad (4)$$

$$p_\chi = \mathcal{P}_\phi (1 + \pi_\phi(t, \mathbf{x})), \quad (5)$$

$$g_{ij} = a(t)^2 (\delta_{ij} + h_{ij}(t, \mathbf{x})), \quad (6)$$

$$\chi = \phi(t) + \varphi(t, \mathbf{x}), \quad (7)$$

$$N = \bar{N}(t) + n(t, \mathbf{x}). \quad (8)$$

Note that the background value of the shift N_i is zero. The action $S^{(0)}$ for the background dynamics is obtained by setting the perturbations to zero:

$$S^{(0)} = \int d^3x dt \left\{ \mathcal{P} \frac{da}{dt} + \mathcal{P}_\phi \frac{d\phi}{dt} - \bar{N} \mathcal{H}_0(t) \right\}, \quad (9)$$

where

$$\mathcal{H}_0(t) = -\frac{\mathcal{P}^2}{24a} + \frac{\mathcal{P}_\phi^2}{2a^3} + a^3 V. \quad (10)$$

The background equations of motion are obtained from $S^{(0)}$ by varying with respect to a , ϕ , \mathcal{P} , and \mathcal{P}_ϕ , resulting in

$$\dot{a} = -\frac{\mathcal{P}}{12a}, \quad (11)$$

$$\dot{\phi} = \frac{\mathcal{P}_\phi}{a^3}, \quad (12)$$

$$\dot{\mathcal{P}} = -\frac{\mathcal{P}^2}{24a^2} + \frac{3}{2} \frac{\mathcal{P}_\phi^2}{a^4} - 3a^2 V, \quad (13)$$

$$\dot{\mathcal{P}}_\phi = -a^3 V_{,\phi}, \quad (14)$$

where $V = V(\phi(t))$, $V_{,\phi} = dV(\phi)/d\phi$ and we introduced the ‘‘dotted’’ derivative, $\dot{a} \equiv \bar{N}^{-1} da/dt$, etc. Furthermore, variation with respect to \bar{N} gives the constraint $\mathcal{H}_0(t) = 0$, or equivalently,

$$\frac{\mathcal{P}^2}{24a} = \frac{\mathcal{P}_\phi^2}{2a^3} + a^3 V. \quad (15)$$

The well-known equations of flat Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology are obtained with the identification

$$H \equiv \frac{\dot{a}}{a} = -\frac{\mathcal{P}}{12a^2}. \quad (16)$$

Indeed, inserting the solutions for the momenta \mathcal{P} and \mathcal{P}_ϕ into (13)–(15), one obtains

$$H^2 = \frac{\rho_\phi}{6} \equiv \frac{1}{6} \left(\frac{\dot{\phi}^2}{2} + V \right) \quad (17)$$

$$\dot{H} = -\frac{\dot{\phi}^2}{4} \quad (18)$$

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0 \quad (19)$$

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = 0, \quad \left(p_\phi = \frac{\dot{\phi}^2}{2} - V \right). \quad (20)$$

Recalling that a *dot* refers to $d/(\bar{N}dt)$, we see that these equations are time reparametrization invariant. For example, their form in cosmological and conformal time is obtained simply by choosing the lapse $\bar{N} = 1$ and $\bar{N} = a$, respectively. Of course, only two out of these four equations are independent. Indeed, from the first equation and any one of the other three, one can derive the remaining two equations. To recover the dependence on the Newton constant G_N one should reinsert $\kappa^2 = 16\pi G_N = 1$ into the rhs of the first two background equations (17) and (18).

In such a model, inflation takes place when the field is rolling slowly on the slope of its potential. We therefore recall the ‘‘slow-roll’’ parameters, which for the purpose of this paper we define as

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{3}{2} \frac{\dot{\phi}^2}{\rho_\phi}, \quad \eta = -\frac{\ddot{\phi}}{H\dot{\phi}}. \quad (21)$$

Notice that in the slow-roll approximation when $\epsilon \ll 1$ and $\eta \ll 1$, ϵ and η reduce to the standard slow-roll definitions, $\epsilon \rightarrow (V_{,\phi}/V)^2$ and $\eta \rightarrow (V_{,\phi\phi}/V) - \epsilon$. For the rest of the paper we will be using H instead of \mathcal{P} and \mathcal{P}_ϕ instead of $\dot{\phi}$.

A. The action for perturbations

We can now proceed to obtain the action for the perturbations by making the replacements (4)–(8) and expanding (1). We ignore terms linear in perturbations since they are multiplied by the background equations of motion (17)–(20), which we assume to hold. We can therefore write the complete action for the perturbations in canonical form as

$$\begin{aligned} \mathcal{S}_{\text{pert}} = \int d^3x dt & (\mathcal{P}_\phi \pi_\phi \partial_t \phi - 2a^3 H \pi^{ij} \partial_t h_{ij} \\ & - \mathcal{H}_{\text{pert}} + nC_0 + N_i C_i) \equiv \mathcal{S}_F + \mathcal{S}_I \end{aligned} \quad (22)$$

$$\begin{aligned} \mathcal{S}_F = \int d^3x dt & \{ \mathcal{P}_\phi \pi_\phi \partial_t \phi - 2a^3 H \pi^{ij} \partial_t h_{ij} - \mathcal{H}_F \\ & + nC_0^{(1)} + N_i C_i^{(1)} \} \end{aligned} \quad (23)$$

$$\mathcal{S}_I = \int d^3x dt \{ -\mathcal{H}_I + nC_0^{\geq 2} + N_i C_i^{\geq 2} \}, \quad (24)$$

which completely specifies the dynamics of perturbations to all orders in the interaction terms. For later convenience we have split the full action $\mathcal{S}_{\text{pert}}$ (22) into the free (quadratic) part \mathcal{S}_F (23) and the interactions \mathcal{S}_I (24), which include cubic, quartic, and other higher order terms in perturbations. $C_0 = C_0^{(1)} + C_0^{\geq 2} \equiv -\mathcal{H}^{(1)}/\bar{N} - \mathcal{H}_{\text{pert}}/\bar{N}$ stands for the Hamiltonian constraint, which is split into the linear part (denoted by the superscript (1)) and quadratic and higher order parts (denoted by the superscript ≥ 2 and the subscript pert). We have also split the momentum constraint as $C_i = C_i^{(1)} + C_i^{\geq 2} \equiv -\mathcal{H}^i$. Analogously, $\mathcal{H}_{\text{pert}} = \mathcal{H}_F + \mathcal{H}_I$ is the full Hamiltonian dictating the time evolution of the perturbations, split in Eqs. (23) and (24) into a free (quadratic) and an interacting part (cubic, quartic, etc.). The Poisson bracket algebra of the system of perturbations can be calculated to close

$$\begin{aligned} \{ \mathcal{H}_{\text{pert}}(\mathbf{x}), C_0(\mathbf{y}) \} = & -(C_i(\mathbf{x}) \partial_i^y - C_i(\mathbf{y}) \partial_i^x) \delta(\mathbf{x} - \mathbf{y}) \\ & + \frac{\partial_t C_0}{\bar{N}} \delta(\mathbf{x} - \mathbf{y}), \end{aligned} \quad (25)$$

$$\{ \mathcal{H}_{\text{pert}}(\mathbf{x}), C_i(\mathbf{y}) \} = C_0(\mathbf{y}) \partial_i^x \delta(\mathbf{x} - \mathbf{y}) + \frac{\partial_t C_i}{\bar{N}} \delta(\mathbf{x} - \mathbf{y}), \quad (26)$$

$$\{ C_0(\mathbf{x}), C_0(\mathbf{y}) \} = (C_i(\mathbf{x}) \partial_i^y - C_i(\mathbf{y}) \partial_i^x) \delta(\mathbf{x} - \mathbf{y}), \quad (27)$$

$$\{ C_i(\mathbf{x}), C_0(\mathbf{y}) \} = C_0(\mathbf{x}) \partial_i^y \delta(\mathbf{x} - \mathbf{y}), \quad (28)$$

$$\{ C_i(\mathbf{x}), C_j(\mathbf{y}) \} = (C_j(\mathbf{x}) \partial_i^y - C_i(\mathbf{x}) \partial_j^y) \delta(\mathbf{x} - \mathbf{y}), \quad (29)$$

where the fundamental Poisson brackets are seen from (22) to be

$$\{ \varphi(\mathbf{y}, t), \pi_\phi(\mathbf{x}, t) \} = \frac{1}{\mathcal{P}_\phi} \delta(\mathbf{x} - \mathbf{y}), \quad (30)$$

$$\{h_{kl}(\mathbf{y}, t), \pi^{ij}(\mathbf{x}, t)\} = -\frac{1}{4a^3H}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta(\mathbf{x} - \mathbf{y}). \quad (31)$$

We see that from a canonical point of view, inflationary perturbations form a *constrained Hamiltonian system* [60,61].²

What is the number of dynamical degrees of freedom in such a system? Following the point of view in [61], we see that the seven fields (φ, h_{ij}) appearing in (22) are not all independent degrees of freedom: n and N_i are Lagrange multipliers without dynamics of their own, imposing under variation the conditions $C_0 = 0$ and $C_i = 0$ which constrain the evolution of the system to take place in a lower dimensional hypersurface of the full 14-dimensional phase space. The functions n and N_i are not determined by the dynamics and are arbitrary; in this particular case they parametrize the freedom in choosing how to break space-time into spatial hypersurfaces and time. Thus, the dynamical equations contain four completely free functions which need to be fixed by the imposition of four further *gauge conditions*: $\mathcal{Q}_\alpha(h_{ij}, \varphi, \pi^{ij}, \pi_\varphi) = 0$. These gauge conditions can be chosen at will, up to the requirement

$$\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = 0, \quad \text{Det}\{\mathcal{Q}_\alpha, C_\beta\} \neq 0. \quad (32)$$

We see that the physical phase space has dimension $14 - 4 - 4 = 6$. Indeed, as is well known and will be rederived below, there is only one dynamical scalar and a transverse traceless tensor propagating in single-field inflation, corresponding to 3 degrees of freedom and a six-dimensional dynamical phase space. Note that once a choice of \mathcal{Q}_α has been made, the Lagrange multipliers $N_\alpha \equiv \{n, N_i\}$ can be determined by the imposition of the consistency relation

$$\dot{\mathcal{Q}}_\alpha = \{\mathcal{H}_{\text{pert}}, \mathcal{Q}_\alpha\} - N_\beta\{C_\beta, \mathcal{Q}_\alpha\} \approx 0, \quad (33)$$

where the symbol \approx implies that the constraints $C_\beta = 0$ are imposed *after* the Poisson brackets are evaluated. Condition (32) then ensures that Eq. (33) can be solved for N_α .

Since the number of dynamical fields in the system is much less than the apparent dimension of its phase space, it proves convenient to explicitly separate out the 3 propagating degrees of freedom from the rest. We do this in the next paragraph, leading to Eq. (45). Let us note that the physical degrees of freedom could be isolated by actually solving the constraints, thus expressing some of the canonical variables in terms of the others, and plugging the solution back in the action. This is feasible for the free theory, where the interactions \mathcal{S}_I in (24) are ignored [57]. If the interactions are included however, the solution of the constraints can only be found by an iteration procedure and leads to rather cumbersome expressions. The approach we

follow in this paper dispenses with the need to explicitly solve the constraints, although by direct comparison it is equivalent to such a solution at the linear level.

Recall now Eqs. (22)–(24), where the action for perturbations was split into the free and interacting parts. Explicating the free action \mathcal{S}_F (23), the free Hamiltonian reads

$$\begin{aligned} \mathcal{H}_F = & 4\bar{N}a^3H^2\left[\frac{1}{2}\pi^{ij}A_{ijkl}\pi^{kl} + \pi^{ij}\left(h_{ij} - \frac{1}{2}\delta_{ij}h\right)\right] \\ & + \bar{N}\frac{\mathcal{P}_\phi^2}{2a^3}(\pi_\varphi^2 - h\pi_\varphi) + \bar{N}\frac{a}{2}(\partial_i\varphi)^2 \\ & + \bar{N}\frac{a^3}{2}[V_{,\phi\phi}\varphi^2 + V_{,\phi}\varphi h] + \bar{N}\left(4a^3H^2 + \frac{\mathcal{P}_\phi^2}{a^3}\right) \\ & \times \frac{h_{ij}h_{ij}}{4} + \bar{N}a\left(\frac{1}{4}h\nabla^2h - \frac{1}{2}h\partial_i\partial_jh_{ij} + \frac{1}{2}h_{ij}\partial_i\partial_lh_{jl}\right. \\ & \left. - \frac{1}{4}h_{jl}\nabla^2h_{jl}\right), \end{aligned} \quad (34)$$

where we have used the notation $A_{ijkl} \equiv \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl}$, and

$$\begin{aligned} C_0^{(1)} = & a^3\left(\frac{\mathcal{P}_\phi^2}{2a^6} - 2H^2\right)h + a(\partial_i\partial_jh_{ij} - \nabla^2h) \\ & - a^3V_{,\phi}\varphi - \frac{\mathcal{P}_\phi^2}{a^3}\pi_\varphi + 4a^3H^2\pi, \end{aligned} \quad (35)$$

$$C_i^{(1)} = -\frac{\mathcal{P}_\phi}{a^2}\partial_i\varphi - 4aH\left(\partial_l\pi^{li} + \partial_lh_{li} + \frac{1}{2}\partial_ih\right), \quad (36)$$

and where $\nabla^2 = \delta_{ij}\partial_i\partial_j$. From now on a superscript/subscript $\geq n$ will indicate that only terms of order $\geq n$ in the perturbations are included, such that $C_\mu^{\geq 2}$ denotes contributions that are quadratic or higher order in the perturbations.

Next, it will be convenient to use the standard scalar-vector-tensor decomposition of the spatial metric perturbation,

$$h_{ij} = \frac{\delta_{ij}}{3}h + \left(\partial_i\partial_j - \frac{\delta_{ij}}{3}\nabla^2\right)\tilde{h} + \partial_{(i}h_{j)}^T + h_{ij}^{TT} \quad (37)$$

with

$$\begin{aligned} \partial_{(i}h_{j)}^T &= \frac{1}{2}(\partial_ih_j^T + \partial_jh_i^T), & \partial_ih_i^T &= 0, \\ \partial_ih_{ij}^{TT} &= 0 = \partial_jh_{ij}^{TT}. \end{aligned} \quad (38)$$

Furthermore, we will decompose the shift into the longitudinal and transverse components as

$$N_i = \partial_iS + N_i^T, \quad \text{with } \partial_iN_i^T = 0. \quad (39)$$

After a series of manipulations, which we present in Appendix B, the free action \mathcal{S}_F can be diagonalized. In summary, if we define

²For an early treatment of inflationary perturbations as a constrained Hamiltonian system, see [62].

$$w = (h - \nabla^2 \tilde{h}) - \frac{3}{\sqrt{\epsilon}} \varphi, \quad (40)$$

and perform the following linear shifts in all the perturbation fields,

$$n = \tilde{n} + L_0 \quad (41)$$

$$N_i = \tilde{N}_i + L_i \quad (42)$$

$$\pi^{ij} = \rho^{ij} + L_{ij} \quad (43)$$

$$\pi_\varphi = \rho_\varphi + L_\varphi, \quad (44)$$

where L_0 , L_i , L_{ij} , and L_φ are given in Eqs. (B19)–(B23) of Appendix B, we find that, up to boundary terms, the free action takes the form

$$\begin{aligned} \mathcal{S}_F = \int d^3x \bar{N} dt a^3 & \left\{ \frac{\epsilon}{18} \left[\dot{w}^2 - \left(\frac{\partial_i w}{a} \right)^2 \right] \right. \\ & + \frac{1}{4} \left[(\dot{h}_{ij}^{TT})^2 - \left(\frac{\partial_i h_{ij}^{TT}}{a} \right)^2 \right] - 2H^2 \epsilon \rho_\varphi^2 \\ & - 2H^2 \rho^{ij} A_{ijkl} \rho^{kl} - \frac{2(3 - \epsilon)H^2}{\bar{N}^2} \tilde{n}^2 - \frac{1}{2a^4 \bar{N}^2} \\ & \left. \times \left(\tilde{N}_j^T \nabla^2 \tilde{N}_j^T - \frac{4}{3 - \epsilon} (\nabla^2 \tilde{S})^2 \right) \right\}. \quad (45) \end{aligned}$$

In this form, the action clearly shows that the perturbations which propagate are one scalar degree of freedom w and the 2 degrees of freedom of the transverse traceless tensor h_{ij}^{TT} . All the other fields are *auxiliary* fields without dynamics of their own. The first two terms of (45) are well known from the gauge-invariant theory of cosmological perturbations [58].³ The fields w and h_{ij}^{TT} in Eq. (45) are gauge invariant under *linear* gauge transformations and time independent on long wavelengths.⁴ It is interesting to note that all the auxiliary fields, \tilde{n} , \tilde{N}_i^T , and \tilde{S} as well as the shifted momenta ρ^{ij} and ρ_φ , are also invariant under such linearized transformations—see Eq. (B27) in Appendix B and the discussion that precedes it.

The variation of the action (45) with respect to (w.r.t.) π_φ , π^{ij} , \tilde{n} , \tilde{N}_j^T , and \tilde{S} yields the on-shell relations, $\rho_\varphi = 0$, $\rho^{ij} = 0$, $\tilde{n} = 0$, $\tilde{N}_j^T = 0$, and $\tilde{S} = 0$. The relations $\rho_\varphi = 0$ and $\rho^{ij} = 0$ translate into the relation between the momenta π^{ij} and π_φ and the time derivatives of h_{ij} and φ . On the other hand, $\tilde{n} = 0$, $\nabla^2 \tilde{N}_j^T = 0$, and $\nabla^2 \tilde{S} = 0$ translate into the solutions of the linear constraints. From Eqs. (B19), (B20), and (B23) we see that the lapse pertur-

bation n and the shift N_i are then given in terms of the fields (φ, h_{ij}) and their one time derivative or one or two spatial derivatives. Thus, the free action (45) contains all the elements of the well-known gauge-invariant treatment of linear inflationary perturbations and the transition between the Hamiltonian and Lagrangian formulation of the problem.

Interactions are encoded in \mathcal{S}_I given in Eq. (24), with

$$C_0^{\geq 2} = -\frac{1}{\bar{N}} (\mathcal{H}_F + \mathcal{H}_I) \quad (46)$$

and

$$\begin{aligned} C_i^{\geq 2} = & -\frac{\mathcal{P}_\phi}{a^2} (\tilde{g}^{il})_{\geq 1} \partial_l \varphi - \frac{\mathcal{P}_\phi}{a^2} \pi_\varphi \tilde{g}^{il} \partial_l \varphi - 4aH \left\{ (\tilde{g}^{il})_{\geq 1} \right. \\ & \left. \times \left(\partial_j h_{jl} - \frac{1}{2} \partial_l h \right) + \tilde{g}^{il} \left(\partial_j h_{lr} - \frac{1}{2} \partial_l h_{jr} \right) \pi^{jr} \right\}, \quad (47) \end{aligned}$$

where \mathcal{H}_F and \mathcal{H}_I are given in Eq. (34) and Eq. (D1) of Appendix D, respectively. After all the variables are expressed in terms of the redefined fields, \tilde{n} , \tilde{N}_i , ρ^{ij} , and ρ_φ , it becomes clear that \mathcal{S}_I cannot be written solely in terms of w , h_{ij}^{TT} and the auxiliary fields which we will collectively call σ . Indeed, if φ is exchanged for w in \mathcal{S}_I , $\varphi \rightarrow \frac{\sqrt{\epsilon}}{3} \times (h - \nabla^2 \tilde{h} - w)$, one obtains an interaction action (24) which is a functional not only of w , h_{ij}^{TT} , and σ , but also of h , \tilde{h} , and h_i^T [for a more explicit form of \mathcal{S}_I see Eqs. (D1), (46), and (47)]. Thus, the interaction terms seem to contain four more fields than those appearing in \mathcal{S}_F (45) which determines the free dynamics. This is of course related to the aforementioned arbitrariness in the time evolution of the system. The imposition of the four gauge conditions $\mathcal{Q}_\alpha = 0$ cures this arbitrariness by expressing h , \tilde{h} , and h_i^T in terms of the fields appearing in \mathcal{S}_F . This observation also demonstrates the necessity to consider *nonlinearized* gauge transformations when interactions are included. Notice that \mathcal{S}_F is explicitly invariant under linearized transformations, and hence it would seem that the action as a whole is not since \mathcal{S}_I explicitly depends on the gauge chosen, even at the linear level. This contradicts the well-known exact gauge invariance of the action.⁵ The resolution of this seeming paradox lies in the observation that the exact gauge invariance of the action implies that for interaction terms of order up to n , one *must be* able to absorb any such apparently gauge dependent terms by nonlinear *field redefinitions*: gauge transformations of order $n - 1$. An explicit example for cubic interactions has been worked out in [13]. With this observation, quantities that have been calculated in different gauges can be compared using the appropriate field redefinitions.

³The scalar perturbation is usually expressed in terms of the Sasaki-Mukhanov variable $v = a(\varphi - z(h - \nabla^2 \tilde{h}))$ with $z = \mathcal{P}_\phi / [6a^3 H]$, and the corresponding free action, when written in conformal time, is shown in Eq. (B28) of Appendix B.

⁴If the gauge $\varphi = 0$ is chosen, w represents the curvature perturbation and is also constant during inflation on long wavelengths to all orders in perturbation theory [13].

⁵Up to boundary terms which we ignore in this paper.

B. The path integral

The results of the previous section can be used to obtain a convenient quantization scheme for the perturbations, including the interactions. According to the above discussion, inflationary perturbations are described by a constrained Hamiltonian system with action (22). In general, the quantization of such systems is easily formulated in terms of a path integral [61]. Indeed, the transition amplitude from state $|\mathcal{A}; t_{\mathcal{A}}\rangle$ at $t = t_{\mathcal{A}}$ to $|\mathcal{B}; t_{\mathcal{B}}\rangle$ at time $t_{\mathcal{B}}$ can be written as

$$\begin{aligned} \langle \mathcal{B}; t_{\mathcal{B}} | \mathcal{A}; t_{\mathcal{A}} \rangle &= \int_{(h_{ij}, \varphi)(t_{\mathcal{A}}) = \mathcal{A}}^{(h_{ij}, \varphi)(t_{\mathcal{B}}) = \mathcal{B}} \\ &\times [\mathcal{D}h_{ij} \mathcal{D}\varphi \mathcal{D}\pi^{ij} \mathcal{D}\pi_{\varphi} \mathcal{D}n \mathcal{D}N_i] \\ &\times \prod_{\alpha} \delta[\mathcal{Q}_{\alpha}] |\text{Det}\{\mathcal{Q}_{\alpha}, C_{\beta}\}| e^{i\mathcal{S}_{\text{pert}}}, \quad (48) \end{aligned}$$

where $\mathcal{S}_{\text{pert}}$ is given in (22)–(24), integrated over time $t' \in [t_{\mathcal{A}}, t_{\mathcal{B}}]$, and we have schematically denoted the initial (\mathcal{A}) and final (\mathcal{B}) field configurations by the limits of the path integral. The transition amplitude (48) is the fundamental quantity for calculating quantum correlators of any kind and the starting point for calculating the correct Lagrangian path integral. According to [61], it is *independent* of the choice of gauge conditions \mathcal{Q}_{α} , as long as the boundary states are appropriately defined in an invariant manner. We shall address this issue in a forthcoming publication. Note the appearance of a nontrivial measure containing $|\text{Det}\{\mathcal{Q}_{\alpha}, C_{\beta}\}|$, the determinant of the Poisson brackets between the constraints C_{α} and the gauge conditions \mathcal{Q}_{α} . Equation (48) is a self-contained starting point for doing perturbative calculations. By splitting the action as above in (22)–(24)

$$\mathcal{S}_{\text{pert}} = \mathcal{S}_F + \mathcal{S}_I, \quad (49)$$

it is possible to evaluate the path integral (48) perturbatively, expanding in powers of the interaction action, after including an appropriate representation of the determinant. The free action \mathcal{S}_F then determines the propagators while \mathcal{S}_I gives the various interaction vertices, completely defining a diagrammatic expansion.

The focal point of the previous section has been the free action (23) and (34)–(36) that can be brought to the diagonal form (45) from which propagators can be easily obtained. The field redefinitions (41)–(44) are linear shifts in all the fields of the action which do not affect the form of the functional measure in the path integral. More precisely, they contribute time dependent—but field independent—factors which cancel in our later computations. The transition amplitude can therefore also be written in terms of the redefined fields as

$$\begin{aligned} \langle \mathcal{B}; t_{\mathcal{B}} | \mathcal{A}; t_{\mathcal{A}} \rangle &= \int_{(h_{ij}^{TT}, w)(t_{\mathcal{A}}) = \mathcal{A}}^{(h_{ij}^{TT}, w)(t_{\mathcal{B}}) = \mathcal{B}} \\ &\times [\mathcal{D}w \mathcal{D}h_{ij}^{TT} \mathcal{D}\sigma \mathcal{D}h \mathcal{D}\tilde{h} \mathcal{D}h_j^T] \\ &\times \prod_{\alpha} \delta[\mathcal{Q}_{\alpha}] |\text{Det}\{\mathcal{Q}_{\alpha}, C_{\beta}\}| \\ &\times e^{i\mathcal{S}_F[w, h_{ij}^{TT}, \sigma] + i\mathcal{S}_I}, \quad (50) \end{aligned}$$

making it evident that the physical propagating degrees of freedom of the system are the quanta of w and h_{ij}^{TT} . $\mathcal{D}\sigma \equiv \mathcal{D}\rho^{ij} \mathcal{D}\rho_{\varphi} \mathcal{D}\tilde{n} \mathcal{D}\tilde{N}_i^T \mathcal{D}\tilde{S}$ collectively denotes integration over the 11 auxiliary fields. One can note that the full action is quadratic in ρ_{ij} and ρ_{φ} which could in principle be integrated out. However, it proves convenient to keep them along with \tilde{n} , \tilde{N}_i^T , and \tilde{S} . The “propagation” of all these auxiliary fields is determined by (45) and is trivial. We here use the term propagation for the auxiliary fields rather loosely since they are not attached to wave operators. Since w and h_{ij}^{TT} are the dynamical degrees of freedom, it is clear that the initial and final states should be determined in terms of their free quanta.

To complete the derivation of the more workable expression for the path integral, we are left with the task of evaluating the determinant $|\text{Det}\{\mathcal{Q}_{\alpha}, C_{\beta}\}|$ in (50). Denoting

$$\Omega_{\alpha\beta} \equiv \{\mathcal{Q}_{\alpha}, C_{\beta}\}|_{\mathcal{Q}_{\gamma}=0}, \quad (51)$$

where the gauge condition $\mathcal{Q}_{\gamma} = 0$ has been imposed after evaluating the Poisson brackets, we can represent the determinant as a path integral over anticommuting scalar fields—Faddeev-Popov ghosts— $\bar{\eta}_{\alpha}$ and η_{α} ⁶

$$|\text{Det}\Omega_{\alpha\beta}| = \int [\mathcal{D}\bar{\eta}_{\gamma} \mathcal{D}\eta_{\gamma}] e^{i \int d^4x \bar{\eta}_{\alpha} \Omega_{\alpha\beta} \eta_{\beta}}. \quad (52)$$

The transition amplitude is then written as

$$\begin{aligned} \langle \mathcal{B}; t_{\mathcal{B}} | \mathcal{A}; t_{\mathcal{A}} \rangle &= \int_{(h_{ij}^{TT}, w)(t_{\mathcal{A}}) = \mathcal{A}}^{(h_{ij}^{TT}, w)(t_{\mathcal{B}}) = \mathcal{B}} \\ &\times [\mathcal{D}w \mathcal{D}h_{ij}^{TT} \mathcal{D}\sigma \mathcal{D}h_j^T \mathcal{D}h \mathcal{D}\tilde{h} \mathcal{D}\bar{\eta} \mathcal{D}\eta] \\ &\times \prod_{\alpha} \delta[\mathcal{Q}_{\alpha}] \\ &\times e^{i\mathcal{S}_F + i \int d^4x \bar{\eta}_{\alpha} \Omega_{\alpha\beta}^F \eta_{\beta} + i\mathcal{S}_I + i \int d^4x \bar{\eta}_{\alpha} \Omega_{\alpha\beta}^I \eta_{\beta}}, \quad (53) \end{aligned}$$

where we have split $\Omega_{\alpha\beta} = \Omega_{\alpha\beta}^F + \Omega_{\alpha\beta}^I$ into a free part, determining the propagation of $\bar{\eta}_{\alpha}$ and η_{α} , and an interaction part which contains the other fields and couples them to $\bar{\eta}_{\alpha}$ and η_{α} . Two common gauge choices \mathcal{Q}_{α} and the resulting matrices $\Omega_{\alpha\beta}$ are discussed in the Appendix. We now have a complete description of the transition

⁶See, for example, Eqs. (5.1.31–32) in Ref. [63].

amplitude amenable to perturbative calculations. Equation (53) is the central result of this paper.

III. THE FUNCTIONAL IN-IN FORMALISM AND EXPECTATION VALUES

The previous section described the derivation of a path-integral representation for the transition amplitude between states of inflationary perturbations in the past and the future, separated by time t . The perturbative evaluation of such amplitudes between the far past and the far future leads to the usual Feynman diagrams for the calculation of S-matrix elements, i.e. scattering amplitudes between in-states and out-states, useful in deriving predictions for the scattering experiments of particle physics. However, in cosmological applications the quantities of interest are not scattering amplitudes between predefined states (a boundary value problem), but expectation values of operators at a time t , given an initial state $|\text{in}, t_{\mathcal{A}}\rangle$ at time $t_{\mathcal{A}}$ (an initial value problem). For the purpose of this work, this state is defined in terms of the freely propagating modes of w and h_{ij}^{TT} at very early times: $t_{\mathcal{A}} \rightarrow t_{-\infty}$, and is taken to be an appropriately defined vacuum state $|\Omega, t_{-\infty}\rangle$. More generally, one has to begin the evolution from a finite time and with an initial state chosen such that it includes corrections due to the interactions, such that all field correlators are finite at the initial hypersurface. In this setting, the calculation of expectation values leads to a modification of the standard diagrammatic rules of quantum field theory (QFT). Here, we sketch the functional in-in formalism for obtaining expectation values and arrive at the appropriate diagrammatic rules. For more details, see [64,65] as well as the Appendix of [44].

A. The in-in generating functional

In analogy with standard QFT, one defines the in-in generating functional $Z[J_-, J_+]$ which contains *two* arbitrary and independent sources J_+ and J_- :

$$Z[J_-, J_+] = \sum_{\alpha} \langle \Omega, t_{-\infty} | \alpha, t_{\text{out}} \rangle_{J_-} \langle \alpha, t_{\text{out}} | \Omega, t_{-\infty} \rangle_{J_+}, \quad (54)$$

where $\langle \alpha, t_{\text{out}} | \Omega, t_{-\infty} \rangle_J$ denotes the transition amplitude from the aforementioned vacuum state at time $t_{-\infty}$ to a state $|\alpha, t_{\text{out}}\rangle$ at time t_{out} in the presence of a source J . Expression (54) can be interpreted as going forward in time in the presence of a source J_+ and then returning back in time in the presence of a source J_- . This explains the term ‘‘closed-time-path formulation’’ that is sometimes also used. However, in what follows such an interpretation is not needed. The final time t_{out} can be taken to be any time later than all the times of interest in the calculation; here we take t_{out} to be in the far future, $t_{\text{out}} \rightarrow +\infty$. Obviously, $Z[J, J] = 1$.

The transition amplitudes appearing in (54) can be expressed in terms of path integrals. Collectively denoting the relevant fields by ξ , we have that

$$Z[J_-, J_+] = \int [\mathcal{D}\xi_+ \mathcal{D}\xi_-] e^{iS[\xi_+] - iS[\xi_-] + iJ_+ \xi_+ - iJ_- \xi_-}, \quad (55)$$

with boundary conditions appropriate for the state $|\Omega, t_{-\infty}\rangle$ at $t_{-\infty}$ and the constraint that $\xi_+(t_{\text{out}}) = \xi_-(t_{\text{out}})$. Once $Z[J_-, J_+]$ has been calculated, expectation values can be obtained by taking variational derivatives:

$$\begin{aligned} & \frac{\delta}{-i\delta J_-(y_1)} \cdots \frac{\delta}{-i\delta J_-(y_m)} \\ & \times \frac{\delta}{i\delta J_+(x_1)} \cdots \frac{\delta}{i\delta J_+(x_n)} Z[J_-, J_+] \Big|_{J_+ = J_- = 0} \\ & = \langle \Omega; t_{-\infty} | \{ \bar{T} \xi(y_1) \cdots \xi(y_m) \} \\ & \times \{ T \xi(x_1) \cdots \xi(x_n) \} | \Omega; t_{-\infty} \rangle, \end{aligned} \quad (56)$$

where T denotes operator time ordering, $T\xi(x)\xi(x') = \theta(x^0 - x'^0)\xi(x)\xi(x') + \theta(x'^0 - x^0)\xi(x')\xi(x)$, and \bar{T} operator anti-time-ordering, $\bar{T}\xi(x)\xi(x') = \theta(x'^0 - x^0)\xi(x)\xi(x') + \theta(x^0 - x'^0)\xi(x')\xi(x)$.

The exact calculation of $Z[J_+, J_-]$ is impossible for interacting theories but it can be evaluated perturbatively. Writing the action as a free part plus interactions

$$S[\xi] = \int d^4x \frac{1}{2} \xi \mathcal{D}_0 \xi + S_I[\xi], \quad (57)$$

we have⁷

$$\begin{aligned} Z[J_-, J_+] &= e^{iS_I[\delta/(i\delta J_+)] - iS_I[\delta/(-i\delta J_-)]} \int [\mathcal{D}\xi_+ \mathcal{D}\xi_-] \\ & \times \exp \left\{ \int d^4x \left[\frac{1}{2} (\xi_+, \xi_-) \begin{pmatrix} i\mathcal{D}_0 & 0 \\ 0 & -i\mathcal{D}_0 \end{pmatrix} \right. \right. \\ & \left. \left. \times \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} + (\xi_+, \xi_-) \begin{pmatrix} iJ_+ \\ -iJ_- \end{pmatrix} \right] \right\} \\ & = e^{iS_I[\delta/(i\delta J_+)] - iS_I[\delta/(-i\delta J_-)]} \\ & \times \exp \left\{ \int d^4x d^4x' \frac{1}{2} (iJ_+, -iJ_-)(x) \right. \\ & \left. \times \begin{pmatrix} i\Delta_{++} & i\Delta_{+-} \\ i\Delta_{-+} & i\Delta_{--} \end{pmatrix} (x; x') \begin{pmatrix} iJ_+ \\ -iJ_- \end{pmatrix} (x') \right\}, \end{aligned} \quad (58)$$

where \mathcal{D}_0 denotes the operator acting on ξ as implied by the free action (45). The last equality is derived by shifting the fields

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} \rightarrow \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} - \begin{pmatrix} i\Delta_{++} & i\Delta_{+-} \\ i\Delta_{-+} & i\Delta_{--} \end{pmatrix} \begin{pmatrix} iJ_+ \\ -iJ_- \end{pmatrix}, \quad (59)$$

where the Keldysh 2×2 matrix of propagators is defined as the matrix inverse of \mathcal{D}_0 ,

⁷Formula (58) applies also for the (fermionic) ghost fields, as can be seen e.g. from Eq. (5.1.32) in Ref. [63], where $i\Delta$ are the ghost propagators.

$$\begin{pmatrix} \mathcal{D}_0 & 0 \\ 0 & -\mathcal{D}_0 \end{pmatrix}(x) \begin{pmatrix} i\Delta_{++} & i\Delta_{+-} \\ i\Delta_{-+} & i\Delta_{--} \end{pmatrix}(x; x') = i\delta^4(x - x'). \quad (60)$$

The J -independent path integrals in (58) contribute phase factors which cancel. The diagonal elements in (60) are the (Wightman) Green functions of the operator \mathcal{D}_0 which appear in usual S-matrix calculations in QFT, while the off-diagonal elements are solutions of the homogeneous equations, $\mathcal{D}_0 i\Delta_{-+} = \mathcal{D}_0 i\Delta_{+-} = 0$. These off-diagonal elements in (60) are necessary in order to impose the boundary condition $\xi_+ = \xi_-$ at time t_{out} on the J -dependent parts of ξ_+ and ξ_- .

The explicit form of the Δ propagators can be found by using (56) and (58), after setting $S_I \rightarrow 0$. We see that for propagating fields the appropriate elements of the Keldysh (bosonic) propagators in (60) are

$$\begin{aligned} i\Delta_{++}(x; x') &= \langle \Omega | T \xi(x) \xi(x') | \Omega \rangle \\ &= \theta(x^0 - x'^0) i\Delta_{-+}(x; x') + \theta(x'^0 - x^0) \\ &\quad \times i\Delta_{+-}(x; x') \end{aligned} \quad (61)$$

$$\begin{aligned} i\Delta_{--}(x; x') &= \langle \Omega | \bar{T} \xi(x) \xi(x') | \Omega \rangle \\ &= \theta(x^0 - x'^0) i\Delta_{+-}(x; x') + \theta(x'^0 - x^0) \\ &\quad \times i\Delta_{-+}(x; x'), \end{aligned} \quad (62)$$

where the Wightman functions $i\Delta_{+-}$ and $i\Delta_{-+}$ are defined as

$$\begin{aligned} i\Delta_{+-}(x; x') &= \langle \Omega | \xi(x') \xi(x) | \Omega \rangle, \\ i\Delta_{-+}(x; x') &= \langle \Omega | \xi(x) \xi(x') | \Omega \rangle, \end{aligned} \quad (63)$$

and $|\Omega\rangle$ denotes the physical vacuum state.⁸ If the fields are nonpropagating, the Wightman functions vanish, $\Delta_{+-} = 0 = \Delta_{-+}$ and Eq. (60) is easily inverted, $i\Delta_{\pm\pm} = \pm(\mathcal{D}_0)^{-1}(x) i\delta^4(x - x')$.

B. Propagators

Let us now focus on the problem at hand and, in particular, on the propagators that correspond to \mathcal{S}_F given in (45). The propagators for w ,

⁸In statistical field theory pure states are replaced by more general mixed states (see e.g. [66,67]), and the evolution of a system is generated by the density operator $\hat{\rho}(t)$. In this case the Green functions are defined by the more general relation, $i\Delta_{++}(x; x') = \text{Tr}[\hat{\rho}(t) T \xi(x) \xi(x')]$, etc., where the definitions must be picture independent. For example, if the density operator $\hat{\rho}(t)$ is in the Schrödinger picture, the fields $\xi(x)$ must be in the Heisenberg picture.

$$\begin{aligned} i\Delta_{++}^w(x; x') &= \langle \Omega | T w(x) w(x') | \Omega \rangle, \\ i\Delta_{--}^w(x; x') &= \langle \Omega | \bar{T} w(x) w(x') | \Omega \rangle, \\ i\Delta_{-+}^w(x; x') &= \langle \Omega | w(x) w(x') | \Omega \rangle, \\ i\Delta_{+-}^w(x; x') &= \langle \Omega | w(x') w(x) | \Omega \rangle, \end{aligned} \quad (64)$$

and for h_{ij}^{TT} ,

$$\begin{aligned} i\Delta_{++}^{ijkl}(x; x') &= \langle \Omega | T h_{ij}^{TT}(x) h_{kl}^{TT}(x') | \Omega \rangle, \\ i\Delta_{--}^{ijkl}(x; x') &= \langle \Omega | \bar{T} h_{ij}^{TT}(x) h_{kl}^{TT}(x') | \Omega \rangle, \\ i\Delta_{-+}^{ijkl}(x; x') &= \langle \Omega | h_{ij}^{TT}(x) h_{kl}^{TT}(x') | \Omega \rangle, \\ i\Delta_{+-}^{ijkl}(x; x') &= \langle \Omega | h_{ij}^{TT}(x') h_{kl}^{TT}(x) | \Omega \rangle \end{aligned} \quad (65)$$

can be obtained by solving the equations

$$\begin{aligned} \left(-\partial_t z^2 a^3 \frac{\partial_t}{N} + \bar{N} z^2 a \nabla^2 \right) i\Delta_{\pm\pm}^w(x; x') &= \pm i\delta^4(x - x'), \\ \left(-\partial_t z^2 a^3 \frac{\partial_t}{N} + \bar{N} z^2 a \nabla^2 \right) i\Delta_{\pm\mp}^w(x; x') &= 0 \end{aligned} \quad (66)$$

and⁹

$$\begin{aligned} \left(-\partial_t a^3 \frac{\partial_t}{N} + \bar{N} a \nabla^2 \right) i\Delta_{\pm\pm}^{ijkl}(x; x') &= \pm (P_{ik} P_{jl} + P_{il} P_{jk} \\ &\quad - P_{ij} P_{kl}) i\delta^4(x - x'), \\ \left(-\partial_t a^3 \frac{\partial_t}{N} + \bar{N} a \nabla^2 \right) i\Delta_{\pm\mp}^{ijkl}(x; x') &= 0, \end{aligned} \quad (67)$$

where $z \equiv \sqrt{\epsilon}/3$, $P_{ij} = \delta_{ij} - \partial_i \partial_j / \nabla^2$ denotes the transverse projector, and the operator ∇^{-2} is defined by $\nabla_{\mathbf{x}}^{-2} \delta(\mathbf{x} - \mathbf{y}) = i\Delta_0(\mathbf{x}; \mathbf{y}) \equiv -1/[4\pi|\mathbf{x} - \mathbf{y}|]$. The projectors on the right-hand side of (67) project out the potentially unphysical (scalar and vector) degrees of freedom, which are not transverse or which are not traceless, assuring the correct counting of the physical degrees of freedom. Because of the infrared problems which plague Eqs. (66) and (67) in accelerating space-times, the corresponding solutions are not simple and need to be handled with due care [52,53,68]. Here we just note that, when the fields in (64) and (65) are Fourier transformed to the spatial momentum space,

⁹The normalization on the right-hand side can be checked by noting that, if the graviton were not transverse or traceless, i.e. just an ordinary 3×3 symmetric matrix, then the propagator equation would read

$$\begin{aligned} \frac{1}{2} \left[-\partial_t a^3 \frac{\partial_t}{N} + \bar{N} a \nabla^2 \right] i\Delta_{\pm\pm}^{ijkl}(x; x') \\ \rightarrow \pm \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) i\delta^4(x - x'). \end{aligned}$$

Equation (67) is obtained from this equation by removing the longitudinal and the trace degrees of freedom.

$$w(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} [w(t, k)a_{\mathbf{k}} + w^*(t, k)a_{-\mathbf{k}}^+],$$

$$h_{ij}^{TT}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} \sum_{\alpha=+, \times} \epsilon_{ij}^{\alpha}(\mathbf{k}) [h(t, k)a_{\alpha, \mathbf{k}} + h^*(t, k)a_{\alpha, -\mathbf{k}}^+] \quad (68)$$

and written in conformal time ($\bar{N} \rightarrow a(\eta)$, $t \rightarrow \eta$), Eqs. (66) and (67) simplify to

$$(-\partial_{\eta} z^2 a^2 \partial_{\eta} - z^2 a^2 \mathbf{k}^2) i\Delta_{\pm\pm}^w(\eta; \eta'; \mathbf{k}) = \pm i\delta(\eta - \eta'),$$

$$(-\partial_{\eta} \eta^2 \partial_{\eta} - a^2 \mathbf{k}^2) i\Delta_{\pm\pm}^{ijkl}(\eta; \eta'; \mathbf{k}) = \pm (\bar{P}_{ik}\bar{P}_{jl} + \bar{P}_{il}\bar{P}_{jk} - \bar{P}_{ij}\bar{P}_{kl}) i\delta(\eta - \eta'),$$

$$\bar{P}_{ij} = \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}. \quad (69)$$

We arrived at these results by choosing the physical state $|\Omega\rangle$ that is destroyed by the annihilation operators $a_{\mathbf{k}}$ and $a_{\alpha, \mathbf{k}}$ in (68), $a_{\mathbf{k}}|\Omega\rangle = 0$, $a_{\alpha, \mathbf{k}}|\Omega\rangle = 0$. The solutions of Eqs. (69) can be expressed in terms of the mode functions $w(\eta, k)$ and $h(\eta, k)$ as

$$i\Delta_{++}^w = \theta(\eta - \eta') i\Delta_{-+}^w + \theta(\eta' - \eta) i\Delta_{+-}^w,$$

$$i\Delta_{+-}^w = w^*(\eta, k)w(\eta', k),$$

$$i\Delta_{-+}^w = w(\eta, k)w^*(\eta', k),$$

$$i\Delta_{++}^{ijmn} = \theta(\eta - \eta') i\Delta_{-+}^{ijmn} + \theta(\eta' - \eta) i\Delta_{+-}^{ijmn},$$

$$i\Delta_{+-}^{ijmn} = \sum_{\alpha} \epsilon_{ij}^{\alpha}(\mathbf{k}) \epsilon_{kl}^{\alpha}(\mathbf{k}) h^*(\eta, k) h(\eta', k),$$

$$i\Delta_{-+}^{ijmn} = \sum_{\alpha} \epsilon_{ij}^{\alpha}(\mathbf{k}) \epsilon_{kl}^{\alpha}(\mathbf{k}) h(\eta, k) h^*(\eta', k),$$

where $w(\eta, k)$ and $h(\eta, k)$ satisfy

$$\left(-\partial_{\eta}^2 - \mathbf{k}^2 + \frac{(az)''}{az}\right) (azw(\eta, k)) = 0,$$

$$\left(-\partial_{\eta}^2 - \mathbf{k}^2 + \frac{a''}{a}\right) (ah(\eta, k)) = 0, \quad (70)$$

where ϵ_{ij}^{α} denotes the graviton polarization tensor.¹⁰ These equations can be solved for certain choices of the scale factor $a = a(\eta)$. For example, when $\epsilon = -\dot{H}/H^2 = \text{constant}$, $a = [-(1 - \epsilon)H_0\eta]^{1/(1-\epsilon)}$, where $H_0 = H(\eta_0)$, such that $a''/a = (2 - \epsilon)/[(1 - \epsilon)^2\eta^2]$ and (70) can be solved in terms of the Hankel functions,

$$w(k, \eta) = \frac{1}{az} \sqrt{\frac{\pi|\eta|}{4}} [\alpha_w(k)H_{\nu}^{(1)}(k|\eta|) + \beta_w(k)H_{\nu}^{(2)}(k|\eta|)],$$

$$\nu = \frac{3 - \epsilon}{2(1 - \epsilon)}, \quad |\alpha_w(k)|^2 - |\beta_w(k)|^2 = 1,$$

$$h(k, \eta) = \frac{1}{a} \sqrt{\frac{\pi|\eta|}{2}} [\alpha_h(k)H_{\nu}^{(1)}(k|\eta|) + \beta_h(k)H_{\nu}^{(2)}(k|\eta|)],$$

$$|\alpha_h(k)|^2 - |\beta_h(k)|^2 = 1, \quad (71)$$

and where $\nu_w = \nu_h \equiv \nu = (3 - \epsilon)/[2(1 - \epsilon)]$, and the coefficients $(\alpha_w(k), \beta_w(k))$ and $(\alpha_h(k), \beta_h(k))$ of positive and negative frequency mode functions are chosen such that the real space propagators are infrared finite [42,53]. Alternatively, one can place the Universe in a large comoving box [43,45,52,68], or in a universe with spatially positively curved sections, which effectively remove the far infrared modes, thus rendering the state infrared finite. For nonconstant ϵ , solutions can be obtained in the slow-roll approximation, by using the de Sitter mode functions around the time of first Hubble crossing [13], or the methods of [69].

The two-point correlators for all of the auxiliary fields, ρ^{ij} , ρ_{φ} , \tilde{n} , \tilde{N}_i^T , and \tilde{S} , can be straightforwardly obtained. It is clear that the off-diagonal elements (Wightman functions) of (60) vanish, while the diagonal elements are either

proportional to delta functions or involve spatially nonlocal operators. In particular, the Green functions for the auxiliary fields as implied by (45) are¹¹

¹⁰The polarization tensor ϵ_{ij}^{α} satisfies $\sum_{\alpha=+, \times} \epsilon_{ij}^{\alpha} \epsilon_{kl}^{\alpha} = (1/2) \times [\bar{P}_{ik}\bar{P}_{jl} + \bar{P}_{il}\bar{P}_{jk} - \bar{P}_{ij}\bar{P}_{kl}]$ and $\sum_{ij} \epsilon_{ij}^{\alpha} \epsilon_{ij}^{\beta} = \delta^{\alpha\beta}$.

¹¹Equation (73) is obtained by noting that the inverse of A_{ijkl} is $(A^{-1})_{ijkl} = (1/4)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl})$ in the sense that $A_{ijkl}(A^{-1})_{klmn} = \frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})$. That this indeed is the correct definition of the inverse can be checked by calculating, for example, the propagator for the (scalar) trace $\rho \equiv \delta_{ij}\rho^{ij}$ of ρ^{ij} , for which the free action (45) yields $\int d^4x [2\bar{N}a^3 H^2 \rho^2/3]$. The corresponding propagator is, of course, $i\Delta_{\pm\pm}^{\rho}(x; x') = \pm[3/(4\bar{N}a^3 H^2)] i\delta^4(x - x')$, which agrees with the double trace of Eq. (73), obtained by applying $\delta_{ij}\delta_{kl}$. One can analogously check that the propagators for the other components of ρ^{ij} are correct.

$$i\Delta_{\pm\pm}^{\rho_\varphi}(x; x') = \mp \frac{1}{4\bar{N}a^3H^2\epsilon} i\delta^4(x - x') \quad (72)$$

$$i(\Delta_{\pm\pm}^{\rho})_{ijkl}(x; x') = \mp \frac{1}{16\bar{N}a^3H^2} (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl}) i\delta^4(x - x') \quad (73)$$

$$i\Delta_{\pm\pm}^{\bar{n}}(x; x') = \mp \frac{\bar{N}}{4(3 - \epsilon)a^3H^2} i\delta^4(x - x') \quad (74)$$

$$i(\Delta_{\pm\pm}^T)_{ij}(x; x') = \mp a\bar{N}\nabla_x^{-2} P_{ij} i\delta^4(x - x') \quad (75)$$

$$i\Delta_{\pm\pm}^{\bar{s}}(x; x') = \pm \frac{1}{4} a\bar{N}(3 - \epsilon)\nabla_x^{-4} i\delta^4(x - x'), \quad (76)$$

with all the $i\Delta_{+-} = 0$ for the auxiliary fields.

Let us now turn to the ghost fields. Their propagators satisfy the equations of motion of the general form,

$$\begin{aligned} \Omega_{\alpha\beta}^F(i\Delta_{\pm\pm}^{\eta})_{\beta\gamma}(x; x') &= \pm i\delta_{\alpha\gamma} \delta^4(x - x'), \\ \Omega_{\alpha\beta}^F(i\Delta_{\pm\mp}^{\eta})_{\beta\gamma}(x; x') &= 0. \end{aligned} \quad (77)$$

Whether the ghosts are propagating or not depends of course on the choice of gauge conditions and, in particular, whether the latter contain time derivatives. In the former case we have

$$\begin{aligned} (i\Delta_{\alpha\beta}^{\eta})_{++}(x; x') &= \langle \Omega | T\eta_{\alpha}(x)\bar{\eta}_{\beta}(x') | \Omega \rangle \\ &= \theta(x^0 - x'^0)(i\Delta_{\alpha\beta}^{\eta})_{-+}(x; x') \\ &\quad + \theta(x'^0 - x^0)(i\Delta_{\alpha\beta}^{\eta})_{+-}(x; x'), \\ (i\Delta_{\alpha\beta}^{\eta})_{--}(x; x') &= \langle \Omega | \bar{T}\eta_{\alpha}(x)\bar{\eta}_{\beta}(x') | \Omega \rangle \\ &= \theta(x^0 - x'^0)(i\Delta_{\alpha\beta}^{\eta})_{+-}(x; x') \\ &\quad + \theta(x'^0 - x^0)(i\Delta_{\alpha\beta}^{\eta})_{-+}(x; x'), \end{aligned} \quad (78)$$

where

$$\begin{aligned} (i\Delta_{\alpha\beta}^{\eta})_{+-}(x; x') &= -\langle \Omega | \bar{\eta}_{\beta}(x')\eta_{\alpha}(x) | \Omega \rangle, \\ (i\Delta_{\alpha\beta}^{\eta})_{-+}(x; x') &= \langle \Omega | \eta_{\alpha}(x)\bar{\eta}_{\beta}(x') | \Omega \rangle. \end{aligned} \quad (79)$$

Note that because they are anticommuting, their corresponding time ordered and anti-time-ordered propagators differ slightly from the bosonic counterparts (61) and (62). Since common gauge choices do not lead to propagating ghosts, the corresponding propagators have simpler form than the one given in (78) and (79). Consider, for example, the *tensor gauge*

$$\mathcal{Q}_0 = h, \quad \mathcal{Q}_i = \partial_j \left(h_{ij} - \frac{\delta_{ij}}{3} h \right). \quad (80)$$

Imposing the gauge conditions $\mathcal{Q}_{\alpha} = 0$ sets $h = \tilde{h} = h_i^T = 0$ and, hence, the metric perturbation contains only the (transverse traceless) tensor part of the spatial metric, while w represents the perturbation of the inflaton. In linear

theory it corresponds to the uniform spatial expansion gauge, or the comoving gauge.

From Eq. (C6) in Appendix C, it follows that the free part of the ghost operator $\Omega_{\alpha\beta}^F$ is nondynamical in the tensor gauge; it reads

$$\Omega^F = \begin{pmatrix} -6H & -2a^{-2}\partial_j^x \\ 0 & -a^{-2}\left(\delta_{ij}\nabla_x^2 + \frac{1}{3}\partial_i^x\partial_j^x\right) \end{pmatrix} \delta(\mathbf{x} - \mathbf{y}). \quad (81)$$

The ghost equations, when written in a condensed matrix notation $\Omega^F \cdot (i\Delta^{\eta})_{\pm\pm} = \pm i\delta^4$, $\Omega^F \cdot (i\Delta^{\eta})_{\pm\mp} = 0$, imply

$$\begin{aligned} (i\Delta^{\eta})_{\pm\pm}(x; x') &= \begin{pmatrix} (i\Delta_{00}^{\eta})_{\pm\pm}(x; x') & (i\Delta_{0j}^{\eta})_{\pm\pm}(x; x') \\ (i\Delta_{i0}^{\eta})_{\pm\pm}(x; x') & (i\Delta_{ij}^{\eta})_{\pm\pm}(x; x') \end{pmatrix} \\ &= \begin{pmatrix} \mp \frac{1}{6H} & \pm \frac{1}{4H} \frac{\partial_j^x}{\nabla_x^2} \\ 0 & \mp a^2 \left(\delta_{ij} - \frac{1}{4} \frac{\partial_i^x \partial_j^x}{\nabla_x^2} \right) \frac{1}{\nabla_x^2} \end{pmatrix} i\delta^4(x - x') \end{aligned} \quad (82)$$

and $(i\Delta^{\eta})_{\pm\mp}(x; x') = 0$.

The second gauge we consider here is the *uniform field gauge*,

$$\mathcal{Q}_0 = \varphi, \quad \mathcal{Q}_i = \partial_j \left(h_{ij} - \frac{\delta_{ij}}{3} h \right). \quad (83)$$

In this gauge $\mathcal{Q}_{\alpha} = 0$ fixes $\varphi = \tilde{h} = h_i^T = 0$ and hence the metric perturbation contains the tensor and the spatial trace perturbation. This means that w now represents the perturbation of the local spatial volume, or the spatial curvature perturbation. From Eq. (C8) we see that, also in the uniform field gauge, the free ghost operator is nondynamical,

$$\Omega^F = \begin{pmatrix} -2\sqrt{\epsilon}H & 0 \\ 0 & -a^{-2}\left(\delta_{ij}\nabla_x^2 + \frac{1}{3}\partial_i^x\partial_j^x\right) \end{pmatrix} \delta(\mathbf{x} - \mathbf{y}), \quad (84)$$

implying the following two-point ghost correlators:

$$\begin{aligned} (i\Delta^{\eta})_{\pm\pm}(x; x') &= \begin{pmatrix} \mp \frac{1}{2\sqrt{\epsilon}H} & 0 \\ 0 & \mp a^2 \left(\delta_{ij} - \frac{1}{4} \frac{\partial_i^x \partial_j^x}{\nabla_x^2} \right) \frac{1}{\nabla_x^2} \end{pmatrix} \\ &\quad \times i\delta^4(x - x'); \\ (i\Delta^{\eta})_{\pm\mp}(x; x') &= 0. \end{aligned} \quad (85)$$

Notice that the gauge (83) is not suitable when one is interested in the de Sitter limit $\epsilon \rightarrow 0$, since the ghost propagator (85) appears singular in that limit. That problem can be fixed by simply replacing \mathcal{Q}_0 in (83) by $\mathcal{Q}_0 = \varphi/\dot{\phi}$. In order to make the whole action (22)–(24) regular, one also needs to redefine the canonical momentum of φ as

$\mathcal{P}_\phi \pi_\phi \rightarrow \pi_\phi$, and similarly $-(\sqrt{\epsilon}/3)w \rightarrow \tilde{\phi}$, such that the corresponding propagators (72) and (66) become regular. When these changes are exacted, the de Sitter limit of our path integral (53) will be regular in the uniform field gauge.

C. Diagrammatic rules

After expanding (58) in powers of S_I , we are led to a diagrammatic expansion for $Z[J_+, J_-]$, analogous to that of the vacuum-to-vacuum amplitude in the presence of a source J of standard QFT. Diagrams are now composed of external lines coupling $+$ and $-$ currents at external points to $+$ and $-$ vertices, as well as internal lines coupling $+$ and $-$ vertices with each other. Each vertex carries a $+i$ or $-i$ factor respectively, along with the relevant interaction factors. Vertices and currents of the same valence ($+$, $+$ and $-$, $-$) are joined by lines corresponding to Δ_{++} and Δ_{--} propagators, respectively, while those of opposite valence ($+$, $-$ and $-$, $+$) are joined by Δ_{+-} and Δ_{-+} propagators, respectively. Finally, one integrates over the coordinates of internal points.

It is important to keep in mind that, as mentioned above, the two-point correlators of the auxiliary fields and of the ghosts in common gauges satisfy $\Delta_{+-} = \Delta_{-+} = 0$ while the Δ_{++} and Δ_{--} are either delta functions or involve nonlocal spatial operators of the type $\delta(t-t')/\|\mathbf{x}-\mathbf{x}'\|$ or even $\delta(t-t')\nabla^{-2}\|\mathbf{x}-\mathbf{x}'\|^{-1} = -(4\pi)^{-1}\delta(t-t') \times \int d^3\mathbf{z}\|\mathbf{x}-\mathbf{z}\|^{-1}\|\mathbf{z}-\mathbf{x}'\|^{-1}$ as in Eqs. (75) and (76). Since there is no actual propagation in spacetime, these correlators always correspond to effective vertices, either local or nonlocal.

Once $Z[J_-, J_+]$ has been expressed as a sum of diagrams to the desired order in the interaction or in loops, variational derivatives will then give expectation values according to (56). Note that in case all fields in (56) are taken at the same time, the valence of the currents is immaterial; here we will be using J_+ , so that, e.g.,

$$\langle \Omega | \xi(t, \mathbf{x}) \xi(t, \mathbf{y}) | \Omega \rangle = \frac{\delta}{i\delta J_+(t, \mathbf{x})} \frac{\delta}{i\delta J_+(t, \mathbf{y})} \times Z[J_-, J_+]_{J_- = J_+ = 0}, \quad (86)$$

and

$$\langle \Omega | \xi(t, \mathbf{x}) \xi(t, \mathbf{y}) \xi(t, \mathbf{z}) | \Omega \rangle = \frac{\delta}{i\delta J_+(t, \mathbf{x})} \frac{\delta}{i\delta J_+(t, \mathbf{y})} \frac{\delta}{i\delta J_+(t, \mathbf{z})} \times Z[J_-, J_+]_{J_- = J_+ = 0}. \quad (87)$$

All of the above mirrors the corresponding discussion for standard QFT transition amplitudes. We can thus summarize the diagrammatic rules for the calculation of an n -point function $\langle \Omega | \xi(t, \mathbf{x}) \xi(t, \mathbf{y}) \dots \xi(t, \mathbf{z}) | \Omega \rangle$ as follows (see the Appendix of [44] for a complete definition of such rules):

- (i) draw the usual Feynman diagrams with all external points having a $+$ valence;

- (ii) all internal vertices are either $+$ or $-$ and carry a $+i$ or $-i$ factor, respectively, along with all the relevant interaction operators;
- (iii) connect external points with vertices and vertices with each other by the propagators of the appropriate valence;
- (iv) integrate over the temporal and spatial coordinates of all the vertices.

D. The tensor gauge and cosmological perturbations

The N -point functions of the trace of the spatial metric $\langle \Omega | h(t, \mathbf{x}_1) h(t, \mathbf{x}_2) \dots h(t, \mathbf{x}_N) | \Omega \rangle$ on comoving spacelike hypersurfaces are significant cosmological observables, particularly in relation with the non-Gaussianity of cosmological perturbations. During inflation, such quantities can be directly computed in the uniform field gauge, since in that gauge $w \rightarrow h$. However, the interaction terms in S_I are simplest in the tensor gauge and the question arises of how to relate results in these two different gauges. The issue is that, although the transition amplitude (53) is independent of the choice of gauge conditions, simply coupling w to a current and calculating expectation values does not automatically lead to a gauge-invariant expression. This is obvious since w is not invariant under nonlinear transformations and adding a Jw term in the action explicitly breaks its gauge invariance. A more physical way to understand this is to note that a time slice defined by some fixed value t of the background time corresponds to *physically different* hypersurfaces in different gauges. Thus, one cannot simply compare $\langle w^N(t) \rangle$ for fixed t in different gauges because it refers to physically different quantities evaluated on different timelike hypersurfaces. Such difficulties can be circumvented however if we define the observables of interest and the timelike hypersurfaces on which they are to be calculated in a geometrical fashion [70].

So, let us choose the tensor gauge to obtain the interaction terms. The physical interpretation of w is that it represents the quanta of the scalar field: $w = -(3/\sqrt{\epsilon})\phi$, while the metric perturbations are transverse traceless tensors. Interactions between these degrees of freedom are specified by S_I in this gauge, given explicitly in Sec. IV below. The transition amplitude that enters in (54) is then written as

$$\langle \Omega; t_{+\infty} | \Omega; t_{-\infty} \rangle = \int [\mathcal{D}w \mathcal{D}h_{ij}^{TT} \mathcal{D}\sigma \mathcal{D}\bar{\eta} \mathcal{D}\eta] \times e^{iS_F + i \int d^4x \bar{\eta}_\alpha \Omega_{\alpha\beta}^F \eta_\beta + iS_I + i \int d^4x \bar{\eta}_\alpha \Omega_{\alpha\beta}^I \eta_\beta}, \quad (88)$$

where the operator $\Omega_{\alpha\beta}$ is the one appropriate for the particular gauge choice—see (81) and (C6)—and the vacuum states have been obtained by considering large time intervals and the appropriate $i\epsilon$ prescription. The expectation value of any product of w 's can now be calculated using the diagrammatic rules described above, remember-

ing that the meaning of correlators in this gauge is

$$\begin{aligned} & \langle \Omega | w(t, \mathbf{x}_1) w(t, \mathbf{x}_2) \dots w(t, \mathbf{x}_N) | \Omega \rangle \\ &= \left(-\frac{3}{\sqrt{\epsilon}} \right)^N \langle \Omega | \varphi(t, \mathbf{x}) \varphi(t, \mathbf{x}_2) \dots \varphi(t, \mathbf{x}_N) | \Omega \rangle. \end{aligned} \quad (89)$$

Let us now consider the hypersurfaces defined by the condition $\varphi = 0$, which are of course different from the $t = \text{constant}$ hypersurfaces in the tensor gauge we are using. We are interested in the determinant $\text{Det}[\gamma_{ij}]_{\varphi=0}$ of the 3-metric γ_{ij} induced on these hypersurfaces which measures their local expansion. Writing

$$\text{Det}(\gamma_{ij})_{\varphi=0} = a(t)^2 e^{2\zeta}, \quad (90)$$

we can calculate the expectation value of any N -point function of ζ from

$$\begin{aligned} & \langle \Omega | \zeta(t, \mathbf{x}_1) \zeta(t, \mathbf{x}_2) \dots \zeta(t, \mathbf{x}_N) | \Omega \rangle \\ &= \zeta \left(\frac{\delta}{i\delta J_+(t, \mathbf{x}_1)} \right) \zeta \left(\frac{\delta}{i\delta J_+(t, \mathbf{x}_2)} \right) \dots \zeta \left(\frac{\delta}{i\delta J_+(t, \mathbf{x}_N)} \right) \\ & \quad \times Z[J_-, J_+]_{J_-=J_+=0}, \end{aligned} \quad (91)$$

where on the rhs $\zeta(w)$ has been expressed as the appropriate function of w and all occurrences of w have been replaced by $\delta/[i\delta J_+(t, \mathbf{x})]$. For example, to second order in perturbations, ζ is related to w in the tensor gauge by (cf. e.g. [13])

$$\begin{aligned} 36\zeta &\simeq 6w + \left(\epsilon - \frac{1}{2}\eta \right) w^2 + \frac{1}{H} \dot{w} w - \frac{1}{4} \frac{1}{a^2 H^2} \left((\partial_i w)(\partial_i w) \right. \\ & \quad \left. - \frac{\partial_i \partial_j}{\nabla^2} ((\partial_i w)(\partial_j w)) \right) + \frac{\epsilon}{H} (\partial_i w) \left(\frac{\partial_i}{\nabla^2} \dot{w} \right) \\ & \quad - \frac{\epsilon}{H} \frac{\partial_i \partial_j}{\nabla^2} \left((\partial_i w) \left(\frac{\partial_j}{\nabla^2} \dot{w} \right) \right) - \frac{3}{2H} h_{ij}^{TT} \partial_i \partial_j w. \end{aligned} \quad (92)$$

When restricted to long wavelengths, formula (91) could be considered as a quantum mechanical generalization of the ΔN formula. Analogous expressions can be used for relating any quantity of interest to results obtained in the tensor gauge.

IV. THE THREE-POINT AND FOUR-POINT FUNCTIONS OF INFLATON PERTURBATIONS

As an illustration of the formalism developed above, we will derive expressions for the tree-level three-point and four-point functions of the inflaton fluctuations. We shall work in the tensor gauge (80). All of the relevant cubic vertices are presented in Appendix D in Eqs. (D3)–(D6).

Here we list the specific couplings that are relevant for computation of three- and four-point functions.

From Eqs. (D3)–(D6) we can easily extract the terms involving w^3 :¹²

$$\begin{aligned} S_{www} &= \int d^3x dt \frac{\bar{N} a^3}{18} \left\{ \left(\frac{\epsilon^2}{2} - \frac{\epsilon^3}{12} \right) w \dot{w}^2 + \frac{\epsilon^2}{3} w (\partial_i \dot{w}) \left(\frac{\partial_i}{\nabla^2} \dot{w} \right) \right. \\ & \quad + \frac{\epsilon^3}{12} w \left(\frac{\partial_i \partial_j}{\nabla^2} \dot{w} \right) \left(\frac{\partial_i \partial_j}{\nabla^2} \dot{w} \right) + \left(\frac{\epsilon^3}{3} - \frac{\epsilon^2 \eta}{2} \right) H w^2 \dot{w} \\ & \quad + \left(-\frac{\epsilon^3}{2} + \frac{\epsilon^2 V_{,\phi\phi}}{6H^2} + \frac{\epsilon^{3/2} V_{,\phi\phi\phi}}{9H^2} + \frac{\epsilon^2 \eta^2}{6} \right) H^2 w^3 \\ & \quad \left. + \frac{\epsilon^2}{6} w \left(\frac{\nabla w}{a} \right)^2 \right\}. \end{aligned} \quad (93)$$

Notice that the $w^2 \dot{w}$ term can be integrated by parts to obtain terms that can be combined with the w^3 terms. The terms in Eqs. (D3)–(D6) of relevance for the tensor three-point function involving three gravitons are

$$\begin{aligned} S_{hhh} &= \int d^3x dt \bar{N} a^3 \left\{ -\frac{1}{2} h_{ij}^{TT} h_{jl}^{TT} h_{li}^{TT} - 3H h_{ij}^{TT} h_{jl}^{TT} h_{li}^{TT} \right. \\ & \quad - \frac{2(5-\epsilon)H^2}{3} h_{ij}^{TT} h_{jl}^{TT} h_{li}^{TT} + \frac{h_{ij}^{TT}}{a^2} \left(\frac{1}{4} (\partial_i h_{kl}^{TT})(\partial_j h_{kl}^{TT}) \right. \\ & \quad \left. \left. + \frac{1}{2} (\partial_i h_{jk}^{TT})(\partial_k h_{il}^{TT}) - \frac{3}{2} (\partial_i h_{ik}^{TT})(\partial_l h_{jk}^{TT}) \right) \right\}. \end{aligned} \quad (94)$$

There are also the cubic terms that involve two scalars and one graviton and those that involve one scalar and two gravitons,

¹²The result (93) can be compared to Eq. (3.8) of Ref. [13], which, when expressed in our language, $\phi \rightarrow 2\sqrt{\epsilon}H$, $\dot{\rho} \rightarrow H$, $\varphi \rightarrow -(\sqrt{\epsilon}/3)w$, $\dot{\varphi} \rightarrow (-\sqrt{\epsilon}/3)(\dot{w} + (\epsilon - \eta)Hw)$, $\nabla^2 \chi \rightarrow (\epsilon/3)\dot{w}$, reduces to

$$\begin{aligned} S_{www}|_{\text{Maldacena}} &= \int d^3x dt \frac{a^3}{9} \left\{ \left(\frac{\epsilon^2}{2} - \frac{\epsilon^3}{6} \right) w \dot{w}^2 + \frac{\epsilon^2}{3} w (\partial_i \dot{w}) \right. \\ & \quad \times \left(\frac{\partial_i}{\nabla^2} \dot{w} \right) + \frac{\epsilon^3}{6} w \left(\frac{\partial_i \partial_j}{\nabla^2} \dot{w} \right) \left(\frac{\partial_i \partial_j}{\nabla^2} \dot{w} \right) \\ & \quad + \left(\frac{\epsilon^3}{6} - \frac{\epsilon^2 \eta}{2} \right) H w^2 \dot{w} + \left(-\epsilon^3 + \frac{\epsilon^2 V_{,\phi\phi}}{6H^2} \right. \\ & \quad \left. + \frac{\epsilon^{3/2} V_{,\phi\phi\phi}}{18H^2} + \frac{\epsilon^2 (\epsilon + \eta)^2}{6} \right) H^2 w^3 \\ & \quad \left. + \frac{\epsilon^2}{6} w \left(\frac{\nabla w}{a} \right)^2 \right\}. \end{aligned}$$

We extracted $1/9$ as opposed to $1/18$ in (93) in order to compensate for the different definition of the Planck constant in Ref. [13], where $8\pi G_N = 1$ (recall that here $16\pi G_N = \kappa^2 = 1$). By comparing with Eq. (93) we see that the leading order $[\mathcal{O}(\epsilon^2)]$ terms agree, while there are some disagreements in the subleading terms (recall that $V_{,\phi\phi}$ and $V_{,\phi\phi\phi}/\sqrt{\epsilon}$ are both linear in slow-roll parameters). Since it involves subdominant terms, this discrepancy is quantitatively unimportant. Nevertheless, it would be of interest to find the source of the disagreement.

$$\mathcal{S}_{wwh} = \int d^3x dt \bar{N} a^3 \left\{ -\frac{\epsilon^2}{36} (\dot{w} h_{ij}^{TT} + w h_{ij}^{TT}) \frac{\partial_i \partial_j}{\nabla^2} \dot{w} + \frac{\epsilon^2}{36} h_{ij}^{TT} \left(\frac{\partial_j \partial_l}{\nabla^2} \dot{w} \right) \left(\frac{\partial_l \partial_i}{\nabla^2} \dot{w} \right) + \frac{\epsilon}{18} h_{ij}^{TT} \left(\frac{\partial_i w}{a} \right) \left(\frac{\partial_j w}{a} \right) \right\} \quad (95)$$

$$\begin{aligned} \mathcal{S}_{whh} = \int d^3x dt \bar{N} a^3 & \left\{ \frac{\epsilon}{24} w h_{ij}^{TT} h_{ij}^{TT} - \frac{\epsilon}{6} \dot{w} h_{ij}^{TT} h_{ij}^{TT} + \frac{\epsilon}{6} \left(\frac{\partial_i \partial_j}{\nabla^2} \dot{w} \right) h_{jl}^{TT} h_{li}^{TT} - \frac{\epsilon}{12} \left(\frac{\partial_i}{\nabla^2} \dot{w} \right) (\partial_i h_{jl}^{TT}) h_{jl}^{TT} + \frac{\epsilon H}{3} w h_{ij}^{TT} h_{ij}^{TT} \right. \\ & - \frac{\epsilon H}{6} \dot{w} h_{ij}^{TT} h_{ij}^{TT} + \frac{\epsilon H}{3} \left(\frac{\partial_i \partial_j}{\nabla^2} \dot{w} \right) h_{jl}^{TT} h_{li}^{TT} + \frac{\epsilon(7 + \epsilon - 2\eta)\epsilon H^2}{6} w h_{ij}^{TT} h_{ij}^{TT} \\ & \left. - \frac{\epsilon}{6} w h_{ij}^{TT} \left(\frac{\nabla^2}{a^2} h_{ij}^{TT} \right) - \frac{\epsilon}{8} w \left(\frac{\partial_l h_{ij}^{TT}}{a} \right) \left(\frac{\partial_l h_{ij}^{TT}}{a} \right) \right\}. \end{aligned} \quad (96)$$

Notice that the last term in (95) will contribute as $\mathcal{O}(\epsilon^2)$ to the four-point function [71]. For the calculation of the scalar four-point functions, the terms involving w^2 and any of the auxiliary fields $\{\tilde{S}, \tilde{n}, \tilde{N}_i^T, \rho^{ij}, \rho_\varphi\}$ are also needed:

$$\begin{aligned} \mathcal{S}_{ww(aux)} = \int d^3x dt \bar{N} a^3 & \left\{ \frac{\tilde{n}}{\bar{N}} \left[\left(-\frac{\epsilon}{18} + \frac{\epsilon^2}{36} \right) \dot{w}^2 - \frac{\epsilon(\epsilon - \eta)}{9} H w \dot{w} + \left(\frac{\epsilon^2}{2} - \frac{\epsilon^2 \eta}{9} - \frac{\epsilon \eta^2}{18} \right) H^2 w^2 - \frac{\epsilon V_{,\phi\phi}}{18} w^2 - \frac{\epsilon}{18} \left(\frac{\partial_i w}{a} \right) \left(\frac{\partial_i w}{a} \right) \right. \right. \\ & - \frac{\epsilon^2}{36} \left(\frac{\partial_i \partial_j}{\nabla^2} \dot{w} \right) \left(\frac{\partial_i \partial_j}{\nabla^2} \dot{w} \right) - \frac{\epsilon^2 H}{9} (\partial_i w) \left(\frac{\partial_i}{\nabla^2} \dot{w} \right) \left. \right] + \frac{\nabla^2 \tilde{S}}{(3 - \epsilon) \bar{N} a^2 H} \left[\left(\frac{\epsilon}{18} - \frac{\epsilon^2}{36} \right) \dot{w}^2 + \left(\frac{(1 - \epsilon)\epsilon^2}{18} - \frac{\epsilon \eta}{9} \right) H w \dot{w} \right. \\ & + \left(-\frac{\epsilon^2}{2} - \frac{\epsilon \eta}{6} + \frac{2\epsilon^3}{9} + \frac{\epsilon^2 \eta}{6} + \frac{\epsilon \eta^2}{18} \right) H^2 w^2 + \frac{\epsilon V_{,\phi\phi}}{18} w^2 + \frac{\epsilon}{18} \left(\frac{\partial_i w}{a} \right) \left(\frac{\partial_i w}{a} \right) + \frac{\epsilon^2}{36} \left(\frac{\partial_i \partial_j}{\nabla^2} \dot{w} \right) \left(\frac{\partial_i \partial_j}{\nabla^2} \dot{w} \right) \\ & + \frac{\epsilon^2 H}{9} (\partial_i w) \left(\frac{\partial_i}{\nabla^2} \dot{w} \right) \left. \right] + \frac{\epsilon^2 H}{9} w \rho^{ij} \left(\frac{\partial_i \partial_j}{\nabla^2} \dot{w} \right) - \frac{\epsilon^2 H^2}{9} \rho w^2 + \rho_\varphi \left[-\frac{\epsilon^2 H}{9} w \dot{w} + \frac{\epsilon^2 \eta H^2}{9} w^2 + \frac{\epsilon^2 H}{9} (\partial_i w) \left(\frac{\partial_i}{\nabla^2} \dot{w} \right) \right] \\ & \left. + \frac{\epsilon^2 (\partial_i \tilde{N}_j^T + \partial_i \partial_j \tilde{S})}{18 \bar{N} a^2} w \left(\frac{\partial_i \partial_j}{\nabla^2} \dot{w} \right) - \frac{\epsilon (\tilde{N}_i^T + \partial_i \tilde{S})}{9 \bar{N} a^2} \dot{w} (\partial_i w) \right\}. \end{aligned} \quad (97)$$

The purely quartic scalar terms can be read off from Eqs. (D12)–(D14) in Appendix D:

$$\mathcal{S}_{wwww} = \int d^3x dt \bar{N} a^3 \left\{ -\frac{\epsilon^2 V_{,\phi\phi\phi\phi}}{1944} w^4 - \frac{\epsilon^{5/2} V_{,\phi\phi\phi\phi}}{972} w^4 \right\}. \quad (98)$$

With the above interaction vertices one can, for example, compute the scalar contributions to the three-point and four-point functions, as we describe below.

A. The three-point function $\langle w^3 \rangle$

We can now use the diagrammatic rules described above to obtain the tree-level three-point function. The relevant diagrams are

$$\begin{array}{c} + \\ | \\ + \text{---} \text{---} + \\ | \\ + \end{array} + \begin{array}{c} + \\ | \\ - \text{---} \text{---} + \\ | \\ + \end{array} \quad (99)$$

where the vertex is determined by (93). Keeping the leading order [$\mathcal{O}(\epsilon^2)$] slow-roll terms in Eq. (93),

$$\mathcal{S}_{www} = \int d^3x dt \bar{N} a^3 \left\{ \frac{\epsilon^2}{36} w \dot{w}^2 + \frac{\epsilon^2}{54} w (\partial_i \dot{w}) \left(\frac{\partial_i}{\nabla^2} \dot{w} \right) + \frac{\epsilon^2}{108} w \left(\frac{\partial_i w}{a} \right) \left(\frac{\partial_i w}{a} \right) \right\}, \quad (100)$$

and recalling Eqs. (50), (54)–(56), and (87), we see that the first interaction term in (100) contributes as

$$\begin{aligned}
\langle \Omega | w(t, \mathbf{x}) w(t, \mathbf{y}) w(t, \mathbf{z}) | \Omega \rangle^{(1)} &= i \int_{-\infty}^{+\infty} d^3 \mathbf{u} \bar{N} d\tau a^3 \frac{\epsilon^2}{36} \left\{ \langle \Omega | T w(t, \mathbf{x}) w(\tau, \mathbf{u}) | \Omega \rangle \frac{\partial}{\bar{N} \partial \tau} \langle \Omega | T w(t, \mathbf{y}) w(\tau, \mathbf{u}) | \Omega \rangle \frac{\partial}{\bar{N} \partial \tau} \right. \\
&\quad \times \langle \Omega | T w(t, \mathbf{z}) w(\tau, \mathbf{u}) | \Omega \rangle + \text{perms} - \langle \Omega | w(\tau, \mathbf{u}) w(t, \mathbf{x}) | \Omega \rangle \frac{\partial}{\bar{N} \partial \tau} \langle \Omega | w(\tau, \mathbf{u}) w(t, \mathbf{y}) | \Omega \rangle \\
&\quad \left. \times \frac{\partial}{\bar{N} \partial \tau} \langle \Omega | w(\tau, \mathbf{u}) w(t, \mathbf{z}) | \Omega \rangle + \text{perms} \right\} \\
&= \int_{-\infty}^t d^3 \mathbf{u} d\tau a^3 \frac{\epsilon^2}{36 \bar{N}} \left\{ \Delta_{+-}^w(\tau, \mathbf{u}; t, \mathbf{x}) \partial_\tau \Delta_{+-}^w(\tau, \mathbf{u}; t, \mathbf{y}) \partial_\tau \Delta_{+-}^w(\tau, \mathbf{u}; t, \mathbf{z}) + \text{perms} \right. \\
&\quad \left. - \Delta_{-+}^w(\tau, \mathbf{u}; t, \mathbf{x}) \partial_\tau \Delta_{-+}^w(\tau, \mathbf{u}; t, \mathbf{y}) \partial_\tau \Delta_{-+}^w(\tau, \mathbf{u}; t, \mathbf{z}) + \text{perms} \right\}, \tag{101}
\end{aligned}$$

which is just the retarded contribution. Including in the similar manner the other two leading order contributions from (100), we finally get for the scalar three-point function,

$$\begin{aligned}
\langle \Omega | w(t, \mathbf{x}) w(t, \mathbf{y}) w(t, \mathbf{z}) | \Omega \rangle &= \int_{-\infty}^t d^3 \mathbf{u} d\tau \frac{a^3 \epsilon^2}{18 \bar{N}} \left\{ \Delta_{+-}^w(\tau, \mathbf{u}; t, \mathbf{x}) \left[\frac{1}{2} \partial_\tau \Delta_{+-}^w(\tau, \mathbf{u}; t, \mathbf{y}) \partial_\tau \Delta_{+-}^w(\tau, \mathbf{u}; t, \mathbf{z}) \right. \right. \\
&\quad \left. \left. + \frac{1}{3} \frac{\partial}{\partial u^i} \partial_\tau \Delta_{+-}^w(\tau, \mathbf{u}; t, \mathbf{y}) \frac{\partial}{\partial u^i} \frac{1}{\nabla_u^2} \partial_\tau \Delta_{+-}^w(\tau, \mathbf{u}; t, \mathbf{z}) + \frac{\bar{N}^2}{6a^2} \frac{\partial}{\partial u^i} \Delta_{+-}^w(\tau, \mathbf{u}; t, \mathbf{y}) \right. \right. \\
&\quad \left. \left. \times \frac{\partial}{\partial u^i} \Delta_{+-}^w(\tau, \mathbf{u}; t, \mathbf{z}) \right] + \text{perms} - \Delta_{-+}^w(\tau, \mathbf{u}; t, \mathbf{x}) \left[\frac{1}{2} \partial_\tau \Delta_{-+}^w(\tau, \mathbf{u}; t, \mathbf{y}) \partial_\tau \Delta_{-+}^w(\tau, \mathbf{u}; t, \mathbf{z}) \right. \right. \\
&\quad \left. \left. + \frac{1}{3} \frac{\partial}{\partial u^i} \partial_\tau \Delta_{-+}^w(\tau, \mathbf{u}; t, \mathbf{y}) \frac{\partial}{\partial u^i} \frac{1}{\nabla_u^2} \partial_\tau \Delta_{-+}^w(\tau, \mathbf{u}; t, \mathbf{z}) + \frac{\bar{N}^2}{6a^2} \frac{\partial}{\partial u^i} \Delta_{-+}^w(\tau, \mathbf{u}; t, \mathbf{y}) \right. \right. \\
&\quad \left. \left. \times \frac{\partial}{\partial u^i} \Delta_{-+}^w(\tau, \mathbf{u}; t, \mathbf{z}) \right] + \text{perms} \right\}, \tag{102}
\end{aligned}$$

where perms denote the six permutations of $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$. When Eq. (102) is transformed to the spatial Fourier space, and inflationary mode functions are used for the scalar propagators, one can calculate the induced bispectrum, which in turn can be related to the cosmic background temperature fluctuations. As expected, the expression (102) leads to a manifestly real result since $\Delta_{+-}^w = -(\Delta_{-+}^w)^*$, and coincides with that obtained in the operator formalism [13] via

$$\begin{aligned}
\langle \Omega | w(t, \mathbf{x}) w(t, \mathbf{y}) w(t, \mathbf{z}) | \Omega \rangle \\
= i \int_{-\infty}^t d^3 \mathbf{u} d\tau \langle \Omega | [w(t, \mathbf{x}) w(t, \mathbf{y}) w(t, \mathbf{z}), \mathcal{S}_I(\tau, \mathbf{u})] | \Omega \rangle,
\end{aligned}$$

where \mathcal{S}_I is the interaction action in the tensor gauge and all fields are taken to be free Heisenberg fields. Thus, we have reproduced the results already obtained using the operator formalism and canonical quantization [13].

B. The four-point function $\langle w^4 \rangle$

The quartic interaction terms in (98) contribute diagrams of the form

$$\begin{aligned}
\begin{array}{ccc}
+ & & + \\
\diagdown & & \diagup \\
+ & & + \\
\diagup & & \diagdown \\
+ & & +
\end{array}
+
\begin{array}{ccc}
+ & & + \\
\diagdown & & \diagup \\
- & & - \\
\diagup & & \diagdown \\
+ & & +
\end{array}
\tag{103}
\end{aligned}$$

However, the leading order contribution to the four-point function comes from (97), corresponding to diagrams like

$$\begin{array}{ccc}
+ & & + \\
\diagdown & & \diagup \\
+ & & + \\
\diagup & & \diagdown \\
+ & & +
\end{array}
+
\begin{array}{ccc}
+ & & + \\
\diagdown & & \diagup \\
- & & - \\
\diagup & & \diagdown \\
+ & & +
\end{array}, \tag{104}$$

where the dashed line represents the propagation of auxiliary fields. As is evident from the form of the auxiliary propagators, such diagrams correspond to effective four-point interactions. Since $\Delta_{\pm\mp} = 0$ for all of the auxiliary fields, there can be no $+ -$ internal lines involving auxiliary fields. The leading order $\mathcal{O}(\epsilon^1)$ terms from Eq. (97), determining the vertices of the diagrams (104), read

$$\begin{aligned}
S_{ww(aux)} &= \int d^3 x dt \bar{N} a^3 \left\{ -\frac{\epsilon}{18} \frac{\bar{n}}{\bar{N}} \left[\dot{w}^2 + \left(\frac{\partial_i w}{a} \right) \left(\frac{\partial_i w}{a} \right) \right] \right. \\
&\quad \left. + \frac{\epsilon}{18} \frac{\nabla^2 \tilde{\mathcal{S}}}{(3 - \epsilon) \bar{N} a^2 H} \left[\dot{w}^2 + \left(\frac{\partial_i w}{a} \right) \left(\frac{\partial_i w}{a} \right) \right] \right. \\
&\quad \left. - \frac{\epsilon}{9} \frac{(\tilde{N}_i^T + \partial_i \tilde{\mathcal{S}})}{\bar{N} a^2} \dot{w} (\partial_i w) \right\}. \tag{105}
\end{aligned}$$

Diagrams like (104) contribute products of the form

$$\begin{aligned}
 & -\frac{1}{2}\langle\Omega|\hat{\mathcal{O}}S_{ww(aux)}S_{ww(aux)}|\Omega\rangle \\
 & = -\frac{1}{2}\int d^3\mathbf{u}\int_{-\infty}^{\infty}d\tau\bar{N}a^3\langle\Omega|\hat{\mathcal{O}}\left[-\frac{\epsilon^2}{81}(\dot{w}\partial_i w)\frac{1}{\nabla^2}(\dot{w}\partial_i w)\right. \\
 & \quad +\frac{\epsilon^2(1+\epsilon)}{324}(\dot{w}\partial_i w)\frac{\partial_i\partial_j}{\nabla^4}(\dot{w}\partial_j w)+\frac{\epsilon^2}{324H} \\
 & \quad \left.\times\left[(\dot{w})^2+\left(\frac{\partial_j w}{a}\right)^2\right]\frac{\partial_i}{\nabla^2}(\dot{w}\partial_i w)\right]|\Omega\rangle, \quad (106)
 \end{aligned}$$

where $\hat{\mathcal{O}}$ is an operator and we used the auxiliary propagators (74)–(76). Note that the contributions from the two auxiliary fields linking the terms $\bar{n}-\nabla^2\tilde{S}/[(3-\epsilon)a^2H]$ cancel each other out. We see that the exchange of auxiliary fields induces an effective four-point interaction given by the expression inside the curly brackets of (106). This effective interaction corresponds to the leading order terms of the quartic interaction terms derived in [72,73].

The contribution to the four-point function is obtained when $\hat{\mathcal{O}}\rightarrow w_+(t,\mathbf{x})w_+(t,\mathbf{y})w_+(t,\mathbf{z})w_+(t,\mathbf{v})$. The result is

$$\begin{aligned}
 \langle\Omega|w(t,\mathbf{x})w(t,\mathbf{y})w(t,\mathbf{z})w(t,\mathbf{v})|\Omega\rangle & = \int d^3\mathbf{u}\int_{-\infty}^t d\tau\bar{N}a^3\left\{\frac{\epsilon^2}{162}\left(\frac{\partial}{\bar{N}\partial\tau}\Delta_{+-}^w(\tau,\mathbf{u};t,\mathbf{x})\frac{\partial}{\partial u^i}\Delta_{+-}^w(\tau,\mathbf{u};t,\mathbf{y})\right)\frac{1}{\nabla_{\mathbf{u}}^2}\right. \\
 & \quad \times\left(\frac{\partial}{\bar{N}\partial\tau}\Delta_{+-}^w(\tau,\mathbf{u};t,\mathbf{z})\frac{\partial}{\partial u^i}\Delta_{+-}^w(\tau,\mathbf{u};t,\mathbf{v})\right)-\frac{\epsilon^2(1+\epsilon)}{648}\left(\frac{\partial}{\bar{N}\partial\tau}\Delta_{+-}^w(\tau,\mathbf{u};t,\mathbf{x})\right. \\
 & \quad \times\frac{\partial}{\partial u^i}\Delta_{+-}^w(\tau,\mathbf{u};t,\mathbf{y})\left.\right)\frac{1}{\nabla_{\mathbf{u}}^2}\frac{\partial}{\partial u^i}\frac{\partial}{\partial u^j}\left(\frac{\partial}{\bar{N}\partial\tau}\Delta_{+-}^w(\tau,\mathbf{u};t,\mathbf{z})\frac{\partial}{\partial u^j}\Delta_{+-}^w(\tau,\mathbf{u};t,\mathbf{v})\right) \\
 & \quad -\frac{\epsilon^2}{648H}\left[\left(\frac{\partial}{\bar{N}\partial\tau}\Delta_{+-}^w(\tau,\mathbf{u};t,\mathbf{x})\frac{\partial}{\bar{N}\partial\tau}\Delta_{+-}^w(\tau,\mathbf{u};t,\mathbf{y})\right)+\left(\frac{\partial}{a\partial u^j}\Delta_{+-}^w(\tau,\mathbf{u};t,\mathbf{x})\right.\right. \\
 & \quad \left.\left.\times\left(\frac{\partial}{a\partial u^j}\Delta_{+-}^w(\tau,\mathbf{x};t,\mathbf{y})\right)\right]\frac{1}{\nabla_{\mathbf{u}}^2}\frac{\partial}{\partial u^i}\left(\frac{\partial}{\bar{N}\partial\tau}\Delta_{+-}^w(\tau,\mathbf{u};t,\mathbf{z})\frac{\partial}{\partial u^i}\Delta_{+-}^w(\tau,\mathbf{u};t,\mathbf{v})\right)\right. \\
 & \quad \left.-(+\rightarrow-+)+\text{perms}\right\}, \quad (107)
 \end{aligned}$$

where *perms* denote the 24 permutations of $\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{v}\}$. This leading order contribution is of the order $\mathcal{O}(\epsilon^2)$. There is also the tree-level contribution coming from the graviton exchange. The leading order vertex in Eq. (95) is of the order $\mathcal{O}(\epsilon^1)$, and hence formally it also contributes at the order $\mathcal{O}(\epsilon^2)$. Here we do not discuss this contribution any further, and refer the reader to Ref. [71] where the graviton exchange contribution to the four-point function has been estimated. Finally, the terms in Eq. (98) yield contributions that are of the order $\mathcal{O}(\epsilon^4)$; since their contribution to the four-point function is suppressed with respect to (107), we do not calculate them here.

V. DISCUSSION

In this paper we developed a path-integral formulation for inflationary perturbations in single-field inflation. Using a phase space formulation of the system as a starting point, we were able to bring the free action into a particularly simple form which allows for a straightforward calculation of the various propagators, recovering in the process standard results of linear gauge-invariant cosmological perturbation theory. The interaction terms can then be obtained without the necessity of first solving the energy and momentum constraints as has been done so far in the literature. A notable feature is the appearance of two types of auxiliary fields in the path integral beyond the physical propagating degrees of freedom: a set of commuting non-

dynamical fields related to the existence of the constraints, as well as a set of anticommuting Faddeev-Popov ghost fields induced by the imposition of gauge conditions. The resulting path integral is independent of the choice of gauge, provided the asymptotic states are defined in terms of the linearized fields and the corresponding linearized gauge transformations.

We then briefly described how to obtain N -point expectation values and commented on the meaning of two different gauges and their relation to cosmological observables in this formalism. They are the uniform field gauge and the tensor gauge which have been widely used in past considerations of non-Gaussianity but in the context of an interaction picture operator approach. Correlators can be expressed in a systematic expansion in diagrams in the in-in formalism. We found that, when quantum loops are taken into account, anticommuting ghost fields must be included in the computation. Furthermore, internal lines in diagrams should also include the commuting auxiliary fields and we demonstrated their role in the computation of four-point functions, in which the leading order contributions indeed come from diagrams with internal lines involving auxiliary fields. This way of obtaining the effective four-point interactions seems simpler than having to go through the solution of the constraints at second order and the substitution of these solutions back in the action [72,73].

So far, we have only considered standard single-field inflation but the generalization to multifield and more complicated models should be straightforward as long as the canonical formulation is known. We have also explicitly considered only tree diagrams in the tensor gauge, noting that the standard nonlinear redefinitions (gauge transformations) can be used to obtain results for the curvature perturbation on comoving slices. Of course we could have obtained this result by working directly in the uniform field gauge, at the price of more complicated interaction terms. Relating results in the two gauges would be particularly interesting if quantum loop corrections are taken into account and questions of renormalization arise. This is a regime where the role of the anticommuting ghosts would become crucial. Currently, such questions are only addressed using the (essentially classical) ΔN formalism on long wavelengths. Other issues involve the appearance of infrared divergences, backreaction, and a more rigorous definition of stochastic inflation. A path-integral formulation might prove very useful for such considerations which will be the focus of future work.

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APPENDIX A

In the following Appendices we give details on the derivation of Eqs. (34) and (D1) (Appendices A and B); the Poisson brackets needed to calculate (51) (Appendix C); and the general interaction Hamiltonian as well as an intermediate result needed for cubic and quartic vertices evaluated in Sec. IV (Appendix D).

Let us consider the spatial Ricci scalar contribution the action (1)–(3). A spatial metric perturbation $\delta g_{ij} = a^2 h_{ij}$ on a flat FLRW background can be written as

$$g_{ij} = a^2(\delta_{ij} + h_{ij}). \quad (\text{A1})$$

Its matrix inverse $g^{ij} = a^{-2}\tilde{g}^{ij}$ is then

$$\tilde{g}^{ij} = \delta^{ij} - h^{ij} + h^{il}h_l^j - h^{il}h_l^k h_k^j + h^{il}h_l^k h_k^m h_m^j + \mathcal{O}(h^5), \quad (\text{A2})$$

where the indices are raised with the Kronecker delta, $h^i_j = \delta^{il}h_{lj}$, $h^{ij} = \delta^{il}h_l^j$. Equation (A2) can be also writ-

ten in matrix notation as

$$\tilde{\mathbf{g}} = \mathbf{I} - \mathbf{h} + \mathbf{h} \cdot \mathbf{h} - \mathbf{h} \cdot \mathbf{h} \cdot \mathbf{h} + \dots + (-1)^n \mathbf{h}^n + \mathcal{O}(\mathbf{h}^{n+1}). \quad (\text{A3})$$

We will find the following notation useful: a numerical subscript n will indicate that terms of order n and higher in h_{ij} are included. For example, $\tilde{\mathbf{g}}_{\geq n} \equiv (-1)^n \mathbf{h}^n + \mathcal{O}(\mathbf{h}^{n+1})$, etc. To calculate the determinant $\tilde{g}^{\pm 1/2} = [\text{Det}(\mathbf{I} + \mathbf{h})]^{\pm 1/2}$, we can use

$$\begin{aligned} [\text{Det}(\mathbf{I} + \mathbf{h})]^{\pm 1/2} &= \exp\left\{\pm \frac{1}{2} \text{Tr} \ln(\mathbf{I} + \mathbf{h})\right\} \\ &= \sum_{n=0}^{\infty} \frac{(\pm 1)^n}{2^n n!} \left(\text{Tr} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \mathbf{h}^m \right)^n. \end{aligned} \quad (\text{A4})$$

For example, in accordance with this notation, we have ($h = \text{Tr}[\mathbf{h}]$)

$$\begin{aligned} (\tilde{g}^{\pm 1/2})_{\geq 2} &= \left[\mp \frac{1}{4} \text{Tr}(\mathbf{h}^2) + \frac{1}{8} h^2 \right] + \left[\pm \frac{1}{6} \text{Tr}(\mathbf{h}^3) \right. \\ &\quad \left. - \frac{1}{8} h \text{Tr}(\mathbf{h}^2) \pm \frac{1}{48} h^3 \right] + \left[\mp \frac{1}{8} \text{Tr}(\mathbf{h}^4) \right. \\ &\quad \left. + \frac{1}{32} \text{Tr}(\mathbf{h} \cdot \mathbf{h})^2 + \frac{1}{12} h \text{Tr}(\mathbf{h}^3) \right. \\ &\quad \left. \mp \frac{1}{32} h^2 \text{Tr}(\mathbf{h}^2) + \frac{1}{384} h^4 \right] + \mathcal{O}(h^5). \end{aligned} \quad (\text{A5})$$

We will also be using

$$A_{ijkl} \equiv \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl}. \quad (\text{A6})$$

We can now write the spatial connection as

$$\Gamma_{jl}^i = \frac{1}{2} \tilde{g}^{ik} [\partial_j h_{kl} + \partial_l h_{jk} - \partial_k h_{jl}]. \quad (\text{A7})$$

The corresponding Ricci scalar $R = g^{ij} R_{ij}$, with $R_{ij} = \partial_k \Gamma_{ij}^k - \partial_j \Gamma_{ik}^k + \Gamma_{kl}^l \Gamma_{ij}^k - \Gamma_{kj}^l \Gamma_{il}^k$, is then

$$\begin{aligned} R &= \frac{\tilde{g}^{ij}}{a^2} \left\{ \frac{1}{2} (\partial_k \tilde{g}^{kl}) (\partial_i h_{lj} + \partial_j h_{il} - \partial_l h_{ij}) + \frac{1}{2} \tilde{g}^{kl} (\partial_k \partial_i h_{lj} \right. \\ &\quad \left. + \partial_k \partial_j h_{il} - \partial_k \partial_l h_{ij}) - \frac{1}{2} (\partial_j \tilde{g}^{kl}) (\partial_i h_{kl}) - \frac{1}{2} \tilde{g}^{kl} (\partial_j \partial_i h_{kl}) \right. \\ &\quad \left. + \frac{1}{4} \tilde{g}^{lm} \tilde{g}^{kn} (\partial_k h_{ml}) (\partial_i h_{nj} + \partial_j h_{in} - \partial_n h_{ij}) \right. \\ &\quad \left. - \frac{1}{4} \tilde{g}^{lm} \tilde{g}^{kn} (\partial_k h_{mj} + \partial_j h_{km} - \partial_m h_{kj}) \right. \\ &\quad \left. \times (\partial_i h_{nl} + \partial_l h_{in} - \partial_n h_{il}) \right\}. \end{aligned} \quad (\text{A8})$$

Now making use of

$$\partial_l \tilde{g}^{ij} = -\tilde{g}^{in} (\partial_l h_{nm}) \tilde{g}^{mj}, \quad (\text{A9})$$

and upon summing various identical terms, Eq. (A8) can be written in the more symmetric form as

$$\begin{aligned}\sqrt{g}R &= a\sqrt{\tilde{g}}\tilde{g}^{ij}\{\tilde{g}^{kl}[\partial_k\partial_i h_{lj} - \partial_i\partial_j h_{kl}] \\ &+ \tilde{g}^{km}\tilde{g}^{ln}[-(\partial_k h_{mn})(\partial_i h_{lj}) - \frac{1}{4}(\partial_l h_{jm})(\partial_i h_{nk}) \\ &- \frac{1}{4}(\partial_l h_{jm})(\partial_k h_{in}) - \frac{1}{4}(\partial_n h_{ij})(\partial_l h_{km}) \\ &+ (\partial_i h_{ij})(\partial_k h_{mn}) + \frac{3}{4}(\partial_i h_{kl})(\partial_j h_{mn})\}],\end{aligned}\quad (\text{A10})$$

where [see Eq. (A5)]

$$\sqrt{g} = a^3\sqrt{\tilde{g}} \quad \sqrt{\tilde{g}} = 1 + \frac{1}{2}h + (\sqrt{\tilde{g}})_{\geq 2}. \quad (\text{A11})$$

Formula (A10) contains vertices to all orders generated by the $\sqrt{g}R$ term, and it is written such that it is relatively easy to extract its contribution to an arbitrary n -point vertex. For example, the linear contribution in h_{ij} is

$$[\sqrt{g}R]_{\text{lin}} = a[\partial_i\partial_j h_{ij} - \partial_i^2 h], \quad (\text{A12})$$

while the quadratic contribution reads

$$\begin{aligned}[\sqrt{g}R]_{\text{quad}} &= a\left[\frac{h}{2}(\partial_i\partial_j h_{ij} - \nabla^2 h) - h_{kl}(2\partial_k\partial_i h_{li} - \nabla^2 h_{kl})\right. \\ &+ h_{ij}\partial_i\partial_j h - \frac{3}{2}(\partial_k h_{kl})(\partial_i h_{il}) - \frac{1}{4}(\partial_i h)(\partial_i h) \\ &\left.+ (\partial_i h)(\partial_j h_{ji}) + \frac{3}{4}(\partial_i h_{ji})(\partial_i h_{jl})\right] \\ &= a\left[-\frac{1}{4}h\partial_i^2 h + \frac{1}{2}h\partial_i\partial_j h_{ij} - \frac{1}{2}h_{ij}\partial_i\partial_l h_{jl}\right. \\ &\left.+ \frac{1}{4}h_{jl}\nabla^2 h_{jl}\right] + (\text{tot.der.}),\end{aligned}\quad (\text{A13})$$

where in the last line we ignored several total spatial derivative terms (tot.der.), which drop out when inserted into the (free part of the) action (1).

APPENDIX B

We now present steps in the derivation of the free action (45) and (B17) below. Let us first analyze the momentum part of the action (1)–(3) quadratic in perturbations [see Eq. (34)], which can be written in the form

$$\begin{aligned}S_{\pi}^{(2)} &= \int d^3x\bar{N}dt\left[-\frac{\mathcal{P}_{\phi}^2}{2a^3}(\pi_{\phi}^2 + I_{\phi}\pi_{\phi})\right. \\ &\left.- 4a^3H^2\left[\pi^{ij}\frac{A_{ijkl}}{2}\pi^{kl} + I_{ij}\pi^{ij}\right]\right],\end{aligned}\quad (\text{B1})$$

where

$$\begin{aligned}I_{\phi} &= -h - \frac{2a^3}{\mathcal{P}_{\phi}}\dot{\phi} + \frac{2}{\bar{N}}n \\ I_{ij} &= \frac{1}{2H}\dot{h}_{ij} + 2h_{ij} - \frac{\delta_{ij}}{2}h - \frac{\delta_{ij}}{\bar{N}}n - \frac{1}{\bar{N}a^2H}\partial_{(i}N_{j)} \\ I &\equiv \delta_{ij}I_{ij} = \frac{1}{2H}\dot{h} + \frac{1}{2}h - \frac{3}{\bar{N}}n - \frac{1}{\bar{N}a^2H}\nabla^2 S.\end{aligned}\quad (\text{B2})$$

It is now convenient to complete the square in Eq. (B1) by the appropriate shifts of the momenta π^{ij} and π_{ϕ} of the

gravitational and scalar fields, respectively. From Eq. (B1) we easily see that the shifted momenta are

$$\rho^{ij} = \pi^{ij} + \frac{1}{2}(I_{ij} - \delta_{ij}I) \quad (\text{B3})$$

$$\rho_{\phi} = \pi_{\phi} + \frac{1}{2}I_{\phi}. \quad (\text{B4})$$

With these we get for the momentum terms (B1)

$$\begin{aligned}S_{\pi}^{(2)} &= \int d^3x\bar{N}dt\left\{-\frac{\mathcal{P}_{\phi}^2}{2a^3}\rho_{\phi}^2 - 4a^3H^2\rho^{ij}\frac{A_{ijkl}}{2}\rho^{kl} + \frac{\mathcal{P}_{\phi}^2}{8a^3}I_{\phi}^2\right. \\ &\left.+ a^3H^2(I_{ij}I_{ij} - I^2)\right\}.\end{aligned}\quad (\text{B5})$$

The linear shifts (B3) and (B4) decouple the momenta from the inflaton and the metric perturbations.

Next we observe from (B5) that the momenta shifts (B3) and (B4) have introduced terms quadratic in the lapse perturbation n . Along with the original linear terms in the action (23), they read

$$S_n^{(2)} = \int d^3xdt\left\{\left[-\frac{6a^3H^2}{\bar{N}} + \frac{\mathcal{P}_{\phi}^2}{2a^3\bar{N}}\right]n^2 + nI_n\right\}, \quad (\text{B6})$$

where

$$\begin{aligned}I_n &= 2a^3H\dot{h} - \mathcal{P}_{\phi}\dot{\phi} - a^3V_{,\phi}\phi + a(\partial_i\partial_j h_{ij} - \nabla^2 h) \\ &\quad - \frac{4aH}{\bar{N}}\nabla^2 S.\end{aligned}\quad (\text{B7})$$

Just like in the case of the momentum terms above, we now complete the square for the lapse. The contribution is

$$\begin{aligned}S_n^{(2)} &= \int d^3x\bar{N}dt\left\{-\frac{a^3V}{\bar{N}^2}\tilde{n}^2 + \frac{1}{4a^3V}I_n^2\right\}, \\ \tilde{n} &= n - \frac{\bar{N}}{2a^3V}I_n.\end{aligned}\quad (\text{B8})$$

From the above manipulations we have the shift N_i also contributing to the quadratic action

$$S_{N_i}^{(2)} = \int d^3xdt\left\{\frac{1}{a\bar{N}}([\partial_{(i}N_{j)}]^2 - \alpha(\partial_i N_i)^2 + J_{ij}\partial_{(i}N_{j)})\right\}, \quad (\text{B9})$$

where

$$\alpha = \frac{1}{3}\left(1 - \frac{\mathcal{P}_{\phi}^2}{a^6V}\right) \quad (\text{B10})$$

and

$$\begin{aligned}J_{ij} &= \bar{N}a^2\left[-\dot{h}_{ij} + \delta_{ij}\left(\alpha\dot{h} - \frac{2H}{a^2V}(\partial_k\partial_l h_{kl} - \nabla^2 h)\right)\right. \\ &\quad \left.+ \frac{2H\mathcal{P}_{\phi}}{a^3V}\dot{\phi} + \left(\frac{\mathcal{P}_{\phi}}{a^3} + \frac{2HV_{,\phi}}{V}\right)\phi\right], \\ J &= \delta_{ij}J_{ij},\end{aligned}\quad (\text{B11})$$

where in fact only the vector and scalar parts of J_{ij} in (B11) contribute. Analogously to the lapse, the shift contribution

to the quadratic action can be written (up to boundary terms) in the form

$$S_{N_i}^{(2)} = \int d^3x \bar{N} dt \left\{ \frac{1}{a\bar{N}^2} ([\partial_{(i}\bar{N}_{j)}^T]^2 + (1-\alpha)(\nabla^2\tilde{S})^2) - \frac{a^3}{4} [\partial_{(i}\dot{h}_{j)}^T]^2 - \frac{a^3}{4(1-\alpha)V} \left[-\frac{1}{3}(\dot{\phi}^2\dot{h} + 2V\nabla^2\tilde{h}) + 2\dot{\phi}H\dot{\phi} + \frac{4H}{3a^2}\nabla^2(h - \nabla^2\tilde{h}) + (\dot{\phi}V + 2HV_{,\phi})\varphi \right]^2 \right\}, \quad (\text{B12})$$

where

$$S_{h_{ij},\varphi}^{(2)} = \int d^3x \bar{N} dt \frac{a^3}{4} \left\{ -2H^2 \left[5\text{Tr}[\mathbf{h} \cdot \mathbf{h}] - \frac{3}{2}h^2 \right] - \frac{1}{a^2} [h\partial_i^2 h - 2h\partial_i\partial_j h_{ij} + 2h_{ij}\partial_i\partial_l h_{jl} - h_{jl}\partial_i^2 h_{jl}] - \frac{\dot{\phi}^2}{2} \left[\text{Tr}[\mathbf{h} \cdot \mathbf{h}] + \frac{1}{2}h^2 \right] - \frac{1}{2a^2} (\partial_i\varphi)^2 + V \left[\text{Tr}[\mathbf{h} \cdot \mathbf{h}] - \frac{1}{2}h^2 \right] - 2V_{,\phi\phi}\varphi^2 - 2V_{,\phi}h\varphi + [\text{Tr}[\mathbf{h} \cdot \mathbf{h}] - h^2] + 8H \left[\text{Tr}[\mathbf{h} \cdot \dot{\mathbf{h}}] - \frac{1}{2}h\dot{h} \right] + 2H^2 [8\text{Tr}[\mathbf{h} \cdot \mathbf{h}] - 3h^2] + 2 \left[\dot{\phi}^2 + \dot{\phi}h\dot{\phi} + \frac{1}{4}\dot{\phi}^2 h^2 \right] + \frac{1}{V} \left[2H\dot{h} - \dot{\phi}\dot{\phi} + \frac{1}{a^2} (\partial_i\partial_j h_{ij} - \partial_i^2 h) - V_{,\phi}\varphi \right]^2 - [\partial_{(i}\dot{h}_{j)}^T]^2 - \frac{1}{4H^2 V} \left[2H\dot{\phi}\dot{\phi} - \frac{2}{3} \left(\frac{\dot{\phi}^2}{2}\dot{h} + V\partial_i^2\dot{h} \right) + \frac{4}{3}H \frac{\partial_i^2(h - \partial_j^2\tilde{h})}{a^2} + (\dot{\phi}V + 2HV_{,\phi})\varphi \right]^2 \right\}. \quad (\text{B15})$$

This action can be significantly simplified when expressed in terms of the transverse traceless tensor h_{ij}^{TT} defined in Eqs. (37) and (38) and the gauge-invariant combination of scalar matter and scalar gravitational potentials,

$$\tilde{\varphi} = \varphi - z(h - \nabla^2\tilde{h}), \quad z = \frac{\mathcal{P}_\phi}{6a^3H}, \quad (\text{B16})$$

which is also known as the Sasaki-Mukhanov variable. An important step in the derivation of the quadratic action is a partial integration of the terms that contain one time derivative acting on perturbations φ and h_{ij} . Furthermore, since vectors are not sourced, their contribution to (B15) cancels out, and the contributions from scalars and tensors separate. Following this procedure, one eventually arrives at the quadratic action,

$$S_F = \int d^3x \bar{N} dt a^3 \left\{ -\frac{\mathcal{P}_\phi^2}{2a^6} \rho_\phi^2 - 4H^2 \rho^{ij} \frac{A_{ijkl}}{2} \rho^{kl} - \frac{V}{\bar{N}^2} \tilde{n}^2 + \frac{1}{a^4 \bar{N}^2} \left([\partial_{(i}\bar{N}_{j)}^T]^2 + \frac{4H^2}{V} (\nabla^2\tilde{S})^2 \right) + \frac{1}{2} \left[\dot{\tilde{\varphi}}^2 - \left(\frac{\nabla\tilde{\varphi}}{a} \right)^2 + \left(3H\frac{\dot{z}}{z} + \frac{\ddot{z}}{z} \right) \tilde{\varphi}^2 \right] + \frac{1}{4} \left[(\dot{h}_{ij}^{TT})^2 - \left(\frac{\nabla h_{ij}^{TT}}{a} \right)^2 \right] \right\}, \quad (\text{B17})$$

where

$$3H\frac{\dot{z}}{z} + \frac{\ddot{z}}{z} = -\frac{\dot{\phi}^2 V}{4H^2} - \frac{\dot{\phi}}{H} V_{,\phi} - V_{,\phi\phi}, \quad (\text{B18})$$

and the shifts L_0 , L_i , L_{ij} , and L_φ of the shifted momenta

$$\nabla^2\tilde{S} = \nabla^2 S + \frac{1}{6(1-\alpha)} [J - 2a^2\bar{N}\nabla^2\tilde{h}] \quad (\text{B13})$$

$$\partial_{(i}\bar{N}_{j)}^T = \partial_{(i}N_{j)}^T - \frac{a^2\bar{N}}{2} \partial_{(i}\dot{h}_{j)}^T. \quad (\text{B14})$$

When all the terms from Eqs. (B5), (B8), (B12), and (34) are combined, we get the intermediate, complex expression for the action of quadratic perturbations in h_{ij} and φ (here we drop the contributions from the momenta and constraints),

and constraints (41)–(44) in the first line of (B17) can be obtained from Eqs. (B2), (B3), (B7), (B13), and (B14) and read

$$L_0 = \frac{\bar{N}}{2a^3V} \left\{ 2a^3H\dot{h} - \mathcal{P}_\phi\dot{\phi} - a^3V_{,\phi}\varphi + a(\partial_i\partial_j h_{ij} - \nabla^2 h) + \frac{aV}{6\bar{N}H} (J - 2\bar{N}a^2\nabla^2\tilde{h}) \right\} - \frac{2H}{a^2V} \nabla^2\tilde{S} \quad (\text{B19})$$

$$L_i = \frac{a^2\bar{N}V}{12H^2} \partial_i\dot{h} - \frac{V}{24H^2} \frac{\partial_i}{\nabla^2} J + \frac{a^2\bar{N}}{2} \dot{h}_i^T \quad (\text{B20})$$

$$L_{ij} = -\frac{1}{4H} \dot{h}_{ij} - h_{ij} + \frac{1}{2\bar{N}a^2H} (\partial_{(i}L_{j)}) + \partial_{(i}\bar{N}_{j)}) + \frac{\delta_{ij}}{2} \left[\frac{1}{2H} \dot{h} + h - \frac{2}{\bar{N}} L_0 - \frac{1}{\bar{N}a^2H} \partial_k L_k - \frac{2}{\bar{N}} \tilde{n} - \frac{1}{\bar{N}a^2H} \left(1 - 4\frac{H^2}{V} \right) \nabla^2\tilde{S} \right] \quad (\text{B21})$$

$$L_\varphi = \frac{1}{2} h + \frac{a^3}{\mathcal{P}_\phi} \dot{\phi} - \frac{L_0}{\bar{N}} - \frac{\tilde{n}}{\bar{N}} + \frac{2H}{\bar{N}a^2V} \nabla^2\tilde{S}, \quad (\text{B22})$$

with

$$J = \bar{N}a^2 \left[-\frac{\mathcal{P}_\phi^2}{a^6V} \dot{h} + 3 \left(\frac{2H\mathcal{P}_\phi}{a^3V} \dot{\phi} + \left(\frac{\mathcal{P}_\phi}{a^3} + \frac{2HV_{,\phi}}{V} \right) \varphi - \frac{2H}{a^2V} (\partial_k\partial_l h_{kl} - \nabla^2 h) \right) \right]. \quad (\text{B23})$$

We shall now argue that the action (B17) is gauge invariant. To show this we need to consider how the metric and scalar transform under infinitesimal coordinate transformations, $x^\mu \rightarrow x^\mu + \xi^\mu(x)$, under which the metric transforms as $g_{\mu\nu} \rightarrow g_{\mu\nu} - 2\nabla_{(\mu}\xi_{\nu)}$, $\xi_\mu = g_{\mu\nu}\xi^\nu$. To evaluate the covariant derivative in the metric transformation, we need the Levi-Civita connection to the zeroth order in perturbations. The nonvanishing components are

$$\Gamma_{00}^{(0)0} = \dot{N}; \quad \Gamma_{0j}^{(0)i} = \Gamma_{j0}^{(0)i} = \bar{N}H\delta_{ij}; \quad \Gamma_{ij}^{(0)0} = \frac{a^2H}{\bar{N}}\delta_{ij}.$$

On the other hand, the metric components to linear order in perturbations are

$$g_{00}^{(1)} = -2\bar{N}n; \quad g_{0i}^{(1)} = a^2\delta_{ij}N^j = N_i; \quad g_{ij}^{(1)} = a^2h_{ij}, \quad (\text{B24})$$

where h_{ij} is decomposed as in Eqs. (37) and (38). These then imply the following linear transformation rules:

$$\begin{aligned} n &\rightarrow n + \dot{\xi}_0 - \frac{\dot{N}}{\bar{N}}\xi_0; & N_i^T &\rightarrow N_i^T - \bar{N}\dot{\xi}_i^T + 2\bar{N}H\xi_i^T; \\ S &\rightarrow S - \bar{N}\dot{\xi} - \xi_0 + 2\bar{N}H\xi; & \varphi &\rightarrow \varphi + \frac{\dot{\phi}}{\bar{N}}\xi_0; \\ h &\rightarrow h - 2\frac{\nabla^2}{a^2}\xi + \frac{6H}{\bar{N}}\xi_0; & h_i^T &\rightarrow h_i^T - \frac{2}{a^2}\xi_i^T; \\ & & h_{ij}^{TT} &\rightarrow h_{ij}^{TT}, \end{aligned} \quad (\text{B25})$$

where $\xi_i = \xi_i^T + \partial_i\xi$ and $\partial_i\xi_i^T = 0$. In particular, $h - \nabla^2\tilde{h} \rightarrow h - \nabla^2\tilde{h} + (6H/\bar{N})\xi_0$. Now, from the transformations (B25) and Eqs. (B16), (B13), (B14), (B11), (B7), and (B8) it follows that h_{ij}^{TT} , $\tilde{\varphi}$, \tilde{n} , \tilde{N}_i^T , \tilde{S} , and also the corresponding terms in the action (B17), are gauge invariant.

An important question is whether the whole action (B17) is gauge invariant, also when the momentum terms are included. That the answer is yes can be argued as follows. When the Hamilton equations for the momenta, obtained by varying the action (1), are solved in terms of the fields and inserted back into the action (1), one obtains the standard Einstein-Hilbert action. This then suggests that one can define the coordinate transformations for the momenta such that also the action (1) becomes covariant. When this program is carried through at linear order in coordinate transformations, one gets the transformation rules for the momenta which render the free action (B17) gauge invariant. To see how this works in detail, note first that varying the action with respect to ρ_φ and ρ^{ij} yields $\rho_\varphi = \pi_\varphi + I_\varphi/2 = 0$ and $\rho^{ij} = \pi^{ij} + (I_{ij} - \delta_{ij}I)/2 = 0$, cf. Eqs. (B3) and (B4), where I_φ , I_{ij} , and $I = \delta_{ij}I_{ij}$ are defined in Eqs. (B2). These equations are just the Hamilton equations of the linearized theory and must be gauge invariant, implying that ρ^{ij} and ρ_φ are gauge invariant. From the transformation rules

$$\begin{aligned} I_\varphi &\rightarrow I_\varphi + \frac{2}{a^2}\nabla^2\xi + \frac{2V_{,\phi}}{\bar{N}\dot{\phi}}\xi_0; \\ I_{ij} &\rightarrow I_{ij} - \frac{4}{a^2}\partial_{(i}\xi_{j)} + \frac{1}{a^2}\delta_{ij}\nabla^2\xi + \frac{1}{\bar{N}Ha^2}\partial_i\partial_j\xi_0 \\ &\quad + \frac{1}{\bar{N}H}(H^2 + \dot{H})\delta_{ij}\xi_0, \end{aligned} \quad (\text{B26})$$

and based on the Hamilton equations for the (unshifted) momenta $\pi_\varphi = -I_\varphi/2$ and $\pi^{ij} = -(I_{ij} - \delta_{ij}I)/2$, we conclude that the momenta transform as

$$\begin{aligned} \pi_\varphi &\rightarrow \pi_\varphi - \frac{\nabla^2}{a^2}\xi - \frac{V_{,\phi}}{\bar{N}\dot{\phi}}\xi_0; \\ \pi^{ij} &\rightarrow \pi^{ij} + \frac{2}{a^2}\partial_{(i}\xi_{j)} - \frac{1}{2\bar{N}a^2H}(\partial_i\partial_j - \delta_{ij}\nabla^2)\xi_0 \\ &\quad + \frac{1}{\bar{N}H}(H^2 + \dot{H})\delta_{ij}\xi_0. \end{aligned} \quad (\text{B27})$$

From this, one can easily reconstruct how the scalar, vector, and tensor components of $\pi^{ij} = (\delta_{ij}/3)\pi + (\partial_i\partial_j - \delta_{ij}\nabla^2/3)\tilde{\pi} + \partial_{(i}\pi_{j)}^T + \pi_{ij}^{TT}$ transform:

$$\begin{aligned} (\pi^{ij})^{TT} &\rightarrow (\pi^{ij})^{TT}; & (\pi^i)^T &\rightarrow (\pi^i)^T + \frac{2}{a^2}\xi_i^T; \\ \tilde{\pi} &\rightarrow \tilde{\pi} + \frac{2}{a^2}\xi - \frac{1}{2\bar{N}a^2H}\xi_0; \\ \pi &\rightarrow \pi + \frac{2}{a^2}\nabla^2\xi - \frac{1}{\bar{N}a^2H}\nabla^2\xi_0 + \frac{3}{\bar{N}H}[H^2 + \dot{H}]\xi_0. \end{aligned}$$

The implications of these transformation rules for the cosmological perturbation theory are discussed in Sec. II B.

Next, the action (B17) is time reparametrization invariant. Thus choosing, for example, $\bar{N} = 1$ or $\bar{N} = a$ gives the action in physical (cosmological) or conformal time, respectively. One often defines the Mukhanov variable [58] $v = a\tilde{\varphi}$. For example, when written in conformal time ($\bar{N} = a$, $t \rightarrow \eta$) the Lagrangian of the scalar part of the action (B17) will acquire an additional $[a''/a]v^2$ contribution (here a *prime* refers to a derivative w.r.t. conformal time η):

$$S_v^{(2)} = \int d^3x d\eta \frac{1}{2} \left[v'^2 - (\partial_i v)^2 + \frac{(az)''}{az} v^2 \right]. \quad (\text{B28})$$

Another convenient variable is

$$w = -\frac{\tilde{\varphi}}{z} = (h - \nabla^2\tilde{h}) - \frac{\varphi}{z}, \quad (\text{B29})$$

in terms of which the scalar part of the quadratic action (B17) simplifies to

$$\begin{aligned} S_w^{(2)} &= \int d^3x \bar{N} dt a^3 z^2 \frac{1}{2} \left[\dot{w}^2 - \left(\frac{\partial_i w}{a} \right)^2 \right], \\ z &= \frac{\dot{\phi}}{6H} = \sqrt{\frac{\epsilon}{3}} \text{sign}[\dot{\phi}]. \end{aligned} \quad (\text{B30})$$

APPENDIX C

We now give the expressions for the Poisson brackets of the constraints $\{\mathcal{Q}_\alpha(\mathbf{x}), C_\beta(\mathbf{y})\}$, which appear in the measure of the path integral (53) and the constraints $C_\beta = C_\beta^{(1)} + C_\beta^{\geq 2}$ are given in Eqs. (35), (36), (46), and (47), with the free and interaction Hamiltonian given in Eqs. (34) and (D1), respectively. Before we proceed, we first present the Poisson brackets between the fields and the constraints,

$$\begin{aligned} \{h_{ij}(\mathbf{x}, t), C_0(\mathbf{y}, t), \} &= -2H(\tilde{g}^{-1/2})\{\delta_{ij} + [-h_{ij} + h\delta_{ij} \\ &\quad - 2\pi^{ij} + \pi\delta_{ij}] + [hh_{ij} - h_{il}h_{lj} \\ &\quad - 4\pi^{il}h_{lj} + \pi h_{ij} + \pi^{kl}h_{kl}\delta_{ij}] \\ &\quad + [-2h_{il}\pi^{lk}h_{kj} + h_{kl}\pi^{lk}h_{ij}]\} \\ &\quad \times \delta(\mathbf{x} - \mathbf{y}) \end{aligned} \quad (\text{C1})$$

$$\begin{aligned} \{h_{ij}(\mathbf{x}, t), C_l(\mathbf{y}, t)\} &= -\frac{2}{a^2}\{\delta_{l(i}\partial_{j)}^x - \Gamma_{ij}^l\}\delta(\mathbf{x} - \mathbf{y}); \\ \Gamma_{ij}^l &= \tilde{g}^{lk}\left(\partial_{(i}h_{j)k} - \frac{1}{2}\partial_k h_{ij}\right) \end{aligned} \quad (\text{C2})$$

$$\{\varphi(\mathbf{x}, t), C_0(\mathbf{y}, t), \} = -\frac{\mathcal{P}_\phi}{a^3}(\tilde{g}^{-1/2})(1 + \pi_\varphi)\delta(\mathbf{x} - \mathbf{y}) \quad (\text{C3})$$

$$\{\varphi(\mathbf{x}, t), C_i(\mathbf{y}, t), \} = -\frac{1}{a^2}\tilde{g}^{ij}\partial_j\varphi\delta(\mathbf{x} - \mathbf{y}). \quad (\text{C4})$$

We are now ready to consider different gauges.

(a) ‘‘Tensor gauge’’: $\mathcal{Q}_0 = h$, $\mathcal{Q}_i = \partial_j(h_{ij} - \frac{\delta_{ij}}{3}h)$:

$$\begin{aligned} \{\mathcal{Q}_0(\mathbf{x}), C_0(\mathbf{y})\} &= -2H(\tilde{g}^{-1/2})[3 + (2h + \pi) + (-h_{ij}h_{ij} + h^2 - \pi^{ij}h_{ij} + \pi h) + (-2h_{ik}\pi^{kl}h_{li} + \pi^{ij}h_{ij}h)]\delta(\mathbf{x} - \mathbf{y}) \\ \{\mathcal{Q}_0(\mathbf{x}), C_i(\mathbf{y})\} &= -\frac{2}{a^2}\left[\partial_i^x - \tilde{g}^{il}\left(\partial_k h_{lk} - \frac{1}{2}\partial_l h\right)\right]\delta(\mathbf{x} - \mathbf{y}) \\ \{\mathcal{Q}_i(\mathbf{x}), C_0(\mathbf{y})\} &= -2H\tilde{g}^{-1/2}\left\{-\left(h_{ij} - \frac{\delta_{ij}}{3}h\right) - 2\left(\pi^{ij} - \frac{\delta_{ij}}{3}\pi\right) + (h + \pi)\left(h_{ij} - \frac{\delta_{ij}}{3}h\right) - \left(h_{il}h_{lj} - \frac{\delta_{ij}}{3}h_{kl}h_{kl}\right) \right. \\ &\quad \left. - 4\left(\pi^{il}h_{lj} - \frac{\delta_{ij}}{3}\pi^{kl}h_{kl}\right) - 2\left(h_{il}\pi^{lr}h_{rj} - \frac{\delta_{ij}}{3}h_{kr}\pi^{rs}h_{sk}\right) + \pi^{kl}h_{kl}\left(h_{ij} - \frac{\delta_{ij}}{3}h\right)\right\}(\mathbf{y}, t)\partial_j^x\delta(\mathbf{x} - \mathbf{y}) \\ \{\mathcal{Q}_i(\mathbf{x}), C_j(\mathbf{y})\} &= -\frac{1}{a^2}\left\{\delta_{ij}\nabla_x^2 + \frac{1}{3}\partial_i^x\partial_j^x - \tilde{g}^{jk}(2\partial_{(i}h_{j)k} - \partial_k h_{il})(\mathbf{y}, t)\partial_i^x + \frac{1}{3}\tilde{g}^{jk}(2\partial_l h_{lk} - \partial_k h)(\mathbf{y}, t)\partial_i^x\right\}\delta(\mathbf{x} - \mathbf{y}). \end{aligned} \quad (\text{C5})$$

Since the tensor gauge conditions $\mathcal{Q}_\alpha = 0$ set the spatial scalars and vectors to zero, $h = 0 = \tilde{h} = h_i^T$, from the relations (C5) one easily obtains the ghost operators $\Omega_{\alpha\beta} = \{\mathcal{Q}_\alpha, C_\beta\}_{\mathcal{Q}_\alpha=0}$ of Eq. (51):

$$\begin{aligned} \Omega_{00} &= -2H(\tilde{g}^{-1/2})^{TT}[3 + \rho + L - h_{ij}^{TT}h_{ij}^{TT} - (\rho^{ij} + L_{ij})h_{ij}^{TT} - 2h_{ik}^{TT}(\rho^{kl} + L_{kl})h_{ij}^{TT}]\delta(\mathbf{x} - \mathbf{y}), \\ \Omega_{0i} &= -\frac{2}{a^2}\partial_i^x\delta(\mathbf{x} - \mathbf{y}), \\ \Omega_{i0} &= -2H(\tilde{g}^{-1/2})^{TT}\left\{-h_{ij}^{TT} - 2\left(\rho^{ij} + L_{ij} - \frac{\delta_{ij}}{3}(\rho + L)\right) + (\rho + L)h_{ij}^{TT} - \left(h_{il}^{TT}h_{lj}^{TT} - \frac{\delta_{ij}}{3}h_{kl}^{TT}h_{kl}^{TT}\right) \right. \\ &\quad \left. - 4\left((\rho^{il} + L_{il})h_{lj}^{TT} - \frac{\delta_{ij}}{3}(\rho^{kl} + L_{kl})h_{kl}^{TT}\right) - 2\left(h_{il}^{TT}(\rho^{lr} + L_{lr})h_{rj}^{TT} - \frac{\delta_{ij}}{3}h_{kr}^{TT}(\rho^{rs} + L_{rs})h_{sk}^{TT}\right) \right. \\ &\quad \left. + (\rho^{kl} + L_{kl})h_{kl}^{TT}h_{ij}^{TT}\right\}(\mathbf{y}, t)\partial_j^x\delta(\mathbf{x} - \mathbf{y}), \\ \Omega_{ij} &= -\frac{1}{a^2}\left\{\delta_{ij}\nabla_x^2 + \frac{1}{3}\partial_i^x\partial_j^x - (\tilde{g}^{jk})^{TT}(\partial_i h_{kl}^{TT} + \partial_l h_{ik}^{TT} - \partial_k h_{il}^{TT})(\mathbf{y}, t)\partial_i^x\right\}\delta(\mathbf{x} - \mathbf{y}), \end{aligned} \quad (\text{C6})$$

where $L = L_{ij}\delta_{ij}$ is defined in (B21), where $h = 0 = \tilde{h} = h_i^T$ and $\varphi \rightarrow -[\mathcal{P}_\phi/(6a^3H)]w$ are to be exacted, resulting in the shift functions given in Eqs. (D6)–(D10) below. Finally, from Eqs. (A2) and (A5) we infer

$$\begin{aligned}
(\tilde{g}^{ij})^{TT} &= \delta^{ij} - h_{ij}^{TT} + h_{il}^{TT} h_{lj}^{TT} - h_{il}^{TT} h_{lk}^{TT} h_{kj}^{TT} + h_{il}^{TT} h_{lk}^{TT} h_{km}^{TT} h_{mj}^{TT} + \mathcal{O}((h_{ij}^{TT})^5), \\
(\tilde{g}^{-1/2})^{TT} &= 1 + \frac{1}{4} h_{ij}^{TT} h_{ij}^{TT} - \frac{1}{6} h_{ij}^{TT} h_{jl}^{TT} h_{li}^{TT} + \frac{1}{8} h_{ij}^{TT} h_{jl}^{TT} h_{lk}^{TT} h_{ki}^{TT} + \frac{1}{32} h_{ij}^{TT} h_{ij}^{TT} h_{kl}^{TT} h_{kl}^{TT} + \mathcal{O}((h_{ij}^{TT})^5).
\end{aligned}$$

(b) ‘‘Uniform field gauge’’: $\mathcal{Q}_0 = \varphi$, $\mathcal{Q}_i = \partial_j(h_{ij} - \frac{\delta_{ij}}{3}h)$: In this gauge the relevant Poisson brackets follow from Eqs. (C3) and (C4),

$$\{\mathcal{Q}_0(\mathbf{x}, t), C_0(\mathbf{y}, t)\} = -\frac{\mathcal{P}_\phi}{a^3} \tilde{g}^{-1/2} (1 + \pi_\varphi) \delta(\mathbf{x} - \mathbf{y}), \quad \{\mathcal{Q}_0(\mathbf{x}, t), C_i(\mathbf{y}, t)\} = -\frac{1}{a^2} \tilde{g}^{ij} (\partial_j \varphi) \delta(\mathbf{x} - \mathbf{y}), \quad (C7)$$

and the other Poisson brackets $\{\mathcal{Q}_i(\mathbf{x}, t), C_j(\mathbf{y}, t)\}$ are identical to the corresponding tensor gauge expressions in Eq. (C5). In this gauge $\varphi = 0 = \dot{h} = h_i^T$, such that $h_{ij} = (\delta_{ij}/3)h + h_{ij}^{TT}$, and the corresponding ghosts operators are

$$\begin{aligned}
\Omega_{00} &= -\frac{\mathcal{P}_\phi}{a^3} (\tilde{g}^{-1/2})^{\text{ufg}} (1 + \rho_\varphi + L_\varphi) \delta(\mathbf{x} - \mathbf{y}), \quad \Omega_{0i} = 0, \\
\Omega_{i0} &= -2H(\tilde{g}^{-1/2})^{\text{ufg}} \left\{ -h_{ij}^{TT} - 2\left(\rho^{ij} + L_{ij} - \frac{\delta_{ij}}{3}(\rho + L)\right) + \left(\frac{1}{3}h + \rho + L\right)h_{ij}^{TT} - \left(h_{il}^{TT}h_{lj}^{TT} - \frac{\delta_{ij}}{3}h_{kl}^{TT}h_{kl}^{TT}\right) \right. \\
&\quad - \frac{4h}{3}\left(\rho^{ij} + L_{ij} - \frac{\delta_{ij}}{3}(\rho + L)\right) - 4\left(\rho^{il} + L_{il}\right)h_{lj}^{TT} - \frac{\delta_{ij}}{3}\left(\rho^{kl} + L_{kl}\right)h_{kl}^{TT} - 2\left[\left(\frac{\delta_{il}}{3}h + h_{il}^{TT}\right)(\rho^{lr} + L_{lr}) \right. \\
&\quad \times \left.\left(\frac{\delta_{rj}}{3}h + h_{rj}^{TT}\right) - \frac{\delta_{ij}}{3}\left(\frac{\delta_{kr}}{3}h + h_{kr}^{TT}\right)(\rho^{rs} + L_{rs})\left(\frac{\delta_{sk}}{3}h + h_{sk}^{TT}\right)\right] + \left[\frac{1}{3}(\rho + L)h + (\rho^{kl} + L_{kl})h_{kl}^{TT}\right]h_{ij}^{TT} \left. \right\} \\
&\quad \times (\mathbf{y}, t) \partial_i^x \delta(\mathbf{x} - \mathbf{y}), \\
\Omega_{ij} &= -\frac{1}{a^2} \left\{ \delta_{ij} \nabla_x^2 + \frac{1}{3} \partial_i^x \partial_j^x - \frac{1}{3} (\tilde{g}^{jk})^{\text{ufg}} (\partial_i h \delta_{kl} + \delta_{ki} \partial_l h - \partial_k h \delta_{il}) (\mathbf{y}, t) \partial_l^x - (\tilde{g}^{jk})^{\text{ufg}} (\partial_i h_{lk}^{TT} + \partial_l h_{ik}^{TT} \right. \\
&\quad \left. - \partial_k h_{il}^{TT}) (\mathbf{y}, t) \partial_l^x - \frac{1}{9} [(\tilde{g}^{jk})^{\text{ufg}} \partial_k h] (\mathbf{y}, t) \partial_i^x \right\} \delta(\mathbf{x} - \mathbf{y}), \quad (C8)
\end{aligned}$$

where the superscripts in $(\tilde{g}^{-1/2})^{\text{ufg}}$ and $(\tilde{g}^{ij})^{\text{ufg}}$ signify that only h and h_{ij}^{TT} contribute to h_{ij} , i.e. $h_{ij} = (\delta_{ij}/3)h + h_{ij}^{TT}$.

APPENDIX D

The interaction Hamiltonian containing cubic and higher order interactions as inferred from Eqs. (2) and (A10) is of the general form:

$$\begin{aligned}
\mathcal{H}_I &= 4\bar{N}a^3 H^2 \left[-\frac{3}{2} (\tilde{g}^{-1/2})_{\geq 3} - (\tilde{g}^{-1/2})_{\geq 2} h + h_{ij} (\tilde{g}^{-1/2})_{\geq 1} \frac{1}{2} A_{ijkl} h_{kl} + \pi^{ij} (-(\tilde{g}^{-1/2})_{\geq 2} \delta_{ij} + (\tilde{g}^{-1/2})_{\geq 1} (h_{ij} - \delta_{ij} h) \right. \\
&\quad \times \left. + \tilde{g}^{-1/2} (h_{ik} h_{kj} - h h_{ij}) + \pi^{ij} \left((\tilde{g}^{-1/2})_{\geq 1} \frac{1}{2} A_{ijkl} + \tilde{g}^{-1/2} \left(2h_{jl} \delta_{ik} - \delta_{ij} h_{kl} + h_{ik} h_{jl} - \frac{1}{2} h_{ij} h_{kl} \right) \right) \pi^{kl} \right] \\
&\quad + \bar{N} a^3 \left((\tilde{g}^{1/2})_{\geq 2} \sum_{n=1}^{\infty} \frac{V^{(n)}}{n!} \varphi^n + \frac{1}{2} h \sum_{n=2}^{\infty} \frac{V^{(n)}}{n!} \varphi^n + \sum_{n=3}^{\infty} \frac{V^{(n)}}{n!} \varphi^n + (\tilde{g}^{1/2})_{\geq 3} V \right) + \bar{N} \frac{\mathcal{P}_\phi^2}{2a^3} ((\tilde{g}^{-1/2})_{\geq 1} \pi_\varphi^2 \\
&\quad + 2(\tilde{g}^{-1/2})_{\geq 2} \pi_\varphi + (\tilde{g}^{-1/2})_{\geq 3}) + \frac{a\bar{N}}{2} ((\tilde{g}^{1/2})_{\geq 1} \delta^{ij} + \tilde{g}_{\geq 1}^{ij} + (\tilde{g}^{1/2})_{\geq 1} (\tilde{g}^{ij})_{\geq 1}) \partial_i \varphi \partial_j \varphi - a\bar{N} ((\tilde{g}^{1/2})_{\geq 1} ((\tilde{g}^{ij})_{\geq 1} \delta^{kl} \\
&\quad + \delta^{ij} (\tilde{g}^{kl})_{\geq 1}) + \tilde{g}^{1/2} (\tilde{g}^{ij})_{\geq 1} (\tilde{g}^{kl})_{\geq 1}) [\partial_k \partial_i h_{lj} - \partial_i \partial_j h_{kl}] - a\bar{N} (\tilde{g}^{1/2} \tilde{g}^{ij} \tilde{g}^{km} \tilde{g}^{ln} - \delta^{ij} \delta^{km} \delta^{ln}) \left[-(\partial_k h_{mn}) (\partial_i h_{lj}) \right. \\
&\quad \left. - \frac{1}{4} (\partial_i h_{jm}) (\partial_i h_{nk}) - \frac{1}{4} (\partial_i h_{jm}) (\partial_k h_{in}) - \frac{1}{4} (\partial_n h_{ij}) (\partial_l h_{km}) + (\partial_l h_{ij}) (\partial_k h_{mn}) + \frac{3}{4} (\partial_i h_{kl}) (\partial_j h_{mn}) \right], \quad (D1)
\end{aligned}$$

where $\tilde{g}_{ij}^{(\geq n)}$ and $(\tilde{g}^{\pm 1/2})^{(\geq n)}$ are given in Eqs. (A2) and (A5), respectively. The cubic part of the interaction action (24) is of the form

$$\mathcal{S}_{\text{cubic}} = \int d^3x dt \{ -\mathcal{H}_{\text{cubic}} + nC_0^{(2)} + N_i C_i^{(2)} \}, \quad (\text{D2})$$

where $C_0^{(2)}$ and $C_i^{(2)}$ are the quadratic part of Eqs. (46), (47), and (34). When written in tensor gauge (80), the cubic part of the interaction Hamiltonian density (D1) reads

$$\begin{aligned} -\mathcal{H}_{\text{cubic}} = & \bar{N}a^3 \left\{ 2H^2 \left[-\frac{3-\epsilon}{3} h_{ij}^{TT} h_{jl}^{TT} h_{li}^{TT} + \frac{1}{2} (\rho + L) h_{ij}^{TT} h_{ij}^{TT} - 2h_{ij}^{TT} (\rho^{jl} + L_{jl}) h_{li}^{TT} - 4(\rho^{ij} + L_{ij}) h_{jl}^{TT} (\rho^{li} + L_{li}) \right. \right. \\ & + 2(\rho + L)(\rho^{ij} + L_{ij}) h_{ij}^{TT} - \frac{\epsilon}{2} (\rho_\varphi + L_\varphi) h_{ij}^{TT} h_{ij}^{TT} \left. \right] + \frac{h_{ij}^{TT}}{a^2} \left[\frac{1}{4} (\partial_i h_{kl}^{TT}) (\partial_j h_{kl}^{TT}) + \frac{1}{2} (\partial_l h_{jk}^{TT}) (\partial_k h_{il}^{TT}) \right. \\ & \left. \left. - \frac{3}{2} (\partial_l h_{ik}^{TT}) (\partial_l h_{jk}^{TT}) \right] + \frac{\epsilon}{18a^2} h_{ij}^{TT} (\partial_i w) (\partial_j w) + \frac{(3-\eta)\epsilon H^2}{6} h_{ij}^{TT} h_{ij}^{TT} w + \frac{\epsilon^{3/2} V_{,\phi\phi\phi}}{162} w^3 \right\}, \quad (\text{D3}) \end{aligned}$$

while the cubic constraint contributions read

$$\begin{aligned} nC_0^{(2)} = & (\tilde{n} + L_0) a^3 \left\{ -4H^2 (\rho^{ij} + L_{ij}) (\rho^{ij} + L_{ij}) \right. \\ & + 2H^2 (\rho + L)^2 - 4H^2 (\rho^{ij} + L_{ij}) h_{ij}^{TT} \\ & - 2H^2 \epsilon (\rho_\varphi + L_\varphi)^2 - \frac{\epsilon}{18a^2} (\partial_i w)^2 - \frac{\epsilon V_{,\phi\phi}}{18} w^2 \\ & + \frac{1}{a^2} h_{ij}^{TT} \nabla^2 h_{ij}^{TT} + \frac{3}{4a^2} (\partial_l h_{ij}^{TT}) (\partial_l h_{ij}^{TT}) \\ & \left. - H^2 (1 + \epsilon) h_{ij}^{TT} h_{ij}^{TT} \right\} \quad (\text{D4}) \end{aligned}$$

$$\begin{aligned} N_i C_i^{(2)} = & (\tilde{N}_i + L_i) a^3 \left\{ \frac{2\epsilon H}{3a^2} (\rho_\varphi + L_\varphi) \partial_i w - \frac{2\epsilon H}{3a^2} h_{ij}^{TT} \partial_j w \right. \\ & \left. - \frac{2H}{a^2} (\rho^{lj} + L_{lj}) (2\partial_j h_{il}^{TT} - \partial_i h_{jl}^{TT}) \right\}. \quad (\text{D5}) \end{aligned}$$

The shift functions (B19)–(B23) are

$$L_0 = -\frac{\tilde{N}\epsilon}{6} w - \frac{\nabla^2 \tilde{S}}{(3-\epsilon)a^2 H} \quad (\text{D6})$$

$$L_i = \frac{a^2 \tilde{N}\epsilon}{6} \frac{\partial_i}{\nabla^2} \dot{w} \quad (\text{D7})$$

$$L_\varphi = -\frac{\dot{w}}{6H} + \frac{\eta w}{6} - \frac{\tilde{n}}{\bar{N}} + \frac{\nabla^2 \tilde{S}}{(3-\epsilon)\bar{N}a^2 H} \quad (\text{D8})$$

$$\begin{aligned} L_{ij} = & -\frac{\dot{h}_{ij}^{TT}}{4H} - h_{ij}^{TT} - \frac{\epsilon}{12H} \left(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2} \right) \dot{w} + \frac{\epsilon \delta_{ij}}{6} w \\ & + \frac{\partial_i \tilde{N}_j^T + \partial_i \partial_j \tilde{S}}{2\tilde{N}a^2 H} - \frac{\delta_{ij} \tilde{n}}{\bar{N}} - \frac{(1-\epsilon)\delta_{ij} \nabla^2 \tilde{S}}{2(3-\epsilon)\bar{N}a^2 H} \quad (\text{D9}) \end{aligned}$$

$$L = \delta_{ij} L_{ij} = -\frac{\epsilon}{6H} \dot{w} + \frac{\epsilon}{2} w - \frac{3\tilde{n}}{\bar{N}} + \frac{\nabla^2 \tilde{S}}{(3-\epsilon)\bar{N}a^2 H}. \quad (\text{D10})$$

Finally, for the calculation of four-point functions, the quartic contribution from the interaction action (24) is needed,

$$\mathcal{S}_{\text{quartic}} = \int d^3x dt \{ -\mathcal{H}_{\text{quartic}} + nC_0^{(3)} + N_i C_i^{(3)} \}, \quad (\text{D11})$$

where $C_0^{(3)}$ and $C_i^{(3)}$ are the cubic parts of Eqs. (46) and (47). In tensor gauge they are

$$nC_0^{(3)} = -\left(\frac{\tilde{n}}{\bar{N}} + \frac{L_0}{\bar{N}} \right) \mathcal{H}_{\text{cubic}} \quad (\text{D12})$$

$$\begin{aligned} N_i C_i^{(3)} = & (\tilde{N}_i + L_i) a^3 \left\{ -\frac{2\epsilon H}{3a^2} (\rho_\varphi + L_\varphi) h_{ij}^{TT} \partial_j w \right. \\ & + \frac{2\epsilon H}{3a^2} h_{il}^{TT} h_{lj}^{TT} \partial_j w + \frac{2H}{a^2} h_{im}^{TT} (\rho^{lj} + L_{lj}) \\ & \left. \times (2\partial_j h_{ml}^{TT} - \partial_m h_{lj}^{TT}) \right\}, \quad (\text{D13}) \end{aligned}$$

while the quartic part of the Hamiltonian (D1) becomes

$$\begin{aligned}
-\mathcal{H}_{\text{quartic}} = & \bar{N}a^3 \left\{ \frac{3}{4} H^2 \left[h_{ij}^{TT} h_{jl}^{TT} h_{lk}^{TT} h_{ki}^{TT} + \frac{1}{4} h_{ij}^{TT} h_{ij}^{TT} h_{kl}^{TT} h_{kl}^{TT} \right] - \frac{2}{3} H^2 (\rho + L) h_{ij}^{TT} h_{jl}^{TT} h_{li}^{TT} + H^2 \left[-h_{ij}^{TT} h_{ij}^{TT} - (\rho^{ij} + L_{ij}) h_{ij}^{TT} \right. \right. \\
& - (\rho^{ij} + L_{ij})(\rho^{ij} + L_{ij}) + \frac{1}{2} (\rho + L)^2 \left. \right] h_{kl}^{TT} h_{kl}^{TT} + 2H^2 [-2(\rho^{ij} + L_{ij}) h_{jk}^{TT} (\rho^{kl} + L_{kl}) h_{li}^{TT} \\
& + (\rho^{ij} + L_{ij}) h_{ij}^{TT} (\rho^{kl} + L_{kl}) h_{kl}^{TT}] + \frac{1}{a^2} [-h_{mn}^{TT} h_{mn}^{TT} (\delta_{ij} h_{kl}^{TT} + h_{ij}^{TT} \delta_{kl}) - h_{in}^{TT} h_{nj}^{TT} h_{kl}^{TT} - h_{ij}^{TT} h_{kn}^{TT} h_{nl}^{TT}] \\
& \times [\partial_k \partial_i h_{ij}^{TT} - \partial_i \partial_j h_{kl}^{TT}] + \frac{1}{a^2} \left[\delta_{ij} \delta_{km} h_{lp}^{TT} h_{pn}^{TT} + \delta_{ij} h_{kp}^{TT} h_{pm}^{TT} \delta_{ln} + h_{ip}^{TT} h_{pj}^{TT} \delta_{km} \delta_{ln} + \delta_{ij} h_{km}^{TT} h_{nl}^{TT} + h_{ij}^{TT} \delta_{km} h_{nl}^{TT} \right. \\
& + h_{ij}^{TT} h_{km}^{TT} \delta_{nl} - \frac{1}{4} h_{pr}^{TT} h_{pr}^{TT} \delta_{ij} \delta_{km} \delta_{ln} \left. \right] \left[\partial_k h_{mn}^{TT} \partial_i h_{ij}^{TT} - \frac{1}{4} \partial_i h_{jm}^{TT} \partial_i h_{nk}^{TT} - \frac{1}{4} \partial_i h_{jm}^{TT} \partial_k h_{in}^{TT} - \frac{1}{4} \partial_n h_{ij}^{TT} \partial_i h_{km}^{TT} \right. \\
& + \partial_i h_{ij}^{TT} \partial_k h_{mn}^{TT} + \frac{3}{4} \partial_i h_{kl}^{TT} \partial_j h_{mn}^{TT} \left. \right] - 2\epsilon H^2 \left[\frac{1}{4} (\rho_\varphi + L_\varphi)^2 h_{ij}^{TT} h_{ij}^{TT} - \frac{1}{3} (\rho_\varphi + L_\varphi) h_{ij}^{TT} h_{jl}^{TT} h_{li}^{TT} + \frac{1}{8} h_{ij}^{TT} h_{jl}^{TT} h_{lk}^{TT} h_{ki}^{TT} \right. \\
& + \frac{1}{32} h_{ij}^{TT} h_{ij}^{TT} h_{kl}^{TT} h_{kl}^{TT} \left. \right] + \frac{\epsilon}{18} \left[\frac{1}{4} h_{kl}^{TT} h_{kl}^{TT} \delta_{ij} - h_{il}^{TT} h_{lj}^{TT} \right] \left(\frac{\partial_i w}{a} \right) \left(\frac{\partial_j w}{a} \right) + \left[-\frac{\epsilon^2 V_{,\phi\phi\phi\phi}}{1944} w^4 + \frac{\epsilon V_{,\phi\phi}}{72} w^2 h_{ij}^{TT} h_{ij}^{TT} \right. \\
& \left. - \frac{(3-\eta)\epsilon H^2}{9} w h_{ij}^{TT} h_{jl}^{TT} h_{li}^{TT} + (3-\epsilon) H^2 \left(\frac{1}{4} h_{ij}^{TT} h_{jk}^{TT} h_{kl}^{TT} h_{li}^{TT} - \frac{1}{16} h_{ij}^{TT} h_{ij}^{TT} h_{kl}^{TT} h_{kl}^{TT} \right) \right] \left. \right\}. \tag{D14}
\end{aligned}$$

These formulas are used in Sec. IV to construct some of the cubic and quartic vertices of the theory.

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