



CDT meets Hořava–Lifshitz gravity

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ABSTRACT

The theory of causal dynamical triangulations (CDT) attempts to define a nonperturbative theory of quantum gravity as a sum over spacetime geometries. One of the ingredients of the CDT framework is a global time foliation, which also plays a central role in the quantum gravity theory recently formulated by Hořava. We show that the phase diagram of CDT bears a striking resemblance with the generic Lifshitz phase diagram appealed to by Hořava. We argue that CDT might provide a unifying nonperturbative framework for anisotropic as well as isotropic theories of quantum gravity.

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1. Introduction

A major unsolved problem in theoretical physics is to reconcile the classical theory of general relativity with quantum mechanics. Recently there has been a resurgence in interest in using mundane quantum field theory to address this question. Progress over the last ten years in the use of renormalization group (RG) techniques [1] suggests that the so-called asymptotic safety scenario, originally put forward by S. Weinberg [2], may be realized, namely, the existence of a nontrivial ultraviolet fixed point, where one can define a theory of quantum gravity.

In tandem with this approach, the method of Causal Dynamical Triangulation (CDT) has been developed, likewise with the aim of defining and constructing a nonperturbative quantum gravity theory [3–7] (for recent reviews, see [8]). CDT provides a lattice framework in which a variety of nonperturbative field-theoretical aspects of quantum gravity can be studied, including in principle predictions from other candidate theories. Despite the fact that the CDT and the RG approaches use rather different sets of tools, they might be two sides of the same coin. Locating a suitable UV fixed point in causal dynamical triangulations would provide strong evidence that this is indeed the case and that “asymptotic safety” is on the right track.

More recently, P. Hořava has suggested yet another field-theoretical approach to quantum gravity in the continuum [9], since dubbed Hořava–Lifshitz gravity, where the four-dimensional diffeomorphism symmetry of general relativity is explicitly broken. Assuming a global time-foliation, time and space are treated differently, in the sense that only suitable *second-order* derivatives in time appear to render the quantum theory unitary, while higher-order spatial derivatives ensure renormalizability.

A common key ingredient in both CDT and Hořava–Lifshitz gravity is a global time foliation, with the difference that in CDT this is not directly associated with a violation of diffeomorphism symmetry, since the dynamics is defined directly on the quotient space of metrics modulo diffeomorphisms. This raises the question whether new insights can be gained by analyzing and interpreting CDT quantum gravity in a generalized, anisotropic framework along the lines of Hořava–Lifshitz gravity. The reference frame until now has been a covariant one, assuming that any UV fixed point found in the CDT formulation could be identified with that found in the covariant renormalization group approach, appealing to the general sparseness of fixed points.¹ At the same time, we have presented general arguments in favour of a reflection-positive transfer matrix in the (Euclideanized version of) CDT [10,11]. Thus the conditions for a unitary quantum field theory at the UV fixed point are also met. The philosophy behind formulating gravity at a Lifshitz point was that unitarity in a theory of quantum gravity should be

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¹ Of course, one should also show that a lattice fixed point reproduces the critical exponents of the RG treatment.

the prime requirement, rather than treating space and time on the (almost) equal footing required by special relativity. We conclude that the CDT approach not only shares the time-foliated structure of spacetime, but also the enforcement of unitarity by construction with Hořava–Lifshitz gravity.

This led us to asking whether CDT may be able to capture aspects of the latter, despite the fact that no higher-order spatial derivative terms are put in by hand in the CDT action. Some support for this idea comes from the fact that one UV result which can be compared explicitly, namely, the nontrivial value of the spectral dimension of quantum spacetime, appears to coincide in both approaches [12,13]. Interestingly, also the renormalization group approach was able to reproduce the same finding, after the spectral dimension had first been measured in simulations of CDT quantum gravity, a result taken at the time as possible corroboration of the equivalence between the CDT and RG approaches [14].²

In view of the considerations outlined above, we have returned to a closer analysis of the basic CDT phase diagram. In what follows, we will report on some striking similarities between the phase diagram of causal dynamically triangulated gravity and the Lifshitz phase diagram promoted in Hořava–Lifshitz gravity. They become apparent when one identifies “average geometry”, presumably related to the conformal mode of the geometry in some way, with the order parameter ϕ of an effective Lifshitz theory. We find that the phase structure allows potentially for both an anisotropic and an isotropic UV fixed point, opening the exciting prospect that CDT can serve as a nonperturbative lattice foundation for both the renormalization group approach and Hořava–Lifshitz gravity, in the same way as theories on fixed lattices provide us with nonperturbative definitions of quantum field theories in the formulation of K. Wilson.

2. Causal dynamical triangulation

We will merely sketch the setup used in CDT, and refer to [11, 4,7] for more complete descriptions and to [16] for the rationale behind the formulation. We attempt to define the path integral of quantum gravity by summing over a class of piecewise linear spacetime geometries, much in the same way as one can define the path integral in ordinary quantum mechanics by dividing the time into intervals of length a , considering paths which are linear between $t_n = na$ and $t_{n+1} = (n+1)a$, and then taking the limit of vanishing “lattice spacing”, $a \rightarrow 0$.

Let us introduce a time slicing labeled by discrete lattice times t_n . The spatial hypersurface labeled by t_n has the topology of S^3 and is a piecewise flat triangulation, obtained by gluing together identical, equilateral tetrahedra with link length a_s , to be identified with the short-distance lattice cut-off. We now connect the three-dimensional triangulation of S^3 at t_n with that at time t_{n+1} by means of four types of four-simplices: four-simplices of type (4, 1), which share four vertices (in fact, an entire tetrahedron) with the spatial hypersurface at t_n and one vertex with the hypersurface at t_{n+1} ; four-simplices of type (1, 4), where the roles of t_n and t_{n+1} are interchanged; four-simplices of type (3, 2), which share three vertices (in fact, an entire triangle) with the hypersurface at t_n , and two vertices with the hypersurface at t_{n+1} (belonging to the same spatial link); lastly, four-simplices of type (2, 3), defined analogously but with t_n and t_{n+1} interchanged.

These four-simplices have a number of links (and corresponding triangles and tetrahedra) connecting vertices in hypersurfaces t_n

and t_{n+1} . We take all of these links to be time-like with (squared) length $a_t^2 = \alpha a_s^2$. The four-simplices are glued together such that the “slab” between hypersurfaces labeled by t_n and t_{n+1} has the topology $S^3 \times [0, 1]$. We say that the hypersurfaces are separated by a proper distance $\sqrt{\alpha}a_t$, but this is not strictly speaking true if one takes the piecewise flat geometries (despite their curvature singularities) seriously as classical spacetimes. However, what is true is that all links connecting neighbouring hypersurfaces have proper length $\sqrt{\alpha}a_t$.

In the path integral we sum over all geometrically distinct piecewise linear geometries of this type, and with a fixed number of time steps. As an action we use the Einstein–Hilbert action, which has a natural realization on piecewise linear geometries, first introduced by Regge. The geometries allow a rotation to Euclidean geometries simply by rotating $\alpha \rightarrow -\alpha$ in the lower-half complex plane. The action changes accordingly and becomes the Euclidean Einstein–Hilbert Regge action of the thus “Wick-rotated” piecewise flat Euclidean spacetime. Its functional form becomes extremely simple because we use only two different kinds of building blocks, which contribute in discrete units to the four-volume and the scalar curvature. In this way the Euclidean action becomes a function of “counting building blocks”, namely,

$$S_E = \frac{1}{G} \int \sqrt{g}(-R + 2\Lambda) \rightarrow -(\kappa_0 + 6\Delta)N_0 + \kappa_4(N_4^{(4,1)} + N_4^{(3,2)}) + \Delta(2N_4^{(4,1)} + N_4^{(3,2)}), \quad (1)$$

where N_0 is the number of vertices, $N_4^{(4,1)}$ the number of four-simplices of type (4, 1) or (1, 4), and $N_4^{(3,2)}$ the number of four-simplices of type (3, 2) or (2, 3) in the given triangulated spacetime history. For later use we denote the total number $N_4^{(4,1)} + N_4^{(3,2)}$ of four-simplices by N_4 . Furthermore, the parameter κ_0 in (1) is proportional to the inverse bare gravitational coupling constant, while κ_4 is related to the bare cosmological coupling constant. Finally, Δ is an asymmetry parameter which in a convenient way encodes the dependence of the action on the relative time–space scaling α introduced above, and is handy when studying the relation to Hořava–Lifshitz gravity. Vanishing $\Delta = 0$ implies $\alpha = 1$, and increasing Δ away from zero corresponds to decreasing α , i.e. the time-like links shrink in length when Δ is increased.

The rotation to Euclidean space is necessary in order to use Monte Carlo simulations as a tool to explore the theory nonperturbatively. For simulation-technical reasons it is preferable to keep the total number N_4 of four-simplices fixed during a Monte Carlo simulation, which implies that κ_4 effectively does not appear as a coupling constant. Instead we can perform simulations for different four-volumes if needed. To summarize, we are dealing with a statistical system of fluctuating four-geometry, whose phase diagram as function of the two bare coupling constants κ_0 and Δ we are going to explore next.

3. The CDT phase diagram

The CDT phase diagram was described qualitatively as part of the first comprehensive study of four-dimensional CDT quantum gravity [4]. For the first time, we are presenting here the real phase diagram (Fig. 1), based on computer simulations with $N_4 = 80000$. Because there are residual finite-size effects for universes of this size, one can still expect minor changes in the location of the transition lines as $N_4 \rightarrow \infty$. The dotted lines in Fig. 1 represent mere extrapolations, and lie in a region of phase space which is difficult to access due to inefficiencies of our computer algorithms.

² Inspired by the seemingly universal value of the UV spectral dimension, more general arguments about the underlying UV nature of spacetime have been put forward [15].

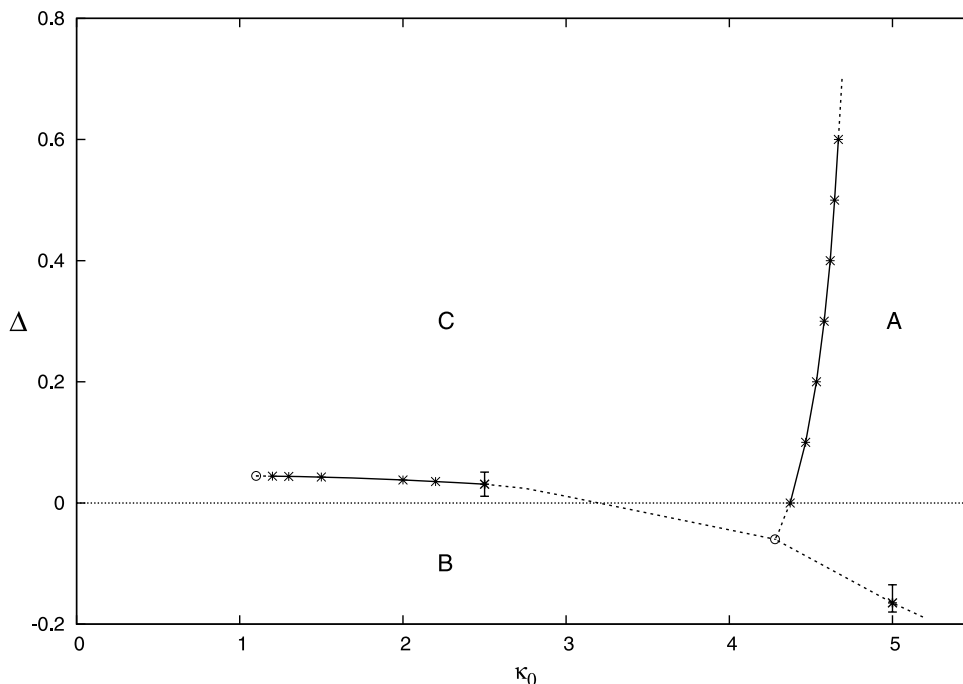


Fig. 1. The phase diagram of four-dimensional quantum gravity, defined in terms of causal dynamical triangulations, parametrized by the inverse bare gravitational coupling κ_0 and the asymmetry Δ .

There are three phases, labeled A, B and C in [4]. In phase C, which had our main interest in [7,6], we observed a genuine four-dimensional universe in the sense that as a function of the continuum four-volume V_4 (linearly related to the number of four-simplices), the time extent scaled as $V_4^{1/4}$ and the spatial volume as $V_4^{3/4}$. Moving into phase A, these scaling relations break down. Instead, we observe a number of small universes arranged along the time direction like “pearls on a string”, if somewhat uneven in size. They can split or merge along the time direction as a function of the Monte Carlo time used in the simulations. These universes are connected by thin “necks”, i.e. slices of constant integer time t_n , where the spatial S^3 -universes are at or close to the smallest three-volume permitted (consisting of five tetrahedra glued together), to prevent “time” from becoming disconnected.

By contrast, phase B is characterized by the “vanishing” of the time direction, in the sense that only one spatial hypersurface has a three-volume appreciably larger than the minimal cut-off size of five just mentioned. One might be tempted to conclude that the resulting universe is three-dimensional, just lacking the time direction of the extended universe found in phase C. However, the situation is more involved; although we have a large three-volume collected at a single spatial hypersurface, the corresponding spatial universe has almost no extension. This follows from the fact (ascertained through measurement) that it is possible to get in just a few steps from one tetrahedron to any other by moving along the centres of neighbouring tetrahedra or, alternatively, from one vertex to any other along a chain of links. The Hausdorff dimension is therefore quite high, and possibly infinite. Let us assume for the moment that it is indeed infinite; then the universe in phase B has neither time nor spatial extension, and there is no geometry in any classical sense.

We can now give the following qualitative characterization of the three phases in terms of what we will provisionally call “average geometry”. The universe of phase C exhibits a classical four-dimensional background geometry on large scales, such that

$\langle geometry \rangle \neq 0$. One may even argue that $\langle geometry \rangle = \text{const.}$ in view of the fact that according to the mini-superspace analysis of [6,7,5] and allowing for a finite rescaling of the renormalized proper time, the universe can be identified with the round S^4 , a maximally symmetric de Sitter space of constant scalar curvature. By contrast, in phase B the universe presumably has no extension or trace of classicality, corresponding to $\langle geometry \rangle = 0$. Lastly, in phase A, the geometry of the universe appears to be “oscillating” in the time direction. The three phases are separated by three phase transition lines which meet in a triple point as illustrated in Fig. 1.

We have chosen this particular qualitative description to match precisely that of a Lifshitz phase diagram [17]. In an effective Lifshitz theory, the Landau free energy density $F(x)$ as function of an order parameter $\phi(x)$ takes the form³

$$F(x) = a_2\phi(x)^2 + a_4\phi(x)^4 + a_6\phi(x)^6 + \dots + c_2(\partial_\alpha\phi)^2 + d_2(\partial_\beta\phi)^2 + e_2(\partial_\beta^2\phi)^2 + \dots, \quad (2)$$

where for a d -dimensional system $\alpha = m + 1, \dots, d, \beta = 1, \dots, m$. Distinguishing between “ α ”- and “ β ”-directions allows one to take anisotropic behaviour into account. For a usual system, $m = 0$ and a phase transition can occur when a_2 passes through zero (say, as a function of temperature). For $a_2 > 0$ we have $\phi = 0$, while for $a_2 < 0$ we have $|\phi| > 0$ (always assuming $a_4 > 0$). However, one also has a transition when anisotropy is present ($m > 0$) and d_2 passes through zero. For negative d_2 one can then have an oscillating behaviour of ϕ in the m “ β ”-directions. Depending on the sign of a_2 , the transition to this so-called modulated or helical phase can occur either from the phase where $\phi = 0$, or from the phase where $|\phi| > 0$. We conclude that the phases C, B, and A of CDT quantum gravity depicted in Fig. 1 can be put into a one-to-one correspondence with the ferromagnetic, paramagnetic and helical

³ See, for example, [18] for an introduction to the content and scope of “Landau theory”.

$$N_4=120k, \kappa_0=4.7110, \Delta=0.6$$

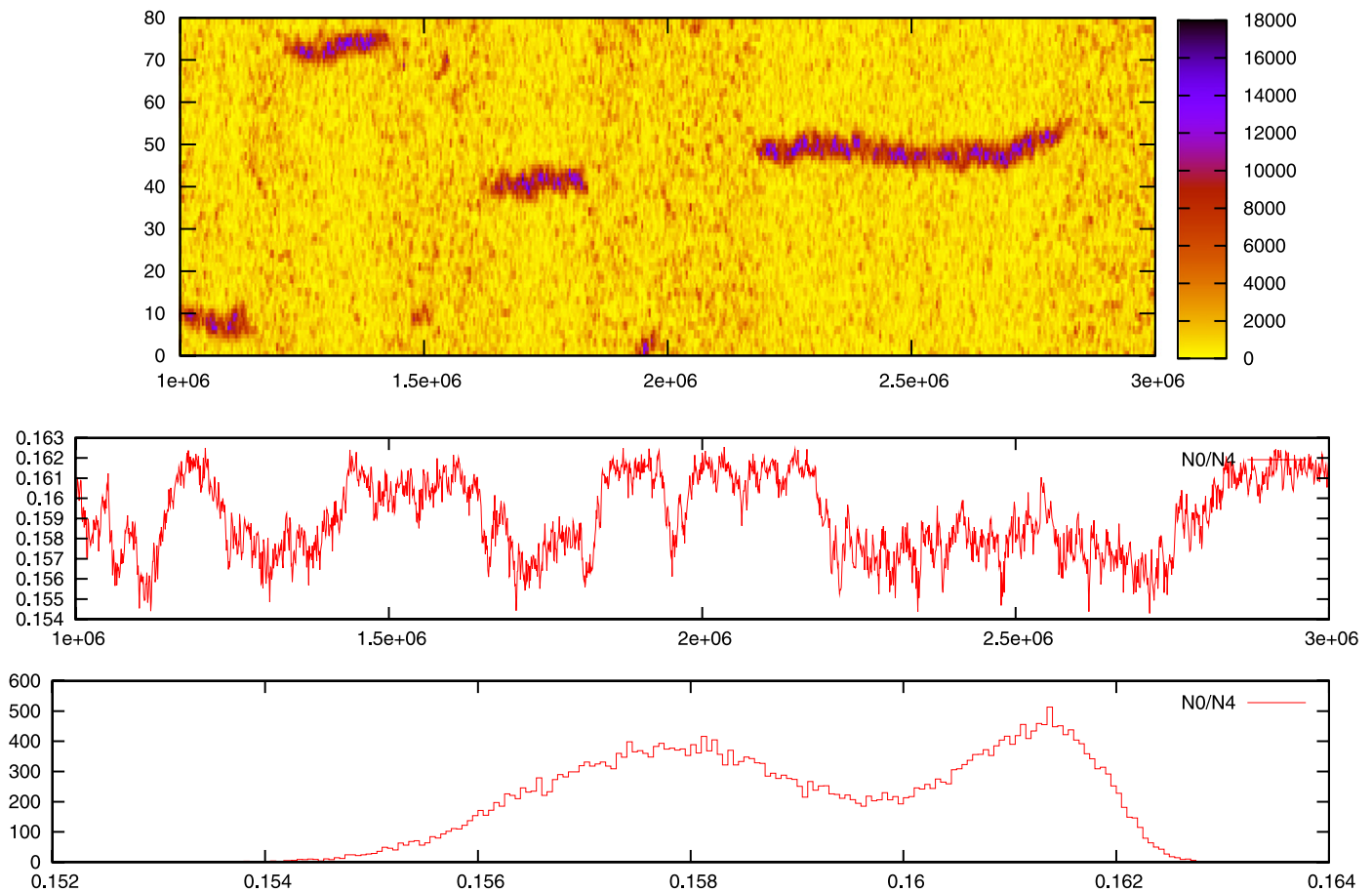


Fig. 2. Evidence for a first-order signal at the A–C phase transition. (For interpretation of the references to color in this figure, the reader is referred to the web version of this Letter.)

phases of the Lifshitz phase diagram.⁴ The triple point where the three phases meet is the so-called Lifshitz point.

The critical dimension beyond which the mean-field Lifshitz theory alluded to above is believed to be valid is $d_c = 4 + m/2$. In lower dimensions, the fluctuations play an important role and so does the number of components of the field ϕ . This does not necessarily affect the general structure of the phase diagram, but can alter the order of the transitions. Without entering into the details of the rather complex general situation, let us just mention that for $m = 1$ fluctuations will often turn the transition along the A–C phase boundary into a first-order transition. Likewise, most often the transition between phases B and C is of second order.

We have tried to determine the order of the transitions in the CDT phase diagram. Based on our numerical investigation so far, the A–C transition appears to be a first-order transition, while the B–C transition may be either of first or second order. This points to significant similarities with the Lifshitz results mentioned above, although we would like to stress that at this stage this is only at the level of analogy. We have not yet derived a gravitational analogue of the Landau free energy (2) governing the effective Lifshitz model. Also, a difference which may turn out to be important is that in our case the B–C phase transition line seems to end, and the endpoint may play a special role, as we will discuss below.

The two graphs at the bottom of Fig. 2 illustrate the behaviour of N_0/N_4 at the A–C phase transition line. Since we can approach this line by changing the coupling constant κ_0 while keeping Δ fixed, the quantity conjugate to κ_0 (N_4 is fixed), namely, the ratio N_0/N_4 , is a natural candidate for an order parameter. The graph at the centre of Fig. 2 shows N_0/N_4 as a function of Monte Carlo time. One sees clearly that it jumps between two values, corresponding to the distinct nature of geometry in phases A and C. We have checked that the geometry indeed “jumps” in the sense that no smoothly interpolating typical configurations have been found. Lastly, we have also established that the jump becomes more and more pronounced as the four-volume N_4 of the universe increases, further underlining the archetypical first-order behaviour at this transition line.

The top graph in Fig. 2 shows the location of the universe along the vertical “proper-time axis” ($t_n \in [0, 80]$, and to be periodically identified) as a function of Monte Carlo time, plotted along the horizontal axis. The value of the spatial three-volume $V_3(t_n)$ in the slice labeled by t_n is colour-coded; the darker, the bigger the volume at time t_n . We can distinguish two types of behaviour as a function of Monte Carlo time, (i) presence of an extended universe centred at and fluctuating weakly around some location on the proper-time axis; (ii) absence of such a universe with a well-defined “centre-of-volume”. The former is associated with the presence of a distinct dark band in the figure, which disappears abruptly as a function of Monte Carlo time, only to reappear at some different location t_n later on in the simulation. Comparing with the middle graph, it is clear that these abrupt changes in

⁴ For definiteness, we are using here a “magnetic” language for the Lifshitz diagram. However, the Lifshitz diagram can also describe a variety of other systems, for instance, liquid crystals.

$N_4=40k, \kappa_0=2.20, \Delta=0.0220$

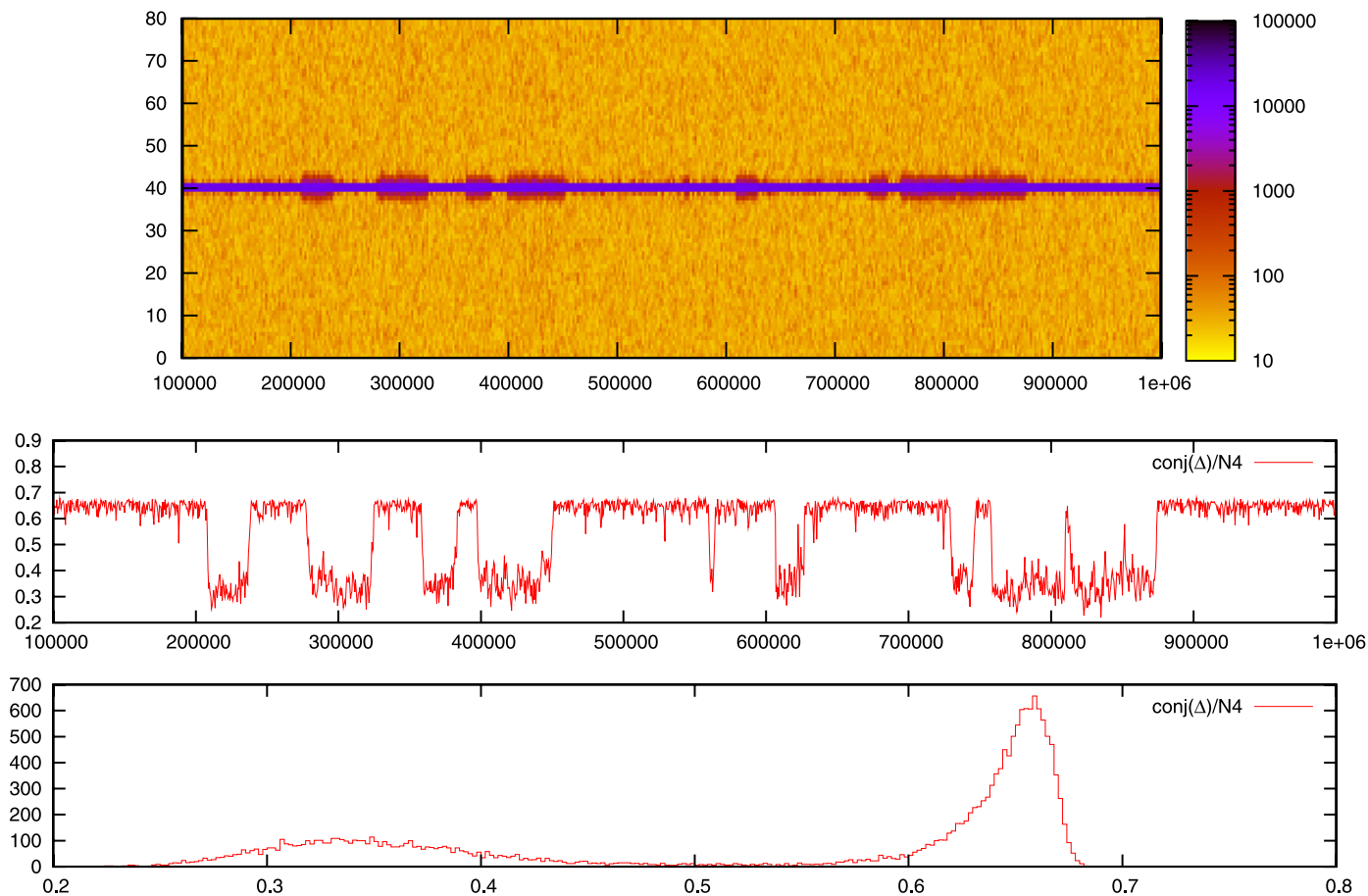


Fig. 3. Evidence for a first-order signal at the B–C phase transition – or is it? (For interpretation of the references to color in this figure, the reader is referred to the web version of this Letter.)

geometry correlate perfectly with the changes of the order parameter N_0/N_4 . When N_0/N_4 is small, we witness the extended de Sitter universe of phase C, whose “equator” coincides with the dark blue/red line of the colour plot. Conversely, at the larger values of N_0/N_4 characteristic of phase A this structure disappears, to be replaced by an array of universes too small to be individually identifiable on the plot. When jumping back to phase C the centre-of-volume of the single, extended universe reappears at a different location in time. Finally, the bottom graph in Fig. 2 illustrates the double-peak structure of the distribution of the values taken by the order parameter N_0/N_4 .

Our measurements to determine the character of the B–C transition are depicted in an analogous manner in Fig. 3. Since we are varying Δ to reach this transition from inside phase C, we have chosen the variable conjugate to Δ in the action (1) (up to a constant normalization N_4), $(-6N_0 + N_4^{(4,1)})/N_4$, as our order parameter. Looking at the graph at the centre, we see that this parameter exhibits the same jumping behaviour as a function of Monte Carlo time characteristic of a first-order transition. Small values of the parameter indicate the system is in phase C, while large values correspond to phase B. The time extent of the universe diminishes as one approaches the phase transition line from phase C, but it does not go to zero. It jumps to zero only when we cross the line. Some indication of this behaviour is given by the colour-coded three-volume profile $V_3(t)$ as function of the Monte Carlo time in the top graph of Fig. 3. In phase B, the entire “universe” is concentrated in a single slice, while in phase C it has a

nontrivial time extension. The bottom graph in Fig. 3 again shows the double-peak structure of the order parameter.

Looking at Fig. 3 and comparing it with the previous Fig. 2, the evidence for a first-order transition at the B–C phase boundary seems even more clear cut than in the case of the A–C transition. However, there is one set of measurements which potentially calls this result into question. By studying the strength of this signal systematically as a function of the total four-volume N_4 , we found that it becomes weaker with increasing N_4 ; the sharp jumps seen in Fig. 3 become blurred and the two peaks of the order parameter start to merge. We should therefore keep an open mind to the possibility that the observed behaviour is an artifact of using systems which are too small to see their true nature. Unfortunately it is presently not possible for us to use much larger systems, because the local Monte Carlo algorithm becomes quite inefficient as one approaches the transition, and the autocorrelation time grows rapidly with the four-volume.

We have not studied the A–B transition in any detail since it seems not interesting from a gravitational point of view, where we want to start out in a phase with an extended, quasi-classical four-dimensional universe, i.e. in phase C.

Finally, the order parameters used above, given in terms of the conjugates to the coupling constants varied in the action (1) cannot be identified directly with the “average geometry” alluded to in the introduction, although they must clearly capture some aspects of it. Some further speculations on the nature of the order parameter can be found in Section 5 below.

4. Relation to Hořava–Lifshitz and RG gravity

When we approach the B–C phase transition line from phase C by varying the coupling constant Δ , the time extent of the universe becomes smaller, when measured in units of discrete time steps (recall that we are keeping the four-volume constant). However, as argued in [19], the real time extent does not change all that much, since there is a compensating effect of “stretching” of individual four-simplices in the time direction due to an increase in α , the relative scaling between space- and time-like lattice spacings defined by $a_t^2 = \alpha a_s^2$ [11], and set by hand. (As mentioned above, for given values of κ_0 and κ_4 one can calculate Δ as a function of α .) In addition, one needs to keep in mind that the precise effective action underlying the form of the phase diagram in Fig. 1 is truly nonperturbative, since it results from the interplay of the bare action (1) and the measure (i.e. the entropy of the geometries as a function of $N_0, N_4^{(3,2)}, N_4^{(4,1)}$), and is therefore difficult to calculate. As a consequence, it may happen that the effective action changes *form* when we change κ_0 or even Δ . Since the CDT geometries, unlike their Euclidean counterparts, explicitly break the isotropy between space and time, a natural deformation could be “Hořava–Lifshitz-like”. In this case a potential effective Euclidean continuum action, including the measure, and expressed in terms of standard metric variables could be of the form

$$S_H = \frac{1}{G} \int d^3x dt N \sqrt{g} ((K_{ij}K^{ij} - \lambda K^2) + (-\gamma R^{(3)} + 2\Lambda + V(g_{ij}))), \quad (3)$$

where K_{ij} denotes the extrinsic curvature, g_{ij} the three-metric on the spatial slices, $R^{(3)}$ the corresponding 3d scalar curvature, N the lapse function, and finally V a “potential” which in Hořava’s formulation would contain higher orders of spatial derivatives, potentially rendering S_H renormalizable. The kinetic term depending on the extrinsic curvature is the most general such term which is at most second order in time derivatives and consistent with spatial diffeomorphism invariance. The parameter λ appears in the (generalized) DeWitt metric, which defines an ultralocal metric on the classical space of all three-metrics.⁵ The parameter γ can be related to a relative scaling between time and spatial directions. When $\lambda = \gamma = 1$ and $V = 0$ we recover the standard (Euclidean) Einstein–Hilbert action.

Making a simple mini-superspace ansatz with compact spherical slices, which assumes homogeneity and isotropy of the spatial three-metric g_{ij} , and fixing the lapse to $N = 1$, the Euclidean action (3) becomes a function of the scale factor $a(t)$ (see also [20–22]), that is,

$$S_{mini} = \frac{2\pi^2}{G} \int dt a(t)^3 \left(3(1 - 3\lambda) \frac{\dot{a}^2}{a^2} - \gamma \frac{6}{a} + 2\Lambda + \tilde{V}(a) \right). \quad (4)$$

The first three terms in the parentheses define the IR limit, while the potential term $\tilde{V}(a)$ contains inverse powers of the scale factor a coming from possible higher-order spatial derivative terms.

The results reported in [7,6,5] are compatible with the functional form of the mini-superspace action (4), but we were not able to determine $\tilde{V}(a)$, which could be important for small values of the scale factor. As outlined in [7], because of the nature of the dependence of the *renormalized* coupling constants on the bare

parameters κ_0, Δ , resolving shorter distances seems to necessitate a closer approach to the phase transition lines, with UV behaviour found along those lines. The A–C line is first order and thus not of interest. The order of the A–B line has not been determined, but since it cannot be approached from inside phase C which has good IR properties, it is currently not of interest either. This leaves the B–C transition line, whose character has not yet been established. If it is a first-order line, we are left with two potentially interesting points: the endpoint P_0 of the transition, where the phase transition is often of higher order than along the line itself, and the Lifshitz triple point P_t , where the transition also might be of higher order. On the other hand, if the line is second order, we can approach it anywhere.

One defining aspect of Hořava–Lifshitz gravity is the assumption that the scaling dimensions of space and time differ in the ultraviolet regime. This difference is used to construct a theory containing higher spatial derivatives in such a way that it is renormalizable. How would one observe such a difference in the present lattice approach? Consider a universe of time extent T , spatial extension L and total four-volume $V_4(T, L)$. By measuring T and L we can establish the mutual relations

$$T \propto V_4^{1/d_t}, \quad L \propto (V_4^{1-1/d_t})^{1/d_s} \propto T^{(d_t-1)/d_s}. \quad (5)$$

Well inside phase C we measured $d_t = 4$ and $d_s = 3$, in agreement with what is expected for an ordinary four-dimensional spacetime. If the dimension $[T]$ of time was z times the dimension $[L]$ of length we would have

$$z = \frac{d_s}{d_t - 1}. \quad (6)$$

We have seen that well inside phase B both d_s and d_t must be large, if not infinite. If the B–C phase transition is second order, it may happen that z goes to a value different from 1 when we approach the transition line. To investigate this possibility, we have tried to determine z as a function of the parameter Δ as $\Delta \rightarrow 0$. For $\Delta > 0.3$ one obtains convincingly $d_t \approx 4$ and $d_s \approx 3$ and thus $z \approx 1$, but for smaller Δ the quality of our results does not allow for any definite statements. Autocorrelation times seem to become very long and there may be large finite-volume effects, which obscure the measurements and which are precisely based on finite-size scaling. Hopefully the latter are more a function of our algorithms than indicating a need to go to much larger four-volumes.

5. Discussion

We have shown that the CDT phase diagram bears a striking resemblance to a Lifshitz phase diagram if we identify “average geometry” with the Lifshitz field ϕ in the heuristic sense discussed above. In our earlier papers, we tentatively interpreted phase A as being dominated by the wrong-sign kinetic term of the conformal factor of the continuum Euclidean Einstein–Hilbert action. It appears to be a Lorentzian remnant of a degeneracy found in the old Euclidean quantum gravity model based on (Euclidean) dynamical triangulations, which for large values of κ_0 exhibits a branched-polymer phase, likewise interpreted as caused by the dominance of the conformal factor (see, for example, [23]).

Eq. (2) strongly suggests an identification of the Lifshitz field ϕ with the conformal factor or some function thereof, such that the transition from phase C to phase A in the Lifshitz diagram is associated with a sign swap of the corresponding kinetic term from negative to positive. An effective action for the conformal mode coming out of a nonperturbative gravitational path integral would consist of (i) a contribution from the bare action (where the kinetic

⁵ The value of λ governs the signature of the generalized DeWitt metric

$$G_\lambda^{ijkl} = \frac{1}{2} \sqrt{\det g} (g^{ik}g^{jl} + g^{il}g^{jk} - 2\lambda g^{ij}g^{kl}),$$

which is positive definite for $\lambda < 1/3$, indefinite for $\lambda = 1/3$ and negative definite for $\lambda > 1/3$.

conformal term has the “wrong”, negative sign), (ii) a contribution from the measure, and (iii) contributions from integrating out other field components and, where applicable, other matter fields. It has been argued previously that the Faddeev–Popov determinants contributing to the gravitational path integral after gauge-fixing contribute effectively to the conformal kinetic term with the opposite, positive sign [24,25]. For example, when working in proper-time gauge, to imitate the time-slicing of CDT, Euclidean metrics can be decomposed according to⁶

$$ds^2 = d\tau^2 + e^{2\phi(\tau,x)} g_{ij}(\tau,x) dx^i dx^j, \quad (7)$$

giving rise to a term $-1/G^{(b)} e^{3\phi} \sqrt{\det g(\partial_\tau \phi)^2}$ in the bare gravity Lagrangian density, where $G^{(b)}$ is the bare Newton’s constant. According to [25], one expects that the leading contribution from the associated Faddeev–Popov determinant has the same functional form, but with a plus instead of a minus sign, and with a different dependence on $G^{(b)}$. The presence of contributions of opposite sign to the effective action for the conformal mode $\phi(\tau,x)$ can therefore lead to two different behaviours, depending on the value of $G^{(b)}$. Identifying the κ_0 of our lattice formulation – an a priori freely specifiable parameter – with the inverse of $G^{(b)}$, this mechanism can account exactly for the observed behaviour with regard to the transition between phases A and C.⁷

However, we have at this stage not constructed a gravity analogue of the Landau free energy density (2) incorporating all observed features of the CDT phase diagram. Also, as already mentioned above, due to finite-size effects and/or the inefficiency of our computer algorithms in this region we have not yet been able to establish the order of the B–C phase transition. In a Lifshitz diagram it is often of second order. If this scenario was realized in CDT quantum gravity too, any point on the line could potentially be associated with a continuum UV limit, thus implying the need for fine-tuning at least one other coupling constant (apart from the cosmological constant).

Engaging in a bit of speculation, an interesting possibility would be if the triple Lifshitz point corresponded to an asymmetric scaling between space and time, like in the Hořava model, while the endpoint of the B–C line represented an isotropic point associated with the RG asymptotic safety picture. Deciding which scenario is actually realized will require substantially longer simulations or improved updating algorithms, something we are working on presently. The analysis of the phase structure in the framework of causal dynamical triangulations we have performed here may turn out to provide a universal template for understanding non-perturbative theories of higher-dimensional, dynamical geometry, including “true” quantum gravity.

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⁶ The conformal decomposition of the spatial three-metric is essentially unique if one requires g_{ij} to have constant scalar curvature.

⁷ Related mechanisms have earlier been considered in the context of purely Euclidean dynamical triangulations by J. Smit [26].